

Logical Consequences

Theory and Applications: An Introduction

Luis M. Augusto

© Individual author and College Publications 2017
All rights reserved.

ISBN 978-1-84890-236-7

College Publications
Scientific Director: Dov Gabbay
Managing Director: Jane Spurr

<http://www.collegepublications.co.uk>

Printed by Lightning Source, Milton Keynes, UK

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form, or by any means, electronic, mechanical, photocopying, recording or otherwise without prior permission, in writing, from the publisher.

Contents

Preface	xii
I. Introduction	1
0.1. Some introductory remarks on logical consequence	3
0.2. On this and other books	5
II. Theoretical aspects of logical consequence	11
1. <i>Sine quibus non</i>: Basic mathematical notions	13
1.1. Sets, operations, and relations	13
1.2. Order relations, graphs, and lattices	14
1.3. Algebraic structures	22
2. Logical systems, logics, and logical consequences	29
2.1. The formal language L^* and the logical system L	29
2.2. Proof theory and proof systems	32
2.2.1. Frege and Hilbert systems	33
2.2.2. Gentzen-style systems	35
2.2.2.1. Natural deduction	35
2.2.2.2. Sequent calculi	38
2.3. Model theory and semantics	40
2.3.1. Truth tables	43
2.3.2. Semantic and analytic tableaux	44
2.4. The consequence operation	48
2.4.1. General aspects of a consequence operation ★	48
2.4.2. The Tarskian conditions	48
2.4.3. Mathematical interpretation of the consequence operation	49
2.4.4. Inference systems, theories, and deductive systems	51
2.4.5. Tarski and the consequence operation	54
2.5. The consequence relation	55
2.5.1. The Tarskian consequence relation	55
2.5.2. The generalized consequence relation	58

Contents

- 2.5.3. The syntactical consequence relation 58
- 2.5.4. The semantical consequence relation 61
- 2.6. Equivalence of derivability and entailment 63
- 2.7. Logical consequence and the definition of a logic 64
- 2.8. Further semantics 65
 - 2.8.1. Relational semantics 65
 - 2.8.2. Algebraic and matrix semantics 69

III. Logical consequences and their applications 77

3. Classical deductive consequence 79

- 3.1. The deduction theorem and deductive systems 79
 - 3.1.1. Well-determined logics 82
- 3.2. Tarski-style conditions for deductive classicality 83
 - 3.2.1. Classical \blacksquare -consequence relations and operations . 84
 - 3.2.1.1. Classical syntactical \blacksquare -consequence relations 85
 - 3.2.1.2. Classical syntactical \blacksquare -consequence operations 86
 - 3.2.1.3. Classical semantical \blacksquare -consequence relations 86
 - 3.2.1.4. Classical semantical \blacksquare -consequence operations 88
 - 3.2.2. Classical \forall - and \exists -consequence relations and operations 88
- 3.3. Non-monotonic and defeasible consequence 89

4. Non-classical deductive consequences 97

- 4.1. Non-classicality and deduction 97
- 4.2. Many-valued consequence 98
 - 4.2.1. Some main many-valued logical systems 98
 - 4.2.1.1. Łukasiewicz logics 99
 - 4.2.1.2. Kleene's 3-valued logical system 101
 - 4.2.1.3. Bochvar's 3-valued logical system 102
 - 4.2.1.4. Fuzzy logics 104
 - 4.2.2. Logical consequence in many-valued logics 108
- 4.3. Intuitionistic consequence 114
- 4.4. Modal consequence 122
 - 4.4.1. Necessity and possibility 122
 - 4.4.2. Translations and extensions 129
 - 4.4.2.1. Temporal logic 130

4.4.2.2. Dynamic logic	136
4.5. Paraconsistent consequence	144
4.5.1. Relevance	146
4.5.2. Preservationism	149
4.6. Substructural consequence	154
4.6.1. Relevance, again	156
4.6.2. Linear Logic	159
4.6.3. The Lambek calculus	164
5. Non-deductive logical consequences	167
5.1. Abductive consequence	169
5.2. Inductive consequence	177
5.3. Probabilistic consequence	188
Bibliography	195
Index	209

List of Figures

0.2.1.A (fragment of an infinite) hierarchy of theories based on order relations.	8
0.2.2.A fragment of the hierarchy of modal logics based on order relations.	9
1.2.1.Hasse diagram for the strict partial order in Example 1.2.1.	15
1.2.2.A directed acyclic graph.	17
1.2.3.Transitive reduction of the DAG of Fig. 1.2.2.	18
1.2.4.Join table of 2^A	20
1.2.5.Meet table of 2^A	21
1.2.6.The lattice $(\mathcal{S}, \cup, \cap)$	21
1.2.7.The non-distributive lattices \mathcal{L}_1 and \mathcal{L}_2	22
2.2.1. \mathcal{NK} proof of $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$. . .	38
2.2.2.A \mathcal{LK} proof of $\forall x (A(x) \rightarrow B) \Rightarrow \exists x (A(x) \rightarrow B)$	40
2.3.1. $\Vdash (\phi \rightarrow \psi) \rightarrow (\neg(\psi \wedge \chi) \rightarrow \neg(\chi \wedge \phi))$: an analytic tableau proof.	47
2.5.1.A proof in the form of labeled trees.	59
2.8.1.Graphical depiction of the accessibility relations in Example 2.8.1.	68
2.8.2.Interpretation of a propositional language by means of a matrix.	73
4.4.1.Main normal modal systems and respective axioms.	127
4.4.2.The operators of LTL.	133
4.4.3.Systems with initial states satisfying $\mathbf{A}\Box\phi$, $\mathbf{E}\Box\phi$, and $\mathbf{A}\Diamond\phi$, respectively.	137

Preface

Wishing to anchor my research more and more in logic, pure or applied (where *or* is to be read in the logical, inclusive, sense), I aimed to increase substantially my knowledge of (classes of) logics. For both the pure and applied aspects I had in mind, tackling the central notion of logical consequence appeared as the best way to achieve my objective.

My work in this topic benefited from a short stay in early 2015 as a visiting researcher at the University of Barcelona, where, at the libraries of the Faculties of Mathematics and of Philosophy, I was able to collect much of the material that allowed me to take the turnstiles by the horns.

Melvin Fitting, Peter Schotch, and Yde Venema read parts of the manuscript and their observations helped me to improve it considerably. My sincere thanks to them.

I wish also to express my thanks to Dov M. Gabbay for including this book in this excellent series of College Publications, and to Jane Spurr for impeccable assistance during the publication process.

Madrid, February 2017

Luis M. Augusto