Idealisation and Mathematisation in Cassirer's Critical Idealism

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1 Idealist Philosophy of Science?

In Anglo-American philosophy there is a strong conviction that idealism on the one hand, and science and serious philosophy of science on the other hand, do not go well together. Often, idealism plays the role of a strawman to whom all the vices are attributed that one wants to criticise. As a particular savage example of this kind of anti-idealism let me mention Israel Scheffler's characterisation of Thomas Kuhn as an irresponsible and even immoral idealist:

> The current attacks (of Kuhn, T. M.) challenge . . . the very opposition between science and speculative idealism, from which scientifically minded philosophies have sprung. The attacks threaten further the underlying moral motivation of these philosophies, their upholding of the ideal of responsibility in the sphere of belief as against willfullness, authoritarianism, and inertia. The issues are fundamental, indeed more fundamental than is generally realized, precisely because a powerful moral vision has implicitly been called into question. [Scheffler, 1967, pp. 7–8]

Perhaps Scheffler's attack can be seen as a remote rehearsal of Moore's and Russell's anti-idealist revolt against British idealism around the turn of the last century. Be that as it may, in the Anglo-American philosophical scene the opinion is wide-spread that all variants of idealism subscribe to the doctrine that 'reality is fundamentally mental'. As typical one may take Haack's characterisation according to which '[A]n idealist holds that everything there is, is mental: that the world is a construction out of our ideas', [Haack, 2002, p. 70]. A bit more specifically Russell maintained that idealists believe that all propositions are of the subject-predicate form, and therefore idealists do not appreciate the merits of modern relational logic. As will be shown in the following, Cassirer may serve as a brilliant counter-example to the claims of Russell and Haack: it is difficult to find a philosopher who praised the achievements of relational logic more ardently than Cassirer.

Laws and Models in Science, 137–157.

Despite, or perhaps just because of its oversimplified character, conceptions of idealism like Haack's are rather common ones. Consequently, in contemporary analytic philosophy of science, idealist positions are rarely mentioned. For instance, in most books that intend to give a history of philosophy of science in the twentieth century neither Cassirer nor any other idealist is mentioned at all.

If idealism were what its caricature maintains it to be this would be alright, but actually things are more complicated. To show why, one may start with a bit of history of philosophy of science: nobody will deny the importance of Logical Empiricism for philosophy of science of the 20th century. Some authors even contend that contemporary philosophy of science is to be conceived of as a successor discipline of Logical Empiricism. As has been shown by detailed studies by Coffa, Friedman, Richardson and others, most logical empiricists started their philosophical careers as neo-Kantian idealists. A case in question is Carnap: not only that Carnap began as a neo-Kantian idealist, even worse, in *The Logical Structure of the World* [Carnap, 1928] he openly confessed to have sympathies for idealist doctrines. For instance, he characterized the constitutional theory which may safely be called the core of the *Aufbau* program, as deeply influenced by transcendental idealism:

> The merits of having discovered the necessary basis of the constitutional system thereby belongs to two entirely different . . . philosophical tendencies. *Positivism* has stressed that the sole material for cognition lies in the undigested experimental given . . . Transcendental idealism, however, especially of neo-Kantian tendency (Rickert, Cassirer, Bauch), has rightly emphasised that these elements do not suffice; order-posits must be added, our *basic relations*. [Carnap, 1928, Section 75]

Thus, in one of the key works of logical empiricist philosophy of science we find a large dose of idealism. This and other evidences show that there is something in idealism that does not go away as easily as many may wish. Summarising, then, I'd like to assent to Crispin Wright who put the problem with idealism in the following way:

> For all the vilification and caricature which its critics have meted out over the years, the idealist tradition in philosophy has proved sufficiently durable to encourage the belief that, at least locally, there are insights for which it is striving, but for which—its persistently controversial character suggests—we have yet to find definitive means of expressions. [Wright, 1992, p. 3]

After these introductory remarks let me come to my point, the role of idealisation and mathematisation in Cassirer's 'critical idealism'. More precisely I propose to reconsider a central thesis of Cassirer's that deals with the role of idealisation and mathematisation in the sciences, in particular, in physics. This is done not only for an interest in history of philosophy of science. Rather, I contend that Cassirer's thesis might be of some interest for the contemporary agenda of philosophy of science dealing with the roles of idealisation, mathematisation, laws, and models. Cassirer put forward his thesis in the early paper *Kant und die Moderne Mathematik* ([Cassirer, 1907], KUMM in the following). The title of this bulky article is a bit misleading, the issue is not so much Kant but the problem of how neo-Kantian philosophy of science should assess the recent developments of logic and mathematics, in particular the growing importance of the theory of relations for logic, mathematics and philosophy in general. KUMM may be considered as the programmatic precursor of Cassirer's first *opus magnum Substance and Function*, [Cassirer, 1910].

It goes without saying that the thesis of Cassirer's I want to discuss in the following is not an isolated assertion but embedded in a rather complex theoretical context, to wit, his theory of the formation of scientific concepts. Indeed, according to him, philosophy of science is to be conceived as the theory of the formation of scientific concepts. Since this paper is not the appropriate place to develop the essentials of Cassirer's account in an orderly manner, the following six theses may be distilled from *Substance and Function* and later works (cf. Cassirer [1929; 1937]).

1. Scientific knowledge does not cognize objects as ready-made entities. Rather, knowledge is organized objectually in the sense that in the continuous stream of experience invariant relations are fixated.

2. The unity of a concept is not to be found in a fixed group of properties, but in the rule, which lawfully represents the mere diversity as a sequence of elements. The meaning of a concept depends on the system of concepts in which it occurs. It is not completely determined by one single system, but rather by the continuous series of systems unfolding in the course of history. Scientific knowledge is a 'fact in becoming' ('Werdefaktum').

3. Scientific concepts and conceptual systems do not yield pictures of reality, rather, concepts provide guide lines for the conceptualisation of the world. The fundamental concepts of theoretical physics are blueprints for possible experiences.

4. Factual and theoretical components of scientific knowledge cannot be neatly separated. In a scientific theory 'real' and 'non-real' components are inextricably interwoven. Not a single concept is confronted with reality but a whole system of concepts.

5. Our experience is always conceptually structured. There is no non-conceptually structured 'given'. Rather, the 'given' is an artifact of a bad metaphysics.

6. The concepts of mathematics and the concepts of the empirical sciences are of the same kind.

In the following, I'd like to concentrate on (6). Implicitly, however, we have to deal with the other theses of Cassirer's as well. As a start, it may be expedient to quote (6) more fully as follows:

> What 'critical idealism' seeks and what it must demand is a *logic of objective knowledge* (gegenständliche Erkenntnis). Only when we have understood that the *same foundational syntheses* (Grundsynthesen) on which logic and mathematics rest also govern the scientific construction of experiential knowledge, that they first make it possible for us to speak of a strict, lawful ordering among appearances and therewith of their objective meaning: only then the true justification of the principles is attained. [Cassirer, 1907, p. 44]

This thesis will be referred to as the 'sameness thesis' (henceforth ST). I'd like to contend that ST lies at the heart of the 'critical idealist' philosophy of science Cassirer first presented in *Substance and Function* and later elaborated throughout his entire philosophical career (cf. Cassirer [1929; 1937]).

The outline of this paper is as follows: in Section 2 some preliminary comments on ST are put forward in order to forestall some unnecessary misunderstandings. In order to set the stage for a proper assessment of ST in Section 3 we reconsider some paradigmatic examples of the introduction of ideal elements that may be considered as point of departure for Cassirer's account. In Section 4 we deal with idealisation in mathematical physics in order to render plausible ST for the realms of mathematics and physics. In Section 5 we conclude with some general remarks on the place of Cassirer's 'critical idealism' in the landscape of 20th century's philosophy of science.

2 Some Preliminary Comments on ST

At first glance, one may be tempted to read the 'sameness thesis' ST as a sort of vulgar idealism which *identifies* mathematics and physics. This would be a misunderstanding. According to Cassirer, philosophy as philosophy of science has to concentrate *neither on mathematics*, as an ideal science, *nor on physics* as an empirical science, but rather:

> If one is allowed to express the relation between philosophy and science in a blunt and paradoxical way, one may say: The eye of philosophy must be directed neither on mathematics nor on physics; it is

> to be directed solely on the connection of the two realms. [Cassirer, 1907, p. 48]

For Cassirer's philosophy of science the central point of reference is neither mathematics—as a science of ideal objects, nor physics—as a purely empirical science. Cassirer did *not* aim at the futile reduction of physics to mathematics or an identification of both. Rather, he was looking for a common root from which both physics *and* mathematics spring. This common root is identified as the idealising method of the introduction of ideal elements.

Today, when dealing with idealisation in science it is usually taken for granted that there is a strict separation between the mathematical and the physical realm. Implicitly it is assumed that within mathematics there is no place for idealisation. Mathematics already is on the ideal side, so to speak. Under this assumption, the problem of the idealisational character of scientific knowledge is said to be solely concerned with the problem of the role of idealisation in the empirical realm. For instance, Leszek Nowak and his school have set up a detailed classification of the various methods of idealisation, but they are concerned only with the various forms of idealisations in the empirical sciences. They never consider mathematics as a domain for which idealisation could be relevant (cf. [Nowakowa and Nowak, 2000]).

According to Cassirer such a theory of idealisation starts too late: for him, idealisation has a role in mathematics *and* in the empirical sciences. Hence, a theory of idealisation in science has to take into account both mathematics *and* the empirical sciences. If one wants to understand the role of idealisation in empirical science one should study how it works in mathematics and in empirical science. Moreover, one should not tackle this problem armed with 'philosophical' presuppositions of what are the philosophically correct methods of idealisation. The methods of idealisation should be studied empirically, so to speak, no philosophical intuition will give us the answer what the common foundational syntheses are on which logic, mathematics and empirical science are based. Rather, this has to be revealed by studying the history of science. For Cassirer this meant to study the history of the formation of scientific concepts. Hence, philosophy of science has to pay attention to the ongoing evolution of science, it has to investigate and explicate the formation of scientific concepts in the real history of science.[1] In a nutshell, then, ST contends:

[1]This entails a specific difficulty for contemporary philosophy of science, in particular philosophy of mathematics. Today, the latter is not too much interested in 'real mathematics' and its historical development, but rather in its so-called logical foundations. The more advanced topics of mathematics are assumed to contribute nothing to the philosophical understanding. In contrast, Cassirer was interested in contentful mathematics, since a basic assumption of his philosophy of science was that science and mathematics can only be understood by studying its real history. Hence, his answer of what are the common foundational syntheses requires something more than elementary mathematics.

> *The common foundational syntheses on which mathematical and physical knowledge are based, are provided by the method of 'ideal elements'.*

The point Cassirer wants to make is that mathematicians and physicists are *both* using the methods of ideal elements. This claim is in stark contrast with modern views according to which the mathematicians are, so to speak, already *inside* the sphere of ideal objects, while the physicists, when they are offering their idealized laws and models that do not hold good in the real world, somehow attempt to enter the ideal domain but, strictly speaking, never succeed in getting in.

ST claims that this dichotomy is misleading. Mathematics should not be characterized as a realm of ideal objects while physics is said to be confined to the crude and non-ideal empirical sphere. Principally, the domain of mathematics is not too different from that of physics. Also within mathematics a lot of idealising is necessary to formulate and prove interesting theorems in a sea of ephemeral phenomena. In other words, one has to provide appropriate settings [2] in order to be able to do some interesting work in mathematics. This involves some sort of idealising. In order to render plausible this thesis one has to explain what is the role of ideal elements in the realm of geometry, or where idealisation takes place in the realm of numbers. Cassirer gave detailed answers to these questions, but today most philosophers seem to ignore them. For instance, in a recent discussion on the influence of neo-Kantian idealism on Carnap's *Aufbau* Alan Richardson comes to the conclusion that Cassirer failed to say clearly what is to be understood by these foundational syntheses, and the allegedly 'common foundational syntheses' remained mere verbiage [Richardson, 1997]. I think that Richardson's verdict is too hasty. Although a comprehensive answer to what are the common foundational syntheses may be said to be missing in KUMM, a full-fledged answer can be found in *Substance and Function*.

This does not mean that this answer is still fully satisfying today. The reason is not that Cassirer hadn't offered a good account of the methodology of ideal elements. Quite the contrary. Since his days the methodology of ideal elements has made great progress and therefore his account needs an update. In order to show that Cassirer's account of the method of ideal elements is not obsolete for understanding the role of idealisation one has to go beyond elementary mathematics. Standard elementary mathematics is sanitised in such a way that the the role of idealising in it is hidden from the eye. This should not be too surprising: for quite a long time the idealisational character of physical knowledge has been ignored by philosophy as well. Concentrating on some toy theories encapsulated in sentences like 'Copper expands when heated' or 'All swans are white' the indispensable role of idealisation hardly springs to the eye. Hence, with respect to matters of ideali-

[2] This expression has been coined by Wilson [1992].

sation, a parallelity between 'real' mathematics and 'real' physics arises: in both fields its role has been neglected by philosophy of science. Whilst philosophers of science have made some progress to get into contact with 'real physics', the analogous re-orientation towards 'real mathematics' in philosophy of mathematics is still in its beginnings.[3] In this respect, a reconsideration of Cassirer's philosophy of science that deliberately dealt with mathematics and physics might be helpful.[4]

3 Ideal Elements in Mathematics

Till the beginnings of the 19th century, a non-expert, for instance a philosopher, might have been justified to conceptualise the domain of geometry as an unalterable sphere of ideal objects such as ideal points, ideal lines etc. From that time onwards, however, it became more and more evident that Euclidean geometry was less than perfect and ideal. Seen from a mathematical perspective, it could be said to have certain conceptual defects which called for fixing. To formulate it in a somewhat paradoxical way: too many theorems one wanted to be true, turned out not to be true. Perhaps the simplest example was provided by projective geometry of the plane. From a mathematical point of view it had long been known that between points and lines there existed a certain useful *duality*: for a given theorem it was sometimes possible to obtain a new theorem by switching the terms 'point' and 'line': for instance, given the proposition that every two points determine a single line, the dual proposition was that every two lines determine a point by their intersection. Or, a triangle could be defined by its three vertices as well as by its three intersecting sides.[5]

Unfortunately, in Euclidean geometry a dual of a theorem was not always a theorem. For instance, although two points always determine a unique line, two lines not always determine a point since two parallels do not intersect. The method of ideal elements was to fix deficiencies of this kind. It introduced new 'ideal points' located on a new 'ideal line' that rendered the originally incomplete duality perfect.[6] This could be done in several ways. One method was to conceive an ideal

[3] To be sure, there are promising exceptions, for instance, Corfield's recent book 'Towards a Philosophy of Real Mathematics' [Corfield, 2003].

[4] Before we go on it should be noted that Cassirer's emphasis of the important role of ideal elements in mathematics is not a special feature of his 'critical idealism' or the neo-Kantianism of the Marburg school. In the 19th century, the issue of ideal elements was a common topic discussed by philosophers of mathematics and mathematicians alike (cf. [Wilson, 1992]). This continued till the first decades of the 20th century. After the rise of logicism and formalism the talk about ideal elements was no longer to be considered a serious theme in philosophy of mathematics. In mathematics itself, mathematicians continued to talk about them, but, as it seems, philosophers no longer were interested to listen.

[5] As a less trivial example one may mention the dual theorems of Pascal and Brianchon, see[Smith, 1959, pp. 331–336].

[6] For an insightful discussion of 'ideal elements' in the case of complex projective geometry, see [Wilson, 1992]. Here, I restrict myself to some brief remarks on the more elementary case of real projective geometry. An easily accessible, more detailed discussion can be found in[Torretti, 1978].

point as an equivalence class of parallel lines in such a way that any two parallel lines intersect at this ideal point. This sounded paradoxical, since the new ‘point’, being an equivalence class of lines, seemed to be larger than the ‘line’ it was to be a part of. Nevertheless, this method worked, and the new points and lines could be shown to fulfil the tasks they were designed for.[7]

When they were introduced, these ideal elements aroused some suspicion among the more conservative minded mathematicians. Ideal points and ordinary points had not the same ontological dignity, so to speak. Only gradually ideal points and ideal lines became citizens on a par with ‘ordinary’ points and lines. An important step on the road to full recognition was the construction of models of projective spaces. For instance, the points of the real projective plane could be conceptualised as the set of lines through the origin of the real 3-dimensional vector space, and correspondingly the lines of the projective plane could be conceived of as planes through the origin.[8] As a result of this and other developments, in the second half of the 19th century geometry was no longer considered as the investigation of an immutable domain of ideal objects but rather as an unfolding theory of generalised spatial structures defined by appropriate ‘idealising’ constructions. Geometry was no longer characterized as a theory of space in a narrow sense, but as a general theory of *Ordnungssetzungen* (*order posits*) (cf. [Carnap, 1928]).

The philosophical upshot of this evolution was that even in geometry the role of Kantian pure intuition was in decline. Instead of *Anschauung*, general principles of theory construction began to play a central role such as principles of duality and completion. More generally, considerations of practical and theoretical fruitfulness became dominant (cf. [Wilson, 1992; Tappenden, 1995]). This change was grist to the mill of the neo-Kantians. In contrast to orthodox Kantians, the neo-Kantians took the new developments as a confirmation of their Anti-Kantian claim that the Kantian idea of pure intuition had to be abandoned in the light of modern science and mathematics. According to Cassirer, the development of mathematics in the 19th century made a naive intuition-oriented view of geometry untenable. He whole-heartedly welcomed the new developments in geometry.[9]

The methodology of ideal elements was not confined to geometry. Also the domain of algebra underwent a growing variety of idealisational procedures which unfolded the original narrow domain of numbers in a multifaceted way. Maybe the best-known example is the idealisational procedure of *Dedekind cuts* which Cassirer chose as his paradigmatic case.

Let us consider the rational number **Q** as the objects ‘antecedently understood’, to use an apt terminology of Hempel. As is generally agreed, rational numbers are

[7]Cf. Wilson’s discussion of von Staudt’s method of ‘concept-objects’.

[8]Cf. [Torretti, 1978, Chapter 2.3].

[9]His discussion of the philosophical relevance of projective geometry in *Substance and Function* may still be considered as one of the best that can be found in the philosophical literature.

useful concepts, but for certain applications they are less than optimal, for instance for measuring or solving polynomial equations. One is in need of more numbers, the so called irrational numbers. In other words, with respect to the problems of solving polynomial equations, the domain of rational numbers does not qualify as an appropriate setting. In order to overcome this unpleasant situation one has to construct new 'ideal' or 'imaginary' numbers that provide solutions for equations for which 'real' solutions do not exist.

In order to construct the missing irrational numbers we may consider the set of 'cuts' of **Q**, i.e., the set of partitions of **Q** in two mutually disjoint and jointly exhaustive subsets such that all elements of the lower cut are strictly smaller than all elements of the upper cut. Obviously, a cut is determined by either its lower or upper cut alone. Hence, we are entitled to carry out the completion of **Q** by the lower cuts. Denoting the power set of all subsets of **Q** by $P\mathbf{Q}$, a lower cut $q \in P\mathbf{Q}$ may be precisely defined as follows: q is a lower cut iff it satisfies the following two conditions:

1. q is a downset, i.e., for all $a \leq b, b \in q \Rightarrow a \in q$.
2. q has no maximal element.

Denote the set of lower cuts of **Q** by $C\mathbf{Q}$. Then there is a canonical embedding of **Q** –e→ $C\mathbf{Q}$ by mapping a rational number $r \in \mathbf{Q}$ to the cut $q(r)$ defined by $e(r) := \{a; a < r\}$. This means, rational numbers can be identified with lower cuts whose upper cuts have a minimal element, to wit r itself. Hence the set of rational numbers **Q** can be identified with its image $e(\mathbf{Q}) \subseteq C(\mathbf{Q})$. The interesting point is that $C(\mathbf{Q})$ is larger than $e(\mathbf{Q})$—not all lower cuts have the form $e(r)$.[10] A typical example is the cut $\{a; a^2 < 2\}$ which is to be identified with the irrational number $\sqrt{2}$. The set $\{a; a^2 < 2\}$ is not a rational cut since its upper cut $\{a; a^2 < 2\}$ has no minimal element. The next step is to identify $C(\mathbf{Q})$ with the set **R** of real numbers by showing that the arithmetic of rational numbers **Q** can be extended to the arithmetic of $C(\mathbf{Q})$ in such a way that the elements of $C(\mathbf{Q})$ play indeed the arithmetic roles they are designed to play. Summing up we may say that by embedding **Q** into $C(\mathbf{Q})$ one has completed the domain **Q** of rational numbers by certain 'ideal elements' in such a way that the completed domain $C(\mathbf{Q})$ behaves arithmetically better than **Q**.

Although for the completed domain of real numbers **R** arithmetic works more smoothly than for rational numbers, even **R** leaves something to desire, since it is not algebraically complete. In order to get a fully satisfying theory of polynomial equations for which the fundamental theorem of algebra holds, the algebraically incomplete field **R** should be replaced by the algebraically complete field **C** of

[10] Actually, the cardinality of $C(\mathbf{Q})$ is much larger than that of **Q**: **Q** is countable but $C(\mathbf{Q})$ has the cardinality of the continuum.

complex numbers. This may be done by considering complex numbers as pairs (a, b) of real numbers $a, b \in R$, such that the original real numbers are identified with pairs of type $(a, 0)$. Again, this process may be conceived as an idealising completion process, although of a different kind than Dedekind's. For our purposes these differences are not important, the upshot for a theory of idealisation is that the introduction of new 'ideal' or 'imaginary' elements in geometry or numbers, not only enlarges the domain of objects to be considered, but, and this is the most important characteristics, it enhances the global conceptual features of the domain to be considered. Put in a nutshell, it leads to construction of new, more appropriate settings for doing geometry and algebra. Thus, these theories are not to be confined to their fixed domains of platonic entities, rather they are to be conceived as open fields of idealising constructions. These idealising constructions introduce a wealth of new objects that render untenable any 'intuitionist' conception of mathematics as the theory of an intuitively given unalterable domain of timeless platonic entities. Although it may well be the case that mathematics once started in some intuitive domain, it certainly did not remain confined to it.

One of the deepest philosophical insights of Cassirer's idealist philosophy of science was that constructions such as Dedekind's are not only mathematically interesting technical achievements. Rather, these idealisational constructions are to be considered as the prototypes of idealisational constructions essential for 20th century's mathematics in general.[11] Evidence for this sweeping claim is that 'idealisation' and 'completion' in the sense of finding 'appropriate settings' for the problems one is studying is now a routine part of the mathematicians's work. Typical are the following remarks of the mathematician Horst Herrlich;[12] After having listed nine assertions concerning set-theoretical topology, he remarks:

> Although we would like the above statements to be true, we know that none of them is [true]... in the category *Top* of topological spaces and continuous maps. However, there exist settings—more appropriate it would seem—in which the above statements are valid. The category *Top* can be decently embedded in larger, more convenient categories such that ... the above statements are not only true but, in fact, special cases of more general theorems. [Herrlich, 1976, p. 265]

[11] For instance, the proof of one of the most famous theorems of 20th century mathematics, Stone's representation theorem, may be considered as a generalisation of Dedekind's cut construction.

[12] I'd like to emphasise that Herrlich's remarks have no 'philosophical' intentions. His article may be characterized as a mixture of a survey and a research paper. Philosophers of mathematics certainly do not belong to its intended audience. The general tenor of Herrlich's paper is in no way original, analogous remarks for other research areas may easily be found in many places.

Then he goes on and proposes other categories [13] than *Top* as more appropriate settings for doing topology. These categories are more appropriate settings for doing topology since in them the above mentioned 'assertions concerning set-theoretical topology' become provable theorems. Thus, category theory as a general theory of local mathematical frameworks offers a framework in which mathematicians can discuss problems of appropriate setting in a way that goes beyond a subjectivist presentation of personal whims and preferences. Of course, there is still room for negotiation. It may turn out that there are good reasons for wishing one of two incompatible theorems to be true. An interesting case is the axiom of choice (AC) and the axiom of determination (AD): for many domains of mathematics the axiom of choice (AC) seems to be virtually indispensable (e.g. topology) while for others, the axiom of determination (AD), which may be considered as the opposite to (AC), appears more attractive. Examples like these can be easily multiplied. They show that mathematics of 20th century has been fully aware of the importance of the method of ideal elements, or, more generally, of idealisation. Unfortunately, this issue has yet to find the attention it deserves from the side of philosophy of mathematics:

> The official position, dominant since the start of this (the 20th) century, maintains that any self-consistent domain is equally worthy of mathematical investigation; preference for a given domain is justified only by aesthetic considerations, personal whim or its potential physical applications. [Wilson, 1992, p. 152]

Although the 'official position' meanwhile may have lost some of its strength, having given room for some less global, more local and specific considerations, it is remarkable how far ahead Cassirer's account was compared with that of most philosophers of mathematics who still stick to the 'official position'. [14] His account may be characterized as a kind of a general pragmatics of idealisation and completion. As a last mathematical evidence for the importance of idealising completions in mathematics I'd like to mention Stone's representation theorem that provided the first non-trivial relation between topology and logic.

Let B be any Boolean algebra. Every $b \in B$ defines an ideal, to wit, the set $r(b) := \{a; a \leq b, b \in B\}$. Denoting the set of ideals of B by IDEAL(B) one obtains a map $r : B \longrightarrow$ IDEAL(B). One can show that in general IDEAL(B)

[13] A category may be informally characterised as a local universe of mathematical discourse, for a detailed discussion of categories and their role in mathematics see[Adámek *et al.*,].

[14] It is a pity that in the recent work on the issue of 'appropriate settings' for mathematical theories Cassirer's contributions are completely ignored. Wilson [1992] asserts in a footnote that it is worth mentioning that various philosophers attempted to extend the 'principle of continuity' (i.e., the introduction of ideal elements) into general philosophy of language, e.g. Cassirer in [*Substance and Function*]. This is a somewhat strange remark: *Substance and Function* deals with many things, but certainly not with a 'general philosophy of language'.

is larger than B, since there are ideals that are not of the form $r(b)$. These 'non-principal' ideals correspond to the non-rational numbers in the analogous Dedekind completion of rational numbers. In other words, IDEAL(B) may be conceived as a completion of B. As was shown by Stone in the thirties, this completion defines a bridge from the theory of Boolean algebras to the theory of totally disconnected Boolean (or Stone) topological spaces in the sense that IDEAL(B) uniquely defines a topological space whose lattice of open and closed sets is isomorphic to B (cf. [Davey and Priestley, 1990]). Thereby one obtains a 'concrete' set-theoretical representation of the 'abstract' Boolean algebra B.

Stone's theorem has been considered as one of the most important theorems of 20th century's mathematics (cf. [**?**]). This is not the place to discuss this assessment in any detail but certainly strongly evidences that idealising completions are of utmost importance for contemporary mathematics. Guided by ST Cassirer did not confine the method of ideal elements to mathematics. According to him, the same kind of idealising completions can be found in the empirical sciences. This will be discussed in the next section.

4 Idealisation in Mathematical Physics

Critical idealism argued for the importance of idealisations in physics and other empirical sciences from an empirical perspective, so to speak. It is a fact that the advanced sciences make heavy use of mathematics, and it is the task of philosophy of science to make sense of this fact. For instrumentalist and empiricist currents of philosophy of science the employment of advanced mathematics in all areas of science presents a conceptual difficulty since according to them scientific concepts have only the task of reproducing the given facts of perception in abbreviated form [Cassirer, 1910, p. 148]. If this were really the case, the task of philosophy of physics would be achieved, if every concept of a physical theory had been dissolved into a sum of perceptions such that this sum could be used to recover the full realm of empirical facts falling under that concept (cf. [Cassirer, 1910, p. 151]). But such a replacement of mathematical concepts by perceptual or observational ones is virtually impossible:

> The theories of physics gain their definiteness from the mathematical form in which they are expressed. The function of numbering and measuring is indispensable even in order to produce the raw material of 'facts' that are to be reproduced and unified in theory.
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> ...
>
> [For] it is precisely the complex mathematical concepts, such as possess no possibility of direct sensuous realisation, that are continually used in the constructions of mechanics and physics. Conceptions,

> which are completely alien to intuition in their origin and logical properties, and transcend it in principle, lead to fruitful applications within intuition itself. This relation finds its most pregnant expression in the analysis of the infinite, yet is not limited to the latter. [Cassirer, 1910, p. 116]

Exactly this intertwinement of 'factual' and 'theoretical' elements is the base of which empirical theories [Cassirer, 1910, p. 130].[15] A typical example is the representation of moving bodies in space. As long as space is conceived just as a sum of visual or tactile impressions it does not allow 'motions' in the sense of physics:

> Motion, in the universal scientific sense, is nothing but a certain relation into which space and time enter. Space and time themselves, however, are assumed as members of this relation not in their immediate, psychological and 'phenomenal' properties, but in their strict *mathematical* meaning..... [Motion] demands the continuous and homogeneous space of pure geometry as a foundation; continuity and homogeneity, however, never belong to the coexistence of the sensuous impression itself, but only to those forms of manifold, into which we constructively transform it by certain intellectual postulates. In this way, from the very beginning motion is cast in a conceptual framework. [Cassirer, 1910, p. 118]

In brief, motion is a fact of conception, not of perception. However, it is important to note that the idealising method of empirical science should not simply be conceived as a replacement of the directly observable experiences by their ideal limit cases. This would suggest that the objects empirical science is dealing with are in line with the objects of perception. Thereby idealisation would boil down to not much more than approximation. Idealisation could be characterized as a continuation of empirical observation. Cassirer emphatically insists that this is not the case. The ideal elements to be introduced are not just some other things we add to the 'real' things. Rather, they express a certain way we deal with the 'real'

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[15] Margenau, once a colleague of Cassirer's in Yale, was one of the few working scientists of the 20th century who took seriously neo-Kantian philosophy of science (cf. [Margenau, 1950]). He described the entanglement of mathematical and empirical components of physical knowledge paradigmatically as follows: '... we observe a falling body, or many different falling bodies; we then take the typical body into mental custody and endow it with the abstract properties expressed in the law of gravitation. It is no longer the body we originally perceived, for we have added properties which are neither immediately evident nor empirically necessary. If it be doubted that these properties are in a sense arbitrary we need merely recall the fact that there is an alternate, equally or even more successful physical theory—that of general relativity—which ascribes to the typical bodies the power of influencing the metric of space, i.e. entirely different properties from those expressed in Newton's law of gravitation'[Margenau, 1935, p. 57].

things. This may become plausible, if we consider in some detail the construction of points as limiting elements of other more basic 'empirical' objects called regions.

The underlying problem may be described as follows: In physical geometry and other sciences the talk of points as basic elements is ubiquitous. Nevertheless, in 'reality' one never meets points. They are idealisational constructs. The measured values of a physical magnitude never assume points as their values simply because this would amount to absolute precision which real science never reaches. Instead, in real science, the measured values are assumed to be located in some more or less extended intervals. Usually, these intervals are considered as sets of points. Thus, even if one admits that our measuring methods never reach the points, we are accustomed to consider them as the basic building blocks of space, time, space-time, and other generalised spatial structures used in science. From a strictly empirical point of view, however, points appear to be rather contrived entities. It would be too simple, however, to consider them just as convenient mathematical fictions that 'somehow' play the role they are assumed to play. This would amount to a strict separation between the domain of empirical reality on the one hand and the domain of mathematics on the other hand whereby it becomes impossible to bring them together again by somehow establishing a link between them by stipulation. The question is how to avoid the standard dichomotic account.

An answer, or at least some hints, where one may look for an answer, can be found in Whitehead's method of 'extensive abstraction' [Whitehead, 1929]. Whitehead was probably the first who attempted to replace points as fundamental entities of spatial and temporal structures by objects that were empirically more accessible. In the case of space and spacetime these entities may be characterized as *regions*. Intuitively, a spatial region may be described as a more or less well-shaped part of space. Whitehead's programme was to take regions instead of points as the basic fundamental building blocks and to construct points and their geometric relations from regions and their relations. Whitehead only gave an informal sketch of how this might work but his account can be reconstructed in a formally rigorous way (cf. [Mormann, 1998]).

The interesting point is that this construction of points from regions can be conceived as a generalisation of Dedekind's method of cuts. That is to say, the insertion of points as ideal or limit elements of the realm of regions is an idealising completion process analogous to the construction of real numbers from rational ones. Ignoring the technical details it goes like this: Assume W to be a complete Boolean algebra of regions: This means that for two regions a and b a relation $\leq$ is defined such that $(W, \leq)$ satisfies the axioms of a complete Boolean algebra. The relation '$a \leq b$' is to be read as 'The region a is part of the region b'. One observes that the existence of points is not presupposed. In order to introduce points, one needs a further relation $\ll$ such that $a \ll b$ is to be intuitively read as 'a is an *interior* part of b' in the sense that a does not touch the boundary of b.[16]

A set q of regions is a 'round ideal' iff it is a downset of regions, and for $a \in q$ there is always a region b such that $a \ll b \in q$. The set of round ideals $\text{IDEAL}_r(W)$ of W corresponds to the Dedekind C(**Q**) of **Q**, while W corresponds to **Q**. An embedding W–e→ $\text{IDEAL}_r(W)$ is given by $b \longrightarrow \{a; a \ll b\}$. $\text{IDEAL}_r(W)$ can be topologically represented in the sense that there is a topological space[17] $(pt(\text{IDEAL}_r(B)), O(pt(\text{IDEAL}_r(B)))$ such that a region $a \in W$ is represented by a regular open subset $e(a) \in O(pt(\text{IDEAL}_r(W))) = \text{IDEAL}_r(W)$. Although this construction is technically more complicated than that of Dedekind cuts, it follows the same pattern. The upshot is that by this idealising completion regions can be represented by point sets. This is a necessary condition if one wants to speak of 'motion' in a scientific sense. More precisely, the completion of W by $\text{IDEAL}_r(W)$ allows to conceive processes as continuous mappings $I \longrightarrow X$ of a time interval I into a topological space X defined by $\text{IDEAL}_r(W)$.

Before leaving the general discussion of idealising completions it may be expedient to emphasise once more the formal similarity of the three examples considered so far. All can be described as embeddings X–e→$I(X)$ of the domain X into an ideal completion $I(X)$:

1. The completion of **Q** to **R** by a map **Q** $\longrightarrow$ **R**.
2. The completion of **B** to IDEAL**(B)** by a map **B** $\longrightarrow$ IDEAL(**B**).
3. The completion of **W** to IDEAL_r(**W**) by a map **W** $\longrightarrow$ IDEAL_r(**W**)**.**

Of course, not just every map will $X \longrightarrow I(X)$ will do as an honest completion, certain structural requirements have to be satisfied.[18] Thus one may speak of a general theory of completions in which the examples (1)–(3) can be considered as paradigmatic cases,[19] This corroborates Cassirer's claim that Dedekind-like completion methods are among the important foundational syntheses on which mathematics *and* empirical sciences are grounded. This corroboration is the more compelling the more one delves into the intricacies of modern mathematics and science. Without going into the details of this general theory of completion methods,

[16] Actually it is sufficient to take the relation $\ll$ of interior parthood as the only primitive relation, since the standard parthood relation $\leq$ can be defined in terms of $\ll$ (cf.[Mormann, 1998]).

[17] A topological space $(X, O(X))$ is a set X endowed with a set $O(X)$ of open subsets of X satisfying certain properties. Details can be found in any textbook on topology. For a succinct presentation, see [Davey and Priestley, 1990]. If the topology is understood, a topological space $(X, O(X))$ is denoted by X.

[18] A comprehensive general discussion of the many kinds of completions occurring in various areas of mathematics may be found in (cf. [Adámek *et al.*, , III.12]).

[19] It is a nice terminological coincidence that in these examples idealisations go together with ideals. This is not always the case. For instance, in topological contexts, idealisations are often described as compactifications, which have not direct idealising connotations.

one may note that idealising completions can be conceived of as *representations*: the incomplete and 'non-ideal' manifold X is *represented* by the completed 'ideal' manifold $I(X)$. For instance, an incomplete perceptual manifold of experiences is represented by a completed conceptual manifold. This kind of construction is ubiquitous in physics and other empirical sciences.

The Whiteheadean construction of physical space is a typical example but in no way the only one. Quite the contrary, physical space is only one among a wealth of spatial constructions employed in the empirical sciences. One may even contend that it does not provide the best example for this kind of constructions, since its overly simple appearance may hide its conceptually constructed character. This misapprehension is avoided when one considers the construction of state spaces of empirical theories in general. State spaces are defined with respect to certain systems, i.e., a state space is always a state space of a system or a kind of systems. The concept of a *system* is taken as primitive. Examples of systems are provided by mechanical or thermodynamic systems such as a particles, projectiles, pendulums, planets, gases, liquids, lasers. Even entities as large as galaxies may be considered as systems, or the universe itself taken as the largest possible system. Generally, a system S is an appropriately chosen chunk of the world taken to be the object of theoretical investigation. Systems are assumed to be possibly in different states. For instance, an atom considered as a system in the sense of quantum theory may be in an excited state or not. In order to be accessible to theoretical considerations at all, a class of *possible states* of S must be selected. This class of possible states is denoted by $\Sigma(S)$ or simply Σ if S is understood. $\Sigma(S)$ is called the state space of S.

Here, 'possible" is to be understood in the weak sense of logical possibility. That is to say, some of the elements of Σ may well turn out to be really impossible, i.e., physically impossible states for S. For instance, in first approximation, the state space $\Sigma(o)$ of a material object o may be taken to be the whole universe, even if most places in the universe are physically inaccessible for o. The state space of a system S serves only as general stage on which its story is rehearsed. It is not assumed that S has to occupy all possible locations during the play. Quite the contrary. It is a crucial task of the theory to select certain areas as the 'really possible" states and to classify their complement as a sort of no-go area for S. Thereby, a modal component is introduced in the theory's framework. For a rough and preliminary distinction between possible and impossible states purely set theoretical methods will suffice, but for a more refined determination, in particular for the distinction between possible and impossible *processes*, more refined (geometrical) structures of the state space come into play. Let us consider some examples. The first is Cassirer's and once again rehearses the case of 'physical space':

> The individual positions of Mars, which Kepler took as a basis ... do not in themselves alone contain the thought of the orbit of Mars;

> and all the heaping up of particular positions could not lead to this thought, if there were not active from the beginning ideal presuppositions though which the gaps of actual perception are supplemented. What sensation offers is and remains a plurality of luminous points in the heavens; it is only the pure mathematical concept of the ellipses, which has to have been previously conceived, which transforms this discrete aggregate into a continuous system. Every assertion concerning the unitary path of a moving body involves the assumption of an infinity of possible places; however, the infinite obviously cannot be perceived as such, but first arises in intellectual synthesis and in the anticipation of a universal law. Motion is gained as a scientific fact only after we produce by this law a determination that includes the totality of the space and time points, which can be constructively generated, in so far as this determination coordinates to every moment of continuous time one and only one position of the body in space. [Cassirer, 1910, p. 118–119]

If even for physical space—conceived as a state space—'real' and 'not-real' elements are inseparately interwoven, this holds *a fortiori* for general state spaces [Cassirer, 1910, p. 117]. Physical space is only a first evidence that idealising elements always play an essential role. For instance, consider an elementary thermodynamical system S characterized by the two quantities of volume and pressure only. As a first approximation of the state space $\Sigma(S)$ one may take a 2-dimensional Euclidean plan **E** having an orthogonal base consisting of the two vectors V (volume) and P (pressure). Since negative volume and pressure do not make sense the 'really possible' states of S are to be found only the first quadrant of **E**. Actually, further constraints will play a role. If we assume the ideal gas law to hold, the product $S(V){\cdot}S(P)$ must be constant for all 'really" possible states of S. Hence, the state space $\Sigma(S)$ of possible states of S is the hyperbola defined by the equation $S(V){\cdot}S(P)$ = constants.

State spaces are not empirical objects given by nature. Rather they depend on the theories used. Depending on the theory different state spaces for the 'same' S may be obtained.[20] In any case, the first step for theoretically understanding the behaviour of any empirical system S consists in providing an appropriate state space. In other words, a system S enters the theoretical realm only if it S represented by an appropriate state space. Now, as is already suggested by the term *space*, $\Sigma(S,T)$ is usually not simply a set but a space, i.e., a set endowed with some geometric structure. This structure is employed to differentiate between really possible and really impossible states of the system.

[20]Hence, one might have written $\Sigma(S,T)$ instead of $\Sigma(S)$.

What is really interesting, however, is not the state a system occupies or does not occupy, but rather the processes a system may run through. State spaces provide useful representations of empirical processes by representing them as paths in the state space. Mathematically a path is defined as a map $f : I \longrightarrow \Sigma(S,T)$ of the unit interval I into the state space $\Sigma(S,T)$. Only a few paths represent processes allowed according to the laws of the theory, most paths represent forbidden ones, i.e., those that are impossible according the theory. It may be considered as the essential task of the theory to distinguish the possible and the impossible ones. In order to do this the geometric structure of the representing state space has to be brought into play. Typically, constraints on admissible paths are defined by geometric concepts such as vector fields, differential and tensor forms. All these devices are constructions based on the introduction of ideal elements. The continuous motion of a body in physical space is just the most elementary case, but I hope to have shown that it already suffices to grasp the philosophical essence of the huge variety of spatial constructions used in science.

Up to now we have discussed only conceptual idealisations, i.e., idealisations that amount to an embedding or completion of an empirical manifold of perceptions or experiences into a conceptual framework that is richer and better structured than the empirical manifold we started with. This might suggest that idealisation in the sense of Cassirer is merely a conceptual activity 'in the head'. I think, this would amount to a misunderstanding or at least a too narrow interpretation succumbing to the old anti-idealist prejudice according to which idealism is always concerned with the 'mental' (cf. [Haack, 2002]). Instead, I'd like to propose to understand idealising as an activity to construct 'appropriate settings' (cf. [Wilson, 1992]) for certain scientific purposes. According to Cassirer, it is a matter of science what counts as appropriate settings and how they are constructed, and what are the means for their constructions. It may well be the case that science will invent hitherto unknown methods of idealising, i.e., constructing 'appropriate settings' for its purposes. Examples are the various methods of simulation that recently have been developed and play an ever more important in many areas of science.[21] There is no reason to assume that for Cassirer's philosophy of science the arsenal of idealising methods is a priori restricted to purely conceptual idealisations. Rather, idealising may involve machines, simulations and possible other devices that bring about appropriate settings for interesting stable phenomena. Thus we may speak, somewhat paradoxically, of material idealisations. I

[21]Another option is to conceive the so called 'nomological machines' of Cartwright as idealising devices [Cartwright, 1995]. Indeed, as Pickering showed sometime ago, there are far-reaching similarities between the building of highly complicated nomological machines like bubble chambers of particle physics and the construction of conceptual systems such as Hamilton's quaternions (cf.[Pickering, 1996]). On a more elementary level, one could read Lakatos's discussion of establishing a context for a valid proof for Euler's theorem as a series of attempts to build a smoothly running conceptual machine (cf. [**?**]).

think this is not a too far-fetched generalisation of Cassirer's account. For him, the most important feature of scientific conceptualisation has always been the serial character of concepts through which they lawfully connect a manifold of isolated experiences [Cassirer, 1910, p. 148]. From an algorithmic perspective, a serial concept is nothing but a conceptual machine that produces for a given input an output following its internal rules. According to ST this holds for scientific concepts in general, they may be empirical or mathematical ones. More precisely, physical concepts only continue what is evident for mathematical concepts:

> ... the physical concepts only carry forward the process that is begun in the mathematical concepts, and which here gains full clarity. The meaning of the mathematical concept cannot be comprehended, as long as we seek any sort of presentational correlate for it in the given; the meaning only appears when we recognize the concept as the expression of a *pure relation*, upon which rests the unity and continuous connection of the members of a manifold. The function of the physical concept also is first evident in this interpretation. The more it disclaims every independent perceptible content and everything pictorial, the more clearly its logical and systematic function is shown. [Cassirer, 1910, p. 166]

Thus, the sameness thesis ST may be conceived as providing an appropriate epistemological perspective for dissolving the problem of the applicability of mathematics in empirical science. After all, the applicability of mathematics appears as a miracle only if the spheres of mathematics and the real world are totally separated. This separation, however, is nothing but the result of a bad metaphysics that erroneously reifies methodological differences. From the perspective of ST, the concepts used in both area are based on the same foundational syntheses. More specifically, conceptualisation in mathematics as well as in empirical science amounts to idealising constructions of appropriate settings (completed idealized manifolds) for which lawful regularities and stable phenomena obtain.

5 Concluding Remarks

For a general assessment of Cassirer's 'critical idealist' philosophy of science it is expedient not only to discuss some of its more specific technical theses such as ST but also to offer an attempt to locate critical idealism on the general map of 20th century's philosophy of science. In this respect I'd propose to conceive critical idealism as a kind of moderate conventionalism. This is suggested by Cassirer's contention that ideal constructions may be characterized as conventions. From the critical idealist's perspective, conventionalism is the thesis

> that thought does not proceed merely receptively and imitatively, but develops a characteristic and original spontaneity. This spontaneity is

> not unlimited and unrestrained; it is connected, although not with the individual perception, with the system of perceptions in their order and connection. [Cassirer, 1910, p. 187]

Occasionally, Cassirer even adopts an explicitly instrumentalist stance:

> The objects of physics: matter and force, atom and ether can no longer be misunderstood as so many new realities for investigation, and realities whose inner essence is to be penetrated, when once they are recognized as instruments produced by thought for the purpose of comprehending the confusion of phenomena as an ordered and measurable whole. [Cassirer, 1910, p. 166]

An important feature of the critical idealist's modest conventionalism is its relational holistic component: although the manifolds of our experiences are atomic in the sense that sensuous experiences can be isolated from each other, the idealised completed manifolds of our conceptualisations are connected in the sense that a theory arising from such an idealised manifold is not confronted with isolated experiences but with a relationally conceived reality as a whole.

Locating neo-Kantian philosophy of science in the neighbourhood of conventionalism is not intended to offer a full description of this philosophical current but at least it is a starting point for overcoming the allegedly unbridgeable abyss between idealist and analytic philosophy of science, which has hindered a fruitful discussion between both currents for such a long time.

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