Originally in *Journal of Idealistic Studies* 38(3): 209-220 (2008). Please quote from published version.

THE EUCLIDEAN MOUSETRAP:

SCHOPENHAUER'S CRITICISM OF THE SYNTHETIC METHOD IN GEOMETRY

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Abstract

In his doctoral dissertation *On the Principle of Sufficient Reason*, Arthur Schopenhauer there outlines a critique of Euclidean geometry on the basis of the changing nature of mathematics, and hence of demonstration, as a result of Kantian idealism. According to Schopenhauer, Euclid treats geometry *synthetically*, proceeding from the simple to the complex, from the known to the unknown, "synthesizing" later proofs on the basis of earlier ones. Such a method, although proving the case logically, nevertheless fails to attain the *raison d'être* of the entity. In order to obtain this, a separate method is required, which Schopenhauer refers to as "analysis", thus echoing a method already in practice among the early Greek geometers, with however some significant differences. In this essay, I here discuss Schopenhauer's criticism of synthesis in Euclid's *Elements*, and the nature and relevance of his own method of analysis.

The influence of philosophy upon the development of mathematics is readily seen in the practice among mathematicians of offering a demonstration or "proof" of the many theorems and problems which they encounter. This practice finds its origin among the early Greek geometricians and arithmeticians, during a time in which philosophy and mathematics intermingled at an unprecedented level, and a period in which rationalism enjoyed preeminence. Indeed, the ancient emphasis upon rationalism, upon the necessity of offering a logical account for all knowledge, resulted in the development of two distinct although integrally related methods for demonstration in mathematics, finding particular application in geometry, and referred to separately as analysis and synthesis.

Analysis then is akin to the method of foundations. Through it, the geometrician attempts to resolve each separate problem, much like in philosophy, into its basic elements. Although this method seems to have followed synthesis in its development among the earliest of Greek geometers, it nonetheless represents something of the method of beginnings, for it proceeds from the hypothesis of the solution to an already established and known foundation, implying an element of intuition and discovery.[1] There was then an alternative method, a more rationally oriented as opposed to intuitive one, referred to as synthesis. Synthesis offered itself as a practical tool both for confirmation of the initial results of an analysis as well as for aiding in the expansion of the science into ever more complex domains, for the geometer, having now laid down the foundations, could then construct his edifice upon it. Synthesis, as opposed to analysis, helped to satisfy the need for rational verification and logical precision among the ancient Greek mathematicians, mirroring later developments in philosophy at the time, as the syllogisms of Aristotle, and Plato's recommendation in the *Theaetetus* (201d) that knowledge be true belief *with an account*.[2] Considering the emphasis upon rationalism among the early Greek mathematicians and philosophers, it is not surprising that synthesis should become the standard method for demonstration, recommended by Plato and the Academy, and put into practical application by Euclid in his *Elements*.[3]

As Arthur Schopenhauer would later point out, Kant's Copernican revolution necessarily changes all of this. Schopenhauer indicates this in his works through criticism of the method of synthesis employed by Euclid, referring to the latter's proof of the Pythagorean theorem in the *Elements* as a "mousetrap demonstration". For Schopenhauer, mathematical entities as ideal objects for the subject,[4] implies now that any discovery of such entities requires an examination or "analysis" of the intuitive ground from which these entities first arise and are encountered within cognition. Schopenhauer considers this method of analysis superior to the former synthetic method in use among the ancient geometers, inasmuch as analysis not only leads to verification of the existence or non-existence of any entity, but also, in revealing the intuitive ground from which it arises, it further reveals *why* the entity is as it is. This stands in striking contrast to the synthetic method. For in departing from the intuitive ground of mathematics into rational abstraction, this method, although offering logical certainty, nevertheless leads to the loss of the *raison d'être* of the entity. The larger consequences of this are worked out in the sections which follow, the main thesis centering around a discussion of the significance of Schopenhauer's criticism of the method of synthesis in Euclid, and of Schopenhauer's understanding and interpretation of the method of analysis.[5]

In the section which follows entitled, "Analysis and synthesis", I there describe the meaning of these two methods as they were understood and applied by the ancient Greek geometers. In the next section, "Euclid's Mousetrap Demonstration", I proceed to an examination of Schopenhauer's criticism of Euclid's proof of the Pythagorean theorem in the *Elements* through the synthetic method. The reasons for Schopenhauer's rejection of this method are there detailed. In the following section, "Schopenhauer's analytic method", I consider Schopenhauer's understanding of analysis, and how it relates to and yet differs from the same method as applied by the ancient Geometers. In the final section, "Concluding observations", I discuss the significance of Schopenhauer's critical remarks in relation to idealism and later rejections, particularly among such philosophers as Friedrich Nietzsche, of rationalism.

Analysis and synthesis

What are the methods of synthesis and analysis, and how were they applied by the early Greek geometers? Perhaps the most renowned as well as lucid description of these methods can be found in Pappus of Alexandria's *Collections*.[6] There, the two methods are discussed side by side, each as developing upon the results of the other. Analysis is first stated as the method by which the geometer proceeds directly from the hypothesis of the problem itself, that is, from the solution as unknown, and from there works his way regressively to a known ground. In other words, through analysis the geometer reduces the problem to its elements (*stoicheia*), which included axioms (first principles), definitions, postulates, as well as theorems and problems already established within the treatise.[7] In synthesis on the other hand, the geometer proceeds in precisely the opposite direction, this time starting on the basis of what is already known or established, and from there proceeding to demonstrate the logical coherence of the unknown with the known. If it is shown to be coherent, then the unknown is now incorporated within the science as an established problem or theorem, that is, a known, consistent with the science, as a piece which completes a puzzle, or a part cohering with the whole. Analysis is therefore the method of regression, synthesis of progression. With the former, problems are 'deconstructed' or resolved into their constituent elements, with the latter, problems are constructed and synthesized into more complex structures.

To offer a very basic (albeit hypothetical) example of these two methods, consider a science having only the "point" as its singular element, and a single hypothesis in the form of a proposition (*protasis*): "*Given point A, to draw a straight line AB*".[8] The geometer could now proceed to demonstrate the existence or non-existence of this hypothesis in two distinct ways. He could attempt this first through analysis, starting *from* the solution (the unknown), viz., "Suppose the problem solved and the line AB drawn." The geometer would then proceed regressively, resolving the problem into a known, and (since this science has but one known) eventually, to the "point" itself. Alternatively, the geometer could attempt to proceed synthetically, this time starting from a known and proceeding to the unknown (the solution), and hence he would begin from the "point" and attempt to demonstrate that a line can be drawn, viz., "Supposing point A and point B, etc." The differences between these two methods should be evident enough from the above example. With analysis, one proceeds from the surface to the bottom, and with synthesis, from the bottom to the surface. With the former, the problem is resolved into the known, whereas with the later, the problem is composed, albeit somewhat artificially, from the known. The importance of this distinction is essential, for as Schopenhauer will later point out in his criticism of the method of synthesis, establishing proof of the unknown on the basis of construction from the known offers merely logical certainty for that entity now proved. The reason why this is so should be initially evident: with analysis the ground is revealed and uncovered, whereas with synthesis, the resulting construction, although revealing *consistency* with the ground, nevertheless covers and conceals it.

Regardless of the above distinction, it should be noted that these two methods can and were often used side by side for the solution of the same problem. This last point is easily made evident from the point and line example above. A geometer could start with an analysis, resolve the problem into a point, and then proceed in the reverse from the point to a synthetic demonstration of the line. Following Plato's time, many geometrical texts made use of *both* methods. As a result of this, there are two senses in which one may refer to synthesis. In the first sense, synthesis is precisely what was explained in the above example of proceeding from a point to a synthetic demonstration of a line. In the second sense, synthesis may also refer to the *general* method by which a geometer proceeds from one problem or theorem to the next, using the demonstrations of the previous steps, as the foundation for the next. For example, after proceeding from a point to the demonstration (synthesis) of a line, the geometer would then take the line as his starting point, and from there proceed to a demonstration (synthesis) of a square. The geometer thus constructs a great edifice upon the most simple foundations, and in such cases, the text is itself considered a "synthetic" treatise. This follows despite the fact that the geometer can apply an analysis on a new theorem or problem *at any stage* within the treatise. The difference of course would be that such an analysis would likely be based upon previous stages of synthesis, for now the geometer would not regress to actual first principles, but rather to preceding theorems and problems already established within the treatise. Accordingly, Euclid's *Elements* is traditionally considered a synthetic treatment of geometry in both of the above senses, as Thomas Heath notes in his *History of Greek Mathematics*:

> The elements is a synthetic treatise in that it goes directly forward the whole way, always proceeding from the known to the unknown, from simple and particular to the more complex and general; hence *analysis*, which reduces the unknown or the more complex to the known, has no place in the exposition, though it would play an important part in the discovery of the proofs.[9]

There are at least two reasons why the early Greek geometers made use of both methods of analysis and synthesis for a single problem or theorem. The first reason was due to the fact that, following initial developments in geometry, later encounters with extremely abstruse problems, led to the difficulty of discerning precisely *how* and *where* to initiate the next synthetic demonstration.[10] Starting then from the solution itself (analysis), the geometers would work their way backward until finding a known which they might use as the basis for a subsequent synthesis—which would require nothing more than now reversing the process. This is what Heath means when he states above that analysis would play a part in the *discovery* of many of the proofs found in Euclid. A geometer could easily apply an analysis to a problem in order to discover the way to the proof, *independently*, and then supply the reverse process, the synthesis, within the treatise itself.

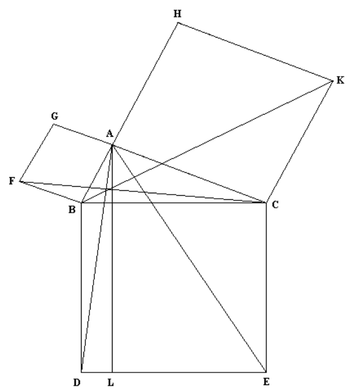
The second reason for using both methods was due to the need to avoid the possibility of false hypotheses, as Aristotle discusses in his *Prior Analytics*.[11] Within geometry, false hypotheses present the problem of inconvertibility, that is to say, a solution one way, through analysis, may not be convertible the other way, through synthesis. That would be akin to the paradox that a line could be reduced to a point, but a point could not lead to the synthesis of a line. In encountering such false hypotheses in geometry, early Greek mathematicians therefore began the practice of applying a synthesis in order to "confirm" the initial analysis, thereby avoiding the error of accepting such hypotheses as true.

Euclid's Mousetrap Demonstration

Turning now to the larger discussion of Schopenhauer's criticism of the method of synthesis, it is found that one object of this critique is Euclid's proof of the Pythagorean theorem in proposition 47 of the first book of the *Elements*. There, Euclid begins with the statement: "*In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*" Euclid then begins with his proof:[12]

> Let ABC be a right-angled triangle having the angle BAC right; I say that the square on BC is equal to the squares on BA, AC. For let there be described on BC the square BDEC, and on BA, AC the squares GB, HC; through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

In the first place, it should be noted that for his initial construction (*sumperasma*) of the proof, Euclid makes use of proposition 46—previously demonstrated in the *Elements*—and of itself based upon previously demonstrated propositions. Accordingly, Euclid proceeds upon the basis of the results of already established principles and proofs within the text itself. The treatise is therefore evidently *synthetic*. Furthermore, the proof is itself also synthetic as opposed to analytic. Euclid begins through construction of various parallel lines and squares, and from there proceeds to demonstrate the theorem through the inner coherence of the construction with the right triangle itself. Accordingly, on the basis of what is known (parallel lines, squares, parallelograms), Euclid synthetically constructs his proof. The final demonstration itself arises brilliantly through parallelograms BL and CL within the square BDEC, brought into relation to the squares GB and HC. The following diagram illustrates the process:[13]

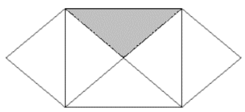


Euclid's proof of the Pythagorean theorem is thus exemplary of the application of the synthetic method in both senses, i.e. as illustrating the method of the *Elements* itself, and also the method of proof within it. He proceeds through construction of problems and theorems on the basis of those previously solved, as well as the basic elements of geometry established at the beginning, and moving from the known to the unknown, he synthesizes ever more complex conclusions.

For Arthur Schopenhauer, although Euclid has successfully proven *that* the theorem is true, there is nonetheless missing a sense of *why* a right triangle should necessarily result in Pythagoras' theorem. This follows from Schopenhauer's belief that Euclid, although offering a brilliant proof, nevertheless loses something quite essential in the process, referring to that proof comically as, "Euclid's mousetrap demonstration".[14] The basic idea behind his criticism is that indeed, Euclid proves the theorem unquestionably, but the problem is that by nature of the proof, the *content* of the thing in question, the *why* of the theorem, is irretrievably lost. For Schopenhauer, an alternative method is thus required, one which can reveal the *why* of the entity for every case, and this he identifies with analysis, concluding: "It is generally the analytic method that I desire for the expounding of mathematics, instead of the synthetic method Euclid made use of."[15]

Schopenhauer's analytic method

In antithesis to the Euclidean synthetic method for the demonstration of the Pythagorean theorem, Schopenhauer proposes what he considers a more lucid and direct proof, in the form of a singular image, one which he thinks reveals intuitively the sufficient reason *why* the Pythagorean theorem *is as it is*, without appeal to the sleight of hand tricks Euclid is forced to resort to as a result of his method:[16]



What is revealed in this image? For Schopenhauer we immediately recognize the nature of a "square" and through it "four equal sides". The two diagonal lines which intersect the square are just as evidently equal, and the resulting inner triangles, equal and right. Similarly, the two smaller squares at the diagonals, as complementing the symmetry of the image, further indicate that they are twice the area of each of the inner triangles inasmuch as two such triangles occupy them. Finally, the larger square, having a length equal to the hypotenuse of the inner triangles, thus encompasses an area equal to the sum of the two smaller squares. The Pythagorean theorem is thereby proved immediately and directly, as if by insight into the *raison d'être* of the entity itself. Indeed, from the above image one can "see" precisely *why* the theorem is so.[17]

For Schopenhauer, the above image offers direct and intuitive insight into the nature of the properties of the Pythagorean theorem. This is due to the fact that for him, mathematics is based upon an originally cognitive and *intuitive* formal representation (*Vorstellung*) which arises through the perceptual faculty of the understanding (*Verstand*), and is then brought into conceptual *abstraction* within the faculty of reason (Vernunft)—a quite Kantian notion.[18] The essential difference between the intuitive and the abstract is that whereas the datum of the former is much more primordial, direct, and particular, the datum of the latter is essentially derived, indirect, and universal.[19] All our knowledge of the world arises initially through intuitive experience, and second, through our rational abstraction of this experience, and Schopenhauer refers to these separate sources as *roots*. Although he names four roots, one abstract and three further roots as a subdivision of the intuitive, for the purposes of this essay, it is necessary to consider only two: the abstract root of knowing through reason, and the intuitive root of being through the *formal* representation of time and space.[20]

We may furthermore give an account for each entity as it arises in and through experience, i.e. a *why* or sufficient reason for its existence, but in order to do so properly and for knowledge, it is first of all necessary to identify the correct root of each thing.[21] This further implies that any attempt to account for an originally intuitive entity on the basis of abstraction leads to the loss of the sufficient reason, for in so doing, confirmation of the existence of that entity is sought on the basis of a mere shadow or reflection of it, that is to say, "representations of representations".[22] In its most basic sense, Schopenhauer's criticism essentially points out the idea that intuitive datum may not be *properly* verified on the basis of the abstract, and this is precisely what Euclid does, according to him, in his *Elements*.

For Schopenhauer, any demonstration within geometry which proceeds on the basis of the *abstract* root of knowing within reason as opposed to the *intuitive* root of being, wherein we obtain all our knowledge of mathematics, although indeed it may "prove" the logical coherence of the construct to the actual entity (as a shadow implies the existence of that to which it corresponds), the *raison d'être* of the entity is nevertheless lost. This is precisely what occurs in Euclid's mousetrap demonstration of the Pythagorean theorem. There, the theorem is proved on a *logical* and *abstract* basis, *for* rational knowledge, but fails to reveal the inner nature of the right triangle. That is why Schopenhauer states of Euclid's proof that it offers for knowledge merely "logical certainty", and further that, "while it no doubt conveys the conviction that the theorem which has been demonstrated is true, it nevertheless gives no insight as to *why* that which it asserts is what it is."[23]

Schopenhauer's criticism follows from the general recommendations of Aristotle in his *Posterior Analytics*: knowledge which reveals the *why* of an entity is superior to knowledge which reveals merely *that* it is so.[24] This is precisely what analysis should accomplish, according to Schopenhauer's understanding of it. For with analysis, every separate problem or theorem is regressed back to the "thing itself", so to speak, back to the original intuition from which it arose, and hence verified on the basis of the ground of being in space and time. That is why Schopenhauer feels that the above image which he supplies sufficiently demonstrates Pythagoras' theorem. Furthermore, for cases wherein such images are not readily available or are overly complex, the proper steps according to Schopenhauer, require merely, "an analysis of the process of thought in the first discovery of a geometrical proof".[25] In other words, the said entity in question is analyzed until the original intuition into its ground within being, its root, is identified.

In this last sense however, Schopenhauer's method of analysis certainly differs from the manner in which the ancient geometers understood and applied it. As explained in previous sections, analysis for the ancient geometers implied *either* a regress to a principle of geometry *or* to previous theorems and problems already established on the basis of prior syntheses. In the latter case, the ground of being of the problem still remains concealed. To reveal the ground, the analysis would have to be further carried over until reaching an axiom, but then the actual entity in question would be lost, since only mere fragments and elements of it would be revealed. In the former case, only certain very fundamental axioms were accepted without further interrogation by the geometers, as it were 'intuitively'. Indeed, such axioms were considered both indemonstrable but yet indispensible to geometry, since without them, no further progress in the science would be possible. With Schopenhauer on the other hand, *every* geometrical entity is axiomatic, as representative of a spatial-construct rooted within our cognitive intuition of space, and hence he states, "The reason of being is certainly not as evident in all cases…still I am persuaded that it might be brought to evidence in every theorem, however complicated, and that the proposition can always be reduced to some such simple intuition."[26]

Concluding observations

Schopenhauer's criticism of Euclidean geometry is significant on a number of levels. In the first place, although Kant himself mentioned the necessary change that must undergo mathematics following his Copernican turn, Schopenhauer was really the first to point out the fact that *certitude* in mathematics necessarily changes, and with it, the nature of demonstration. This is however based upon the assumption that Kant was right, and indeed that mathematical entities are essentially grounded within the cognitive apparatus of the subject, and hence that its objects are ideal. But what if this assumption is wrong? Does that necessarily do away with Schopenhauer's criticism or is their still some basis for it?

There is a case to be made that regardless of the ontological status of mathematics, particularly in geometry, our knowledge of such entities seems initially intuitive, although the actual nature of "intuition" becomes now the essential question. Points, lines, circles, squares, the Pythagorean theorem, the Golden section, the 5 regular solids—all of these entities have a certain initially sensuous appeal to the origin of their existence, inasmuch as our encounter with them arises first and foremost through our experience of the world. Indeed, there is something to be learned from Schopenhauer's remarks regarding the consequences of relying too heavily upon purely logical forms of demonstration to the exclusion of the intuitive:

> The fact that Geometry only aims at effecting *convictio*, and that this, as I have said, leaves behind it a disagreeable impression, but gives no insight into the reason of being—which insight, like all knowledge, is satisfactory and pleasing—may perhaps be one of the reasons for the great dislike which many otherwise eminent heads have for mathematics.[27]

If Schopenhauer's arguments for the use of analysis on the basis of idealism or "intuitive" grounds seems outdated, as I'm sure it does to many readers today, there is nonetheless a further historical point to be made on the basis of the dominance of early Greek rationalism, particularly through Plato and the Academy, and of how much this led to the predominance of the synthetic method, both as it is seen in Euclid and other ancient geometers, as well as its use and influence within the other sciences. Perhaps this method is merely a distant remnant from an era dominated by the prejudice of rationalism and the need to offer a logical account for knowledge? Perhaps in doing so, it led to neglect of other possible avenues for knowledge, certitude, and verification? Of course, this latter argument leads to Nietzsche's own critical remarks within his works regarding the influence of Platonism upon metaphysics and the history of philosophy, though undoubtedly, the roots of these remarks can be found first in Schopenhauer "as educator". Despite the truth or falsity then of Schopenhauer's claims regarding mathematics and geometrical demonstration, something more fundamental seems to be at stake. This is mirrored in his statements below, later echoed by Nietzsche:

> The Eleatics first discovered the difference indeed often the antagonism, between the perceived, φαινόμενον, and the conceived, νοούμενον, and used it in many ways for their philosophemes, and also for sophisms…It was recognized that perception through the senses was not to be trusted unconditionally, and it was hastily concluded that only rational thinking established truth…this rationalism, which arose in opposition to empiricism, kept the upper hand, and Euclid modeled mathematics in accordance with it.[28]

The further significance of Schopenhauer's above remarks, I leave to the reader to reflect upon and to consider. As a final point, it should be noted that the larger question of how Schopenhauer's criticism of Euclid reflects upon the nature of certitude as it has been understood from the time of the ancient Greek geometers, the emphasis upon the scientific method, and the predominance of rationalism is a question which, although initially raised by such philosophers as Nietzsche, has yet to be *fully* decided.

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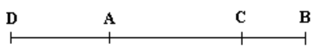
[1] The method of *apagogic* or *reduction ad absurdum*, of itself an analytic method, was used by the early Pythagoreans. See for example John Burnet's discussion of early Greek rationalism and its influence upon Plato, in his *Greek Philosophy: Thales to Plato* (London, MacMillan and Company: 1950), pp. 219-220.
[2] Ibid., pp. 230-233.
[3] Ibid. p. 219, "Book XIII of Euclid…is in a preeminent sense the work of the Academy".
[4] As seen in Schopenhauer's statement: "*No object without a subject*". Arthur Schopenhauer, *The World as Will and Representation*, trans. E. F. J. Payne (New York: Dover Publications, 1969), vol. I, p. 434.
[5] Schopenhauer focuses upon the nature of geometrical demonstration, since he considers all forms of demonstration in arithmetic to be entirely analytic by nature. This follows inasmuch as number, according to him, is essentially an abstraction of time relations, and the nature of time being essentially successive (unlike space), we do not confront the same difficulties there, as we do in our abstractions of spatial-relations. See for e.g., Schopenhauer, *The World as Will and Representation*, vol. I, pp. 75-76.

[6] Pappus states, "Analysis then takes that which is sought as if it were admitted and passes from it through successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards. But in synthesis, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis." From Euclid, *The Thirteen Books of the Elements*, trans. and comm. Thomas L. Heath (New York: Dover Publications, 1956), vol. I, p. 138.
[7] For a discussion of the nature of such 'elements' as the axioms, definitions, postulates, etc., see Aristotle *Anal. post.* I, 10. Also, Heath's comments in Euclid, *Elements*, pp. 117-124.
[8] This example is of course merely for purpose of illustration, since the early geometers normally assumed points and straight lines as part of the normal definitions, without offering any proof of their existence or non-existence. A more concrete example can be seen in a separate proof for Pythagoras' *golden section* provided by Heath, *Elements*, vol. III, pp. 442-443. The proof and the two methods which it follows are provided in full below (XIII, I):
"If a straight line be cut in extreme and mean ratio the square on the greater segment added to the half of the whole is five times the square on the half."
Let AB be divided in extreme and mean ration at C, AC being the greater segment;



And let AD = ½AB. I say that (sq. on CD) = 5(sq. on AD).
(Analysis)
For since (sq. on CD) = 5(sq. on AD), and (sq. on CD) = (sq. on CA) + (sq. on AD) + 2(rect. CA, AD), therefore (sq. on CA) + 2(rect. CA, AD) = 4(sq. on AD). But (rect. BA, AC) = 2(rect. CA, AD) and (sq. on CA) = (rect. AB, BC). Therefore (rect. BA, AC) + (rect. AB, BC) = 4(sq. on AD), or (sq. on AB) = 4(sq. on AD);
And this is true, since AD = ½AB.
(Synthesis)
Since (sq. on AB) = 4(sq. on AD), and (sq. on AB) = (rect. BA, AC) + (rect. AB, BC), therefore 4(sq. on AB) = 2(rect. DA, AC) + (sq. on AC). Adding to each the square on AD, we have: (sq. on CD) = 5(sq. on AD).
[9] Thomas Heath, *A History of Greek Mathematics* (Boston: Adamant Media Corporation, 2006), vol. I, p. 371.
[10] Heath notes (Euclid, *Elements*, vol. I, p. 140), "It is in relation to problems that the ancient analysis has the greatest significance, because it was the one general method which the Greeks used for solving all 'the more abstruse problems'."
[11] Heath notes (Euclid, *Elements*, vol. I, p. 140), "…in practice the Greeks secured what was wanted by always insisting on the *analysis* being confirmed by subsequent *synthesis*, that is, the laboriously worked backwards the whole way…reversing the order of analysis, which process would undoubtedly bring to light any flaw which had crept into the argument through the accidental neglect of the necessary precautions."
[12] Euclid, *Elements*, vol. I, pp. 349-350.
[13] The remainder of the proof is supplied here for the satisfaction of the reader's curiosity: "Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG [proposition i.14]. For the same reason BA is also a straight line with AH. And, since the angle DBC is equal to the angle FBA, for each is right: let the angle ABC be added to each; therefore the whole angle DBA is equal to the whole angle FBC [common notion 2]. And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively, and the angle ABD is equal to the angle FBC; therefore the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC [proposition i.4]. Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels BD, AL [proposition i.41]. And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC [proposition i.41]. But the doubles of equals are equal to one another [common notion 1]. Therefore the parallelogram BL is also equal to the square GB. Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC; Therefore the whole square BDEC is equal to the two squares GB, HC. And the square BDEC is described on BC, and the squares GB, HC on BA, AC. Therefore the square on the side BC is equal to the squares on the sides BA, AC. Therefore etc. Q. E. D."

[14] Arthur Schopenhauer, *On the Principle of Sufficient Reason*, trans. Karl Hillebrand (New York: Prometheus Books, 2006), p.164. In order to understand this comical jest at Euclid, one should have in mind an image of the mousetrap of the 18th century. Unlike the wooden plate with the steel bar and single spring used today, mousetraps during Schopenhauer's time were generally constructed from 2 to 3 blocks of wood, one of the pieces being hoisted by a string, attached to which would be a bait-tray underneath. In the event that a mouse took the bait, the string would be released, and the raised block would fall sharply, killing the poor animal. Looking again at the above diagram from Euclid's proof, one catches a glimpse of that old mousetrap of the 18th century.

[15] Schopenhauer, *The World as Will and Representation*, vol. I, p. 73. Regarding Schopenhauer's views of demonstration and the intuitive nature of all our knowledge, Brian Magee states in *The Philosophy of Schopenhauer* (New York: Oxford University Press, 1983), p. 39, "We may be inclined, for as long as we do not think about it, to suppose that human knowledge about the world has come into existence through chains of reasoning, and is embodied in their conclusions, but in reality all the information we have is already embodied in the premises from which those very chains of reasoning begin – if we know anything about the world we know it not because it has been demonstrated or proved but because it has been directly experienced or perceived, or else because it follows *by logical processes which contribute nothing at all in the way of empirical content* from what has been directly experienced or perceived."

[16] Schopenhauer uses this image both in his *On the Principle of Sufficient Reason*, p. 164, and in *The World as Will and Representation*, vol. I, p. 73. Although insight into the Pythagorean theorem through the supplied image is in this case more evidently direct, it should be noted, and indeed Schopenhauer points this out, that in most cases such direct intuition is not always possible, and many times it is necessary to analyze a problem or theorem more thoroughly until a more direct intuition into it is revealed (i.e. one is able to grasp its principle). In such cases the process of analysis would of course require much more than a simple diagram, involving a transformation of the problem into more fundamental elements which are solvable. I refer the reader to Schopenhauer's work, *On the Principle of Sufficient Reason*. There (pp. 162-164) he discusses proposition 16 of Euclid's *Elements*, illustrating the synthetic method and contrasting this with a clear example of his own use of analysis.

[17] It has been suggested that through similar images, Pythagoras himself came to his insight regarding the properties of the right triangle and the theorem. See Euclid, *Elements*, vol. I, p. 352.

[18] Schopenhauer, *The World as Will and Representation*, vol. I, p. 6.

[19] Schopenhauer, *The World as Will and Representation*, vol. I, p. 53.

[20] Schopenhauer, *On the Principle of Sufficient Reason*, p. 182.

[21] This distinction between the intuitive and the abstract as united beneath sufficient reason, is noted in relation to its historical context at the beginning of Schopenhauer's *On the Principle of Sufficient Reason*, p. 28, wherein he states that: "…the one being its application to judgments, which, to be true, must have a reason; the other, its application to changes in material objects, which must always have a cause. In both cases we find the principle of sufficient reason authorizing us to ask *why*?"

[22] Schopenhauer, *On the Principle of Sufficient Reason*, p. 114. Schopenhauer is here referring to the nature of abstract concepts as opposed to intuitions. For him, both are essentially representations, but since concepts are essential abstractions on the basis of the intuitive, they are therefore second hand representations.

[23] Ibid., p.159. Regarding the nature of logical certainty, Gerard Mannion writes in his book, Schopenhauer*, Religion and Morality: The Humble Path to Ethics* (Burlington, VT: Ashgate, 2003), p. 55, "The principle of *knowing* corresponds to what many see as the business of logical entailment, namely abstract concepts which are related to one another by the judgment we make. We seek the reason or ground for the judgment which follows. Schopenhauer did not simply mean logical entailment in the formal sense, but it is one of the types and the best example with which to illustrate this category as it was understood by him. The principle of *being* is what many would understand to be the business of mathematics, whereby the relationship of things is understood and explained with reference to temporal and spatial factors."

[24] Aristotle, *Anal. post.* I, 27.

[25] Schopenhauer, *The World as Will and Representation*, vol. I, p. 73.

[26] Schopenhauer, *On the Principle of Sufficient Reason,* p. 161.

[27] Schopenhauer, *On the Principle of Sufficient Reason*, p. 164.

[28] Schopenhauer, *The World as Will and Representation*, vol. I, p. 71. See for example Nietzsche, *The Gay Science*, trans. Walter Kaufmann (New York: Random House, 1974), pp. 169-170, "Those exceptional thinkers, like the Eleatics, who nevertheless posited and clung to the opposites of the natural errors, believed that it was possible to *live* in accordance with these opposites…they had attribute to themselves, fictitiously, impersonality and changeless

duration; they had to deny the role of the impulses of knowledge; and quite generally they had to conceive of reason as a completely free and spontaneous activity."