# Testing Convexity and Acyclicity, and New Constructions for Dense Graph Embeddings 

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# ABSTRACT <br> Testing Convexity and Acyclicity, and New Constructions for Dense Graph Embeddings 

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Property testing, especially that of geometric and graph properties, is an ongoing area of research. In this thesis, we present a result from each of the two areas. For the problem of convexity testing in high dimensions, we give nearly matching upper and lower bounds for the sample complexity of algorithms have one-sided and two-sided error, where algorithms only have access to labeled samples independently drawn from the standard multivariate Gaussian. In the realm of graph property testing, we give an improved lower bound for testing acyclicity in directed graphs of bounded degree.

Central to the area of topological graph theory is the genus parameter, but the complexity of determining the genus of a graph is poorly understood when graphs become nearly complete. We summarize recent progress in understanding the space of minimum genus embeddings of such dense graphs. In particular, we classify all possible face distributions realizable by minimum genus embeddings of complete graphs, present new constructions for genus embeddings of the complete graphs, and find unified constructions for minimum triangulations of surfaces.

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## 1 Introduction

The bulk of this thesis consists of two distinct areas of inquiry: property testing and topological graph theory. In the first half, we describe results concerning property testing. Let $\mathcal{C}$ be a class of objects, e.g. Boolean functions, and let $\mathcal{P}$ be a property (e.g. monotonicity). We may think of $\mathcal{P}$ as simply a subset of the class of objects $\mathcal{C}$. Suppose we have a distance function dist : $\mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}$. We say that an object $c \in \mathcal{C}$ is $\varepsilon$-far from $\mathcal{P}$ for some $\varepsilon>0$ if

$$
\operatorname{dist}(c, \mathcal{P}):=\inf _{c^{\prime} \in \mathcal{P}} \operatorname{dist}\left(c, c^{\prime}\right)>\varepsilon
$$

Property testing concerns the following type of promise problem:

Given a class $\mathcal{C}$, property $\mathcal{P}$, and distance function $\operatorname{dist}(-,-)$, determine with high probability if an object $c \in \mathcal{C}$ is in $\mathcal{P}$ or $\varepsilon$-far from $\mathcal{P}$.

Of particular interest are algorithms that are able to correctly distinguish between these two cases with a sublinear number of queries or samples, i.e., by inspecting a small fraction of the input. This goal is typically impossible when trying to test a property exactly, but in many properties, the farness condition induces strong structural results that enable us to achieve this goal of efficiency. When an algorithm always accepts if $c$ is indeed in $\mathcal{P}$, we say that it has one-sided error, otherwise it has two-sided error.

We describe our work on the problem of high-dimensional convexity testing, where the unknown object is some body $S \subseteq \mathbb{R}^{n}$ and one wishes to determine if $S$ is convex or far from convex. In our model, algorithms are only given access to samples $(\boldsymbol{x}, S(\boldsymbol{x})) \in \mathbb{R}^{n} \times\{0,1\}$, where $\boldsymbol{x}$ is a point drawn from the standard multivariate Gaussian, and $S(\boldsymbol{x})$ is 1 if and only if $\boldsymbol{x} \in S$. We give upper and lower bounds that show that with regards to the dimension
$n$, the number of samples required is roughly $2^{n}$ for one-sided error, and $2^{\sqrt{n}}$ for two-sided error.

Another fruitful area is the testing of graph properties. The two most common models for graphs are the adjacency matrix and the incidence list representations. In the former, the algorithm is allowed to query whether there is an edge between two specific vertices. Given two graphs $G_{1}=\left(V, E_{1}\right), G_{2}=\left(V, E_{2}\right)$ on the same set of vertices of size $N$, the distance between the two graphs in this model is

$$
\operatorname{dist}\left(G_{1}, G_{2}\right)=\frac{\left|E_{1} \triangle E_{2}\right|}{N^{2}}
$$

where $\triangle$ denotes the symmetric difference. Roughly speaking, this is the number of edges one needs to delete or add to $G_{1}$ to obtain $G_{2}$, normalized by the total possible number of edges.

When the graphs of interest are sparse (i.e., have $o\left(N^{2}\right)$ many edges), this model is no longer suitable, as all such graphs are $\varepsilon$-close to the empty graph. An alternative model in this regime is the incidence list representation for bounded-degree graphs, which allows the algorithm to query the $i$ th edge of specific vertex, where $i$ is at most some constant degree bound $d$. Analogously, distance in this model is measured as

$$
\operatorname{dist}\left(G_{1}, G_{2}\right)=\frac{\left|E_{1} \triangle E_{2}\right|}{d N}
$$

The same notions carry over for directed graphs, where edges are given an orientation. We present a new lower bound for testing acyclicity in directed graphs of bounded degree. Namely, we show that any algorithm that has one-sided error with query access to outgoing edges needs at least $\Omega\left(N^{5 / 9-\delta}\right)$ queries, for any constant $\delta>0$.

The second half of this thesis details work on understanding genus embeddings of dense graphs. A well-studied property of graphs is that of planarity, i.e., being embeddable in the plane (or equivalently, the sphere). One of the more natural generalizations of planarity is the genus of a graph, which is defined to be the surface $S_{g}$ of minimum genus such that the graph embeds in $S_{g}$. While the class of planar graphs has several known characterizations, the situation for higher-genus surfaces is murkier. The problem of computing the genus is NP-hard in general, even for cubic graphs, but it is conjectured that not only should the genus be computable in polynomial time for very dense graphs, but that it should be exactly equal to the so-called Euler lower bound, a formula derived as an immediate consequence of the Euler polyhedral equation.

The most well-known evidence for this conjecture is that in the densest possible case, the Map Color Theorem of Ringel, Youngs, and others states that the genus of the complete graphs $K_{n}$ is

$$
\left\lceil\frac{(n-3)(n-4)}{12}\right\rceil
$$

The complete graphs, and many other families of dense graphs, have genera matching the Euler lower bound. However, it is currently not known whether the conjecture is true even for the complete graphs minus three edges. We provide more evidence towards this conjecture by exhibiting new constructions for minimum genus embeddings of dense graphs. Let $S$ be a surface and consider all the triangular embeddings of simple graphs in $S$. A minimum triangulation of $S$ is such a triangular embedding with the least number of vertices. In this thesis we present the following:

- a classification of all possible face distributions of genus embeddings of complete
graphs [Sun18],
- unified constructions for minimum triangulations and genus embeddings of complete graphs [Sun19b], and
- simplified constructions for genus embeddings of complete graphs, including a novel approach to constructing triangular embeddings of $K_{12 s}$ [Sun19a].

Much of our work concerns symmetric embeddings of dense graphs derived from the theory of current graphs. We present some new techniques for designing current graphs, and we also outline a search algorithm that enabled us to find many of the constructions presented herein. When used to give new genus embeddings of graphs whose genera were previously known, our families of current graphs are significantly simpler than those found in the existing literature.

The material for this thesis is derived from several papers of the author [CFSS17, Sun19a, Sun18, Sun19b] and ongoing work of the author with X. Chen, T. Randolph, and R. Servedio.

### 1.1 Related work

### 1.1.1 Property Testing

Testing convexity and related properties. Various kinds of high-dimensional objects have been studied, including probability distributions (e.g., [BKR04, RS05, ACS10, BFRV11, ADK15]) and Boolean functions (e.g., [BLR93, PRS02, Bla09, MORS10, KMS15]) A distinct line of work has focused on testing geometric properties, see e.g. [CSZ00, CS01, Ras03, BMR16b, BMR16c], mostly in the low-dimensional case (especially the two-dimensional
case).
Our work on testing presented here, namely that on the problem of high-dimensional convexity testing, falls under both of the topics mentioned above. Convexity is a fundamental property in high-dimensional geometry (see e.g. [GW93, Bal97, Sza06]) and has been studied in the property testing of images (i.e., the two-dimensional case) [Ras03, BMR16c, BMR16b, BMR16a], but little is known about high-dimensional convexity testing.

The aforementioned work on convexity testing [Ras03, BMR16a, BMR16b, BMR16c] was restricted to the 2-dimensional case under the uniform distribution over the unit square [BMR16a, BMR16b] or a discretization thereof [Ras03, BMR16c]. The model of [BMR16a, BMR16b] is more closely related to ours: [BMR16b] showed that $\Theta\left(\varepsilon^{-4 / 3}\right)$ samples are necessary and sufficient for one-sided sample-based testers, while [BMR16c] gave a onesided general tester (which can make adaptive queries to the unknown set) for 2-dimensional convexity with only $O(1 / \varepsilon)$ queries.

The high-dimensional case has received considerably less attention, and the only prior work we are aware of that is that of Rademacher and Vempala [RV05]. In their setting, algorithms are allowed to make membership queries on the unknown set $S$ and also receive random points guaranteed to be in $S$. Their main result is an algorithm which uses $(c n / \varepsilon)^{n}$ samples of the latter type of query to learn the unknown set $S$.

In $\mathbb{R}^{n}$, the standard multivariate Gaussian is a natural choice of measure, and several previous works have studied learning and property testing over this distribution, such as the work on testing halfspaces of [MORS10, BBBY12] and the work on testing surface area of [KNOW14, Nee14].

Sample-based testing. Various problems have been tackled with the weaker model of allowing testers to have access only to random samples, including the aforementioned works of Berman et al. [BMR16c, BMR16b, BMR16a] which study sample-based testing of convexity over two-dimensional domains. The model of sample-based testing was originally introduced by Goldreich, Goldwasser, and Ron almost two decades ago [GGR98], where it was referred to as "passive testing;" it has received significant attention over the years $\left[\mathrm{KR} 00, \mathrm{GGL}{ }^{+} 00\right.$, BBBY12, GR16], with an uptick in research activity in this model very recently [AHW16, BY16, BMR16c, BMR16b, BMR16a]. In earlier work on sample-based testing, Balcan et al. [BBBY12] gave a characterization of the sample complexity of (two-sided) sample-based testing, in terms of a quantity called the "passive testing dimension." Our two-sided upper and lower bounds (Theorem 2.3) may be interpreted as giving a bound on the passive testing dimension of the class of convex sets in $\mathbb{R}^{n}$ with respect to $\mathcal{N}(0,1)^{n}$.

Graph property testing. The adjacency matrix and incidence list query models were introduced by Goldreich et al. [GGR98] and Goldreich and Ron [GR97], respectively. Many natural graph properties in the incidence list model have query complexity that is dependent on $N$, the number of vertices. This is in stark contrast to the adjacency matrix model [GGR98], where the term "testable" refers to properties that have testers with query complexity independent of $N$. For example, bipartiteness testing has query complexity roughly $\tilde{\Theta}\left(N^{1 / 2}\right)$ [GR97, GR99]. Perhaps surprisingly, the query complexity of testing $H$ -minor-freeness [BSS10, HKNO09, LR15] is now known to be quasipolynomial in the error $\varepsilon$, with no dependence on the number of vertices.

Bender and Ron [BR02] consider the same bounded-degree model, except for directed
graphs. There are two major variants for directed graphs: either the algorithm can query both outgoing and incoming arcs of a vertex, or just the outgoing arcs. Clearly the former is at least as powerful as the latter, and Hellweg and Sohler [HS12] showed that some properties, like strong connectivity, can be tested much more efficiently when given access to incoming arc queries. Czumaj et al. [CPS16] showed that constant-query algorithms in the former model can be simulated with a sublinear number of queries in the latter model. Bender and Ron showed that any algorithm testing acyclicity in digraphs requires at least $\Omega\left(N^{1 / 3}\right)$ queries, even for two-sided error and access to both types of queries. No nontrivial upper bound is known.

### 1.1.2 Embeddings of dense graphs

The property of planarity in graphs is well-studied, with ties to the origins of graph theory. There are many known characterizations, e.g. Kuratowski's theorem [Kur30], which states that a graph is planar if and only if it does not contain a subdivision of $K_{5}$ or $K_{3,3}$. One of the most natural generalizations of planarity is the genus of a graph, which is the smallest integer $k$ such that the graph embeds in the surface of genus $k$. The Robertson-Seymour theorem [RS04] shows the existence of a generalized Kuratowski's theorem, but an explicit list of forbidden graphs is known only for the plane and the projective plane [Arc81].

The Map Color Theorem was largely proven during the 1960s by Ringel, Youngs, and others [Gus63, RY69b, RY69c, RY69a, You70a, You70b, TWY67, TWY70, May69], using the theory of current graphs to exhibit a single minimum genus embedding of each complete graph. Various attempts at simplifying parts of the proof (e.g. Guy and Youngs [GY73a] and the work of Jungerman) are chronicled in Ringel's monograph [Rin74]. It is now
known that complete graphs have exponentially many nonisomorphic minimum genus embeddings [BGGŠ00, KV01, KV02, Kor08, GRŠ07] but many seemingly straightforward and natural follow-ups remain open.
"Case 0 ," the family of complete graphs on $K_{12 s}$ is treated separately from the other cases in the Map Color Theorem because of its resolution using nonabelian current graphs. Terry et al. [TWY67] elegantly leverage the representation theory of finite fields in their current graph constructions, but if one wishes to write down their derived embeddings explicitly, it is not known how to efficiently construct such a representation deterministically (see [AL86]). Pengelley and Jungerman [PJ79] and Korzhik [Kor08] attempted to alleviate this by finding solutions using cyclic current groups, but these families of index 4 current graphs are much more complicated than the original approach of Terry et al. [TWY67].

Thomassen [Tho89, Tho97, Tho93] showed that computing the genus of a graph is NPhard, solving a longstanding open problem of Garey and Johnson [GJ02]. However, the wealth of minimum genus embeddings of complete graphs suggests that perhaps the problem for computing the genus of dense graphs might be easier. Indeed, Mohar and Jing [MJ18] gave an efficient polynomial-time approximation scheme for the genus of such graphs.

A problem "dual" to the Map Color Theorem is the problem of minimum triangulations for surfaces, where one wishes to determine the smallest number of vertices needed to triangulate the surface with a simple graph. Jungerman and Ringel [JR80] gave a complete solution to this problem using a complicated combination of current graphs.

## 2 Sample-based high-dimensional convexity testing

In this chapter we focus on sample-based testing algorithms for convexity. Recall that in this model, a testing algorithm has access to independent draws $(\boldsymbol{x}, S(\boldsymbol{x})) \in \mathbb{R}^{n} \times\{0,1\}$, where $\boldsymbol{x}$ is drawn from $\mathcal{N}(0,1)^{n}$ and $S \subseteq \mathbb{R}^{n}$ is the unknown set being tested for convexity (so in particular the algorithm cannot select points to be queried) with $S(\boldsymbol{x})=1$ if $\boldsymbol{x} \in S$. We say such an algorithm is an $\varepsilon$-tester for convexity if it accepts $S$ with probability at least $2 / 3$ when $S$ is convex and rejects with probability at least $2 / 3$ when it is $\varepsilon$-far from convex.

Our ambient underlying space is $\mathbb{R}^{n}$ equipped with the standard Gaussian measure $\mathcal{N}(0,1)^{n}$, so the distance $\operatorname{dist}(S, C)$ between two subsets $S, C \subseteq \mathbb{R}^{n}$ is $\operatorname{Pr}_{\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}}[\boldsymbol{x} \in S \triangle C]$, where $S \triangle C$ denotes their symmetric difference.

We consider sample-based testers for convexity that are allowed both one-sided (i.e., the algorithm always accepts $S$ when it is convex) and two-sided error. In each case, for constant $\varepsilon>0$ we give nearly matching upper and lower bounds on sample complexity. Our results are as follows:

Theorem 2.1 (One-sided lower bound). Any one-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ for some $\varepsilon<1 / 2$ must use $2^{\Omega(n)}$ samples.

Theorem 2.2 (One-sided upper bound). For any $\varepsilon>0$, there is a one-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ which uses $(n / \varepsilon)^{O(n)}$ samples.

Theorem 2.3 (Two-sided lower bound). There exists a positive constant $\varepsilon_{0}$ such that any two-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ for some $\varepsilon \leq \varepsilon_{0}$ must use $2^{\Omega(\sqrt{n})}$ samples.

| Model | Sample complexity bound | Reference |
| :--- | :--- | :--- |
| One-sided | $2^{\Omega(n)}$ samples (for $\varepsilon<1 / 2$ ) | Theorem 2.1 |
|  | $2^{O(n \log (n / \varepsilon))}$ samples | Theorem 2.2 |
| Two-sided | $2^{\Omega(\sqrt{n})}$ samples (for $\varepsilon<\varepsilon_{0}$ ) | Theorem 2.3 |
|  | $2^{O\left(\sqrt{n} \log (n) / \varepsilon^{2}\right)}$ samples | Theorem 2.4 |

Table 1: Sample complexity bounds for sample-based convexity testing.

Theorem 2.4 (Two-sided upper bound). For any $\varepsilon>0$, there is a two-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ which uses $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ samples.

We will prove Theorems 2.1, 2.2, 2.3 and 2.4 in Sections 2.5, 2.3, 2.4 and 2.6, respectively. These results are summarized above in Table 1.

One-sided lower bound. Our one-sided lower bound has a simple proof using only elementary geometric and probabilistic arguments. It follows from the fact (see Lemma 2.30) that if $q=2^{\Theta(n)}$ many points are drawn independently from $\mathcal{N}(0,1)^{n}$, then with probability $1-o(1)$ no one of the points lies in the convex hull of the $q-1$ others. This can easily be shown to imply that more than $q$ samples are required (since given only $q$ samples, with probability $1-o(1)$ there is a convex set consistent with any labeling and thus a one-sided algorithm cannot reject).

Two-sided lower bound. At a high-level, the proof of our two-sided lower bound uses the following standard approach. We first define two distributions $\mathcal{D}_{\text {yes }}$ and $\mathcal{D}_{\text {no }}$ over sets in $\mathbb{R}^{n}$ such that (i) $\mathcal{D}_{\text {yes }}$ is a distribution over convex sets only, and (ii) $\mathcal{D}_{\mathrm{no}}$ is a distribution such that $\boldsymbol{S} \leftarrow \mathcal{D}_{\mathrm{no}}$ is $\varepsilon_{0}$-far from convex with probability at least $1-o(1)$ for some positive constant $\varepsilon_{0}$. We then show that every sample-based, $q$-query algorithm $A$ with $q=2^{0.01 n}$
must have

$$
\begin{equation*}
\underset{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes } ; \boldsymbol{x}}}{\operatorname{Pr}}[A \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))]-\underset{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }} ; \boldsymbol{x}}{\mathbf{P r}}[A \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))] \leq o(1), \tag{1}
\end{equation*}
$$

where $\boldsymbol{x}$ denotes a sequence of $q$ points drawn from $\mathcal{N}(0,1)^{n}$ independently and $(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))$ denotes the $q$ labeled samples from $\boldsymbol{S}$. Theorem 2.3 follows directly from (1).

To draw a set $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$, we sample a sequence of $N=2^{\sqrt{n}}$ points $\mathbf{y}_{1}, \ldots, \mathbf{y}_{N}$ from the sphere $S^{n-1}(r)$ of radius $r$ for some $r=\Theta\left(n^{1 / 4}\right)$. Each $\mathbf{y}_{i}$ defines a halfspace $\boldsymbol{h}_{i}=\left\{x: x \cdot \mathbf{y}_{i} \leq\right.$ $\left.r^{2}\right\} . \boldsymbol{S}$ is then the intersection of all $\boldsymbol{h}_{i}$ 's. (This is essentially a construction used by Nazarov [Naz03] to exhibit a convex set that has large Gaussian surface area, and used by [KOS07] to lower bound the sample complexity of learning convex sets under the Gaussian distribution.) The most challenging part of the two-sided lower bound proof is to show that, with $q$ points $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{q} \leftarrow \mathcal{N}(0,1)^{n}$, the $q$ bits $\boldsymbol{S}\left(\boldsymbol{x}_{1}\right), \ldots, \boldsymbol{S}\left(\boldsymbol{x}_{q}\right)$ with $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$ are "almost" independent. More formally, the $q$ bits $\boldsymbol{S}\left(\boldsymbol{x}_{1}\right), \ldots, \boldsymbol{S}\left(\boldsymbol{x}_{q}\right)$ with $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$ have $o(1)$-total variation distance from $q$ independent bits with the $i$ th bit drawn from the marginal distribution of $\boldsymbol{S}\left(\boldsymbol{x}_{i}\right)$ as $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$. On the other hand, it is relatively easy to define a distribution $\mathcal{D}_{\text {no }}$ that satisfies (ii) and at the same time, $\boldsymbol{S}\left(\boldsymbol{x}_{1}\right), \ldots, \boldsymbol{S}\left(\boldsymbol{x}_{q}\right)$ when $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ has $o(1)$-total variation distance from the same product distribution. (1) follows by combining the two parts.

Structural result. Our algorithms rely on a new structural result which we establish for convex sets in $\mathbb{R}^{n}$. Roughly speaking, this result gives an upper bound on the Gaussian volume of the "thickened surface" of any bounded convex subset of $\mathbb{R}^{n}$; it is inspired by, and builds on, the classic result of Ball [Bal93] that gives an upper bound for the Gaussian surface area of any convex subset of $\mathbb{R}^{n}$.

One-sided upper bound. Our one-sided testing algorithm employs a "grid-based" approach to decompose the relevant portion of $\mathbb{R}^{n}$ (namely, those points which are not too far from the origin) into a collection of disjoint cubes. It draws samples and identifies a subset of these cubes as a proxy for the "thickened surface" of the target set; by the structural result sketched above, if the Gaussian volume of this thickened surface is too high, then the one-sided algorithm can safely reject (as the target set cannot be convex). Otherwise the algorithm does random sampling to probe for points which are inside the convex hull of positive examples it has received but are labeled negative (there should be no such points if the target set is indeed convex, so if such a point is identified, the one-sided algorithm can safely reject). If no such points are identified, then the algorithm accepts.

Two-sided upper bound. Finally, the main tool we use to obtain our two-sided testing algorithm is a learning algorithm for convex sets with respect to the normal distribution over $\mathbb{R}^{n}$. The main result of $[\mathrm{KOS} 07]$ is an (improper) algorithm which learns the class of all convex subsets of $\mathbb{R}^{n}$ to accuracy $\varepsilon$ using $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ independent samples from $\mathcal{N}(0,1)^{n}$. Using the structural result mentioned above, we show that this can be converted into a proper algorithm for learning convex sets under $\mathcal{N}(0,1)^{n}$, with essentially no increase in the sample complexity. Given this proper learning algorithm, a two-sided algorithm for testing convexity follows from the well-known result of [GGR98] which shows that proper learning for a class of functions implies (two-sided) testability.

### 2.1 Preliminaries and Notation

Notation. We use boldfaced letters such as $\boldsymbol{x}, \boldsymbol{f}, \mathbf{A}$, etc. to denote random variables (which may be real-valued, vector-valued, function-valued, set-valued, etc; the intended type will be clear from the context). We write " $\boldsymbol{x} \leftarrow \mathcal{D}$ " to indicate that the random variable $\boldsymbol{x}$ is distributed according to probability distribution $\mathcal{D}$. Given $a, b, c \in \mathbb{R}$ we use $a=b \pm c$ to indicate that $b-c \leq a \leq b+c$.

Geometry. For $r>0$, we write $S^{n-1}(r)$ to denote the origin-centered sphere of radius $r$ in $\mathbb{R}^{n}$ and $\operatorname{Ball}(r)$ to denote the origin-centered ball of radius $r$ in $\mathbb{R}^{n}$, i.e.,

$$
S^{n-1}(r)=\left\{x \in \mathbb{R}^{n}:\|x\|=r\right\} \quad \text { and } \quad \operatorname{Ball}(r)=\left\{x \in \mathbb{R}^{n}:\|x\| \leq r\right\}
$$

where $\|x\|$ denotes the $\ell_{2}$-norm $\|\cdot\|_{2}$ of $x \in \mathbb{R}^{n}$. We also write $S^{n-1}$ for the unit sphere $S^{n-1}(1)$.

Recall that a set $C \subseteq \mathbb{R}^{n}$ is convex if $x, y \in C$ implies $\alpha x+(1-\alpha) y \in C$ for all $\alpha \in[0,1]$. We write $\mathcal{C}_{\text {convex }}$ to denote the class of all convex sets in $\mathbb{R}^{n}$. Recall that convex sets are Lebesgue measurable. Given a set $C \subseteq \mathbb{R}^{n}$ we write $\operatorname{Conv}(C)$ to denote the convex hull of $C$.

For sets $A, B \subseteq \mathbb{R}^{n}$, we write $A+B$ to denote the Minkowski sum $\{a+b: a \in A$ and $b \in$ $B\}$. For a set $A \subseteq \mathbb{R}^{n}$ and $r>0$ we write $r A$ to denote the set $\{r a: a \in A\}$. Given a point $a$ and a set $B \subseteq \mathbb{R}^{n}$, we use $a+B$ and $B-a$ to denote $\{a\}+B$ and $B+\{-a\}$ for convenience. For a convex set $C$, we write $\partial C$ to denote its boundary, i.e. the set of points $x \in \mathbb{R}^{n}$ such that for all $\delta>0$, the set $x+\operatorname{Ball}(\delta)$ contains at least one point in $C$ and at least one point outside $C$.

Probability. We use $\mathcal{N}(0,1)^{n}$ to denote the standard $n$-dimensional Gaussian distribution with zero mean and identity covariance matrix. We also recall that the probability density function for the one-dimensional Gaussian distribution is

$$
\varphi(x)=\frac{1}{\sqrt{2 \pi}} \cdot \exp \left(-x^{2} / 2\right)
$$

Sometimes we denote $\mathcal{N}(0,1)^{n}$ by $\mathcal{N}^{n}$ for convenience. The squared norm $\|\boldsymbol{x}\|^{2}$ of $\boldsymbol{x} \leftarrow$ $\mathcal{N}(0,1)^{n}$ is distributed according to the chi-squared distribution $\chi_{n}^{2}$ with $n$ degrees of freedom. The following tail bound for $\chi_{n}^{2}$ (see [Joh01]) will be useful:

Lemma 2.5 (Tail bound for the chi-squared distribution). Let $\mathbf{X} \leftarrow \chi_{n}^{2}$. Then we have

$$
\operatorname{Pr}[|\mathbf{X}-n| \geq t n] \leq e^{-(3 / 16) n t^{2}}, \quad \text { for all } t \in[0,1 / 2)
$$

All target sets $S \subseteq \mathbb{R}^{n}$ to be tested for convexity are assumed to be Lebesgue measurable and we write $\operatorname{Vol}(S)$ to denote $\operatorname{Pr}_{\boldsymbol{x} \leftarrow \mathcal{N}^{n}}[\boldsymbol{x} \in S]$, the Gaussian volume of $S \subseteq \mathbb{R}^{n}$. Given two Lebesgue measurable subsets $S, C \subseteq \mathbb{R}^{n}$, we view $\operatorname{Vol}(S \triangle C)$ as the distance between $S$ and $C$, where $S \triangle C$ is the symmetric difference of $S$ and $C$. Given $S \subseteq \mathbb{R}^{n}$, we abuse the notation and use $S$ to denote the indicator function of the set, so we may write " $S(x)=1$ " or " $x \in S$ " to mean the same thing.

We say that a subset $\mathcal{C}$ of $\mathcal{C}_{\text {convex }}$ is a $\tau$-cover of $\mathcal{C}_{\text {convex }}$ if for every $C \in \mathcal{C}_{\text {convex }}$, there exists a set $C^{\prime} \in \mathcal{C}$ such that $\operatorname{Vol}\left(C \triangle C^{\prime}\right) \leq \tau$.

Given a convex set $C$ and a real number $h>0$, we let $C_{h}$ denote the set of points in $\mathbb{R}^{n}$ whose distance from $C$ do not exceed $h$. We recall the following theorem of Ball [Bal93] (also see [Naz03]).

Theorem 2.6 ([Bal93]). For any convex set $C \subseteq \mathbb{R}^{n}$ and $h>0$, we have

$$
\frac{\operatorname{Vol}\left(C_{h} \backslash C\right)}{h} \leq 4 n^{1 / 4}
$$

Sample-based property testing. Given a point $x \in \mathbb{R}^{n}$, we refer to $(x, S(x)) \in \mathbb{R}^{n} \times$ $\{0,1\}$ as a labeled sample from a set $S \subseteq \mathbb{R}^{n}$. A sample-based testing algorithm for convexity is a randomized algorithm which is given as input an accuracy parameter $\varepsilon>0$ and access to an oracle that, each time it is invoked, generates a labeled sample $(\boldsymbol{x}, S(\boldsymbol{x}))$ from the unknown (Lebesgue measurable) target set $S \subseteq \mathbb{R}^{n}$ with $\boldsymbol{x}$ drawn independently each time from $\mathcal{N}(0,1)^{n}$. When run with any Lebesgue measurable $S \subseteq \mathbb{R}^{n}$, such an algorithm must output "accept" with probability at least $2 / 3$ (over the draws it gets from the oracle and its own internal randomness) if $S \in \mathcal{C}_{\text {convex }}$ and must output "reject" with probability at least $2 / 3$ if $S$ is $\varepsilon$-far from being convex, meaning that for every $C \in \mathcal{C}_{\text {convex }}$ it is the case that $\operatorname{Vol}(S \triangle C) \geq \varepsilon$. (We also refer to an algorithm as an $\varepsilon$-tester for convexity if it works for a specific accuracy parameter $\varepsilon$.) Such a testing algorithm is said to be one-sided if whenever it is run on a convex set $S$ it always outputs "accept;" equivalently, such an algorithm can only output "reject" if the labeled samples it receives are not consistent with any convex set. A testing algorithm which is not one-sided is said to be two-sided.

Throughout the rest of this section we reserve the symbol $S$ to denote the unknown target set (a measurable subset of $\mathbb{R}^{n}$ ) that is being tested for convexity. If $S(x)=1$ then we say that $x$ is a positive point, and if $S(x)=0$ we say $x$ is a negative point.

Given a finite set $T$ of labeled samples $(x, b)$ with $x \in \mathbb{R}^{n}$ and $b \in\{0,1\}$, we say $x$ is a positive point in $T$ if $(x, 1) \in T$ and is a negative point in $T$ if $(x, 0) \in T$. We use $T^{+}$to denote the set of positive points $\{x:(x, 1) \in T\}$, and $T^{-}$to denote the set of negative points

$$
\{x:(x, 0) \in T\} .
$$

### 2.2 A structural result on the boundaries of convex bodies

For a bounded convex set $C$ in $\mathbb{R}^{n}$ (i.e., $\sup _{c \in C}\|c\| \leq K$ for some real $K$ ) we may view $\partial C+\operatorname{Ball}(\alpha)$ as the " $\alpha$-thickened boundary" of $C$. In this section, we use Theorem 2.6 of [Bal93] to give an upper bound on the volume of the $\alpha$-thickened boundary of such a set:

Theorem 2.7. If $C \subset \mathbb{R}^{n}$ is convex and $\sup _{c \in C}\|c\| \leq K$ for some $K>1$, then we have

$$
\operatorname{Vol}(\partial C+\operatorname{Ball}(\alpha)) \leq 20 n^{5 / 8} K \sqrt{\alpha}, \quad \text { for any } 0<\alpha<n^{-3 / 4}
$$

Having such a bound will be useful to us in two different contexts. First, it plays an important role in the proof of correctness of our one-sided algorithm for testing convexity (see Section 2.3). Second, as an easy consequence of the theorem, we get an algorithm which, for any $\tau>0$, constructs a $\tau$-cover of $\mathcal{C}_{\text {convex }}$ (this is Corollary 2.34 , which we defer to later as its proof employs a "gridding" argument which we introduce in Section 2.3). This cover construction algorithm plays an important role in our two-sided algorithm for testing convexity (see Section 2.6).

### 2.2.1 Some calculations in convex geometry

Here we collect some technical results on convex bodies needed for proving Theorem 2.7.

Lemma 2.8. If $C \subset \mathbb{R}^{n}$ is convex and contains no ball of radius $\rho$, then we have

$$
\operatorname{Vol}(C+\operatorname{Ball}(\alpha)) \leq 2(n \rho+\alpha)
$$

Proof. By John's theorem [Joh48] (see also Theorem 3.1 of [Bal97]), there is a unique ellipsoid contained in $C$ that has maximal Euclidean volume; let us denote this by $E(C)$. Since $C$ does not contain a ball of radius $\rho, E(C)$ must have some axis $u$ which has length less than $\rho$. Let us translate $C$ so that the center of $E(C)$ lies at the origin. Again by John's theorem (see the discussion in [Bal97] on pages 13 and 16), we have that $C \subseteq n E(C)$. Now consider the set $H$ of all points $v \in \mathbb{R}^{n}$ whose projection onto the $u$ direction has magnitude at most $n \rho+\alpha$. This is a "thickened hyperplane" which contains $C+\operatorname{Ball}(\alpha)$, and its Gaussian volume is given by

$$
\operatorname{Vol}(H)=\int_{-(n \rho+\alpha)}^{(n \rho+\alpha)} \varphi(x) d x
$$

where $\varphi(x)$ is the density function of a univariate normal distribution as defined in Section 2.1. We know that $\phi$ is bounded from above by 1 so this integral is at most $2(n \rho+\alpha)$. It is also easy to see that the same volume upper bound must hold upon undoing the translation of $C$ back to its original position, and the lemma is proved.

Lemma 2.9. Let $C$ be a bounded convex subset of $\mathbb{R}^{n}$ that contains $\operatorname{Ball}(\rho)$, the origincentered ball of radius $\rho$, for some $\rho>\alpha$. Then the Euclidean distance between any point in $(1-(\alpha / \rho)) C$ and $\partial C$ is at least $\alpha$.

Proof. This is essentially Lemma 2.2 of [Ker92]; for completeness we give the simple proof here.

Let $\beta=\alpha / \rho$. Let $z \in \partial C$ be a point on the boundary of $C$. Since $C$ is convex and contains the origin, there exists a vector $v$ for which $v \cdot z=1$ but for all $x \in C$ we have $v \cdot x \leq 1$ (intuitively, one can think of $v$ as defining a supporting hyperplane at $z$ of the convex body $C$ ). Then for any $y \in(1-\beta) C$ we have $v \cdot y \leq 1-\beta$, which implies that
$v(z-y) \geq \beta$. Since $\rho v /\|v\| \in \operatorname{Ball}(\rho) \subseteq C$, it must be the case that $v \cdot(\rho v /\|v\|)=\rho\|v\| \leq 1$, which means that $\|v\| \leq 1 / \rho$ and thus $(\operatorname{as} v(z-y) \geq \beta)\|z-y\| \geq \alpha$.

Lemma 2.10. Let $C \subset \mathbb{R}^{n}$ be a convex set that satisfies $\sup _{c \in C}\|c\| \leq K$ for some $K>1$. Then for any $0<\beta<1$, every point $v \in \partial C+\operatorname{Ball}(\alpha)$ is within Euclidean distance $2 K \beta+\alpha$ of a point in $(1-\beta) C$.

Proof. We have that $v=c+y$ for some $c \in \partial C$ and $y$ with $\|y\| \leq \alpha$. While $v$ may not lie in $C$ (as $C$ might be an open set), we know for any $\varepsilon>0$ there is a point $c^{\prime} \in C$ and $\left\|c^{\prime}-c\right\| \leq \varepsilon$. Take such a point $c^{\prime}$ with $\varepsilon=\beta K$. Then $(1-\beta) c^{\prime} \in(1-\beta) C$ and
$\left\|(1-\beta) c^{\prime}-v\right\|=\left\|(1-\beta) c^{\prime}-c-y\right\| \leq\left\|c^{\prime}-c\right\|+\beta\left\|c^{\prime}\right\|+\|y\| \leq \beta K+\beta K+\alpha=2 \beta K+\alpha$.

This finishes the proof of the lemma.

### 2.2.2 Proof of Theorem 2.7

Let $C \subset \mathbb{R}^{n}$ be a bounded convex set that satisfies $\sup _{c \in C}\|c\| \leq K$ for some $K>1$.
The proof has two cases and uses Lemmas 2.8, 2.9, and 2.10 to be proved later.

Case I: $C$ contains no ball of radius $\rho:=\sqrt{\alpha} / n^{3 / 8}$. In this case we have

$$
\begin{array}{rlrl}
\operatorname{Vol}(\partial C+\operatorname{Ball}(\alpha)) \leq \operatorname{Vol}(C+\operatorname{Ball}(\alpha)) & \leq 2(n \rho+\alpha) & & (\text { Lemma 2.8 }) \\
& \leq 3 n^{5 / 8} \sqrt{\alpha} & \left(\text { using } \alpha<n^{-3 / 4}\right) \\
& <20 n^{5 / 8} K \sqrt{\alpha} & & (\text { using } K>1)
\end{array}
$$

Case II: $C$ contains some ball of radius $\rho$. We let $z^{*}$ be the center of such a ball and let $D=\partial C+\operatorname{Ball}(\alpha)$. To upper-bound $\operatorname{Vol}(D)$, we define a set that contains $D$ and then upper-bound its volume.

To this end, we first shift $C$ to get $C^{\prime}=C-z^{*}$ (so that the ball of radius $\rho$ is now centered at the origin). By the triangle inequality we have $\sup _{c \in C^{\prime}}\|c\| \leq 2 K$. Let $\beta=n^{3 / 8} \sqrt{\alpha}=\alpha / \rho$, and observe that since $\alpha<n^{-3 / 4}$ we have $\beta<1$. Let $D^{\prime}=D-z^{*}=\partial C^{\prime}+\operatorname{Ball}(\alpha)$. By Lemma 2.9, we have

$$
C_{0}^{\prime}:=(1-\beta) C^{\prime}=(1-\beta)\left(C-z^{*}\right)
$$

contains no point of $D^{\prime}$, and then by Lemma 2.10 the set $C_{1}^{\prime}:=\left(C_{0}^{\prime}\right)_{h}$ with $h=4 \beta K+\alpha$ contains all of $D^{\prime} .{ }^{1}$ As a result, $D^{\prime} \subseteq C_{1}^{\prime} \backslash C_{0}^{\prime}$ and it suffices to upperbound $\operatorname{Vol}\left(z^{*}+C_{1}^{\prime} \backslash C_{0}^{\prime}\right)$, which is at most $4 h n^{1 / 4}$ by Theorem 2.6 , since $C_{0}^{\prime}$ is convex. Combining everything together, we have

$$
\operatorname{Vol}(D) \leq \operatorname{Vol}\left(z^{*}+C_{1}^{\prime} \backslash C_{0}^{\prime}\right) \leq(4 \beta K+\alpha)\left(4 n^{1 / 4}\right) \leq 20 n^{5 / 8} K \sqrt{\alpha}
$$

(again using $K>1$ and $\alpha<n^{-3 / 4}$ for the last inequality).

### 2.3 One-sided upper bound: Proof of Theorem 2.2

Recall Theorem 2.2:

Theorem. For any $\varepsilon>0$, there is a one-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ which uses $(n / \varepsilon)^{O(n)}$ samples.

In Section 2.3 .1 we show that it suffices to test convex bodies contained in a large ball $B$ centered at the origin (rather than all of $\mathbb{R}^{n}$ ) and give some useful preliminaries. Section 2.3.2 then builds on Theorem 2.7 (the upper bound on the volume of the "thickened boundary"
${ }^{1}$ Recall that $\left(C_{0}^{\prime}\right)_{h}$ is the set of all points that have distance at most $h$ to $C_{0}^{\prime}$. Also note that the coefficient of $\beta K$ in our choice of $h$ is 4 instead of 2 since we have $\sup _{c \in C^{\prime}}\|c\| \leq 2 K$ instead of $K$.
of any bounded convex body) to give an upper bound, in the case that $S$ is convex and contained in $B$, on the total volume of certain "boundary cubes" (defined in Section 2.3.1). In Section 2.3.3 we present the one-sided testing algorithm and establish its correctness, thus proving Theorem 2.2.

### 2.3.1 Setup

Let $n^{\prime}$ be the following parameter (that depends on both $n$ and $\varepsilon$ ):

$$
n^{\prime}:=(n+4 \sqrt{n \ln (4 / \varepsilon)})^{1 / 2} .
$$

Let $\mathcal{C}_{\text {convex }}^{\prime}$ denote the set of convex bodies in $\mathbb{R}^{n}$ that are contained in $\operatorname{Ball}\left(n^{\prime}\right)$, equivalently,

$$
\mathcal{C}_{\text {convex }}^{\prime}=\left\{C \cap \operatorname{Ball}\left(n^{\prime}\right): C \in \mathcal{C}_{\text {convex }}\right\} .
$$

We prove the following claim that helps us focus on testing of $\mathcal{C}_{\text {convex }}^{\prime}$ instead $\mathcal{C}_{\text {convex }}$.

Claim 2.11. Suppose that there is a one-sided sample-based $\varepsilon$-testing algorithm $A^{\prime}$ which, given any Lebesgue measurable target set $S$ contained in $\operatorname{Ball}\left(n^{\prime}\right)$, uses $(n / \varepsilon)^{O(n)}$ samples drawn from $\mathcal{N}(0,1)^{n}$ to test whether $S \in \mathcal{C}_{\text {convex }}^{\prime}$ versus $S$ is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}^{\prime}$. Then this implies Theorem 2.2.

Proof. Given $A^{\prime}$ for $\mathcal{C}_{\text {convex }}^{\prime}$, we consider an algorithm $A$ which works as follows to test whether an arbitrary Lebesgue measurable subset $S$ of $\mathbb{R}^{n}$ is convex or $\varepsilon$-far from $\mathcal{C}_{\text {convex }}$ : algorithm $A$ runs $A^{\prime}$ with parameter $\varepsilon / 2$, but with the following modification: each time $A^{\prime}$ receives from the oracle a labeled sample $(x, b)$ with $x \notin \operatorname{Ball}\left(n^{\prime}\right)$, it replaces the label $b$ with 0 and gives the modified labeled sample to $A^{\prime}$. When the run of $A^{\prime}$ is complete $A$ returns the output of $A^{\prime}$.

If $S \subseteq \mathbb{R}^{n}$ is the target set, then it is clear that the above modification results in running $A^{\prime}$ on $S \cap \operatorname{Ball}\left(n^{\prime}\right)$. If $S$ is convex, then $S \cap \operatorname{Ball}\left(n^{\prime}\right)$ is also convex. As $A^{\prime}$ commits only one-sided error, it will always output "accept," and hence so will $A$. On the other hand, suppose that $S$ is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}$. We claim that $\operatorname{Vol}\left(\operatorname{Ball}\left(n^{\prime}\right)\right) \geq 1-\varepsilon / 4$ (this will be shown below); given this claim, it must be the case that $S \cap \operatorname{Ball}\left(n^{\prime}\right)$ is at least (3 $\left.3 / 4\right)$-far from $\mathcal{C}_{\text {convex }}$ and at least $(3 \varepsilon / 4)$-far from $\mathcal{C}_{\text {convex }}^{\prime}$ as well. Consequently $A^{\prime}$ will output "reject" with probability at least $2 / 3$, and hence so will $A$.

To bound $\operatorname{Vol}\left(\operatorname{Ball}\left(n^{\prime}\right)\right)$, observe that it is the probability that an $\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}$ has

$$
\|\boldsymbol{x}\|^{2} \leq n+4 \sqrt{n \ln (4 / \varepsilon)} .
$$

It follows from Lemma 2.5 that the probability is at least $1-\varepsilon / 4$ as claimed.

Given Claim 2.11, it suffices to prove the following slight variant of Theorem 2.2:

Theorem 2.12. There is a one-sided sample-based $\varepsilon$-testing algorithm $A^{\prime}$ which, given any Lebesgue measurable target set $S$ contained in $\operatorname{Ball}\left(n^{\prime}\right)$, uses $(n / \varepsilon)^{O(n)}$ samples from $\mathcal{N}(0,1)^{n}$ to test whether $S \in \mathcal{C}_{\text {convex }}^{\prime}$ versus $S$ is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}^{\prime}$.

In the rest of this section we prove Theorem 2.12. We start with some terminology and concepts that we use in the description and analysis of our algorithm. Some of the notions that we introduce below, such as the notions of "boundary" cubes and "internal" cubes, are inspired by related notions that arise in earlier works such as [Ker92, Ras03].

Fix $\ell:=\varepsilon^{3} / n^{4}$ in the rest of the section, and let Cube $_{0}$ denote the following set

$$
\text { Cube }_{0}:=[-\ell / 2, \ell / 2)^{n} \subset \mathbb{R}^{n}
$$

of side length $\ell$ that is centered at the origin. We say that a cube is a subset of $\mathbb{R}^{n}$ of the form $\mathrm{Cube}_{0}+\ell \cdot\left(i_{1}, \ldots, i_{n}\right)$, where each $i_{j} \in \mathbb{Z}$, which contains at least one point of $\operatorname{Ball}\left(2 n^{\prime}\right)$. We use CubeSet to denote the set of all such cubes.

It is easy to see that

$$
\operatorname{Ball}\left(n^{\prime}\right) \subset \bigcup \operatorname{CubeSet} \subset \operatorname{Ball}\left(2 n^{\prime}+\ell \sqrt{n}\right) \subset \operatorname{Ball}\left(3 n^{\prime}\right)
$$

Fix an $S \subseteq \operatorname{Ball}\left(n^{\prime}\right)$ as the target set being tested for membership in $\mathcal{C}_{\text {convex }}^{\prime}$. Additionally fix a finite set $T=\left\{\left(x^{1}, S\left(x^{1}\right)\right), \ldots,\left(x^{M}, S\left(x^{M}\right)\right)\right\}$ of labeled samples according to $S$, for some positive integer $M$. (The set $T$ will correspond to the set of labeled samples that the testing algorithm receives.) We classify cubes in the CubeSet based on $T$ in the following way:

- A cube Cube is said to be an external cube if Cube $\cap T^{+}=\emptyset$ (i.e., no positive point of $T$ lies in Cube). We let $E C$ denote the union of all the external cubes.
- Any cube which is not an external cube (equivalently, any cube that contains at least one positive point of $T$ ) is said to be a positive cube.
- We say that two cubes Cube, Cube ${ }^{\prime}$ are adjacent if for any $\kappa>0$ there exist $x \in$ Cube and $y \in$ Cube $^{\prime}$ that have Euclidean distance at most $\kappa$ (in other words, two cubes are adjacent if their closure "touch anywhere, even only at a vertex;" note that each cube is adjacent to itself). If a cube is both (i) a positive cube and (ii) is adjacent to a cube (including itself) that contains at least one negative point of $T$, then we call it a boundary cube. We use $B C$ to denote the union of all boundary cubes.


Figure 1: A 2D example of the different types of cubes induced by a set of labeled samples. The target set $S$ is a disk, and the solid and hollow dots are positive and negative samples, respectively. The hollow, hatched, and shaded boxes are external, boundary, and internal cubes, respectively.

- We say that a positive cube which is not a boundary cube is an internal cube. (Equivalently, a cube is internal if and only if it contains at least one positive point and all the points in $T$ that are contained in any of its adjacent cubes, including itself, are positive.) We use $I C$ to denote the union of all internal cubes.

We note that since each cube is either external, internal, or boundary, the set $\operatorname{Ball}\left(n^{\prime}\right)$ is contained in the (disjoint) union of $E C, B C$ and $I C$. Figure 1 illustrates the different types of cubes.

We will use the following useful property of internal cubes:

Lemma 2.13. Suppose a finite set of labeled samples $T$ is such that every cube in CubeSet contains at least one point of $T$. Then every internal cube is contained in $\operatorname{Conv}\left(T^{+}\right)$.

The lemma is a direct consequence of the following claim by setting $H=T^{+}$:

Claim 2.14. Let $H \subseteq \mathbb{R}^{n}$ be any set that contains at least one point in each cube that is adjacent to $\mathrm{Cube}_{0}$. Then $\mathrm{Cube}_{0}$ is contained in $\operatorname{Conv}(H)$.

Proof. We prove the claim by induction on the dimension $n$. When $n=1$ the claim is trivial since $\mathrm{Cube}_{0}$ is simply the interval $[-\ell / 2, \ell / 2)$ and by assumption, there is at least one point of $H$ in $[-3 \ell / 2,-\ell / 2)$ and at least one point of $H$ in $[\ell / 2,3 \ell / 2)$.

For $n>1$, let $P=\left\{p \in H \mid p_{n} \geq \ell / 2\right\}$ and $P^{\prime}=\left\{p^{\prime} \in H \mid p_{n}^{\prime} \leq-\ell / 2\right\}$ be two subsets of H. Intuitively, the convex hulls of $P$ and $P^{\prime}$ "cover" Cube $_{0}$ on both sides (by induction), so the convex hull of their union will contain the whole Cube $_{0}$. More formally, let $x$ be any point in Cube ${ }_{0}$. By projecting $P, P^{\prime}$ and $x$ onto the first $n-1$ dimensions and using the inductive hypothesis ${ }^{2}$, we can find points $y \in \operatorname{Conv}(P)$ and $y^{\prime} \in \operatorname{Conv}\left(P^{\prime}\right)$ such that $y_{i}=y_{i}^{\prime}=x_{i}$ for all $i \in[n-1]$. Since we have $p_{n} \geq 1 / 2$ and $p_{n}^{\prime} \leq-1 / 2$ for all $p \in P$ and $p^{\prime} \in P^{\prime}$, respectively, it follows directly that $y_{n} \geq 1 / 2$ and $y_{n}^{\prime} \leq-1 / 2$. As $x \in \mathrm{Cube}_{0}, x$ is on the line segment between $y$ and $y^{\prime}$ and thus is in the convex hull of $H$. Hence all of $\mathrm{Cube}_{0}$ is contained in $\operatorname{Conv}(H)$.

### 2.3.2 Bounding the total volume of boundary cubes

Before presenting our algorithm we record the following useful corollary of Theorem 2.7, which allows the one-sided tester to reject bodies as non-convex if it detects too much volume in boundary cubes. (Note that we do not assume below that $T$ satisfies the condition of Lemma 2.13, i.e., that $T$ has at least one point in each cube in CubeSet, though this will

[^0]be the case when we use it later.)

Corollary 2.15. Let $S$ be a convex set in $\mathcal{C}_{\text {convex }}^{\prime}$ and $T$ be any finite set of labeled samples according to $S$, which defines sets $E C, I C$ and $B C$ as discussed earlier. Then we have

$$
\operatorname{Vol}(B C) \leq 20 n^{5 / 8} n^{\prime} \sqrt{2 \ell \sqrt{n}}=o(\varepsilon)
$$

Proof. Let Cube be a boundary cube. Then by definition, there is a positive point of $T$ (call it $t$ ) in Cube, and there is a Cube' adjacent to Cube that contains a negative point of $T$ (call it $t^{\prime}$ ). It follows that there must be a boundary point of $\partial S$ (call it $t^{*}$ ) in the segment between $t$ and $t^{\prime}$, and we have Cube $\in t^{*}+\operatorname{Ball}(2 \ell \sqrt{n})$. It follows that $B C \subseteq \partial S+\operatorname{Ball}(2 \ell \sqrt{n})$, and hence

$$
\operatorname{Vol}(B C) \leq \operatorname{Vol}(\partial S+\operatorname{Ball}(2 \ell \sqrt{n})) \leq 20 n^{5 / 8} n^{\prime} \sqrt{2 \ell \sqrt{n}}=o(\varepsilon)
$$

by Theorem 2.7 (and using $\ell \sqrt{n} \ll n^{-3 / 4}$ by our choice of $\ell=\varepsilon^{3} / n^{4}$ ).

### 2.3.3 The one-sided testing algorithm

Now we describe and analyze the one-sided testing algorithm $A^{\prime}$ mentioned in Theorem 2.12. Algorithm $A^{\prime}$ works by performing $O(1 / \varepsilon)$ independent runs of the algorithm $A^{*}$, which we describe in Figure 2. If any of the runs of $A^{*}$ output "reject" then algorithm $A^{\prime}$ outputs "reject," and otherwise it outputs "accept."

In words, Algorithm $A^{*}$ works as follows: first, in Step 1 it draws enough samples so that (with very high probability) it will receive at least one sample in each cube (if the lowprobability event that this does not occur takes place, then the algorithm outputs "accept" since it can only reject if it is impossible for $S$ to be convex). If the region "close to the boundary" of $S$ (as measured by $\operatorname{Vol}(B C)$ in Step 3) is too large, then the set cannot be

Algorithm $A^{*}$ : Given access to independent draws $(\boldsymbol{x}, S(\boldsymbol{x}))$ where $\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}$ and the target set $S$ is a Lebesgue measurable set that is contained in $\operatorname{Ball}\left(n^{\prime}\right)$.

1. Draw a set $\mathbf{T}$ of $s:=(n / \varepsilon)^{O(n)}$ labeled samples $(\boldsymbol{x}, S(\boldsymbol{x}))$, where each $\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}$.
2. If any cube does not contain a point of $\mathbf{T}$, then halt and output "accept."
3. If $\operatorname{Vol}(B C) \geq \varepsilon / 4$ (the volume of the union of boundary cubes), halt and output "reject."
4. Define $\mathbf{I} \subseteq \mathbb{R}^{n}$ to be $\operatorname{Conv}\left(\mathbf{T}^{+}\right)$, the convex hull of all positive points in $\mathbf{T}$.
5. Draw a single fresh labeled sample $(\mathbf{y}, S(\mathbf{y}))$, where $\mathbf{y} \leftarrow \mathcal{N}(0,1)^{n}$. If $\mathbf{y} \in \mathbf{I}$ but $S(\mathbf{y})=0$ then halt and output "reject." Otherwise, halt and output "accept."

Figure 2: Description of the algorithm $A^{*}$
convex (by Corollary 2.15) and the algorithm rejects. Finally, the algorithm checks a freshly drawn point; if this point is in the convex hull of the positive samples but is labeled negative, then the set cannot be convex and the algorithm rejects. Otherwise, the algorithm accepts.

To establish correctness and prove Theorem 2.12 we must show that (i) algorithm $A^{*}$ never rejects if the target set $S$ is a Lebesgue measurable set that belongs to $\mathcal{C}_{\text {convex }}^{\prime}$, and (ii) if $S$ is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}^{\prime}$ then algorithm $A^{*}$ rejects with probability at least $\Omega(\varepsilon)$. Part (i) is trivial as $A^{*}$ only rejects if either (a) $\operatorname{Vol}(B C) \geq \varepsilon / 4$ or (b) step 5 identifies a negative point in the convex hull of the positive points in T. For both cases we conclude (using Corollary 2.15 for (a)) that $S \notin \mathcal{C}_{\text {convex }}^{\prime}$.

For (ii) suppose that $S$ is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}^{\prime}$. Let $E$ be the following event (over the draw of $\mathbf{T}$ ):

Event E: Every cube in CubeSet contains at least one point of $\mathbf{T}$ (so the
algorithm does not accept in Step 2) and moreover, every Cube with

$$
\frac{\operatorname{Vol}(\text { Cube } \cap S)}{\operatorname{Vol}(\text { Cube })} \geq \epsilon / 4
$$

contains at least one positive point in $\mathbf{T}$ and thus, is not external.

It is easy to show that the probability mass of each cube in CubeSet is at least $(\varepsilon / n)^{O(n)}$ (since its volume is $(\varepsilon / n)^{O(n)}$ and the density function of the Gaussian is at least $(1 / \varepsilon)^{O(n)}$ using our choice of $n^{\prime}$ ), it follows from a union bound over CubeSet that, for a suitable choice of $s=(n / \varepsilon)^{O(n)}$ (with a large enough coefficient in the exponent), $E$ occurs with probability $1-o(1)$. Assuming that $E$ occurs, we show below that either $\operatorname{Vol}(B C) \geq \varepsilon / 4$ or $A^{*}$ rejects in Step 5 with probability $\Omega(\varepsilon)$.

For this purpose, we assume below that both $E$ occurs and $\operatorname{Vol}(B C)<\varepsilon / 4$. Note that the set $I$ is convex and is contained in $\operatorname{Ball}\left(n^{\prime}\right)$. Thus it belongs to $\mathcal{C}_{\text {convex }}^{\prime}$ and consequently $\operatorname{Vol}(I \triangle S) \geq \varepsilon\left(\right.$ since $S$ is assumed to be $\varepsilon$-far from $\left.\mathcal{C}_{\text {convex }}^{\prime}\right)$, which implies that

$$
\operatorname{Vol}(S \backslash I)+\operatorname{Vol}(I \backslash S) \geq \varepsilon
$$

It suffices to show that $\operatorname{Vol}(S \backslash I) \leq \epsilon / 2$, since $\operatorname{Vol}(I \backslash S)$ is exactly the probability that algorithm $A^{*}$ rejects in Step 5. To see that $\operatorname{Vol}(S \backslash I) \leq \epsilon / 2$, observe that by Lemma 2.13, $\operatorname{Vol}(S \backslash I)$ is at most $\operatorname{Vol}(S \cap B C)+\operatorname{Vol}(S \cap E C)$. On the one hand, $\operatorname{Vol}(S \cap B C) \leq \operatorname{Vol}(B C)<$ $\varepsilon / 4$ by assumption. On the other hand, given the event $E$, every external cube has at most $(\varepsilon / 4)$-fraction of its volume in $S$ and thus, $\operatorname{Vol}(S \cap E C) \leq \varepsilon / 4$ (as the total volume of $E C$ is at most 1). Hence $\operatorname{Vol}(S \backslash I) \leq \varepsilon / 2$, concluding the proof of Theorem 2.12.

### 2.4 Two-sided lower bound

We recall Theorem 2.3:

Theorem. There exists a positive constant $\varepsilon_{0}$ such that any two-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ for some $\varepsilon \leq \varepsilon_{0}$ must use $2^{\Omega(\sqrt{n})}$ samples.

Let $q=2^{0.01 \sqrt{n}}$ and let $\varepsilon_{0}$ be a positive constant to be specified later. To prove Theorem 2.3, we show that no sample-based, $q$-query (randomized) algorithm $A$ can achieve the following goal:

Let $S \subset \mathbb{R}^{n}$ be a target set that is Lebesgue measurable. Let $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{q}$ be a sequence of $q$ samples drawn from $\mathcal{N}(0,1)^{n}$. Upon receiving $\left(\left(\boldsymbol{x}_{i}, S\left(\boldsymbol{x}_{i}\right)\right): i \in[q]\right), A$ accepts with probability at least $2 / 3$ when $S$ is convex and rejects with probability at least $2 / 3$ when $S$ is $\varepsilon_{0}$-far from convex.

Recall that a pair $(x, b)$ with $x \in \mathbb{R}^{n}$ and $b \in\{0,1\}$ is a labeled sample. Thus, a samplebased algorithm $A$ is simply a randomized map from a sequence of $q$ labeled samples to \{"accept", "reject"\}.

### 2.4.1 Proof Plan

Assume for contradiction that there is a $q$-query (randomized) algorithm $A$ that accomplishes the task above. In Section 2.4.2 we define two probability distributions $\mathcal{D}_{\text {yes }}$ and $\mathcal{D}_{\text {no }}$ such that (1) $\mathcal{D}_{\text {yes }}$ is a distribution over convex sets in $\mathbb{R}^{n}\left(\mathcal{D}_{\text {yes }}\right.$ is a distribution over certain convex polytopes that are the intersection of many randomly drawn halfspaces), and (2) $\mathcal{D}_{\text {no }}$ is a probability distribution over sets in $\mathbb{R}^{n}$ that are Lebesgue measurable ( $\mathcal{D}_{\mathrm{no}}$ is actually supported over a finite number of measurable sets in $\mathbb{R}^{n}$ ) such that $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ is $\varepsilon_{0}$-far from convex with probability at least $1-o(1)$.

Given a sequence $x=\left(x_{1}, \ldots, x_{q}\right)$ of points, we abuse the notation and write

$$
S(x)=\left(S\left(x_{1}\right), \ldots, S\left(x_{q}\right)\right)
$$

and use $(x, S(x))$ to denote the sequence of $q$ labeled samples $\left(x_{1}, S\left(x_{1}\right)\right), \ldots,\left(x_{q}, S\left(x_{q}\right)\right)$. It then follows from our assumption on $A$ that

$$
\begin{aligned}
& \quad \underset{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes } ;} ; \boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}}{ }[A \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))] \geq 2 / 3 \quad \text { and } \\
& \underset{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }} ; \boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}}{\operatorname{Pr}}[A \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))] \leq 1 / 3+o(1) .
\end{aligned}
$$

where we use $\boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ to denote a sequence of $q$ points sampled independently from $\mathcal{N}^{n}$ and we usually skip the $\leftarrow\left(\mathcal{N}^{n}\right)^{q}$ part in the subscript when it is clear from the context. Since $A$ is a mixture of deterministic algorithms, there exists a deterministic sample-based, $q$-query algorithm $A^{\prime}$ (equivalently, a deterministic map from sequences of $q$ labeled samples to \{"Yes", "No" $\}$ ) with

$$
\begin{equation*}
\underset{\boldsymbol{S} \leftarrow \mathcal{D r e s}_{\text {y }} ; \boldsymbol{x}}{\operatorname{Pr}}\left[A^{\prime} \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))\right]-\underset{\boldsymbol{S} \leftarrow \mathcal{D}_{\mathrm{no}} ; \boldsymbol{x}}{\operatorname{Pr}}\left[A^{\prime} \text { accepts }(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))\right] \geq 1 / 3-o(1) . \tag{2}
\end{equation*}
$$

Let $\mathcal{E}_{\text {yes }}$ (or $\mathcal{E}_{\text {no }}$ ) be the distribution of $(\boldsymbol{x}, \boldsymbol{S}(\boldsymbol{x}))$, where $\boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ and $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$ (or $S \leftarrow \mathcal{D}_{\mathrm{no}}$, respectively). Both of them are distributions over sequences of $q$ labeled samples. Then the LHS of (2), for any deterministic sample-based, $q$-query algorithm $A^{\prime}$, is at most the total variation distance between $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}$. We prove the following key lemma, which leads to a contradiction.

Lemma 2.16. The total variation distance between $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}$ is o(1).

To prove Lemma 2.16, it is convenient for us to introduce a third distribution $\mathcal{E}_{\text {no }}^{*}$ over sequences of $q$ labeled samples, where $(\boldsymbol{x}, \mathbf{b}) \leftarrow \mathcal{E}_{\mathrm{no}}^{*}$ is drawn by first sampling a sequence of
$q$ points $\boldsymbol{x}=\left(\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{q}\right)$ from $\mathcal{N}^{n}$ independently and then for each $\boldsymbol{x}_{i}$, its label $\mathbf{b}_{i}$ is set to be 1 independently with a probability that depends only on $\left\|\boldsymbol{x}_{i}\right\|$ (see Section 2.4.2). Lemma 2.16 follows from the following two lemmas by the triangle inequality.

Lemma 2.17. The total variation distance between $\mathcal{E}_{n o}$ and $\mathcal{E}_{n o}^{*}$ is o(1).

Lemma 2.18. The total variation distance between $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{n o}^{*}$ is $o(1)$.

The rest of the section is organized as follows. We define the distributions $\mathcal{D}_{\text {yes }}, \mathcal{D}_{\text {no }}$ (which are used to define $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}$ ) as well as $\mathcal{E}_{\text {no }}^{*}$ in Section 2.4.2 and prove the necessary properties about $\mathcal{D}_{\text {yes }}$ and $\mathcal{D}_{\text {no }}$ as well as Lemma 2.17. We prove Lemma 2.18 in Sections 2.4.3 and 2.4.4.

### 2.4.2 The Distributions

Let $r=\Theta\left(n^{1 / 4}\right)$ be a parameter to be fixed later, and let $N=2^{\sqrt{n}}$. We start with the definition of $\mathcal{D}_{\text {yes }}$. A random set $\boldsymbol{S} \subset \mathbb{R}^{n}$ is drawn from $\mathcal{D}_{\text {yes }}$ using the following procedure:

1. We sample a sequence of $N$ points $\mathbf{y}_{1}, \ldots, \mathbf{y}_{N}$ from $S^{n-1}(r)$ independently and uniformly at random. Each point $\mathbf{y}_{i}$ defines a halfspace

$$
\boldsymbol{h}_{i}=\left\{x \in \mathbb{R}^{n}: x \cdot \mathbf{y}_{i} \leq r^{2}\right\} .
$$

2. The set $\boldsymbol{S}$ is then the intersection of $\boldsymbol{h}_{i}, i \in[N]$ (this is always nonempty as indeed $\operatorname{Ball}(r)$ is contained in $\boldsymbol{S})$.

It is clear from the definition that $S \leftarrow \mathcal{D}_{\text {yes }}$ is always a convex set.

Next we define $\mathcal{E}_{\mathrm{no}}^{*}\left(\right.$ instead of $\left.\mathcal{D}_{\mathrm{no}}\right)$, a distribution over sequences of $q$ labeled samples $(\boldsymbol{x}, \mathbf{b})$. To this end, we use $\mathcal{D}_{\text {yes }}$ to define a function $\rho: \mathbb{R}_{\geq 0} \rightarrow[0,1]$ as follows:

Due to the symmetry of $\mathcal{D}_{\text {yes }}$ and $\mathcal{N}^{n}$, the value $\rho(t)$ is indeed the probability that a point $x \in \mathbb{R}^{n}$ at distance $t$ from the origin lies in $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$. To draw a sequence of $q$ labeled samples $(\boldsymbol{x}, \mathbf{b}) \leftarrow \mathcal{E}_{\mathrm{no}}^{*}$, we first independently draw $q$ random points $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{q} \leftarrow \mathcal{N}^{n}$ and then independently set each $\mathbf{b}_{i}=1$ with probability $\rho\left(\left\|\boldsymbol{x}_{i}\right\|\right)$ and $\mathbf{b}_{i}=0$ with probability $1-\rho\left(\left\|\boldsymbol{x}_{i}\right\|\right)$.

Given $\mathcal{D}_{\text {yes }}$ and $\mathcal{E}_{\mathrm{no}}^{*}$, Lemma 2.18 shows that information-theoretically no sample-based algorithm can distinguish a sequence of $q$ labeled samples $(\boldsymbol{x}, \mathbf{b})$ with $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}, \boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$, and $\mathbf{b}=\boldsymbol{S}(\boldsymbol{x})$ from a sequence of $q$ labeled samples drawn from $\mathcal{E}_{\text {no }}^{*}$. While the marginal distribution of each labeled sample is the same for the two cases, the former is generated in a correlated fashion using the underlying random convex $S \leftarrow \mathcal{D}_{\text {yes }}$ while the latter is generated independently.

Finally we define the distribution $\mathcal{D}_{\text {no }}$, prove Lemma 2.17, and show that a set drawn from $\mathcal{D}_{\text {no }}$ is far from convex with high probability. To define $\mathcal{D}_{\mathrm{no}}$, we let $M \geq 2^{\sqrt{n}}$ be a large enough integer to be specified later. With $M$ fixed, we use

$$
0=t_{0}<t_{1}<\cdots<t_{M-1}<t_{M}=2 \sqrt{n}
$$

to denote a sequence of numbers such that the origin-centered ball $\operatorname{Ball}(2 \sqrt{n})$ is partitioned into $M$ shells $\operatorname{Ball}\left(t_{i}\right) \backslash \operatorname{Ball}\left(t_{i-1}\right), i \in[M]$, and all the $M$ shells have the same probability mass under $\mathcal{N}^{n}$. By spherical coordinates, it means that the following integral takes the
same value for all $i$ :

$$
\begin{equation*}
\int_{t_{i-1}}^{t_{i}} \phi(x, 0, \ldots, 0) x^{n-1} d x \tag{3}
\end{equation*}
$$

where $\phi$ denotes the density function of $\mathcal{N}^{n}$. We show below that when $M$ is large enough, we have

$$
\begin{equation*}
\left|\rho(x)-\rho\left(t_{i}\right)\right| \leq 2^{-\sqrt{n}}, \tag{4}
\end{equation*}
$$

for any $i \in[M]$ and any $x \in\left[t_{i-1}, t_{i}\right]$. We will fix such an $M$ and use it to define $\mathcal{D}_{\text {no }}$. (Our results are not affected by the size of $M$ as a function of $n$; we only need it to be finite, given n.)

To show that (4) holds when $M$ is large enough, we need the continuity of the function $\rho$, which follows directly from the explicit expression for $\rho$ given later in (6).

Lemma 2.19. The function $\rho: \mathbb{R}_{\geq 0} \rightarrow[0,1]$ is continuous.

Since $\rho$ is continuous, it is continuous over $[0,2 \sqrt{n}]$. Since $[0,2 \sqrt{n}]$ is compact, $\rho$ is also uniformly continuous over $[0,2 \sqrt{n}]$. Also note that $\max _{i \in[M]}\left(t_{i}-t_{i-1}\right)$ goes to 0 as $M$ goes to $+\infty$. It follows that (4) holds when $M$ is large enough.

With $M \geq 2^{\sqrt{n}}$ fixed, a random set $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ is drawn as follows. We start with $\boldsymbol{S}=\emptyset$ and for each $i \in[M]$, we add the $i$ th shell $\operatorname{Ball}\left(t_{i}\right) \backslash \operatorname{Ball}\left(t_{i-1}\right)$ to $\boldsymbol{S}$ independently with probability $\rho\left(t_{i}\right)$. Thus an outcome of $\boldsymbol{S}$ is a union of some of the shells and $\mathcal{D}_{\text {no }}$ is supported over $2^{M}$ different sets.

Recall the definition of $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}$ using $\mathcal{D}_{\text {yes }}$ and $\mathcal{D}_{\text {no }}$. We now prove Lemma 2.17.

Proof of Lemma 2.17. Let $x=\left(x_{1}, \ldots, x_{q}\right)$ be a sequence of $q$ points in $\mathbb{R}^{n}$. We say $x$ is bad if either (1) at least one point lies outside of $\operatorname{Ball}(2 \sqrt{n})$ or (2) there are two points that
lie in the same shell of $\mathcal{D}_{\text {no }}$; we say $x$ is good otherwise. We first claim that $\boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ is bad with probability $o(1)$. To see this, we have from Lemma 2.5 that event (1) occurs with probability $o(1)$, and from $M \geq 2^{\sqrt{n}}$ and $q=2^{0.01 \sqrt{n}}$ that event (2) occurs with probability $o(1)$. The claim follows from a union bound.

Given that $\boldsymbol{x} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ is good with probability $1-o(1)$, it suffices to show that for any good $q$-tuple $x$, the total variation distance between (1) $\boldsymbol{S}(x)$ with $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ and (2) $\mathbf{b}=\left(\mathbf{b}_{1}, \ldots, \mathbf{b}_{q}\right)$ with each bit $\mathbf{b}_{i}$ being 1 with probability $\rho\left(\left\|x_{i}\right\|\right)$ independently, is $o(1)$. Let $\ell_{i} \in[M]$ be the index of the shell that $x_{i}$ lies in. Since $x$ is good (and thus, all points lie in different shells), $\boldsymbol{S}(x)$ has the $i$ th bit being 1 independently with probability $\rho\left(t_{\ell_{i}}\right)$; for the other distribution, the probability is $\rho\left(\left\|x_{i}\right\|\right)$. Using the subadditivity of total variation distance (i.e., the fact that the $d_{\mathrm{TV}}$ between two sequences of independent random variables is upper bounded by the sum of the $d_{\mathrm{TV}}$ between each pair) as well as (4), we have $d_{\mathrm{TV}}(\boldsymbol{S}(x), \mathbf{b}) \leq q \cdot 2^{-\sqrt{n}}=o(1)$. This finishes the proof.

The next lemma shows that $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ is $\varepsilon_{0}$-far from convex with probability $1-o(1)$, for some positive constant $\varepsilon_{0}$. In the proof of the lemma we fix both the constant $\varepsilon_{0}$ and our choice of $r=\Theta\left(n^{1 / 4}\right)$. (We remind the reader that $\rho$ and $\mathcal{D}_{\mathrm{no}}$ both depend on the value of $r$.

Lemma 2.20. There exist a real value $r=\Theta\left(n^{1 / 4}\right)$ with $e^{r^{2} / 2} \geq N / n$ and a positive constant $\varepsilon_{0}$ such that a set $\boldsymbol{S} \leftarrow \mathcal{D}_{n o}$ is $\varepsilon_{0}$-far from convex with probability at least $1-o(1)$.

Proof. We need the following claim but delay its proof to the end of the subsection:

Claim 2.21. There exist an $r=\Theta\left(n^{1 / 4}\right)$ with $e^{r^{2} / 2} \geq N / n$ and a constant $c \in(0,1 / 2)$ such
that

$$
c<\rho(x)<1-c, \quad \text { for all } x \in[\sqrt{n}-10, \sqrt{n}+10] \text {. }
$$

Let $K \subset[M]$ denote the set of all integers $k$ such that $\left[t_{k-1}, t_{k}\right] \subseteq[\sqrt{n}-10, \sqrt{n}+10]$ (note that $K$ is a set of consecutive integers). Observe that (1) the total probability mass of all shells $k \in K$ is at least $\Omega(1)$ (by Lemma 2.5), and (2) the size $|K|$ is at least $\Omega(M)$ (which follows from (1) and the fact that all shells have the same probability mass).

Consider the following 1-dimensional scenario. We have $|K|$ intervals $\left[t_{k-1}, t_{k}\right]$ and draw a set $\boldsymbol{T}$ by including each interval independently with probability $\rho\left(t_{k}\right)$. We prove the following claim:

Claim 2.22. The random set $\boldsymbol{T}$ satisfies the following property with probability at least $1-o(1):$ For any interval $I \subseteq \mathbb{R}_{\geq 0}$, either $I$ contains $\Omega(M)$ intervals $\left[t_{k-1}, t_{k}\right]$ that are not included in $\boldsymbol{T}$, or $\bar{I}$ contains $\Omega(M)$ intervals $\left[t_{k-1}, t_{k}\right]$ included in $\boldsymbol{T}$.

Proof. First note that it suffices to consider intervals $I \subseteq \cup_{k \in K}\left[t_{k-1}, t_{k}\right]$ and moreover, we may further assume that both endpoints of $I$ come from endpoints of $\left[t_{k-1}, t_{k}\right], k \in K$. (In other words, for a given outcome $T$ of $\boldsymbol{T}$, if there exists an interval $I$ that violates the condition, i.e., both $I$ and $\bar{I}$ contain fewer than $\Omega(M)$ intervals, then there is such an interval $I$ with both ends from end points of $\left[t_{k-1}, t_{k}\right]$ ). This assumption allows us to focus on $|K|^{2} \leq M^{2}$ many possibilities for $I$ (as we will see below, our argument applies a union bound over these $K^{2}$ possibilities).

Given a candidate such interval $I$, we consider two cases. If $I$ contains $\Omega(M)$ intervals $\left[t_{k-1}, t_{k}\right], k \in K$, then it follows from Claim 2.21 and a Chernoff bound that $I$ contains at least $\Omega(M)$ intervals not included in $\boldsymbol{T}$ with probability $1-2^{-\Omega(M)}$. On the other hand, if
$\bar{I}$ contains $\Omega(M)$ intervals, then the same argument shows that $\bar{I}$ contains $\Omega(M)$ intervals included in $\boldsymbol{T}$ with probability $1-2^{-\Omega(M)}$. The claim follows from a union bound over all the $|K|^{2}$ possibilities for $I$.

We return to the $n$-dimensional setting and consider the intersection of $S \leftarrow \mathcal{D}_{\text {no }}$ with a ray starting from the origin. Note that the intersection of the ray and any convex set is an interval on the ray. As a result, Claim 2.22 shows that with probability at least $1-o(1)$ (over the draw of $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {no }}$ ), the intersection of any convex set with any ray either contains $\Omega(M)$ intervals $\left[t_{k-1}, t_{k}\right]$ such that shell $k \in K$ is not included in $\boldsymbol{S}$, or misses $\Omega(M)$ intervals $\left[t_{k-1}, t_{k}\right]$ such that shell $k \in K$ is included in $\boldsymbol{S}$. Since by (1) above shells $k \in K$ together have $\Omega(1)$ probability mass under $\mathcal{N}^{n}$ and each shell contains the same probability mass, we have that with probability $1-o(1), \boldsymbol{S}$ is $\varepsilon_{0}$-far from any convex set for some constant $\varepsilon_{0}>0$. (A more formal argument can be given by performing integration using spherical coordinates and applying (3).)

Proof of Claim 2.21. We start with the choice of $r$. Let

$$
\alpha=\sqrt{n}-10 \quad \text { and } \quad \beta=\sqrt{n}+10
$$

Let $\operatorname{cap}(t)$ denote the fractional surface area of the spherical cap $S^{n-1} \cap\left\{x: x_{1} \geq t\right\}$, i.e.,

$$
\operatorname{cap}(t)=\operatorname{Pr}_{\boldsymbol{x} \leftarrow S^{n-1}}\left[\boldsymbol{x}_{1} \geq t\right]
$$

So cap is a continuous, strictly decreasing function over $[0,1]$. Since $\operatorname{cap}(0)=1 / 2$ and $\operatorname{cap}(1)=0$, there is a unique $r \in(0, \alpha)$ such that $\operatorname{cap}(r / \alpha)=1 / N=2^{-\sqrt{n}}$. Below we show that $r=\Theta\left(n^{1 / 4}\right)$ and fix it in the rest of the proof. First recall the following explicit
expression (see e.g. [KOS07]):

$$
\operatorname{cap}(t)=a_{n} \int_{t}^{1}\left(\sqrt{1-z^{2}}\right)^{n-3} d z
$$

where $a_{n}=\Theta\left(n^{1 / 2}\right)$ is a parameter that only depends on $n$. Also recall the following inequalities from [KOS07] about $\operatorname{cap}(t)$ :

$$
\begin{equation*}
\operatorname{cap}(t) \leq e^{-n t^{2} / 2}, \text { for all } t \in[0,1] ; \quad \operatorname{cap}(t) \geq \Omega\left(t \cdot e^{-n t^{2} / 2}\right), \text { for } t=O\left(1 / n^{1 / 4}\right) \tag{5}
\end{equation*}
$$

By our choice of $\alpha$ and the monotonicity of the cap function, this implies that $r=\Theta\left(n^{1 / 4}\right)$ and

$$
\begin{aligned}
1 / N=\operatorname{cap}(r / \alpha) & \geq \Omega\left(1 / n^{1 / 4}\right) \cdot e^{-n(r / \alpha)^{2} / 2} \\
& \geq \Omega\left(1 / n^{1 / 4}\right) \cdot e^{-\left(r^{2} / 2\right)(1+O(1 / \sqrt{n}))} \\
& =\Omega\left(1 / n^{1 / 4}\right) \cdot e^{-r^{2} / 2}
\end{aligned}
$$

(using $r=\Theta\left(n^{1 / 4}\right)$ for the last inequality), and thus, we have $e^{r^{2} / 2} \geq N / n$.
Next, using the function cap we have the following expression for $\rho$ :

$$
\begin{equation*}
\rho(x)=\left(1-\operatorname{cap}\left(\frac{r}{x}\right)\right)^{N} . \tag{6}
\end{equation*}
$$

As a side note, $\rho$ is continuous and thus, Lemma 2.19 follows. Since cap is strictly decreasing, we have that $\rho$ is strictly decreasing as well. To finish the proof it suffices to show that there is a constant $c \in(0,1 / 2)$ such that $\rho(\alpha)<1-c$ and $\rho(\beta) \geq c$. The first part is easy since

$$
\rho(\alpha)=(1-1 / N)^{N} \approx e^{-1}
$$

by our choice of $r$. In the rest of the proof we show that

$$
\begin{equation*}
\operatorname{cap}\left(\frac{r}{\beta}\right) \leq a \cdot \operatorname{cap}\left(\frac{r}{\alpha}\right)=\frac{a}{N}, \tag{7}
\end{equation*}
$$

for some positive constant $a$. It follows immediately that

$$
\rho(\beta)=\left(1-\operatorname{cap}\left(\frac{r}{\beta}\right)\right)^{N} \geq\left(1-\frac{a}{N}\right)^{N} \geq\left(e^{-2 a / N}\right)^{N}=e^{-2 a}
$$

using $1-x \geq e^{-2 x}$ for $0 \leq x \ll 1$, and this finishes the proof of the claim.


Figure 3: A plot of the integrand $\left(\sqrt{1-z^{2}}\right)^{(n-3)}$. Area $A$ is $\operatorname{cap}(r / \beta)-\operatorname{cap}(r / \alpha)$ and area $B$ is $\operatorname{cap}(r / \alpha)$. The rectangles on the right are an upper bound of $A$ and a lower bound of $B$.

Finally we prove (7). Let

$$
w=\frac{r}{\alpha}-\frac{r}{\beta}=\Theta\left(\frac{1}{n^{3 / 4}}\right)
$$

since $r=\Theta\left(n^{1 / 4}\right)$. Below we show that

$$
\begin{equation*}
\int_{r / \beta}^{r / \alpha}\left(\sqrt{1-z^{2}}\right)^{n-3} d z \leq a^{\prime} \cdot \int_{r / \alpha}^{r / \alpha+w}\left(\sqrt{1-z^{2}}\right)^{n-3} d z \tag{8}
\end{equation*}
$$

for some positive constant $a^{\prime}$. It follows that

$$
\operatorname{cap}\left(\frac{r}{\beta}\right)-\operatorname{cap}\left(\frac{r}{\alpha}\right) \leq a^{\prime} \cdot \operatorname{cap}\left(\frac{r}{\alpha}\right)
$$

and implies (7) by setting $a=a^{\prime}+1$. For (8), note that the ratio of the $[r / \beta, r / \alpha]$-integration over the $[r / \alpha, r / \alpha+w]$-integration is at most

$$
\left(\frac{\sqrt{1-(r / \beta)^{2}}}{\sqrt{1-(r / \beta+2 w)^{2}}}\right)^{n-3}
$$

as the length of the two intervals are the same and the function $\left(\sqrt{1-z^{2}}\right)^{n-3}$ is strictly decreasing. Figure 3 illustrates this calculation. Let $\tau=r / \beta=\Theta\left(1 / n^{1 / 4}\right)$. We can rewrite
the above as

$$
\left(\frac{1-\tau^{2}}{1-(\tau+2 w)^{2}}\right)^{(n-3) / 2}=\left(1+\frac{4 \tau w+4 w^{2}}{1-(\tau+2 w)^{2}}\right)^{(n-3) / 2}=\left(1+O\left(\frac{1}{n}\right)\right)^{(n-3) / 2}=O(1)
$$

This finishes the proof of the claim.

### 2.4.3 Distributions $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}^{*}$ are close

In the rest of the section we show that the total variation distance between $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\mathrm{no}}^{*}$ is $o(1)$ and thus prove Lemma 2.18. Let $z=\left(z_{1}, \ldots, z_{q}\right)$ be a sequence of $q$ points in $\mathbb{R}^{n}$. We use $\mathcal{E}_{\text {yes }}(z)$ to denote the distribution of labeled samples from $\mathcal{E}_{\text {yes }}$, conditioning on the samples being $z$, i.e., $(z, \boldsymbol{S}(z))$ with $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$. We let $\mathcal{E}_{\text {no }}^{*}(z)$ denote the distribution of labeled samples from $\mathcal{E}_{\text {no }}^{*}$, conditioning on the samples being $z$, i.e., $(z, \mathbf{b})$ where each $\mathbf{b}_{i}$ is 1 independently with probability $\rho\left(\left\|z_{i}\right\|\right)$. Then

$$
\begin{equation*}
d_{\mathrm{TV}}\left(\mathcal{E}_{\text {yes }}, \mathcal{E}_{\mathrm{no}}^{*}\right)=\mathbf{E}_{\mathbf{z} \leftarrow\left(\mathcal{N}^{n}\right)^{q}}\left[d_{\mathrm{TV}}\left(\mathcal{E}_{\text {yes }}(\mathbf{z}), \mathcal{E}_{\text {no }}^{*}(\mathbf{z})\right)\right] . \tag{9}
\end{equation*}
$$

We split the proof of Lemma 2.18 into two steps. We first introduce the notion of typical sequences $z$ of $q$ points and show in this subsection that with probability $1-o(1), \mathbf{z} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ is typical. In the next subsection we show that $d_{\mathrm{TV}}\left(\mathcal{E}_{\text {yes }}(z), \mathcal{E}_{\mathrm{no}}^{*}(z)\right)$ is $o(1)$ when $z$ is typical. It follows from (9) that $d_{\mathrm{TV}}\left(\mathcal{E}_{\text {yes }}, \mathcal{E}_{\text {no }}^{*}\right)$ is $o(1)$. We start with the definition of typical sequences.

Given a point $z \in \mathbb{R}^{n}$, we are interested in the fraction of points $y$ (in terms of the area) in $S^{n-1}(r)$ such that $z \cdot y>r^{2}$. This is because if any such point $y$ is sampled in the construction of $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$, then $z \notin \boldsymbol{S}$. This is illustrated in Figure 4. We refer to the set of such points $y$ as the (spherical) cap covered by $z$ and we write $\operatorname{cover}(z)$ to denote it. (Note that $\operatorname{cover}(z)=\emptyset$ if $\|z\| \leq r$.)

Given a subset $H$ of $S^{n-1}(r)$ (such as $\operatorname{cover}(z)$ ), we use fsa $(H)$ to denote the fractional surface area of $H$ with respect to $S^{n-1}(r)$. Using Figure 4 and elementary geometry, we have the following connection between the fractional surface area of $\operatorname{cover}(z)$ and the cap function (for $S^{n-1}$ ):

$$
\begin{equation*}
\operatorname{fsa}(\operatorname{cover}(z))=\operatorname{cap}(r /\|z\|) \tag{10}
\end{equation*}
$$

We are now ready to define typical sequences.


Figure 4: The fractional surface area of $\operatorname{cover}(z), \mathrm{fsa}(\operatorname{cover}(z))$, is the fraction of $S^{n-1}(r)$ to the right of the dashed line. By similarity of triangles $0 a z$ and $0 b a$, scaling down to the unit sphere, we get (10).

Definition. We say a sequence $z=\left(z_{1}, \ldots, z_{q}\right)$ of $q$ points in $\mathbb{R}^{n}$ is typical if

1. For every point $z_{i}$, we have

$$
\begin{equation*}
\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right)\right) \in\left[e^{-0.51 r^{2}}, e^{-0.49 r^{2}}\right] . \tag{11}
\end{equation*}
$$

2. For every $i \neq j$, we have

$$
\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{j}\right)\right) \leq e^{-0.96 r^{2}}
$$

The first condition of typicality essentially says that every $z_{i}$ is not too close to and not too far away from the origin (so that we have a relatively tight bound on the fractional surface area of the cap covered by $z_{i}$ ). The second condition says that the caps covered by two points $z_{i}$ and $z_{j}$ have very little intersection. We prove the following lemma:

Lemma 2.23. $\mathbf{z} \leftarrow\left(\mathcal{N}^{n}\right)^{q}$ is typical with probability at least $1-o(1)$.

Proof. We show that $\mathbf{z}$ satisfies each of the two conditions with probability $1-o(1)$. The lemma then follows from a union bound.

For the first condition, we let $c^{*}=0.001$ be a sufficiently small constant. We have from Lemma 2.5 and a union bound that every $\mathbf{z}_{i}$ satisfies $\left(1-c^{*}\right) \sqrt{n} \leq\left\|\mathbf{z}_{i}\right\| \leq\left(1+c^{*}\right) \sqrt{n}$ with probability $1-o(1)$. When this happens, we have (11) for every $\mathbf{z}_{i}$ using (5) and the upper bound of $\operatorname{cap}(t) \leq e^{-n t^{2} / 2}$.

For the second condition, we first note that the argument used in the first part implies that

$$
\mathbf{E}_{\boldsymbol{z}_{i} \leftarrow \mathcal{N}^{n}}\left[\mathrm{fsa}\left(\operatorname{cover}\left(\boldsymbol{z}_{i}\right)\right)\right] \leq e^{-0.49 r^{2}}
$$

Let $x_{0}$ be a fixed point in $S^{n-1}(r)$. Viewing the fractional surface area as the following probability

$$
\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right)\right)=\operatorname{Pr}_{x \leftarrow S^{n-1}(r)}\left[\boldsymbol{x} \in \operatorname{cover}\left(z_{i}\right)\right],
$$

we have

$$
\begin{align*}
e^{-0.49 r^{2}} & \geq \mathbf{E}_{\boldsymbol{z}_{i} \leftarrow \mathcal{N}^{n}}\left[\mathrm{fsa}\left(\operatorname{cover}\left(\boldsymbol{z}_{i}\right)\right)\right]  \tag{12}\\
& =\mathbf{E}_{\boldsymbol{z}_{i}}\left[\underset{\boldsymbol{x} \leftarrow S^{n-1}(r)}{\operatorname{Pr}}\left[\boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{i}\right)\right]\right] \\
& =\underset{\boldsymbol{x}, \mathbf{z}_{i}}{\operatorname{Pr}}\left[\boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{i}\right)\right]=\underset{\mathbf{z}_{i}}{\operatorname{Pr}}\left[x_{0} \in \operatorname{cover}\left(\mathbf{z}_{i}\right)\right],
\end{align*}
$$

where the last equation follows by sampling $\boldsymbol{x}$ first and spherical and Gaussian symmetry.
Similarly we can express the fractional surface area of $\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{j}\right)$ as

$$
\text { fsa }\left(\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{j}\right)\right)=\operatorname{Pr}_{\boldsymbol{x} \leftarrow S^{n-1}(r)}\left[\boldsymbol{x} \in \operatorname{cover}\left(z_{i}\right) \text { and } \boldsymbol{x} \in \operatorname{cover}\left(z_{j}\right)\right] .
$$

We consider the expectation over $\mathbf{z}_{i}$ and $\mathbf{z}_{j}$ drawn independently from $\mathcal{N}^{n}$ :

$$
\begin{aligned}
& \mathbf{E}_{\mathbf{z}_{i}, \mathbf{z}_{j}}\left[\operatorname{fsa}\left(\operatorname{cover}\left(\mathbf{z}_{i}\right) \cap \operatorname{cover}\left(\mathbf{z}_{j}\right)\right)\right] \\
& =\mathbf{E}_{\mathbf{z}_{i}, \mathbf{z}_{j}}\left[\operatorname{Pr}_{\boldsymbol{x} \leftarrow S^{n-1}(r)}\left[\boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{i}\right) \text { and } \boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{j}\right)\right]\right] \\
& =\operatorname{Pr}_{\boldsymbol{x}, \mathbf{z}_{i}, \mathbf{z}_{j}}\left[\boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{i}\right) \text { and } \boldsymbol{x} \in \operatorname{cover}\left(\mathbf{z}_{j}\right)\right]=\underset{\mathbf{z}_{i}}{\operatorname{Pr}}\left[x_{0} \in \operatorname{cover}\left(\mathbf{z}_{i}\right)\right] \cdot \operatorname{Pr}_{\mathbf{z}_{j}}^{\operatorname{Pr}}\left[x_{0} \in \operatorname{cover}\left(\mathbf{z}_{j}\right)\right],
\end{aligned}
$$

where the last equation follows by sampling $\boldsymbol{x}$ first, independence of $\mathbf{z}_{i}$ and $\mathbf{z}_{j}$, and symmetry.
By (12), the expectation of $\mathrm{fsa}\left(\operatorname{cover}\left(\mathbf{z}_{i}\right) \cap \operatorname{cover}\left(\mathbf{z}_{j}\right)\right)$ is at most $e^{-0.98 r^{2}}$, and hence by Markov's inequality, the probability of it being at least $e^{-0.96 r^{2}}$ is at most $e^{-0.02 r^{2}}$. Using $e^{r^{2}} \geq(N / n)^{2}$ and a union bound, the probability of one of the pairs having the fsa at least $e^{-0.96 r^{2}}$ is at most

$$
q^{2} \cdot e^{-0.02 r^{2}} \leq 2^{0.02 \sqrt{n}} \cdot(n / N)^{0.04}=o(1)
$$

since $q=2^{0.01 \sqrt{n}}$ and $N=2^{\sqrt{n}}$. This finishes the proof of the lemma.

We prove the following lemma in Section 2.4.4 to finish the proof of Lemma 2.18.

Lemma 2.24. For every typical sequence $z$ of $q$ points, we have $d_{T V}\left(\mathcal{E}_{\text {yes }}(z), \mathcal{E}_{\text {no }}^{*}(z)\right)=o(1)$.

### 2.4.4 Proof of Lemma 2.24

Fix a typical $z=\left(z_{1}, \ldots, z_{q}\right)$. Our goal is to show that the total variation distance of $\mathcal{E}_{\text {yes }}(z)$ and $\mathcal{E}_{\text {no }}^{*}(z)$ is $o(1)$. To this end, we define a distribution $\mathcal{F}$ over pairs $(\mathbf{b}, \mathbf{d})$ of strings in $\{0,1\}^{q}$
(as a coupling of $\mathcal{E}_{\text {yes }}(z)$ and $\mathcal{E}_{\text {no }}^{*}(z)$ ), where the marginal distribution of $\mathbf{b}$ as $(\mathbf{b}, \mathbf{d}) \leftarrow \mathcal{F}$ is the same as $\mathcal{E}_{\text {yes }}(z)$ and the marginal distribution of $\mathbf{d}$ is the same as $\mathcal{E}_{\text {no }}^{*}(z)$. Our goal follows by establishing

$$
\begin{equation*}
\operatorname{Pr}_{(\mathbf{b}, \mathbf{d}) \leftarrow \mathcal{F}}[\mathbf{b} \neq \mathbf{d}]=o(1) . \tag{13}
\end{equation*}
$$

To define $\mathcal{F}$, we use $\mathbf{M}$ to denote the $q \times N\{0,1\}$-valued random matrix derived from $z$ and $\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}$ (recall that $\boldsymbol{S}$ is the intersection of $N$ random halfspaces $\boldsymbol{h}_{j}, j \in[N]$ ): the $(i, j)$ th entry $\mathbf{M}_{i, j}$ of $\mathbf{M}$ is 1 if $\boldsymbol{h}_{j}\left(z_{i}\right)=1$ (i.e., $z_{i} \in \boldsymbol{h}_{j}$ ) and is 0 otherwise. We use $\mathbf{M}_{i, *}$ to denote the $i$ th row of $\mathbf{M}, \mathbf{M}_{*, j}$ to denote the $j$ th column of $\mathbf{M}$, and $\mathbf{M}^{(i)}$ to denote the $i \times N$ sub-matrix of $\mathbf{M}$ that consists of the first $i$ rows of $\mathbf{M}$. (We note that $\mathbf{M}$ is derived from $\boldsymbol{S}$ and they are defined over the same probability space. So we may consider the (conditional) distribution of $S \leftarrow \mathcal{D}_{\text {yes }}$ conditioning on an event involving $\mathbf{M}$, and we may consider the conditional distribution of M conditioning on an event involving $\boldsymbol{S}$.)

We now define the distribution $\mathcal{F}$. A pair $(\mathbf{b}, \mathbf{d}) \leftarrow \mathcal{F}$ is drawn using the following randomized procedure. The procedure has $q$ rounds and generates the $i$ th bits $\mathbf{b}_{i}$ and $\mathbf{d}_{i}$ in the $i$ th round:

1. In the first round, we draw a random real number $\mathbf{r}_{1}$ from $[0,1]$ uniformly at random. We set $\mathbf{b}_{1}=1$ if $\mathbf{r}_{1} \leq \operatorname{Pr}_{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}}\left[\boldsymbol{S}\left(z_{1}\right)=1\right]$ and set $\mathbf{b}_{1}=0$ otherwise. We then set $\mathbf{d}_{1}=1$ if $\mathbf{r}_{1} \leq \rho\left(\left\|z_{1}\right\|\right)$ and set $\mathbf{d}_{1}=0$ otherwise. (Note that for the first round, the two thresholds are indeed the same so we always have $\mathbf{b}_{1}=\mathbf{d}_{1}$.) At the end of the first round, we also draw a row vector $\mathbf{N}_{1, *}$ according to the distribution of $\mathbf{M}_{1, *}$ conditioning on $\boldsymbol{S}\left(z_{1}\right)=\mathbf{b}_{1}$.
2. In the $i$ th round, for $i$ from 2 to $q$, we draw a random real number $\mathbf{r}_{i}$ from $[0,1]$
uniformly at random. We set $\mathbf{b}_{i}=1$ if we have

$$
\mathbf{r}_{i} \leq \operatorname{Pr}_{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}}\left[\boldsymbol{S}\left(z_{i}\right)=1 \mid \mathbf{M}^{(i-1)}=\mathbf{N}^{(i-1)}\right]
$$

and set $\mathbf{b}_{i}=0$ otherwise. We then set $\mathbf{d}_{i}=1$ if $\mathbf{r}_{i} \leq \rho\left(\left\|z_{i}\right\|\right)$ and set $\mathbf{d}_{i}=0$ otherwise.
At the end of the $i$ th round, we also draw a row vector $\mathbf{N}_{i, *}$ according to the distribution of $\mathbf{M}_{i, *}$ conditioning on $\mathbf{M}^{(i-1)}=\mathbf{N}^{(i-1)}$ and $\boldsymbol{S}\left(z_{i}\right)=\mathbf{b}_{i}$.

It is clear that the marginal distributions of $\mathbf{b}$ and $\mathbf{d}$, as $(\mathbf{b}, \mathbf{d}) \leftarrow \mathcal{F}$, are $\mathcal{E}_{\text {yes }}$ and $\mathcal{E}_{\text {no }}^{*}$ respectively.

To prove (13), we introduce the following notion of nice and bad matrices.

Definition. Let $M$ be an $i \times N\{0,1\}$-valued matrix for some $i \in[q]$. We say $M$ is nice if

1. $M$ has at most $\sqrt{N}$ many 0 -entries; and
2. Each column of $M$ has at most one 0 -entry.

We say $M$ is bad otherwise.

We prove the following two lemmas and use them to prove (13).

Lemma 2.25. $\operatorname{Pr}_{S \leftarrow \mathcal{D}_{\text {yes }}}[\mathbf{M}$ is bad $]=o(1 / q)$.

Note that when $\mathbf{M}$ is nice, we have by definition that $\mathbf{M}^{(i)}$ is also nice for every $i \in[q]$.

Lemma 2.26. For any nice $(i-1) \times N\{0,1\}$-valued matrix $M^{(i-1)}$, we have

$$
\begin{equation*}
\underset{S \leftarrow \mathcal{D}_{\text {yes }}}{\mathbf{P r}}\left[\boldsymbol{S}\left(z_{i}\right)=1 \mid \mathbf{M}^{(i-1)}=M^{(i-1)}\right]=\rho\left(\left\|z_{i}\right\|\right) \pm o(1 / q) . \tag{14}
\end{equation*}
$$

Before proving Lemma 2.25 and 2.26, we first use them to prove (13). Let $\mathbf{I}_{i}$ denote the indicator random variable that is 1 if $(\mathbf{b}, \mathbf{d}) \leftarrow \mathcal{E}$ has $\mathbf{b}_{i} \neq \mathbf{d}_{i}$ and is 0 otherwise, for each
$i \in[q]$. Then (13) can be bounded from above by $\sum_{i \in[q]} \operatorname{Pr}\left[\mathbf{I}_{i}=1\right]$. To bound each $\operatorname{Pr}\left[\mathbf{I}_{i}=1\right]$ we split the event into

$$
\sum_{M^{(i-1)}} \operatorname{Pr}\left[\mathbf{N}^{(i-1)}=M^{(i-1)}\right] \cdot \operatorname{Pr}\left[\mathbf{I}_{i}=1 \mid \mathbf{N}^{(i-1)}=M^{(i-1)}\right],
$$

where the sum is over all $(i-1) \times N\{0,1\}$-valued matrices $M^{(i-1)}$, and further split the sum into two sums over nice and bad matrices $M^{(i-1)}$. As $\mathbf{N}^{(i-1)}$ has the same distribution as $\mathbf{M}^{(i-1)}$, it follows from Lemma 2.25 (and the fact that $\mathbf{M}$ is bad when $\mathbf{M}^{(i-1)}$ is bad) that the sum over bad $M^{(i-1)}$ is at most $o(1 / q)$. On the other hand, it follows from Lemma 2.26 that the sum over nice $M^{(i-1)}$ is $o(1 / q)$. As a result, we have $\operatorname{Pr}\left[\mathbf{I}_{i}=1\right]=o(1 / q)$ and thus, $\sum_{i \in[q]} \operatorname{Pr}\left[\mathbf{I}_{i}=1\right]=o(1)$.

We prove Lemmas 2.25 and 2.26 in the rest of the section.

Proof of Lemma 2.25. We show that the probability of $\mathbf{M}$ violating each of the two conditions in the definition of nice matrices is $o(1 / q)$. The lemma then follows by a union bound.

For the first condition, since $z$ is typical the probability of $\mathbf{M}_{i, j}=0$ is

$$
\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right)\right) \leq e^{-0.49 r^{2}}
$$

By linearity of expectation, the expected number of 0 -entries in $\mathbf{M}$ is at most

$$
q N \cdot e^{-0.49 r^{2}}=o(\sqrt{N} / q)
$$

using $e^{r^{2} / 2} \geq N / n, N=2^{\sqrt{n}}$ and $q=2^{0.01 \sqrt{n}}$. It follows directly from Markov's inequality that the probability of $\mathbf{M}$ having more than $\sqrt{N}$ many 0 -entries is $o(1 / q)$.

For the second condition, again since $z$ is typical, the probability of $\mathbf{M}_{i, j}=\mathbf{M}_{i^{\prime}, j}=0$ is

$$
\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{i}^{\prime}\right)\right) \leq e^{-0.96 r^{2}}
$$

By a union bound, the probability of $\mathbf{M}_{i, j}=\mathbf{M}_{i^{\prime}, j}=0$ for some $i, i^{\prime}, j$ is at most

$$
q^{2} N \cdot e^{-0.96 r^{2}}=o(1 / q)
$$

This finishes the proof of the lemma.

Finally we prove Lemma 2.26. Fix a nice $(i-1) \times N$ matrix $M$ (we henceforth omit the superscript $(i-1)$ since the number of rows of $M$ is fixed to be $i-1)$. Recall that $\boldsymbol{S}\left(z_{i}\right)=1$ if and only if $\boldsymbol{h}_{j}\left(z_{i}\right)=1$ for all $j \in[N]$. As a result, we have

$$
\operatorname{Pr}_{\boldsymbol{S} \leftarrow \mathcal{D}_{\text {yes }}}\left[\boldsymbol{S}\left(z_{i}\right)=1 \mid \mathbf{M}^{(i-1)}=M\right]=\prod_{j \in[N]} \mathbf{P r}_{\boldsymbol{h}_{j}}\left[\boldsymbol{h}_{j}\left(z_{i}\right)=1 \mid \mathbf{M}_{*, j}^{(i-1)}=M_{*, j}\right] .
$$

On the other hand, letting $\tau=\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right)\right)=\operatorname{cap}\left(r /\left\|z_{i}\right\|\right)$, we have $\rho\left(\left\|z_{i}\right\|\right)=(1-\tau)^{N}$.
In the next two claims we compare

$$
\underset{\boldsymbol{h}_{j}}{\operatorname{Pr}}\left[\boldsymbol{h}_{j}\left(z_{i}\right)=1 \mid \mathbf{M}_{*, j}^{(i-1)}=M_{*, j}\right]
$$

with $1-\tau$ for each $j \in[N]$ and show that they are very close. The first claim works on $j \in[N]$ with no 0 -entry in $M_{*, j}$ and the second claim works on $j \in[N]$ with one 0 -entry in $M_{*, j}$. (These two possibilities cover all $j \in[N]$ since the matrix $M$ is nice.) Below we omit $\mathbf{M}_{*, j}^{(i-1)}$ in writing the conditional probabilities.

Claim 2.27. For each $j \in[N]$ with no 0 -entry in the $j$ th column $M_{*, j}$, we have

$$
\underset{\boldsymbol{h}_{j}}{\operatorname{Pr}}\left[\boldsymbol{h}_{j}\left(z_{i}\right)=1 \mid M_{*, j}\right]=(1-\tau)\left(1 \pm \frac{o(1)}{q N}\right) .
$$

Proof. Let $\delta$ be the probability of $\boldsymbol{h}_{j}\left(z_{i}\right)=0$ conditioning on $M_{*, j}$ (which is all-1). Then

$$
\delta=\frac{\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right)-\bigcup_{j<i} \operatorname{cover}\left(z_{j}\right)\right)}{1-\mathrm{fsa}\left(\bigcup_{j<i} \operatorname{cover}\left(z_{j}\right)\right)}
$$

Using $e^{-0.51 r^{2}} \leq \mathrm{fsa}\left(\operatorname{cover}\left(z_{j}\right)\right) \leq e^{-0.49 r^{2}}$ and $\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{j}\right)\right) \leq e^{-0.96 r^{2}}$, we have

$$
\delta \leq \frac{\tau}{1-q \cdot e^{-0.49 r^{2}}}<\tau\left(1+2 q \cdot e^{-0.49 r^{2}}\right)=\tau+2 \tau q \cdot e^{-0.49 r^{2}} .
$$

Using $\tau \leq e^{-0.49 r^{2}}$ and $e^{r^{2} / 2} \geq N / n$, we have

$$
1-\delta \geq 1-\tau-2 \tau q \cdot e^{-0.49 r^{2}} \geq 1-\tau-o(1 /(q N)) \geq(1-\tau)(1-o(1 /(q N)))
$$

On the other hand, we have $\delta \geq \tau-q \cdot e^{-0.96 r^{2}}$ and thus,

$$
1-\delta \leq 1-\tau+q \cdot e^{-0.96 r^{2}} \leq 1-\tau+o(1 /(q N))=(1-\tau)(1+o(1 /(q N)))
$$

This finishes the proof of the claim.

Claim 2.28. For each $j \in[N]$ with one 0 -entry in the $j$ th column $M_{*, j}$, we have

$$
\underset{\boldsymbol{h}_{j}}{\operatorname{Pr}}\left[\boldsymbol{h}_{j}\left(z_{i}\right)=1 \mid M_{*, j}\right] \geq 1-O\left(e^{-0.45 r^{2}}\right) .
$$

Proof. Let $i^{\prime}$ be the point with $M_{i^{\prime}, j}=1$ and $\delta$ be the conditional probability of $\boldsymbol{h}_{j}\left(z_{i}\right)=0$.
Then

$$
\delta \leq \frac{\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}\right) \cap \operatorname{cover}\left(z_{i^{\prime}}\right)\right)}{\mathrm{fsa}\left(\operatorname{cover}\left(z_{i}^{\prime}\right)-\bigcup_{j<i: j \neq i^{\prime}} \operatorname{cover}\left(z_{j}\right)\right)} \leq \frac{e^{-0.96 r^{2}}}{e^{-0.51 r^{2}}-q \cdot e^{-0.96 r^{2}}}=O\left(e^{-0.45 r^{2}}\right)
$$

by our choice of $q$. This finishes the proof of the claim.

We combine the two claims to prove Lemma 2.26.

Proof of Lemma 2.26. Let $h$ be the number of 0 -entries in $M$. We have $h \leq \sqrt{N}$ since $M$ is
nice. By Claims 2.27, the conditional probability of $\boldsymbol{S}\left(z_{i}\right)=1$ is at most

$$
\begin{aligned}
\left((1-\tau)\left(1+o\left(\frac{1}{q N}\right)\right)\right)^{N-h} & =\rho\left(\left\|z_{i}\right\|\right) \cdot \frac{1}{(1-\tau)^{h}} \cdot\left(1+o\left(\frac{1}{q N}\right)\right)^{N-h} \\
& \leq \rho\left(\left\|z_{i}\right\|\right) \cdot(1+2 \tau)^{h} \cdot\left(1+o\left(\frac{1}{q N}\right)\right)^{N} \\
& \leq \rho\left(\left\|z_{i}\right\|\right) \cdot \exp (2 \tau h+o(1 / q)) \\
& =\rho\left(\left\|z_{i}\right\|\right) \cdot \exp (o(1 / q))=\rho\left(\left\|z_{i}\right\|\right)+o(1 / q)
\end{aligned}
$$

Similarly, the conditional probability of $\boldsymbol{S}\left(z_{i}\right)=1$ is at least

$$
\begin{aligned}
& \left((1-\tau)\left(1-o\left(\frac{1}{q N}\right)\right)\right)^{N-h}\left(1-O\left(e^{-0.45 r^{2}}\right)\right)^{h} \\
& \quad \geq \rho\left(\left\|z_{i}\right\|\right) \cdot\left(1-o\left(\frac{1}{q N}\right)\right)^{N-h}\left(1-O\left(e^{-0.45 r^{2}}\right)\right)^{h} \\
& \quad \geq \rho\left(\left\|z_{i}\right\|\right) \cdot(1-o(1 / q)) \geq \rho\left(\left\|z_{i}\right\|\right)-o(1 / q)
\end{aligned}
$$

This finishes the proof of the lemma.

### 2.5 One-sided lower bound

We recall Theorem 2.1:

Theorem. Any one-sided sample-based algorithm that is an $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ for some $\varepsilon<1 / 2$ must use $2^{\Omega(n)}$ samples.

We say a finite set $\left\{x^{1}, \ldots, x^{M}\right\} \subset \mathbb{R}^{n}$ is shattered by $\mathcal{C}_{\text {convex }}$ if for every $\left(b_{1}, \ldots, b_{M}\right) \in$ $\{0,1\}^{M}$ there is a convex set $C \in \mathcal{C}_{\text {convex }}$ such that $C\left(x^{i}\right)=b_{i}$ for all $i \in[M]$. Theorem 2.1 follows from the following lemma:

Lemma 2.29. There is an absolute constant $c>0$ such that for $M=2^{c n}$, it holds that

$$
\underset{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}\left[\left\{\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M}\right\} \text { is shattered by } \mathcal{C}_{\text {convex }}\right] \geq 1-o(1) .
$$

Proof of Theorem 2.1 using Lemma 2.29. Suppose that $A$ were a one-sided sample-based algorithm for $\varepsilon$-testing $\mathcal{C}_{\text {convex }}$ using at most $M$ samples. Fix a set $S$ that is $\varepsilon$-far from $\mathcal{C}_{\text {convex }}$ to be the unknown target subset of $\mathbb{R}^{n}$ that is being tested. Since $S$ is $\varepsilon$-far from convex, it must be the case that

$$
\begin{equation*}
\underset{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}\left[A \text { rejects when run on }\left(\boldsymbol{x}^{1}, S\left(\boldsymbol{x}^{1}\right)\right), \ldots,\left(\boldsymbol{x}^{M}, S\left(\boldsymbol{x}^{M}\right)\right)\right] \geq 2 / 3 . \tag{15}
\end{equation*}
$$

But Lemma 2.29 together with the one-sidedness of $A$ imply that

$$
\underset{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}\left[\forall\left(b^{1}, \ldots, b^{M}\right) \in\{0,1\}^{M}, A \text { rejects on }\left(\boldsymbol{x}^{1}, b^{1}\right), \ldots,\left(\boldsymbol{x}^{M}, b^{M}\right)\right] \leq o(1),
$$

as $A$ can only reject if the labeled samples are not consistent with any convex set, which implies that $A$ cannot reject when $\left\{\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M}\right\}$ is shattered by $\mathcal{C}_{\text {convex }}$. This contradicts (15).

In the next subsection we prove Lemma 2.29 for $c=1 / 500$.

### 2.5.1 Proof of Lemma 2.29

Let $M=2^{c n}$ with $c=1 / 500$. We prove the following lemma:

Lemma 2.30. For $\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M}$ drawn independently from $\mathcal{N}(0,1)^{n}$, with probability $1-o(1)$ it is the case that for all $i \in[M]$, no $\boldsymbol{x}^{i}$ lies in $\operatorname{Conv}\left(\left\{\boldsymbol{x}^{j}: j \in[M] \backslash i\right\}\right)$.

If $\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M}$ are such that no $\boldsymbol{x}^{i}$ lies in $\operatorname{Conv}\left(\left\{\boldsymbol{x}^{j}: j \in[M] \backslash i\right\}\right)$, then given any $\left(b^{1}, \ldots, b^{M}\right)$, by taking $C=\operatorname{Conv}\left(\left\{\boldsymbol{x}^{i}: b^{i}=1\right\}\right)$ we see that there is a convex set $C$ such that $C\left(\boldsymbol{x}^{i}\right)=b^{i}$ for all $i \in[M]$. Thus to establish Lemma 2.29 it suffices to prove Lemma 2.30. Intuitively, like in the condition for typical sequences, the sample points will all be roughly located on the sphere of radius $\sqrt{n}$ centered at the origin.

To prove Lemma 2.30, it suffices to show that for each fixed $j \in[M]$ we have

$$
\begin{equation*}
\operatorname{Pr}_{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}\left[\boldsymbol{x}^{j} \in \operatorname{Conv}\left(\left\{\boldsymbol{x}^{k}: k \in[M] \backslash\{j\}\right\}\right)\right] \leq M^{-2} \tag{16}
\end{equation*}
$$

since given this a union bound implies that

$$
\underset{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}{\mathbf{P r}}\left[\text { for some } j \in[M], \boldsymbol{x}^{j} \text { lies in } \operatorname{Conv}\left(\left\{\boldsymbol{x}^{k}: k \in[M] \backslash\{j\}\right\}\right)\right] \leq M^{-1}=o(1) .
$$

By symmetry, to establish (16) it suffices to show that

$$
\begin{equation*}
\operatorname{Pr}_{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}\left[\boldsymbol{x}^{M} \in \operatorname{Conv}\left(\left\{\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M-1}\right\}\right)\right] \leq M^{-2} \tag{17}
\end{equation*}
$$

In turn (17) follows from the following inequalities $\left(v \in \mathbb{R}^{n}\right.$ is a fixed unit vector in the second)

$$
\begin{equation*}
\underset{\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}[\|\boldsymbol{x}\| \leq \sqrt{n} / 10]<\frac{1}{2} M^{-2} \quad \text { and } \quad \underset{\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}[\boldsymbol{x} \cdot v \geq \sqrt{n} / 10]<\frac{1}{2} M^{-3} . \tag{18}
\end{equation*}
$$

The first inequality follows directly from Lemma 2.5 using $c=1 / 500$. For the second, by the spherical symmetry of $\mathcal{N}(0,1)^{n}$ we may take $v=(1,0, \ldots, 0)$. Recall the standard Gaussian tail bound

$$
\operatorname{Pr}_{\boldsymbol{z} \leftarrow \mathcal{N}(0,1)}[\boldsymbol{z} \geq t] \leq e^{-t^{2} / 2}
$$

for $t \geq 0$. This gives us that

$$
\operatorname{Pr}_{\boldsymbol{x} \leftarrow \mathcal{N}(0,1)^{n}}[\boldsymbol{x} \cdot v \geq \sqrt{n} / 10] \leq e^{-n / 200}<\frac{1}{2} M^{-3}
$$

again using that $M=2^{c n}$ and $c=1 / 500$.
Finally, to see that (17) follows from (18), we observe first that by the first inequality we may assume that $\left\|\boldsymbol{x}^{M}\right\|>\sqrt{n} / 10$ (at the cost of failure probability at most $M^{-2} / 2$ towards
(17)); fix any such outcome $x^{M}$ of $\boldsymbol{x}^{M}$. By a union bound over $\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M-1}$ and the second inequality, we have

$$
\underset{\boldsymbol{x}^{i} \leftarrow \mathcal{N}(0,1)^{n}}{\operatorname{Pr}}\left[\text { any } i \in[M-1] \text { has } \boldsymbol{x}^{i} \cdot \frac{x^{M}}{\left\|x^{M}\right\|} \geq \sqrt{n} / 10\right]<\frac{1}{2} M^{-2} .
$$

But if every $\boldsymbol{x}^{i}$ has $\boldsymbol{x}^{i} \cdot\left(x^{M} /\left\|x^{M}\right\|\right)<\sqrt{n} / 10<\left\|x^{M}\right\|$, then $x^{M} \notin \operatorname{Conv}\left(\left\{\boldsymbol{x}^{1}, \ldots, \boldsymbol{x}^{M-1}\right\}\right)$.

### 2.6 Two-sided upper bound

Recall Theorem 2.4:

Theorem. For any $\varepsilon>0$, there is a two-sided sample-based $\varepsilon$-tester for convexity over $\mathcal{N}(0,1)^{n}$ using $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ samples.

We begin by recalling some definitions from learning theory. Let $\mathcal{C}$ be a class of subsets of $\mathbb{R}^{n}$ (such as $\mathcal{C}_{\text {convex }}$ ). We say an algorithm learns $\mathcal{C}$ to error $\varepsilon$ with confidence $1-\delta$ under $\mathcal{N}(0,1)^{n}$ if, given a set of labeled samples $(\boldsymbol{x}, S(\boldsymbol{x}))$ from an unknown set $S \in \mathcal{C}$ with $\boldsymbol{x}$ 's drawn independently from $\mathcal{N}(0,1)^{n}$, the algorithm outputs with probability at least $1-\delta$ a hypothesis set $H \subseteq \mathbb{R}^{n}$ with $\operatorname{Vol}(S \triangle H) \leq \varepsilon$. We say it is a proper learning algorithm if it always outputs a hypothesis $H$ that belongs to $\mathcal{C}$. Next we recall the main algorithmic result of [KOS07]:

Theorem 2.31 (Theorem 5 of $[\mathrm{KOS} 07]$ ). There is an algorithm $A$ that learns the class $\mathcal{C}_{\text {convex }}$ of all convex subsets of $\mathbb{R}^{n}$ to error $\varepsilon$ with confidence $1-\delta$ under $\mathcal{N}(0,1)^{n}$ using

$$
n^{O\left(\sqrt{n} / \varepsilon^{2}\right)} \cdot \log (1 / \delta)
$$

samples ${ }^{3}$ drawn from $\mathcal{N}(0,1)^{n}$.
${ }^{3}$ Theorem 5 as stated in [KOS07] gives a sample complexity upper bound of $n^{O\left(\sqrt{n} / \varepsilon^{4}\right)}$

Next we recall the result of Goldreich, Goldwasser and Ron which relates proper learnability of a class $\mathcal{C}$ to the testability of $\mathcal{C}$.

Theorem 2.32 (Proposition 3.1.1 of [GGR98], adapted to our context). Let $\mathcal{C}$ be a class of subsets of $\mathbb{R}^{n}$ that has a proper learning algorithm $A$ which uses $m_{A}(n, \varepsilon, \delta)$ samples from $\mathcal{N}(0,1)^{n}$ to learn $\mathcal{C}$ to error $\varepsilon$ with confidence $1-\delta$. Then there is a property testing algorithm $A_{\text {test }}$ for $\mathcal{C}$ under the distribution $\mathcal{N}(0,1)^{n}$ that uses

$$
m_{A}(n, \varepsilon / 2, \delta / 2)+O(\log (1 / \delta) / \varepsilon)
$$

samples drawn from $\mathcal{N}(0,1)^{n}$.

By Theorem 2.32, to obtain Theorem 2.4 it suffices to have a proper learning analogue of Theorem 2.31. We establish the required result, as a corollary of Theorem 2.31, in the next subsection:

Corollary 2.33. There is a proper learning algorithm $A^{\prime}$ for the class $\mathcal{C}_{\text {convex }}$ of all convex subsets of $\mathbb{R}^{n}$ that uses $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)} \cdot \log (1 / \delta)$ samples from $\mathcal{N}(0,1)^{n}$ to learn to error $\varepsilon$ with confidence $1-\delta$.

We remark that while algorithm $A$ from Theorem 2.31 runs in time $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ and uses $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ samples, the algorithm $A^{\prime}$ of Corollary 2.33 presented below has a much larger running time (at least $(n / \varepsilon)^{O(n)}$ ); however, its sample complexity is essentially no larger than that of algorithm $A$.
for agnostic learning, but inspection of the proof gives the theorem as stated here, with an upper bound of $n^{O\left(\sqrt{n} / \varepsilon^{2}\right)}$ for non-agnostic learning.

### 2.6.1 Proof of Corollary 2.33

The idea behind the proof of Corollary 2.33 is simple. Let $S \subseteq \mathbb{R}^{n}$ be the unknown target convex set that is to be learned. Algorithm $A^{\prime}$ first runs algorithm $A$ with error parameter $\varepsilon / 5$ and confidence parameter $\delta / 2$ to obtain, with probability $1-(\delta / 2)$, a hypothesis $H \subseteq \mathbb{R}^{n}$ with $\operatorname{Vol}(H \triangle S) \leq \varepsilon / 5$.

In the rest of the algorithm we find with high probability a convex set $C^{*}$ with $\operatorname{Vol}(H \triangle$ $\left.C^{*}\right) \leq 4 \varepsilon / 5$ and thus, we have $\operatorname{Vol}\left(S \triangle C^{*}\right) \leq \varepsilon / 5+4 \varepsilon / 5=\varepsilon$. (Note that this part of the algorithm does not require any labeled samples $(\boldsymbol{x}, S(\boldsymbol{x}))$ from the oracle for $S$.)

For this purpose let $\mathcal{C}_{\text {cover }} \subset \mathcal{C}_{\text {convex }}$ be a finite $(\varepsilon / 5)$-cover of $\mathcal{C}_{\text {convex }}$. (We show in Corollary 2.34 below that there is an algorithm the finds a finite $(\varepsilon / 5)$-cover of $\mathcal{C}_{\text {convex }}$.) Next, the algorithm $A^{\prime}$ enumerates over all elements $C \in \mathcal{C}_{\text {cover }}$ and for each such $C$ uses random sampling from $\mathcal{N}(0,1)^{n}$ to estimate $\operatorname{Vol}(H \triangle C)$ to within an additive error of $\varepsilon / 5$, with success probability $1-\delta /\left(2\left|\mathcal{C}_{\text {cover }}\right|\right)$ for each $C$. (Note that this does not require any labeled samples $(\boldsymbol{x}, S(\boldsymbol{x}))$ from the oracle for $S$, since $A^{\prime}$ can generate its own draws from $\mathcal{N}(0,1)^{n}$ and for each such $\boldsymbol{x}$ it can compute $H(\boldsymbol{x})$ and $C(\boldsymbol{x})$ on its own.) $A^{\prime}$ outputs the $C^{*} \in \mathcal{C}_{\text {cover }}$ for which the estimate of $\operatorname{Vol}\left(H \triangle C^{*}\right)$ is smallest.

The fact that this works follows a standard argument. Since

$$
\operatorname{Vol}(H \triangle S) \leq \varepsilon / 5 \quad \text { and } \quad \operatorname{Vol}\left(S \triangle C^{\prime}\right) \leq \varepsilon / 5
$$

for some set $C^{\prime} \in \mathcal{C}_{\text {cover }}$, it holds that $\operatorname{Vol}\left(H \triangle C^{\prime}\right) \leq 2 \varepsilon / 5$ and hence the estimate of $\operatorname{Vol}\left(H \triangle C^{\prime}\right)$ will be at most $3 \varepsilon / 5$. Thus the element $C^{*}$ of $\mathcal{C}_{\text {cover }}$ that is selected will have its estimated value of $\operatorname{Vol}\left(H \triangle C^{*}\right)$ being at most $3 \varepsilon / 5$, which implies that its actual value of $\operatorname{Vol}\left(H \triangle C^{*}\right)$ will be at most $4 \varepsilon / 5$ (since each estimate is within $\pm \varepsilon / 5$ of the true value).

Given the above analysis, to finish the proof of Corollary 2.33 it suffices to establish the following corollary of structural results proved in Sections 2.2 and 2.3.1, which shows that indeed it is possible for $A^{\prime}$ to enumerate over the elements of $\mathcal{C}_{\text {cover }}$ as described above:

Corollary 2.34. There is an algorithm that, on inputs $\varepsilon$ and $n$, outputs a finite $\varepsilon$-cover of $\mathcal{C}_{\text {convex }}$.

Proof. We recall the material and parameter settings from Section 2.3.1. Since every convex set in $\mathbb{R}^{n}$ is $(\epsilon / 4)$-close to a set in $\mathcal{C}_{\text {convex }}^{\prime}$, it suffices to describe a finite family $\mathcal{C}$ of convex sets $C_{1}, C_{2}, \ldots$ such that every $C \in \mathcal{C}_{\text {convex }}^{\prime}$ is $(3 \epsilon / 4)$-close to some $C_{i}$ in $\mathcal{C}$. We claim that

$$
\mathcal{C}=\left\{\operatorname{Conv}\left(\cup_{\text {Cube } \in Q} \text { Cube }\right) \mid Q \subseteq \text { CubeSet }\right\}
$$

is such a family. To see this, fix any convex body $C \in \mathcal{C}_{\text {convex }}^{\prime}$. Let

$$
Q_{C}=\{\text { Cube } \in \text { CubeSet } \mid \text { Cube } \subseteq C\},
$$

the set of cubes that are entirely contained in $C$. Note that $\operatorname{Conv}\left(Q_{C}\right)$ is a subset of $C$. If a Cube contains at least one point in $C$ and at least one point outside $C$, then every point in Cube has distance at most $\ell \sqrt{n}$ from the boundary of $C$ (since any two points in a given Cube have distance at most $\ell \sqrt{n})$. Thus, the missing volume $C \backslash \operatorname{Conv}\left(Q_{C}\right)$ is completely contained in $\partial C+\operatorname{Ball}(\ell \sqrt{n})$, whose Gaussian volume, by Theorem 2.7, is at most $20 n^{5 / 8} n^{\prime} \sqrt{\ell \sqrt{n}} \ll 3 \varepsilon / 4$.

## 3 An improved lower bound for testing acyclicity in sparse graphs

Goldreich and Ron [GR97] defined property testing in the incidence list model for boundeddegree graphs, whose extension to directed graphs was introduced by Bender and Ron [BR02]. A directed graph (or digraph) $D=(V, E)$ consists of vertices $V$ and arcs $E$ connecting those vertices. An arc $(u, v)$, where $u, v \in V$ is considered to be originating at $u$ and terminating at $v$. We use $N=|V|$ to denote the number of vertices. The indegree (resp. outdegree) of a vertex $v$ is the number of arcs that originate (resp. terminate) at $v$. A source (resp. sink) is a vertex with indegree (resp. outdegree) 0 . A graph is said to be acyclic if it does not contain any directed cycle, and a feedback arc set is a set of arcs whose deletion results in an acyclic digraph.

In the bounded-degree digraph model of Bender and Ron [BR02], a query to a digraph $D=(V, E)$ of bounded indegree and outdegree $d$ is a triple $q=(v, i, b)$, where $v \in V$ is a vertex, $i \in\{1, \ldots, d\}$ is an integer, and $b \in\{\mathrm{in}$, out $\}$ is a bit, and the answer of that query is the $i$ th arc (and the other incident vertex) originating or terminating (depending on $b$ ) at vertex $v$, if it exists. The distance between two digraphs $D_{1}$ and $D_{2}$ on the same vertex set is defined to be

$$
\operatorname{dist}\left(D_{1}, D_{2}\right)=\frac{\left|E\left(D_{1}\right) \triangle E\left(D_{2}\right)\right|}{d N}
$$

We can think of the distance as the fraction of edges one needs to delete from or add to $D_{1}$ to obtain $D_{2}$. The focus of this section is on the monotone property of acyclicity, so for a given directed graph $D$, it is $\varepsilon$-far from acyclic if one needs to delete $\varepsilon d N$ many edges to
break all directed cycles in $D$.
As with convexity testing, an algorithm $\mathcal{A}$ is an $\varepsilon$-tester for acyclicity if it accepts a digraph $D$ with probability at least $2 / 3$ when $D$ is acyclic, and rejects with probability at least $2 / 3$ when it is $\varepsilon$-far from acyclic. For algorithms that commit only one-sided error, they must accept if $D$ is acyclic.

### 3.1 Distributions of random graphs

Bender and Ron [BR02] showed that for two-sided error, any algorithm needs at least $\Omega\left(N^{1 / 3}\right)$ queries. Implicit in their work is an additional $\Omega\left(N^{1 / 2}\right)$ lower bound for one-sided error. They introduced the following two distributions on random digraphs:

- $\mathcal{D}_{1}$, which are all acyclic, consisting of $L$ many layers $R_{1}, R_{2}, \ldots, R_{L}$ of $W$ vertices, where $L W=N$. For each layer $R_{i}$, there are $d W$ outgoing arcs to layer $R_{i+1}$, determined by $d$ random matchings between $R_{i}$ and $R_{i+1}$.
- $\mathcal{D}_{2}$, which are $\varepsilon_{0}$-far from acyclic (for some constant $\varepsilon_{0}>0$ ) with high probability, consisting of two layers $B_{0}$ and $B_{1}$ of $N / 2$ vertices. For each layer $B_{i}$, there are $d N / 2$ outgoing arcs to layer $B_{1-i}$, again determined by $d$ random matchings.

Bender and Ron [BR02] proved that for $W=N^{2 / 3}, L=N^{1 / 3}$ any algorithm that can distinguish between these two distributions with high probability requires at least $\Omega\left(N^{1 / 3}\right)$ queries. The "NO" distribution $\mathcal{D}_{2}$ alone immediately yields a stronger lower bound for one-sided error via birthday paradox arguments:

Proposition 3.1 (Bender and Ron [BR02, Lemma 7]). Any algorithm $\mathcal{A}$ that makes $Q=$
$o\left(N^{1 / 2}\right)$ queries to a graph $G$ randomly drawn from the distribution $\mathcal{D}_{2}$ does not find a cycle with high probability.

Before we sketch the proof of their intermediate result, we introduce some notation. Let $\mathcal{A}$ be a (possibly randomized) algorithm for testing acyclicity using $Q=Q(N)$ queries. Formally, algorithm $\mathcal{A}$ is a mapping from query-answer histories $\left(q_{1}, a_{1}\right), \ldots,\left(q_{t}, a_{t}\right)$ to the next query $q_{t+1}$ if $t<Q$ and \{"accept", "reject" $\}$ if $t=Q$. Define the knowledge graph $K G\left(e_{1}, \ldots, e_{i}\right)$ to be the subgraph formed by the arcs $e_{1}, \ldots, e_{i}$ and their incident vertices. Typically, we take the arcs to be the set of all answers $a_{1}, \ldots, a_{t}$ so far after the $t$ th query during the runtime of $\mathcal{A}$ but later on it will be beneficial to consider the knowledge graph on a subset of the queries. We assume that no algorithm will make the same query twice, as it reveals no new information about the graph.

If $\mathcal{A}$ is truly a tester for acyclicity that makes only one-sided error, then there needs to be a directed cycle in the knowledge graph $K G\left(a_{1}, \ldots, a_{Q}\right)$, where $Q=o\left(N^{1 / 2}\right)$ as stated in Proposition 3.1. As it turns out, not only does the algorithm not find a directed cycle, but it does not even find a cycle in the underlying undirected graph:

Proof Sketch of Proposition 3.1. Let $\left(q_{1}, a_{1}\right),\left(q_{2}, a_{2}\right), \ldots,\left(q_{Q}, a_{Q}\right)$ be the query-answer history for algorithm $\mathcal{A}$. Since each query adds at most one vertex from each of the two layers $B_{1}$ and $B_{2}$, after $Q$ queries, the knowledge graph $K G\left(a_{1}, \ldots, a_{Q}\right)$ has at most $Q$ vertices in each layer at any time. The probability that the answer of the $t$ th query, for each $t=1, \ldots, Q$, is already a vertex in $K G\left(a_{1}, \ldots, a_{t-1}\right)$ is at most $Q /(N / 2-Q)$ (in the denominator, some potential endpoints are ruled out by the perfect matching condition), so by the union bound,
the probability that this happens at least once over all $Q$ queries is at most

$$
\sum_{t=1}^{Q} \frac{Q}{N / 2-Q}=O\left(Q^{2} / N\right)=o_{N}(1)
$$

Equivalently, the above calculation shows that for each of the $Q$ queries, a new previously unseen vertex is added to the knowledge graph. We call such a knowledge graph tree-like, and analyzing subsequences of queries that produce tree-like knowledge graphs is fundamental to our main result.

By a similar argument, a random walk, where we repeatedly choose one of the outgoing arcs of a vertex at random, will uncover an already visited vertex in roughly $\Theta\left(N^{1 / 2}\right)$ queries. Our approach is to defend against these types of algorithms, so we propose another distribution of graphs, similar to a construction proposed by Bender and Ron [BR02], that is built with the following principles in mind:

- Start with a directed graph which is $\varepsilon$-far from acyclic.
- Add additional arcs to the existing vertices to "traps" i.e. vertices where all outgoing paths lead to sinks.

Our lower bound applies to the case when the algorithm only has access to outgoing arcs. To simplify the analysis, for some vertices we choose its outneighbors uniformly at random with replacement, unlike in the distributions $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$, where the arcs come from perfect matchings. Since the algorithm does not have access to incoming arcs, we permit the indegree to be arbitrarily high, though it should be straightforward to avoid this relaxation. Our improvement on Proposition 3.1 is the following:

Theorem 3.2. Testing acyclicity with one-sided error and query access to only outgoing arcs requires at least $\Omega\left(N^{5 / 9-\delta}\right)$ queries, for any $\delta>0$.

### 3.2 A hard digraph distribution for outgoing-only queries

The proof of the aforementioned result requires analyzing the following distribution, which resembles an amalgamation of distributions $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$. Define $\mathcal{D}^{\prime}$ to be the following distribution of random graphs:

- Each graph in $\mathcal{D}^{\prime}$ has $3 N$ vertices grouped into subsets $B, R_{1}, R_{2}, \ldots$ :
- $N$ vertices labeled blue, belonging to $B$.
- $2 N$ vertices labeled red, arranged into layers $R_{1}, R_{2}, \ldots, R_{2 L}$ of $W$ vertices each, where $L$ and $W$ are parameters to be decided later. ${ }^{4}$
- Each vertex, except those in $R_{2 L}$, have $d$ outgoing arcs:
- For each vertex in $B$, the endpoints of its $d$ arcs are chosen uniformly at random, with replacement, from $B \cup R_{1} \cup \cdots \cup R_{L}$.
- For each vertex in $R_{i}, i=1, \ldots, 2 L-1$, the endpoints of its $d$ arcs are chosen uniformly at random, with replacement, from $R_{i+1}$.

An illustration of a random graph drawn from $\mathcal{D}^{\prime}$ is shown in Figure 5. Vertices may have, by a standard balls-into-bins argument, superconstant indegree, so such a graph would not be permissible in model where one can query incoming arcs, as the indegree needs to be

[^1]

Figure 5: An $\varepsilon$-far from acyclic graph (left) with arcs that lead to an acyclic graph of dead ends (right).
bounded by an absolute constant. We first show that this distribution is indeed $\varepsilon_{0}$-far from acyclic for some $\varepsilon_{0}>0$, regardless of the choice of parameters $W$ and $L$ :

Lemma 3.3. Let $d \geq 128$ and $\varepsilon_{0}=1 / 48$. Then with probability at least $1-2^{-N}$, a random graph drawn from $\mathcal{D}^{\prime}$ is $\varepsilon_{0}$-far from acyclic.

The lemma is proven by considering the fact that every acyclic graph has a topological sort, while every $\varepsilon$-far from acyclic graph does not have any topological sort with few errors.

Claim 3.4. An $N$-vertex directed graph $(V, E)$ is $\varepsilon$-far from acyclic if and only if for all (bijective) vertex orderings $\pi: V \rightarrow\{1, \ldots, N\}$, the number of "backedges" (i.e. directed edges $\left(v, v^{\prime}\right)$ such that $\left.\pi(v)>\pi\left(v^{\prime}\right)\right)$ is greater than $\varepsilon d N$.

Proof. We prove the contrapositive in both directions: $(\Rightarrow)$ Deleting all the backedges leaves an acyclic graph, showing that the graph is $\varepsilon$-close to acyclic. $(\Leftarrow)$ Given a feedback arc set,
after deleting it we can find a topological sort of the resulting acyclic graph. The ordering resulting from the topological sort induces a set of backedges that is a subset of the feedback arc set.

Given any vertex ordering $\pi$ as described above, by splitting the ordered sequence of vertices in half $\pi$ induces an ordered balanced partition of vertices $\left(V_{f}, V_{\ell}\right)$, or more formally,

$$
\begin{aligned}
V_{f} & =\pi^{-1}(\{1, \ldots, N / 2\}), \\
V_{\ell} & =\pi^{-1}(\{N / 2+1, \ldots, N\}) .
\end{aligned}
$$

Claim 3.5. Suppose that for all balanced partitions $\left(V_{f}, V_{\ell}\right)$ of the vertices of a directed graph $G$ into two equal-sized halves, the number of edges $E\left(V_{\ell}, V_{f}\right):=\left|\left\{\left(v, v^{\prime}\right) \mid v \in V_{\ell}, v^{\prime} \in V_{f}\right\}\right|$ is at least $\varepsilon d N$. Then $G$ is $\varepsilon$-far from acyclic.

Proof. Every ordering of vertices $\pi$ induces such a balanced partition $\left(V_{f}, V_{\ell}\right)$, and all such edges are backedges. The result follows from Claim 3.4.

We use the above claim to argue that almost every graph in $\mathcal{D}^{\prime}$ is far from acyclic:

Proof of Lemma 3.3. It suffices to consider only the induced subgraph on the blue vertices of a random graph drawn from $\mathcal{D}^{\prime}$, which we may generate in the following way:

- For each vertex $v$ of the $N$ blue vertices, repeat $d$ times:
- With probability $1 / 2$, choose a random vertex $v^{\prime} \in B$ and add a directed edge $\left(v, v^{\prime}\right)$.

This simulates the previous random graph generation process because the blue vertices form exactly half of the potential outneighbors of a blue vertex.

Fix a specific balanced partition $\left(V_{f}, V_{\ell}\right)$ of the $N$ blue vertices. Then the number of edges in $E\left(V_{\ell}, V_{f}\right)$ is the sum of $d\left|V_{\ell}\right|=d N / 2$ Bernoulli random variables each with probability $1 / 4$ (for each outgoing edge from a vertex in $V_{\ell}$, there is a $1 / 2$ chance of it even being present, and a $1 / 2$ chance its endpoint is a vertex in $V_{f}$ if it is present). By a Chernoff bound, the probability that the total number of such edges is less than

$$
\frac{1}{2}\left(\frac{d N}{2} \cdot \frac{1}{4}\right)=\frac{d N}{16}
$$

is at most

$$
e^{-(d N / 8)(1 / 2)^{2}(1 / 2)}=e^{-d N / 64}<2^{-d N / 64}
$$

which is at most $2^{-2 N}$ for $d \geq 128$. The number of balanced partitions is at most

$$
\binom{N}{N / 2} \leq \sum_{i=0}^{N}\binom{N}{i}=2^{N}
$$

so by taking a union bound over all possible partitions, we find that the probability that all partitions have at least $d N / 16$ edges from $V_{\ell}$ to $V_{f}$ (implying $D$ is $(1 / 16)$-far from acyclic with high probability) is at least $1-2^{-N}$.

Lifting back to the original graph, the number of vertices is $3 N$, which is triple the size of $B$, so if just the subgraph induced on blue vertices is $(1 / 16)$-far from acyclic, then the entire graph is $(1 / 3)(1 / 16)=(1 / 48)$-far from acyclic.

### 3.3 An improved lower bound for acyclicity testing

We assume in this section that algorithms only have query access to the outgoing arcs of a vertex - that is, a query is now of the form $q=(v, i)$, where there is no longer a bit indicating whether or not to query an outgoing or incoming arc. Our analysis focuses on what we call
surprise edges, arcs $a_{t}$ that terminate at vertices already in the existing knowledge graph $K G\left(a_{1}, \ldots, a_{t-1}\right)$. A one-sided algorithm must detect a directed cycle in order to determine if a graph is $\varepsilon$-far from acyclic, in which case the cycle must have been discovered through a surprise edge. However, note that not all surprise edges form cycles, even in the underlying undirected graph.

The runtime of the algorithm is divided up into blocks of queries called epochs, where each epoch ends when one of the following two conditions occurs:

- A surprise edge has been discovered, or
- $L$ queries have been made in this epoch.

Thus, the first condition forces the knowledge graph of an epoch to be a directed forest (i.e. a graph with no cycles in its underlying undirected graph). The knowledge graph is almost tree-like, except that the algorithm can discover sinks within an epoch. The second condition, which we sometimes refer to as a time-out, implies that if at any point one discovers a sink vertex, that component in the knowledge graph does not contain any blue vertices. Thus any component of the knowledge graph containing a blue vertex is indeed tree-like. We also define a blue epoch to be an epoch that ends in a surprise edge between two blue vertices. Note that the algorithm is not told the color of the vertices as part of the answer to a query.

The proof proceeds in three steps. First, we bound the number of epochs and blue epochs (Lemma 3.6). Then, we show that within any given epoch, there is a small chance that one uncovers a directed path of blue vertices of logarithmic length (Lemma 3.7). Finally, with a rough estimate on maximum tree sizes (Proposition 3.8), we combine the two aforemen-
tioned results to determine the maximum number of "ancestors" a vertex can have. After determining appropriate values for $L$ and $W$, the "dimensions" of the set of red vertices, we can show that with $\Omega\left(N^{5 / 9-\delta}\right)$ queries, one cannot find a cycle with high probability.

Formally, one defines a random process $\mathcal{P}$ that incrementally maintains a random graph uniformly at random from $\mathcal{D}^{\prime}$ and answer queries from an algorithm $\mathcal{A}$. Since all arcs are independent from one another, for our purposes, we can simply think of $\mathcal{P}$ as randomly choosing a suitable vertex uniformly at random from the necessary vertex sets $B, R_{1}, \ldots$ whenever $\mathcal{A}$ queries the graph. That is, the response of $\mathcal{P}$ does not depend on the current knowledge graph of the query-answer history of $\mathcal{A}$.

Lemma 3.6. Suppose $\mathcal{A}$ is an algorithm that makes $Q$ queries, where $Q=\omega\left(N^{1 / 2}\right)$ and $Q=o(W)$. With probability $1-o_{N}(1)$, the total number of epochs is at most

$$
O\left(Q^{2} / W+Q / L\right)
$$

and the total number of blue epochs is at most

$$
O\left(Q^{2} / N\right)
$$

Proof. Suppose an algorithm $\mathcal{A}$ has the history $\left(q_{1}, a_{1}\right), \ldots,\left(q_{t-1}, a_{t-1}\right)$, and makes a query $q_{t}=\left(v_{t}, i_{t}\right)$ with answer $a_{t}$. We consider several cases:

1. If $v_{t} \in B$, then of its $2 N$ potential neighbors, at most $2(t-1)<2 Q$ of them are in the knowledge graph $K G\left(a_{1}, \ldots, a_{t-1}\right)$. The probability that $a_{t}$ is a surprise edge is at $\operatorname{most} Q / N$.
2. If $v_{t} \in R_{\ell}$, for $\ell=1, \ldots, 2 L-1$, then $v_{t}$ has $W$ potential neighbors, so the probability that $a_{t}$ is a surprise edge is at most $2 Q / W$.
3. If $v_{t} \in R_{2 L}$, then no arc is returned and there is no chance of a surprise edge.

In all cases, the probability is at most $2 Q / W$, hence by a Chernoff bound, the probability that there are more than, say, $4 Q^{2} / W$ surprise edges over $Q$ queries is at most

$$
e^{-(1 / 3)\left(2 Q^{2} / W\right)}=o_{N}(1) .
$$

The number of time-out epochs is at most $Q / L$, so the total number of epochs is at most $4 Q^{2} / W+Q / L$ with probability $1-o_{N}(1)$. The same analysis holds for the number of blue epochs, except we only need to consider surprise edges between two blue vertices.

In a given knowledge graph, we define a long blue path to be a directed path of length $100 \log N$ on all blue vertices.

Lemma 3.7. Suppose algorithm $\mathcal{A}$ has a query-answer history $\left(q_{1}, a_{1}\right), \ldots,\left(q_{t}, a_{t}\right)$ and makes at most $M \leq L$ additional queries $q_{1}^{\prime}, q_{2}^{\prime}, \ldots, q_{M}^{\prime}$ with answers $a_{1}^{\prime}, a_{2}^{\prime}, \ldots, q_{M}^{\prime}$ in an epoch. Then with probability $1-N^{-50}$, there does not exist a long blue path in $K G\left(a_{1}^{\prime}, \ldots, a_{M}^{\prime}\right)$.

Proof Sketch. The knowledge graph $K G\left(a_{1}^{\prime}, \ldots, a_{M}^{\prime}\right)$ on the additional queries can have multiple components, and since the number of additional queries is at most $L$, any component that contains sink vertices cannot have any blue vertices, as the shortest path from a blue vertex to a sink (which is at depth $2 L$ in the set of red vertices) is greater than $L$. Hence we may ignore any component with sinks.

For each of the remaining components, which are directed trees, there are no sink vertices or surprise edges, so $\mathcal{A}$ cannot differentiate between different choices of outneighbors for its queries on these components. The probability that the endpoint of a queried arc is blue is at most $N /(2 N-2 Q)$, which is at most $2 / 3$ for sufficiently large $N$, for $Q=o(N)$.

Since the knowledge graph is a forest, each vertex $v$ in $K G\left(a_{1}^{\prime}, \ldots, a_{M}^{\prime}\right)$ is the endpoint of a unique maximal directed path. Thus, the probability that the unique directed path of length $100 \log N$ (if it exists) terminating at $v$ is a long blue path is at most

$$
(2 / 3)^{100 \log N} \leq N^{-51}
$$

Taking the union bound over all vertices in the knowledge graph completes the proof.

We say that a vertex $u$ is an ancestor of $v$ if there is a directed path in the knowledge graph from $u$ to $v$. If the algorithm finds a cycle, it must be through a surprise edge from a vertex to one of its ancestors. We will be able to bound the number of ancestors of any vertex using the following combinatorial argument:

Proposition 3.8. Let $T$ be a rooted tree of height at most $h$ with $\ell$ leaves. Then $|V(T)| \leq h \ell$.

Proof. Take the union of all paths from leaves to the root.

With all ingredients in place, we are able to balance our parameters to achieve the desired lower bound. Recall Theorem 3.2:

Theorem. Testing acyclicity with one-sided error and query access to only outgoing arcs requires at least $\Omega\left(N^{5 / 9-\delta}\right)$ queries, for any $\delta>0$.

Proof. By Lemma 3.6, we have with high probability that the number of epochs and blue epochs is at most

$$
O\left(Q^{2} / W+Q / L\right) \text { and } O\left(Q^{2} / N\right)
$$

respectively. To minimize the total number of epochs, we set $Q=\Theta\left(W^{2} / N\right)$, noting that $Q$ is indeed smaller than $W$ for all $W=o(N)$.

Lemma 3.7 implies that with high probability, the longest path of blue vertices in the knowledge graph, ignoring surprise edges, can only grow by at most $100 \log N$ after each epoch, so after the $O(Q / L)$ epochs, the longest such path has length at most $O(Q \log N / L)$. Consider the induced subgraph of the knowledge graph on the set of ancestors of a vertex $v$. If we delete all surprise edges, we obtain a directed forest, where the sinks are either $v$ or the origin of a blue surprise edge. Since there are no surprise edges, the height of any component is at most $O(Q \log N / L)$. Hence, by Proposition 3.8 the total number of ancestors at any point in the algorithm's runtime is at most

$$
O(Q \log N / L) \cdot O\left(Q^{2} / N\right)=O\left(Q^{3} \log N / N L\right)
$$

Recall that a query will discover a directed cycle in the knowledge graph exactly when the queried arc goes from a blue vertex to one of its ancestors. Thus, the probability that any query $(v, i)$ finds a cycle is at most

$$
\frac{\# \text { ancestors of } v}{N}=O\left(\frac{Q^{3} \log N}{N^{2} L}\right)
$$

so by taking a union bound over all $Q$ queries, we find that if we set $Q=\Theta\left(N^{5 / 9-\delta}\right)$, $W=\Theta\left(N^{7 / 9-\delta / 2}\right)$, and $L=N / W=\Theta\left(N^{2 / 9+\delta / 2}\right)$ for any small $\delta>0$, the probability that any algorithm finds a cycle after $Q$ queries is at most

$$
O\left(Q^{3} \log N / N^{2} L\right) \cdot Q=O\left(N^{-9 \delta / 2} \log N\right)=o_{N}(1)
$$



Figure 6: Starting from the subgraph induced by the ancestors of a vertex $v$, deleting all surprise edges leaves a directed forest, allowing us to bound the total number of vertices.

## 4 Embeddings of dense graphs

In this chapter, all surfaces (manifolds of dimension two) are closed and orientable. For more information on topological graph theory, see Gross and Tucker [GT87] or Ringel [Rin74]. The classification of surfaces tells us that the complete list of surfaces is $S_{0}, S_{1}, S_{2}, \ldots$, where $S_{g}$ denotes the surface of genus $g$, the sphere with $g$ handles or equivalently the $g$-holed torus. Let $\phi: G \rightarrow S$ be an embedding of a graph $G=(V, E)$ in a surface. The embedding $\phi$ is cellular if the complement of its image $S \backslash \phi(G)$ decomposes into a disjoint union of disks, which we call the faces of the embedding. We describe a face by giving a cyclic ordering of (possibly nondistinct) vertices $\left[v_{1}, v_{2}, \ldots, v_{k}\right]$. We say that a face is $k$-sided or is of length $k$, where $k$ is the number of elements in the cyclic ordering. We sometimes call a 3 -sided face a triangle, a 4-sided face a quadrilateral, and so on.

We restrict ourselves to cellular embeddings, ones where the complement of the image $S \backslash \phi(G)$ decomposes into a disjoint union of open disks. The Euler polyhedral equation states that if $\phi: G \rightarrow S_{g}$ is a cellular embedding, then

$$
|V(G)|-|E(G)|+|F(G, \phi)|=2-2 g
$$

where $F(G, \phi)$ denotes the number of faces in the embedding $\phi$. For simple graphs, where the shortest cycle is of length 3, we can obtain bounds on the maximum number of edges, namely

Proposition 4.1. Let $\phi: G \rightarrow S_{g}$ be an embedding, then

$$
|E(G)| \leq 3|V(G)|+6 g-6
$$

where equality holds if the embedding is triangular, i.e., where every face is a triangle.

Let the (minimum) genus $\gamma(G)$ of a graph $G$ be the smallest value $g$ such that $G$ embeds in $S_{g}$. Rearranging yields:

Corollary 4.2. The genus of a simple graph $G$ is at least

$$
\gamma(G) \geq \frac{|E|-3|V|+6}{6}
$$

where equality holds if the embedding is triangular.

Since the genus is always an integer, we can sharpen this bound slightly to

$$
\gamma(G) \geq\left\lceil\frac{|E|-3|V|+6}{6}\right\rceil .
$$

This is known as Euler lower bound, which is satisfied by many families of dense graphs. We say that an embedding of a graph is tight if its genus equals the graph's Euler lower bound. Thus, if we are able to find a tight embedding of a graph, that embedding is necessarily of minimum genus. The most notable result in this area is the Map Color Theorem ${ }^{5}$ of Ringel, Youngs, and others that states that the complete graphs $K_{n}$, simple graphs on $n$ vertices where every pair of vertices are connected by an edge, achieve this lower bound. In particular, the genus is

Theorem 4.3 (Map Color Theorem [Rin74]).

$$
\gamma\left(K_{n}\right)=\left\lceil\frac{(n-3)(n-4)}{12}\right\rceil .
$$

[^2]Tight genus embeddings of most of the complete graphs were found using the theory of current graphs, which produce symmetric embeddings. Roughly speaking, it seems like higher degree should mean more "degrees of freedom" for finding triangular or near-triangular embeddings. However, no general theorems along these lines are known.

Thomassen [Tho89, Tho97] showed that the graph genus problem is NP-complete, even for cubic graphs. In particular, he suggests the following possibility.

Conjecture 4.4 (Thomassen [Tho97]). There exist absolute constants $0<c_{0}<c_{1}<1$ such that the graph genus problem is NP-complete for graphs on $n$ vertices with maximum degree at most $c_{0} n$, but in $P$ for graphs on $n$ vertices with minimum degree at least $c_{1} n$.

A stronger statement for the latter part replaces polynomial time computability with a simple formula:

Conjecture 4.5 (Mohar and Thomassen [MT01, Problem 4.4.10]). There exists an absolute constant $c_{1}<1$ such that all graphs on $n$ vertices with minimum degree at least $c_{1} n$ have genus equal to the Euler lower bound.

This conjecture is far from being resolved-for example, the genus of the complete graph $K_{n}$ with three edges deleted is not known for all $n$.

### 4.1 Combinatorics of graph embeddings

Each edge $e$ has two ends $e^{+}$and $e^{-}$that each have an incident vertex (which, in the case of when $e$ is a self-loop, may be the same vertex). A rotation at vertex $v$ is a cyclic permutation of the edge ends incident with $v$. A rotation system $\Phi$ of a graph $G$ is a collection of rotations for each vertex of $G$. In the case of a simple graph, we only need to specify a cyclic ordering of
the neighbors of $v$. Rotation systems of simple graphs are often written as a table of symbols, so we sometimes refer to the rotation at $v$ as row $v$. For an embedding $\phi: G \rightarrow S$ in an orientable surface, we can obtain a rotation system by considering the clockwise order of edges incident with each vertex, for some orientation of the surface $S$. The Heffter-Edmonds principle states that this is a one-to-one correspondence - each rotation system induces an embedding that is essentially unique.

The surface can be constructed in a group-theoretic way. Consider the involution $\theta$ : $e^{+} \mapsto e^{-}$for all edges $e$ and regard $\Phi$ as a permutation of the set of edge ends. Then, the cycles of the composition $\Phi \circ \theta$ define the boundaries of the faces. Intuitively, this permutation is essentially tracing around the boundaries of the faces. In all our drawings, we take the convention where rotations are specified in clockwise order, which induces a counterclockwise orientation on the faces. One special case of the Heffter-Edmonds principle immensely helpful in the study of tight embeddings of complete graphs is a simple rule for determining whether an embedding is triangular:

Proposition 4.6 (see Ringel [Rin74]). An embedding is triangular if and only if the corresponding rotation system satisfies the following property: for all vertices $i, j, k$, if the rotation at vertex $i$ is of the form

$$
i . \quad \ldots \quad j \quad k \quad \ldots,
$$

then the rotation at $j$ is of the form

$$
\begin{array}{llllll}
j . & \ldots & k & i & \ldots
\end{array}
$$

For example, the following rotation system is of the complete graph $K_{7}$, and by Propositions 4.6 and 4.1, it must be a triangular embedding in the torus:

| 0. | 3 | 2 | 6 | 4 | 5 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1. | 4 | 3 | 0 | 5 | 6 | 2 |
| 2. | 5 | 4 | 1 | 6 | 0 | 3 |
| 3. | 6 | 5 | 2 | 0 | 1 | 4 |
| 4. | 0 | 6 | 3 | 1 | 2 | 5 |
| 5. | 1 | 0 | 4 | 2 | 3 | 6 |
| 6. | 2 | 1 | 5 | 3 | 4 | 0 |

However, not every complete graph will triangulate some surface, as seen in Corollary 4.2 when the right-hand side is not an integer. Nonetheless, we can triangulate the nontriangular faces with additional edges without increasing the genus. Substituting " $K_{n}$ plus $t$ edges" into Proposition 4.1 yields

$$
6 n+12 g-12=2\left(\left|E\left(K_{n}\right)\right|+t\right)=n(n-1)+2 t .
$$

To remove the genus parameter $g$, we take the resulting equation modulo 12 :

$$
2 t \equiv-(n-3)(n-4) \quad(\bmod 12)
$$

The analysis now breaks down into twelve Cases (with a capital "C") depending on the residue $n \bmod 12$. We observe the following:

- If $n \equiv 0,3,4,7(\bmod 12), t \equiv 0(\bmod 6)$.
- If $n \equiv 2,5(\bmod 12), t \equiv 5(\bmod 6)$.
- If $n \equiv 1,6,9,10(\bmod 12), t \equiv 3(\bmod 6)$.
- If $n \equiv 8,11(\bmod 12), t \equiv 2(\bmod 6)$.

One way of stating the Map Color Theorem is to say that there exist triangular embeddings where we actually have equality for the number of extra edges $t$. For positive values of $t$, we can consider how the extra $t$ edges arise as diagonals in the nontriangular faces of the embedding, classifying the possible face distributions for tight embeddings of the complete graphs (Section 4.5). For negative values of $t$, we can see if there are dense subgraphs of complete graphs which have triangular embeddings. These turn out to be minimum triangulations of surfaces (Section 4.6) and feature some connections with the former problem.

### 4.2 Current graphs and their derived embeddings

Embeddings of dense graphs are difficult to reason about directly, so whenever possible we would like to simplify the search by looking for symmetric embeddings. In the triangular embedding of $K_{7}$ in the torus from the previous section, if we think of the vertices as elements of $\mathbb{Z}_{7}$, the integers modulo 7 , the embedding can be succinctly described by the rotation at vertex 0 : to generate the rotation at vertex $k$, we add $k$ to each element in the rotation of 0 . This is sometimes referred to as the additivity rule. One effective visual description of these types of embeddings, where a few rotations can be used to generate the remaining rows, comes from Gustin [Gus63], who called his tool current graphs. For a complete description of the techniques present in this thesis, see Ringel [Rin74], especially $\S 2.3$ and $\S 9.1$.

A current graph $(D, \phi, \alpha)$ is an embedded, arc-labeled directed graph $D=(V, E)$. The embedding $\phi: D \rightarrow S$ is into an orientable surface, typically with few faces. The number of faces is the index of the current graph, and in this thesis we consider current graphs
of index at most 4. The arc-labeling $\alpha: E \rightarrow \Gamma$ has labels, known as currents, that are elmeents of a current group $\Gamma$-here we only consider cyclic current groups $\mathbb{Z}_{n}$, i.e., the integers modulo $n$. The face boundary walks of the current graph, which we call circuits, are labeled [0], [1], .., $[k-1]$.

The excess of a vertex is the sum of the incoming currents minus the sum of the outgoing currents, and we say a vertex satisfies Kirchhoff's current law (KCL) if its excess is 0. For generating triangular embeddings of dense graphs, most vertices will satisfy KCL. However, there are some exceptional vertices, known as vortices. Most vortices will induce long nontriangular faces, which we subdivide with a new vertex. In our current graphs, each face corner incident with such a vortex is labeled with a lowercase letter as a mnemonic for where the nontriangular faces will appear-we use the same letter to label the subdivision vertex.

The $\log$ of a circuit records the currents encountered along the walk in the following manner: if we traverse arc $e$ along its orientation, we write down $\alpha(e)$; otherwise, we write down $-\alpha(e)$; if we encounter a vortex, we record the label at the corresponding face corner. If we encounter two consecutive instances of the order 2 element of $\mathbb{Z}_{n}$ (as in vortices of type (A4) described below), for $n$ even, we combine them into one instance in the log. We follow the convention where the degree 1 vertex is omitted in the drawing of the current graph.

Various types of vortices are employed in the present work. For index 1 current graphs, those embedded with a single face, we make use of the following types of vortices. Each face corner incident with such a vortex is labeled with distinct letters.
(A1) Vortex of degree 1 whose excess generates the current group $\mathbb{Z}_{n}$.
(A2) Vortex of degree 2 whose excess generates the index 2 subgroup of the current group $\mathbb{Z}_{2 n}$ and whose incident arcs have odd currents.
(A3) Vortex of degree 3 whose excess generates the index 3 subgroup of the current group $\mathbb{Z}_{3 n}$ and whose incident arcs have currents either all congruent to $1(\bmod 3)$ or all congruent to $2(\bmod 3)$.
(A4) Vortex of degree 1 whose excess generates the order 2 subgroup of the current group $\mathbb{Z}_{2 n}$.

A generalization of vortex type (A1) for index 3 current graphs is the following, where each face corner of such a vortex is given the same label:
(B1) Vortex of degree 3 whose excess generates the order 3 subgroup of the current group $\mathbb{Z}_{3 n}$ and whose incident arcs have currents either all congruent to $1(\bmod 3)$ or all congruent to $2(\bmod 3)$.

We remark that the conditions on the currents for vortices of type (A3) are the same as that of type (B1). Finally, for any index, we use vortices which generate many short cycles of length 2 or 3 . For this reason, no letters are assigned to these vortices.
(C1) Vortex of degree 1 whose excess is an order 2 or 3 element of $\mathbb{Z}_{n}$. No face corners are labeled.

The derived embedding of the current graph has, ignoring vortex letters, $\left|\mathbb{Z}_{n}\right|$ vertices, one for each element of $\mathbb{Z}_{n}$. We refer to these vertices as the numbered vertices and all other vertices as lettered vertices. Using the logs of circuits $[0], \ldots,[k-1]$, we generate the rotation
at vertex $i$ by taking the $\log$ of circuit $[i \bmod k]$ and adding $i$ to each element of $\mathbb{Z}_{n}$. For the vortex letters, we apply the following modifications:

- For vortices of type (A1) and (B1), leave their letters unchanged.
- For vortices of type (A2) with face corner labels $a$ and $b$, swap the positions of $a$ and $b$ for $i$ odd.
- For vortices of type (A3) with face corner labels $a, b$, and $c$, suppose without loss of generality that the $\log$ is of the form
and that the incoming currents are all congruent to $1(\bmod 3)$.
- If $i \equiv 0(\bmod 3)$, leave the letters unchanged.
- If $i \equiv 1(\bmod 3)$, replace them as

$$
i . \quad \ldots \quad b \quad \ldots \quad c \quad \ldots \quad a \quad \ldots
$$

- If $i \equiv 2(\bmod 3)$, replace them as

$$
\begin{array}{lllllllll}
i . & \ldots & c & \ldots & a & \ldots & b & \ldots
\end{array}
$$

- For vortices of type (A4) with face corner labels $a$, replace with $a_{0}$ and $a_{1}$ for even and odd $i$, respectively.

To obtain the rotations at these lettered vertices, we choose them so that the embedding near those vertices is "locally" triangular. This can be done with the assistance of Proposition 4.6. With these types of vortices in mind, we list our "construction principles," which
describe the properties that all of our current graphs $(D, \phi, \alpha)$, of index $k$ and with current group $\mathbb{Z}_{n}$, satisfy:
(P1) Every vertex where KCL is satisfied has degree 3.
(P2) Every element $\gamma \in \mathbb{Z}_{n} \backslash\{0\}$ appears in the $\log$ of each circuit exactly once.
(P3) If circuit [a] traverses arc $e$ along its orientation and circuit [b] traverses $e$ in the opposite direction, then $\alpha(e) \equiv b-a(\bmod k)$.
(P4) All vortices are of type (A1-4), (B1), or (C1).

These conditions guarantee that the entire embedding is triangular, all the numbered vertices are incident with one another, and all the lettered vertices are either adjacent to all the numbered vertices, or in the case of vortices of type (A4), the two lettered vertices are adjacent to disjoint halves of the numbered vertices. In the case of index 1 current graphs, principle (P3) is satisfied automatically.

Figure 7 gives an example of a current graph illustrating vortex types (A1), (A3), (A4), (C1), and the construction principles. The rotations at solid vertices are oriented clockwise, and the rotations at hollow vertices are oriented counterclockwise. The log of its one circuit is

$$
\left[\begin{array}{ccccccccccccccccccccccccc}
{[0] .} & 11 & x & 7 & a & 8 & w & 13 & 1 & 15 & 9 & 6 & 5 & u & 16 & y & 2 & v & 10 & c & 14 & 17 & 12 & 3 & 4
\end{array}\right) b .
$$

For the numbered vertices, the rules for generating the rotations at numbered vertices yield the rotations


Figure 7: A current graph used by Ringel and Youngs [RY69b] and the boundary of the single face of its embedding. Solid and hollow vertices correspond to clockwise and counterclockwise rotations, respectively.
0. $\begin{array}{lllllllllllllllllllllllll}11 & x & 7 & a & 8 & w & 13 & 1 & 15 & 9 & 6 & 5 & u & 16 & y_{0} & 2 & v & 10 & c & 14 & 17 & 12 & 3 & 4 & b\end{array}$



4. $15 \begin{array}{lllllllllllllllllllllll} & x & 11 & c & 12 & v & 17 & 5 & 1 & 13 & 10 & 9 & w & 2 & y_{0} & 6 & u & 14 & b & 0 & 3 & 16 & 7\end{array} 8$
and so on. Some examples of rotations at lettered vertices are

$$
\begin{array}{rrrrrrrrrrrrrrrrrrr}
a . & 0 & 7 & 11 & 3 & 10 & 14 & 6 & 13 & 17 & 9 & 16 & 2 & 12 & 1 & 5 & 15 & 4 & 8 \\
x . & 0 & 11 & 4 & 15 & 8 & 1 & 12 & 5 & 16 & 9 & 2 & 13 & 6 & 17 & 10 & 3 & 14 & 7 \\
y_{0} & 0 & 16 & 14 & 12 & 10 & 8 & 6 & 4 & 2 & & & & & & & & &
\end{array}
$$

An example of an index 1 current graph that has a vortex of type (A2) is in Figure 8. The log of its one circuit is the cyclic sequence

$$
\left[\begin{array}{lrrrrrrrrrrrrrrrrrr}
{[0] .} & x & 1 & y & 31 & z & 29 & 24 & 20 & 2 & 21 & 25 & 15 & 6 & 16 & 22 & 7 & \ldots \\
28 & 8 & 5 & 23 & 17 & 10 & 26 & 9 & 14 & 12 & 4 & 11 & 13 & w & 19 & 30 & 18 & 27 . \tag{19}
\end{array}\right.
$$



Figure 8: An index 1 current graph with current group $\mathbb{Z}_{32}$ generating a triangular embedding of $K_{36}-K_{4}$. Solid and hollow vertices represent clockwise and counterclockwise rotations, respectively.

A partial picture of the rotation system showing how the vortex letters $x$ and $z$ are switched around looks like the following:

| 0. | 3 | $x$ | 1 | $y$ | 31 | $z$ | 29 | 24 | 20 | 2 | 21 | 25 | $\cdots$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | :--- |
| 1. | 4 | $z$ | 2 | $y$ | 0 | $x$ | 30 | 25 | 21 | 3 | 22 | 26 |  |
| 2. | 5 | $x$ | 3 | $y$ | 1 | $z$ | 31 | 26 | 22 | 4 | 23 | 27 |  |
| 3. | 6 | $z$ | 4 | $y$ | 2 | $x$ | 0 | 27 | 23 | 5 | 24 | 28 |  |
| 4. | 7 | $x$ | 5 | $y$ | 3 | $z$ | 1 | 28 | 24 | 6 | 25 | 29 |  |
| $\vdots$ |  |  |  |  |  |  |  |  |  |  |  |  | $\ddots$ |

which induces the following rotation for, e.g., vertex $x$ :

$$
\begin{array}{lllllllllll}
x . & \ldots & 31 & 30 & 1 & 0 & 3 & 2 & 5 & 4 & \ldots
\end{array}
$$

Finally, an example of an index 3 current graph is shown in Figure 9. Its logs are

$$
\begin{array}{lrrrrrrrrrrrrrrrr}
{[0] .} & 1 & a & 8 & 5 & 9 & 4 & 13 & 12 & 14 & b & 7 & 10 & 6 & 11 & 2 & 3 \\
{[1] .} & 14 & 2 & 6 & 4 & 13 & 9 & 11 & 5 & 12 & 7 & 10 & 3 & 8 & b & 1 & a \\
{[2] .} & 1 & 13 & 9 & 11 & 2 & 6 & 4 & 10 & 3 & 8 & 5 & 12 & 7 & a & 14 & b
\end{array}
$$



Figure 9: A current graph for $K_{17}-K_{2}$ from Ringel [Rin74]. The ends labeled A and B are to be identified.

Constructions for minimum genus embeddings of complete or near-complete graphs usually proceed in two steps:

- A regular step which involves finding a suitable current graph for triangularly embedding a graph that is close to complete (e.g. an embedding of a complete graph minus three edges).
- An additional adjacency step which modifies the embedding and the graph so it becomes complete (e.g. using a handle to add the three missing edges).

Unless explicitly stated otherwise, one can check using the Euler lower bound that every embedding derived from a current graph, and all the resulting embeddings via surgical operations, are of minimum genus. We detail the types of surgical operations used in the latter step in the next section.

### 4.3 Handle additions and the outline for additional adjacencies

Like in previous work, our additional adjacency solutions make use of three different operations for adding a handle, which are described in Constructions 4.7, 4.8, and 4.12 in
primal form. Many of these augmentations were previously known to Ringel and others (see, e.g., [JR80, GR76, RY69b]) in dual form. In particular, the constructions in Proposition 4.10 and Lemma 4.14 have not appeared in existing literature.

In prose, we describe the modifications to the embeddings in terms of rotation systems, so their correctness can be checked by tracing the faces and applying the Heffter-Edmonds principle. Our drawings, on the other hand, describe an alternate topological interpretation using surgery on the embedded surfaces. While these operations work more generally, we assume that all graphs in this section are simple and their embeddings are triangular.

Construction 4.7. Modifying the rotation at vertex $v$ from

$$
\begin{array}{llllllllll}
v . & x_{1} & \ldots & x_{i} & y_{1} & \ldots & y_{j} & z_{1} & \ldots & z_{k}
\end{array}
$$

to

$$
v . \quad x_{1} \quad \ldots \quad x_{i} \quad z_{1} \quad \ldots . z_{k} \quad y_{1} \quad \ldots \quad y_{j},
$$

as in Figure 10 increases the genus by 1 and induces the 9-sided face

$$
\left[x_{1}, z_{k}, v, y_{1}, x_{i}, v, z_{1}, y_{j}, v\right]
$$

Construction 4.8. Modifying the rotation at vertex $v$ from

$$
\begin{array}{lllllllllllll}
v . & x_{1} & \ldots & x_{i} & y_{1} & \ldots & y_{j} & z_{1} & \ldots & z_{k} & w_{1} & \ldots & w_{l}
\end{array}
$$

to

$$
\begin{array}{lllllllllllll}
v . & x_{1} & \ldots & x_{i} & w_{1} & \ldots & w_{l} & z_{1} & \ldots & z_{k} & y_{1} & \ldots & y_{j}
\end{array}
$$



Figure 10: Rearranging the rotation at vertex $v$ (a) increases the genus and creates room (b) to add new edges.
as in Figure 11 increases the genus by 1 and induces the two 6-sided faces

$$
\left[x_{1}, w_{l}, v, z_{1}, y_{j}, v\right] \text { and }\left[w_{1}, z_{k}, v, y_{1}, x_{i}, v\right] \text {. }
$$

Remark. While the drawings in Figures 10 and 11 are drawn asymmetrically, the operations are in fact invariant under cyclic shifts of the subsets $x_{1}, \ldots, x_{i} ; y_{1}, \ldots, y_{j}$, etc.

Several Cases of the Map Color Theorem are solved by first finding triangular embeddings of $K_{n}-K_{3}$. The first consequence of Construction 4.7 is to transform such an embedding into a genus embedding of a complete graph.

Proposition 4.9 (Ringel [Rin61]). If there exists a triangular embedding $\phi: K_{n}-K_{3} \rightarrow S_{g}$, then there exist a genus embedding of $K_{n}$ in the surface $S_{g+1}$.

Proof. If the three nonadjacent vertices are $a, b, c$, pick any other vertex $v$ and apply Construction 4.7 with $x_{1}=a, y_{1}=b, z_{1}=c$. In the resulting nontriangular face, the nonadjacent vertices can be connected like in Figure 12(a).


Figure 11: Rearranging four groups of neighbors (a) yields two hexagonal faces (b).

For Cases 8 and 11, we will construct triangular embeddings of the graph $K_{n}-K_{1,4}$. These missing edges can be added in using one handle if the embedding satisfies an additional constraint:

Proposition 4.10. Let $K_{n}-K_{1,4}$ be a complete graph with the edges $\left(u, q_{1}\right), \ldots,\left(u, q_{4}\right)$ deleted. If there exists a triangular embedding $\phi:\left(K_{n}-K_{1,4}\right) \rightarrow S_{g}$ with a vertex $v$ with rotation

$$
\begin{array}{cccccccc}
v . & \ldots & q_{1} & q_{2} & \ldots & q_{3} & q_{4} & \ldots,
\end{array}
$$

then there exists a genus embedding of $K_{n}$ in the surface $S_{g+1}$.

Proof. Note that vertices $u$ and $v$ are adjacent, so assume without loss of generality that $u$
appears in the rotation of $v$ in between $q_{4}$ and $q_{1}$. Apply Construction 4.7 with

$$
x_{i}=q_{1}, y_{1}=q_{2}, y_{j}=q_{3}, z_{1}=q_{4}, z_{k}=u
$$

and connect the missing edges in the 9 -sided face, as in Figure 12(b).

(a)

(b)

Figure 12: Two possibilities for adding edges after invoking Construction 4.7: a $K_{3}$ subgraph (a), and a $K_{1,4}$ subgraph (b).

This constraint is relatively easy to satisfy, since there are a few possible permutations for $q_{1}, \ldots, q_{4}$, in addition to the fact that $v$ is an arbitrary vertex. In fact, when we only need to add back three edges, this is always possible:

Corollary 4.11 (Ringel et al. [RY69b, GR76]). If there exists a triangular embedding $\phi$ : $K_{n}-K_{1,3} \rightarrow S_{g}$, then there exist a genus embedding of $K_{n}$ in the surface $S_{g+1}$.

Proof. One can always find such a vertex $v$ by choosing a vertex on one of the triangles incident with, say, the edge $\left(q_{1}, q_{2}\right)$.

A third type of handle operation is to merge two faces with a handle without modifying the rotations at any vertices. To do this, we excise a disk from two faces and identify the resulting boundaries. In Figure 13, adding the handle between faces $F_{1}$ and $F_{2}$ causes the embedding to become noncellular, as the resulting region is an annulus. However, once we start adding edges between the two boundary components of the annulus, the embedding becomes cellular again.

Construction 4.12. Let $F_{1}=\left[u_{1}, u_{2}, \ldots, u_{i}\right]$ and $F_{2}=\left[v_{1}, v_{2}, \ldots, v_{j}\right]$ be two faces. Inserting the edge $\left(u_{1}, v_{1}\right)$ in the following way

$$
\begin{array}{lllll}
u_{1} . & \ldots & u_{i} & u_{2} & \ldots \\
v_{1} . & \ldots & v_{j} & v_{2} & \ldots
\end{array} \Rightarrow \begin{array}{llllll}
u_{1} \cdot & \ldots & u_{i} & v_{1} & u_{2} & \ldots \\
v_{1} . & \ldots & v_{j} & u_{1} & v_{2} & \ldots
\end{array}
$$

as in Figure 13 increases the genus by 1 and induces the $(i+j+2)$-sided face

$$
\left[u_{1}, u_{2}, \ldots, u_{i}, u_{1}, v_{1}, v_{2}, \ldots, v_{j}, v_{1}\right]
$$



Figure 13: Adding a handle between two faces, then adding an edge to transform the annulus into a cell.

The most elementary operation one can do is to simply add one edge to create a genus embedding:

Proposition 4.13. If there exists a triangular embedding $\phi: K_{n}-K_{2} \rightarrow S_{g}$, then there exist a genus embedding of $K_{n}$ in the surface $S_{g+1}$.

The forthcoming additional adjacency solutions are to be applied on triangular embeddings of graphs of the form $K_{n}-K_{\ell}$, which is the graph formed by taking the complete graph $K_{n}$ and deleting all the pairwise adjacencies between $\ell$ vertices. We label the vertices missing adjacencies with the letters $a, b, c, \ldots, h$. The remaining vertices will be assigned numbers and are represented here as letters later in the alphabet $\left(u, v, p_{i}, \ldots\right)$. These constructions all make use of edge flips, which is the act of deleting an edge from a triangular embedding and replacing it with the other diagonal from the resulting quadrilateral.

The intermediate embeddings from these constructions are also useful, as they will typically be triangular as well. If $G$ is a simple graph on $n$ vertices and $\binom{n}{2}-t$ edges that has a triangular embedding in some surface $S$, we say the embedding is an $(n, t)$-triangulation.

Lemma 4.14. If there exists a triangular embedding of $K_{n}-K_{5}$ with numbered vertices $u$ and $v$ whose rotations are of the form
and

$$
v . \quad \ldots \quad p_{\sigma(1)} \quad p_{\sigma(2)} \ldots p_{\sigma(3)} \quad p_{\sigma(4)} \ldots,
$$

where $\sigma:\{1, \ldots, 4\} \rightarrow\{1, \ldots, 4\}$ is some permutation, then there exist $(n, 10)-$ and $(n, 4)$ triangulations and a tight embedding of $K_{n}$.

Proof. The initial embedding is an ( $n, 10$ )-triangulation. First, delete the edges $\left(u, p_{1}\right)$, $(u, b),\left(u, p_{2}\right)$ in exchange for $(a, b),(a, c),(b, c)$ and apply edge flips on $\left(u, p_{3}\right)$ and $\left(u, p_{4}\right)$ to
obtain $(c, d)$ and $(d, e)$, as in Figure 14(a). If we merge the faces $[a, c, b]$ and $[u, e, d]$ with a handle, we can recover the deleted edge $(u, b)$ and add in the remaining edges between lettered vertices following Figure $14(\mathrm{~b})$. The missing edges $\left(u, p_{1}\right), \ldots,\left(u, p_{4}\right)$ in this $(n, 4)$ triangulation can be reinserted with one handle using Proposition 4.10, setting $p_{\sigma(i)}=q_{i}$, to get a tight embedding of $K_{n}$.

(a)

(b)

Figure 14: Various edge flips are applied in the neighborhood of vertex $u$ (a) so that one handle suffices for connecting all the lettered vertices.

Lemma 4.15 (Guy and Ringel [GR76]). If there exists a triangular embedding of $K_{n}-K_{6}$ with a numbered vertex $u$ whose rotations are of the form

$$
\begin{array}{cccccccccccccc}
u . & \ldots & a & p_{1} & b & \ldots & c & p_{2} & d & \ldots & e & p_{3} & f & \ldots
\end{array}
$$

then there exist $(n, 15)-,(n, 9)$-, and $(n, 3)$-triangulations and a tight embedding of $K_{n}$.

Proof. We first modify the embedding near vertex $u$ using edge flips to gain the edges $(a, b)$, $(c, d)$, and $(e, f)$, as in Figure 15(a). If we apply Construction 4.7 to vertex $u$, we obtain a

9-sided face incident with all six vertices $a, b, \ldots, f$. In Figure 15(b) and (c), we give one way to insert the nine missing edges between these lettered vertices with the help of a handle.


Figure 15: Three pairs of lettered vertices are connected with some edge flips (a), after which a handle adds some of the missing adjacencies (b). The remaining edges between lettered vertices are added using another handle merging faces I and II (c).

The missing edges $\left(u, p_{1}\right),\left(u, p_{2}\right),\left(u, p_{3}\right)$ can be added back using Corollary 4.11, yielding a tight embedding of $K_{n}$.

Lemma 4.16. If there exists a triangular embedding of $K_{n}-K_{8}$ with numbered vertices $u$
and $v$ whose rotations are of the form

$$
\begin{array}{llllllllllllllllll}
u . & \ldots & a & p_{1} & b & \ldots & c & p_{2} & d & \ldots & e & p_{3} & f & \ldots & g & p_{4} & h & \ldots
\end{array}
$$

and

$$
v . \quad \ldots \quad p_{\sigma(1)} \quad p_{\sigma(2)} \ldots p_{\sigma(3)} \quad p_{\sigma(4)} \ldots,
$$

where $\sigma:\{1, \ldots, 4\} \rightarrow\{1, \ldots, 4\}$ is some permutation, then there exist $(n, 28)-,(n, 22)-$, $(n, 16)-,(n, 10)-$, and ( $n, 4$ )-triangulations and a tight embedding of $K_{n}$.

Proof. The first four handles of our additional adjacency approach is the same as that of Ringel and Youngs' solution for Case 2 of the Map Color Theorem [RY69b] (also see Ringel [Rin74, §7.5]), with different vertex names. We perform an edge flip on each edge $\left(u, p_{i}\right)$ for $i=1, \ldots, 4$, gaining the edges $(a, b),(c, d),(e, f)$, and $(g, h)$. Now, the rotation at vertex $u$ is of the form

$$
\begin{array}{lllllllllllllll}
u . & \ldots & a & b & \ldots & c & d & \ldots & e & f & \ldots & g & h & \ldots
\end{array}
$$

These edge flips are depicted in Figure 16. Applying Construction 4.8 to this resulting rotation yields two nontriangular faces

$$
[h, g, v, d, c, v] \text { and }[f, e, v, b, a, v] .
$$

In these faces, we induce two quadrilateral faces by adding the edges $(d, g),(c, h),(b, e)$, and $(a, f)$, as in Figure 17(a). Three more handles are used to add all the remaining edges between lettered vertices $a, \ldots, h$ as shown in Figure 17(bc). At this point, the embedding is of the graph $K_{n}-K_{1,4}$ and is still triangular, so we replace the deleted edges ( $u, p_{i}$ ) with one handle using Proposition 4.10 to obtain a tight embedding of $K_{n}$.


Figure 16: Initial edge flips to join some of the vortex letters.

The embeddings after adding the second through fourth handles are all triangular. After adding only the first handle, the two quadrilateral faces in Figure 17(a) can be triangulated arbitrarily to form an ( $n, 22$ )-triangulation.

It seems that nowhere in the literature, including in the original proof of the Map Color Theorem, is there a construction of a genus embedding of $K_{n}$ derived from an ( $n, 4$ )triangulation. Even though we outlined a natural approach in Proposition 4.10 for converting an ( $n, 4$ )-triangulation to a genus embedding of $K_{n}$, no prior such unification was known.

### 4.3.1 An extension of Construction 4.7

We introduce another handle operation with some edge deletions that will be helpful for finding embeddings of graphs with a specified distribution on its face sizes:

Construction 4.17. Modifying the rotation at vertex $v$ from

$$
\begin{array}{llllllllllll}
v . & a & x_{1} & \ldots & x_{i} & b & y_{1} & \ldots & y_{j} & c & z_{1} & \ldots \\
z_{k}
\end{array}
$$



Figure 17: After connecting some of the lettered vertices with a handle (a), another handle can be introduced in between the faces I and II (b). Using faces generated from this handle (III and IV, V and VI), we can add all the remaining edges using two additional handles (c).
to

$$
\begin{array}{llllllllll}
v . & x_{1} & \ldots & x_{i} & z_{1} & \ldots & z_{k} & y_{1} & \ldots & y_{j},
\end{array}
$$

as in Figure 18 increases the genus by 1 and induces the 12-sided face

$$
\left[x_{1}, a, z_{k}, v, y_{1}, b, x_{i}, v, z_{1}, c, y_{j}, v\right] .
$$

The idea behind this modification is that there is an increase in the number of ways of rearranging the deleted edges after adding the handle. The choice of vertices in the


Figure 18: A modification of Figure 10 with more room.
construction in Proposition 4.9 is not unique - in fact, there are eight possible choices. One can consider Construction 4.17 as a way of considering all eight choices simultaneously. To see this increased flexibility, we again consider how to add the missing edges to a triangular embedding of $K_{n}-K_{3}$, this time using Construction 4.17. Suppose the rotation at vertex $v$ is of the form in Construction 4.17, with the edges $(a, b),(a, c)$, and $(b, c)$ missing. Inside the 12 -sided face, we can add back those edges. At this point, there are many choices for how to add back the remaining missing edges $(v, a),(v, b)$, and $(v, c)$, one of which is shown in Figure 19.

### 4.4 Handle subtraction for minimum triangulations

While many constructions for genus embeddings of graphs involve augmenting an embedding with additional handles, we can also consider removing edges to decrease the genus. This idea of handle subtraction factors heavily into Jungerman and Ringel's approach to constructing minimum triangulations. As the number of triangular embeddings of graphs on $n$ vertices one needs to construct grows linearly in $n$, it would be helpful if a single method could


Figure 19: Adding in a $K_{3}$ with a handle after deleting some edges. One possible way of restoring the deleted edges is shown with dashed lines.
dispense with all but a constant number of those embeddings.
The triangular embeddings in Section 4.6 of $K_{12 s+3+k}-K_{k}$ and the embeddings en route to constructing a genus embedding of $K_{12 s+3+k}$ are

$$
\left(12 s+3+k,\binom{k}{2}-6 h\right) \text {-triangulations, }
$$

where $h$ is a nonnegative integer less than the number of added handles. To construct triangular embeddings of graphs on the same number of vertices, but with more missing edges, we turn to the main idea of Jungerman and Ringel [JR80]: we enforce a specific structure in the current graph that allows us to "subtract" handles. The fragment shown in Figure 20 is what we refer to as an arithmetic 3-ladder. If the step size $h$ in the arithmetic sequence is divisible by the index of the current graph, then it is possible to find triangular embeddings in smaller-genus surfaces in the following manner:

Lemma 4.18 (Jungerman and Ringel [JR80]). Let $(D, \phi, \alpha)$ be an index 3 current graph with current group $\mathbb{Z}_{3 m}$ that satisfies all construction principles. Suppose further that it has an arithmetic 3-ladder with step size divisible by 3. If the derived embedding of the current


Figure 20: An arithmetic 3-ladder and a circuit passing through it.
graph has $|V|$ vertices and $|E|$ edges, then for each $k=0, \ldots, m$, there exists a triangular embedding of a graph with $|V|$ vertices and $|E|-6 k$ edges.

Proof Sketch. Following Figure 20, the rotation at vertices 0 and $h$ are of the form

$$
\begin{array}{cccccccccc}
0 . & \ldots & -t-h & g-h & r & g & -t & g+h & r+h & \ldots \\
h . & \ldots & -t & g & r+h & g+h & & & & \ldots
\end{array}
$$

Here we used the fact that $h$ is divisible by 3 . We may infer, by repeated application of Proposition 4.6, the following partial rotation system, for $i=0,1, \ldots, m$ :

$$
\begin{array}{ccccccc}
0 . & \ldots & g & -t & g+h & r+h & \ldots \\
g . & \ldots & r+h & h & -t & 0 & \ldots \\
r+h . & \ldots & 0 & g+h & h & g & \ldots \\
& & & & & &  \tag{20}\\
h . & \ldots & -t & g & r+h & g+h & \ldots \\
-t . & \ldots & g+h & 0 & g & h & \ldots \\
g+h . & \ldots & h & r+h & 0 & -t & \ldots
\end{array}
$$

If we delete the middle two columns, the rotation system becomes

$$
\begin{array}{ccccc}
0 . & \ldots & g & r+h & \ldots \\
g . & \ldots & r+h & 0 & \ldots \\
r+h . & \ldots & 0 & g & \ldots \\
& & & & \\
h . & \ldots & -t & g+h & \ldots \\
-t . & \ldots & g+h & h & \ldots \\
g+h . & \ldots & h & -t & \ldots
\end{array}
$$

This new embedding has six fewer edges, and is still triangular by Proposition 4.6, hence it must be a triangular embedding in a surface with one fewer handle by Proposition 4.1.

More generally, we obtain other handles that can be subtracted in the same manner, using the additivity rule. That is, we can find another subtractible handle by adding a multiple of 3 to every element of (20). The six edges from each of $m$ handles can be deleted simultaneously, as none of the handles share any faces.

One way to visualize this operation is to interpret it as the reverse of Construction 4.12, like in Figure 21. One can check that in all instances in this thesis, the number of handles we can subtract in a given embedding is greater than the number needed to realize the minimum triangulation with the fewest number of edges, i.e., the $(n, t)$-triangulation where $t \approx n-6$.


Figure 21: The six deleted edges form a cycle that is, roughly speaking, surrounded by two triangles.

### 4.5 Face distributions of complete graph embeddings

One of our goals is to classify the different possible face distributions for, primarily, minimum genus embeddings of complete graphs. ${ }^{6}$ The face distribution of an embedding is the sequence $f_{1}, f_{2}, \ldots$ where $f_{i}$ is the number of faces of length $i$. For example, the triangular embedding of $K_{7}$ in the torus has face distribution

$$
0,0,14,0,0, \ldots
$$

For $n \not \equiv 0,3,4,7(\bmod 12)$ there exists a triangular embedding of $K_{n}$ in some surface, so there is only one possible face distribution for a minimum genus embedding. For the residue classes $n \not \equiv 0,3,4,7(\bmod 12)$, we try to partition the $t$ "chordal" edges into the faces to get embeddings for each possible face distribution permitted by the Euler polyhedral equation. For example, if $n=14$, then $t=5$. As seen in Figure 22, distributing all five additional edges into the same face gives us an 8-sided face, but we could distribute the edges in a different way to get one 6 -sided face and one 5 -sided face.


Figure 22: For a minimum genus embedding, the missing chords needed to make the embedding triangular could be distributed among faces in a few different ways.

Instead of writing out face distributions in full and counting all the triangular faces, we

[^3] forwardly from known constructions.
say that an embedding of a simple graph is of type $\left(a_{1}, \ldots, a_{i}\right)$, if it has faces of length $a_{1}, a_{2}, \ldots, a_{i}$, where $a_{1} \geq a_{2} \geq \cdots \geq a_{i}>3$ and all the other faces are triangular. In this terminology, $K_{14}$ could have embeddings of type (8) and $(6,5)$. In general, if $t=$ $b_{1}+\cdots+b_{j}$ is a partition of $t$ into positive integers $b_{i}$, we are looking for an embedding of type $\left(b_{1}+3, b_{2}+3, \ldots, b_{j}+3\right)$. Thus, we need to find the following embedding types:

- For $n \equiv 2,5(\bmod 12)$, types $(8),(7,4),(6,5),(6,4,4),(5,5,4),(5,4,4,4)$, and $(4,4,4,4,4)$.
- For $n \equiv 1,6,9,10(\bmod 12)$, types $(6),(5,4),(4,4,4)$.
- For $n \equiv 8,11(\bmod 12)$, types $(5)$ and $(4,4)$.

We appeal to Proposition 4.1 to show that regardless of the graph, embeddings of these types are of minimum genus:

Proposition 4.19. Suppose an embedding $\phi$ of a simple graph $G$ is of type $\left(a_{1}, \ldots, a_{i}\right)$, where

$$
\left(a_{1}-3\right)+\cdots+\left(a_{i}-3\right) \leq 5
$$

Then $\phi$ is a tight embedding.

Proof. The inequality is equivalent to the statement that there are at most 5 extra edges. Proposition 4.1 loosely states that each handle allows for 6 extra edges, so the number of edges of $G$ exceeds the number of edges in a triangular embedding in any surface of smaller genus.

We state our main result for face distributions in this language.

Theorem 4.20. For all $n \geq 3, n \neq 5,8$ and for every partition of $t=t(n)$ into positive integers

$$
t=b_{1}+b_{2}+\cdots+b_{j}
$$

for $b_{1} \geq b_{2} \geq \cdots \geq b_{j}$, there exists an embedding of type $\left(b_{1}+3, b_{2}+3, \ldots, b_{j}+3\right)$ of $K_{n} . K_{5}$ only has minimum genus embeddings of type (8), (7, 4), (6, 4, 4), (5, 5, 4), and (4, 4, 4, 4, 4), and $K_{8}$ only has minimum genus embeddings of type $(4,4)$.

This characterization can been seen as a step in understanding the embedding polynomials (as introduced by Gross and Furst [GF87]) of the complete graphs-we fully determine which coefficients corresponding to minimum genus embeddings are nonzero. Our result answers a question of Archdeacon and Craft [Arc95], who asked whether or not every complete graph has a nearly triangular minimum genus embedding, i.e., one with at most one nontriangular face. We show that

Corollary 4.21. For $n \geq 3, n \neq 8$, there exists a nearly triangular tight embedding of $K_{n}$.

For the Cases where $t=2$ or 3 , it turns out that practically all of the difficulty is in finding the nearly triangular embedding, i.e. the embeddings of types (5) and (6). Using those embeddings, it is straightforward to obtain the other types. We say a face is simple if it is not incident with the same vertex more than once.

Lemma 4.22. Let $G$ be a simple graph with minimum degree 2. For any orientable embedding of $G$, all 5-sided faces are simple. All 6-sided faces have at most one repeated vertex-in particular, it is of the form $\left[a, b, x, c, d, x^{\prime}\right]$, where only $x$ and $x^{\prime}$ are possibly nondistinct.

Proof. Suppose some vertex $v$ appears twice in some 5 -sided face. The face cannot be of the form $[\ldots v, v \ldots]$, otherwise there would be a self-loop at $v$. On the other hand, the face also
cannot be of the form $[\ldots v, w, v \ldots]$ for some vertex $w$, because otherwise $w$ would have degree 1 , or there would be more than one edge incident with $v$ and $w$.

By the same reasoning, the two instances of a repeated vertex on a 6 -sided face must appear "opposite" each other. Suppose two vertices $a$ and $b$ appeared twice on the same face. Without loss of generality, the face must be of the form $\left[a, b, c, a, b, c^{\prime}\right]$. However, this would imply that the embedding is on a nonorientable surface, since the edge $(a, b)$ is traversed twice in the same direction. ${ }^{7}$

Proposition 4.23. If $K_{n}$ has an orientable embedding of type (5) (resp. type (6)), then it has an embedding of type $(4,4))$ (resp. types $(5,4)$ and $(4,4,4)$ ).

Proof. In the embedding of type (5), the 5-sided face $f$ is simple by Lemma 4.22 , so if $f$ is of the form $[\ldots a, b, c \ldots]$, $a$ must be different from $c$, and the edge $(a, c)$ is not incident with this face. If we delete the edge $(a, c)$ and add it back in as a chord of $f$, we get an embedding of type $(4,4)$.

Applying Lemma 4.22 again, suppose the 6 -sided face in an embedding of type (6) is of the form $\left[a, v, w, a^{\prime}, x, y\right]$, where $a$ and $a^{\prime}$ are possibly not distinct. Like in the previous case, we alter the positions of edges $(v, w)$ and $(x, y)$, like in Figure 23, so that they become chords. The result is an embedding of type $(4,4,4)$. Applying this procedure to just one of the edges yields an embedding of type $(5,4)$.

The idea of changing the location of an existing edge to a nontriangular face is prevalent in this section. We call such an operation a chord exchange $\pm(u, v)$ or say that we are exchanging the chord $(u, v)$.

[^4]

Figure 23: Changing an embedding of type (6) into one of type (4, 4, 4). The dashed and thickened lines represent the old and new locations of the edges, respectively.

The current graphs we will encounter contain ladder-like subgraphs, like in the middle of the current graph in Figure 7. The additional adjacency steps only use part of a current graph, so we ignore the unneeded parts by replacing ladders with boxes, as in Figure 24. All the vertices replaced by the box satisfy KCL, and the currents are assigned such that construction principle (P3) holds. In this paper, the rotations have already been specified, but the originators of this notation, Korzhik and Voss [KV02], used the box to mean any set of rotations that produce a one-face embedding. Additionally, we may omit some current assignments on edges outside of these boxes for clarity. Typically they can be recovered by KCL.


Figure 24: Instead of drawing the ladder subgraph on the left, we replace it with a box indicating the number of "rungs."

We prove Theorem 4.20 across the next several subsections in roughly increasing order of the difficulty of the additional adjacency solution.

### 4.5.1 Cases 2 and 5

For $n \equiv 2,5(\bmod 12)$, we expect to find a nearly triangular embedding with an 8 -sided face. Fortunately, we can leverage existing constructions for these Cases:

Theorem 4.24 (Jungerman [Jun75], Ringel [Rin74, p.83]). For $s \geq 1$, there exists a triangular embedding of $K_{12 s+2}-K_{2}$.

Theorem 4.25 (Youngs [You70a] or Ringel [Rin74, §9.2]). For $s \geq 0$, there exists a triangular embedding of $K_{12 s+5}-K_{2}$.

From one of these embeddings, arbitrarily adding the missing edge using Construction 4.12 causes two triangular faces to combine into an 8 -sided face. Figure 25 shows this operation along with how the orientation of the two participating faces affect the final nontriangular face. Achieving the other face distributions requires a few small modifications. We prove the following using those embeddings:


Figure 25: Adding an edge with the help of one handle merges two triangular faces together. The pairs of thick dashed arrows labeled with the same letters are identified together.

Proposition 4.26. For $s \geq 1$, there exists embeddings of type (8), (7, 4), $(6,5),(6,4,4)$, $(5,5,4),(5,4,4,4)$, and $(4,4,4,4,4)$ of $K_{12 s+2}$ and $K_{12 s+5}$.

Proof. Let $x$ and $y$ be the two nonadjacent vertices. The general approach is to exchange chords in the 8 -sided face, which does not increase the genus of the embedding. Some of these constructions are illustrated in Figure 26.
(Types $(7,4),(6,4,4),(5,5,4)$, and $(4,4,4,4,4))$ Since $s \geq 1, x$ and $y$ have at least 12 neighbors. We can find faces $[x, a, b]$ and $[y, c, d]$ such that $a, b, c, d$ are all distinct vertices. After merging these two faces with a handle and adding the edge $x y$, the resulting 8 -sided face will be $[x, a, b, x, y, c, d, y]$. Exchanging the following sets of chords yields the following embeddings:

- type $(7,4): \pm(a, y)$,
- type $(6,4,4): \pm(a, d)$,
- type $(5,5,4): \pm(a, c)$,
- type $(5,4,4,4): \pm(a, d), \pm(b, y)$, and
- type $(4,4,4,4,4): \pm(a, d), \pm(b, c)$.
(Type $(6,5)$ ) We assert that there exist faces $[x, a, b],[y, b, c]$, where $a \neq c$. Since $s \geq 1$, vertex $b$ has at least 13 neighbors. Without loss of generality, we may assume that the rotation at $b$ is of the form

$$
b . \quad \ldots \quad y \quad c \quad \ldots \quad a \quad x \quad \ldots,
$$



Figure 26: Finding embeddings of types $(5,4,4,4)$ and $(6,5)$.
where there are at least two other vertices in between $y$ and $x$ in the cyclic sequence. Hence, these triangles incident with $b$ are the desired faces. Adding the edge $(x, y)$ using those two faces and exchanging the chord $(a, b)$ yields an embedding of type $(6,5)$.

Remark. One might ask why we need $a \neq c$ for the type $(6,5)$ construction. If they are the same vertex, then the edge $(a, b)$ appears twice on the 8 -sided face. Deleting that edge causes the genus to decrease and the face to split in two.

We note that $K_{5}$, despite there being a triangular embedding of $K_{5}-K_{2}$, does not realize all its predicted face distributions. An exhaustive enumeration produced the following:

Proposition 4.27 (see Gagarin et al. [GKN03] or White [Whi01, p.270]). $K_{5}$ has embeddings of type (8), $(7,4),(6,4,4),(5,5,4)$, and $(4,4,4,4,4)$, but no embeddings of type $(6,5)$ or $(5,4,4,4)$.

It can be verified that the constructions in Proposition 4.26 for the latter two cases cannot be applied to the essentially unique planar embedding of $K_{5}-K_{2}$. Each vertex has too few neighbors.

### 4.5.2 Case 9

In the previous section, we found nearly triangular embeddings by taking a triangular embedding and adding a single edge. Jungerman's solution for Case 9 also has a simple additional adjacency solution that involves only one extra edge. We say that $G_{n}$ is a split-complete graph if we can label its vertices $1,2, \ldots, n-1, x_{0}, x_{1}$ such that

- $1, \ldots, n-1$ are all pairwise adjacent, and
- the neighbors of $x_{0}$ and the neighbors of $x_{1}$ form a partition of $\{1, \ldots, n-1\}$.

The aforementioned solution of Jungerman employed a beautiful construction for splitcomplete graphs.

Theorem 4.28 (see Ringel $[\operatorname{Rin} 74, \S 6.5]$ ). For $s \geq 0$, there exists a triangular embedding of a split-complete graph $G_{12 s+9}$.

In the proof of the Map Color Theorem, embeddings were expressed in dual form, where the vertices were regarded as "countries" drawn on surfaces. The countries $x_{0}$ and $x_{1}$ were then connected with a handle and then merged into one "cylindrical region." Upon closer examination, the resulting embedding in primal form is in fact nearly triangular.

Proposition 4.29. If there exists a triangular embedding of a split-complete graph $G_{n}$, then there exists an embedding of type (6) of $K_{n}$.

Proof. Add the edge $\left(x_{0}, x_{1}\right)$ arbitrarily as we did for Cases 2 and 5 . Note that the newly added edge $\left(x_{0}, x_{1}\right)$ appears twice in the resulting 8 -sided face. Locally contracting this edge leaves a 6-sided face, as in Figure 27.


Figure 27: Adding a handle to add an edge, and then contracting it to get a 6 -sided face.

Corollary 4.30. For $s \geq 0$, there exists a nearly triangular tight embedding of $K_{12 s+9}$.

Corollary 4.31. For $s \geq 0$, there exist embeddings of type (6), $(5,4)$, and $(4,4,4)$ of $K_{12 s+9}$.

### 4.5.3 Case 6

Theorem 4.32. For $s \geq 0$, there exists a nearly triangular tight embedding of $K_{12 s+6}$.

Proof. For $s=0$, such an embedding can be found by deleting a vertex from the triangular embedding of $K_{7}$ in the torus. For $s=1$, Mayer [May69] constructed a split-complete $G_{18}$, so applying Proposition 4.29 yields the desired embedding. The larger-order cases are covered by combining triangular embeddings of $K_{12 s+6}-P_{3}$ (Proposition 4.35 for $s=2$, Theorem 4.34 for $s \geq 3$ ), with Lemma 4.36.

Corollary 4.33. For $s \geq 0$, there exist embeddings of type (6), $(5,4)$, and $(4,4,4)$ of $K_{12 s+6}$.

The original proof of Case 6 by Youngs et al. had a few ad hoc solutions and a general construction for $s \geq 4$. For $s \geq 2$, Youngs [You70a] gives a current graph construction for triangular embeddings of $K_{12 s+6}-K_{3}$. The theory of current graphs is most suited for deleting a $K_{3}$ subgraph, but the Euler polyhedral equation does not rule out triangular embeddings of other graphs with the same number of edges and vertices. Gross [Gro75]
obtains triangular embeddings for some of these "nearly complete" graphs by modifying Youngs' constructions.

Theorem 4.34 (Gross [Gro75]). For $s \geq 3$, there exists a triangular embedding of $K_{12 s+6}-$ $H$, where $H \in\{A, B, C, D, E\}$ is any of the five graphs on three edges in Figure 28.


A


B


C


E

Figure 28: The graphs on three edges.

Before applying these embeddings for our task at hand, we extend this result one step further by filling in the case $s=2$. Youngs [You70a] also devised a current graph construction for $K_{30}-K_{3}$, which did not appear until Ringel's book [Rin74]. We modify this embedding to get triangular embeddings of the other graphs.

Proposition 4.35. There exists a triangular embedding of $K_{30}-H$, for all

$$
H \in\{A, B, C, D, E\} .
$$

Proof. The current graph given by Ringel [Rin74, p.155] uses the group $\mathbb{Z}_{27}$ and produces the following three logs:


The derived embedding is that of $K_{30}-A$. When row $a$ is of the form $\ldots c b d \ldots$ and $(c, d)$ is not an edge in the graph, Gross [Gro75] uses the notation $-(a, b)+(c, d)$ to denote the edge flip where we delete the edge $(a, b)$ and add the edge $(c, d)$ in the resulting quadrilateral. One can check that after applying the following groups of edge flips, we realize triangular embeddings of the four other graphs:

- $K_{30}-B:-(0,10)+(x, y)$
- $K_{30}-C:-(0,10)+(x, y),-(1,26)+(x, z)$
- $K_{30}-D:-(0,10)+(x, y),-(8,10)+(x, z),-(10, x)+(y, z)$
- $K_{30}-E:-(1,26)+(x, z),-(11,16)+(1,26),-(6, x)+(11,16)$

The graph we focus on in particular is $K_{n}-E$, where $E=P_{3}$ is the path graph on three edges. Carefully adding these edges back yields a nearly triangular embedding.

Lemma 4.36. If there exists a triangular embedding of $K_{n}-P_{3}$, there exists an embedding of type (6) of $K_{n}$.

Proof. Suppose the missing edges are $(a, b),(b, c)$, and $(c, d)$. The edges $(a, c)$ and $(b, d)$ are in the graph, so there are triangular faces $[a, c, x]$ and $\left[d, b, x^{\prime}\right]$ for some (possibly nondistinct) vertices $x$ and $x^{\prime}$. With one handle, we can add back the missing edges following Figure 29, leaving the 6 -sided face $\left[a, b, x^{\prime}, d, c, x\right]$.


Figure 29: Adding a $P_{3}$ subgraph using a specific pair of faces to get a 6 -sided face.

Remark. The approach of flipping edges in a triangulation to get the graph $K_{n}-P_{3}$ seems better suited for index 3 current graphs (see Youngs [You70a]), where the vortices can be nearly adjacent to each other in the log of the face boundary. The known current graph constructions for Cases 1 and 10 enjoy no such benefit.

### 4.5.4 Case 10

Theorem 4.37. For $s \geq 0$, there exists a nearly triangular tight embedding of $K_{12 s+10}$.

Proof. For $s=0$, we apply Lemma 4.36 to the triangular embedding of $K_{10}-P_{3}$ given in the Appendix. A unified solution is given for $s \geq 1$ in Theorem 4.39.

Corollary 4.38. For $s \geq 0$, there exist embeddings of type (6), (5, 4), and $(4,4,4)$ of $K_{12 s+10}$.

For $s \geq 0$, Ringel [Rin74, §2.3] gives a current graph generating $K_{12 s+10}-K_{3}$, the $s=2$ case being illustrated in Figure 30. Luckily for us, the current assignments, which follow the same alternating pattern in the rungs of the ladder in Figure 30, can be used to produce a nearly triangular embedding.

Theorem 4.39. For $s \geq 1$, there exists a nearly triangular tight embedding of $K_{12 s+10}$.


Figure 30: The current graph for $s=2$, which produces a triangular embedding of $K_{34}-K_{3}$.

Proof. We use the same current graph as Ringel [Rin74], except we flip the rotation at the vertex adjacent to vortex $z$, as shown in Figure 31. Our solution to the additional adjacency problem hinges on the fact that the current $2 s+1$ flowing into vortex $x$ is twice that of $-(5 s+3)$, the current flowing into vortex $z$. Let $c=-(5 s+3)=7 s+4$. Then $2 s+1=2 c$ and $3 s+2=-3 c$ in the group $\mathbb{Z}_{12 s+7}$.

$$
\mathbb{Z}_{12 s+7}
$$



Figure 31: A current graph generating $K_{12 s+10}-K_{3}$ with the pertinent currents marked.

After rewriting the currents in Figure 31, the log of this current graph and some partial rotations become

$$
\begin{array}{rlllllllllllll}
0 . & -3 c & y & 3 c & 1 & c & z & -c & \ldots & -2 c-1 & 2 c & x & -2 c & \ldots \\
c+1 . & & & & & & & & \ldots & -c & 3 c+1 & x & \ldots & \\
2 c . & & & \ldots & 2 c+1 & 3 c & z & \ldots & & & & & & \\
2 c+1 . & & & \ldots & 3 c+1 & z & c+1 & \ldots & & & & &
\end{array}
$$

In addition, the rotation at vertex $x$ reads

After applying Construction 4.17 to vertices 0 and $x, y, z$, we obtain the 12 -sided face

$$
[x, 2 c, 0,3 c, y,-3 c, 0,-c, z, c, 0,-2 c]
$$

as in Figure 32. We can exchange the chords $(x, c)$ and $(x, 3 c)$, generating the 5 -sided face $[x,-c, c, 3 c, 5 c]$. There remains only one way of adding back the edges $(0, y)$ and $(0, z)$. With the two remaining quadrilateral faces, we add $(0, x)$ to $[0,-2 c, x, c]$, and on the other face, we start a sequence of chord exchanges

$$
\pm(2 c, 3 c) \pm(2 c+1, z) \pm(c+1,3 c+1) \pm(-c, x)
$$

These swaps are depicted in Figure 33. Since the last edge was incident with the 5 -sided face, we get a nearly triangular embedding of $K_{12 s+10}$.


Figure 32: Obtaining another 5-sided face using some simple chord exchanges.


Figure 33: Exchanging chords to get a 6-sided face.

### 4.5.5 Case 1

Theorem 4.40. For $s \geq 1$, there exists a nearly triangular tight embedding of $K_{12 s+1}$.

Proof. The minimum genus embedding of $K_{13}$ given by Ringel [Rin74, p.82] already happens to be nearly triangular. The remaining cases are handled by Theorem 4.42 using current graphs.

Corollary 4.41. For $s \geq 1$, there exist embeddings of type (6), $(5,4)$, and $(4,4,4)$ of $K_{12 s+1}$.

Gustin (see Ringel [Rin74, §6.3]) found the first complete solution for triangular embeddings of $K_{12 s+1}-K_{3}$. Those current graphs are most elegantly described using the group $\mathbb{Z}_{2} \times \mathbb{Z}_{6 s-1}$, but since our general solution does not make use of this representation, we have relabeled Gustin's current graph for $s=2$, as shown in Figure 34 .


Figure 34: Gustin's current graph relabeled.

Theorem 4.42. For $s \geq 2$, there exists a nearly triangular tight embedding of $K_{12 s+1}$.

Proof. In addition to the current graph in Figure 34, we also make use of Figure 35, which gives a new triangular embedding of $K_{12 s+1}-K_{3}$ for all $s \geq 3$. The elements 1,3 , and $6 s-3$ are all generators of $\mathbb{Z}_{12 s-2}$, so the vortices are all of type (A1). We note that when $s=3$, the ladder portion has exactly one rung labeled $9=6 s-9$.


Figure 35: Current graphs for Case $1, s \geq 3$. The box in the upper half (a) is replaced by the ladder in the bottom half (b).

For $s=2$, the embedding produced from the current graph in Figure 34 is of the form

| 0. | 17 | 9 | $z$ | 13 | $\ldots$ | 3 | $y$ | 19 | 21 | $x$ | 1 | 20 | 14 | $\ldots$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3. |  |  |  |  |  |  |  | $\ldots$ | 2 | $x$ | 4 | $\ldots$ |  |  |
| 4. |  |  |  |  |  |  |  |  | $\ldots$ | 5 | 2 | 18 | $\ldots$ |  |
| 18. | 13 | 5 | $z$ | $\ldots$ |  |  |  |  |  |  |  |  |  |  |

In the general case, we are interested in the following parts:

$$
\begin{array}{rcccccccccccccccc}
0 . & 6 s-3 & z & 6 s+1 & \ldots & 6 s+4 & 6 & 6 s+5 & \ldots & 3 & y & -3 & \ldots & -1 & x & 1 & \ldots \\
6 s-3 . & & & & & & & & \ldots & 6 s & y & 6 s-6 & \ldots & & & & \\
6 s-6 . & & & \ldots & 0 & 6 s & 1 & \ldots & & & & & & & &
\end{array}
$$

In both cases, the relative positions of the letters $x, y$, and $z$ in the rotation of 0 is the same, so applying Construction 4.17 on vertex 0 and vertices $x, y, z$, we get the 12 -sided face

$$
[z, 6 s-3,0,-3, y, 3,0,1, x,-1,0,6 s+1] .
$$

The sequences of chord exchanges

$$
\pm(x, 3) \pm(2,4) \pm(5,18) \pm(z, 13)
$$

for $s=2$ and

$$
\pm(y, 6 s-3) \pm(6 s-6,6 s) \pm(0,1)
$$

for $s \geq 3$ removes one of the edges incident with the 12 -sided face. If we add the remaining edges according to Figure 36, we are left with a single 6 -sided face, indicating that the resulting embeddings are nearly triangular.

Remark. To the best of our knowledge, all previously published families of current graphs for orientable triangulations of $K_{12 s+1}-K_{3}$ split into two subfamilies depending on the parity of $s$. Our current graphs in Figure 35 form a solution which handles all $s \geq 3$ irrespective of parity. Another such family of current graphs is presented in Section 4.7.2 that extends to the $s=2$ case, though we were unable to use it to prove Theorem 4.42.


Figure 36: The end result of the Case 1 additional adjacency for $s=2$ (a) and $s \geq 3$ (b). The dashed edges are deleted at the end of sequences of chord exchanges.

### 4.5.6 Case 8

Theorem 4.43. For $s \geq 1$, there exists a nearly triangular tight embedding of $K_{12 s+8}$. For $s=0$, there does not exist such an embedding.

Proof. Corollary 4.46 shows nonexistence for $K_{8}$. All remaining values of $s$ are covered by Theorem 4.47 with a unified additional adjacency step.

Corollary 4.44. For $s \geq 1$, there exist embeddings of type $(5)$ and $(4,4)$ of $K_{12 s+8}$. All minimum genus embeddings of $K_{8}$ are of type $(4,4)$.

Proof. For the exceptional case $s=0$, the graph $G_{9}$ in Theorem 4.28 (see Ringel [Rin74, p.79]) has two vertices $x_{0}$ and $x_{1}$ of degree 4 . Deleting both those vertices leaves an embedding of $K_{8}$ of type $(4,4)$.

We first show the nonexistence of a nearly triangular embedding of $K_{8}$, which was also verified by an exhaustive computer search. The proof relies on another nonexistence result
for minimum triangulations.

Theorem 4.45 (Huneke [Hun78]). If a simple graph $G$ triangulates $S_{2}$, then $G$ must have at least 10 vertices.

Corollary 4.46. $K_{8}$ does not have a nearly triangular tight embedding.

Proof. Suppose such an embedding exists. The Map Color Theorem states that the minimum genus of $K_{8}$ is 2 , and furthermore, a nearly triangular embedding in the surface $S_{2}$ would have a simple 5 -sided face as a consequence of Proposition 4.1 and Lemma 4.22. Subdivide the face by adding a new vertex $v$ inside the face and add edges to connect $v$ to the vertices on the boundary of the face. Now, we have a triangular embedding of the simple graph $K_{9}-D$ in $S_{2}$, where $D=K_{1,3}$ is shown in Figure 28. However, Theorem 4.45 states that no such embedding exists.

In the additional adjacency steps of both Cases 8 and 11, edge flips are used to sacrifice one existing edge to gain a previously missing edge. For example, suppose we had the following partial table of a triangular embedding:

$$
\begin{array}{cccccc}
a . & \ldots & c & b & d & \ldots \\
c . & \ldots & e & d & f & \ldots \\
e . & \ldots & g & f & h & \ldots
\end{array}
$$

and $(g, h)$ is not an edge of the graph. Then we can do the edge flips

$$
\begin{aligned}
& -(e, f)+(g, h) \\
& -(c, d)+(e, f) \\
& -(a, b)+(c, d)
\end{aligned}
$$

to add $(g, h)$ at the cost of $(a, b)$. For brevity, we write this operation as the sequence of edge flips

$$
-(a, b) \pm(c, d) \pm(e, f)+(g, h)
$$

The notation suggests that we can view this operation alternatively as deleting the edge $(a, b)$, exchanging the chords $(c, d)$ and $(e, f)$, and then finally adding $(g, h)$.

Theorem 4.47. For $s \geq 1$, there exists a nearly triangular tight embedding of $K_{12 s+8}$.

Proof. We will use the novel family of current graphs in Figure 39 for all $s \geq 3$. For $s=2$, we appeal to Ringel and Youngs [RY69c] for the current graph in Figure 37, and for $s=1$, we use the index 3 current graph in Figure 38. The resulting triangulations have vertices $0,1, \ldots, 12 s+5, x, y_{0}, y_{1}$, where all the numbered vertices are adjacent, $x$ is adjacent to all the numbered vertices, and $y_{0}$ and $y_{1}$ are adjacent to all the even and odd numbered vertices, respectively. We use one handle to connect $y_{0}, y_{1}$, and $x$. Then, contracting the edge $\left(y_{0}, y_{1}\right)$ yields a minimum genus embedding of $K_{12 s+8}$. Initially, however, there is no vertex adjacent to all three lettered vertices.


Figure 37: The current graph of Ringel and Youngs [RY69c] for $K_{32}$.

For $s \geq 2$, the $\log$ of the index 1 current graph is of the form

$$
\left[\begin{array}{lllllllllllllll}
{[0] .} & 6 s+4 & y_{0} & 6 s+2 & \ldots & -1 & x & 1 & \ldots & 6 s+9 & 5 & \ldots & 6 s+6 & 12 s+4 & 4
\end{array} \ldots\right.
$$



Figure 38: An index 3 current graph for Case $8, s=1$.


Figure 39: A new family of current graphs for $s \geq 3$.

In all cases, including $s=1$, employing the additivity rule yields the following partial rotations:

$$
\begin{array}{rlrrrr}
6 s-1 . & \ldots & 6 s-2 & x & 6 s & \ldots \\
6 s . & \ldots & 0 & 6 s-2 & 6 s+4 & \ldots
\end{array}
$$

and the following partial row for $12 s+1$ :

$$
12 s+1 . \quad \ldots .6 s-1 \quad y_{1} \quad 6 s-3 \quad \ldots .12 s \quad x \quad 12 s+2 \ldots .6 s+4 \quad 0 \quad \ldots
$$

We can perform the sequence of edge flips

$$
-(6 s-1, x) \pm(6 s, 6 s-2) \pm(0,6 s+4)+\left(y_{0}, 12 s+1\right)
$$

to produce a vertex adjacent to all three of $x, y_{0}$, and $y_{1}$ in preparation for Construction 4.17. The rotation at vertex $12 s+1$ is now of the form

$$
12 s+1 . \quad \ldots .6 s-1 \quad y_{1} \quad 6 s-3 \quad \ldots .12 s \quad x \quad 12+2 \ldots .6 s+4 \quad y_{0} \quad 0 \quad \ldots
$$

as illustrated in Figure 40. If we apply Construction 4.17 to vertex $12 s+1$ and neighbors $y_{0}, y_{1}, x$, we obtain the 12 -sided face

$$
\left[y_{0}, 6 s+4,12 s+1,6 s-3, y_{1}, 6 s-1,12 s+1,12 s+2, x, 12 s, 12 s+1,0\right] .
$$

Adding the edge $\left(y_{0}, y_{1}\right)$ in this face and contracting it to make a new vertex $y$ yields one 4 -sided face and one 8 -sided face, and the remaining edges $(x, y),(y, 12 s+1),(x, 12 s+1)$, and $(x, 6 s-1)$ can be added back in, pursuant to Figure 41, to produce an embedding of type (5).

Remark. We made use of a current graph of Ringel and Youngs [RY69c], but we did not include any of their other constructions. In fact, their family of current graphs for $s \geq 4$ is


Figure 40: The rotation at vertex $12 s+1$ after the initial modifications.


Figure 41: Using a handle to connect $x$ with $y$ and to replace the missing edge $(x, 6 s-1)$. The edge $\left(y_{0}, y_{1}\right)$ is contracted and the amalgamated vertex is renamed $y$.
also applicable for the additional adjacency solution presented here. Our family of current graphs for $s \geq 3$, while slightly more complicated in terms of the underlying graph, benefits from a significantly simpler current assignment, where the generalization is, like Figure 35 for Case 1, a zigzag pattern where the vertical arcs form an arithmetic sequence. This pattern is "smooth" in the sense of Guy and Youngs [GY73b] as it corresponds to the canonical graceful labeling of a path. In addition, our solution handles the odd and even $s$ cases simultaneously, and it extends downwards to $s=3$, for which Ringel and Youngs [RY69c] needed a special
solution incompatible with our additional adjacency solution.

### 4.5.7 Case 11

Theorem 4.48. For $s \geq 0$, there exists a nearly triangular tight embedding of $K_{12 s+11}$.

Proof. The embedding of $K_{11}$ given by Mayer [May69], after deleting two extra edges, is nearly triangular. ${ }^{8}$ The embedding of $K_{23}$ we give in the Appendix was also found starting from [May69]. Two sequences of chord exchanges, starting with $(8,22)$ and $(10,16)$, eventually "collide" at two edges incident with the same face, resulting in a 5 -sided face.

The general case $s \geq 2$ is proved in Theorem 4.50.

Corollary 4.49. For $s \geq 0$, there exist embeddings of type (5) and $(4,4)$ of $K_{12 s+11}$.

Theorem 4.50. For $s \geq 2$, there exists a nearly triangular tight embedding of $K_{12 s+11}$.

Proof. Ringel and Youngs [RY69a] found current graphs with the structure of Figure 42 for $s \geq 2$. The current graphs produce triangular embeddings of $K_{12 s+11}-K_{5}$, so the goal is to add in the edges between the lettered vertices using two handles. Near the vortices, the logs of both current graphs are

$$
[0] . \quad x \quad 6 s+5 \quad 12 s+4 \quad a \quad 12 s+5 \quad y \quad 1 \quad b \quad 12 s+2 . \ldots c c c c c c
$$

Before adding handles, several local edge additions and deletions are made to the triangular embedding of $K_{12 s+11}-K_{5}$. We omit the exact details of these modifications, which are identical to those in Ringel and Youngs [RY69a] (see also Ringel [Rin74, p.100]). In summary, the resulting embedding now has the edges $(a, y),(b, y)$, and $(a, x)$ at the expense of
${ }^{8}$ The embedding given in Ringel [Rin74, p.81], results from deleting the "wrong" edge of each doubled pair, leaving an embedding of type $(4,4)$.


Figure 42: The geometry of two general current graphs for Case 11, depending on the parity of $s$.
$(0,12 s+4),(0,6 s+5),(c, 12 s+4)$, and $(b, 4)$. The embedding also has a single nontriangular face $[a, 12 s+4,6 s+5, x]$, as seen in Figure 43.


Figure 43: Modifications to the rotation at vertex 0 . The shaded quadrilateral face on the right will be used again later on.

Applying Construction 4.17 to vertex 0 and nonadjacent vertices $a, b, c$, we obtain the

12-sided face

$$
[0,12 s+5, a, x, 0,12 s+2, b, y, 0,2, c, 4]
$$

while losing the edges $(0, a),(0, b)$ and $(0, c)$. In this face, we add the chords $(0, a),(0, b)$, $(0, c),(a, b),(b, c),(c, y),(b, 4),(b, x)$ as in Figure 44(a). The handle creates the face $[0, c, y]$, and from the previous modifications, there is the quadrilateral $[x, a, 12 s+4,6 s+5]$. Using Construction 4.12, we can merge the two faces to add the edges $(a, c),(c, x),(x, y),(0,6 s+5)$, $(0,12 s+4)$, and $(c, 12 s+4)$ as in Figure 44(b).

(a)

(b)

Figure 44: The gained edges from two handles. Note that the second handle in part (b) makes use of the shaded faces from Figure 43 and part (a).

Now, all the missing edges have been added and we are left with an embedding of $K_{12 s+11}$ with two quadrilateral faces

$$
[0,6 s+5, x, y] \text { and }[0,12 s+2, b, x]
$$

Exchanging the chord ( $0, x$ ) yields an embedding of type (5), completing the construction.

### 4.5.8 Maximum genus embeddings

The (orientable) maximum genus $\gamma_{M}(G)$ is the largest integer $g$ such that $G$ has a cellular embedding in $S_{g}$. Archdeacon and Craft [Arc95] also ask if $K_{n}$ has a nearly triangular maximum genus embedding. Nordhaus et al. [NSW71] show that $K_{n}$ is upper-embeddable, meaning it has an embedding with one or two faces, depending on the parity of $|V(G)|-$ $|E(G)|$. In particular, the maximum genus embedding has one face exactly when $n \equiv 1,2$ $(\bmod 4)$. The one-face embeddings are already nearly triangular in a trivial way, so we need a construction just for two-face embeddings.

A special case of Xuong's characterization [Xuo79] of maximum genus states that a graph $G$ is upper-embeddable if and only if there is a spanning tree $T$ such that $G-T$ has at most one component with an odd number of edges. To construct the one- or two-face embedding, the edges of $G-T$ are partitioned into pairs such that the edges of each pair share a vertex. Starting with an arbitrary embedding of the spanning tree $T$ in the plane (which has one face), we add the pairs one by one, as in Figure 45. After each addition, the resulting embedding still has one face. If there is an edge left over (i.e. one of the edges of the odd-sized component), it is added arbitrarily into the embedding, resulting in a two-face embedding.

We note that the final embedding of $G$ restricted to $T$ is the same as the original embedding of $T$ that we started with. This observation is enough for constructing a nearly triangular two-face embedding.

Proposition 4.51. For $n \equiv 0,3(\bmod 4)$, there exists a two-face embedding of $K_{n}$ where one of the faces is a triangle.


Figure 45: Starting from a one-face embedding, we can add two incident edges to get another one-face embedding.

Proof. Label the vertices $1, \ldots, n$. Delete the edge $(2,3)$ and let the spanning tree $T$ be all the edges incident with vertex 1 . Then, $\left(K_{n}-(2,3)\right)-T$ is connected and has an even number of edges. Let the rotation at vertex 1 simply be

$$
\text { 1. } 2 \quad 3 \quad \ldots \quad n \text {. }
$$

Adding in all the edge pairs in the manner described above preserves the rotation at 1 , resulting in an embedding with one face of the form $[\ldots 2,1,3 \ldots]$. We can then insert the edge $(2,3)$ into the embedding to get one triangular face $[2,1,3]$ and one long nontriangular face.

### 4.6 Unified minimum triangulations and complete graph embeddings

Besides face distributions of complete graphs, which can be seen as finding triangular embeddings of the complete graphs with additional edges, one could also consider deleting a small number of edges from a complete graph and seeing if the resulting dense graphs have triangular embeddings in surfaces. When $t$ is at most $n-6$, then one can show (see Junger-
man and Ringel [JR80]) using the Euler polyhedral equation that an ( $n, t)$-triangulation is necessarily a minimum triangulation. For almost all surfaces, such an embedding exists:

Theorem 4.52 (Jungerman and Ringel [JR80]). For all pairs of integers $(n, t) \neq(9,3)$, where

$$
\begin{aligned}
& n \geq 4,0 \leq t \leq n-6 \\
& (n-3)(n-4) \equiv 2 t \quad(\bmod 12)
\end{aligned}
$$

there exists an ( $n, t$ )-triangulation.

The lone exception, $(n, t)=(9,3)$, was seen earlier in Theorem 4.45. The same connection between nearly triangular embeddings and minimum triangulations will be used later on to show the existence of a (23,16)-triangulation.

Like in the proof of Map Color Theorem, the problem breaks down into the 12 Cases. Indeed, in several Cases, the current graphs used in the proof of the Map Color Theorem [Rin74] for $K_{n}$ have the dual purpose of also providing all the necessary minimum triangulations on the same number of vertices $n$. However, not all Cases have been combined in this manner.

In this section, the central goal is to provide, for some Case $k$, a single family of current graphs that simultaneously solves the minimum triangulations problem and yields a genus embedding of the complete graphs on $12 s+k$ vertices. As mentioned earlier, Jungerman and Ringel's constructions for minimum triangulations typically break down into multiple families of current graphs. For each of Cases $6,8,9$, and 11, we provide a family of index 3 current graphs that share most of the structure for the standard solution for Case 5 of the Map Color Theorem.

In general, our constructions will proceed in the following way: using an index 3 current graph, we generate an ( $n, t$ )-triangulation. We wish to find other embeddings of graphs on the same number of vertices using the following operations:

- Handle subtraction, which deletes edges from a triangular embedding to produce a triangular embedding in a lower-genus surface. (Section 4.4)
- Additional adjacency, which adds edges using extra handles and edge flips. (Section 4.3)

By subtracting handles, we obtain all the necessary $\left(n, t^{\prime}\right)$-triangulations, for $t^{\prime}>t$, and over the course of the additional adjacency step for constructing a genus embedding of $K_{n}$, we construct the remaining $\left(n, t^{\prime \prime}\right)$-triangulations, for $t^{\prime \prime}<t$.

### 4.6.1 The Bose ladder

A sketch of the standard proof of Case 5 of the Map Color Theorem (see Ringel [Rin74, §9.2] or Youngs [You70a]) is given first, as we reuse significant parts of its structure for our current graphs. The case $s=1$ was given earlier in Figure 9, and the higher order cases are given in Figures 46 and 47. The construction also works trivially for $s=0$ as well. These current graphs produce triangular embeddings of $K_{12 s+5}-K_{2}$.


Figure 46: A current graph for $K_{29}-K_{2}$.


Figure 47: The family of current graphs for $K_{12 s+5}-K_{2}$, for general $s$. The omitted current on a circular arc is the same as those on the horizontal arcs above and below it.

The general shape of the family of current graphs is a long ladder whose "rungs" alternate between simple vertical arcs and so-called "globular rungs," where two vertices have a pair of parallel edges between them. As we parse from left to right, the vertical arcs, except for the arc connecting the two vortices, alternate in direction and form the arithmetic sequence consisting of the nonzero multiples of 3 in $\mathbb{Z}_{12 s+3}$. The zigzag pattern induced on the horizontal arcs is essentially the canonical graceful labeling of a path graph on $4 s+1$ vertices (see, e.g., Goddyn et al. [GRŠ07] for more information on this connection), where the vertical arcs correspond to the edge labels on the path graph. The horizontal arcs come in pairs that share the same current and are oriented in opposite directions. The currents on these arcs exhaust all the elements of the form $3 k+1$ in $\mathbb{Z}_{12 s+3}$.

To see that construction principle (P2) is satisfied, the circuit [0] traverses each pair of horizontal arcs twice in the rightward direction, so each element $3 k+1$ and its inverse
appear in the log of the circuit. Circuits [1] and [2] pass through only one arc of each pair of horizontal arcs - they each pass through the inverse of that current on one of the parallel arcs in a nearby globular rung. For the multiples of 3, note that for each such element and its inverse, exactly one of them appears as vertical arcs on a globular rung, and one appears on a simple rung. For the former, both circuits [1] and [2] will make use of such arcs in both directions, and for the latter, circuit [0] will pass through in both directions.

We utilize this family of current graphs in the following way: many families of current graph constructions consist of

- A fixed portion, which contains vortices and some salient currents for additional adjacency solutions. The underlying directed graph stays the same, while the currents may vary as a function of $s$.
- A varying portion, which subsumes all remaining currents not present in the fixed portion. The size of this ingredient varies as a function of $s$, and the currents are arranged in a straightforward pattern.

In the construction for Case 5, we might consider the vortices and its incident edge ends as the fixed portion, and the rest of the graph (see Figure 48) as the varying portion. The solutions for Case 3 and 5 of the Map Color Theorem are, coincidentally or not, intimately connected to Bose's construction for Steiner triple systems on $6 s+3$ elements (see Grannell et al. [GGŠ98]) and are prized for their simplicity. For these reasons, we consider this varying portion, which we call the Bose ladder, to be the best possible choice for index 3 current graphs.

The approach of Youngs et al. [You70a, GY73a, GR76] was to first finalize the fixed


Figure 48: The Bose ladder is essentially the current graphs for Case 5 with two vertices deleted.
portion and then solve certain labeling problems (so-called "zigzag" and "chord" problems) to deal with the varying portion. We tackle the problem in reverse, opting to massage the fixed portion around a preset varying portion, which we choose to be a contiguous subset of the Bose ladder. Starting with the arc labeled 1 that runs between the two vortices, we successively peel off rungs of the Bose ladder until we have enough material for our desired fixed portion.

We expect this procedure to become more difficult as the number of vortices increasesnot only do we need appropriate currents that feed into the vortices, but there becomes an imbalance between the currents which are not divisible by 3 and those which are. Each vortex will use three currents of the former type, leaving a surplus of those of the latter type. To correct this effect, we make use of the double bubble in Figure 49, which is essentially two globular rungs joined together. By tracing out the partial circuits and invoking construction principle (P3), we find that all six currents entering the highest and lowest vertices must be divisible by 3 , while the four remaining arcs may be labeled with an element not divisible by 3 depending on which circuits touch this gadget. The double bubble and its generalization have appeared in other work regarding current graphs of index greater than 1 , such as Korzhik and Voss [KV02] and Pengelley and Jungerman [PJ79].

In all of our current graph constructions in this section, we use the cyclic group $\mathbb{Z}_{12 s+3}$ unless we specify otherwise. While we often simplify the labels by reversing the directions of some arcs, e.g. replacing a label like $12 s+1$ with 2 , the ends which connect to the Bose ladder are kept unchanged, i.e., left as a current which is congruent to $1(\bmod 3)$.


Figure 49: The "double bubble" motif appears in all of our general constructions.

### 4.6.2 Case 5

As a warmup, let us consider how to find minimum triangulations for Case 5. The original solution in Figure 47 does not have any arithmetic 3 -ladders, but we can modify it by swapping two of the rungs in the Bose ladder, namely the two with vertical arcs labeled 6 and $12 s-3$, as in Figure 50. In this drawing and all forthcoming figures, we only describe the fixed portion of the family of current graphs-at the ellipses, we complete the picture by attaching the corresponding segment of the Bose ladder, as mentioned earlier. Exactly where to truncate the Bose ladder is determined by the currents at the ends of the fixed portion.

The idea of pairing the rungs is crucial in Youngs' method [You70a] for constructing index 3 current graphs. In their proof of minimum triangulations for Case 5, Jungerman and Ringel [JR80] took this idea to the extreme and switched all pairs of rungs so that all of the


Figure 50: A slight modification to the Bose ladder results in another family of $(12 s+5,1)$ triangulations from which we can subtract handles to produce other minimum triangulations.
globular rungs appeared on one side of the ladder, but as seen in our example, implementing all these exchanges is not necessary.

We note that to the left of the vortices in our drawing in Figure 50, the directions of the arcs are inverted from that of Figure 47. Most of our infinite families involve attaching a Bose ladder with a "Möbius twist," i.e., the final current graph is a long ladder-like graph whose top-left and bottom-left ends become identified with the bottom-right and top-right ends, respectively.

### 4.6.3 Case 6

The family of current graphs in Figure 51 applies for all $s \geq 2$ and has an arithmetic 3ladder, giving a simpler and more unified construction for Case 6 of the Map Color Theorem (after applying Proposition 4.9), in addition to providing a single family of current graphs, irrespective of parity, for Case 6 of Theorem 4.52. The case $s=1$ is particularly pesky-in the original proof of the Map Color Theorem, the minimum genus embedding of $K_{18}$ was found using purely ad hoc methods by Mayer [May69]. One might ask if an index 3 current graph exists for $K_{18}-K_{3}$, but an exhaustive computer search suggests that one does not exist. In Section 4.7.3, we present another solution for Case 6 of the Map Color Theorem,
$s \geq 2$, that almost achieves the 18 -vertex case.


Figure 51: A current graph for $K_{12 s+6}-K_{3}$ for $s \geq 2$.

### 4.6.4 Case 9

We improve on the construction of Guy and Ringel [GR76] with the family of index 3 current graphs seen in Figure 52. These current graphs produce triangular embeddings of $K_{12 s+9}-K_{6}$ for all $s \geq 2$, and the vertical rungs labeled 3, 6, 9 form an arithmetic 3-ladder. The circuits [1] and [2] have the six vortices packed as close together as possible. In particular, the log of circuit [1] reads
so we may apply Lemma 4.15 with, e.g., $u=1$, to obtain $(12 s+9,9)$ - and $(12 s+9,3)$ triangulations and a tight embedding of $K_{12 s+9}$.

For $s=1$ we use the special current graph in Figure 53. It is essentially one of the inductive constructions used by Jungerman and Ringel [JR80], with the additional observation that the current graph used has an arithmetic 3-ladder.


Figure 52: A current graph for $K_{12 s+9}-K_{6}$ for $s \geq 2$. Additional fragments of circuits besides the guidelines at the left and right ends indicate components used in the additional adjacency solution.


Figure 53: A current graph for $K_{21}-K_{3}$ with an arithmetic 3-ladder.

### 4.6.5 Case 8

The family of current graphs in Figure 54 yields triangular embeddings of $K_{12 s+8}-K_{5}$ and has the necessary arithmetic 3-ladder for producing the minimum triangulations on fewer edges. The logs of this current graph are of the form

$$
\left.\begin{array}{rlllllllllll} 
& {[0] .} & \ldots & 6 s+1 & 12 s & \ldots & 12 s-3 & 6 s-2 & \ldots & \\
& {[2] .} & \ldots & a & 6 s+2 & b & 12 s+1 & c & 6 s-1 & d & 12 s-2 & e
\end{array}\right] .
$$

These translate, by additivity, to the rotations

$$
\begin{array}{rllllllllllll} 
& 3 . & \ldots & 6 s+4 & 0 & \ldots & 12 s & 6 s+1 & \ldots & \\
\text { 2. } & \ldots & a & 6 s+4 & b & 0 & c & 6 s+1 & d & 12 s & e & \ldots
\end{array}
$$

By applying Lemma 4.14 with $u=2, v=3,\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=(6 s+5,0,6 s+1,12 s)$, we can construct a $(12 s+8,4)$-triangulation and a tight embedding of $K_{12 s+8}$.


Figure 54: A family of index 3 current graphs for $K_{12 s+8}-K_{5}, s \geq 2$.

Remark. Our additional adjacency solution makes use of some of the arcs forming the arithmetic 3-ladder. However, there is no conflict since handle subtractions and additional adjacencies are never applied simultaneously.

### 4.6.6 Case 11

For $s \geq 3$, we found the family of current graphs in Figure 55 that generate triangular embeddings of $K_{12 s+11}-K_{8}$. On the bottom right is an arithmetic 3-ladder with labels
$9,12,15$. By examining the circuit [1], we obtain the rotations

$$
\begin{aligned}
& \text { 1. } \begin{array}{rrrrrrrrlllllllll} 
& \ldots & a & 6 s+8 & b & \ldots & c & 5 & d & \ldots & e & 12 s-1 & f & \ldots & g & 6 s+2 & h
\end{array} \ldots \\
& 12 s+1 .
\end{aligned} \ldots
$$

Applying Lemma 4.16 with $u=1, v=12 s+1,\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=(6 s+8,5,12 s-1,6 s+2)$ yields the remaining triangulations and a tight embedding of $K_{12 s+11}, s \geq 3$.


Figure 55: Index 3 current graphs for $K_{12 s+11}-K_{8}, s \geq 3$.

For $s=1,2$, we first find a current graph with group $\mathbb{Z}_{12 s+6}$ that generates a triangular embedding of $K_{12 s+11}-K_{5}$. For $s=1$, consider the index 3 current graph in Figure 56. The rotations at vertices 1 and 12 are of the form

$$
\text { 12. } \ldots .5 \begin{array}{lllll} 
& \ldots & \ldots & 9 & \ldots,
\end{array}
$$

so applying Lemma 4.14 with $u=1, v=12,\left(p_{1}, p_{2}, p_{3}, p_{4}\right)=(5,8,3,9)$ yields (23,10)and (23,4)-triangulations, and a tight embedding of $K_{23}$. For $s=2$, the current graph in Figure 57 generates a triangular embedding of $K_{35}-K_{5}$. Similar to the $s=1$ case, we use the rotations

$$
\begin{array}{lllllllllllll}
2 . & \ldots & a & 10 & b & 6 & c & 7 & d & 4 & e & \ldots \\
& 19 . & \ldots & 7 & 10 & \ldots & 6 & 3 & \ldots,
\end{array}
$$

and Lemma 4.14 to find the $(35,10)$ - and $(35,4)$-triangulations, and a tight embedding of $K_{35}$. The remaining minimum triangulations can be found using the arithmetic 3-ladder.


Figure 56: An index 3 current graph for $K_{23}-K_{5}$.

The following result relates nearly triangular embeddings to minimum triangulations:

Proposition 4.53. If there exists a nearly triangular tight embedding $\phi: K_{n} \rightarrow S_{g}$ of the complete graph $K_{n}$ with a simple nontriangular face, then there exists a minimum triangulation of the surface $S_{g}$ on $n+1$ vertices.

Proof. The length $\ell$ of the nontriangular face must be at least 5 by Theorem 4.20. Subdividing the nontriangular face with a new vertex yields an $(n+1, t)$-triangulation, with

$$
t=n-\ell \leq(n+1)-6
$$



Figure 57: An index 3 current graph for $K_{35}-K_{5}$.

In particular, we had used Theorem 4.45 to show that $K_{8}$ does not have a nearly triangular embedding in $S_{2}$ with the same proof. We use the nearly triangular genus embedding of $K_{22}$ given in 4.5.4 to construct the remaining $(23,16)$-triangulation.

Finally, we give a unification of the 11-vertex case using an asymmetric ad hoc embedding of $K_{11}-C_{4}$ :

| 0. | 1 | 10 | 8 | 4 | 2 | 9 | 7 | 5 | 3 | 6 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1. | 0 | 6 | 4 | 8 | 5 | 9 | 3 | 7 | 2 | 10 |
| 2. | 0 | 4 | 10 | 1 | 7 | 6 | 5 | 8 | 3 | 9 |
| 3. | 0 | 5 | 10 | 4 | 7 | 1 | 9 | 2 | 8 | 6 |
| 4. | 0 | 8 | 1 | 6 | 9 | 5 | 7 | 3 | 10 | 2 |
| 5. | 0 | 7 | 4 | 9 | 1 | 8 | 2 | 6 | 10 | 3 |
| 6. | 0 | 3 | 8 | 10 | 5 | 2 | 7 | 9 | 4 | 1 |
| 7. | 0 | 9 | 6 | 2 | 1 | 3 | 4 | 5 |  |  |
| 8. | 0 | 10 | 6 | 3 | 2 | 5 | 1 | 4 |  |  |
| 9. | 0 | 2 | 3 | 1 | 5 | 4 | 6 | 7 |  |  |
| 10. | 0 | 1 | 2 | 4 | 3 | 5 | 6 | 8 |  |  |

The missing edges are $(7,8),(8,9),(9,10)$, and $(10,7)$, which can be added with one handle using Construction 4.12 as in Figure 58. Note that this construction does not really make use of any specific structure in the embedding, as we can always find a face incident with a given edge. We thus formulate this additional adjacency approach more generally:

Proposition 4.54. If there exists a triangular embedding of $K_{n}-C_{4}$, then there exists a tight embedding of $K_{n}$.


Figure 58: A generic method for adding a $C_{4}$ with one handle, applied to the triangular embedding of $K_{11}-C_{4}$.

### 4.7 Simplified constructions for the Map Color Theorem

Several Cases in the original proof of the Map Color Theorem were resolved with difficult solutions. In previous sections, we already found some simpler families of current graphs (e.g., Case 8, Figure 39)—here we give some refinements of those constructions when restricting ourselves to just the Map Color Theorem (Cases 1 and 6), in addition to a new approach to finding triangular embeddings of $K_{12 s}$ (Case 0).


Figure 59: The fixed part of the current graph (a) contains the salient currents for adding one handle, and the simple ladder (b) inside the box.

### 4.7.1 Case 0

For $s \geq 4$, consider the current graphs in Figure 59 with current groups $\mathbb{Z}_{12 s-4}$. By examining their logs near the vortices and the curved arcs in part (a), we find that the rotation at vertex 0 is of the form

$$
\text { 0. } x 1 y 12 s-5 z \ldots 6 s+226 s+3 \ldots 6 s-266 s-3 \ldots 6 s-5 w \ldots
$$

For $s=3$, the rotation (1) of the current graph in Figure 8 differs from this general form in that the two elements adjacent to 6 are swapped. That is, it is of the form
0. $x 1$ y $12 s-5 z \ldots 6 s+226 s+3 \ldots 6 s-366 s-2 \ldots 6 s-5 w \ldots$

Figure 60 illustrates how this partial information allows us to add the missing edges using
edge flips and a handle. In particular, adding the edge $(x, y)$ at the cost of the edge $(0,1)$ allows us to install a handle that merges three faces containing the four lettered vertices. We then add the missing edges near this handle or via sequences of edge flips. The minor discrepancy between the logs for $s=3$ and $s \geq 4$ manifests in one of the quadrilaterals in Figure 60-that quadrilateral is merely mirrored for $s=3$, so the corresponding edge flip is still permissible and the rest of the additional adjacency solution is identical. The resulting embeddings of $K_{12 s}$ are triangular for all $s \geq 3$, completing the construction. For $s=6$, the family of current graphs has an arithmetic 3-ladder.


Figure 60: The embedding near vertex 0 is augmented using a handle, which is represented by excising two disks and identifying their boundaries. For the edge flips, the dashed edges are replaced by the thick solid edges. The quadrilateral inside the box has the reverse orientation for $s=3$.

Remark. The arcs labeled 9 and $6 s-12$ in Figure 59(a) extend the arithmetic sequence in Figure 59(b), but unfortunately the rotations assigned to their endpoints must differ from
the pattern in Figure 59(b). Thus, generalizing this family of current graphs to the $s=3$ case is impossible.

For $s=1$, Ringel [Rin74] gives an example of an index 4 current graph that generates a triangular embedding of $K_{12}$ using the group $\mathbb{Z}_{12}$. Pengelley and Jungerman [PJ79] and Korzhik [Kor08] generalized that solution to all $K_{12 s}$, but their solutions were complicated. Since our simple index 1 solution works for $s \geq 3$, we only need to supply the missing case $s=2$, which is actually absent in both of the aforementioned papers. The logs of the current graph in Figure 61 are

$$
\begin{array}{lrrrrrrrrrrrrrrrrrrrrrr}
{[0] .} & 19 & 16 & 4 & 1 & 21 & 20 & 12 & 8 & 3 & 5 & 2 & 1 & 11 & 22 & 10 & 17 & 7 & 14 & 6 & 15 & 9 & 18 \\
23 \\
{[1] .} & 5 & 8 & 20 & 23 & 3 & 4 & 12 & 16 & 21 & 19 & 22 & 11 & 13 & 2 & 14 & 7 & 17 & 10 & 18 & 9 & 15 & 6 \\
1 \\
{[2] .} & 19 & 20 & 21 & 1 & 4 & 23 & 5 & 6 & 15 & 9 & 18 & 8 & 16 & 10 & 17 & 7 & 14 & 12 & 2 & 13 & 11 & 22 \\
\hline \text { [3]. } & 5 & 4 & 3 & 23 & 20 & 1 & 19 & 18 & 9 & 15 & 6 & 16 & 8 & 14 & 7 & 17 & 10 & 12 & 22 & 11 & 13 & 2
\end{array}
$$

$\mathbb{Z}_{24}$


Figure 61: This index 4 current graph, which generates a triangular embedding of $K_{24}$, was discovered using techniques found in Pengelley and Jungerman [PJ79]. The ends labeled "A" and "B" are identified to form a cylindrical digraph.

### 4.7.2 Case 1

In Section 4.5.5, we found nearly triangular embeddings of $K_{12 s+1}$ for $s \geq 3$ using a single family of current graphs with a large simple zigzag. In fact, there even exist families of current graphs for generating triangular embeddings of $K_{12 s+1}-K_{3}$ that include the $s=2$ case as well, such as the one in Figure 62. This is the simplest known proof of Case 1 for $s \geq 2$ of the original Map Color Theorem, and as remarked by Ringel [Rin74, p.96], there cannot exist an index 1 current graph with three vortices of type (A1) for $s=1$ because $\mathbb{Z}_{10}$ does not have enough generators.

(a)

(b)

Figure 62: Current graphs producing triangular embeddings of $K_{12 s+1}-K_{3}$ for all $s \geq 2$.

### 4.7.3 Case 6

In Figure 63, we give another index 3 construction for triangular embeddings of $K_{12 s+6}-K_{3}$ using as much of the Bose ladder as possible. The corresponding segment of the Bose ladder has $4 s-5$ rungs - if we had a family of current graphs where the varying portion was a Bose ladder with one more rung, then for $s=1$ an index 3 current graph would exist (with 0 rungs from the Bose ladder). Since our experimental results shows that no current graph exists for $s=1$, this construction maximizes the number of rungs used from the Bose ladder. As a side note, this family of current graphs uses the same "gadgets" for building current graphs known to Ringel et al.


Figure 63: Another construction for triangular embeddings of $K_{12 s+6}-K_{3}$.

### 4.8 Search algorithms for current graphs

Here we briefly sketch a search algorithm for finding index 1 current graphs that generate triangular embeddings. An algorithm for finding triangular embeddings of a graph simply tries to add triangles using depth-first search until the triangles form a closed surface - the same procedure can be applied when the rotation system is derived by additivity from the $\log$ of a single circuit. The algorithm first finds a rotation for vertex 0 . Then, a current graph is reconstructed from this rotation.

Suppose we tried to put $k$ after $j$ in the rotation at vertex 0 . Since we want the embedding to be triangular, invoking Proposition 4.6 causes the rotations at vertices $j$ and $k$ to be of the form

```
j. ... k 0 _..
k. ... 0 j ...
```

Since the rotations at $j$ and $k$ were generated from the rotation at vertex 0 , we may "reduce" these rows to obtain the following constraints on the rotation at vertex 0 :

$$
\begin{array}{ccccc}
0 . & \ldots & (k-j) & -j & \ldots \\
0 . & \ldots & -k & (j-k) & \ldots
\end{array}
$$

Remark. This is essentially the justification for why Kirchhoff's current law needs to hold at all vertices of degree 3. Having these simultaneous constraints is equivalent to having a vertex of degree 3 satisfying KCL.

Thus, fixing part of the rotation system to get a triangular face incident with vertex 0 gives rise to two other triangles. Let us call this procedure adding a reduced triangle. Each edge $(0, v)$ for $v=1, \ldots, n-1$ will be incident with a left and right triangle. That is, if the rotation is of the form

$$
\begin{array}{llllll}
0 . & \ldots & i & j & k & \ldots,
\end{array}
$$

then $[0, i, j]$ and $[0, j, k]$ are the left and right triangles of the edge $(0, j)$, respectively. Our subroutine for finding a complete rotation is described in Figure 64.

After we have found a suitable rotation at vertex 0 using LOGSEARCH, we need to find the current graph matching that rotation. The fact that such a current graph even exists was proven purely combinatorially by Youngs [You66]. However, a topological viewpoint greatly

Algorithm LOGSEARCH: Given a partial rotation $\widehat{R}$, return a full rotation $R$.

1. Find an edge $(0, i)$ with no right triangle. If none exists, output $R:=\widehat{R}$.
2. For each edge $(0, j)$ with no left triangle:
(a) Add the reduced triangle $[0, i, j]$ to $\widehat{R}$ to get $\widehat{R_{j}}$.
(b) If $\widehat{R_{j}}$ has a cycle but is not a cyclic permutation, discard.
(c) If $R:=\mathbf{L O G S E A R C H}\left(\widehat{R_{j}}\right)$ is not NONE, return $R$.
3. Return NONE.

Figure 64: An algorithm for finding a generating row.

Algorithm FOLDER: Given a rotation $R$, return a current graph.

1. Initialize a labeled directed cycle graph $C_{|R|}$, with the arcs labeled in the same order as they appear in $R$.
2. For each pair of inverse currents $\varepsilon,-\varepsilon$, identify their $\operatorname{arcs}$ in $C_{|R|}$ to get a labeled directed graph $D$.
3. Set the rotations at the vertices of $D$ so that its $\log$ is $R$.
4. Return the resulting current graph.

Figure 65: Recovering the current graph from a rotation.
simplifies the argument. Recall that an index 1 current graph is one which is embedded in a surface so that there is exactly one face, which we may interpret as a polygon. The surface and the embedded graph can be reconstructed by appropriately identifying the sides of the polygon, as in Figure 65.

As an example, consider the following log (from Ringel [Rin74, p.27]), which generates a triangular embedding of $K_{19}$ using the group $\mathbb{Z}_{19}$ :

$$
\left[\begin{array}{cccccccccccccccccc}
{[0] .} & 9 & 7 & 4 & 17 & 10 & 18 & 5 & 16 & 12 & 2 & 6 & 1 & 11 & 14 & 13 & 15 & 3
\end{array}\right.
$$

Running FOLDER on this rotation yields the current graph in Figure 66.


Figure 66: The one face embedding on the left is folded on itself to create the current graph on the right.

Many of the families of current graphs found in this thesis were found using these algorithms and their generalizations to arbitrary index, non-complete graphs, and current graphs with vortices.

## 5 Conclusion

In this thesis, we presented results on sample-based high-dimensional convexity testing, acyclicity testing in bounded degree graphs, and embeddings of dense graphs. One might ask if there are improved algorithms for "active," query-based testers for convexity testing, but no substantial algorithmic improvements or lower bounds are known. In our two-sided algorithm, we presented a computationally inefficient approach to transforming an improper learner into a proper learner. It would be desirable to improve runtime of this algorithm while achieving the same sample complexity, perhaps by exhibiting a proper learning algorithm for convex bodies.

The most striking open question about acyclicity testing is whether or not there is a sublinear algorithm, even in the setting where the algorithm can make outgoing and incoming queries with two-sided error. The existence of expanders of logarithmic girth [LPS88] implies that naive breadth-first search does not work. Our lower bound of $\Omega\left(N^{5 / 9-\delta}\right)$ only applies to the restricted setting of one-sided error, outgoing-only queries, and unbounded indegree. We believe that the last condition can be removed with essentially the same analysis, but for lower bounds on algorithms with two-sided error, one would need a new idea beyond tree-like knowledge graphs, for which Bender and Ron's [BR02] analysis appears to be best possible.

Section 4.6 details a unification of approximately $5 / 12$ of the Map Color Theorem, and we conjecture that for sufficiently large complete graphs, one can generalize the ideas presented there to all Cases of both the Map Color Theorem and the minimum triangulations problem. Many attempts to unify the proof of the Map Color Theorem have been attempted, and our
use of the Bose ladder has been the most successful one thus far.
With the computational tools developed in this thesis, it seems that the greatest difficulty in finding genus embeddings of families of dense graphs possessing cyclic symmetry is not in the general case, but for "medium-sized" graphs-for small enough graphs, one can simply use brute force, while for large enough graphs, eventually one should be able to find suitable current graphs. Search results on current graphs also suggest some sort of "contiguity": roughly speaking, if there exists a current graph for $s$, there should exist one for $s+1$. Our infinite families are built using ladder-like graphs, but perhaps there exists a more inductive way of finding current graphs.

Current graphs are best suited for symmetric graphs, but Conjecture 4.5 covers many asymmetric graphs. However, the veracity of Conjecture 4.5 is unknown even in various special cases that could be solved using current graphs, such as the complete graphs $K_{n}$ minus three edges. Perhaps current graphs are not the best tool for resolving this conjecture, but at present they are the only way of constructing genus embeddings of many of the complete graphs. The technology of probabilistic constructions, like those for block designs [Kee14], may be applicable to graph embeddings.

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## Appendix

Here we detail the exceptional cases $n=10,23$ in the proof of Theorem 4.20 that did not admit a current graph solution. These rotation systems all satisfy Proposition 4.6.

The following is a triangular embedding of $K_{10}-P_{3}$, which by Lemma 4.36, can be converted into a nearly triangular embedding of $K_{10}$.

| 0. | 2 | 6 | 5 | 7 | 4 | 3 | 8 | 9 |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1. | 3 | 5 | 6 | 9 | 4 | 8 | 7 |  |  |
| 2. | 0 | 9 | 7 | 5 | 8 | 4 | 6 |  |  |
| 3. | 0 | 4 | 5 | 1 | 7 | 9 | 6 | 8 |  |
| 4. | 0 | 7 | 6 | 2 | 8 | 1 | 9 | 5 | 3 |
| 5. | 0 | 6 | 1 | 3 | 4 | 9 | 8 | 2 | 7 |
| 6. | 0 | 2 | 4 | 7 | 8 | 3 | 9 | 1 | 5 |
| 7. | 0 | 5 | 2 | 9 | 3 | 1 | 8 | 6 | 4 |
| 8. | 0 | 3 | 6 | 7 | 1 | 4 | 2 | 5 | 9 |
| 9. | 0 | 8 | 5 | 4 | 1 | 6 | 3 | 7 | 2 |

The following is a nearly triangular embedding of $K_{23}$. The pentagonal face has been subdivided with a new lettered vertex $p$ to make the entire embedding triangular-deleting that vertex reveals the desired embedding.

1. $23 \begin{array}{lllllllllllllllllllll}23 & 12 & 17 & 6 & 9 & 2 & 7 & 18 & 20 & 8 & 5 & 16 & 14 & 3 & 11 & 22 & 21 & 15 & 13 & 4 & 10\end{array}$
2. $1 \begin{array}{lllllllllllllllllllllll} & 9 & 20 & 15 & 4 & 11 & 5 & 13 & 3 & 16 & 19 & 6 & 21 & 22 & 17 & 14 & 10 & 8 & 18 & 23 & 12 & 7\end{array}$
3. $1 \begin{array}{lllllllllllllllllllllll} & 14 & 23 & 5 & 17 & 15 & 10 & 22 & 16 & 2 & 13 & 18 & 6 & 8 & 20 & 9 & 19 & 4 & 12 & 21 & 7 & 11\end{array}$
4. $1 \begin{array}{llllllllllllllllllllll}13 & 13 & 22 & 18 & 9 & 11 & 2 & 15 & 23 & 6 & 16 & 8 & 7 & 14 & 17 & 21 & 20 & 5 & 12 & 3 & 19 & 10\end{array}$
$\begin{array}{llllllllllllllllllllllll}5 . & 1 & 8 & 12 & 4 & 20 & p & 15 & 18 & 22 & 19 & 17 & 3 & 23 & 7 & 21 & 9 & 6 & 14 & 13 & 2 & 11 & 10 & 16\end{array}$
5. $1 \begin{array}{lllllllllllllllllllllll} & 17 & 10 & 20 & 16 & 4 & 23 & 13 & 21 & 2 & 19 & 15 & p & 12 & 11 & 7 & 22 & 8 & 3 & 18 & 14 & 5 & 9\end{array}$
$\begin{array}{lllllllllllllllllllllll}7 . & 1 & 2 & 12 & 13 & 15 & 19 & 14 & 4 & 8 & 17 & 20 & 22 & 6 & 11 & 3 & 21 & 5 & 23 & 16 & 9 & 10 & 18\end{array}$
6. $1 \begin{array}{llllllllllllllllllllllll} & 20 & 3 & 6 & 22 & 23 & 14 & 9 & 17 & 7 & 4 & 16 & 21 & 18 & 2 & 10 & 13 & 19 & 11 & 15 & 12 & 5\end{array}$
$\begin{array}{lllllllllllllllllllllll}9 . & 1 & 6 & 5 & 21 & 13 & 17 & 8 & 14 & 22 & 12 & 23 & 10 & 7 & 16 & 15 & 11 & 4 & 18 & 19 & 3 & 20 & 2\end{array}$
7. $1 . \begin{array}{llllllllllllllllllllll} & 4 & 19 & 16 & 5 & 11 & 21 & 12 & 22 & 3 & 15 & 20 & 6 & 17 & 13 & 8 & 2 & 14 & 18 & 7 & 9 & 23\end{array}$
8. $1 \begin{array}{llllllllllllllllllllll} & 3 & 7 & 6 & 12 & 14 & 21 & 10 & 5 & 2 & 4 & 9 & 15 & 8 & 19 & 18 & 17 & 16 & 13 & 20 & 23 & 22\end{array}$
9. $1 \begin{array}{lllllllllllllllllllllll} & 19 & 13 & 7 & 2 & 23 & 9 & 22 & 10 & 21 & 3 & 4 & 5 & 8 & 15 & 14 & 11 & 6 & p & 20 & 18 & 16 & 17\end{array}$
10. $1 \begin{array}{llllllllllllllllllllll} & 15 & 7 & 12 & 19 & 8 & 10 & 17 & 9 & 21 & 6 & 23 & 18 & 3 & 2 & 5 & 14 & 20 & 11 & 16 & 22 & 4\end{array}$
11. $1 \begin{array}{llllllllllllllllllllll} & 16 & 20 & 13 & 5 & 6 & 18 & 10 & 2 & 17 & 4 & 7 & 19 & 21 & 11 & 12 & 15 & 22 & 9 & 8 & 23 & 3\end{array}$
12. $1 \begin{array}{llllllllllllllllllllllll} & 21 & 23 & 4 & 2 & 20 & 10 & 3 & 17 & 22 & 14 & 12 & 8 & 11 & 9 & 16 & 18 & 5 & p & 6 & 19 & 7 & 13\end{array}$
13. $1 . \begin{array}{lllllllllllllllllllll} & 5 & 10 & 19 & 2 & 3 & 22 & 13 & 11 & 17 & 12 & 18 & 15 & 9 & 7 & 23 & 21 & 8 & 4 & 6 & 20 \\ 14\end{array}$
14. $1 \begin{array}{llllllllllllllllllllll} & 12 & 16 & 11 & 18 & 21 & 4 & 14 & 2 & 22 & 15 & 3 & 5 & 19 & 23 & 20 & 7 & 8 & 9 & 13 & 10 & 6\end{array}$
15. $1 \begin{array}{llllllllllllllllllllll} & 7 & 10 & 14 & 6 & 3 & 13 & 23 & 2 & 8 & 21 & 17 & 11 & 19 & 9 & 4 & 22 & 5 & 15 & 16 & 12 & 20\end{array}$
16. $1 \begin{array}{llllllllllllllllllllll} & 23 & 17 & 5 & 22 & 20 & 21 & 14 & 7 & 15 & 6 & 2 & 16 & 10 & 4 & 3 & 9 & 18 & 11 & 8 & 13 & 12\end{array}$
17. $1 \begin{array}{lllllllllllllllllllllllllllllll} & 18 & 12 & p & 5 & 4 & 21 & 19 & 22 & 7 & 17 & 23 & 11 & 13 & 14 & 16 & 6 & 10 & 15 & 2 & 9 & 3 & 8\end{array}$
18. $1 \begin{array}{llllllllllllllllllllll} & 22 & 2 & 6 & 13 & 9 & 5 & 7 & 3 & 12 & 10 & 11 & 14 & 19 & 20 & 4 & 17 & 18 & 8 & 16 & 23 & 15\end{array}$
19. $1 \begin{array}{lllllllllllllllllllllll} & 11 & 23 & 8 & 6 & 7 & 20 & 19 & 5 & 18 & 4 & 13 & 16 & 3 & 10 & 12 & 9 & 14 & 15 & 17 & 2 & 21\end{array}$
20. $1 \begin{array}{llllllllllllllllllllll} & 10 & 9 & 12 & 2 & 18 & 13 & 6 & 4 & 15 & 21 & 16 & 7 & 5 & 3 & 14 & 8 & 22 & 11 & 20 & 17 & 19\end{array}$
p. $\begin{array}{llllll}6 & 15 & 5 & 20 & 12\end{array}$

[^0]:    ${ }^{2}$ Observe that after projecting out the last coordinate, the assumed property of $H$ (that it has at least one sample point in each adjacent cube) will still hold in $n-1$ dimensions.

[^1]:    ${ }^{4}$ We end up choosing $W \approx N^{7 / 9}$, but in the analysis it is helpful to think of $W$ and $L$ as $N^{\alpha}$ and $N^{1-\alpha}$ for some $\alpha \in(1 / 2,1)$. That is, the set of red vertices is wider than it is long.

[^2]:    ${ }^{5}$ Using the Euler equation, one can give an upper bound on the so-called chromatic number of a surface. The bulk of the proof amounts to computing the minimum genus of the complete graphs to show that the inequality is tight.

[^3]:    ${ }^{6}$ The problem for nonorientable surfaces is also treated in [Sun18], which follows straight-

[^4]:    ${ }^{7}$ The union of the face and the edge $(a, b)$ is homeomorphic to a Mobius band.

