ALGORITHMS AND STABILITY ANALYSIS FOR OPTIMIZATION PROBLEMS WITH SPARSITY

by

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Abstract

The optimization models with sparsity arise in many areas of science and engineering, such as compressive sensing, image processing, statistical learning and machine learning. In this thesis, we study a general ℓ_0-minimization model, which can be used to deal with many practical applications. It is therefore important to study both the theoretical property and efficient algorithms for this sparsity model.

We characterize the nonuniqueness of the sparsest solutions of this model and show the existence of a lower bound for the nonzero absolute entries of the solutions of this model. We define the notation of an optimal weight, which ensures that the solution of the weighted ℓ_1-minimization coincides with the sparsest solution of the corresponding ℓ_0-model. The existence of an optimal weight for the weighted ℓ_1-minimization problem will be shown as well.

Two types of re-weighted ℓ_1-algorithms will be developed in this thesis from both the perspectives of primal and dual spaces, respectively. The primal re-weighted ℓ_1-algorithms will be derived through the 1st order approximation of the so-called merit functions for sparsity. The so-called dual re-weighted ℓ_1-algorithms for the general ℓ_0-model will be developed based on the reformulation of the general ℓ_0-model as a certain bilevel programming problem under the assumption of strict complementarity. Following the development of these algorithms, we conduct numerical experiments to demonstrate the efficiency of the primal and dual re-weighted ℓ_1-algorithms and compare with some existing algorithms.

We also establish a general stability result for a class of ℓ_1-minimization approach which is broad enough to cover many important special cases. We introduce the concept of restricted weak RSP of order k which is a generalized version of the weak RSP of order k. Unlike the existing stability results developed under the null space property and

restricted isotonic property, we use a classic Hoffman's theorem to establish a restricted-weak-RSP-based stability result for this class of ℓ_1-minimization approach.

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Chapter 1

Introduction

Many practical applications in science and engineering, for example, compressive sensing, signal and image processing, machine learning and statistical regressions can be formulated as the optimization problem with sparsity. The purpose of this thesis is to study some theoretical properties (including stability) and efficient convex optimization algorithms for a class of sparse optimization problems, called the general ℓ_0-minimization problem. Two types of algorithms are developed for this sparsity model: primal re-weighted ℓ_1-algorithms and dual re-weighted ℓ_1-algorithms. The numerical behaviours of our algorithms are also investigated. Using a certain matrix condition, a stability result for a class of convex optimization algorithms for the sparsity problems is also shown, which includes several existing results as special cases.

1.1 Model

We consider a new sparsity model, which is called general ℓ_0-minimization in this thesis:

$$
\begin{aligned}
(P_0) \quad \min_{x \in R^n} \quad & \|x\|_0 \\
\text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \\
& Bx \leqslant b,
\end{aligned}
\tag{1.1}
$$

where $A \in R^{m\times n}$ and $B \in R^{l\times n}$ are two matrices with $m \ll n$ and $l \leq n$, $y \in R^m$ and $b \in R^l$ are two given vectors, and $\epsilon \geq 0$ is a given parameter, and $\|x\|_0$ represents the number of nonzero components of the vector x, and $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ is ℓ_2-norm on R^n. In compressive sensing (CS), the parameter ϵ is used to estimate the level of measurement error $e = y - Ax$. The existence of error e is natural since the measurements y may not be accurate and thus might not exactly equal to Ax. Clearly, the purpose of (1.1) is to find the sparsest point in the convex set T, which is defined as

$$T = \{x : \|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b\}. \tag{1.2}$$

The constraint $Bx \leq b$ is motivated by some practical applications. For instance, many signal recovery models might not only include $y = Ax$ or $\|y - Ax\|_2 \leqslant \epsilon$ as constraints, but also need to impose extra conditions in order to reflect certain special structures of the target signals. This motivates us to consider the general ℓ_0-minimization model (1.1). In the next section, we will point out that this model covers several important applications in such areas as compressive sensing, 1-bit compressive sensing and statistical regression, and it is closely related to machine learning and low-rank matrix recovery. We also consider another general sparsity model as follows:

$$\begin{aligned} \min_{x \in R^n} \quad & \|x\|_0 \\ \text{s.t.} \quad & a_1 \|y - Ax\|_2 + a_2 \left\|U^T(Ax - y)\right\|_\infty + a_3 \left\|U^T(Ax - y)\right\|_1 \leq \varepsilon, \\ & Bx \leq b, \end{aligned} \tag{1.3}$$

where ε is a given parameter, and $A \in R^{m \times n}$ $(m \ll n)$ and $U \in R^{m \times h}$ are two matrices with full row rank, and a_1, a_2 and a_3 are three given parameters satisfying $a_i \in [0, 1]$ and $\sum_{i=1}^{3} a_i = 1$, and $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ and $\|x\|_\infty = \max_{i=1}^{n} |x_i|$ are ℓ_1- and ℓ_∞-norms on R^n, respectively. Both of the problems (1.1) and (1.3) are two state-of-the-art sparsity problems which have never been studied before.

1.1.1 Two special cases

Monotone sparse model. When B and b are given as

$$B = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \in R^{(n-1)\times n}, \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^{n-1}, \tag{1.4}$$

then (1.1) is reduced to the following structured sparsity model:

$$\min_x \left\{\|x\|_0 : \ \|y - Ax\|_2 \leqslant \epsilon, \ x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n\right\}. \tag{1.5}$$

The aim of this model is to find the sparsest point with monotonically descent entries in the convex set

$$T_1 = \{x : \|y - Ax\|_2 \leqslant \epsilon, x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n\}.$$

Moreover, when A is the identity matrix and B and b are given as above, the model (1.1) is closely related to the following problem:

$$\min_x \left\{ \sum_{i=1}^{n} (y_i - x_i)^2 : \ \text{s.t. } x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n, \ \|x\|_0 \leq r \right\}.$$

Without the sparsity constraint $\|x\|_0 \leq r$, this problem is called the isotonic regression problem (Tibshirani, Hoefling and Tibshirani [84], Hastie, Tibshirani and Wainwright [82]).

Nonnegative sparse model. If B and b are given as

$$B = -I \in R^{n\times n} \quad \text{and} \quad b = (0, ..., 0)^T \in R^n, \tag{1.6}$$

then (1.1) is reduced to the following nonnegative sparse model:

$$\min_x \{\|x\|_0 : \|y - Ax\|_2 \leqslant \epsilon, x \geq 0\}, \tag{1.7}$$

which is to hunt for the sparsest points in the convex set

$$T_2 = \{x : \|y - Ax\|_2 \leqslant \epsilon, x \geq 0\}.$$

The model (1.7) is closely related to the Nonnegative Garrote with sparsity [82], i.e.,

$$\left\{\min_x \sum_{i=1}^m \left(y_i - \sum_{j=1}^n x_j a_{ij}\right)^2 : x \geq 0, \|x\|_0 \leqslant \tau\right\},$$

where a_{ij} is the entry in the ith row and jth column of A and τ is a given parameter.

1.2 Some applications

We shall briefly introduce several applications which can be formulated as or be closely related to the model (1.1).

1.2.1 Compressive sensing (CS)

Compressive sensing (compressed sensing or compressive sampling) has attracted plenty of attention in signal and image processing [18,20,21,33]. It was first studied by Donoho, Candès, Tao, Romberg and others [18,20,31,33] and was one of the important and extensively studied areas in the past decade in science and engineering [7,18,30,33,68,86,97,99]. The sparsity is a key assumption in CS theory and its applications, and it has become an important tool to deal with many applications in the field of information science and applied mathematics. Before further discussion, let us define the k-sparse vectors.

Definition 1. [21] Let $x \in R^n$ be a vector. If the number of non-zero components of x does not exceed k, i.e.,

$$\|x\|_0 := |\{i : x_i \neq 0\}| \leq k,$$

then x is called a k-sparse vector, where $\|x\|_0$, called the ℓ_0-norm, denotes the number of nonzero components of x.

In signal processing, the measurements $y \in R^m$ can take the form

$$y = Ax,$$

where $x \in R^n$ is the signal to recover, and $A \in R^{m\times n}$ is the measurement matrix. When $m < n$, (i.e., the number of measurements is lower than the signal dimension), the above linear system is underdetermined. Therefore it is impossible to reconstruct a signal from such a linear system unless certain additional information is available. Compressive sensing assumes that the signal to recover is sparse. Therefore to recover the signal, one may solve the ℓ_0-minimization problem:

$$\begin{aligned} \min_x \quad & \|x\|_0 \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{1.8}$$

where $A \in R^{m\times n}$ with $m \ll n$ and $y \in R^m$ are given measurements. Based on this, finding the original sparse signal by an underdetermined system of linear equations can be actually achieved by developing efficient algorithms for this model. However, measurements are often inaccurate or incomplete in many situations due to measurement errors, missing values or unavoidable noises. In such cases, we hope that the error between the measurements y and the information data Ax can be bounded in a certain way. As a result, the following sparsity optimization model with ℓ_2-norm constraint is also commonly used in CS:

$$\begin{aligned} \min_x \quad & \|x\|_0 \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon. \end{aligned} \tag{1.9}$$

The following two sparse optimization problems are closely related to the problems (1.8) and (1.9):

$$\begin{aligned} \min_x \quad & \|y - Ax\|_2 \\ \text{s.t.} \quad & \|x\|_0 \leqslant \tau, \end{aligned} \tag{1.10}$$

and

$$\min_x \quad \|y - Ax\|_2 + \lambda \|x\|_0$$

where τ and λ are two given positive parameters. Replacing $\|x\|_0$ by $\|x\|_1$ leads to two relaxation problems [43, 82] which are widely used in statistical regression and CS. Obviously, both the standard CS problems (1.8) and (1.9) are the special cases of (1.1).

1.2.2 1-bit compressive sensing (1-bit CS)

The 1-bit compressive sensing has also attracted some attention in the field of signal processing [9, 10, 50, 57, 58, 88, 100]. As shown above, a sparse signal is possible to be recovered even when the number of measurements is less than the signal length. However, the fine measurements impose a heavy burden on the measurement system, and require more processing time and storages. The 1-bit CS requires only one bit of measurements, for instance, only the sign of measurements. Zhao and Xu [100] have shown that in some situations, it is possible to have a sign recovery of the signal from 1-bit measurements.

A typical 1-bit CS problem is modelled as follows:

$$\begin{aligned} \min_x \quad & \|x\|_0 \\ \text{s.t.} \quad & y = \text{sign}(Ax), \end{aligned} \tag{1.11}$$

where $A \in R^{m \times n}$ is the sensing (measurement) matrix and $y \in \{-1, 0, 1\}^m$ is the sign measurements. Let S_+, S_- and S_0 be the three sets, i.e.,

$$S_+ = \{i : \ y_i = 1\}, \ S_- = \{i : \ y_i = -1\}, \ S_0 = \{i : \ y_i = 0\}.$$

Indexed by S_+, S_- and S_0, the constraints in (1.11) can be rewritten as

$$\text{sign}(A_{S_+,n}x) = e^{S_+}, \ \text{sign}(A_{S_-,n}x) = -e^{S_-} \text{ and } \text{sign}(A_{S_0,n}x) = 0,$$

where $A_{S_+,n}$, $A_{S_-,n}$ and $A_{S_0,n}$ are the submatrices of A which decompose A according to the row indices S_+, S_- and S_0. The vectors e^{S_+} and e^{S_-} denote the vectors of ones. By introducing a positive parameter ρ, Zhao and Xu have shown that [100] the 1-bit CS model (1.11) can be reformulated as the following ℓ_0-minimization model:

$$\begin{aligned} \min_x \quad & \|x\|_0 \\ \text{s.t.} \quad & A_{S_+,n}x \geq \rho e^{S_+}, \ A_{S_-,n}x \leq -\rho e^{S_-}, \ A_{S_0,n}x = 0. \end{aligned} \tag{1.12}$$

Clearly, the model (1.12), the reformulation of 1-bit CS model (1.11), is a special case of our model (1.1), corresponding to the case where (A, B, y, b, ϵ) are of the form

$$A = A_{S_0,n} \in R^{|S_0|\times n}, \ B = \begin{bmatrix} -A_{S_+,n} \\ A_{S_-,n} \end{bmatrix} \in R^{(|S_+|+|S_-|)\times n}, \text{ and } b = \begin{bmatrix} -\rho e^{S_+} \\ -\rho e^{S_-} \end{bmatrix} \in R^{(|S_+|+|S_-|)},$$

$y = 0$ and $\epsilon = 0$.

1.2.3 Fused Lasso

In many situations, noise data are frequently incurred in measurements or sampling data. A high noise level may cause the signal recovery or statistical regression to fail. As one of important methods to deal with the high-level noise data, the so-called Fused Lasso (least absolute shrinkage and selection operator) is introduced (see, e.g. [52, 59, 72, 81, 83]). The Fused Lasso model with sparsity is stated as follows:

$$\min_x \quad \lambda_1 \|x\|_0 + \lambda_2 \|Dx\|_0 + \|y - x\|_2 \tag{1.13}$$

where

$$D = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \in R^{(n-1)\times n}$$

and λ_1 and λ_2 are the nonnegative regularization weights. Replacing $\|\cdot\|_0$ by $\|\cdot\|_1$ in (1.13), the solution to this ℓ_1-minimiation counterpart of (1.13) is called Fused Lasso signal approximator [82]. Let

$$C = \begin{bmatrix} \lambda_1 I \\ \lambda_2 D \end{bmatrix} \in R^{(2n-1)\times n}$$

be a matrix. Clearly, C is a full-column-rank matrix. Let $C^\dagger = (C^T C)^{-1} C^T \in R^{n\times(2n-1)}$. Then (1.13) is equivalent to

$$\min_z \quad \|z\|_0 + \left\|y - C^\dagger z\right\|_2$$

which is closely related to the following model:

$$\begin{array}{ll} \min\limits_z & \|z\|_0 \\ \text{s.t.} & \left\|y - C^\dagger z\right\|_2 \leqslant \epsilon, \end{array} .$$

which is a special case of (1.1). It is worth mentioning that other statistical learning problems can be also formulated or related to ℓ_0-minimization (1.8) (see. e.g. [5, 11, 17, 61, 82]) and the general ℓ_0-minimization (1.1) (see. e.g. [54, 87]).

1.2.4 Low-rank matrix recovery

In many practical situations, one needs to recover a matrix with low rank. The associated problem is called low-rank matrix recovery [8, 12, 41, 43, 60, 91]. The problem is often modelled as the so-called matrix rank minimization problem. Given a linear map

$\mathcal{A}: R^{n_1\times n_2} \to R^m$ and a vector $y \in R^m$, the low-rank minimization problem is stated as follows:

$$\begin{aligned} \min_X \quad & \text{rank}(X) \\ \text{s.t.} \quad & \mathcal{A}(X) = y. \end{aligned} \tag{1.14}$$

Consider the singular value decomposition of $X \in R^{n_1\times n_2}$, i.e.,

$$X = \sum_{i=1}^{n} \sigma_i u_i v_i^*,$$

where n is the minimum number between n_1 and n_2, $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$ are the singular values of X, u_1 and v_1 are the left and right singular vectors. Note that X has rank r if and only if the singular value vector $\sigma(X) = (\sigma_1, ..., \sigma_n)$ is an r-sparse vector. Thus (1.14) can be written as the following ℓ_0-model:

$$\begin{aligned} \min_X \quad & \|\sigma(X)\|_0 \\ \text{s.t.} \quad & \mathcal{A}(X) = y. \end{aligned} \tag{1.15}$$

This problem is more general than the standard ℓ_0-minimization problem in vector forms. Based on the above observations, a generalised version of the low-rank minimization problem can be obtained from the model (1.1) in matrix form.

1.3 Main research work

The research work carried out in this thesis includes the following four aspects:

(i). Properties of the solutions to (1.1);

(ii). Equivalence of weighted ℓ_1-minimization and ℓ_0-minimization via optimal weights;

(iii). Algorithms for solving (1.1);

(iv). Stability of ℓ_1-minimization algorithms.

In what follows, we briefly introduce the background (existing work) in these aspects and the main outcome of our researches.

1.3.1 Properties of solutions to (P_0)

The work carried out in this aspect is included in Chapter 2. Let us first review the properties of the solutions to the standard ℓ_0-minimization (1.8). It is well-known that the uniqueness of the solutions of (1.8) can be guaranteed if $\text{Null}(A) \bigcap \Sigma_{2k} = 0$, where k is the optimal value of (1.8), Null(A) is the null space of the matrix A and Σ_{2k} denotes the set of all $2k$-sparse vectors, i.e., $\Sigma_{2k} = \{x : \|x\|_0 \leq 2k\}$. Based on such a fact, several uniqueness conditions have been developed for the problem (1.8), such as mutual coherence [38,55], Babel function [85], exact recovery condition (ERC) [38,43], NSP [28,89] and RIP [19] also ensure that (1.8) has a unique solution. A good summary of such uniqueness conditions can be found in Zhao [94].

Main work: In Chapter 2, we first develop a necessary condition for the vector x to be the sparsest solution of (1.1). Since the model (1.1) is more complex than the standard ℓ_0-minimization (1.8), the uniqueness of solutions of (1.1) is usually not ensured under the similar conditions for the uniqueness of solutions to the standard ℓ_0-minimization (1.8). In Chapter 2, we mainly show the nonuniqueness conditions for the solutions of (1.1). Under a certain assumption, we also explore the existence of a lower bound for the absolute nonzero components in the solution set of (1.1).

1.3.2 Equivalence of weighted ℓ_1- and ℓ_0-problems via optimal weights

There are several existing conditions for ℓ_1-minimization counterpart of (1.8) being able to exactly solve the standard ℓ_0-minimization (1.8). The conditions NSP of order k [28], RIP of order $2k$ [21], mutual coherence [55] or RSP of order k [92] are sufficient conditions for ℓ_1-minimization to exactly solve the problem (1.8). Moreover, the NSP of order k as well as the RSP of order k is also the necessary conditions for the success of ℓ_1-minimization in solving (1.8). Since the conditions NSP and RIP will be mentioned many times in this thesis, we include their definitions here:

- (NSP of order k) A matrix $A \in R^{m\times n}$ is said to satisfy the null space property (NSP) of order k if for any set $\Lambda \subset \{1,\ldots,n\}$ with $|\Lambda| \leqslant k$, $\|h_\Lambda\|_1 < \|h_{\bar{\Lambda}}\|_1$ is satisfied for any $h \in \text{Null}(A) \setminus \{0\}$, where $\bar{\Lambda} = \{1,...,n\} \setminus \Lambda$, and h_Λ is the subvector of h with the components $h_i, i \in \Lambda$.

- (RIP of order k) If there exists a constant $\delta_k \in (0,1)$ such that for all k-sparse vectors we have $(1-\delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_k)\|x\|_2^2$, then A is said to satisfy the restricted isometry property (RIP) of order k.

Main work: In Chapter 3, we study the weighted ℓ_1-minimization model for the ℓ_0-minimization problem (1.1) and address the issue of when these two problems are equivalent. We introduce the concept of optimal weights which ensures the solution of the weighted ℓ_1-problem is the sparsest solution of (1.1). We show the existence of such an optimal weight for weighted ℓ_1-minimization. This provides a theoretical basis for the development of the algorithms in later chapters. To provide a possible way to find an optimal weight, we investigate the dual problem of the weighted ℓ_1-minimization counterpart of (1.1), and study some properties, such as strong duality and strict complementarity between the two problems. Based on this, we prove that as a certain bilevel programming problem may provide an optimal weight.

1.3.3 Algorithmic development

Let us take (1.8) and (1.9) as examples to briefly introduce some existing approaches for the standard ℓ_0-minimization (1.8) and (1.9). Directly solving (1.8) or (1.9) is generally very difficult since the ℓ_0-norm is a nonlinear, nonconvex and discrete function. Although (1.8) and (1.9) are NP-hard problems [1, 65], an efficient algorithm is needed from the perspectives of both mathematics and applications. Some algorithms have been developed for (1.8) and (1.9) over the past decade, including convex optimization, heuristic methods, non-convex optimization and Bayes' analysis. We briefly introduce the first three types of methods. Let's first recall the following definition:

Definition 2 (Lower Convex Envelope)**.** *The lower convex envelope $\tilde{f}$ of a function f on the interval $[a, b]$ is defined at each point of the interval as the supremum of all convex functions that lie under that function, i.e.,*

$$\tilde{f}(x) = \sup\{g(x) : g(x) \text{ is convex and } g \leqslant f \text{ over}[a, b]\}.$$

Convex optimization methods. The ℓ_1-norm $\|x\|_1$ is the lower convex envelope of the ℓ_0-norm $\|x\|_0$ over the set $\{x : \|x\|_\infty \leqslant 1\}$ [43]. Therefore, the ℓ_1-norm is often called the convex relaxation of the ℓ_0-norm over the above set. By replacing the ℓ_0-norm in (1.9) with the ℓ_1-norm, we obtain the following popular ℓ_1-minimization problem:

$$\begin{aligned} \min_x \quad & \|x\|_1 \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \end{aligned} \tag{1.16}$$

which is called quadratically constrained basis pursuit [25,37,39,42,43,80]. Moreover, the above model is also closely related to the Lasso problem [80, 82]

$$\begin{aligned} \min \quad & \|y - Ax\|_2 \\ \text{s.t.} \quad & \|x\|_1 \leq \tau, \end{aligned} \tag{1.17}$$

and the Dantzig Selector [23]

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & \left\|A^T(y - Ax)\right\|_\infty \leq \varepsilon, \end{aligned} \tag{1.18}$$

where τ and ε are given parameters. Note that when ϵ is 0, the problem (1.16) reduces to the standard ℓ_1-minimization

$$\begin{aligned} \min_x \quad & \|x\|_1 \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{1.19}$$

which is also called basis pursuit, a popular method for solving (1.8) (see, e.g., [26], [38]). Due to its convexity, ℓ_1-minimization has been widely used in CS to recover sparse signals. Another effective convex method for solving ℓ_0-minimization problems is the so-called re-weighted ℓ_1-minimization. Firstly, the weighted ℓ_1-minimization method [24, 46] for (1.8) and (1.9) can be stated respectively as follows:

$$\begin{aligned} \min_x \quad & \|Wx\|_1 \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{1.20}$$

and

$$\begin{aligned} \min_x \quad & \|Wx\|_1 \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \end{aligned} \tag{1.21}$$

where W is a diagonal matrix, i.e., $W = \text{diag}(w)$ where $w \in R^n_+$ is the vector of weights. Re-weighted ℓ_1-minimization is a very useful method for solving sparse optimization problems (see [24], [97]). This method consists of solving a series of individual weighted ℓ_1-minimization problems [3, 4, 24, 46]. Taking (1.20) as example, the method solves a series of the following problems:

$$\begin{aligned} (\ell^k_w) \quad \min_x \quad & (w^k)^T|x| \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{1.22}$$

where k represents the kth iteration and w^k can be updated by certain rules.

Non-convex optimization. Non-convex methods for solving ℓ_0-minimization usually arise from the approximation of the ℓ_0-norm. The ℓ_0-norm can be approximated by some concave merit functions for sparsity with a suitably choice of parameters. For example, the log-type function $\sum_{i=1}^n \log(|x_i| + \varepsilon)$ and ℓ_p-quasinorm function $\|x\|_p^p$, where $0 < p < 1$. The concave merit functions for sparsity can approximate the ℓ_0-norm better than the ℓ_1-norm (see e.g. [21], [8], [97]). We will discuss the merit functions for sparsity in detail and give more specific examples in Chapter 4. Figure 1.1 shows the graphs of

ℓ_1-norm, ℓ_0-norm, log-type function $\log(|x|+1)$, and the quasinorm function $\|x\|_p^p$ with $p = 0.25$ and 0.5 respectively. Clearly, the log-type function and quasinorm functions, in general, approximate the ℓ_0-norm better than the ℓ_1-norm. Because of this, (1.8) or (1.9) can be well approximated by a certain concave minimization through a concave merit function. Such a non-convex optimization can not only approximate ℓ_0-minimization better than ℓ_1-minimization, but also may lead to efficient 1st-order-approximation-based method (see Zhao [94, 97]).

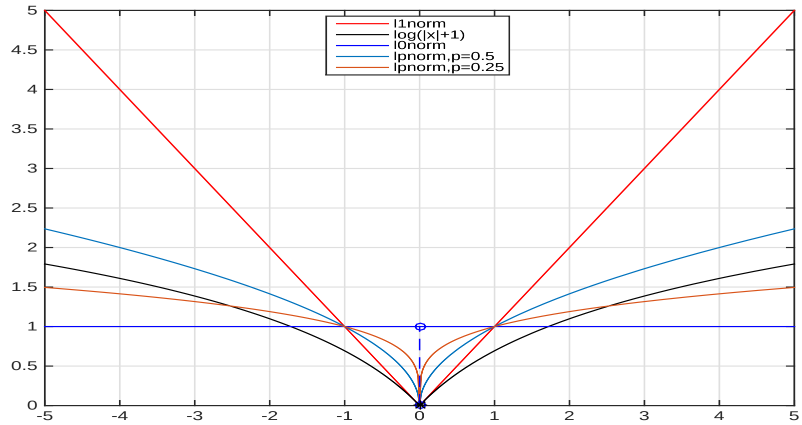


Figure 1.1: The graphs of $\|x\|_0$, $\log(|x|+1)$, $\|x\|_1$, $\|x\|_p^p$, $p = 0.25$ and 0.5 in 1-dimensional space

The first-order method would yield a framework for the updating scheme for w_k in (1.22). For instance, Candès, Wakin and Boyd [24] proposed their re-weighted ℓ_1-algorithm for (1.8) with the following updating scheme:

$$w_i^{k+1} = \frac{1}{|x_i^k| + \varepsilon}, \quad i = 1, ..., n.$$

After that, Needell analyzed re-weighted ℓ_1-minimization for (1.9) in [66] and showed that the error bound for signal recovery can be tighter than that of standard ℓ_1-minimization

(1.16). The re-weighted ℓ_1-minimization (1.22) with

$$w_i^{k+1} = \frac{1}{(|x_i^k| + \varepsilon)^{1-p}}, \quad i = 1, ..., n$$

was analyzed in [42]. Zhao and Li [97] have investigated a unified method for a family of re-weighted ℓ_1-minimization algorithms. The convergence of the re-weighted algorithms was shown under some conditions in [27], [56] and [97]. Both ℓ_1-minimization and re-weighted ℓ_1-minimization are popular methods for signal recovery. The former is very easy to implement and has a good performance on recovering sparse signals while the latter has a better success rate on sparse signal recovery than ℓ_1-minimization when the initial point is suitably chosen, see, e.g., [24, 27, 42, 56, 97].

Heuristic. Some heuristic and greedy algorithms for (1.8) have also been widely studied in the CS literature, such as orthogonal matching pursuit [32, 62, 68, 79, 86], compressive sampling matching pursuit [43, 67], subspace pursuit [7, 43], and thresholding algorithms [7, 30, 38, 43].

Main work: In Chapters 4 & 5, we only consider the convex and nonconvex optimization methods for the ℓ_0-minimization model (1.1). In Chapter 4, we introduce a new merit function and define a set of merit functions for sparsity. By using such merit functions, we develop the primal re-weighted ℓ_1-algorithm, based on the idea of Zhao & Li [97]. Following that, we propose some new convex relaxations to the bilevel programming model developed in Chapter 2, and in the meantime, we also use the relaxation method provided by Zhao and Kočvara [96] and Zhao and Luo [99]. Three types of relaxation models for the bilevel problem will be developed, by which the three types of new dual re-weighted ℓ_1-algorithms will be developed. Chapter 5 presents detailed numerical results regarding the choice of the default parameters and merit functions for the dual algorithms and also provides comparisons of the performance of the proposal algorithms and existing ones. We also implement the primal and dual algorithms for different cases of B and b including: (i) B and b are given as in (1.4); (ii) B, b are given as in (1.6); (iii) $B = 0$ and

$b = 0$; (iv) B is a randomly generated matrix.

1.3.4 Stability analysis

Stability of an algorithm for CS means that when the signal is not exactly k-sparse or the measurements are slightly inaccurate, the error between the solution obtained by an algorithm and the sparse signal can be controlled in terms of the measurement error as well as the error of the k-term approximation of the signal. The error of the best k-term approximation of a vector (x) is defined as follows [28]:

$$\sigma_k(x)_p = \min_{z\in\Sigma_k} \|x - z\|_p\,, \tag{1.23}$$

where $\|\cdot\|_p$ is the ℓ_p-norm for $p \geq 1$ and Σ_k is the set of k-sparse vectors. Clearly, the best k-term approximation of x is obtained by holding the largest k absolute entries of x and setting the remaining components to 0. Obviously, x is k-sparse if and only if its best k-term approximation error is 0.

For the models (1.19) and (1.16), and any $x \in R^n$ in the feasible set of these problems, if the solution $x^{\#}$ of (1.19) or (1.16) satisfies the following bounds:

$$\left\|x - x^{\#}\right\| \leqslant C_0\sigma_k(x)_p \tag{1.24}$$

for (1.19) and

$$\left\|x - x^{\#}\right\| \leqslant C_1\sigma_k(x)_p + C_2\epsilon \tag{1.25}$$

for (1.16), then (1.19) and (1.16) are said to be stable for the recovery of a signal or locating the solution of ℓ_0-minimization problem. In the CS scenarios, to ensure an algorithm being stable, the sensing matrix A must satisfy a certain property, such as the stable NSP of order k [13, 28, 40, 77, 89], RIP of order $2k$ [2, 14–16, 19–21, 42, 64], singular minmial value [78] and weak RSP of order k [92–95, 100]. Meanwhile, the constants C_0, C_1 and C_2 in (1.24) and (1.25) are determined by NSP or RIP constant when the sensing matrix A satisfies the stable NSP of order k [13, 43, 77] or RIP of order $2k$ [21, 22, 43].

However, in [94, 95], these constants are determined by the so-called Robinson's constant when A^T satisfies weak RSP of order k.

Main work: In Chapter 6, we study the stability of a class of ℓ_1-minimization associated with (1.3). For general ℓ_1-minimization methods, it seems difficult to study the stability under the NSP of order k or RIP of order $2k$ since the problem includes more complicated constraints than (1.19) and (1.16). Different from NSP and RIP, the RSP condition is derived from the classic optimality conditions of (1.19), so the RSP condition is more adaptive to the structure of the problem. In Chapter 6, we introduce the so-called restricted weak RSP of order k, under which, a general stability result for the ℓ_1-minimization associated with (1.3) is established. It includes several existing stability results as special case.

1.4 Preliminary, terminology and notation

Definition 3. *A nonnegative function* $\|\cdot\|$*:* $X \longrightarrow [0, +\infty)$ *is called a norm if it satisfies the following properties:*

(A1). $\|x\| = 0$ *if and only if* $x = 0$*;*

(A2). $\|\lambda x\| = |\lambda| \, \|x\|$ *where* λ *is a real number;*

(A3). $\|x + y\| \leq \|x\| + \|y\|$ *for any* x, y*.*

It is well known that $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, $p \geq 1$, is a norm, called ℓ_p-norm on R^n. The following Hölder inequality is stated in terms of ℓ_p-norm.

Lemma 4. *For any* $p, q \in [1, \infty]$ *such that* $1/p + 1/q = 1$ *(admitting* $1/\infty = 0$ *and* $1/0 = \infty$*), one has*

$$\left| x^T y \right| \leq \|x\|_p \, \|x\|_q \, .$$

The matrix norm can be also defined in terms of the ℓ_p-norm of vectors.

Definition 5. *For $1 \leq p, q \leq \infty$, the matrix norm $\|A\|_{p\to q}$ is defined as follows*

$$\|A\|_{p\to q} = \sup_{\|x\|_p \leq 1} \|Ax\|_q .$$

Notation : Unless otherwise stated, the identity matrix of a suitable size is denoted by I. The field of real numbers is denoted by R and the n-dimensional Euclidean space is denoted by R^n. Let R^n_+ be the set of nonnegative vectors in Euclidean space and R^n_- be the set of the nonpositive vectors. Given a vector $u \in R^n$, $|u|$, $(u)^+$ and $(u)^-$ denote the vectors with components $|u|_j = |u_j|$, $[(u)^+]_j = \max\{u_j, 0\}$ and $[(u)^-]_j = \min\{u_j, 0\}$, $j = 1, ..., n$, respectively. The cardinality of the set S is denoted by $|S|$ and the complementary set of $S \subseteq \{1, ..., n\}$ is denoted by $\bar{S}$, i.e., $\bar{S} = \{1, ..., n\} \setminus S$. For a given vector $x \in R^n$, denoted by x_S, the vector supported on S. Given a matrix A, $a_{i,j}$ denotes the entry of A in row i and column j. For two sets $S \subseteq \{1, ..., n\}$ and $T \subseteq \{1, ..., m\}$, $A_{T,S}$ denotes the submatrix of $A \in R^{m\times n}$ with components $a_{i,j}$, $i \in T, j \in S$, and A_S denotes the submatrix of $A \in R^{m\times n}$ obtained by deleting the columns indexed by $\bar{S}$. For a matrix $A = (a_{i,j})$, $|A|$ represents the absolute version of A, i.e., $|A| = (|a_{i,j}|)$. $\mathcal{R}(A^T) = \{A^T y : y \in R^m\}$ denotes the range space of A^T.

Chapter 2

Nonuniqueness and Lower Bounds of Solutions of General ℓ_0-minimization

In this chapter, we discuss some fundamental properties of the general ℓ_0-minimization problem (P_0) given in (1.1). This chapter is organized as follows. In Section 2.1, we show some theoretical properties of this model such as the necessary conditions for a point being the sparsest solution to this model. Section 2.2 demonstrate the nonuniqueness of the sparsest solutions of the model, respectively. In Section 2.3, the existence of a positive lower bound for the nonzero absolute entries of the sparsest points will be proven under some conditions. An approximation and the ℓ_1-minimization counterpart of (1.1) will be introduced in the final section.

2.1 Properties of solutions of model (P_0)

We first develop some necessary conditions for a point to be the solution of (1.1), which are summarized in the following Theorem 6 and Lemma 8.

Theorem 6. *If x^* is a sparsest solution to* (1.1) *where $A \in R^{m\times n}$ and $B \in R^{l\times n}$ with columns a_i $(i = 1, 2, ..., n)$ and b_i $(i = 1, 2, ..., n)$ respectively, then*

$$\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) = \{0\},$$

where $S \subseteq \{1, 2, \ldots, n\}$ is the support set of x^*, A_S denotes the submatrix of A with *columns $a_i, i \in S$, and B_S denotes the submatrix of B with columns $b_i, i \in S$.*

Proof. Let x^* be a sparsest solution of (1.1) and k be the optimal value of (1.1). We prove this by contradiction. If $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) \neq \{0\}$, there exists a nonzero vector $\Delta x \in R^n$ with $\Delta x_S \neq 0$ such that $A_S \Delta x_S = 0$ and $B_S \Delta x_S = 0$. The above linear equalities can be written as

$$\sum_{i \in S} a_i \Delta x_i = 0, \quad \sum_{i \in S} b_i \Delta x_i = 0. \tag{2.1}$$

Since $\Delta x_S \neq 0$, there is a nonzero component Δx_j, where $j \in S$, such that the corresponding a_j and b_j can be represented as the linear combination of the other columns, that is,

$$a_j = -\sum_{i \in S, i \neq j} a_i \frac{\Delta x_i}{\Delta x_j}, \quad b_j = -\sum_{i \in S, i \neq j} b_i \frac{\Delta x_i}{\Delta x_j}. \tag{2.2}$$

We know that x^* satisfies the following system

$$\|y - Az\|_2 \leqslant \epsilon, \quad Bz \leqslant b, \tag{2.3}$$

which can be rewritten as

$$\left\| y - \left(\sum_{i \in S, i \neq j} a_i x_i^* \right) - a_j x_j^* \right\|_2 \leqslant \epsilon, \quad \left(\sum_{i \in S, i \neq j} b_i x_i^* \right) + b_j x_j^* \leqslant b.$$

Substituting a_j and b_j in (2.2) into the above system yields

$$\left\| y - \sum_{i \in S, i \neq j} \left(x_i^* - \frac{\Delta x_i}{\Delta x_j} x_j^* \right) a_i \right\|_2 \leqslant \epsilon, \quad \sum_{i \in S, i \neq j} \left(x_i^* - \frac{\Delta x_i}{\Delta x_j} x_j^* \right) b_i \leqslant b. \tag{2.4}$$

The inequalities in (2.4) imply that the vector $\bar{x}$, with $\|\bar{x}\|_0 \leqslant k - 1$ and constructed

$$\bar{x}_i = \begin{cases} x_i^* - \frac{\Delta x_i}{\Delta x_j} x_j^*, & i \in S, i \neq j \\ 0, & i = j \\ 0, & i \notin S \end{cases},$$

satisfies the conditions in (2.3). This means that $\bar{x}$ is a sparser solution of (1.1) than x^*. This is a contradiction. The desired result follows. $\square$

Note that $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) = \{0\}$ means $\begin{bmatrix} A \\ B \end{bmatrix}_S$ has full column rank. We make the following comments for the condition $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) = \{0\}$.

Remark 7. *Let x^* be an arbitrary sparsest point in T given in* (1.2) *and S be the support of x^*. It can be seen that $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) = \{0\}$ has some equivalent forms. Since $B_S x_S^* \leq b$ can be separated or decomposed by active and inactive constraints, the following conditions can be regarded as the equivalent conditions for $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_S) = \{0\}$:*

- $\mathrm{Null}(A_S) \cap \mathrm{Null}\binom{B_{I,S}}{B_{\bar{I},S}} = \{0\}$*;*
- $\mathrm{Null}(A_S) \cap \mathrm{Null}(B_{I,S}) \cap \mathrm{Null}(B_{\bar{I},S}) = \{0\}$*;*
- $\mathrm{Null}\binom{A_S}{B_{I,S}} \cap \mathrm{Null}(B_{\bar{I},S}) = \{0\}$*;*

where $I \subseteq \{1, 2, \ldots, m\}$ is the index set of active constraints in $B_S x_S^* \leqslant b$ and $\bar{I} = \{1, 2, \ldots, m\} \setminus I$ is the index set of inactive constraints in $B_S x_S^* \leqslant b$, $B_{I,S}$ denotes the submatrix of B with components $b_{i,j}$, $i \in I, j \in S$ and $B_{\bar{I},S}$ denotes the submatrix of B with components $b_{i,j}$, $i \in \bar{I}, j \in S$.

One may ask which condition can ensure $\mathrm{Null}\binom{A_S}{B_{I,S}} = \{0\}$. Let $|I(x)|$ be the cardinality of active constraints in $Bx \leqslant b$ with respect to x. Denote the sparsest solution set by

$$\Lambda = \{x \in R^n : \|x\|_0 = k,\ x \in T\}, \tag{2.5}$$

where k is the optimal value of (1.1). We have the following lemma. The proof of this lemma follows the idea of Lemma 3.3 in [100].

Lemma 8. *Let x^* be a sparsest solution to* (1.1) *and S be the support of x^*. If x^* admits* *the maximum cardinality of* $I(x)$, $x \in \Lambda$, *i.e.,* $|I(x^*)| = \max\{|I(x)| : x \in \Lambda\}$, *then*

$$M^* = \begin{bmatrix} A_S \\ B_{I,S} \end{bmatrix} \tag{2.6}$$

has a full column rank where I *is the index set of active constraints in* $Bx^* \leqslant b$, *i.e.,* $I = I(x^*)$.

Proof. Suppose that x^* is the sparsest solution to (1.1) which satisfies the assumption in Lemma 8. We now prove that $\text{Null}(M^*) = \{0\}$, i.e., M^* has a full column rank. We prove this fact by contradiction.

Assume that $\text{Null}(M^*) \neq \{0\}$. Then there exists a nonzero vector Δx with $\Delta x_{\bar{S}} = 0$ and $\Delta x_S \neq 0$ such that

$$A_S \Delta x_S = 0 \text{ and } B_{I,S} \Delta x_S = 0 \tag{2.7}$$

Then we construct a new vector $\bar{x}(\lambda)$ such that

$$\bar{x}(\lambda) = x^* + \lambda \Delta x$$

where λ is a parameter. Clearly, $\bar{x}(\lambda)$ continuously changes with λ and

$$\text{supp}(\bar{x}(\lambda)) \subseteq \text{supp}(x^*) \text{ and } \|\bar{x}(\lambda)\|_0 \leqslant \|x^*\|_0 \tag{2.8}$$

for all λ. If $\bar{x}(\lambda)$ satisfies the following system:

$$\|y - A_S z_S\|_2 \leqslant \epsilon, \quad B_{I,S} z_S \leqslant b_I, \quad B_{\bar{I},S} z_S \leqslant b_{\bar{I}}, \tag{2.9}$$

then $\bar{x}(\lambda)$ is a feasible solution to (1.1), and hence $\bar{x}(\lambda)$ is a sparsest solution to (1.1) which follows from (2.8) and the fact that x^* is a sparsest solution. We now prove that there exists a nonzero λ such that $\bar{x}(\lambda)$ satisfies the system (2.9). Based on (2.7), the following two constraints are satisfied for all λ:

$$\|y - A_S\bar{x}_S(\lambda)\|_2 \leqslant \epsilon, \quad B_{I,S}\bar{x}_S(\lambda) = b_I. \tag{2.10}$$

We only need to check if $\bar{x}(\lambda)$ satisfies the third inequality in (2.9). First we denote three disjoint sets J_+, J_-, J_0 as follows,

$$J_+ = \{j : (B_{\bar{I},S}\Delta x_S)_j > 0\}, \quad J_- = \{j : (B_{\bar{I},S}\Delta x_S)_j < 0\}, \quad J_0 = \{j : (B_{\bar{I},S}\Delta x_S)_j = 0\}. \tag{2.11}$$

Consider the possible two cases:

(M1) $B_{\bar{I},S}\Delta x_S = 0$. In this case $J_+ \cup J_- = \emptyset$. Since $\Delta x_S \in \mathrm{Null}(M^*)$, we have $\Delta x_S \in \mathrm{Null}(M^*) \cap \mathrm{Null}(B_{\bar{I},S})$. This contradicts to Theorem 6. Thus we actually have only the following case $B_{\bar{I},S}\Delta x_S \neq 0$.

(M2) $B_{\bar{I},S}\Delta x_S \neq 0$. In this case $J_+ \cup J_- \neq \emptyset$. Let $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ be continuously increased from $\lambda_{\min}$ to $\lambda_{\max}$ where

$$\lambda_{\max} = \min_{j \in J_+}\left\{\frac{(b_{\bar{I}} - B_{\bar{I},S}x_S^*)_j}{(B_{\bar{I},S}\Delta x_S)_j}\right\}, \quad \lambda_{\min} = \max_{j \in J_-}\left\{\frac{(b_{\bar{I}} - B_{\bar{I},S}x_S^*)_j}{(B_{\bar{I},S}\Delta x_S)_j}\right\}.$$

Clearly, from (2.11), we see that $\lambda_{\min} < 0$ and $\lambda_{\max} > 0$. For $\lambda \in (0, \lambda_{\max}]$, we have that:

$$(B_{\bar{I},S}\bar{x}_S(\lambda))_i \begin{cases} \leqslant (b_{\bar{I}})_i, & i \in J_+, \\ < (b_{\bar{I}})_i + \lambda * 0 = (b_{\bar{I}})_i, & i \in J_-, \\ < (b_{\bar{I}})_i & i \in J_0. \end{cases}$$

The above second and third inequalities are obvious, and the first inequality follows from the fact that for $i \in J_+$,

$$\begin{aligned}(B_{\bar{I},S}\bar{x}_S(\lambda))_i &= (B_{\bar{I},S}x_S^*)_i + \lambda(B_{\bar{I},S}\Delta x_S)_i \\ &\leqslant (B_{\bar{I},S}x_S^*)_i + \min_{j\in J_+}\{\tfrac{(b_{\bar{I}}-B_{\bar{I},S}x_S^*)_j}{(B_{\bar{I},S}\Delta x_S)_j}\}(B_{\bar{I},S}\Delta x_S)_i \\ &\leqslant (B_{\bar{I},S}x_S^*)_i + \tfrac{(b_{\bar{I}}-B_{\bar{I},S}x_S^*)_i}{(B_{\bar{I},S}\Delta x_S)_i}(B_{\bar{I},S}\Delta x_S)_i \\ &= (b_{\bar{I}})_i,\end{aligned}$$

where '$=$' in the first inequality holds when $\lambda = \lambda_{\max}$ and '$=$' in the second inequality holds when

$$i \in J' = \left\{ j : \frac{(b_{\bar{I}} - B_{\bar{I},S}x_S)_j}{(B_{\bar{I},S}\Delta x_S)_j} = z^*, j \in J_+ \right\}$$

where

$$z^* = \min_j \left\{ \frac{(b_{\bar{I}} - B_{\bar{I},S}x_S)_j}{(B_{\bar{I},S}\Delta x_S)_j} : j \in J_+ \right\}.$$

For $\lambda \in [\lambda_{\min}, 0)$, we have that

$$(B_{\bar{I},S}\bar{x}_S(\lambda))_i \begin{cases} < (b_{\bar{I}})_i + \lambda * 0 = (b_{\bar{I}})_i, & i \in J_+, \\ \leqslant (b_{\bar{I}})_i, & i \in J_-, \\ < (b_{\bar{I}})_i & i \in J_0, \end{cases}$$

where the second inequality follows from the fact that for $i \in J_-$,

$$\begin{aligned}(B_{\bar{I},S}\bar{x}_S)_i &= (B_{\bar{I},S}x_S^*)_i + \lambda(B_{\bar{I},S}\Delta x_S)_i, \\ &\leqslant (B_{\bar{I},S}x_S^*)_i + \max_{j\in J_-}\{\tfrac{(b_{\bar{I}}-B_{\bar{I},S}x_S^*)_j}{(B_{\bar{I},S}\Delta x_S)_j}\}(B_{\bar{I},S}\Delta x_S)_i, \\ &\leqslant (B_{\bar{I},S}x_S^*)_i + \tfrac{(b_{\bar{I}}-B_{\bar{I},S}x_S^*)_i}{(B_{\bar{I},S}\Delta x_S)_i}(B_{\bar{I},S}\Delta x_S)_i, \\ &= (b_{\bar{I}})_i,\end{aligned}$$

where '$=$' in the first inequality holds when $\lambda = \lambda_{\min}$ and '$=$' in the second inequality holds if the index of $(B_{\bar{I},S}\bar{x}_S)_i, i \in J_-$ ensures the value of $\frac{(b_{\bar{I}}-B_{\bar{I},S}x_S)_j}{(B_{\bar{I},S}\Delta x_S)_j}, j \in J_-$ to be maximal. Note that when $\lambda = 0$, $\bar{x}(\lambda) = x^*$. Thus we have

$$B_{\bar{I},S}\bar{x}_S(\lambda) \leqslant b_{\bar{I}}$$

for all $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. Combining this with (2.10), we see that $\bar{x}(\lambda) \neq x^*$ for all $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ satisfying (2.9) and hence $\bar{x}(\lambda)$ is a feasible solution to (1.1). Now starting from $\lambda = 0$, we continuously increase the value $|\lambda|$. In this process, if one component of $\bar{x}_S(\lambda)$ satisfying (2.9) becomes 0, then we find a sparser solution than x^* which contradicts the fact that x^* is a sparsest solution. Thus, without loss of generality, we assume $\text{supp}(\bar{x}(\lambda)) = \text{supp}(x^*)$ when $|\lambda|$ is increased continuously. Note that there exists a $\lambda^* \in [\lambda_{\min}, \lambda_{\max}]$ such that at least one index of inactive constraints in $B_S x_S^* \leq b$ will be added to the index set of active constraints in $B_S\bar{x}_S(\lambda^*) \leqslant b$. That is, the index set of active constraints in $B_S\bar{x}_S(\lambda^*) \leqslant b$ includes I and D, where

$$D = \{j : (B_{\bar{I},S}\bar{x}_S(\lambda^*))_j = (b_{\bar{I}})_j\}, \quad D \neq \emptyset.$$

This means $|I(\bar{x}(\lambda^*))| > |I(x^*)|$ which contradicts the fact that $I(x^*)$ has the maximum cardinality of $I(x)$ amongst all sparsest solutions of (1.1). In addition, from the above case $(M2)$, there are two extreme cases. That is,

(N1) the components of $B_{\bar{I},S}\Delta x_S$ are nonpositvie. In this case, $J_+ = \emptyset$ and $J_- \neq \emptyset$.

(N2) the components of $B_{\bar{I},S}\Delta x_S$ are nonnegative. In this case $J_- = \emptyset$ and $J_+ \neq \emptyset$.

In the first case (**N1**), $\bar{x}(\lambda)$ can be proven to be a sparsest solution of (1.1) when $\lambda \in [\lambda_{\min}, +\infty)$ while in the second case (**N2**), $\bar{x}(\lambda)$ can be proven to be a sparsest solution of (1.1) when $\lambda \in (-\infty, \lambda_{\max}]$. For example, consider the case (**N1**). For $\lambda \in [\lambda_{\min}, 0)$, $B_{\bar{I},S}\bar{x}_S(\lambda)$ has the components with

$$(B_{\bar{I},S}\bar{x}_S(\lambda))_i \leqslant (b_{\bar{I}})_i, i \in J_- \quad \text{and} \quad (B_{\bar{I},S}\bar{x}_S(\lambda))_i < (b_{\bar{I}})_i, i \in J_0,$$

while for $\lambda \in (0, +\infty)$, we have

$$(B_{\bar{I},S}\bar{x}_S(\lambda))_i < (b_{\bar{I}})_i, \quad i \in \bar{I} = (J_- \cup J_0).$$

Thus we have $B_{\bar{I},S}\bar{x}_S(\lambda) \leqslant b_{\bar{I}}$ when $\lambda \in [\lambda_{\min}, +\infty)$. This, combined with (2.10), yields $\bar{x}(\lambda)$ being the sparsest point in T. Similarly, without loss of generality, $\text{supp}(\bar{x}(\lambda)) = \text{supp}(x^*)$ is maintained when λ is increased continuously in $[\lambda_{\min}, +\infty)$. However, there also exists $\lambda^* \in [\lambda_{\min}, +\infty)$ such that $I(\bar{x}(\lambda^*)) > I(x^*)$ which contradicts the fact that x^* admits the maximum cardinality of $I(x), x \in \Lambda$.

In summary, the case $(M1)$ does not exist, and the case $(M2)$, including its two extreme cases (**N1**) and (**N2**), contradicts the fact that $I(x^*)$ has the maximum cardinality amongst $I(x), x \in \Lambda$. This contradiction shows that M^* given in (2.6) has full column rank. □

2.2 Multiplicity of sparsest solutions of model (P_0)

In section 2.2, we will characterize the solutions of (1.1). The sparsest solutions to (1.1) might not be unique when the null space of $(A^T, B^T)^T$ is not reduced to the zero vector. In fact, any slight perturbation of the problem data (A, B, b, ϵ) may lead to the non-uniqueness of the solutions to the modified problem. This means that in most cases, the sparsest solutions for the problem (1.1) are largely non-unique.

In this subsection, we show that (1.1) has infinitely many solutions with the same support under some mild conditions. Let x^* be a sparsest solution to (1.1) and S be the support of x^*. From Theorem 6, we know that

$$\text{Null}\begin{pmatrix} A_S \\ B_{I,S} \end{pmatrix} \cap \text{Null}(B_{\bar{I},S}) = \{0\}, \tag{2.12}$$

where I and $\bar{I}$ are the index sets of active and inactive constraints in $B_S x_S^* \leqslant b$ respectively.

Note that (2.12) can be separated into four cases:

$$\begin{cases} \text{Null}\binom{A_S}{B_{I,S}} \neq \{0\}, \ \text{Null}(B_{\bar{I},S}) = \{0\}, \\ \text{Null}\binom{A_S}{B_{I,S}} \neq \{0\}, \ \text{Null}(B_{\bar{I},S}) \neq \{0\} \quad \text{and } \text{Null}\binom{A_S}{B_S} = \{0\}, \\ \text{Null}\binom{A_S}{B_{I,S}} = \{0\}, \ \text{Null}(B_{\bar{I},S}) \neq \{0\}, \\ \text{Null}\binom{A_S}{B_{I,S}} = \{0\}, \ \text{Null}(B_{\bar{I},S}) = \{0\}. \end{cases} \tag{2.13}$$

Under some conditions, it can be shown that for each case in (2.13), there are many vectors satisfying the following system:

$$\|y - A_S z_S\|_2 \leqslant \epsilon, \quad B_{\bar{I},S} z_S \leqslant b_{\bar{I}}, \quad B_{I,S} z_S \leqslant b_I, \tag{2.14}$$

and therefore (1.1) has infinite sparsest solutions admitting the same support as that of the sparsest solution x^*, as indicated by the following Theorems 9 and 10. Theorem 9 covers the first three cases and Theorem 10 covers the last case in (2.13) respectively.

Theorem 9. *Let x^* be an arbitrary sparsest solution to* (1.1)*, S be the support of x^*. The problem* (1.1) *has infinitely many optimal solutions which have the same support as x^* if the following condition* $(C1)$ *holds:*

- $(C1)$. $\text{Null}\binom{A_S}{B_{I,S}} = \text{Null}(M^*) \neq \{0\}$ *and x^* does not admit the maximum cardinality, i.e.,* $|I(x^*)| \neq \max\{|I(z)| : z \in \Lambda\}$ *where Λ given in* (2.5) *is the optimal solution set of* (1.1).

If the corresponding error vector e^*, i.e., $e^* = y - Ax^*$, satisfies $\|e^*\|_2 < \epsilon$, then (1.1) *has infinitely many optimal solutions which have the same support as x^* if one of the following conditions* $(C2), (C3)$ and $(C4)$ *holds:*

- $(C2)$. $\text{Null}(M^*) = \{0\}$ *and* $\text{Null}(B_S) \neq \{0\}$.

- $(C3)$. $\text{Null}(M^*) = \{0\}$ *and* $\{d : B_{I,S} d > 0\} \cap \text{Null}(B_{\bar{I},S}) \neq \emptyset$.

- $(C4)$. $\text{Null}(M^*) = \{0\}$ *and* $\{d : B_{I,S} d < 0\} \cap \text{Null}(B_{\bar{I},S}) \neq \emptyset$.

Proof. **(C1).** Consider the case $(C1)$ in Theorem 9. It follows from Lemma 8 that the linear dependence of the columns of M^* implies that $I(x^*)$ does not have the maximum cardinality amongst $I(x), x \in \Lambda$. Therefore the condition in $(C1)$ is mild. We can find a nonzero Δx such that $\Delta x_S \in \text{Null}(M^*)$ and $\Delta x_{\bar{S}} = 0$, leading to

$$A_S \Delta x_S = 0 \quad \text{and} \quad B_{I,S} \Delta x_S = 0.$$

Due to (2.12), we know that $B_{\bar{I},S} \Delta x_S \neq 0$ no matter whether the null space of $B_{\bar{I},S}$ is only 0 or not. Let $z(\lambda)$ be a vector which is constructed as

$$z(\lambda) = x^* + \lambda \Delta x$$

where λ is a parameter. It is easy to check that $z_S(\lambda)$ satisfies

$$\|y - A_S z_S(\lambda)\|_2 \leqslant \epsilon \text{ and } B_{I,S} z_S(\lambda) = b_I.$$

Let the sets J_+, J_- and J_0 be still defined as the corresponding sets in (2.11) and $J_+ \cup J_- \neq \emptyset$. Let λ be restricted in $[\lambda_{\min}, \lambda_{\max}]$ where

$$\lambda_{\max} = \min_{j \in J_+} \{ \frac{(b_{\bar{I}} - B_{\bar{I},S} x_S^*)_j}{(B_{\bar{I},S} \Delta x_S)_j} \}, \quad \lambda_{\min} = \max_{j \in J_-} \{ \frac{(b_{\bar{I}} - B_{\bar{I},S} x_S^*)_j}{(B_{\bar{I},S} \Delta x_S)_j} \}.$$

Similar to the case $(M2)$ in the proof of Lemma 8, it can be proven that for all $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, we have

$$B_{\bar{I},S} z_S(\lambda) \leqslant b_{\bar{I}}.$$

Then $z(\lambda)$ is a feasible solution to (1.1) when $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, which together with the fact that x^* is a sparsest solution and $\text{supp}(z(\lambda)) \subseteq \text{supp}(x^*)$, implies for all $\lambda \in [\lambda_{\min}, \lambda_{\max}]$ $z(\lambda)$ is a sparsest solution of (1.1) and hence

$$\text{supp}(x^*) = \text{supp}(z(\lambda)).$$

Since $z(\lambda)$ varies when λ is changed continuously in the interval $[-\lambda_{\max}, \lambda_{\max}]$, it implies that (1.1) has infinitely many sparsest solutions with the same support as x^*.

(C2). Consider the case $(C2)$ in Theorem 9. We choose a nonzero vector μ from the set $\text{Null}(B_S)$ such that

$$B_{\bar{I},S}\mu = 0, \quad B_{I,S}\mu = 0.$$

Due to (2.12), we have $A_S\mu \neq 0$. Let $z(\lambda)$ be a vector with components

$$z_S(\lambda) = x_S^* + \lambda\mu \quad \text{and} \quad z_{\bar{S}}(\lambda) = 0.$$

Then we have $B_{\bar{I},S}z_S(\lambda) < b_{\bar{I}}$ and $B_{I,S}z_S(\lambda) = b_I$ for all λ which imply

$$B_S z_S(\lambda) \leqslant b.$$

Let $|\lambda|$ be restricted in $(0, \lambda'_{\max}]$ with

$$\lambda'_{\max} = \frac{\epsilon - \|e^*\|_2}{\|A_S\mu\|_\infty \sqrt{m}},$$

and $e^* = y - A_S x_S^*$. We have

$$\begin{aligned} \|y - A_S(x_S^* + \lambda\mu)\|_2 &= \|e^* - \lambda A_S\mu\|_2\,, \\ &\leqslant \|e^*\|_2 + |\lambda|\,\|A_S\mu\|_2\,, \\ &\leqslant \|e^*\|_2 + \tfrac{\epsilon - \|e^*\|_2}{\|A_S\mu\|_\infty \sqrt{m}}\,\|A_S\mu\|_2\,, \\ &= \|e^*\|_2 + \tfrac{\epsilon - \|e^*\|_2}{\sqrt{m}}\,\|A_S\mu / \|A_S\mu\|_\infty\|_2\,, \\ &\leqslant \|e^*\|_2 + \tfrac{\epsilon - \|e^*\|_2}{\sqrt{m}}\,\|e^m\|_2 = \epsilon, \end{aligned}$$

where the first inequality follows from the triangle equality and e^m is the vector of ones with m dimension. Combining this with the fact $B_S z_S(\lambda) \leqslant b$ implies that $z(\lambda)$ satisfies (2.14) and hence $z(\lambda)$ is a feasible solution of (1.1) when $\lambda \in [0, \lambda'_{\max}]$. Same as the proof in $(C1)$, it implies that $z(\lambda)$ is a sparsest solution of (1.1) when $\lambda \in [0, \lambda'_{\max}]$, and hence

(1.1) has infinitely many sparsest solutions which have the same support as x^*. Moreover, the active and inactive indices in $Bz(\lambda) \leqslant b$ are the same as that in $Bx^* \leqslant b$.

(C3). Consider the case $(C3)$ in Theorem 9. We can find a nonzero vector ξ from the set $\{d : B_{I,S}d > 0\} \cap \mathrm{Null}(B_{\bar{I},S})$ satisfying

$$B_{\bar{I},S}\xi = 0 \text{ and } B_{I,S}\xi > 0.$$

Since the two cases $A_S\xi = 0$ and $A_S\xi \neq 0$ do not contradict $\mathrm{Null}(M^*) = \{0\}$, we consider both of them. Let $v(\lambda)$ be a vector with components

$$v_S(\lambda) = x_S^* + \lambda\xi \quad \text{and} \quad v_{\bar{S}}(\lambda) = 0,$$

where λ is a parameter. Clearly, $\mathrm{supp}(v(\lambda)) \subseteq \mathrm{supp}(x^*)$ for λ. Now we claim that $v(\lambda)$ is a sparsest solution to (1.1) in both cases of $A_S\xi = 0$ and $A_S\xi \neq 0$ when λ is restricted in certain interval.

- 1). $A_S\xi \neq 0$

  When $\lambda \in [-\lambda''_{\max}, 0)$ with $\lambda''_{\max} = \frac{\epsilon - \|e^*\|_2}{\|A_S\xi\|_\infty \sqrt{m}}$, by the same proof as in $(C2)$, we have

  $$\|y - A_S v_S(\lambda)\|_2 \leqslant \epsilon.$$

  It is easy to check that

  $$B_{I,S}v_S(\lambda) < b_I \quad \text{and} \quad B_{\bar{I},S}v_S(\lambda) < b_{\bar{I}}.$$

  Thus $v(\lambda)$ satisfies (2.14), and therefore $v(\lambda)$ is feasible in T for all $\lambda \in [-\lambda''_{\max}, 0]$. $\mathrm{supp}(v(\lambda)) \subseteq \mathrm{supp}(x^*)$ and the fact that x^* is a sparsest point in T imply that $v(\lambda)$ is a sparsest point in T when $\lambda \in [-\lambda''_{\max}, 0]$, and $\mathrm{supp}(x^*) = \mathrm{supp}(v(\lambda))$.

- 2). $A_S\xi = 0$

In this case, λ can be any negative number so that $v(\lambda)$ is a feasible point in T. Similarly, $v(\lambda)$ is a sparsest solution to (1.1) when $\lambda \in (-\infty, 0]$. Combining 1) and 2) implies that (1.1) has infinitely many sparsest solutions which have the same support as that of x^* for the case $(C3)$.

(C4). This proof is omitted. Note that $\{d : B_{I,S}d > 0\} \cap \text{Null}(B_{\bar{I},S}) \neq \emptyset$ is equivalent to $\{d : B_{I,S}d < 0\} \cap \text{Null}(B_{\bar{I},S}) \neq \emptyset$. Thus we can directly get the desired result. □

Note that the case $(C1)$ corresponds to the first case in (2.13), and the cases $(C2)-(C4)$ correspond to the third case in (2.13). Now we consider the last case in (2.13) and show that (1.8) has infinitely many sparsest solutions with the same support under some mild conditions, as indicated by the following theorem.

Theorem 10. *Let x^* be an arbitrary sparsest solution of* (1.1)*, S be the support of x^*. Assume that* $\text{Null}(M^*) = \{0\}$ *and* $\text{Null}(B_{\bar{I},S}) = \{0\}$*. Then* (1.1) *has infinitely many optimal solutions with the same support as x^* if one of the following conditions holds:*

- $(D1)$. $\{d : B_{I,S}d > 0\} \cap \{d : A_S d = 0\} \neq \emptyset$.
- $(D2)$. $\{d : B_{I,S}d < 0\} \cap \{d : A_S d = 0\} \neq \emptyset$.

If the corresponding error vector e, i.e., $e^* = y - Ax^*$, satisfies $\|e^*\|_2 < \epsilon$, then (1.1) *has infinitely many optimal solutions which have the same support as x^* if one of the following conditions holds:*

- $(D3)$. $\text{Null}(B_{I,S}) \neq \{0\}$.
- $(D4)$. $\{d : B_{I,S}d > 0\} \cap \{d : A_S d \neq 0\} \neq \emptyset$.
- $(D5)$. $\{d : B_{I,S}d < 0\} \cap \{d : A_S d \neq 0\} \neq \emptyset$.

Proof. We start from $(D3)$.

(D3). Since $\text{Null}(M^*) = \{0\}$ and $\text{Null}(B_{I,S}) \neq \{0\}$, for $\forall d_1 \in \text{Null}(B_{I,S})$, we have

$$B_{I,S}d_1 = 0 \text{ and } A_S d_1 \neq 0.$$

Since $\text{Null}(B_{\bar{I},S}) = \{0\}$, we have $B_{\bar{I},S}d_1 \neq 0$. Denote

$$J_0 = \{j : (B_{\bar{I},S}d_1)_j = 0\},\ J_- = \{j : (B_{\bar{I},S}d_1)_j < 0\},\ J_+ = \{j : (B_{\bar{I},S}d_1)_j > 0\}.$$

Clearly, $J_+ \cup J_- \neq \emptyset$ and J_+, J_- and J_0 are disjoint. Let $z(\lambda)$ be a vector with components

$$z_S(\lambda) = x^*_S + \lambda d_1 \text{ and } z_{\bar{S}}(\lambda) = 0.$$

Clearly, $\text{supp}(z(\lambda)) \subseteq \text{supp}(x^*)$ for all λ. Let $|\lambda|$ be restricted in $(0, \min(\lambda_1, \lambda_2)]$ where

$$\lambda_1 = \min_{j \in J_+ \cup J_-} \frac{(b_{\bar{I}} - B_{\bar{I},S}x^*_S)_j}{|(B_{\bar{I},S}d_1)|_j}, \quad \lambda_2 = \frac{\epsilon - \|e^*\|_2}{\|A_S d_1\|_\infty \sqrt{m}}.$$

For $i \in J_+ \cup J_-$,

$$\begin{aligned} (B_{\bar{I},S}z_S(\lambda))_i &= (B_{\bar{I},S}x^*_S)_i + \lambda(B_{\bar{I},S}d_1)_i \leqslant (B_{\bar{I},S}x^*_S)_i + |\lambda(B_{\bar{I},S}d_1)_i| \\ &\leqslant (B_{\bar{I},S}x^*_S)_i + |\lambda||(B_{\bar{I},S}d_1)_i| \leqslant (B_{\bar{I},S}x^*_S)_i + \lambda_1|(B_{\bar{I},S}d_1)_i| \\ &= (B_{\bar{I},S}x^*_S)_i + \tfrac{(b_{\bar{I}} - B_{\bar{I},S}x^*_S)_i}{|(B_{\bar{I},S}d_1)|_i}|(B_{\bar{I},S}d_1)_i| \\ &\leqslant (b_{\bar{I}})_i. \end{aligned}$$

The above fact, combined with $(B_{\bar{I},S}z_S(\lambda))_i < (b_{\bar{I}})_i, i \in J_0$, implies that $B_{\bar{I},S}z_S(\lambda) \leqslant b_{\bar{I}}$. We also have $\|y - A_S z_S(\lambda)\|_2 \leqslant \epsilon$ which has been proven for many times in Theorem 9. These, combined with the fact that $B_{I,S}z_S(\lambda) = b_I$, implies that $z(\lambda)$ is a feasible point in T when $\lambda \in [0, \min(\lambda_1, \lambda_2)]$. Since x^* is a sparsest point in T and $\text{supp}(z(\lambda)) \subseteq \text{supp}(x^*)$, $z(\lambda)$ is a sparsest point in T with the same support as x^* when $\lambda \in [0, \min(\lambda_1, \lambda_2)]$, i.e., $\text{supp}(x^*) = \text{supp}(z(\lambda))$.

(D4). Clearly, there exists a nonzero vector d' such that

$$B_{I,S}d' > 0, \quad A_S d' \neq 0.$$

Since $\text{Null}(B_{\bar{I},S}) = \{0\}$, we have $B_{\bar{I},S}d' \neq 0$. Denote

$$J'_0 = \{j : (B_{\bar{I},S}d')_j = 0\}, \quad J'_- = \{j : (B_{\bar{I},S}d')_j < 0\}, \quad J'_+ = \{j : (B_{\bar{I},S}d')_j > 0\}.$$

Clearly, $J'_+ \cup J'_- \neq \emptyset$ and J'_+, J'_- and J'_0 are disjoint. Let $z'(\lambda)$ be a vector with components

$$z'_S(\lambda) = x^*_S + \lambda d' \text{ and } z'_{\bar{S}}(\lambda) = 0.$$

Obviously, $\text{supp}(z'(\lambda)) \subseteq \text{supp}(x^*)$ for all λ. Let λ be restricted in $[\max(\lambda'_1, \lambda'_2), 0)$ where

$$\lambda'_1 = \max_{j \in J'_-} \frac{(b_{\bar{I}} - B_{\bar{I},S}x^*_S)_j}{(B_{\bar{I},S}d')_j}, \quad \lambda'_2 = -\frac{(\epsilon - \|e^*\|_2)}{\|A_S d'\|_\infty \sqrt{m}}.$$

For $i \in J'_-$, we have

$$\begin{aligned} (B_{\bar{I},S}z'_S(\lambda))_i &= (B_{\bar{I},S}x^*_S)_i + \lambda(B_{\bar{I},S}d')_i, \\ &\leqslant (B_{\bar{I},S}x^*_S)_i + \lambda'_1(B_{\bar{I},S}d')_i, \\ &\leqslant (B_{\bar{I},S}x^*_S)_i + \tfrac{(b_{\bar{I}} - B_{\bar{I},S}x^*_S)_i}{(B_{\bar{I},S}d')_i}(B_{\bar{I},S}d')_i, \\ &\leqslant (b_{\bar{I}})_i. \end{aligned}$$

For $i \in J'_+ \cup J'_0$, we have $(B_{\bar{I},S}z'_S(\lambda))_i < (b_{\bar{I}})_i$. It can be proven that $\|y - A_S z'_S(\lambda)\|_2 \leqslant \epsilon$ for $\lambda \in [\max(\lambda'_1, \lambda'_2), 0)$, which combined with the fact $B_{I,S}z'_S(\lambda) < b_I$ implies that $z'(\lambda)$ is a sparsest point in T with the same support as x^* when $\lambda \in [\max(\lambda'_1, \lambda'_2), 0]$, i.e., $\text{supp}(x^*) = \text{supp}(z'(\lambda))$. Thus (1.1) has infinitely many optimal solutions in this case.

(D1). Clearly, there exists a nonzero vector d'' such that

$$B_{I,S}d'' > 0, \quad A_S d'' = 0.$$

Since $\text{Null}(B_{\bar{I},S}) = \{0\}$, we have $B_{\bar{I},S}d'' \neq 0$. Denote

$$J''_0 = \{j : (B_{\bar{I},S}d'')_j = 0\}, \quad J''_- = \{j : (B_{\bar{I},S}d'')_j < 0\}, \quad J''_+ = \{j : (B_{\bar{I},S}d'')_j > 0\}.$$

Clearly, $J''_+ \cup J''_- \neq \emptyset$ and J''_+, J''_- and J''_0 are disjoint. Let $z''(\lambda)$ be a vector with components

$$z''_S(\lambda) = x^*_S + \lambda d'' \text{ and } z''_{\bar{S}}(\lambda) = 0.$$

Clearly, $\text{supp}(z''(\lambda)) \subseteq \text{supp}(x^*)$ for all λ. Due to $A_S d'' = 0$, $\|y - A_S z''_S(\lambda)\|_2 \leqslant \epsilon$ is satisfied. Let λ be restricted in $[\lambda''_1, 0)$ where

$$\lambda''_1 = \max_{j \in J''_-} \frac{(b_{\bar{I}} - B_{\bar{I},S} x^*_S)_j}{(B_{\bar{I},S} d'')_j}.$$

For $i \in J''_-$, we have

$$\begin{aligned}
(B_{\bar{I},S} z''_S(\lambda))_i &= (B_{\bar{I},S} x^*_S)_i + \lambda (B_{\bar{I},S} d'')_i, \\
&\leqslant (B_{\bar{I},S} x^*_S)_i + \lambda''_1 (B_{\bar{I},S} d'')_i, \\
&\leqslant (B_{\bar{I},S} x^*_S)_i + \tfrac{(b_{\bar{I}} - B_{\bar{I},S} x^*_S)_i}{(B_{\bar{I},S} d'')_i} (B_{\bar{I},S} d'')_i, \\
&\leqslant (b_{\bar{I}})_i.
\end{aligned}$$

For $j \in J''_+ \cup J''_0$, we have $(B_{\bar{I},S} z''_S(\lambda))_i < (b_{\bar{I}})_i$. The fact $B_{I,S} z''_S(\lambda) < b_I$ and $\|y - A_S z''_S(\lambda)\|_2 \leqslant \epsilon$ implies that $z''(\lambda)$ is a sparsest point in T with the same support as x^* when $\lambda \in [\lambda''_1, 0]$, i.e., $\text{supp}(x^*) = \text{supp}(z''(\lambda))$. Thus (1.1) has infinitely many sparsest solutions in this case.

(D2,5). The proof of the case 2 and the case 5 are omitted. Note that $(D2)$ is equivalent to $(D1)$ and that $(D5)$ is equivalent to $(D4)$. Thus the desired results can be obtained immediately. □

Through the above theoretical analysis, we know that (1.1) may have infinitely many sparsest solutions. We also want to know whether the sparsest solution set Λ given in (2.5) is bounded or not. This question will be explored in the subsection 2.3. Now, the example below is given to illustrate the results of Theorems 9 and 10.

Example 11. *Consider the system* $\|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b$ *with* $\epsilon = 10^{-1}$ *and*

$$A = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 4 & -9 \\ 1 & 0 & -2 & 5 \end{bmatrix}, B = \begin{bmatrix} -0.5 & 0 & 1 & -2.5 \\ 0.5 & -0.5 & -1 & 2 \\ -3 & -3 & -2 & 3 \end{bmatrix}, y = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} -0.5 \\ 1 \\ -1 \end{bmatrix}.$$

It can be seen that $(0, 0, 2, 1)^T$ and $(0, 1, -1/2, 0)^T$ are the sparsest solutions to the above convex system. Next, we show that the above two sparsest solutions satisfy the corresponding assumptions in Theorems 9 and 10 respectively, and hence the convex system has infinitely many sparsest solutions with identical support to the two solutions.

- $x = (0, 0, 2, 1)^T$: We have $A_S = \begin{bmatrix} -2 & 5 \\ 4 & -9 \\ -2 & 5 \end{bmatrix}$, $B_{I,S} = \begin{bmatrix} 1 & -2.5 \\ -2 & 3 \end{bmatrix}$ and $B_{\bar{I},S} = \begin{bmatrix} -1 & 2 \end{bmatrix}$. We can see that

  $$\text{Null}(A_S) = \{0\}, \quad \text{Null}(B_{I,S}) = \{0\}, \quad \text{Null}(B_{\bar{I},S}) \neq \{0\},$$

  and

  $$(2, 1)^T \in \{d : B_{I,S}d < 0\} \cap \text{Null}(B_{\bar{I},S}), \quad (-2, -1)^T \in \{d : B_{I,S}d > 0\} \cap \text{Null}(B_{\bar{I},S})$$

  which satisfy $(C4)$ and $(C3)$ in Theorem 9. For both cases, the value of λ in the proof of $(C4)$ or $(C3)$ can be determined, i.e.

  $$\lambda \in (0, 1/10\sqrt{3}] \quad \text{for} \quad (2, 1)^T, \quad \lambda \in [-1/10\sqrt{3}, 0) \quad \text{for} \quad (-2, -1)^T.$$

  Then another sparsest solution can be formed as

  $$(0, 0, 2, 1)^T + \lambda(0, 0, 2, 1)^T, \lambda \in (0, 1/10\sqrt{3}],$$

and hence the system $\|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b$ in this example has infinitely many sparsest solutions. If λ takes its maximum absolute value, then another sparsest solution is $(0, 0, 2 + \frac{1}{5\sqrt{3}}, 1 + \frac{1}{10\sqrt{3}})^T$ since

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - A \begin{bmatrix} 0 \\ 0 \\ 2 + \frac{1}{5\sqrt{3}} \\ 1 + \frac{1}{10\sqrt{3}} \end{bmatrix} \right\|_2 = 10^{-1} = \epsilon, \quad B \begin{bmatrix} 0 \\ 0 \\ 2 + \frac{1}{5\sqrt{3}} \\ 1 + \frac{1}{10\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -0.5 - 1/20\sqrt{3} \\ 0 \\ -1 - 1/10\sqrt{3} \end{bmatrix} < b.$$

- $x = (0, 1, -1/2, 0)^T$: We have $A_S = \begin{bmatrix} 0 & -2 \\ 1 & 4 \\ 0 & -2 \end{bmatrix}$, $B_{I,S} = (0, 1)$ and $B_{\bar{I},S} = \begin{bmatrix} -0.5 & -1 \\ -3 & -2 \end{bmatrix}$.

  It is easy to check

$$\mathrm{Null}(A_S) = \mathrm{Null}(B_{\bar{I},S}) = \{0\} \text{ and } \mathrm{Null}(B_{I,S}) \neq \{0\}$$

  so that this example satisfies $\mathrm{Null}(M^*) = \{0\}$ and $\mathrm{Null}(B_{\bar{I},S}) = \{0\}$. We can find two vectors which meet $(D5)$ and $(D4)$ in Theorem 10, i.e,

$$(4, -1)^T \in \{d : B_{I,S}d < 0\} \cap \{d : A_S d \neq 0\}, \quad B_{\bar{I},S}d \neq 0$$

  and

$$(-4, 1)^T \in \{d : B_{I,S}d > 0\} \cap \{d : A_S d \neq 0\}, \quad B_{\bar{I},S}d \neq 0.$$

  Then the value of λ in the proof of $(D5)$ or $(D4)$ can be determined. Analogously, for all $\lambda \in [\max(-1/10, -1/20\sqrt{3}), 0]$, the vector $(0, 1, -1/2, 0)^T + \lambda(0, -4, 1, 0)^T$ is a sparsest point in T. Note that $\mathrm{Null}(B_{I,S}) \neq \{0\}$, which meets $(D3)$ in Theorem 10. We can find $(1, 0)^T \in \mathrm{Null}(B_{I,S})$, and therefore λ_1 and λ_2 in the proof of $(D3)$ can be determined. Consequently, for all λ such that $|\lambda| \in [0, 1/10\sqrt{3}]$, the vector $(0, 1, -1/2, 0)^T + \lambda(0, 1, 0, 0)^T$ is a sparsest point in T. Thus both cases show that

(1.1) has infinitely many sparsest solutions.

In the next subsection, we will show the following property: For any vector $x \in R^n$ in Λ where Λ is the solution set of (1.1), there exists a positive lower bound γ^* such that

$$|x_i| \geq \gamma^*, \text{ for } x_i \neq 0$$

when Λ is bounded. The sufficient conditions for the boundedness of Λ will also be identified.

2.3 Lower bound for nonzero absolute entries of solutions

We start to discuss the lower bound on the absolute value of nonzero components of vectors in

$$\Lambda := \{x \in R^n : \|x\|_0 = k, \ x \in T\}$$

where k is the optimum value of (1.1). We only consider the case that Λ is bounded.

Lemma 12. *If the sparse solution set Λ is bounded, then there exists a positive lower bound γ^* for the nonzero component x_i of any vector x in Λ, that is,*

$$|x_i| \geq \gamma^*, \ i \in \text{supp}(x). \tag{2.15}$$

Proof. We prove this result by considering only two situations: Λ is finite or Λ is infinite. (i) Let the set Λ be finite and bounded. Denote the cardinality of Λ as L and the sparsest solutions of (1.1) as $\{x^k\}$, where $1 \leqslant k \leqslant L$. Obviously, we can find the minimum value among the nonzero absolute entries of all vectors in Λ and set such a minimal value as γ^*, which is expressed as

$$\gamma^* = \min_{1 \leq k \leq L} \min_{i \in \text{supp}(x^k)} |x_i^k|. \tag{2.16}$$

This implies that the absolute values of the nonzero components of vectors in Λ have a positive lower bound γ^*.

(ii) Let the set Λ be infinite and bounded. In this case, L is an infinite number. Since Λ is bounded, there exists a positive number U such that the absolute value of all entries of vectors in Λ is less or equal than U. We assume that (2.15) does not hold for $x \in \Lambda$. This means there exists a sequence $\{x^k\} \in \Lambda$, such that the minimum nonzero absolute entries of x^k approach to 0, i.e.,

$$\min_{i \in \mathrm{supp}(x^k)} |x_i^k| \to 0 \ \text{ as } \ k \to \infty. \tag{2.17}$$

By Λ is bounded, this implies that

$$|x_i^k| \leqslant U, \quad i \in \mathrm{supp}(x^k).$$

Following by Bolzano-Weierstrass Theorem, the sequence $\{x^k\}$ has at least one convergent subsequence, denoted still by $\{x^k\}$, with a limit point $x^* \in T$ satisfying $\|x^*\|_0 \leqslant k-1$. This is a contradiction, and hence the lower bound is ensured when Λ is infinite and bounded. Combining (i) and (ii) obtains the desired result. □

The above lemma ensures the existence of a positive lower bound for the absolute value of the nonzero components of the vectors in Λ when Λ is bounded. In the following lemma, some sufficient conditions are developed to guarantee that the sparsest solution set Λ is bounded.

Lemma 13. *Let k be the optimal value of* (1.1). *The sparse solution set Λ is bounded if one of the following conditions holds:*

- $(E1)$ *For any $\Pi \subseteq \{1, ..., n\}$ and $|\Pi| = k$, we have*

$$\{\eta : A_\Pi \eta = 0\} \cap \{\eta : B_\Pi \eta \leqslant 0\} = \{0\}. \tag{2.18}$$

- $(E2)$ *Any k columns in A are linearly independent.*

- *(E3) $k < spark(A)$, where $spark(A)$ denote the minimum number of linearly dependent columns in A.*

Proof. First of all, we suppose that the solution set Λ is unbounded. There exists a sequence of the sparsest solutions of (1.1), denoted by $\{x^k\}$, satisfying the following properties:

$$\|x^k\|_\infty \to \infty$$

and there is a fixed index set S_1 such that

$$|x_i^k| \to \infty \text{ for all } i \in S_1, \text{ as } k \to \infty \tag{2.19}$$

and the remaining components $x_i^k, i \in \text{supp}(x^k) \setminus S_1$ are bounded. Denote by $S_2 = \text{supp}(x^k) \setminus S_1$. Based on the fact that x^k satisfies the constraints in (1.1), we have

$$\|A_{S_2} x_{S_2}^k + A_{S_1} x_{S_1}^k - y\|_2 \leqslant \epsilon$$

and

$$B_{S_2} x_{S_2}^k + B_{S_1} x_{S_1}^k \leqslant b.$$

We divide the above two inequalities by $\|x_{S_1}^k\|_2$ to obtain

$$\frac{\|A_{S_2} x_{S_2}^k + A_{S_1} x_{S_1}^k - y\|_2}{\|x_{S_1}^k\|_2} \leqslant \frac{\epsilon}{\|x_{S_1}^k\|_2}, \qquad \frac{B_{S_2} x_{S_2}^k + B_{S_1} x_{S_1}^k}{\|x_{S_1}^k\|_2} \leqslant \frac{b}{\|x_{S_1}^k\|_2}.$$

Then we have

$$\left\| A_{S_2} \frac{x_{S_2}^k}{\|x_{S_1}^k\|_2} + A_{S_1} \bar{\eta} - \frac{y}{\|x_{S_1}^k\|_2} \right\|_2 \leqslant \frac{\epsilon}{\|x_{S_1}^k\|_2}, \qquad B_{S_2} \frac{x_{S_2}^k}{\|x_{S_1}^k\|_2} + B_{S_1} \bar{\eta} \leqslant \frac{b}{\|x_{S_1}^k\|_2},$$

where $\bar{\eta}$ is a unit vector in $R^{|S_1|}$. Note that

$$\lim_{k\to\infty} \frac{x_{S_2}^k}{\|x_{S_1}^k\|_2} = 0, \quad \lim_{k\to\infty} \frac{y}{\|x_{S_1}^k\|_2} = 0, \quad \lim_{k\to\infty} \frac{b}{\|x_{S_1}^k\|_2} = 0, \quad \lim_{k\to\infty} \frac{\epsilon}{\|x_{S_1}^k\|_2} = 0.$$

Thus there exists a unit vector $\bar{\eta} \in R^{|S_1|}$ satisfying

$$A_{S_1}\bar{\eta} = 0, \quad B_{S_1}\bar{\eta} \leqslant 0.$$

This means

$$\left\{\eta : A_{S_1}\eta = 0\right\} \cap \left\{\eta : B_{S_1}\eta \leqslant 0\right\} \neq \{0\}. \tag{2.20}$$

which contradicts to the assumption (2.18). Thus under (2.18), Λ is bounded (see Lemma 12). It is clear that if any k columns of A are linearly independent or $k < spark(A)$, then the set $\{\eta : A_{\Pi}\eta = 0\} = \{0\}$ and thus (2.18) holds. Hence the second and third conditions in Lemma 13 can also ensure Λ to be bounded. □

In summary, the boundedness of Λ can be ensured under some conditions, and a lower bound for nonzero absolute entries of points in Λ exists when Λ is bounded. After discussing the properties of the vectors in Λ, including the multiplicity of the sparsest points and the existence of a positive lower bound for absolute values of nonzero components of the vectors in Λ, in the next subsection we introduce some approximation to the problem (1.1), such as ℓ_0^δ-minimization and ℓ_1-minimization.

2.4 Approximation of general ℓ_0-minimization

Based on the discussions in 2.2, we know that (1.1) usually has many optimal solutions. In this subsection, we further point out that it is possible to find a feasible solution for (1.1) with small "tails" which means the value of some nonzero components in such a feasible solution are nearly close to 0. For example, if x is a sparsest solution to (1.1), then it might be possible to find a feasible point in T, denoted as $\bar{x}$ with components

$$\bar{x}_i \approx x_i, \quad i \in S; \quad |\bar{x}_i| \leq \delta, \quad i \notin S,$$

where δ is a sufficiently small number. Note that $\bar{x}$ can be compressed since the magnitude of the "tail" part is sufficiently small. If the approximate solution we obtained satisfies this property, then these approximate solutions are worthy to be considered instead of being discarded. For the convenience of later discussion, we now give some definitions about the thresholding operator and the ℓ_0^δ-norm.

Definition 14. [43] Given a threshold $\delta > 0$, the thresholding operator $H_\delta(z)$ is defined as

$$H_\delta(z) = \begin{cases} z_i, & |z_i| > \delta, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.21}$$

$H_\delta(z)$ is called δ-thresholding of z, which can be obtained by setting the value of components indexed by $\{i : |z_i| \leqslant \delta\}$ to 0 and keeping the value of components indexed by $\{i : |z_i| > \delta\}$. We also call $H_\delta(z)$ the δ-compression of z. The parameter δ is often a sufficiently small positive number and can be determined by practical demands. Another common compression operator, called soft thresholding operator, is defined as

$$[S_\delta(z)]_i = \text{sign}(z_i)(|z_i| - \delta)_+,$$

which has been widely used (see, e.g. [43] and [82]). Now we introduce ℓ_0^δ-norm:

Definition 15. *Given a $z \in R^n$, $\|z\|_{\widetilde{0},\delta}$ is called ℓ_0^δ-norm of z, where*

$$\|z\|_{\widetilde{0},\delta} = n - |\{i : |z_i| \leqslant \delta\}|, \tag{2.22}$$

where $|\{i : |z_i| \leqslant \delta\}|$ represents the cardinality of the set $\{i : |z_i| \leqslant \delta\}$.

Note that

$$\|z\|_{\widetilde{0},\delta} = \|H_\delta(z)\|_0 \,.$$

The ℓ_0^δ-norm can be used to choose some dense vectors with small "tails" as the candidates of the solution of the problem (1.1) when δ is properly chosen. For example, if $x = (100, 1, 0, 0)^T$ is the optimal solution of (1.1) and $\widetilde{x} = (99.98, 1.01, 10^{-4}, 10^{-5})^T$ is the optimal solution obtained by a decoding algorithm, due to the fact $\|x\|_0 = \|\widetilde{x}\|_{\widetilde{0},\delta}$ when $\delta \in [10^{-4}, 1.01)$, we see that $(99.98, 1.01, 10^{-4}, 10^{-5})^T$ can be compressed to $(99.98, 1.01, 0, 0)^T$

by $H_\delta(\widetilde{x})$ with $\delta \in [10^{-4}, 1.01)$. As a result, $\widetilde{x}$ can be seen as an approximate sparse solution of $(100, 1, 0, 0)^T$. Replacing ℓ_0-norm by ℓ_0^δ-norm yields an approximation model of (1.1):

$$\begin{aligned} (P_{\widetilde{0}}) \quad \min \quad & \|x\|_{\widetilde{0},\delta} \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \\ & Bx \leqslant b, \end{aligned} \tag{2.23}$$

which is referred to as the general ℓ_0^δ-minimization problem. Note that the two problems (1.1) and (2.23) are equivalent when $\delta \to 0$, therefore the problem (2.23) can be also regarded as the relaxation of the problem (1.1). The following ℓ_1-minimization is a convex method for solving the ℓ_0-minimization (1.1):

$$\begin{aligned} (P_1) \quad \min \quad & \|x\|_1 \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \\ & Bx \leqslant b, \end{aligned} \tag{2.24}$$

Clearly, this ℓ_1-minimization is also a convex method for solving (2.23) if δ is sufficiently small. In the next chapter, we focus on the properties of the general weighted ℓ_1-minimization problem for solving (1.1), and we will develop a bilevel optimization problem for determining the so-called optimal weights so that the weighted ℓ_1-minimization can solve the problem (1.1), in theory.

Chapter 3

General Weighted ℓ_1-minimization: Existence and Bilevel Model for Optimal Weights

The weighted ℓ_1-minimization counterpart for (1.1) and its Lagrangian dual are analyzed in this chapter. Some properties of the weighted ℓ_1-problem and its dual problem, such as strong duality and complementary condition, will be discussed in details. Moreover, the strict complementarity condition for the weighted ℓ_1-problem is also developed under some assumption. Following the ideas in Zhao & Kočvara [96] and Zhao & Luo [99] (see also Zhao [94]), using the strict complementarity can prove that locating a sparsest solution of our sparsity model can amount to finding the densest optimal slack variable of the dual problem of the weighted ℓ_1-minimization through the so-called optimal weights. As a result, the general ℓ_0-minimization model can be converted to certain ℓ_0-maximization in dual space with non-convex constraints, which can be reformulated as a certain bilevel programming problem. Finally, under a mild assumption (see Assumption 34 for details), it can be proven that the bilevel programming problem can provide a weight by which the solution of the ℓ_0-model can be obtained via weighted ℓ_1-minimization. Such a weight is referred to as an optimal weight. This provides a theoretical basis for the development of new re-weighted ℓ_1 algorithms for general sparsity models.

This chapter is organized as follows. The weighted ℓ_1-minimization counterpart of (1.1) is introduced in Section 3.2, and the existence of an optimal weight for general weighted ℓ_1-minimization is shown in Section 3.3. The Lagrangian dual problem of the weighted ℓ_1-minimization, strong duality, strict complementarity and optimality condition are discussed in Section 3.4. Finally, the bilevel programming model for determining the optimal weights so that the weighted ℓ_1-minimization can guarantee to solve the sparsity model is presented in Section 3.5.

3.1 Introduction

For weighted ℓ_1-minimization, how to determine a weight to guarantee the exact recovery, sign recovery or support recovery of sparse signals is an important issue in CS theory. We recall the definition of the optimal weight in [96] and [99].

Definition 16 (Optimal Weight)**.** *A weight is called an optimal weight if the solution of the weighted ℓ_1-problem with this weight is one of the sparsest solution of the ℓ_0-minimization problem.*

In this chapter, we study the existence of an optimal weight for the weighted ℓ_1-problem. Let's first review some existing work for weighted ℓ_1-problem. The standard weighted ℓ_1-minimization is stated as follows:

$$\begin{aligned} \min \quad & \|Wx\|_1 \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{3.1}$$

where $A \in R^{m\times n}$ with $m \ll n$ is a sensing matrix in CS scenarios, y is a given vector and W is a diagonal matrix. Note that if W is positive definite, (3.1) is equivalent to (1.19). Due to this, (3.1) can exactly solve (1.8) if AW^{-1} satisfies one of the following conditions: mutual coherence (Donoho and Huo [55], Elad [38]), RIP of order $2k$ (Candès and Tao [21]), NSP of order k (Cohen et al. [28], Zhang [89], [90]), minimal singular value (Tang and Nehorai [78]), or RSP of order k (Zhao [92], [93], [100], [94]). Depending on these conditions, however, the optimal weight for solving (1.8) is not given to have an explicit form. Recently, Zhao and Luo [99] (see also in Zhao and Kočvara [96]) have proved that if z is a sparsest point in the set

$$Q = \{x : y = Ax, x \geq 0\},$$

and if the weight w satisfies

$$w_{\overline{J_+(z)}} > A^T_{\overline{J_+(z)}} A_{J_+(z)} (A^T_{J_+(z)} A_{J_+(z)})^{-1} w_{J_+(z)} \tag{3.2}$$

where $J_+(z) = \{i : z_i > 0\}$, then z is the unique optimal solution to the weight ℓ_1-problem:

$$\begin{aligned} \min \quad & \|Wx\|_1 \\ \text{s.t.} \quad & y = Ax, \\ & x \geq 0. \end{aligned} \tag{3.3}$$

Thus, by a suitable choice of weight, the weighted ℓ_1-problem (3.3) can be used to solve

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & y = Ax, \\ & x \geq 0. \end{aligned} \tag{3.4}$$

Thus the weights satisfying (3.2) can guarantee exact recovery of sparse vectors and belong to the set of optimal weights. In addition, such optimal weights depending on the sparse vector z are still not given explicitly but show the existence of optimal weights for solving (3.4). In the following Sections 3.2 and 3.3, we consider the weight ℓ_1-minimization counterpart of (1.1) and show the existence of an optimal weight for guaranteeing to find the support of the solution of (1.1). In this case, we also say the support recovery of the solution of (1.1). More specifically, the support of a sparse vector is understood as follows.

Definition 17 (Support Recovery)**.** *A solution x of a system is said to have a support recovery by an algorithm if the solution x^* found by this algorithm has the same support* *as that of x, i.e.,*

$$\text{supp}(x^*) = \text{supp}(x).$$

Clearly, a weight ensuring support recovery of sparse vectors belong to the set of optimal weights.

3.2 General weighted ℓ_1-minimization model

Let $W \in R^{n\times n}$ be a diagonal matrix with nonnegative diagonal elements $w_i, i = 1, ..., n$, i.e., $W = \text{diag}(w)$. Let us consider the following weighted problem:

$$\begin{aligned} (P_w) \quad \min \quad & \|Wx\|_1 = w^T|x| \\ \text{s.t.} \quad & \|y - Ax\|_2 \leqslant \epsilon, \\ & Bx \leqslant b, \end{aligned} \tag{3.5}$$

which is the general weighted ℓ_1-minimization problem associated with (1.1). The problem (3.5) has a close relationship to the following problem:

$$\begin{aligned} \min \quad & \lambda \|Wx\|_1 + \|y - Ax\|_2^2 \\ \text{s.t.} \quad & Bx \leqslant b. \end{aligned} \tag{3.6}$$

where λ is a positive parameter. The following lemma shows the relationship between them.

Lemma 18. *If z is a minimiser of* (3.6)*, then there exists $\epsilon \geqslant 0$ in* (3.5) *such that z is also a minimiser of* (3.5)*.*

Proof. Suppose that z is any minimiser of (3.6) and $\epsilon = \|y - Az\|_2$. Let x be any feasible solution to (3.5). Clearly, x is also a feasible solution to (3.6). Since z is a minimiser of (3.6), we have

$$\lambda \|Wz\|_1 + \|y - Az\|_2^2 \leqslant \lambda \|Wx\|_1 + \|y - Ax\|_2^2 \leqslant \lambda \|Wx\|_1 + \|y - Az\|_2^2,$$

which implies that $\|Wz\|_1 \leqslant \|Wx\|_1$. Note that z satisfies both constraints of (3.5). Thus z is a minimiser of (3.5). □

3.3 Existence of optimal weights for finding sparsest points

From a computaional point of view, it is vital to know whether there exists a weight such that the sparsest point in T given in (1.2) is a unique optimal solution to the weighted ℓ_1-minimization (3.5), or the solution of (3.5) and the sparsest point of T share the same support and/or sign. Such weights belongs to the set of optimal weights (Zhao & Kočvara [96], Zhao & Luo [99]), by which we can use the weighted ℓ_1-minimization (3.5) to find a sparsest point in T, or to find the sign or support of the sparsest point in T. Note that the parameter ϵ and other linear inequalities appear in the constraints of the model (3.5). To a large extent, (3.5) has many optimal solutions and may have some solutions which are compressible. Thus we may consider the support recovery by (3.5) instead of exact recovery. We expect that the support of the sparsest point ν in T, given in (1.2), can be exactly recovered by (3.5), i.e., the solution x^* to (3.5) satisfies that

$$\mathrm{supp}(x^*) = \mathrm{supp}(\nu).$$

The existence of a weight for support recovery is guaranteed, as shown by the next result.

Theorem 19. *Let ν be a sparsest point in $T = \{x : \|y - Ax\|_2 \leq \epsilon, Bx \leq b\}$ where $\epsilon > 0$. Let the weight w in (3.5) satisfy*

$$w_i \begin{cases} = 0, & i \in \mathrm{supp}(\nu), \\ > 0, & i \in \overline{\mathrm{supp}(\nu)}. \end{cases} \tag{3.7}$$

Then x^* is an optimal solution of (3.5) if and only if $\mathrm{supp}(\nu) = \mathrm{supp}(x^*)$.

Proof. Suppose that $\bar{x}$ is an arbitrary feasible solution of (3.5) with $W = \mathrm{diag}(w)$ where w is given as in (3.7). We can see that

$$\begin{aligned} \|W\nu\|_1 - \|W\bar{x}\|_1 &= w^T_{\mathrm{supp}(\nu)}|\nu_{\mathrm{supp}(\nu)}| - w^T_{\mathrm{supp}(\nu)}|\bar{x}_{\mathrm{supp}(\nu)}| - w^T_{\overline{\mathrm{supp}(\nu)}}|\bar{x}_{\overline{\mathrm{supp}(\nu)}}| \\ &= -w^T_{\overline{\mathrm{supp}(\nu)}}|\bar{x}_{\overline{\mathrm{supp}(\nu)}}|. \end{aligned}$$

Note that $\|W\nu\|_1 = \|W\bar{x}\|_1$ if $\bar{x}$ has the same support as ν. Otherwise, $\|W\nu\|_1 < \|W\bar{x}\|_1$. Thus, under the weights in (3.7), any solution that has the same support of ν, i.e., $\mathrm{supp}(\bar{x}) = \mathrm{supp}(\nu)$, is an optimal solution of (3.5). Conversely, let x^* be any arbitrary optimal solution to (3.5). We have

$$\begin{aligned} \|Wx^*\|_1 - \|W\bar{x}\|_1 &= w^T_{\mathrm{supp}(\nu)}|x^*_{\mathrm{supp}(\nu)}| + w^T_{\overline{\mathrm{supp}(\nu)}}|x^*_{\overline{\mathrm{supp}(\nu)}}| \\ &\quad -w^T_{\mathrm{supp}(\nu)}|\bar{x}_{(\mathrm{supp}(\nu))}| - w^T_{\overline{\mathrm{supp}(\nu)}}|\bar{x}_{\overline{\mathrm{supp}(\nu)}}|, \\ &= w^T_{\overline{\mathrm{supp}(\nu)}}|x^*_{\overline{\mathrm{supp}(\nu)}}| - w^T_{\overline{\mathrm{supp}(\nu)}}|\bar{x}_{\overline{\mathrm{supp}(\nu)}}| \leqslant 0. \end{aligned}$$

Note that ν is also a feasible point in T. If $\bar{x} = \nu$, the above inequality implies

$$(x^*)_i = 0, \quad i \in \overline{\mathrm{supp}(\nu)}.$$

If not, then $\|Wx^*\|_1 > \|W\nu\|_1$ which contradicts the fact that x^* is optimal to (3.5). Then we have

$$\mathrm{supp}(x^*) \subseteq \mathrm{supp}(\nu).$$

Note that ν is a sparsest point in the feasible set T. Thus x^* must have same support as ν, i.e. $\mathrm{supp}(x^*) = \mathrm{supp}(\nu)$. □

In addition, in Theorem 19, x^* can be equal to ν. If ν is the unique sparsest point in T, then ν is unique to (3.5) with w being given in (3.7). Moreover, if there are many sparsest solutions with the same support as ν, then all these solutions are optimal solutions of (3.5) provided that w is given by (3.7). Note that some components of weights in (3.7) are equal to 0. If w is a positive vector, the following theorem claims that the same result in Theorem 19 remains valid.

Theorem 20. *Let ν be a sparsest point in $T = \{x : \|y - Ax\|_2 \le \epsilon, Bx \le b\}$ where $\epsilon > 0$. Let the weight w in* (3.5) *satisfy*

$$w_i \begin{cases} = 1, & i \in \operatorname{supp}(\nu), \\ = \gamma^{\infty}, & i \in \overline{\operatorname{supp}(\nu)}. \end{cases}, \tag{3.8}$$

where γ^{∞} is a big positive number. If x^* is an optimal solution of (3.5)*, then*

$$\operatorname{supp}(\nu) = \operatorname{supp}(x^*).$$

Proof. Let Z^* be the optimal value of (3.5). Since ν is a feasible point in T, Z^* has an upper bound $w^T_{\operatorname{supp}(\nu)}|v_{\operatorname{supp}(\nu)}| = \left\|v_{\operatorname{supp}(\nu)}\right\|_1$. Note that $w_i = \gamma^{\infty}$ for $i \in \overline{\operatorname{supp}(\nu)}$ implies that the value of x_i^* indexed by $\overline{\operatorname{supp}(\nu)}$ must be 0 provided γ^{∞} is large enough. Otherwise, the optimal value Z^* of (3.5) is ∞. So we have

$$\|v\|_0 \ge \|x^*\|_0 \, .$$

This, combined with the fact that ν is the sparsest among all the points in T, yields that $\|v\|_0 = \|x^*\|_0$. Note that $x_i^* = 0$ for $i \in \overline{\operatorname{supp}(\nu)}$. Thus,

$$\operatorname{supp}(\nu) = \operatorname{supp}(x^*),$$

which means that given a sparsest point in T, there exists a weight satisfying (3.8) such that any optimal solution of (3.5) are the sparsest points in T, and their support is consistent with that given point. □

The following theorem shows the existence of an optimal weight for (3.5) in the case of $\epsilon = 0$.

Theorem 21. *Let ν be a sparsest point in $T = \{x : \|y - Ax\|_2 \le \epsilon, Bx \le b\}$ where $\epsilon = 0$*

and $K(\nu)$ be the support set of ν, i.e.,

$$K(\nu) = \{i : \nu_i \neq 0\}, \quad \overline{K(\nu)} = \{i : \nu_i = 0\}.$$

If $A_{K(\nu)}$ has a full column rank and $w \in R_+^n$ satisfies

$$w_{\overline{K(\nu)}} > |A^T_{\overline{K(\nu)}} A_{K(\nu)} (A^T_{K(\nu)} A_{K(\nu)})^{-1}| w_{K(\nu)} \tag{3.9}$$

then ν is the unique solution to (3.5).

Proof. Let us consider the case that the measurements y in (3.5) are accurate, i.e., $y = Ax$. Let z be any point in T and $z \neq \nu$. Then we have

$$A_{K(\nu)} \nu_{K(\nu)} = y = A_{K(\nu)} z_{K(\nu)} + A_{\overline{K(\nu)}} z_{\overline{K(\nu)}}.$$

Due to the full column rank property of $A_{K(\nu)}$, the above equality can be reduced to

$$\nu_{K(\nu)} = z_{K(\nu)} + (A^T_{K(\nu)} A_{K(\nu)})^{-1} A^T_{K(\nu)} A_{\overline{K(\nu)}} z_{\overline{K(\nu)}}. \tag{3.10}$$

Note that $z_{\overline{K(\nu)}}$ can not be 0. Otherwise, $\nu_{K(\nu)} = z_{K(\nu)}$ which leads to $z = \nu$. Now we verify that ν is the optimal solution to (3.5) under the condition that the weight w satisfies (3.9). We calculate the difference between $w^T|\nu|$ and $w^T|z|$, which is given as follows:

$$w^T|\nu| - w^T|z| = w^T_{K(\nu)} |\nu_{K(\nu)}| - w^T_{K(\nu)} |z_{K(\nu)}| - w^T_{\overline{K(\nu)}} |z_{\overline{K(\nu)}}|.$$

Using (3.10), we have

$$
\begin{aligned}
w^T|\nu| - w^T|z| &\leqslant w^T_{K(\nu)}|z_{K(\nu)}| + w^T_{K(\nu)}|(A^T_{K(\nu)}A_{K(\nu)})^{-1}A^T_{K(\nu)}A_{\overline{K(\nu)}}z_{\overline{K(\nu)}}| \\
&\quad -w^T_{K(\nu)}|z_{K(\nu)}| - w^T_{\overline{K(\nu)}}|z_{\overline{K(\nu)}}|, \\
&\leqslant w^T_{K(\nu)}|(A^T_{K(\nu)}A_{K(\nu)})^{-1}A^T_{K(\nu)}A_{\overline{K(\nu)}}||z_{\overline{K(\nu)}}| - w^T_{\overline{K(\nu)}}|z_{\overline{K(\nu)}}|, \\
&= (|A^T_{\overline{K(\nu)}}A_{K(\nu)}(A^T_{K(\nu)}A_{K(\nu)})^{-1}|w_{K(\nu)} - w_{\overline{K(\nu)}})^T|z_{\overline{K(\nu)}}|,
\end{aligned}
$$

where the first inequality follows from the nonnegativeness of weight w and the triangle inequality. The above inequality, together with (3.9) and the positiveness of $|z_{\overline{K(\nu)}}|$, implies that

$$w^T|\nu| < w^T|z|$$

for $\forall z \in T$ and $\nu \neq z$, which means that v is the unique solution to (3.5). $\square$

The idea of the proof of Theorem 21 follows from Zhao and Luo [99]. We denote the optimal solution set of (3.5) as

$$G = \left\{x \in R^n : w^T|x| = z^*,\ \ \|Ax - y\|_2 \leq \epsilon,\ Bx \leq b\right\},$$

where z^* is the optimal value of (3.5). Based on the above analysis in Theorems 19 and 20, (P_w) given in (3.5) guarantees the support recovery of the sparsest point in T, i.e., there exists a weight satisfying (3.7) or (3.8) such that all the solutions of (3.5) have the identically unique support as that of the sparsest point.

However, these theorems only provide a theoretical choice of the (optimal) weights for (3.5) since the weights shown in these theorems are not given explicitly. Which practical methods can be used to determine an optimal weight in (3.5) to find a sparsest point in T is still a hard issue.

3.4 Duality and complementary conditions

In this section, we present some fundamental properties of (3.5), including strong duality and complementary condition. By introducing two variables $t \in R^n$ and $\gamma \in R^m$ such that

$$|x| \leqslant t \quad \text{and} \quad \gamma = y - Ax,$$

we can rewrite (3.5) as the following programming:

$$\begin{aligned} (P_{w1}) \quad \min_{(x,\gamma,t)} \quad & w^T t \\ \text{s.t.} \quad & \|\gamma\|_2 \leqslant \epsilon, \quad Bx \leqslant b, \\ & \gamma = y - Ax, \quad |x| \leqslant t, \\ & t \geqslant 0, \end{aligned} \tag{3.11}$$

where $w \in R^n_+$ is a given vector. Obviously, (3.11) is equivalent to (3.5), and the solution of (3.11), denoted by (x^*, t^*, γ^*), must have

$$|x^*_{\text{supp(w)}}| = t^*_{\text{supp}(w)}, \; |x^*_{\overline{\text{supp(w)}}}| \leqslant t^*_{\overline{\text{supp}(w)}} \;\text{ and }\; \gamma^* = y - Ax^*.$$

Additionally, if $w \in R^n_{++}$, then the solution (x^*, t^*, γ^*) to (3.11) must satisfy that $|x^*| = t^*$ and $\gamma^* = y - Ax^*$. The relation of the solutions of (3.5) and (3.11) is stated as follows:

Lemma 22. *If x^* is optimal to the problem (3.5), then all vectors (x^*, t^*, γ^*) satisfying*

$$|x^*_{\text{supp(w)}}| = t^*_{\text{supp}(w)}, \; |x^*_{\overline{\text{supp(w)}}}| \leqslant t^*_{\overline{\text{supp}(w)}} \;\text{ and }\; \gamma^* = y - Ax^*,$$

are optimal to the problem (3.11). Moreover, if $(\bar{x}, \bar{t}, \bar{\gamma})$ is optimal to the problem (3.11), then x^* is optimal to the problem (3.5).

3.4.1 Dual problem of (P_{w1})

We start to derive the dual problem of (P_{w1}) given in (3.11). Denote Lagrangian multipliers of (3.11) as $\lambda = (\lambda_1, ..., \lambda_6)$, where

1. $\lambda_1 \in R$ corresponds to $\|\gamma\|_2 \leqslant \epsilon$;
2. $\lambda_2 \in R^l$ corresponds to $Bx \leqslant b$;
3. $\lambda_3 \in R^m$ corresponds to $\gamma = y - Ax$;
4. $\lambda_4 \in R^n$ corresponds to $x \leqslant t$;
5. $\lambda_5 \in R^n$ corresponds to $-t \leqslant x$;
6. $\lambda_6 \in R^n$ corresponds to $t \geqslant 0$.

Then Lagrangian function of (3.11) is given as

$$L(x, \gamma, t, \lambda) = w^T t - \lambda_1(\epsilon - \|\gamma\|_2) - \lambda_2^T(b - Bx) - \lambda_3^T(Ax + \gamma - y) - \lambda_4^T(t - x) - \lambda_5^T(x + t) - \lambda_6^T t, \tag{3.12}$$

and the Lagrangian dual function of (3.11) is

$$g(\lambda) = \inf_{(x,\gamma,t) \in R^{2n+m}} L(x, \gamma, t, \lambda),$$

where $\lambda = (\lambda_1, ..., \lambda_6)$. Let p^* be the optimal value of the problem (3.11). We now derive the value of $g(\lambda)$ together with $\lambda_i \geqslant 0, i = 1, 2, 4, 5, 6$ to find the best lower bound of p^*, and we establish the dual problem of (3.11). Firstly, by rearranging the terms of (3.12), we have

$$\begin{aligned} g(\lambda) = \inf_{(x,\gamma,t) \in R^{2n+m}} \{ & (\lambda_2^T B - \lambda_3^T A + \lambda_4^T - \lambda_5^T)x + (\lambda_1 \|\gamma\|_2 - \lambda_3^T \gamma) \\ & + (w^T - \lambda_4^T - \lambda_5^T - \lambda_6^T)t - \lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \} \end{aligned} \tag{3.13}$$

To determine the minimum value of the terms involving x or t, we first calculate the derivatives of the Lagrangian function with respect to x and t respectively as follows:

$$\frac{\partial L(x,\gamma,t)}{\partial x} = B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5,$$

$$\frac{\partial L(x,\gamma,t)}{\partial t} = w - \lambda_4 - \lambda_5 - \lambda_6.$$

Setting them to 0, the infimum of the terms involving x and t in (3.13) can be obtained as follows,

$$\inf_{x\in R^n}\left\{(\lambda_2^T B - \lambda_3^T A + \lambda_4^T - \lambda_5^T)x\right\} = \begin{cases} 0, & B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0 \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.14}$$

$$\inf_{t\in R^n}\left\{(w^T - \lambda_4^T - \lambda_5^T - \lambda_6^T)t\right\} = \begin{cases} 0, & w = \lambda_4 + \lambda_5 + \lambda_6 \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.15}$$

Now we calculate the infimum of $(\lambda_1 \|\gamma\|_2 - \lambda_3^T\gamma)$. Note that

$$\inf_{\gamma}(\lambda_1 \|\gamma\|_2 - \lambda_3^T\gamma) = -\sup_{\gamma}(\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2). \tag{3.16}$$

By Cauchy-Schwarz inequality, we have

$$\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2 \leq \|\lambda_3\|_2 \|\gamma\|_2 - \lambda_1 \|\gamma\|_2 = \|\gamma\|_2 (\|\lambda_3\|_2 - \lambda_1). \tag{3.17}$$

Consider the case $\|\lambda_3\|_2 \leqslant \lambda_1$. Then (3.17) implies that $(\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2)$ is nonpositive, and the supremum of $(\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2)$ attains at 0 (when $\gamma = 0$). When $\|\lambda_3\|_2 > \lambda_1$, by taking $\gamma = \alpha\lambda_3$ where α is a positive parameter, we see that

$$\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2 = \alpha \|\lambda_3\|_2 (\|\lambda_3\|_2 - \lambda_1),$$

and

$$\sup_{\gamma}(\lambda_3^T\gamma - \lambda_1 \|\gamma\|_2) \to \infty, \text{ as } \alpha \to \infty.$$

Therefore, it follows from (3.16) that

$$\inf_{\gamma \in R^m}(\lambda_1 \|\gamma\|_2 - \lambda_3^T\gamma) = \begin{cases} 0, & \|\lambda_3\|_2 \leq \lambda_1, \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.18}$$

Denote the set D by

$$D = \{\lambda : w = \lambda_4 + \lambda_5 + \lambda_6,\ \|\lambda_3\|_2 \leq \lambda_1,\ B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0\}.$$

By pluging (3.18), (3.15) and (3.14) into (3.13), the value of $\inf\limits_{(x,\gamma,t)\in R^{2n+m}} L(x,\gamma,t,\lambda)$ can be determined as follows

$$\inf_{(x,t,\gamma)\in R^{2n+m}} L(x,t,\gamma,\lambda) = \begin{cases} -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y, & \text{if } \lambda \in D, \\ -\infty, & \text{otherwise.} \end{cases} \tag{3.19}$$

Note that the Lagrange dual function $g(\lambda)$ is finite only on the convex set D. This, combined with $\lambda_i \geqslant 0$ for $i = 1,2,4,5,6$, can give the optimal value p^* of (3.11) a nontrivial lower bound. Since for any feasible solution $(\widetilde{x},\widetilde{t},\widetilde{\gamma})$ in T, we have

$$g(\lambda) = \inf_{(x,t,\gamma)\in R^{2n+m}} L(x,t,\gamma,\lambda) \leqslant L(\widetilde{x},\widetilde{t},\widetilde{\gamma},\lambda) \leqslant w^T\widetilde{t}.$$

Thus $g(\lambda) \leqslant p^*$. In order to obtain the best lower bound of p^*, we maximize $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ over the convex set D in (3.19) and $\lambda_i, i = 1,2,4,5,6 \geqslant 0$. Then the dual problem of

(P_{w1}) can be stated as the following optimization:

$$
\begin{aligned}
(D_{w1}) \quad \max_{\lambda} \quad & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \\
\text{s.t.} \quad & B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6, \\
& w = \lambda_4 + \lambda_5 + \lambda_6.
\end{aligned}
\tag{3.20}
$$

Note that (3.20) can be rewritten more concisely. By eliminating $\|\lambda_3\|_2 \leq \lambda_1$ and the nonnegative slack variable λ_6, the dual problem (3.20) can be simplified to

$$
\begin{aligned}
(D_{w2}) \quad \max_{\lambda} \quad & -\epsilon \|\lambda_3\|_2 - \lambda_2^T b + \lambda_3^T y \\
\text{s.t.} \quad & B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& w \geqslant \lambda_4 + \lambda_5, \ \lambda_i \geqslant 0, i = 2, 4, 5.
\end{aligned}
\tag{3.21}
$$

We see that when $\epsilon > 0$, the optimal solution of (3.20) $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ must satisfy $\|\lambda_3^*\|_2 = \lambda_1^*$. Moreover, $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ is an optimal solution to (3.20) if and only if $(\lambda_i^*), \ i = 2, 3, 4, 5$ is an optimal solution to (3.21) and $\|\lambda_3^*\|_2 = \lambda_1^*$.

3.4.2 Strong duality between (P_{w1}) and (D_{w1})

In the above subsection, we developed the dual problem of (3.11). Next, we further explore the relationship of the solutions of (3.11) and (3.20) such as strong duality and complementarity property. The strong duality between (3.11) and (3.20) is explored in this subsection. The optimality condition and the strictly complementary condition for (3.11) will be discussed under some conditions in subsection 3.4.4.

The strong duality, in general, means that the optimal values of primal problem and dual problem are equal and can be achieved in their feasible sets. It is well known that the strong duality holds for any linear programming problem (Goldman and Tucker [45] and Dantzig [29]). For example, consider the following standard linear program and its dual problem:

$$
\min_x \{c^T x : Ax = b, \ x \geqslant 0\}, \tag{3.22}
$$

$$\max_{(y,s)}\{b^T y : A^T y + s = c,\ s \geqslant 0\}, \tag{3.23}$$

if (3.22) and (3.23) have feasible solutions, then both problems have optimal solutions, and for any optimal pair (x^*, y^*, s^*) to (3.22) and (3.23), there is no gap between the optimal objective values of (3.22) and (3.23), that is

$$c^T x^* = b^T y^*.$$

The problems (3.22) and (3.23) also have strictly complementary solutions. Specifically, if (3.22) and (3.23) have feasible solutions, then there exists a pair of strictly complementary optimal solutions $(\tilde{x}, \tilde{y}, \tilde{s})$ such that

$$\tilde{s}^T \tilde{x} = 0,\ \tilde{x} + \tilde{s} > 0,\ \tilde{x} \geq 0,\ \tilde{s} \geq 0.$$

Since some constraints in (3.11) are non-linear, there is no guarantee for the strong duality or the strictly complementary property for (3.11) and (3.20). However, we may still find some conditions under which the strong duality and strictly complementary property hold for (3.11) and (3.20). Let's first recall the well-known Slater condition (see, e.g., Boyd [8]).

Definition 23 (Slater Condition)**.** *Consider the following convex minimization problem*

$$\begin{aligned} \min_{x} \quad & \psi_0(x) \\ \text{s.t.} \quad & \psi_i(x) \leqslant 0,\ i = 1, ..., m, \\ & h_i(x) = 0,\ i = 1, ..., n, \end{aligned} \tag{3.24}$$

where $\psi_1, ..., \psi_k$ are affine functions, $\psi_{k+1}, ..., \psi_m$ are convex functions, and $h_1, ..., h_n$ are linear functions. Let E be the feasible set of (3.24)*, i.e.,*

$$E = \{x :\ \psi_i(x) \leqslant 0,\ i = 1, ..., m,\ h_i(x) = 0,\ i = 1, ..., n\}.$$

Slater condition means that there exists $x^* \in ri(E)$, where $ri(E)$ is the relative interior *of* E, *i.e.,*

$$\psi_i(x^*) \leqslant 0,\ i = 1, ..., k, \quad \psi_i(x^*) < 0,\ i = k+1, ..., m, \quad h_i(x^*) = 0,\ i = 1, ..., n.$$

It is well-known that (see, e.g. [76] [12]) the Slater condition is a sufficient condition (constriant qulification) for strong duality to hold for convex optimization problems. Moreover, Browein and Lewis [8] pointed out that under the Slater condition, if the optimal value of a primal convex problem is finite, then there exists a dual solution achieving that optimal value. We summarize this property as follows:

Lemma 24. [8,12] Let Slater condition (Definition 23) hold for the convex problem (3.24). Then there is no duality gap between (3.24) and its dual problem. Moreover, if the optimal value of (3.24) is finite, then there exists at least one optimal Lagrangian multiplier such that the dual optimal value can be attained.

Note that (3.11) is a convex problem. Based on the above lemma, if the Slater condition holds in (3.11), i.e., there exists $(x^*, \gamma^*, t^*) \in ri(T)$ such that

$$\|\gamma^*\|_2 < \epsilon,\ Bx^* \leqslant b,\ |x^*| \leqslant t^*,\ y = Ax^* - \gamma^*,\ t^* \geqslant 0,$$

then strong duality holds for (3.11) and (3.20). We state this property in the next theorem.

Theorem 25. *Let two vectors* $y \in R^m$ *and* $b \in R^l$ *be given,* $A \in R^{m\times n}$ and $B \in R^{l\times n}$ *be given matrices and* ϵ *be a given positive number. Strong duality holds for the problem* (3.11) *and its dual* (3.20) *if one of the following conditions is satisfied:*

(F1)

$$\Omega = \{x : Ax = y, Bx \leqslant b\} \neq \emptyset; \tag{3.25}$$

(F2) $\epsilon > \epsilon^*$ *where*

$$\epsilon^* = \min_x \{\|y - Ax\|_2 : Bx \leq b\}.$$

Proof. If (F_1) holds, i.e., $\Omega = \{x : Ax = y, Bx \leqslant b\} \neq \emptyset$, then there exists a vector x^* such that $y = Ax^*$ and $Bx^* \leqslant b$. Thus there exists a vector $(x^*, \gamma^* = 0, t^*)$ satisfying

$$\|\gamma^*\|_2 < \epsilon, Bx^* \leqslant b, |x^*| \leqslant t^*, y = Ax^* - \gamma^*, t^* \geqslant 0.$$

If $(F2)$ holds, then it implies that the set $\{x : \|y - Ax\|_2 < \epsilon, Bx \leq b\}$ is not empty. Then either $(F1)$ or $(F2)$ shows that there is a relative interior point in the feasible set of (3.11). By Definition 23, either of them ensures that the Slater condition holds for (3.11). Hence, by Lemma 24, (3.11) and its dual problem (3.20) satisfy strong duality. □

The hypothesis in Theorem 25 is very mild. The set $\Omega = \{x : Ax = y,\ Bx \leqslant b\}$ is in practice not empty due to the fact that y and b are the measurements of certain signals. The condition $(F2)$ in Theorem 25 can be also satisfied very easily provided ϵ is chosen to be greater than the lower bound of $\|y - Ax\|_2$ subject to $Bx \leqslant b$. In summary, the Slater condition is a very mild sufficient condition for strong duality to hold for the problem (3.11) and (3.20). If the optimal value of (3.11) is finite, together with Slater condition (ensured by $(F1)$ or $(F2)$), Lemma 24 implies that there exists a dual solution such that the dual optimal value can be attained.

3.4.3 Optimality condition for (P_{w1}) and (D_{w1})

It is well-known that for any convex problem with differentiable objective and constraints functions for which strong duality holds, Karush-Kuhn-Tucker (KKT) condition is the necessary and sufficient optimality condition for the convex minimization problem and its dual problem, which is stated in the following theorem.

Theorem 26. [8, 74][KKT condition for convex problem] The Langranian dual function of (3.24) is

$$g(\lambda, \tau) = \inf_x \left(\psi_0(x) + \sum_{i=1}^{m} \lambda_i \psi_i(x) + \sum_{i=1}^{n} \tau_i h_i(x) \right)$$

If strong duality holds for the problem (3.24) and its dual problem, then x^* and (λ^*, τ^*) are the optimal solutions to the problem (3.24) and its dual problem respectively if and only if (x^*, λ^*, τ^*) satisfies the following condition:

- $\psi_i(x^*) \leqslant 0, \; i = 1, ..., m,$
- $h_i(x^*) = 0, \; i = 1, ..., n,$
- $\lambda_i^* \geqslant 0, \; i = 1, ..., m,$
- $\lambda_i^* \psi_i(x^*) = 0, \; i = 1, ..., m,$
- $\nabla\psi_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla\psi_i(x^*) + \sum_{i=1}^n \tau_i^* \nabla h_i(x^*) = 0,$

which are called Karush-Kuhn-Tucker (KKT) conditions.

Note that (3.11) is a convex programming and all the constraints and objective function are differentiable. This, together with Slater condition, yields the optimality condition for (3.11), which is stated in the following theorem.

Theorem 27. *If Slater condition holds for* (3.11)*, then* (x^*, γ^*, t^*) *is optimal to* (3.11) *and* $\lambda_i^*, i = 1, ..., 6$ *is optimal to* (3.20) *if and only if* $(x^*, \gamma^*, t^*, \lambda^*)$ *satisfy the KKT conditions for* (3.11)*, i.e.,*

$$\begin{cases} \gamma^* = y - Ax^*, \|\gamma^*\|_2 \leqslant \epsilon, & x^* \leqslant t^*, -t^* \leqslant x^*, \; Bx^* \leqslant b, \; t^* \geqslant 0, \\ \lambda_i^* \geq 0, i = 1, 2, 4, 5, 6, & \\ \lambda_1^*(\epsilon - \|\gamma^*\|_2) = 0, & \lambda_2^{*T}(b - Bx^*) = 0, \\ \lambda_4^{*T}(t^* - x^*) = 0, & \lambda_6^{*T} t^* = 0, \; \lambda_5^{*T}(x^* + t^*) = 0, \\ L(x, \gamma, t, \lambda^*) \quad = & w^T t - \lambda_1^*(\epsilon - \|\gamma\|_2) - \lambda_2^{*T}(b - Bx) - \lambda_3^{*T}(Ax + \gamma - y) \\ & -\lambda_4^{*T}(t^* - x^*) - \lambda_5^{*T}(x^* + t^*) - \lambda_6^{*T} t^*, \\ \partial_x L(x^*, \gamma^*, t^*, \lambda^*) \quad = & B^T \lambda_2^* - A^T \lambda_3^* + \lambda_4^* - \lambda_5^* = 0, \\ \partial_\gamma L(x^*, \gamma^*, t^*, \lambda^*) \quad = & (\lambda_1^*)\nabla(\|\gamma^*\|_2) - \lambda_3^* = 0, \\ \partial_t L(x^*, \gamma^*, t^*, \lambda^*) \quad = & w - \lambda_4^* - \lambda_5^* - \lambda_6^* = 0. \end{cases} \tag{3.26}$$

3.4.4 Complementary and strictly complementary conditions

It is well known that for any convex problem with strong duality, the optimal primal and dual solutions satisfy the so-called complementary property, as stated in the following theorem.

Theorem 28 (Complementary Slackness Condition)**.** [8] Assume that strong duality holds for (3.24). Let x^* and (λ^*, τ^*), where $\lambda^* \geqslant 0$, be the optimal solutions to (3.24) and its dual problem, respectively. Then λ_i^* and $\psi_i(x^*)$ are complementary, i.e.,

$$\lambda_i^* \psi_i(x^*) = 0, \ i = 1, .., m, \tag{3.27}$$

which is called complementary slackness property.

It means for every i, there exists at least one of λ_i^* and $\psi_i(x^*)$ being 0. The condition (3.27) together with $\lambda_i^* + |\psi_i(x^*)| > 0$ is called strictly complementary condition. The strictly complementary condition might not hold for a convex problem even when strong duality holds for this problem. However, this property always holds for linear programming problems. Consider the standard linear programming (3.22) and its dual problem (3.23). The complementary and strictly complementary conditions for (3.22) and (3.23) can be stated as follows.

Theorem 29. [45] If both problems (3.22) and (3.23) are feasible, then both have optimal solutions, and any pair (x^*, s^*) of optimal solutions to (3.22) and (3.23) are complementary in the sense that $(s^*)^T x^* = 0, s^* \geqslant 0, x^* \geqslant 0$. Moreover, (3.22) and (3.23) have a pair of strictly complementary solutions $\bar{x} \geqslant 0$ and $\bar{s} \geqslant 0$ satisfying

$$\bar{x}^T \bar{s} = 0 \quad \text{and } \bar{x} + \bar{s} > 0.$$

In this subsection, we explore the complementary condition for (3.11) and (3.20) and develop the conditions to ensure the strictly complementary condition for the two problems. From the optimality condition given in (3.26) in Theorem 27, the term $(t^*)^T\lambda_6^* = 0$ and the positiveness of t^* and λ_6^* show that t^* and λ_6^* satisfy the complementary condition.

Theorem 30. *Let Slater condition hold in* (3.11)*. If the optimal value of* (3.11) *is finite, then for any optimal solution pair* $((x^*, t^*, \gamma^*), \lambda^*)$*, where* (x^*, t^*, γ^*) *is optimal to* (3.11) *and* $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ *is optimal to* (3.20)*,* t^* *and* λ_6^* *are complementary in the sense that*

$$(t^*)^T\lambda_6^* = 0, \quad t^* \geqslant 0 \quad \text{and} \quad \lambda_6^* \geqslant 0.$$

That is,

$$t_i^*(\lambda_6^*)_i = 0, \quad t_i^* \geqslant 0 \quad \text{and} \quad (\lambda_6^*)_i \geqslant 0, \ i = 1, .., n. \tag{3.28}$$

The proof is omitted since it is obvious. Clearly, if (x^*, t^*, γ^*) is optimal to (3.11) and w is positive, it must hold $|x^*| = t^*$, and hence by Theorem 30, we have

$$|x_i^*|(\lambda_6^*)_i = 0, \quad (\lambda_6^*)_i \geq 0, \ i = 1, .., n. \tag{3.29}$$

When w is nonnegative, clearly, if (x^*, t^*, γ^*) is optimal to (3.11), we have

$$|x_i^*| = t_i^*, i \in \text{supp}(w); \ |x_i^*| \leqslant t_i^*, i \in \overline{\text{supp}(w)}.$$

For $i \in \text{supp}(w)$, (3.29) is valid. For $i \in \overline{\text{supp}(w)}$, due to the constraints $w = \lambda_4 + \lambda_5 + \lambda_6$ and $\lambda_{4,5,6} \geqslant 0$, $w_i = 0$ implies that $(\lambda_6^*)_i = 0$. This means (3.29) is valid for $i \in \overline{\text{supp}(w)}$. Therefore, we have the following theorem:

Theorem 31. *Let Slater condition hold in* (3.11)*. If the optimal value of* (3.11) *is finite, then for any optimal solution pair* $((x^*, t^*, \gamma^*), \lambda^*)$*, where* (x^*, t^*, γ^*) *is optimal to* (3.11) *and* $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ *is optimal to* (3.20)*,* $|x_i^*|$ *and* $(\lambda_6^*)_i$ *are complementary in the sense that*

$$|x_i^*|(\lambda_6^*)_i = 0 \quad \text{and} \quad (\lambda_6^*)_i \geqslant 0, \ i = 1, ..., n. \tag{3.30}$$

The relation (3.30) implies that

$$\|x^*\|_0 + \|\lambda_6^*\|_0 \leq n,$$

where n is the size of the dimension of x^* or λ_6^*. Suppose $|x^*|$ and λ_6^* are strictly complementary, i.e.,

$$(|x^*|)^T \lambda_6^* = 0, \quad \lambda_6^* \geq 0 \quad \text{and} \quad |x^*| + \lambda_6^* > 0.$$

Then

$$\|x^*\|_0 + \|\lambda_6^*\|_0 = n.$$

Unfortunately, for nonlinear optimization models, the strictly complementary property might not hold. However, for our model (3.5) or (3.11), it might be possible to develop a condition such that the strict complementarity still holds. We now develop this property for the problems (3.11) and (3.20) under the following Assumption 32.

Assumption 32. *Let* $W = \text{diag}(w)$ *satisfy the following properties:*

- $\langle G1 \rangle$ *The problem* (3.5) *with* w *has an optimal solution which is a relative interior point in the feasible set* T*, denoted by* $x^* \in ri(T)$*, such that*

$$\|y - Ax^*\|_2 < \epsilon, \;\; Bx^* \leq b,$$

- $\langle G2 \rangle$ *the optimal value* Z^* *of* (3.5) *is finite and positive, i.e.,* $Z^* \in (0, \infty)$*,*

- $\langle G3 \rangle$ $w_j \in (0, \infty]$ *for all* $1 \leqslant j \leqslant n$*.*

Based on the multiplicity of solutions to (3.5), the first condition in Assumption 32 might be achieved. $(G3)$ and $(G1)$ require that (3.5) with certain positive weight has at least one optimal solution such that the constraint $\|y - Ax\|_2 \leq \epsilon$ is inactive. This assumption also requires the optimal value Z^* to be finite and positive. This assumption is mild and can be satisfied very easily. Moreover, $(G1)$ is equivalent to the fact that there exists is a relative interior optimal solution (x^*, t^*, γ^*) of (3.11) such that

$$\|\gamma^*\|_2 < \epsilon, \ Bx^* \leqslant b, \ |x^*| \leqslant t^*, \ y = \gamma^* + Ax^*, \ t^* \geqslant 0,$$

and $(G2)$ is equivalent to the fact that the optimal value of (3.11) is finite and positive. Using the Example 11, we can see that (3.5) with $W = I$ has an optimal solution $(1/2, 0, -1/4, 0)^T$ such that Assumption 32 is satisfied. Next we prove the following theorem concerning the strict complementarity for (3.11) and (3.20) under Assumption 32.

Theorem 33. *Let two vectors y and b be given, $A \in R^{m\times n}$ and $B \in R^{l\times n}$ be two given matrices, and w be a given weight which satisfies Assumption 32. Then there exists a pair $((x^*, t^*, \gamma^*), \lambda^*)$, where (x^*, t^*, γ^*) is an optimal solution to (3.11) and $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ is an optimal solution to (3.20), such that t^* and λ_6^* are strictly complementary, i.e.,*

$$(t^*)^T\lambda_6^* = 0, \ t^* + \lambda_6^* > 0, \ (t^*, \lambda_6^*) \geq 0.$$

Moreover,

$$P^* = \{i : t_i^* > 0\}$$

and

$$Q^* = \{i : (\lambda_6^*)_i > 0\}$$

are invariant for all pairs of strictly complementary solutions.

Proof. Note that $(G1)$ in Assumption 32 implies the Slater condition for (3.11). This, combined with $(G2)$, indicates from Lemma 24 that the duality gap is 0, and the optimal value Z^* for (3.20) can be attained. Hence (3.20) has optimal solutions. For any given index $j : 1 \leq j \leq n$, we consider a series of minimization problems:

$$\begin{aligned}
\min_{(x,t,\gamma)} \quad & -t_j \\
\text{s.t.} \quad & \|\gamma\|_2 \leqslant \epsilon, \\
& Bx \leqslant b, \\
& \gamma = y - Ax, \\
& |x| \leqslant t, \\
& -w^T t \geq -Z^*, \\
& t \geqslant 0.
\end{aligned} \tag{3.31}$$

The dual problem of (3.31) can be obtained by using the same method for developing the dual problem of (3.11), which is stated as follows:

$$\begin{aligned}
\max_{(\mu,\tau)} \quad & -\mu_1 \epsilon - \mu_2^T b + \mu_3^T y - \tau Z^* \\
\text{s.t.} \quad & B^T \mu_2 - A^T \mu_3 + \mu_4 - \mu_5 = 0, \\
& \|\mu_3\|_2 \leq \mu_1, \\
& \tau w = \mu_4 + \mu_5 + \mu_6 + e^j, \\
& \mu_i \geqslant 0, \ i = 1,2,4,5,6, \ \tau \geq 0,
\end{aligned} \tag{3.32}$$

where e^j is a vector whose jth component is 1 and the remaining components are 0, i.e.,

$$e_i^j = 1, \ i = j; \ e_i^j = 0, \ i \neq j.$$

Next we show that (3.31) and (3.32) satisfy the strong duality property under Assumption 32. It can be seen that (x, t, γ) is a feasible solution to (3.31) if and only if (x, t, γ) is an optimal solution of (3.11), or if x is optimal to (3.5). If w satisfies the conditions in Assumption 32, then there exists an optimal solution $\bar{x}$ of (3.5) such that $\|y - A\bar{x}\|_2 < \epsilon$, $B\bar{x} \leqslant b$, $w^T|\bar{x}| = Z^*$ which means there is a relative interior point $(\bar{x}, \bar{t}, \bar{\gamma})$ of the feasible set of (3.31) satisfying

$$\|\bar{\gamma}\|_2 < \epsilon,\ B\bar{x} \leqslant b,\ \bar{\gamma} = y - A\bar{x},\ |\bar{x}| \leq \bar{t},\ w^T\bar{t} \leq Z^*,\ \bar{t} \geq 0.$$

As a result, the strong duality holds for (3.31) and (3.32) for all j. Moreover, due to $(G2)$ and $(G3)$, w is positive and Z^* is finite, so t_j cannot be ∞. Thus the optimal value of all jth minimization problems (3.31) is finite. It follows from Lemma 24 that for each jth optimization (3.31) and (3.32), the duality gap is 0, and each jth dual problem (3.32) can achieve their optimal value.

We use ξ_j^* to denote the optimal value of the jth primal problem in (3.31). Clearly, ξ_j^* is nonpositive, i.e.,

$$\xi_j^* < 0 \quad \text{or} \quad \xi_j^* = 0.$$

Case 1: $\xi_j^* < 0$. Then (3.11) has an optimal solution (x', t', γ') where the jth component in t' is positive since $t'_j = -\xi_j^*$ and admits the largest value amongst all the optimal solutions of (3.11). By Theorem 30, the complementary condition implies that (3.20) has an optimal solution $\lambda' = (\lambda'_1, ..., \lambda'_6)$ where jth component in λ'_6 is 0. Then we have an optimal solution pair $((x', t', \gamma'), \lambda')$ for (3.11) and (3.20) such that $t'_j > 0$ and $(\lambda'_6)_j = 0$. It means that

$$t'_j = -\xi_j^* > 0 \quad \text{implies} \quad (\lambda'_6)_j = 0.$$

Case 2: $\xi_j^* = 0$. Following from the strong duality between (3.31) and (3.32), we have an optimal solution (μ, τ) of the jth optimization problem (3.32) such that

$$-\mu_1\epsilon - \mu_2^T b + \mu_3^T y = \tau Z^*.$$

First, we consider $\tau \neq 0$. The above equality can be reduced to

$$-\frac{\mu_1}{\tau} - \frac{\mu_2^T}{\tau} b + \frac{\mu_3^T}{\tau} y = Z^*,$$

and we also have

$$B^T\frac{\mu_2}{\tau} - A^T\frac{\mu_3}{\tau} + \frac{\mu_4}{\tau} - \frac{\mu_5}{\tau} = 0,$$

$$\left\|\frac{\mu_3}{\tau}\right\|_2 \leq \frac{\mu_1}{\tau},\ w = \frac{\mu_4}{\tau} + \frac{\mu_5}{\tau} + \frac{\mu_6}{\tau} + \frac{e^j}{\tau}.$$

We set

$$\lambda_1^{'} = \frac{\mu_1}{\tau},\ \lambda_2^{'} = \frac{\mu_2}{\tau},\ \lambda_3^{'} = \frac{\mu_3}{\tau},\ \lambda_4^{'} = \frac{\mu_4}{\tau},\ \lambda_5^{'} = \frac{\mu_5}{\tau},\ \lambda_6^{'} = \frac{\mu_6}{\tau} + \frac{e^j}{\tau}.$$

Due to strong duality of (3.11) and (3.20) again, $\lambda^{'} = (\lambda_1^{'}, ..., \lambda_6^{'})$ is optimal to (3.20). Note that

$$(\lambda_6)_j^{'} = \frac{(\mu_6)_j + 1}{\tau}.$$

Thus $(\lambda_6)_j^{'} > 0$, which follows from $\mu_6 \geq 0$ and $\tau > 0$. So

$$t_j^{\prime} = -\xi_j^* = 0 \ \text{ implies } \ (\lambda_6)_j^{'} > 0.$$

Note that the third constraint in jth optimization of (3.32) requires $\tau \neq 0$ since w, μ_4, μ_5, μ_6 are all non-negative and $e_j^j = 1$ so that the jth component in τw must be greater or equal than 1. Therefore, all jth optimization problems in (3.32) are infeasible if $\tau = 0$. As a result, the optimal solution (μ, τ) of (3.32) must have $\tau \neq 0$ and the case of $\tau = 0$ is impossible to occur. Combining the cases 1 and 2 implies that for each $1 \leq j \leq n$, we have an optimal solution pair $((x^j, t^j, \gamma^j), \lambda^j)$ such that $t_j^j > 0$ or $(\lambda_6^j)_j > 0$. For all jth solution pairs, they all satisfy the following properties:

- (1) (x^j, t^j, γ^j) is optimal to (3.11), and $(\lambda_1^j, \lambda_2^j, \lambda_3^j, \lambda_4^j, \lambda_5^j, \lambda_6^j)$ is optimal to (3.20);
- (2) the jth componment of t^j and the jth component of λ_6^j are strictly complementary, such that $t_j^j(\lambda_6^j)_j = 0,\ t_j^j + (\lambda_6^j)_j > 0$.

Denote $(x^*, t^*, \gamma^*, \lambda^*)$ by

$$x^* = \frac{1}{n}\sum_{j=1}^{n} x^j, \quad t^* = \frac{1}{n}\sum_{j=1}^{n} t^j, \quad \gamma^* = \frac{1}{n}\sum_{j=1}^{n} \gamma^j, \quad \lambda_i^* = \frac{1}{n}\sum_{j=1}^{n} \lambda_i^j, \quad i = 1, 2, \cdots, 6.$$

Since $(x^j, t^j, \gamma^j),\ j = 1, 2, ..., n$ are all optimal solutions of (3.11), then for any j, we have

$$\begin{cases} w^T t^j = Z^*, \\ \|\gamma^j\|_2 \leq \epsilon, \\ Bx^j \leq b, \\ \gamma^j = y - Ax^j, \\ |x^j| \leq t^j,\ t^j \geq 0. \end{cases} \tag{3.33}$$

It is easy to see that

$$w^T t^* = Z^*,\ Bx^* \leq b,\ \gamma^* = y - Ax^*,\ t^* \geq 0.$$

Moreover,

$$\|\gamma^*\|_2 = \left\| \frac{1}{n} \sum_{j=1}^{n} \gamma^j \right\|_2 \leq \sum_{j=1}^{n} \left\| \frac{1}{n} \gamma^j \right\|_2 \leq \epsilon,$$

$$|x^*| = \left| \frac{1}{n} \sum_{j=1}^{n} x^j \right| \leq \frac{1}{n} \sum_{j=1}^{n} |x^j| \leq \frac{1}{n} \sum_{j=1}^{n} t^j = t^*,$$

where the first inequality of each equation above follows from the triangle inequality. Then the vector (x^*, t^*, γ^*) satisfies

$$\begin{cases} w^T t^* = Z^*, \\ \|\gamma^*\|_2 \leq \epsilon, \\ Bx^* \leq b, \\ \gamma^* = y - Ax^*, \\ |x^*| \leq t^*, t^* \geq 0. \end{cases} \tag{3.34}$$

Thus (x^*, t^*, γ^*) is optimal to (3.11), and similarly it can be proven that $\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ is an optimal solution to (3.20). By strong duality, t^* and λ_6^* are complementary, i.e., $(t^*)^T \lambda_6^* = 0$. We now check whether t^* and λ_6^* are strictly complementary or not. Due to the above-mentioned property (2), it is impossible to find a pair (t^*, λ_6^*) such that their

jth component are both 0. Thus, (t^*, λ_6^*) is the strictly complementary solution pair for (3.11) and (3.20).

Next, we demonstrate the support of strictly complementary pairs are invariant. Suppose there are two distinct optimal pairs of the solutions of (3.11) and (3.20), denoted by $(x^k, t^k, \gamma^k, \lambda^k), k = 1, 2$, such that $(t^k, \lambda_6^k), k = 1, 2$ are strictly complementary pairs, where (x^k, t^k, γ^k) are optimal to (3.11) and (λ^k) are optimal to (3.20). Due to Theorem 30, we know that

$$(\lambda_6^1)^T t^2 = 0 \text{ and } (\lambda_6^2)^T t^1 = 0.$$

It means that the supports of all strictly complementary pairs of (3.11) and (3.20) are invariant. Otherwise, there exists an index j such that $t_j^1 > 0$ and $(\lambda_6^2)_j > 0$, leading to a contradiction. □

Since the optimal solution (x^*, t^*, γ^*) to (3.11) must have $t^* = |x^*|$ if $w > 0$, the main results of Theorem 33 also imply that $|x^*|$ and λ_6^* are strictly complementary under Assumption 32. Let Z^* be the optimal value of (3.5). Notice that the optimal solution of (3.5) remains the same when w is replaced by αw for any positive α. When $Z^* \neq 1$, by replacing W by W/Z^*, we can obtain

$$1 = \min_x \{\|(W/Z^*)x\|_1 : \ x \in T\}.$$

We use ζ to denote the set of such weights, i.e.,

$$\zeta = \{w \in R_+^n : \ 1 = \min_x \{\|Wx\|_1, x \in T\}\}, \tag{3.35}$$

where $W = \text{diag}(w)$. Clearly, $\bigcup_{\alpha > 0} \alpha\zeta$ is the set of weights such that (3.5) has a finite optimal value, and ζ is not necessarily bounded. Note that (3.5) is equivalent to (3.11), and under Slater condition, (3.11) and (3.20) satisfy strong duality. Thus under Slater condition, for each $w \in \zeta$, (3.11) and (3.20) satisfies strong duality and the optimal value of (3.20) can be attained. Moreover, given any $w \in \zeta$, by Theorem 31, any optimal solutions of (3.11) and (3.20), denoted by $(x^*(w), t^*(w), \gamma^*(w))$ and $\lambda^*(w) = (\lambda_1^*(w), ..., \lambda_6^*(w))$, satisfy that $|x^*(w)|$ and $\lambda_6^*(w)$ are complementary, i.e.,

$$\|x^*(w)\|_0 + \|\lambda_6^*(w)\|_0 \leqslant n. \tag{3.36}$$

If w^* satisfies Assumption 32, then Slater condition is automatically satisfied for (3.11) with w^* and (3.36) is also valid. Moreover, by Theorem 33, there exists a strictly complementary pair $(|x^*(w^*)|, \lambda_6^*(w^*))$ such that

$$\|x^*(w^*)\|_0 + \|\lambda_6^*(w^*)\|_0 = n.$$

If w^* is an optimal weight (see Definition 3.1), then $\lambda_6^*(w^*)$ must be the densest slack variable among all $w \in \zeta$, and locating a sparse vector can be converted to

$$\lambda_6^*(w^*) = \text{argmax}\{\|\lambda_6^*(w)\|_0 : \ w \in \zeta\}.$$

Note that if there exists a weight in (3.7) in Theorem 19 or (3.8) in Theorem 20 satisfying Assumption 32, then such a weight w^* does exist and can be obtained by solving the problem $\max\{\|\lambda_6^*(w)\|_0 : \ w \in \zeta\}$. Inspired by the above fact, we develop a theorem under Assumption 34 in the next section which claims that finding a sparsest point in T is equivalent to seeking the proper weight w such that the dual problem (3.20) has the densest optimal variable λ_6. Such weights are optimal weights and can be determined by certain bilevel programming. This idea was first introduced by Zhao & Kočvara [96] (and also by Zhao & Luo [99]) to solve the standard ℓ_0-minimization (1.8). In this thesis, we generalise this idea to solve the model (1.1).

3.5 Optimal weights via bilevel programming

In this section, we develop a bilevel programming model related to the sparsity problem (1.1). Before that, we make the following assumption:

Assumption 34. *Let ν be an arbitrary sparsest point in T given in* (1.2). *There exists a weight $\bar{w} \geqslant 0$ such that*

- $\langle H1 \rangle$ *The problem* (3.5) *with* $W = \mathrm{diag}(\bar{w})$ *has an optimal solution $\bar{x}$ such that* $\|\bar{x}\|_0 = \|\nu\|_0$,
- $\langle H2 \rangle$ *there exists an optimal variable in* (3.20) *with $w = \bar{w}$, denoted as $\bar{\lambda}$, such that $\bar{\lambda}_6$ and $\bar{x}$ are strictly complementary,*
- $\langle H3 \rangle$ *the optimal value of* (3.5) *with* $W = \mathrm{diag}(\bar{w})$ *is finite.*

Note that Theorem 20 implies that there exists a weight such that $\langle H1 \rangle$ and $\langle H3 \rangle$ are satisfied. Among these weights, if (3.5) has an optimal solution which is a relative interior point in T, then Theorem 33 indicates that $\langle H2 \rangle$ is automatically satisfied for such a weight. An example for the existence of a weight satisfying Assumption 34 is given in the remark after the following theorem.

Theorem 35. *Let the set Ω, given in* (3.25), *be nonempty and Assumption 34 hold. Consider the bilevel programming*

$$
\begin{aligned}
(P_b) \quad \max_{(w,\lambda)} \quad & \|\lambda_6\|_0 \\
\text{s.t.} \quad & B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, \quad w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\
& -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y = \min_x \{\|Wx\|_1 : x \in T\}, \\
& \lambda_i \geqslant 0, \quad i = 1, 2, 4, 5, 6,
\end{aligned}
\tag{3.37}
$$

where $W = \mathrm{diag}(w)$ *and* $T = \{x \in R^n : \|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b\}$. *If (w^*, λ^*) is an optimal solution to the above optimization problem* (3.37), *then any optimal solution x^* to*

$$
\min_x \{\|W^*x\|_1 : x \in T\}, \tag{3.38}
$$

is a sparsest point in T.

Proof. Let ν be a sparsest point in T. Since the set Ω is not empty, Slater condition holds for (3.5) and (3.11). Suppose that (w^*, λ^*) is an optimal solution of (3.37). We now prove that any optimal solution to (3.38) is a sparsest point in T under Assumption 34. Let w' be a weight satisfying Assumption 34, meaning that (3.5) with $W = \mathrm{diag}(w')$ has an optimal solution x' such that $\|x'\|_0 = \|\nu\|_0$. Moreover, there exists a strictly complementary pair (x', λ'_6) satisfying

$$\|x'\|_0 + \|\lambda'_6\|_0 = n = \|\lambda'_6\|_0 + \|\nu\|_0 \,. \tag{3.39}$$

where the vector $\lambda' = (\lambda'_1, ..., \lambda'_6)$ is the dual optimal solution of (3.20) with $w = w'$, i.e.,

$$\begin{aligned} \max_{\lambda} \quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \\ \text{s.t.} \quad & B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & \|\lambda_3\|_2 \leq \lambda_1, \\ & w^{'} = \lambda_4 + \lambda_5 + \lambda_6, \\ & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{aligned} \tag{3.40}$$

By Theorem 25, the nonemptyness of Ω implies that strong duality holds for the problems (3.40) and (3.11) with w'. Note that the optimal values of (3.11) and (3.5) with w' are equal and finite so that (w', λ') is feasible to (3.37). Let x^* be an arbitrary solution to (3.38). Note that (3.11) with w^* is equivalent to (3.38), to which the dual problem is

$$\begin{aligned} \max_{\lambda} \quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \\ \text{s.t.} \quad & B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & \|\lambda_3\|_2 \leq \lambda_1, \\ & w^* = \lambda_4 + \lambda_5 + \lambda_6, \\ & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{aligned} \tag{3.41}$$

Moreover, $\lambda^* = (\lambda^*_1, ..., \lambda^*_6)$ is feasible to (3.41) and the fourth constraint of (3.37) implies that there is no duality gap between (3.11) with w^* and (3.41). Thus, by strong duality,

$\lambda^* = (\lambda_1^*, ..., \lambda_6^*)$ is an optimal solution to (3.41). Therefore, by Theorem 31, $|x^*|$ and λ_6^* are complementary. Hence, we have

$$\|x^*\|_0 \leqslant n - \|\lambda_6^*\|_0 . \tag{3.42}$$

Since (w^*, λ^*) is optimal to (3.37), we have

$$\|\lambda_6'\|_0 \leqslant \|\lambda_6^*\|_0 . \tag{3.43}$$

Pluging (3.39) and (3.43) into (3.42) yields

$$\|x^*\|_0 \leqslant n - \|\lambda_6^*\|_0 \leqslant n - \|\lambda_6'\|_0 = \|x'\|_0 = \|\nu\|_0 ,$$

which implies

$$\|x^*\|_0 = \|\nu\|_0 ,$$

due to the assumption that ν is the sparsest point in T. Then any optimal solution to (3.39) is a sparsest point in T. □

Given Assumption 34 and the nonempty set Ω, finding a sparsest point in T is tantamountly equal to look for the densest dual solution via the bilevel model (3.37). The above theorem can be seen as a generalized version of a result in Zhao and Kočvara [96] (see also Zhao and Luo [99]). Before we close this chapter, we make some remarks for Assumption 34 and the necessary condition in Theorem 35.

Remark 36. *See Example 11. We know* $(0, 0, 2, 1)$ *is a sparsest point in the feasible set* T*. If we choose weights* $w = (100, 100, 1, 1)$*, then we can see that* $(0, 0, 2, 1)$ *is the unique optimal solution of* (3.5) *which satisfies* $\langle H1 \rangle$ *in Assumption 34. Note that the optimal value* Z^* *is* 3*. Thus,* $\langle H3 \rangle$ *in Assumption 34 is also satisfied. In addition,* $(0, 0, 2, 1)$ *is a relative interior point in the feasible set* T*. This, combined with the fact that weights are positive, implies that Assumption 32 is satisfied, and hence the strictly complemen-*

tary condition is satisfied. Therefore, by Theorem 33, it indicates that there must have a strictly complementary solution pair for (3.5) *and* (3.20)*. Note that* $(0, 0, 2, 1)$ *is the unique optimal solution of* (3.5)*. Thus* $\langle H2 \rangle$ *in Assumption 34 is satisfied. Specifically, we can find an optimal dual solution* $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_6)$ *with* $\bar{\lambda}_6 = (32.27, 31.71, 0, 0)$*. Therefore, the weight* $w = (100, 100, 1, 1)$ *satisfies Assumption 34.*

Remark 37. *Suppose there exist some weights satisfying Assumption 34. For any* $w \in \zeta$*, let* $x^*(w)$ *and* $\lambda^*(w)$ *be the optimal solutions to the problem* (3.5) *and* (3.20)*, respectively. Denote* $\lambda_6^*(w^*)$ *as a densest variable among the set* $\{\lambda_6^*(w) : w \in \zeta\}$*. Clearly,* $(w^*, \lambda^*(w^*))$ *is an optimal solution to* (3.37)*. By the definition of optimal weights, Theorem 35 implies that* w^* *is an optimal weight, i.e.,*

$$w^* \in \text{argmax}\{\|\lambda_6^*(w)\|_0 : w \in \zeta\},$$

by which a sparsest point can be obtained via (3.5)*. If there is no weight satisfying the properties in Assumption 34, a heuristic method for finding a sparse point in* T *can be also developed from* (3.36) *since the increase in* $\|\lambda_6(w)\|_0$ *leads to the decrease of* $\|x(w)\|_0$ *to a certain level.*

Remark 38. *If* w^* *is an optimal weight and also satisfies* $(H2)$ *and* $(H3)$ *in Assumption 34 or satisfies Assumption 32, then there exists a strictly complementary pair* (x^*, λ_6^*) *such that*

$$\|x^*\|_0 + \|\lambda_6^*\|_0 = n \tag{3.44}$$

where x^* *is an arbitrary solution of* (3.38) *and* λ^* *is an optimal solution of* (3.20) *with weight* w^**, and hence* x^* *is a sparsest point in* T *and* (w^*, λ^*) *is feasible to* (3.37)*. We can further show that* (w^*, λ^*) *is actually the optimal solution of* (3.37)*. In fact, let* $(\bar{w}, \bar{\lambda})$ *be an arbitrary solution to* (3.37) *and* $\bar{x}$ *be a solution to* (3.5)*. The equivalence between* (3.5) *and* (3.11) *implies that strong duality holds for* (3.11) *and* (3.20) *with* $\bar{w}$*, and hence* $\bar{\lambda}$ *is* *optimal to* (3.20) *with* $\bar{w}$. *By Theorem 31, we have*

$$\|\bar{x}\|_0 + \|\bar{\lambda_6}\|_0 \leqslant n. \tag{3.45}$$

Since x^* *is a sparsest point in* T, *we have*

$$\|x^*\|_0 \leqslant \|\bar{x}\|_0 \,. \tag{3.46}$$

Combining (3.44), (3.45) *and* (3.46) *yields:*

$$\|\lambda_6^*\|_0 = n - \|x^*\|_0 \geqslant n - \|\bar{x}\|_0 \geqslant \|\bar{\lambda_6}\|_0 \,.$$

Thus (w^*, λ^*) *is optimal to* (3.37).

Theorem 35 indicates that an optimal solution of (3.37) is the optimal dual variable λ_6 with the maximum number of nonzero components. This provides a theoretical basis to obtain an optimal weight, i.e., solving the problem (3.37). However, exactly solving this bilevel problem (3.37) [49] is difficult due to the discrete objective function and the nonconvex constraint. In the next chapter, we introduce a certain relaxation or approximation of this bilevel problem to develop some heuristic methods for solving the sparse optimization problem (1.1).

Chapter 4

Primal and Dual Re-weighted ℓ_1-algorithms

4.1 Introduction

We first briefly review some existing re-weighted ℓ_1-algorithms for sparse optimization problems. Consider the standard ℓ_0-minimization (1.8) and the corresponding weighted ℓ_1-minimization:

$$\begin{aligned} \min \quad & w^T|x| = \|Wx\|_1 \\ \text{s.t.} \quad & y = Ax, \end{aligned} \tag{4.1}$$

where w is a given vector of positive weights and $W = \text{diag}(w)$. Given w, the solution of (4.1) may not be the sparsest solution of the system $y = Ax$. Thus, in order to find a sparse solution, the re-weighted ℓ_1-algorithm is developed, which consists of a series of individuals weighted ℓ_1-minimization like (1.22). In the method proposed by Candès, Wakin and Boyd [24], the ith component of the weight w^{k+1}, denoted by w_i^{k+1}, is updated via the current iterate x_i^k, i.e.,

$$w_i^{k+1} = \frac{1}{|x_i^k| + \varepsilon}, \quad i = 1, ..., n, \tag{4.2}$$

where ε is a small positive number ensuring that w_i^{k+1} is not equal to ∞ when $x_i^k = 0$. The weight (4.2) penalizes the ith component of $|x|$ when x_i^k has a sufficiently small absolute value. By such a choice of weight, the iterate x^{k+1} (the solution of (4.1) with weight

w^{k+1}) satisfies $|x_i^{k+1}| \approx 0$. As a result, the new iterate x^{k+1} has almost the same sparsity pattern as x^k. More general than the method in [24], Zhao and Li [97] have proposed a unified method of the re-weighted ℓ_1-algorithms for solving (1.8). The performance of re-weighted ℓ_1-algorithms may depend on the initial weight w^0. For example, given an initial weight, if the support of the iterate obtained in the initial iteration is totally different from that of the sparsest solution of the linear system $Ax = y$, then the weight given as (4.2) might lead to the next iterate still with incorrect support. This might cause the algorithm to fail in some situations. Thus how to choose a proper weight in the first iteration is an issue for re-weighted ℓ_1-algorithms. In existing re-weighted ℓ_1-algorithms (e.g. [24], [42], [56], [97], [94]), the identity matrix I is chosen as the initial matrix W^0. This means that x^0 is obtained by solving the standard ℓ_1-minimization. Recently, Zhao, Kočvara and Luo [96, 99] have derived a new framework of re-weighted algorithms for solving (1.8) and (3.4) from the perspective of dual space.

In this chapter, we develop re-weighted ℓ_1-algorithms for the general model (P_0) given in (1.1) from the above-mentioned two different viewpoints of primal space and dual space. This chapter is organized as follows. First, Section 4.2 serves as an introduction to the merit functions for sparsity approximating the ℓ_0- and ℓ_0^δ-norms and defines a new merit function. In Section 4.3, we apply the re-weighted ℓ_1-algorthms proposed by Zhao and Li [97] to solve (1.1) and demonstrate their empirical performance by comparing with several existing algorithms. In Section 4.4, by connecting the merit functions to the bilevel programming (3.37), we present three types of relaxations for the bilevel problem (3.37) to develop the dual weighted algorithms and dual re-weighted algorithms for solving the ℓ_0-minimization problem (1.1).

4.2 Merit functions for sparsity

A function is called a merit function for sparsity if it can approximate $\|s\|_0$ in some sense (see Zhao [94] and Zhao and Li [97]). As discussed in Chapter 2, $\sum_{i=1}^{n}(\log(|s_i| + \varepsilon))$ and $\|s\|_p^p, 0 < p < 1$ can be seen as merit functions. Due to the approximation of ℓ_0-norm, concave functions are shown to be the good candidates for the merit functions for sparsity. In fact, Harikumar and Bresler in [51] have shown that when finding a sparsest point of a linear system, the concave function is an ideal choice for approximating the ℓ_0-norm, since convex functions might hinder locating the sparsest point because of their bulgy features. Many researches do indicate that the concave functions, denoted by $\Psi_\varepsilon(s)$, may approximate $\|s\|_0$ better than other functions where ε is sufficiently small [24, 94, 96]. In [97], Zhao and Li have identified a family of merit functions in the form

$$\Psi_\varepsilon(s) = \sum_{i=1}^{n} \varphi_\varepsilon(s_i), \;\; s \in R_+^n, \tag{4.3}$$

satisfying, roughly, the following properties:

- $(P1)$. for any given $s \in R_+^n$, $\Psi_\varepsilon(s)$ tends to $\|s\|_0$ as ε tends to 0, i.e.,

$$\lim_{\varepsilon \to 0} \Psi_\varepsilon(s) = \|s\|_0 \, ;$$

- $(P2)$. $\Psi_\varepsilon(s)$ is twice continuously differentiable with respect to s in all of R_+^n;
- $(P3)$. $\Psi_\varepsilon(s)$ is strictly concave in $s \in R_+^n$ and strictly increasing in every $s_i \in R_+$;
- $(P4)$. $\Psi_\varepsilon(s)$ is separable and coercive in s;
- $(P5)$. for any given $s_i > 0$, $\nabla\varphi_\varepsilon(s_i)$ tends to a finite positive number when $\varepsilon \to 0$, and

$$\lim_{(s_i,\varepsilon)\to(0,0)} \nabla\varphi_\varepsilon(s_i) = \lim_{(s_i,\varepsilon)\to(0,0)} (\nabla\Psi_\varepsilon(s))_i = \infty.$$

Clearly, (4.3) can be rewritten as $\Psi_\varepsilon(|s|) = \sum\limits_{i=1}^{n} \varphi_\varepsilon(|s_i|), \;\; s \in R^n$. Zhao and Li [97] have given a list of merit functions satisfying $(P1)$-$(P5)$, for example,

$$\Psi_\varepsilon(|s|) = n - \frac{\sum_{i=1}^{n} \log(|s_i| + \varepsilon)}{\log \varepsilon}, \;\; s \in R^n, \tag{4.4}$$

$$\Psi_\varepsilon(|s|) = \frac{1}{p}\sum_{i=1}^{n}(|s_i| + \varepsilon)^p, \quad p \in (0,1), \; s \in R^n. \tag{4.5}$$

In this chapter, the merit functions we used are not required to satisfy all the properties $(P1)$-$(P5)$. Instead, we consider a function satisfying $(P1)$ and $(P2)$ and the following conditions:

- $(P3')$. $\Psi_\varepsilon(s)$ is non-strictly concave and strictly increasing with respect to every $s \in R^n_+$.
- $(P4')$. $\Psi_\varepsilon(s)$ is separable in s.

Note that $(P3')$ is weaker than $(P3)$, and $(P4')$ is weaker than $(P4)$. Moreover, we hope that the merit functions satisfy the monotonic property such that the merit function $\Psi_\varepsilon(|s|)$ increases when $\|s\|_0$ increases. Zhao and Luo have defined this property in [99], and we still use the same definition of such monotonic property. This, together with $(P1)$, $(P2)$, $(P3')$ and $(P4')$, yields a family of merit functions we will use to approximate the ℓ_0-norm or ℓ_0^δ-norm in this chapter.

Assumption 39. *Let $\Re(\vartheta_1, \vartheta_2) = \{s \in R^n : \vartheta_1 \leqslant |s_i| \leqslant \vartheta_2 \quad \text{when} \quad s_i \neq 0\}$ be a set where ϑ_1, ϑ_2 are two given positive constants and $\vartheta_1 \leqslant \vartheta_2$. Let $\Psi_\varepsilon(|s|), s \in R^n$ be a merit function satisfying the following properties:*

- *$(P1)$, $(P2)$, $(P3')$ and $(P4')$;*
- *the monotonic property: for any fixed $\vartheta^* \in (0,1)$, there exists a positive number ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, any (s_1, s_2) satisfying*

$$s_1 \in \Re(\vartheta_1, \vartheta_2), \; s_2 \in \Re'(\vartheta_2) = \{s \in R^n : 0 \leqslant |s_i| \leqslant \vartheta_2\} \text{ and } \|s_1\|_0 > \|s_2\|_0,$$

  the following inequality holds:

$$\Psi_\varepsilon(|s_2|) - \Psi_\varepsilon(|s_1|) \leqslant \vartheta^* - 1 < 0. \tag{4.6}$$

In this chapter, we use such merit functions satisfying Assumption 39. Note that $(P3')$ and $(P4')$ are milder than $(P3)$ and $(P4)$, respectively. Thus (4.4) and (4.5) both satisfy $(P1)$, $(P2)$, $(P3')$ and $(P4')$. Zhao and Luo [99] have given examples of merit functions satisfying the monotonic property, for example,

$$\Psi_\varepsilon(|s|) = \sum_{i=1}^{n} \frac{|s_i|}{|s_i| + \varepsilon}, \ \ s \in R^n, \tag{4.7}$$

$$\Psi_\varepsilon(|s|) = \sum_{i=1}^{n} (|s_i| + \varepsilon^{1/\varepsilon})^\varepsilon, \ \ s \in R^n \tag{4.8}$$

and (4.4) where $\varepsilon \in (0, 1)$. It is easy to check the above functions (4.7) and (4.8) satisfy $(P1)$, $(P2)$, $(P3')$ and $(P4')$. Therefore the functions (4.4), (4.7) and (4.8) are the specific examples of merit functions satisfying Assumption 39. We now introduce a new merit function:

$$\Psi_\varepsilon(|s|) = \frac{2}{\pi} \sum_{i=1}^{n} \arctan(\frac{|s_i|}{\varepsilon}), \ \ s \in R^n \tag{4.9}$$

where $\varepsilon > 0$. It can be shown that (4.9) satisfies Assumption 39.

Lemma 40. *The function* (4.9) *satisfies Assumption 39.*

Proof. Obviously, the function (4.9) satisfies $(P1)$, $(P2)$ and $(P4')$. In R^n_+, note that

$$\nabla \Psi_\varepsilon(s) = (\nabla \varphi_\varepsilon(s_1), ..., \nabla \varphi_\varepsilon(s_n))^T = \frac{2}{\pi}\left(\frac{\varepsilon}{s_1^2 + \varepsilon^2}, ..., \frac{\varepsilon}{s_n^2 + \varepsilon^2} \right)^T,$$

$$\nabla^2 \Psi_\varepsilon(s) = \begin{bmatrix} \nabla^2 \varphi_\varepsilon(s_1) & & \\ & \ddots & \\ & & \nabla^2 \varphi_\varepsilon(s_n) \end{bmatrix} = \frac{4}{\pi} \mathrm{diag}\left(-\frac{\varepsilon s_1}{(s_1^2 + \varepsilon^2)^2}, ..., -\frac{\varepsilon s_n}{(s_n^2 + \varepsilon^2)^2} \right).$$

Due to $s_i \geqslant 0$ and $\varepsilon > 0$, we have $\nabla \varphi_\varepsilon(s_i) > 0$ and $\nabla^2 \varphi_\varepsilon(s_i) \preceq 0$ for $i = 1, ..., n$ (i.e., $\nabla^2 \Psi_\varepsilon(s)$ is negative semidefinite) which means $\Psi_\varepsilon(s)$ is concave and strictly increasing with respect to every entry of $s \in R^n_+$. Thus (4.9) satisfies $(P1), (P2), (P3')$ and $(P4')$. We now prove that (4.9) satisfies the monotonic property in Assumption 39. Let ϑ_1, ϑ_2 be two arbitrary numbers such that $0 < \vartheta_1 \leqslant \vartheta_2$, and the two sets $\Re(\vartheta_1, \vartheta_2)$ and $\Re'(\vartheta_2)$

be defined as in Assumption 39. Let $\vartheta^* \in (0,1)$ be a fixed number. We now prove that there exists a small ε^* such that (4.9) satisfies

$$\Psi_\varepsilon(|s_2|) - \Psi_\varepsilon(|s_1|) \leqslant \vartheta^* - 1 < 0,$$

for any $\varepsilon \in (0, \varepsilon^*]$ and any (s_1, s_2) such that $s_1 \in \Re(\vartheta_1, \vartheta_2)$, $s_2 \in \Re^{'}(\vartheta_2)$ and $\|s_1\|_0 > \|s_2\|_0$. Since the function $\arctan(|s_i|/\varepsilon)$ is a continuously increasing function when $|s_i|$ increases, then for any $s_1 \in \Re(\vartheta_1, \vartheta_2)$, we have

$$\arctan\frac{\vartheta_1}{\varepsilon} \leqslant \arctan\frac{|(s_1)_i|}{\varepsilon} \leqslant \arctan\frac{\vartheta_2}{\varepsilon}, \; i = 1, ..., n. \tag{4.10}$$

We can find $\varepsilon^* > 0$ such that

$$\arctan\frac{\vartheta_1}{\varepsilon^*} = \frac{\pi}{2} - \frac{\vartheta^*\pi}{2n},$$

i.e.,

$$\varepsilon^* = \frac{\vartheta_1}{\tan(\frac{\pi}{2} - \frac{\vartheta^*\pi}{2n})}.$$

Clearly, for any $\varepsilon \in (0, \varepsilon^*]$ and for any $|s_1| \in \Re(\vartheta_1, \vartheta_2)$, we have

$$\frac{\pi}{2} - \frac{\vartheta^*\pi}{2n} = \arctan\frac{\vartheta_1}{\varepsilon^*} \leqslant \arctan\frac{\vartheta_1}{\varepsilon} \leqslant \arctan\frac{|(s_1)_i|}{\varepsilon} \leqslant \arctan\frac{\vartheta_2}{\varepsilon} < \frac{\pi}{2}, \; i = 1, ..., n. \tag{4.11}$$

Thus, for any $\varepsilon \in (0, \varepsilon^*]$ and (s_1, s_2) such that $s_1 \in \Re(\vartheta_1, \vartheta_2)$ and $s_2 \in \Re^{'}(\vartheta_2)$, we have

$$\begin{aligned}\Psi_\varepsilon(|s_1|) &= \frac{2}{\pi}\sum_{i=1}^{n}\arctan(\frac{|(s_1)_i|}{\varepsilon}) = \frac{2}{\pi}\sum_{|(s_1)_i| \in [\vartheta_1, \vartheta_2]}\arctan(\frac{|(s_1)_i|}{\varepsilon}) \\ &\geqslant \frac{2}{\pi}\|s_1\|_0(\frac{\pi}{2} - \frac{\vartheta^*\pi}{2n}) = \|s_1\|_0(1 - \frac{\vartheta^*}{n}),\end{aligned}$$

and

$$\Psi_\varepsilon(|s_2|) = \frac{2}{\pi}\sum_{|(s_2)_i| \in (0, \vartheta_2]}\arctan(\frac{|(s_2)_i|}{\varepsilon}) < \|s_2\|_0,$$

which follows from the truth that

$$\arctan \frac{|(s_2)_i|}{\varepsilon} \leqslant \arctan \frac{\vartheta_2}{\varepsilon} < \frac{\pi}{2}, \ \forall s_2 \in \Re^{'}(\vartheta_2).$$

Combining the above inequalities and the condition $\|s_2\|_0 < \|s_1\|_0$ yields

$$\Psi_\varepsilon(|s_2|) - \Psi_\varepsilon(|s_1|) < \|s_2\|_0 - \|s_1\|_0 + \|s_1\|_0 \frac{\vartheta^*}{n} \leqslant -1 + \|s_1\|_0 \frac{\vartheta^*}{n} \leqslant -1 + \vartheta^* < 0,$$

which is the desired result. □

We will focus on the merit functions satisfying Assumption 39. We denote the set of such merit functions by

$$\mathbf{F} = \{\Psi_\varepsilon : \Psi_\varepsilon \quad \text{satisfies} \quad \text{Assumption} \quad 39\}$$

Note that the functions (4.4), (4.7), (4.8) and (4.9) are in the set $\mathbf{F}$. Any convex combination of a finite number of merit functions in $\mathbf{F}$ also satisfies Assumption 39. Thus the set $\mathbf{F}$ is convex. In the next sections, we use the merit functions $\Psi_\varepsilon \in \mathbf{F}$ to develop the re-weighted algorithms for solving (1.1) from the viewpoints of primal and dual spaces.

4.3 Re-weighted ℓ_1-algorithms in primal space

By replacing $\|x\|_0$ with $\Psi_\varepsilon(|x|) \in \mathbf{F}$, we obtain an approximation problem of (1.1):

$$\min\left\{\Psi_\varepsilon(|x|) = \sum_{i=1}^{n} \varphi_\varepsilon(|x_i|) : \|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b\right\}. \tag{4.12}$$

Denote by

$$\widetilde{T} = \{(x, t) : \|y - Ax\|_2 \leqslant \epsilon, Bx \leqslant b, |x| \leqslant t\}.$$

Since $\varphi_\varepsilon(|x_i|)$ is strictly increasing with respect to $|x_i|$, then the above optimization (4.12) can be rewritten as the following problem:

$$\min_{(x,t)}\{\Psi_\varepsilon(t) : (x,t) \in \widetilde{T}\}. \tag{4.13}$$

4.3.1 Approximation by first-order method

By using 1st order Taylor expansion of $\Psi_\varepsilon(t) \in \mathbf{F}$ at point t^k, we have

$$\Psi_\varepsilon(t) = \Psi_\varepsilon(t^k) + \nabla\Psi_\varepsilon^T(t^k)(t - t^k) + o(\|t - t^k\|).$$

The concavity of $\Psi_\varepsilon(t)$ implies

$$\Psi_\varepsilon(t) \leqslant \Psi_\varepsilon(t^k) + \nabla\Psi_\varepsilon^T(t^k)(t - t^k) \text{ for } t \in R^n. \tag{4.14}$$

Thus (4.13) can be approximated by the following optimization:

$$\min_{(x,t)}\{\Psi_\varepsilon(t^k) + \nabla\Psi_\varepsilon^T(t^k)(t - t^k) : (x,t) \in \widetilde{T}\}, \tag{4.15}$$

which is equivalent to $\min_{(x,t)}\{\nabla\Psi_\varepsilon^T(t^k)t : (x,t) \in \widetilde{T}\}$. By setting k as the iteration index, (4.15) can be seen as an iterative scheme to generate the new point (x^{k+1}, t^{k+1}), i.e.,

$$(x^{k+1}, t^{k+1}) \in \underset{(x,t)}{\operatorname{argmin}}\{\nabla\Psi_\varepsilon^T(t^k)t : (x,t) \in \widetilde{T}\}. \tag{4.16}$$

Clearly, due to the fact that $\Psi_\varepsilon(t)$ is strictly increasing with respect to each $t_i \in R_+$ (see $P(3')$), it is evident that the iterate (x^k, t^k) must satisfy $t^k = |x^k|$, which implies that (4.16) is equivalent to

$$x^{k+1} \in \underset{x}{\operatorname{argmin}}\{\nabla\Psi_\varepsilon^T(|x^k|)|x| : x \in T\}. \tag{4.17}$$

4.3.2 Re-weighted ℓ_1-algorithm via first-order approximation

Note that $\nabla\Psi_\varepsilon(|x^k|)$ in (4.17) can be seen as the weight w in (3.5). Therefore (4.17) motivates us to develop the re-weighted algorithm with the weight updating scheme

$$w_i^{k+1} = (\nabla\Psi_\varepsilon(|x^k|))_i,\ i = 1, ..., n \ \text{ or } \ W^{k+1} = \text{diag}(\nabla\Psi_\varepsilon(|x^k|)).$$

to find the sparsest point in the feasible set T. This can be stated as follows:

Algorithm: Primal Re-weighted ℓ_1-algorithm (**PRA**)

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$;
sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$ and error bound $\epsilon \in R_+$;
small positive parameter $\varepsilon \in R_{++}$ and initial weight w^0;
the iteration index k and the largest number of iterations $k_{\max}$.

Iteration: At the current iterate x^{k-1}, solve the weighted ℓ_1-minimization

$$x^k \in \text{argmin}\left\{ \sum_{i=1}^{n} w_i^k |x_i| : \quad x \in T \right\},$$

where $w_i^k = \nabla\Psi_\varepsilon(|x^{k-1}|)_i = \nabla\varphi_\varepsilon(|x_i^{k-1}|),\ i = 1, ..., n.$

Update: $w_i^{k+1} := (\nabla\Psi_\varepsilon(|x^k|))_i = \nabla\varphi_\varepsilon(|x_i^k|)$, $i = 1, ..., n$; Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

In order to let PRA likely have a good performance in solving (1.1) or recovering sparse vectors, we may require that the merit functions $\Psi_\varepsilon(t) \in \mathbf{F}$ satisfy the second part of $(P5)$, i.e., for all $i = 1, ..., n$, $(\nabla\Psi_\varepsilon(t))_i$ tends to infinity as (t_i, ε) tends to 0. In fact, the merit functions (4.4), (4.7), (4.8) and (4.9) all satisfy the above property. For (4.9), note that for any $s_i \in R_+$, we have

$$\lim_{(s_i,\varepsilon)\to(0,0)} \nabla\varphi_\varepsilon(s_i) = \infty$$

since $\lim_{s_i\to 0} \nabla\varphi_\varepsilon(s_i) = \frac{1}{\varepsilon}$. Having such a property can let the merit functions $\Psi_\varepsilon(t) \in \mathbf{F}$ have a better approximation of ℓ_0-norm. Due to this, the algorithms might have a good performance of recovering sparse vectors. In Subsection 4.3.4, we carry out some numercial experiments to show the performance of PRA by using three different merit functions (4.7), (4.8), (4.9) and we compare their performances with the traditional re-weighted algorithm with the weight updating scheme in [24], which is often called CWB.

4.3.3 Convergence of PRA

Clearly, (4.17) is equivalent to (4.16) for any $\Psi_\varepsilon(t) \in \mathbf{F}$. So Algorithm PRA may take another form by replacing

$$x^k \in \text{argmin}\left\{ \sum_{i=1}^{n} w_i^k |x_i| = \sum_{i=1}^{n} (\nabla \Psi_\varepsilon(|x^{k-1}|))_i |x_i|, \quad \text{s.t.} \quad x \in T \right\}$$

with its equivalent version:

$$(x^k, t^k) \in \text{argmin}\left\{ \sum_{i=1}^{n} w_i^k t_i = \sum_{i=1}^{n} (\nabla \Psi_\varepsilon(t^{k-1}))_i t_i, \quad \text{s.t.} \quad (x, t) \in \widetilde{T} \right\}.$$

We now briefly discuss the convergence of PRA with the iteration scheme (4.16).

Mangasarian [63] showed that under suitable assumption, the well-known successive linearization algorithm (SLA), an algorithm for concave minimization over a polyhedral set, terminates at a stationary point after a finite number of steps and generates a series of points with decreasing objective function value. For the noiseless recovery, Lai and Wang [56] showed that the re-weighted ℓ_1-algorithm based on ℓ_q-minimization with $q \in (0, 1)$ creates a sequence convergent to a stationary point under some additional assumptions on the underdetermined linear systems $y = Ax$. Zhao and Li [97] showed that if A^T satisfies the so-called RSP condition, the re-weighted ℓ_1-algorithms for (1.8) can truly converge to a sparse solution when the merit functions for sparsity are properly chosen. Note that the constraints in (4.13) are more general than the constraints in (1.8) or (1.9). Thus, it is more challenging to prove that Algorithm 1 converges to a sparse point. This is a worthwhile future work. Nevertheless, the convergence to a stationary point can be shown under some mild assumptions.

Rinaldi, Schoen and Sciandrone [71], and Rinaldi [70] considered the following concave problem:

$$\min\{g(x,z)+\sum_{i=1}^{n}h_i(z_i)+\sum_{i=1}^{n}f_i(x_i):(x,z)\in F,\ z_i\geq 0\}, \tag{4.18}$$

where F is a compact convex set in the form

$$F=\left\{(x,z)\in R^n\times R^n: S_p(x)+\sum_{i=1}^{n}Q_{pi}(z_i)\leqslant 0\right\}$$

and

- $g: R^n\to R$ is a continuously differentiable function;
- for all i, $h_i, f_i: R\to R$ are concave and continuously differentiable functions;
- for all i, S_p and Q_{pi} are convex and continuously differentiable functions.

They proposed a revised Frank-Wolfe algorithm (called *FW-RD* algorithm) via the 1st order approximation to solve the concave minimization (4.18) over F, and showed its convergence under mild conditions for h_i. The problem (4.13) can be written as (4.18) where we have

- $g(x,z)=0$, $h_i(z_i)=\varphi_\varepsilon(t_i)$, $f_i(x_i)=0$, $z_i=t_i$, $i=1,...,n$;
- $S_1(x)=\|y-Ax\|_2-\epsilon$, $Q_{1i}(z_i)=0, i=1,...,n$, $p=1$;
- $S_{1+j}(x)=x_j$, $Q_{(1+j)i}(t_i)=\begin{cases}0, & i\neq j\\ -t_i, & i=j\end{cases}$, $j=1,...,n, i=1,...,n$;
- $S_{1+n+j}(x)=-x_j$, $Q_{(1+n+j)i}(t_i)=\begin{cases}0, & i\neq j\\ -t_i, & i=j\end{cases}$, $j=1,...,n, i=1,...,n$;
- $S_{1+2n+j}(x)=B_jx-b_j$, $Q_{(1+2n+j)i}(t_i)=0$, $j=1,...,l, i=1,...,n$,

where B_j is the jth row in B. Clearly, $\widetilde{T}$ is a closed convex set but may not be bounded. However, we can add some constraints to give a sufficiently large upper bound for t such that $t_i \leqslant C, i = 1, ...n$, which can be represented as

$$S_{1+2n+l+j}(x) = -C, \ Q_{(1+2n+l+j)i}(t_i) = \begin{cases} 0, & i \neq j \\ t_i, & i = j \end{cases}, j = 1, ..., n, i = 1, ..., n.$$

In that case, $\widetilde{T}$ can be converted to a bounded set, and hence the problem (4.13) with $\widetilde{T}$ can be written as (4.16) from the above observation. We consider the following property for h_i:

$(P7)$ for every i, there exists a number K such that $\nabla h_i(0) \geq K$.

Rinaldi [70] has shown the generic convergence of *FW-RD* algorithms for a large number of concave functions satisfying $(P7)$. With such a function, the *FW-RD* algorithm can guarantee that every limit point of the generated sequence is a stationary point of the problem.

For our model (4.13), $(P7)$ means that for $\forall i, (\nabla \Psi_\varepsilon(0))_i = \nabla \varphi_\varepsilon(0) \geq K$. Clearly, due to $(P3')$, $(P7)$ is automatically satisfied for $\forall \Psi_\varepsilon \in \mathbf{F}$ by choosing $K = 0$. Thus under the compactness of the feasible set $\widetilde{T}$, the convergence of the algorithm PRA can be obtained, that is, there exists a family of merit functions $\Psi_\varepsilon \in \mathbf{F}$ such that PRA converges to a stationary point.

4.3.4 Numerical performance

In this subsection, through different merit functions $\Psi_\varepsilon \in \mathbf{F}$, we demonstrate the numerical performance of PRA for finding a sparse point in T. Since we can not test all of the merit functions in $\mathbf{F}$, we only consider the merit functions (4.7), (4.8) and (4.9). For each case, we also compare the algorithms with ℓ_1-minimization (2.24) and CWB method with weight

$$w_i^{k+1} = \frac{1}{|x_i^k| + \varepsilon}, \quad i = 1, ..., n. \tag{4.19}$$

Since (4.19) is motivated by the merit function (4.4), we do not carry out the numerical experiments for PRA by (4.4). The following table shows the algorithms to be performed.

Table 4.1: Algorithms to be tested

Name	Merit Function	Re-weighted Methods
ℓ_1	$\|x\|_1$	ℓ_1-minimization
CWB	$\sum_{i=1}^{n}\log(\|x_i\|+\varepsilon)$	PRA
REW1	(4.7)	PRA
REW2	(4.8)	PRA
ARCTAN	(4.9)	PRA

The algorithm names listed in the 'Name' column are for the ℓ_1 and PRA methods with those merit functions listed in the central column of the above table, respectively. We now briefly introduce the environment of our experiments. Given the noise level $\epsilon = 10^{-4}$, the parameter $\varepsilon = 10^{-1}$ and the dimension $(m, n) = (50, 200)$ of A. We set three different values of $l = 15, 30, 50$ (the number of the columns in matrix B) to compare the performance of the above-mentioned algorithms for locating the sparse points in T. To determine the convex set T, we generate the data (A, B, x^*) with given sparsity levels of x^*. The elements of A and B, and the nonzero entries of the sparse vector x^* are randomly generated from independent and identically distributed (i.i.d.) random Gaussian variables with zero means and unit variances. As long as (A, B, x^*) is generated, we can determine the measurements y and b as

$$y = Ax^* + \frac{c_1\varepsilon}{\|c\|_2}c, \quad b = Bx^* + d,$$

where $d \in R^l_+$ is randomly generated as the absolute value of Gaussian random variables with zero means and unit variances, and $c \in R^m$ and $c_1 \in R$ are randomly generated Gaussian random variables with zero means and unit variances. Then the set T can be obtained and all examples of T in our experiments are given this way. Clearly, x^* is a feasible point in $T := \{x : \|y - Ax\|_2 \leqslant \varepsilon, Bx \leqslant b\}$. For each sparsity level, we generate 200 trials of (A, B, x^*) and calculate the success frequency of finding the sparsest points in

T for all algorithms being tested. If x^* is the solution found by an algorithm, we count one 'success' for this trial. However, when the convex set T admits multiple sparsest points, an algorithm is still successful in solving ℓ_0-problem if the found solution is one of the sparsest points in T. Therefore, in this case, the criterion 'exact recovery' is not suitable to measure the success frequency of an algorithm. Instead, we can use $\|x^k\|_0 \leqslant \|x^*\|_0$ or $\|x^k - x^*\|_2 / \|x^*\|_2 \leqslant 10^{-5}$ as the criteria of finding the sparse vector x^* where x^k is the point generated by an algorithm after k iterations. In our experiments, we prefer to choose the second criterion $\|x^k - x^*\|_2 / \|x^*\|_2 \leqslant 10^{-5}$ as our default criterion, and we perform at most 5 iterations for each re-weighted algorithm CWB, REW1, REW2 and ARCTAN at each sparsity level. The following three figures show the numerical results of CWB, REW1, REW2 and ARCTAN via calculating the success rate of finding the sparse vector at each sparsity level in the case of $l = 15, 30$ and 50, respectively.

The numerical result in the case of $l = 15$

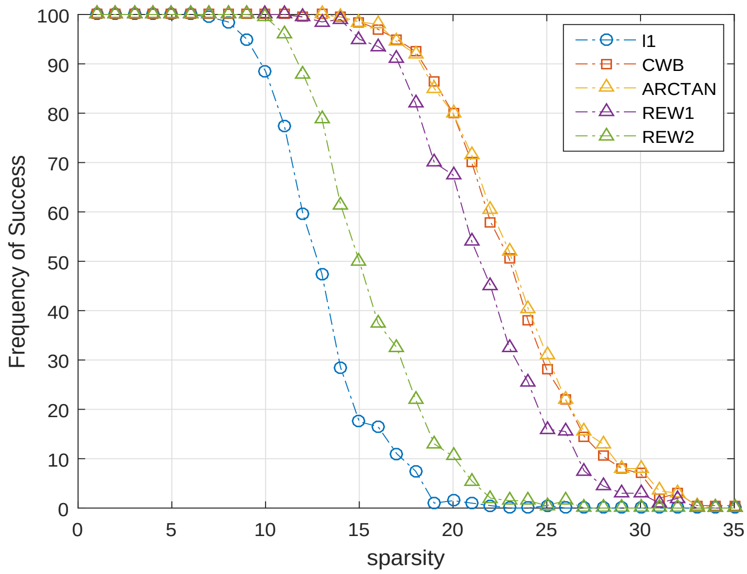


Figure 4.1: The success rate of finding a sparse vector in T via CWB, REW1, REW2 and ARCTAN with Gaussian matrices $A \in R^{50\times 200}, B \in R^{15\times 200}$ and $(\epsilon, \varepsilon) = (10^{-4}, 10^{-1})$. For each sparsity of $\|x\|_0 \leqslant 35$, 200 trials were performed.

The parameters m, n, ε and the noise level ϵ are fixed as

$$(m, n, \varepsilon, \epsilon) = (50, 200, 10^{-1}, 10^{-4})$$

in all the cases of $l = 15, 30, 50$. Figure 4.1 indicates that CWB, REW1, REW2 and ARCTAN outperform l1. For our new algorithms, the experiments show that ARCTAN performs better than others, and it is quite comparable to CWB. Note that CWB, REW1 and ARCTAN are very capable of locating the sparse vectors in T, even when these vectors have high sparsity levels. For example, Figure 4.1 shows that when at the sparsity level 25, REW1 and l1 fail to find sparse vectors but CWB, REW1 and ARCTAN still have reasonably good success rate $16\%, 29\%, 31\%$, respectively.

The numerical result in the case of $l = 30$

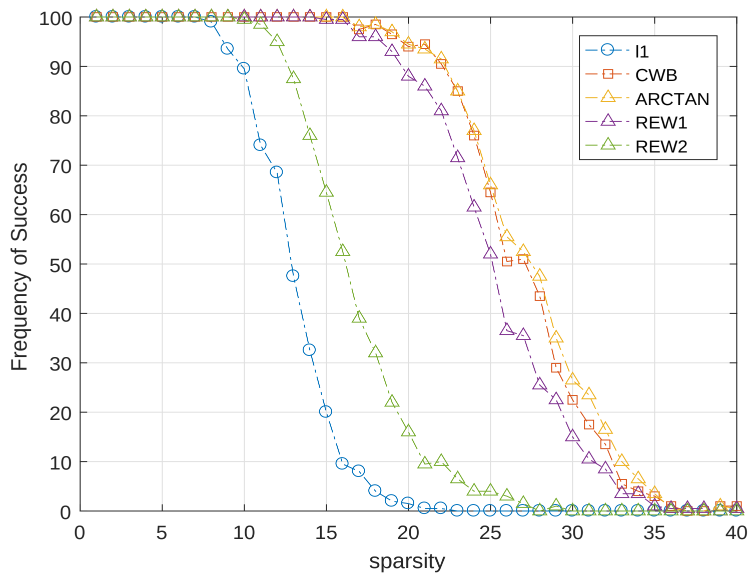


Figure 4.2: The success rate of finding the sparse vectors in T via CWB, REW1, REW2 and ARCTAN with Gaussian matrices $A \in R^{50\times200}$ and $B \in R^{30\times200}$ and $(\epsilon, \varepsilon) = (10^{-4}, 10^{-1})$. For each sparsity of $\|x\|_0 \leqslant 40$, 200 trials were made.

Figures 4.2 and 4.3 show the similar results to Figure 4.1. In addition, Figures 4.2 and 4.3 also show that the curves of CWB, REW1, REW2 and ARCTAN in Figure 4.1 is shifted to the right slightly when l is increased. This makes sense since when l is increased, more information for the measurements can be acquired, the re-weighted ℓ_1-algorithms may gain a better performance in finding the sparse vectors in T.

The numerical result in the case of $l = 50$

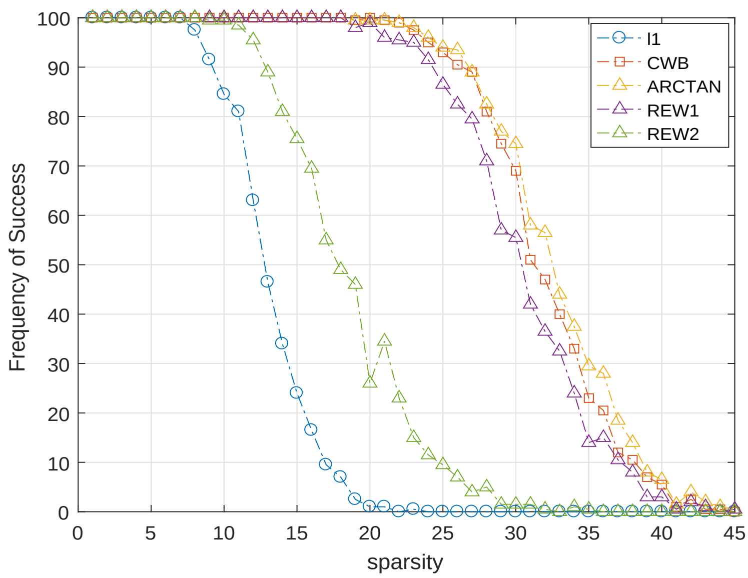


Figure 4.3: The success rate of finding the sparse vectors in T via CWB, REW1, REW2 and ARCTAN with Gaussian matrices $A \in R^{50\times 200}$ and $B \in R^{50\times 200}$ and $(\epsilon, \varepsilon) = (10^{-4}, 10^{-1})$. For each sparsity of $\|x\|_0 \leqslant 45$, 200 trials were performed.

The influence of ε

The above numerical results are based on a fixed parameter ε. Candès et al. [24] developed an updating scheme for the parameter ε in their algorithm with weight (4.19). This might improve the performance of the CWB. Similar to Candès's idea, we use the following updating scheme for ε in PRA:

$$\varepsilon^{k+1} = \max\{(\sigma(x^k))_{\bar{i}}, 10^{-3}\}, \tag{4.20}$$

where $\bar{i}$ denotes the nearest integer to $(m+l)/4(\log(n/(m+l)))$ and $\sigma(x^k)$ is an operator sorting the absolute value of x_i^k from largest to smallest. Adding $\varepsilon^0 = 10^{-3}$ in the initialization step and using the update rule (4.20) in PRA, we obtain the following algorithm:

Algorithm: Revised Primal Re-weighted ℓ_1-algorithm [**RPRA**]

Input:

merit Function $\Psi_\varepsilon \in \mathbf{F}$;
sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$ and error bound $\epsilon \in R_{++}$;
initial weight w^0 and initial parameter for merit function $\varepsilon^0 = 10^{-3}$;
the iteration index k and the largest number of iteration $k_{\max}$;

Iteration: At the current iterate x^{k-1}, solve the weighted ℓ_1-minimization

$$x^k \in \text{argmin}\left\{ \sum_{i=1}^{n} w_i^k |x_i| : \ x \in T \right\},$$

where $w_i^k = \nabla\Psi_\varepsilon(|x^{k-1}|))_i$ and $\varepsilon = \varepsilon^k = \max\{(\sigma(x^{k-1}))_{\bar{i}}, 10^{-3}\}$;

Update:

$w_i^{k+1} = (\nabla\Psi_\varepsilon(|x|^k))_i,\ i = 1, ..., n$ where $\varepsilon = \varepsilon^{k+1} = \max\{(\sigma(x^k))_{\bar{i}}, 10^{-3}\}$;
Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

By adding the rule (4.20), we now compare the performance of the algorithms CWB, REW1 and ARCTAN with ℓ_1-minimization in locating the sparse vectors in T. With updating rule (4.20), the performance of REW1 and ARCTAN for finding the sparse vectors in T were given in Figures 4.4 and 4.5 respectively.

Here we consider the case $l = 40$ and the parameters, except for ε, are taken the same values as that in previous cases. We first compare the performance of each algorithm REW1 and ARCTAN with different parameter $\varepsilon = 0.1, 0.01$ and (4.20). The right number in the legend of Figures 4.4 and 4.5 means the level of the parameter ε. For example, 0.1 means $\varepsilon = 0.1$ and the 'update' means that ε is updated by the scheme (4.20).

Clearly, when $\varepsilon = 0.001$, both algorithms REW1 and ARCTAN cannot compete with the case where $\varepsilon = 0.1$ or ε being updated by (4.20). However, they are still better than ℓ_1-minimization (2.24). In addition, with (4.20), REW1 and ARCTAN are the 'best' among the algorithms that we compare. For example, REW1 has an improvement in its performance when $21 \le \|x\|_0 \le 36$, and so does ARCTAN when $17 \le \|x\|_0 \le 30$.

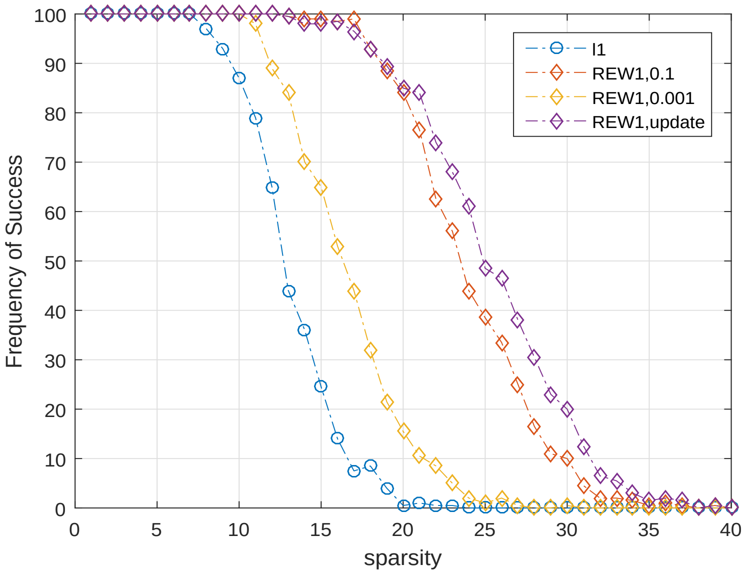


Figure 4.4: Empirical resluts of the frequency of success of REW1 with different ε in locating the sparse vectors in T. The Gaussian matrices are of the size: $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$. For each sparsity level of $\|x\|_0 \leqslant 40$, 200 trials were made.

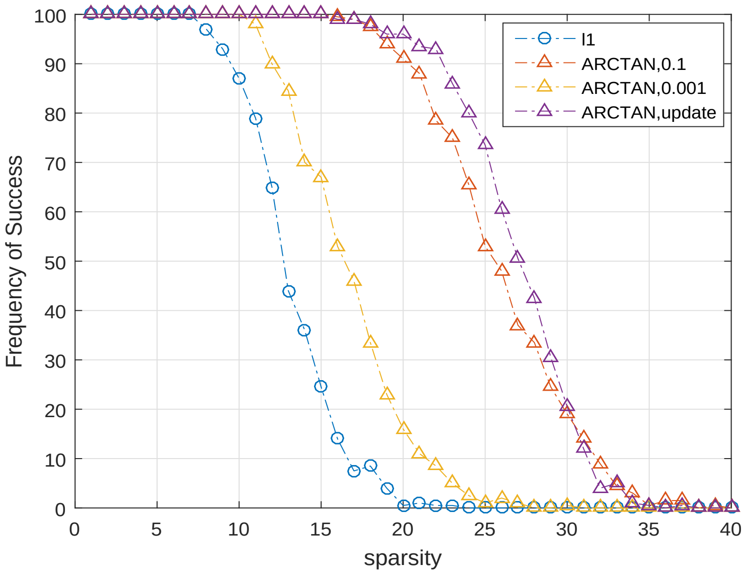


Figure 4.5: Empirical resluts of the frequency of success of ARCTAN with different ε in locating the sparse vectors in T where $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$ are Gaussian matrices. For each sparsity level of $\|x\|_0 \leqslant 40$, 200 trials were made.

Figure 4.6 presents the result for ARCTAN, CWB and REW1 when ε is updated by (4.20). Clearly, all these algorithms remarkably outperform l1, and ARCTAN with (4.20) performs better than CWB with (4.20) as well. In addition, ARCTAN performs slightly better than REW1 when $\|x\|_0 \leqslant 31$ but its curve is slightly lower than REW1 when $31 \leqslant \|x\|_0 \leqslant 38$. Overall, ARCTAN outperforms others in this situation when the updating scheme (4.20) for ε is used.

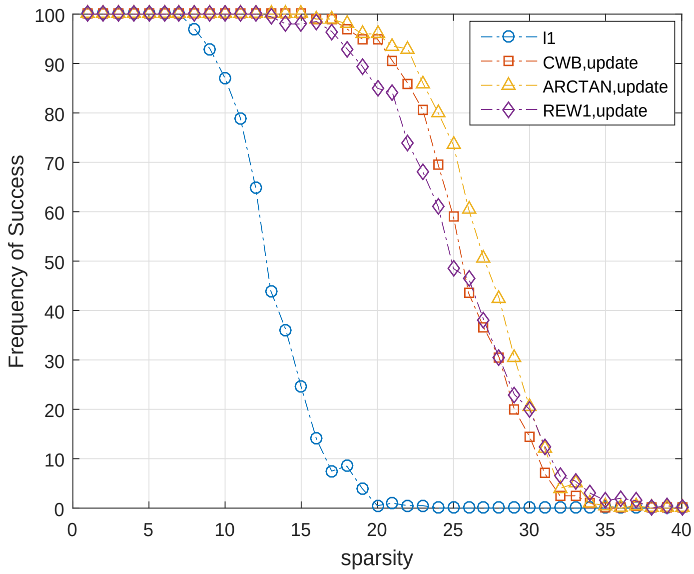


Figure 4.6: Comparsion of success rate of CWB, ARCTAN and REW1 with (4.20) in finding the sparse vectors in T where $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$ are Gaussian matrices. For each sparsity level of $\|x\|_0 \leqslant 40$, 200 trials were made.

4.4 Dual weighted ℓ_1-algorithm for solving (P_0)

As shown in the previous chapter, the solution of weighted ℓ_1-minimization (3.5) with an optimal weight is the sparsest point in T, and such the optimal weight can be determined by solving the bilevel programming (3.37). Note that it is difficult to solve such a bilevel programming (which is NP-hard). However, due to its special structure, it is possible to solve it via certain relaxation. We will develop three relaxation models for solving the bilevel programming (3.37), which are shown in subsections 4.4.1, 4.4.2 and 4.4.3.

4.4.1 Relaxation model 1

Zhao and Luo [99] presented a method to relax a bilevel problem similar to (3.37). Motivated by their idea, we now relax our bilevel model. We focus on relaxing the strong duality constraint $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y = \min_x\{\|Wx\|_1 : x \in T\}$ and the objective function in (3.37). By replacing the objective function $\|\lambda_6\|_0$ in (3.37) by $\Psi_\varepsilon(\lambda_6) \in \mathbf{F}, \lambda_6 \geqslant 0$, we obtain an approximation problem of (3.37), i.e.,

$$
\begin{array}{ll}
\max_{(w,\lambda)} & \Psi_\varepsilon(\lambda_6) \\
\text{s.t.} & B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\
& -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y = \min_x\{\|Wx\|_1 : x \in T\}, \\
& \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6.
\end{array}
\tag{4.21}
$$

Clearly, (3.11) and (3.5) are equivalent and have the same optimal value. We recall the set of the weights ζ given in (3.35). It can be seen that w being feasible to (4.21) implies that (3.11) and (3.20) hold strong duality and have the same finite optimal value, which is equivalent to the fact that $w \in \zeta$ when Slater condition holds for (3.11). Moreover, note that the constraints of (4.21) indicate that for any given $w \in \zeta$, λ satisfying the constraints of (4.21) is optimal to (3.20). Therefore the purpose of (4.21) is to find the densest dual optimal variable λ_6 for all $w \in \zeta$. Thus (4.21) can be rewritten as

$$
\begin{array}{ll}
\max_{(w,\lambda_6)} & \Psi_\varepsilon(\lambda_6) \\
\text{s.t.} & w \in \zeta,\ B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0,\ \|\lambda_3\|_2 \leq \lambda_1, \\
& w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0,\ \lambda_i \geq 0,\ i = 1, 2, 4, 5, 6, \\
& \text{where} \quad \lambda_i, i = 1, 2, ..., 5 \quad \text{is} \quad \text{optimal} \quad \text{to} \\
& \max_\lambda \Big\{ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ \|\lambda_3\|_2 \leq \lambda_1,\ w = \lambda_4 + \lambda_5 + \lambda_6, \\
& B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0,\ \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6 \Big\}.
\end{array}
\tag{4.22}
$$

Denote the feasible set of (3.20) by

$$
\begin{aligned}
D(w) :=\{\lambda : B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0,\ \ \|\lambda_3\|_2 \le \lambda_1,\ w = \lambda_4 + \lambda_5 + \lambda_6 \ge 0,\\
\lambda_i \ge 0,\ i = 1, 2, 4, 5, 6\}.
\end{aligned}
\tag{4.23}
$$

Clearly, the problem (4.22) can be presented as

$$
\begin{aligned}
\max_{(w,\lambda_6)} \quad & \Psi_\varepsilon(\lambda_6)\\
\text{s.t.} \quad & w \in \zeta,\ \lambda \in D(w),\ \text{where } \lambda \text{ is optimal to}\\
& \max_\lambda\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \lambda \in D(w)\}.
\end{aligned}
\tag{4.24}
$$

An optimal solution of (4.24) can be obtained by maximizing $\Psi_\varepsilon(\lambda_6)$ which is based on maximizing $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ over the feasible set of (4.24). Therefore, $\Psi_\varepsilon(\lambda_6)$ and $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ are required to be maximized over the dual constraints $\lambda \in D(w)$ for all $w \in \zeta$. To maximize both the objective functions, we consider the following model as a relaxation of (4.22):

$$
\begin{aligned}
\max_{(w,\lambda)} \quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + \alpha\Psi_\varepsilon(\lambda_6)\\
\text{s.t.} \quad & w \in \zeta, B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0\\
& \|\lambda_3\|_2 \le \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \ge 0\\
& \lambda_i \ge 0, i = 1, 2, 4, 5, 6.
\end{aligned}
\tag{4.25}
$$

where $\alpha > 0$ is a small parameter. There are still two difficulties for solving (4.25). The first one is that $w \in \zeta$ might be unbounded which might lead $\Psi_\varepsilon(\lambda_6)$ to be an infinite value. We may introduce a bounded convex set $\mathcal{W}$ for w into (4.25) so that the optimal value of (4.25) is finite. The second difficulty is that ζ has no explicit form. However, under Slater condition, $w \in \zeta$ can be relaxed to $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le 1$ based on weak duality. Based on the above observations, we obtain the following convex relaxation model of (4.25):

$$
\begin{aligned}
(P_R) \quad \max_{(w,\lambda)} \quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + \alpha\Psi_\varepsilon(\lambda_6) \\
\text{s.t.} \quad & w \in \mathcal{W}, B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0 \\
& \|\lambda_3\|_2 \le \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \ge 0, \\
& -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le 1, \\
& \lambda_i \ge 0, i = 1, 2, 4, 5, 6.
\end{aligned}
\tag{4.26}
$$

Inspired by [96] and [99], we can choose the following bounded convex set:

$$
\mathcal{W} = \left\{ w \in R_+^n : (x^0)^T w \le M, 0 \le w \le M^* e \right\}, \tag{4.27}
$$

where x^0 is the initial point, which can be the solution of the ℓ_1-minimization (2.24), and M, M^* are two given positive numbers such that $1 \leqslant M \leqslant M^*$. Now we develop a new $\mathcal{W}$ as follows:

$$
\mathcal{W} = \left\{ w \in R_+^n : \ w_i \leqslant \frac{M}{|x_i^0| + \sigma_1} \right\}, \tag{4.28}
$$

where both M and σ_1 are two given positive numbers. $(x^0)^T w \le M$ in (4.27) and $w_i \leqslant \frac{M}{|x_i^0|+\sigma_1}$ in (4.28) are motivated by the idea of existing re-weighted algorithm in [24] and [96, 99]. Based on (4.27) and (4.28), we update $\mathcal{W}$ as follows:

$$
\mathcal{W}^k = \left\{ w \in R_+^n : \ (x^{k-1})^T w \le M, \ 0 \le w \le M^* e \right\}, \tag{4.29}
$$

$$
\mathcal{W}^k = \left\{ w \in R_+^n : \ w_i \leqslant \frac{M}{|x_i^{k-1}| + \sigma_1} \right\}. \tag{4.30}
$$

This yields the following two re-weighted ℓ_1-algorithms, called DRA(I) and DRA(II), respectively.

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(I)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_+$;
positive parameters $\varepsilon \in R_{++}$, $\alpha \in R_{++}$, $M \geqslant 1$;
initial weight w^0; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : x \in T\}$ to get x^0 and Z^0;
2. Set $\mathcal{W}^1 = \{w \in R^n_+ : (x^0)^T w \le M, 0 \le w \le M^* e\}$ where $M^* = M(\max(1, 1/Z^0) + 1)$.

Iteration:

At the current iterate x^{k-1}, solve the dual weighted ℓ_1-minimization

$$(w^k, \lambda_6^k) \in \mathrm{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + \alpha\Psi_\varepsilon(\lambda_6) : w \in \mathcal{W}^k,\ \lambda \in D(w),\ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le 1\},$$

where $\mathcal{W}^k = \{w \in R^n_+ : (|x^{k-1}|)^T w \le M,\ 0 \le w \le M^* e\}$ with $M^* = M(\max(1, 1/Z^{k-1}) + 1)$. Then, solve the weighted ℓ_1-minimization

$$x^k \in \mathrm{argmin}\{(w^k)^T|x| : x \in T\}$$

to get Z^k;

Update:

Set $M^* = M(\max(1, 1/Z^k) + 1)$ and $\mathcal{W}^{k+1} = \{w \in R^n_+ : |x^k|^T w \le M, 0 \le w \le M^* e\}$;
Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(II)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurement vectors $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_+$;
positive parameters $\varepsilon \in R_{++}$, $\alpha \in R_{++}$, $\sigma_1 \in R_{++}$, $M \geqslant 1$;
initial weight w^0; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : x \in T\}$ to get x^0;
2. Set $\mathcal{W}^1$ by $\mathcal{W}^1 = \{w \in R^n_+ : (|x_i^0| + \sigma_1)w_i \le M\}$.

Iteration:

At the current iterate x^{k-1}, solve the dual weighted ℓ_1-minimization

$$(w^k, \lambda_6^k) \in \mathrm{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + \alpha\Psi_\varepsilon(\lambda_6) : w \in \mathcal{W}^k,\ \lambda \in D(w),\ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le 1\},$$

where $\mathcal{W}^k = \{w \in R^n_+ : (|x_i^{k-1}| + \sigma_1)w_i \le M\}$. Then, solve the weighted ℓ_1-minimization

$$x^k \in \mathrm{argmin}\{(w^k)^T|x| : x \in T\};$$

Update: Update $\mathcal{W}^{k+1} = \{w \in R^n_+ : (|x_i^k| + \sigma_1)w_i \le M\}$; Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

The updating scheme of M^* in DRA(I) is

$$M^* = M(\max(1, 1/Z^k) + 1) \tag{4.31}$$

where Z^k is the optimal value of the weighted ℓ_1-minimization (3.5) with w^k. This scheme follows the idea in [96] and [99]. Zhao and Luo developed such a scheme for the dual weighted algorithms for (1.8) in [99]. Notice that w is restricted in the bounded set $\mathcal{W}$ so that the optimal value of (4.26) cannot be infinite. Therefore, we can use the bounded or unbounded merit functions in $\Psi \in \mathbf{F}$, for example, (4.4), (4.7), (4.8) and (4.9).

4.4.2 Relaxation model 2

Now we develop the second relaxation of the bilevel programming (3.37). All analysis is under the Slater condition. We start from the model (4.24):

$$\begin{array}{ll} \max_{(w,\lambda_6)} & \Psi_\varepsilon(\lambda_6) \\ \text{s.t.} & w \in \zeta,\ \lambda \in D(w),\ \text{where } \lambda \text{ is optimal to} \\ & \max_\lambda\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \lambda \in D(w)\}, \end{array}$$

where $D(w)$ is given in (4.23). From the above model, we can see that both of $\Psi_\varepsilon(\lambda_6)$ and $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ are required to be maximized over the feasible set of (4.24). Note that under Slater condition, for all $w \in \zeta$, the dual objective $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ must be nonnegative and is homogeneous in $\lambda = (\lambda_1, ..., \lambda_6)$. Moreover, if $w \in \zeta$, then $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ has a nonnegative upper bound due to the weak duality. Inspired by this observation, in order to maximize both $\Psi_\varepsilon(\lambda_6)$ and $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$, we may introduce a small positive α and consider the following approximation:

$$\begin{aligned} \max_{(w,\lambda)} \quad & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \\ \text{s.t.} \quad & w \in \zeta, B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\ & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha \Psi_\varepsilon(\lambda_6), \\ & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{aligned} \tag{4.32}$$

The constraint

$$-\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha \Psi_\varepsilon(\lambda_6) \tag{4.33}$$

implies that $\Psi_\varepsilon(\lambda_6)$ might be maximized when $-\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y$ is maximized if α is small and suitably chosen. Thus (4.32) can be seen as a relaxation problem of (4.24). There are two difficulties for solving the above problem. Similar to the relaxation model 1, ζ is not given explicitly so we need to deal with the constraint $w \in \zeta$. However, unlike the relaxation model 1, we will adopt a relaxation method different from applying weak duality. Another difficulty for solving (4.32) is that $\Psi_\varepsilon(\lambda_6)$ might attain an infinite value when $w_i \to \infty$. This situation might occur due to the choice of the merit functions for sparsity and unboundedness of the set ζ. To overcome these drawbacks, we may use the bounded merit functions such as the merit function (4.7) and merit function (4.9), and we may relax $w \in \zeta$ to $w \in R_+^n$. In this case, even if there are some infinite components in w, $\alpha \Psi_\varepsilon(\lambda_6)$ is still finite. Based on the above observation, we give a new relaxation for (3.37) as follows:

$$\begin{aligned} \max_{(w,\lambda)} \quad & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \\ \text{s.t.} \quad & w \in R_+^n, B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\ & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha \Psi_\varepsilon(\lambda_6), \\ & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{aligned} \tag{4.34}$$

Based on the above optimization problem, a new weighted ℓ_1-algorithm for the model (1.1) is developed:

Algorithm: Dual Weighted ℓ_1-algorithm [**DWA(I)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_{++}$;
small positive parameters $\varepsilon \in R_{++}$ and $\alpha \in R_{++}$;

Step:

1. Solve the problem

$$(w^0, \lambda_6^0) \in \text{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ w \in R^n_+, \ \lambda \in D(w), \ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le \alpha\Psi_\varepsilon(\lambda_6)\},$$

2. Let $x^0 \in \text{argmin}\{(w^0)^T|x| : x \in T\}$.

The set $D(w)$ is the feasible set of (3.20), which is also given in (4.23). The numerical results in Chapter 5 will indicate that the problem (4.34) has a similar ability of finding the sparse vectors in T as the ℓ_1-minimization counterpart of (1.1). The difference is that (2.24) is derived from the primal space while the dual algorithm DWA(I) is developed from the perspective of dual space. Now we develop a re-weighted algorithm for (1.1) based on (4.32). By replacing ζ with a bounded convex set $\mathcal{W}$, we obtain the following new relaxation of the bilevel programming (3.37):

$$\begin{array}{lll} (P_{R1}) & \max_{(w,\lambda)} & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \\ & \text{s.t.} & w \in \mathcal{W}, B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & & \|\lambda_3\|_2 \le \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \ge 0, \\ & & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \le \alpha\Psi_\varepsilon(\lambda_6), \\ & & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{array} \tag{4.35}$$

The set $\mathcal{W}$ can be seen as not only a relaxation of ζ, but also being used to ensure the boundedness of $\Psi_\varepsilon(\lambda_6)$. We can still use (4.30) or (4.29) as the candidate for $\mathcal{W}$. Then, we obtain the re-weighted dual algorithms DRA(III) and DRA(IV).

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(III)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$, sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_{++}$;
positive parameters $\varepsilon \in R_{++}$, $\alpha \in R_{++}$, $M^* \geqslant M \geqslant 1$;
initial set $\mathcal{W}^0 = \{w \in R^n : w \in R^n_+\}$; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve the problem (4.35) with $\mathcal{W}^0$ to get w^0;
2. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : x \in T\}$ to get x^0;
3. Set $\mathcal{W}^1 = \{w \in R^n_+ : |x^0|^T w \leq M, 0 \leq w \leq M^* e\}$.

Iteration:

At the current iterate x^{k-1}, solve the problem

$$(w^k, \lambda_6^k) \in \text{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ w \in \mathcal{W}^k, \ \lambda \in D(w), \ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha\Psi_\varepsilon(\lambda_6)\}$$

where $\mathcal{W}^k = \{w \in R^n_+ : (|x^{k-1}|)^T w \leq M, 0 \leq w \leq M^* e\}$. Then, solve the ℓ_1-minimization $\min\{(w^k)^T|x| : x \in T\}$ to get the vector x^k;

Update:

Set $\mathcal{W}^{k+1} = \{w \in R^n_+ : (|x^k|)^T w \leq M, 0 \leq w \leq M^* e\}$;
Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(IV)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$, sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurement vectors $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_{++}$;
positive parameters $\varepsilon \in R_{++}$, $\alpha \in R_{++}$, $1 \leqslant M$, $\sigma_1 \in R_{++}$;
initial set $\mathcal{W}^0 = \{w \in R^n : w \in R^n_+\}$; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve the problem (4.35) with $\mathcal{W}^0$ to get w^0;
2. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : x \in T\}$ to get x^0;
3. Set $\mathcal{W}^1 = \{w \in R^n_+ : (|x_i^0| + \sigma_1)w_i \leq M\}$,

Iteration:

At the current iterate x^{k-1}, solve the problem

$$(w^k, \lambda_6^k) \in \text{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ w \in \mathcal{W}^k, \ \lambda \in D(w), \ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha\Psi_\varepsilon(\lambda_6)\}$$

where $\mathcal{W}^k = \{w \in R^n_+ : (|x_i^{k-1}| + \sigma_1)w_i \leq M\}$. Then, solve the ℓ_1-minimization $x^k \in \text{argmin}\{(w^k)^T|x| : x \in T\}$;

Update:

Set $\mathcal{W}^{k+1} = \{w \in R^n_+ : (|x_i^k| + \sigma_1)w_i \leq M\}$; Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

The initial steps of the above two algorithms are the same, which is to solve the problem (4.34) and to get the initial weight w^0 and the set $\mathcal{W}^1$. In addition, M can not be too small. If M is a sufficiently small positive number, there might be a gap between the maximum of $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ and the maximum of $\Psi_\varepsilon(\lambda_6)$ over the feasible set. Instead of using $\Psi_\varepsilon(\lambda_6)$ in the step of Iteration in the above algorithms DRA(III) and DRA(IV), we can use bounded and concave composite functions such as the logistic function $P(u) = \frac{1}{1+e^{-u}}$ with $u = \Psi_\varepsilon(\lambda_6)$, which yields the following the optimization problem:

$$
\begin{aligned}
(P_{Rl}) \quad \max_{(w,\lambda)} \quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \\
\text{s.t.} \quad & w \in \mathcal{W}, B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\
& -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \leq \tfrac{\alpha}{1+e^{-\Psi_\varepsilon(\lambda_6)}}, \\
& \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6.
\end{aligned}
\tag{4.36}
$$

The concavity of $\frac{1}{1+e^{-\Psi_\varepsilon(\lambda_6)}}, \lambda_6 \geqslant 0$ needs to be verified. We know that the logistic function is an increasing concave function with respect to $u \geqslant 0$, and we can use the following Theorem 41 to check the concavity of $\frac{1}{1+e^{-\Psi_\varepsilon(\lambda_6)}}$, $\lambda_6 \geqslant 0$.

Theorem 41. [8, 12] Consider the composition f for $h : R \to R$ and $g : R^n \to R$, defined by $f(x) = h(g(x)), \mathbf{dom}\ f = \{x \in \mathbf{dom}\ g \mid g(x) \in \mathbf{dom}\ h\}$. We define the extended-value-extension of a function h as $\breve{h}$ such that assigns $\infty(-\infty)$ to the points not in **dom** h when h is convex (concave). The following table shows the judgement of convexity or concavity for the composition functions:

Table 4.2: Convexity and concavity of composition

h	$\breve{h}$	g	f
convex	nondecreasing	convex	convex
convex	nonincreasing	concave	convex
concave	nonincreasing	convex	concave
concave	nondecreasing	concave	concave

By the definition of the extended-value-extension, we see that the extended-value-extension for $\frac{1}{1+e^{-u}}, u \geqslant 0$ is nondecreasing. Moreover, due to the fact that $\Psi_\varepsilon(\lambda_6)$ is concave and nonnegative for $\lambda_6 \geqslant 0$ and the fact that $\frac{1}{1+e^{-u}}$ is concave for $u \geqslant 0$, Theorem 41 implies that $\frac{1}{1+e^{-\Psi_\varepsilon(\lambda_6)}}$ is a concave function for $\lambda_6 \geqslant 0$. Then (4.36) is a convex programming. Moreover, if we maximize the the dual objective $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$, via the constraint $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \leq \frac{\alpha}{1+e^{-\Psi_\varepsilon(\lambda_6)}}$, then $\Psi_\varepsilon(\lambda_6)$ might be maximized over the feasible set of (4.36). In addition, the logistic function is bounded and thus the choice of the merit function $\Psi_\varepsilon(\lambda_6)$ in (4.36) is flexible. Due to this advantage, the unbounded merit functions, for example, (4.8) and the function $\Psi_\varepsilon(s) = \sum_{i=1}^n \log(s_i + \varepsilon), s \in R_+^n$ can be the candidates of $\Psi_\varepsilon(\lambda_6)$ in (4.36).

From the above discussions, to ensure the efficiency of the relaxation, we may either use a bounded merit function or the logistic function. Also, ζ can be replaced by R_+^n or a bounded convex set $\mathcal{W}$, which leads to the well-defined algorithms.

4.4.3 Relaxation model 3

Now we consider another method to relax the model (4.24). Consider the following inequality:

$$-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma, \tag{4.37}$$

where γ is a given positive number, $f(\lambda_6)$ is a certain function depending on $\varphi_\varepsilon((\lambda_6)_i)$, which satisfies the following properties:

$(I1)$. $f(\lambda_6)$ is convex and continuous with respect to $\lambda_6 \in R_+^n$;

$(I2)$. maximizing $\Psi_\varepsilon(\lambda_6)$ over the feasible set can be equivalently or approximately achieved by minimizing $f(\lambda_6)$.

In many practical cases like $(J1)$-$(J3)$ below, minimising $f(\lambda_6)$ is equivalent to maximising $\Psi_\varepsilon(\lambda_6)$. Replacing $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \leq \alpha\Psi_\varepsilon(\lambda_6)$ in (4.32) by (4.37) leads the following model:

$$\begin{aligned}
\max_{(w,\lambda)} \quad & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \\
\text{s.t.} \quad & w \in \zeta, B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\
& -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma, \\
& \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6.
\end{aligned} \tag{4.38}$$

Clearly, the convexity of $f(\lambda_6)$ guarantees that (4.38) is a convex programming. Moreover, (4.37) and the property $(I2)$ of $f(\lambda_6)$ imply that maximizing $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ is roughly equivalent to minimizing $f(\lambda_6)$ over the feasible set, and thus likely maximizing $\Psi_\varepsilon(\lambda_6)$. The properties $(I1)$ and $(I2)$ ensure that the problem (4.38) is computationally tractable and is a good relaxation of (4.24). There are many functions satisfying the properties $(I1)$ and $(I2)$. For instance, we consider the following functions:

$(J1)$. $e^{-\Psi_\varepsilon(\lambda_6)}$,

$(J2)$. $-\log(\Psi_\varepsilon(\lambda_6) + \sigma_2)$,

$(J3)$. $\frac{1}{\Psi_\varepsilon(\lambda_6)+\sigma_2}$,

$(J4)$. $\frac{1}{n}\sum_{i=1}^n \frac{1}{\varphi_\varepsilon((\lambda_6)_i)+\sigma_2}$, σ_2 is a small positive number.

Now we claim that the functions $(J1)$-$(J4)$ satisfy $(I1)$ and $(I2)$. Clearly, the functions $(J1), (J2)$ and $(J3)$ satisfy $(I2)$. Note that

$$\frac{1}{\Psi_\varepsilon(\lambda_6) + \sigma_2} \leqslant \frac{1}{n}\sum_{i=1}^n \frac{1}{\varphi_\varepsilon((\lambda_6)_i) + \sigma_2}.$$

Thus the minimization of $\frac{1}{n}\sum_{i=1}^n \frac{1}{\varphi_\varepsilon((\lambda_6)_i)+\sigma_2}$ is likely to imply the minimization of $\frac{1}{\Psi_\varepsilon(\lambda_6)}$, which means the maximization of $\Psi_\varepsilon(\lambda_6)$. It is easy to check that the functions $(J1)$-$(J4)$ are continuous in $\lambda_6 \geqslant 0$. To verify the convexity of $(J1)$-$(J3)$, we can use the criterion in the second row of Table 4.2, which implies that $(J1)$-$(J3)$ are convex for $\lambda_6 \geqslant 0$. Note that for any $\varphi_\varepsilon((\lambda_6)_i) > -\sigma_2 > 0$, $i = 1, ..., n$, all functions $\frac{1}{\varphi_\varepsilon((\lambda_6)_i)+\sigma_2}$ are convex. Still based on Theorem 41, $\frac{1}{\varphi_\varepsilon((\lambda_6)_i)+\sigma_2}$ is convex for $(\lambda_6)_i \geqslant 0, i = 1, ..., n$. Therefore their sum is convex for $\lambda_6 \geqslant 0$ as well. Thus all functions $(J1)$-$(J4)$ satisfy the two properties $(I1)$ and $(I2)$. Moreover, the functions $(J1), (J3), (J4)$ have finite values even when $(\lambda_6)_i \to \infty$. By replacing ζ by R^n_+, we obtain a new relaxation of (3.37):

$$\begin{aligned} \max_{(w,\lambda)} \quad & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y \\ \text{s.t.} \quad & w \in R^n_+, B^T \lambda_2 - A^T \lambda_3 + \lambda_4 - \lambda_5 = 0, \\ & \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\ & -\lambda_1 \epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma, \\ & \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6. \end{aligned} \tag{4.39}$$

Based on (4.39), we have the following dual weighted ℓ_1-minimization for solving the sparsity model (1.1): The algorithm DWA(II) can be seen as a dual weighted ℓ_1-algorithm

---

Algorithm: Dual Weighted ℓ_1-algorithm [**DWA(II)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurement vectors $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_{++}$;
positive parameters $\varepsilon \in R_{++}$ and $\gamma \in R_{++}$

Step:

1. Solve the problem

$(w^0, \lambda_6^0) \in \text{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ w \in R^n_+, \ \lambda \in D(w), \ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma\}$;

2. Let $x^0 \in \text{argmin}\{(w^0)^T|x| : x \in T\}$.

---

which is derived from the dual space. Since w is not restricted in DWA(II), this algorithm is expected to have similar ability for recovering sparse signals as ℓ_1-minimization (2.24). We will illustrate this in the next chapter. Using a bounded convex set $\mathcal{W}$ to replace ζ in

(4.38) yields the following relaxation of (3.37):

$$
\begin{aligned}
(P_{R2})\quad \max_{(w,\lambda)}\quad & -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y \\
\text{s.t.}\quad & w \in \mathcal{W}, B^T\lambda_2 - A^T\lambda_3 + \lambda_4 - \lambda_5 = 0, \\
& \|\lambda_3\|_2 \leq \lambda_1, w = \lambda_4 + \lambda_5 + \lambda_6 \geq 0, \\
& -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma, \\
& \lambda_i \geqslant 0, i = 1, 2, 4, 5, 6.
\end{aligned} \tag{4.40}
$$

Based on the two choices of $\mathcal{W}$ in (4.29) and (4.30), we may now develop the dual re-weighted ℓ_1-algorithms DRA(V) and DRA(VI), respectively.

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(V)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$ and error bound $\epsilon \in R_{++}$;
positive parameters $\varepsilon \in R_{++}$, $\gamma \in R_{++}$, $M^* \geqslant M \geqslant 1$;
initial set $\mathcal{W}^0 = \{w \in R^n : w \in R^n_+\}$; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve (4.40) with $\mathcal{W}^0$ to get w^0;
2. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : x \in T\}$ to get x^0;
3. Set $\mathcal{W}^1 = \{w \in R^n_+ : |x^0|^T w \leq M, 0 \leq w \leq M^* e\}$.

Iteration:

1. At the current iterate x^{k-1}, solve the problem

$$(w^k, \lambda_6^k) \in \text{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : w \in \mathcal{W}^k, \lambda \in D(w), -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma\};$$

where $\mathcal{W}^k = \{w \in R^n_+ : |x^{k-1}|^T w \leq M, 0 \leq w \leq M^* e\}$.

2. Solve $\min\{(w^k)^T|x| : x \in T\}$ to get x^k;

Update:

Set $\mathcal{W}^{k+1} := \{w \in R^n_+ : |x^k|^T w \leq M, 0 \leq w \leq M^* e\}$;
Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

Algorithm: Dual Re-weighted ℓ_1-algorithm [**DRA(VI)**]

Input:

merit function $\Psi_\varepsilon \in \mathbf{F}$; sensing matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$;
measurements $y \in R^m$ and $b \in R^l$, error bound $\epsilon \in R_{++}$;
positive parameters $\varepsilon \in R_{++}$, $\gamma \in R_{++}$, $1 \leqslant M$ and $\sigma_1 \in R_{++}$;
initial set $\mathcal{W}^0 = \{w \in R^n : w \in R^n_+\}$; the iteration index k, the largest number of iteration $k_{\max}$;

Initialization:

1. Solve (4.40) with $\mathcal{W}^0$ to get w^0;
2. Solve the weighted ℓ_1-minimization $\min\{(w^0)^T|x| : \ x \in T\}$ to get x^0;
3. Set $\mathcal{W}^1 = \{w \in R^n_+ : (|x^0_i| + \sigma_1)w_i \leq M\}$.

Iteration:

1. At the current iterate x^{k-1}, solve the problem

$$(w^k, \lambda^k_6) \in \operatorname{argmax}\{-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y : \ w \in \mathcal{W}^k, \ \lambda \in D(w), \ -\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y + f(\lambda_6) \leqslant \gamma\};$$

where $\mathcal{W}^k = \{w \in R^n_+ : (|x^{k-1}_i| + \sigma_1)w_i \leq M\}$;

2. Solve $\min\{(w^k)^T|x| : x \in T\}$ to get x^k;

Update:

Set $\mathcal{W}^{k+1} = \{w \in R^n_+ : (|x^k_i| + \sigma_1)w_i \leq M\}$;
Repeat the above iteration until $k = k_{\max}$ (or certain other stopping criterion is met).

Note that the initial steps of DRA(V) and DRA(VI) are the same. It is to solve DWA(II) to get the initial weight w^0 and initial set $\mathcal{W}^1$.

4.4.4 Summary of dual ℓ_1-algorithms

In the above three subsections, we present three ways to relax the model (3.37). Firstly, following the idea in [96] and [99], (3.37) can be relaxed to (4.25). By using two bounded convex set (4.29) and (4.30) respectively, the dual re-weighted ℓ_1-algorithms DRA(I) and DRA(II) are developed to solve the ℓ_0-minimization problem (1.1). The main contribution here is to develop new relaxation of (4.24) and to obtain the convex problems (4.32) and (4.38). In other words, the problems (4.32) and (4.38) are the relaxation versions of the bilevel problem (3.37). Thus the new weighted algorithms DWA(I) and DWA(II) as well as the re-weighted ℓ_1-algorithms DRA(III), DRA(IV), DRA(V) and DRA(VI) are developed from the perspective of dual space. Note that PRA, DRA(I) and DRA(II), similar to existing re-weighted ℓ_1-algorithms, always need an initial iterate, which is often obtained by solving a simple ℓ_1-minimization. Unlike these methods, DRA(III)-DRA(VI) can create an initial iterate by themselves. All developed algorithms are based on the relaxation of the set ζ and the choice of merit functions. The following table shows the details of these algorithms.

Table 4.3: Dual Weighted ℓ_1-algorithms and Dual Re-weighted ℓ_1-algorithms

Relaxation models	Merit functions	ζ	Algorithm types	Algorithms
(P_R)	unbounded/bounded	$\mathcal{W}$ (4.29)	Dual Re-weighted	DRA(I)
(P_R)	unbounded/bounded	$\mathcal{W}$ (4.30)	Dual Re-weighted	DRA(II)
(4.34)	bounded	R^n_+	Dual Weighted	DWA(I)
(P_{R1})	unbounded/bounded	$\mathcal{W}$ (4.29)	Dual Re-weighted	DRA(V)
(P_{R1})	unbounded/bounded	$\mathcal{W}$ (4.30)	Dual Re-weighted	DRA(IV)
(P_{RI})	unbounded/bounded	$\mathcal{W}$ (4.30)	Dual Re-weighted	N/A
(4.39)	unbounded/bounded	R^n_+	Dual Weighted	DWA(II)
(P_{R2})	unbounded/bounded	$\mathcal{W}$ (4.29)	Dual Re-weighted	DRA(V)
(P_{R2})	unbounded/bounded	$\mathcal{W}$ (4.30)	Dual Re-weighted	DRA(VI)

In the next chapter, we will carry out experiments to demonstrate the performance of each dual algorithm proposed in this chapter, and compare their performances with that of several existing algorithms such as CWB and ℓ_1-minimization.

Chapter 5

Numerical Performance of Dual Re-weighted ℓ_1-algorithms

In this chapter, we carry out numerical experiments to demonstrate the performance of the dual weighted ℓ_1-algorithms and dual re-weighted ℓ_1-algorithms listed in Table 4.3. The model (1.1) is a general sparsity model, and it covers some particular models with special matrix B and the vector b, such as the nonnegative sparse model and the monotonic sparse model (see Section 1.1.1 for details). We mainly consider the two cases in our numerical experiments: (1) B and b are given deterministically; (2) B is a random Gaussian matrix. Specifically, for the first case, we take the following cases into our consideration:

(N1) $B = 0$ and $b = 0$ (that is the model (1.9));

(N2) $B = -I$ and $b = 0$ (that is (1.7));

(N3) (B, b) is given by (1.4) (that is (1.5)).

For the second case, we consider the following cases in our experiments:

(N4) $B \in R^{15\times 200}$;

(N5) $B \in R^{50\times 200}$.

For all cases (N1)-(N5), we implement the algorithms DRA(I), DRA(II), DRA(III), DRA(IV), DRA(V) and DRA(VI), and compare their performance in finding the sparse vectors in T with ℓ_1-minimization, CWB and ARCTAN.

This chapter is organised as follows. In Section 5.1.1, we review the environment of our experiments. In Section 5.1.2, the similar performance of DWA(I), DWA(II) and ℓ_1-minimization (2.24) is illustrated. Moreover, the default merit function for DRA(III) and DRA(IV) and the default function $f(\lambda_6)$ for DRA(V) and DRA(VI) are chosen based on the numerical results in Section 5.1.2. In Section 5.1.3, the default parameters of these algorithms are suggested. By using chosen parameters and merit functions in the dual algorithms for each case $(N1)$-$(N5)$, we perform the numerical experiments to demonstrate the behaviours of DRA(I), DRA(II), DRA(III), DRA(IV), DRA(V) and DRA(VI). The results are given in Sections 5.2, 5.3, 5.4, 5.5 and 5.6, respectively. Finally, Section 5.7 reveals the influence of ε on the dual re-weighted ℓ_1-algorithms.

5.1 Merit functions and parameters

In this section, we focus on the case $B = 0$ and $b = 0$ and screen out the 'ideal' merit functions and parameters for each algorithm DRA(III), DRA(IV), DRA(V) and DRA(VI). The default parameters in DRA(I) and DRA(II) are set as that of the algorithms in [99], and the function (4.7) is set as the default merit function when implementing DRA(I) and DRA(II) (also based on the choice in [99]). Note that DWA(I) and DWA(II) are the initial steps of DRA(III) and DRA(IV), and DRA(V) and DRA(VI), respectively. In Subsection 5.1.2, in order to choose suitable merit functions for DRA(III), DRA(IV), DRA(V) and DRA(VI), we compare DWA(I) and DWA(II) via using different merit functions in $\mathbf{F}$ and different functions $f(\lambda_6)$, and we compare their performance with ℓ_1-minimization (2.24). To help choose the suitable parameters for DRA(III), DRA(IV), DRA(V) and DRA(VI), we compare the algorithms with different choices of parameters by fixing other parameters and using a certain merit functions as default. In the process of choosing parameters and merit functions, we set the parameter (merit functions) with highest success rate of finding the sparse vectors in T as our default parameter (merit functions) for the algorithms. Subsection 5.1.3 summarizes the default parameters for each dual re-weighted ℓ_1-algorithm. Before we start, we review the environment of our experiments.

5.1.1 Environment of experiemnts

We only consider the random examples of convex sets T. We first set the noise level ϵ and the parameter ε of merit function. The sparse vector x^* and the entries of A and B (if B is not deterministic) are generated from Gaussian random variables with zero mean and unit variance. For each generated (x^*, A, B), we determine the vectors (measurements) y and b by

$$y = Ax^* + \frac{c_1\epsilon}{\|c\|_2}c, \;\; Bx^* + d = b, \tag{5.1}$$

where $d \in R^l_+$ is generated as absolute Gaussian random variables with zero mean and unit variance, and $c_1 \in R$ and $c \in R^m$ are generated as Gaussian random variables with zero mean and unit variance. Then the convex set T is generated, and all the examples of T are generated this way. Note that by using such a method, T must contain the sparse vector x^*. We still use

$$\frac{\|x' - x^*\|}{\|x^*\|} \leqslant 10^{-5} \tag{5.2}$$

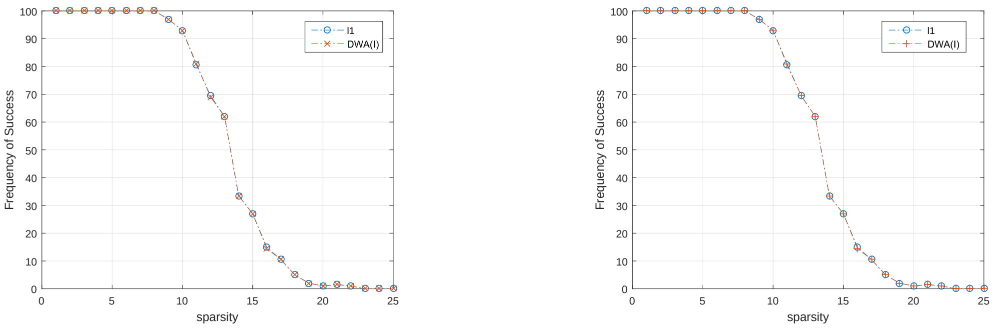
as our default stopping criterion where x' is the solution found by the algorithm, and one success is counted as long as (5.2) is satisfied. In our experiments, we make 200 random examples for each sparsity level. All the algorithms are implemented in Matlab 2017a, and all the convex programming problems in tested algorithms are solved by CVX (Grant and Boyd [47]).

5.1.2 Choosing merit functions

In order to choose a merit functions as default for DRA(III) and DRA(IV), we compare the performance of DWA(I) with different merit functions. From Table 4.3, it is better to choose the bounded merit functions $\Psi_\varepsilon \in \mathbf{F}$ when implementing DWA(I). Thus we can choose the functions (4.4), (4.7) and (4.9). Note that CVX does not recognise arctan function (4.9) so that we choose (4.4) and (4.7) to test. By setting the parameters

$$(\epsilon, \varepsilon, \alpha) = (10^{-4}, 10^{-5}, 10^{-5})$$

and performing 200 random examples for each sparsity level (ranged from 1 to 25), we carry out the experiments for DWA(I) with (4.4) and (4.7), and compare their performances with ℓ_1-minimization, which is shown in Figure 5.1:



$\Psi_\varepsilon(\lambda_6) = \sum_{i=1}^n \frac{(\lambda_6)_i}{(\lambda_6)_i+\varepsilon}$ $\quad\quad$ $\Psi_\varepsilon(\lambda_6) = n - \frac{\sum_{i=1}^n \log((\lambda_6)_i+\varepsilon)}{\log \varepsilon}$

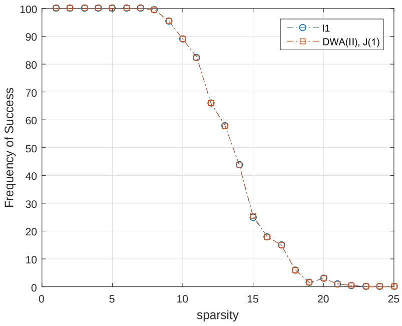
Figure 5.1: The performance of DWA(I) in finding the sparsest points in T via different bounded merit functions $\Psi \in \mathbf{F}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25.

Clearly, in this case, DWA(I) with (4.7) is very comparable to DWA(I) with (4.4) in finding the sparsest points in T. Moreover, both of them are almost identical to the performance of ℓ_1-minimization (2.24). Thus the functions (4.4) and (4.7) can be the candidates of default merit functions for the dual re-weighted ℓ_1-algorithms DRA(III) and DRA(IV).

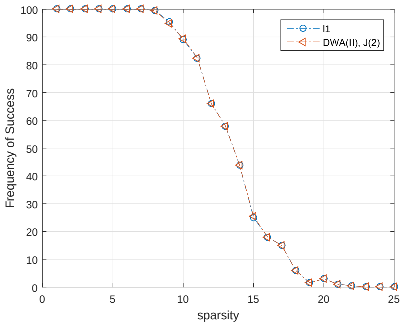
Next, in order to choose default functions $f(\lambda_6)$ for DRA(VI) and DRA(V), we compare the performance of DWA(II) with different $f(\lambda_6)$, including the functions $(J1)$-$(J4)$ given in Subsection 4.4.3. Due to the fact that CVX can not implement $f(s)$ with the merit functions (4.4), (4.8) and (4.9), so we choose $\varphi_\varepsilon((\lambda_6)_i) = \frac{(\lambda_6)_i}{(\lambda_6)_i+\varepsilon}$, $(\lambda_6)_i \in R_+$ in $f(\lambda_6)$. We fix

$$(\epsilon, \varepsilon, \gamma, \sigma_2) = (10^{-4}, 10^{-5}, 1, 1)$$

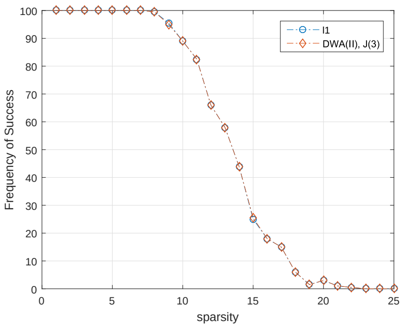
and perform 200 randomly generated examples for each sparsity level from 1 to 25. The numerical results of DWA(II) with $(J1)$-$(J4)$ are shown in Fig 5.2, from which it can be seen that all of them are quite comparable to ℓ_1-minimization (2.24). Moreover, DWA(II) with $(J1)$-$(J3)$ performs slightly better than DWA(II) with $(J4)$. The value of σ_2 may influence the performance of DWA(II) since the sufficiently small σ_2 cause that $\Psi_\varepsilon(\lambda_6)$ might not reach the maximum over the dual feasible set D when maximizing $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$. From this experiment, the functions $(J1)$, $(J2)$ and $(J3)$ can be used as the functions $f(\lambda_6)$ as default for the dual re-weighted ℓ_1-algorithms DRA(V) and DRA(VI).



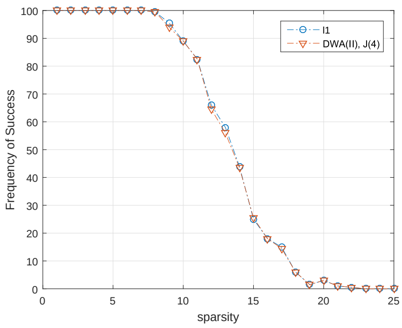
(J1). $f(\lambda_6) = e^{-\Psi_\varepsilon(\lambda_6)}$



(J2). $f(\lambda_6) = -\log(\Psi_\varepsilon(\lambda_6) + \sigma_2)$



(J3). $f(\lambda_6) = \frac{1}{\Psi_\varepsilon(\lambda_6)+\sigma_2}$



(J4). $f(\lambda_6) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\varphi_\varepsilon((\lambda_6)_i)+\sigma_2}$

Figure 5.2: The performance of DWA(II) with different functions $f(\lambda_6)$ in finding the sparsest points in T. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25.

As shown in Figures 5.1 and 5.2, DWA(I) with (4.4) or (4.7), as well as DWA(II) with one of the functions $(J1)$, $(J2)$, $(J3)$ and $(J4)$, have almost the same performance as ℓ_1-minimization. In this chapter, we set (4.7) as the default merit function for DRA(III) and DRA(IV), and set $(J2)$ with the form

$$f(\lambda_6) = \frac{1}{\Psi_\varepsilon(\lambda_6) + \sigma_2}, \quad \Psi_\varepsilon(\lambda_6) = \sum_{i=1}^{n} \frac{(\lambda_6)_i}{(\lambda_6)_i + \varepsilon}, \quad \lambda_6 \in R_+^n \tag{5.3}$$

as the default function $f(\lambda_6)$ for DRA(V) and DRA(VI). We also set $\sigma_2 = 10^{-1}$ as a default parameter. DRA(III) and DRA(IV) with (4.4) and DRA(V) and DRA(VI) with $(J1)$ and $(J3)$ are worthwhile further work.

5.1.3 Choice of parameters

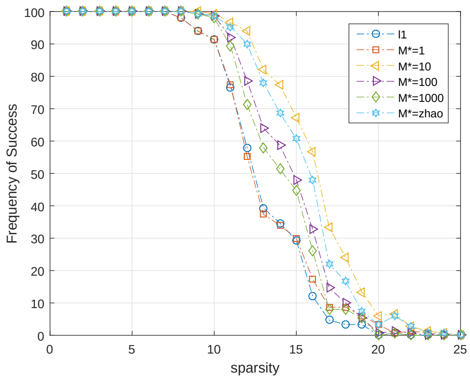
We are now in progress to choose the parameters as default ones for the dual re-weighted ℓ_1-algorithms DRA(III), DRA(IV), DRA(V) and DRA(VI). The following table shows what parameters are needed in these dual algorithms:

Table 5.1: Parameters in dual re-weighted ℓ_1-algorithms

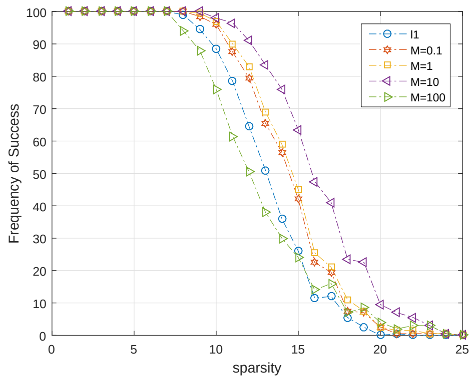
Algorithms	DRA(III)	DRA(IV)	DRA(V)	DRA(VI)
Parameters	(M, M^*, α)	(M, α, σ_1)	(M, M^*, γ)	(M, γ, σ_1)

By comparing the performance of a certain algorithm with different value of a parameter when other parameters are fixed, the parameter with best algorithmic performance can be set as default parameter in the algorithm. All default parameters will be chosen this way. In this subsection, all algorithms are performed for only one iteration. By fixing the noise level ϵ, the parameter ε and the merit function, we test the dual algorithms to find the default parameters. The numerical results for DRA(III), DRA(IV), DRA(V) and DRA(VI) are shown as follows.

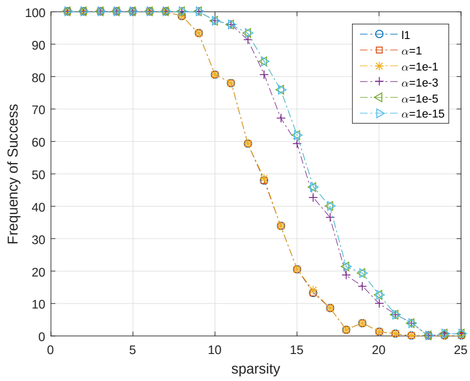
DRA(III)



(i) M^* comparison



(ii) M comparison



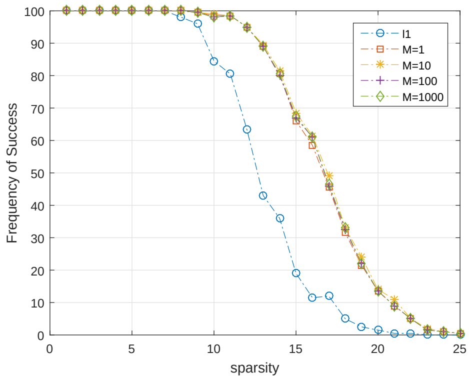
(iii) α comparison

Figure 5.3: Parameter choices for DRA(III). Comparison of the performance of DRA(III) with different M^* in (i), M in (ii) and α in (iii). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25. All re-weighted algorithms are performed only one iteration for each example.

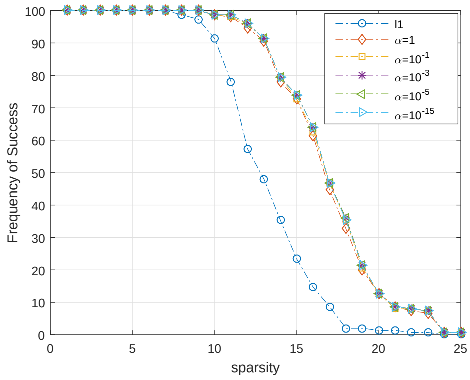
We fix $M = 10$ and $\alpha = 10^{-5}$ and compare the performance of DRA(III)) with different M^* such as $M^* = 1, 10, 100, 1000$ or M^* is updated by (4.31) which is labelled as " $M^* =$ Zhao" in Figure 5.3 (i). Empirical result in (i) indicates that DRA(III)) with $M^* = 10$ has the best performance than the others. By setting $M^* = 10$ and $\alpha = 10^{-5}$, the result (ii) shows the performance of DRA(III)) with $M = 0.1, 1, 10$ and 100. Clearly, DRA(III)) with $M = 10$ outperforms the others. Note that DRA(III)) with $M = 100$ performs worse than ℓ_1-minimization. In fact, due to $w \leqslant M^*e$ in the bounded set $\mathcal{W}^l$ (4.29), the choice like $M \geq M^*$ might lower the success rate of finding the sparsest points in T by DRA(III).

Finally, we compare the performance of DRA(III) with $\alpha = 1, 10^{-1}, 10^{-3}, 10^{-5}$ and 10^{-15}, which is shown in (iii). As expected, DRA(III) with sufficiently small α, such as 10^{-5} and 10^{-15}, have better performance than DRA(III) with $1, 10^{-1}$ and 10^{-3}. In addition, the performance of DRA(III) with 10^{-1} and 1 is almost identical to the performance of ℓ_1-minimization. As a result, we set $(M^*, M, \alpha) = (10, 10, 10^{-5})$ as default parameters for DRA(III).

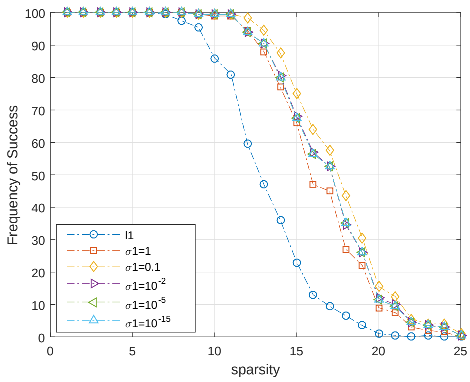
DRA(IV)



(i) M comparison



(ii) α comparison

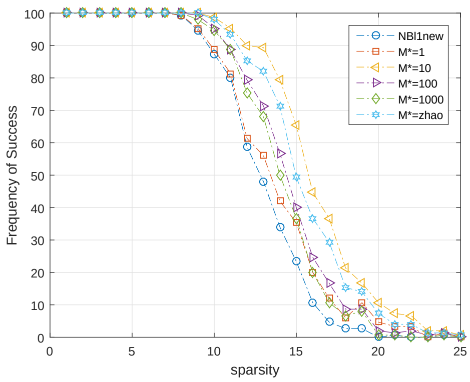


(iii) σ_1 comparison

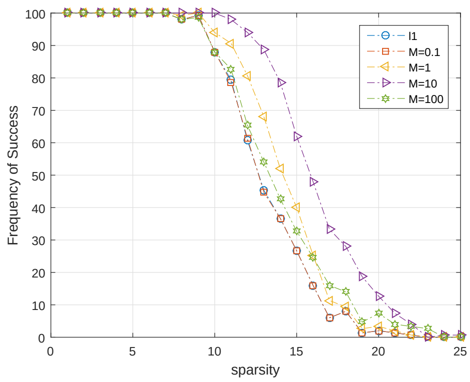
Figure 5.4: Parameter choices for DRA(IV). Comparison of the performance of DRA(IV) with different M in (i), in α (ii) and σ_1 in (iii). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25. All re-weighted algorithms are performed only one iteration for each example.

By fixing $(\alpha, \sigma_1) = (10^{-5}, 10^{-1})$, we compare the performance of DRA(IV) when $M = 1, 10, 10^2$ and 10^3 are taken in DRA(IV). The result was given in Figure 5.4 (i). Clearly, the performance of the DRA(IV) is not sensitive to our choice of M. Moreover, DRA(IV) with $M = 10$ performs slightly better than the others. Thus we set $M = 10$ as default for DRA(IV). We fix $(M, \sigma_1) = (10, 10^{-1})$ and compare the performance of DRA(IV) with $\alpha = 1, 10^{-1}, 10^{-3}, 10^{-5}$ and 10^{-15} in Figure 5.4 (ii). Similar to the results in Figure 5.3 (iii), DRA(IV) is insusceptible to the choice of α if α is small enough. Finally, when $(M, \alpha) = (10, 10^{-5})$, the success frequencies of DRA(IV) with $\sigma_1 = 1, 10^{-1}, 10^{-2}, 10^{-5}$ and 10^{-15} are shown in Figure 5.4 (iii). We can see that DRA(IV) with $\sigma_1 = 10^{-1}$ has the best performance among the others. Thus, we choose $(M, \alpha, \sigma_1) = (10, 10^{-5}, 10^{-1})$ as default parameters for DRA(IV).

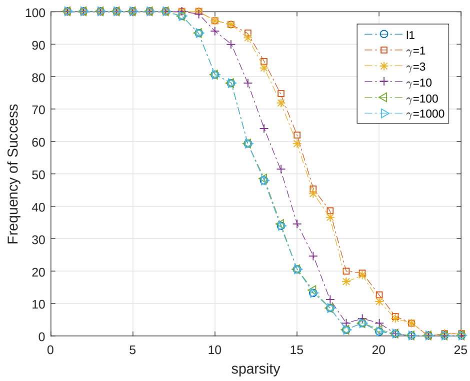
DRA(V)



(i) M^* comparison



(ii) M comparison

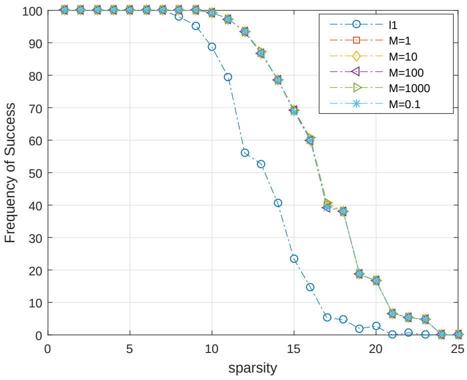


(iii) γ comparison

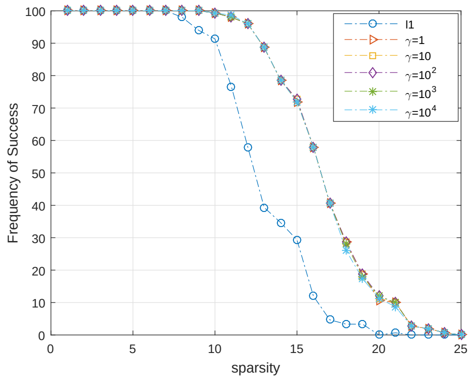
Figure 5.5: Parameter choices for DRA(V). Comparison of the performance of DRA(V) with different M^* in (i), M in (ii) and γ in (iii). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25. All re-weighted algorithms are performed only one iteration for each example.

We compare the performance of DRA(V) with different M^* by fixing $(M, \gamma) = (10, 1)$ in Figure 5.5 (i). Clearly, DRA(V) with $M^* = 10$ outperforms DRA(V) with other choices. From Figures 5.3 (i) and 5.5 (i), we can see that the algorithms with a bounded convex set B, such as DRA(III)) and DRA(V), have a better performance when $M \approx M^*$ than the case when the difference between M and M^* is remarkable. The numerical results of DRA(V) with different M (and fixed $(M^*, \gamma) = (10,\ 1)$) and different γ (and fixed $(M, M^*) = (10,\ 10)$) are shown in Figure 5.5 (ii) and (iii), respectively. Figure 5.5 (ii) shows that DRA(V) with $M = 10$ outperforms DRA(V) with others in the success frequencies of finding the sparsest points in T. In addition, the performance of DRA(V) with $M = 0.1$ in Figure 5.5 (ii) is almost identical to the performance of ℓ_1-minimization. Figure 5.5 (iii) demonstrates that DRA(V) with smaller γ might perform better than DRA(V) with a larger γ. It might be that a small M and a large γ may prevent $\Psi_\varepsilon(\lambda_6)$ from reaching the maximal value over the dual feasible set. Thus we choose $(M^*, M, \gamma) = (10, 10, 1)$ as default parameters for DRA(V).

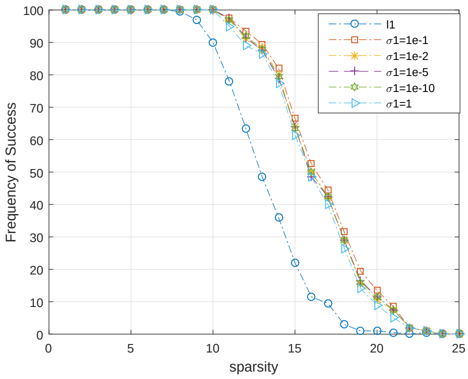
DRA(VI)



(i) M comparison



(ii) γ comparison



(iii) σ_1 comparison

Figure 5.6: Parameter choices for DRA(VI). Comparison of the performance of DRA(VI) with different M in (i), γ in (ii) and σ_1 in (iii). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices $A \in R^{50\times 200}$ for each sparsity level from 1 to 25. All re-weighted algorithms are carried out only one iteration for each example.

Figure 5.6 (i) compares the performance of DRA(VI) with $M = 10^{-1}, 1, 10, 10^2$ and 10^3 when $(\gamma, \sigma_1) = (1, 10^{-1})$, and it demonstrates that DRA(VI) is insensitive to the choice of M. However, DRA(VI) with $M = 10$ performs moderately better than that with $M = 0.1, 1, 100$ and 1000 while this difference is not remarkable. Due to this, we set $M = 10$ as the default parameter for DRA(VI). The performance of DRA(VI) with different γ (and fixed $(M, \sigma_1) = (10, 10^{-1})$) and different σ_1 (and fixed $(M, \gamma) = (10, 10^3)$) are shown in Figure 5.6 (ii) and (iii), respectively. Note that DRA(VI) is insensitive to the choice of γ when $(M, \sigma_1) = (10, 10^{-1})$. We set $\gamma = 10^3$ as the default parameter in DRA(VI). Similarly, we set $\sigma_1 = 10^{-1}$ as the default parameter for DRA(VI) since DRA(VI) with this value has slightly better performance although the its advantage is not remarkable.

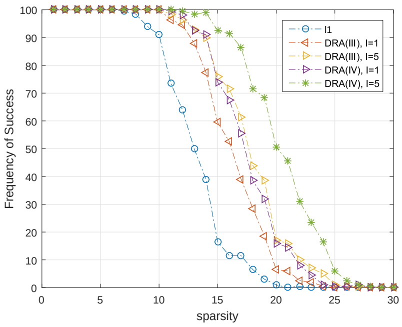
In summery, the default parameters for each dual re-weighted ℓ_1-algorithm are summarized in the following table:

Table 5.2: Default parameters in each dual re-weighted ℓ_1-algorithm

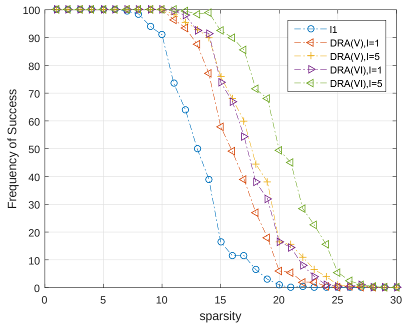
Algorithm/Parameter	α	γ	M	M^*	σ_1	ε
DRA(I)	10^{-8}		10^2	(4.31)		10^{-15}
DRA(II)	10^{-8}		10^2		10^{-1}	10^{-15}
DRA(III)	10^{-5}		10	10		10^{-5}
DRA(IV)	10^{-5}		10		10^{-1}	10^{-5}
DRA(V)		1	10	10		10^{-5}
DRA(VI)		10^3	10		10^{-1}	10^{-5}

In the following sections, we perform numerical experiments to show the behaviours of the dual re-weighed ℓ_1-algorithms in different cases $(N1)$-$(N5)$. The parameters of tested algorithms are chosen as in the above table. We choose the noisy level $\epsilon = 10^{-4}$ for the cases $(N1), (N2), (N4)$ and $(N5)$ and choose $\epsilon = 10^{-1}$ for $(N3)$.

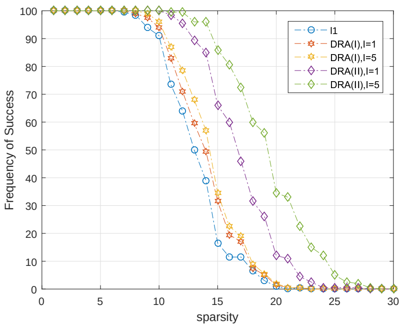
5.2 $B = 0$ and $b = 0$



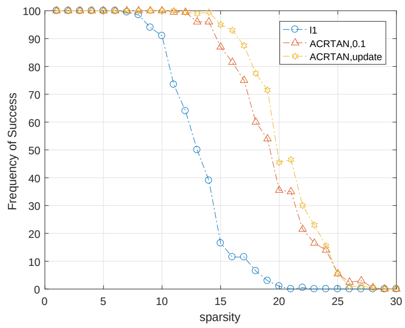
(i) DRA(III)) and DRA(IV)



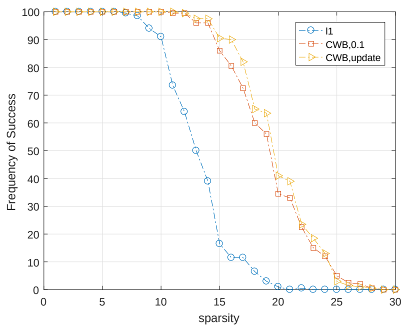
(ii) DRA(V) and DRA(VI)



(iii) DRA(I) and DRA(II)



(iv) ARCTAN with $\varepsilon = 0.1$ and updated rule (4.20)



(v) CWB with $\varepsilon = 0.1$ and updated rule (4.20)

Figure 5.7: (i)-(iii) Comparison of the performance of the dual re-weighted algorithms by using one iteration and five iterations. (iv)-(v) Comparison of the performance of CWB and ARCTAN when $\varepsilon = 0.1$ or ε is updated by (4.20). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 30. All the examples are with random matrix $A \in R^{50 \times 200}$, $B = 0$ and $b = 0$.

For the merit function, we choose (4.7) as $\Psi_\varepsilon(\lambda_6)$ in DRA(I), DRA(II), DRA(III)) and DRA(IV), and we choose (5.3) as $f(\lambda_6)$ in DRA(V) and DRA(VI). The parameters for each dual algorithms are taken as in Table 5.2. We now perform numerical experiments for our dual re-weighted ℓ_1-algorithms in the case of $B = 0$ and $b = 0$. Note that in this case, the model (1.1) is reduced to the sparse model (1.9). The numerical results are given in Figure 5.7 (i)-(iii), each of them compares one type dual re-weighted ℓ_1-algorithms by using different sets $\mathcal{W}$ (4.29) and (4.30). Note that there are five legends in each figure (i)-(iii), corresponding to ℓ_1-minimization, the dual re-weighted ℓ_1-algorithms with one iteration or five iterations. For instance, in (i), we compare DRA(III)) and DRA(IV) which all take either one iteration or five iterations. (DRA(III),1) and (DRA(III),5) represent (DRAIII) take one iteration and five iterations, respectively . It can be seen that the dual re-weighted algorithms are performing better when the number of iteration is increased and all of them outperform ℓ_1-minimization, while the performance of DRA(I) with one or five iterations is similar to the performance of ℓ_1-minimization. (i)-(iii) indicate the same phenomena: the algorithms based on (4.30) might achieve more improvement than the ones based on (4.29) when the number of iteration is increased. In addition, in (i) and (ii), the algorithms based on (4.30) with one iteration perform almost the same as the algorithms based on (4.29) with five iterations. For example, in (ii), the success rate of DRA(VI) with five iterations has improved by nearly 25% compared with those with one iteration for each sparsity from 14 to 20, while DRA(V) has only improved its performance by 10% after increasing the number of iterations, and DRA(VI) with one iteration performs quite similar to DRA(V) with five iterations. We also show the empirical results of CWB and ARCTAN in (iv) and (v), respectively, when $\varepsilon = 0.1$ or ε is updated by (4.20). As expected, the algorithms with ε being updated by (4.20) perform better than those with fixed ε.

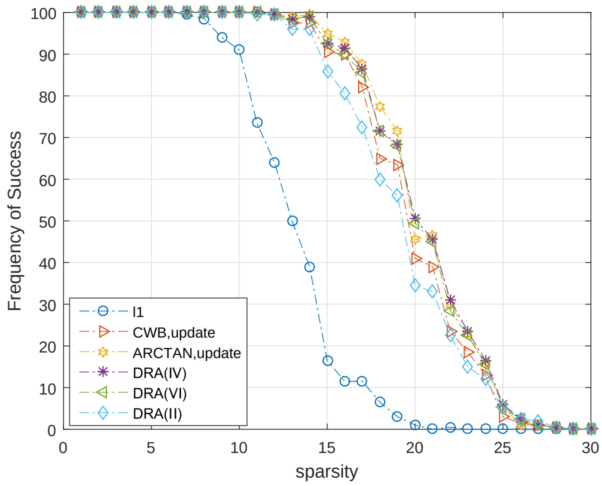
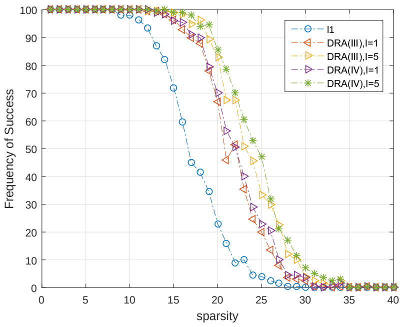


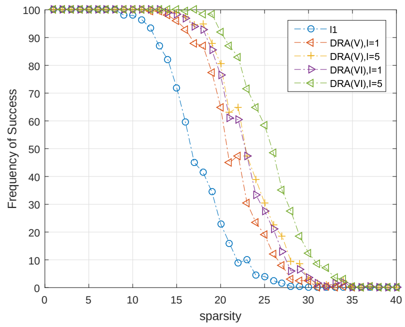
Figure 5.8: Comparison of the performance of the dual re-weighted ℓ_1-algorithms and primal re-weighted algorithms using updated ε in the case of $A \in R^{50\times 200}, B = 0$ and $b = 0$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 30. All re-weighted algorithms are performed five iterations for each example.

We filter the algorithms with the best performance from (i)-(v) in Figure 5.7 and merge them into Figure 5.8. It can be seen that ARCTAN with iterative ε, DRA(II), DRA(IV) and DRA(VI) slightly outperform CWB with iteratively updated ε. It is a surprising results since CWB where ε is updated by (4.20) can be seen as one of the efficient choice for primal re-weighted algorithms, and DRA(IV) and DRA(VI) do not use the iterative scheme for ε. Next, we do numerical experiments on the model (1.1) with $B = -I$ and $B = 0$, which is the nonnegative sparse model (1.7).

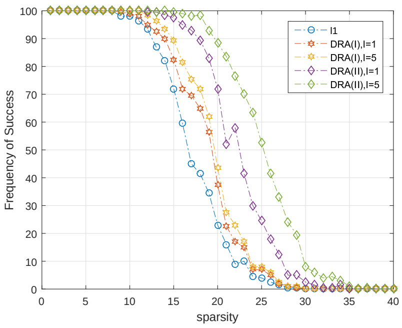
5.3 $B = -I$ and $b = 0$



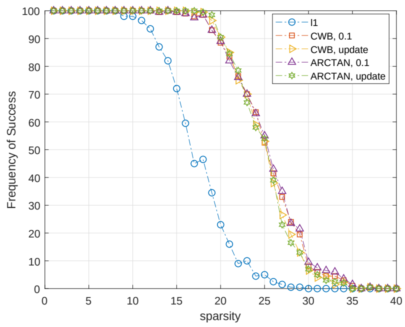
(i) DRA(III) and DRA(IV)



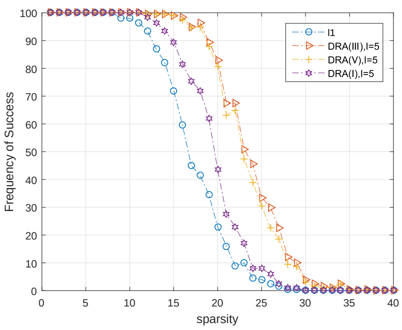
(ii) DRA(V) and DRA(VI)



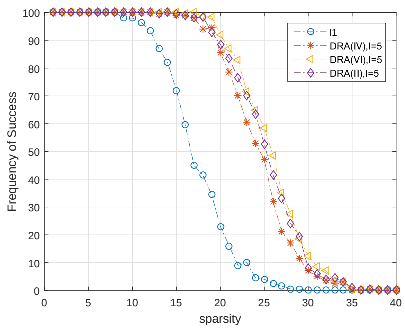
(iii) DRA(I) and DRA(II)



(iv) ARCTAN and CWB with $\varepsilon = 0.1$ or ε updated by (4.20)



(v) The dual algorithms with (4.29)



(vi) The dual algorithms with (4.30)

Figure 5.9: (i)-(iii) Comparison of the performance of the dual re-weighted algorithms with one iteration and five iterations. (iv) Comparison of the performance of primal re-weighted algorithms when $\varepsilon = 0.1$ or ε is updated by (4.20). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 40. All the examples use random matrices of size $A \in R^{50\times 200}$ and $B \in R^{15\times 200}$. (v)-(vi) Comparison of the performance of dual re-weighted algorithms with (4.29) or (4.30)

As demonstrated in Figures 5.9 (i), (ii) and (iii), for all tested dual re-weighted ℓ_1-algorithms, the success frequencies in finding the sparse points in T have improved when the number of iterations is increased from 1 to 5. As shown in precious section, we have three relaxation models for (3.37) and each of them can be relaxed to two different algorithms when using different bounded convex set $\mathcal{W}^l$. Figures 5.9 also shows that the dual re-weighted algorithm with $\mathcal{W}^l$ in (4.29) outperforms the one using $\mathcal{W}^l$ in (4.30) no matter the number of iterations is 1 or 5. For our dual re-weighted algorithms, $\mathcal{W}$ in (4.29) or (4.30) has enhanced the ability to find the sparse points in T. For the dual re-weighted ℓ_1-algorithms with (4.29), Figure (v) indicates that DRA(III) has a better performance than DRA(I) and DRA(V). For the dual re-weighted ℓ_1-algorithms with (4.30), Figure (v) indicates that DRA(VI) has a better performance than DRA(II) and DRA(IV). Different from the case '$B = 0$ and $b = 0$', Figures 5.9 (iv) demonstrates that using the rule (4.20) in ARCTAN and CWB might not enhance the success rate of finding sparse points in T.

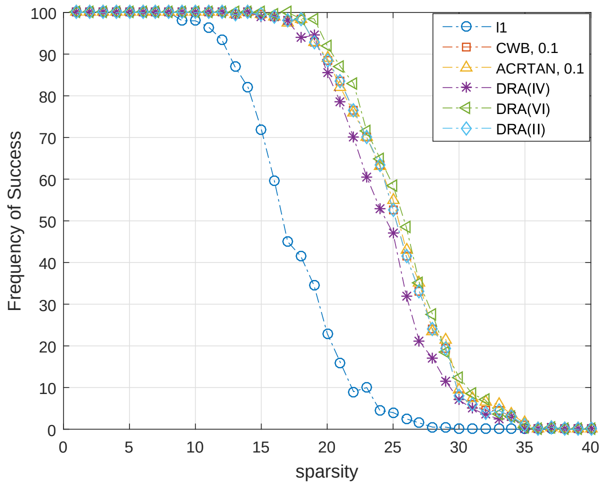
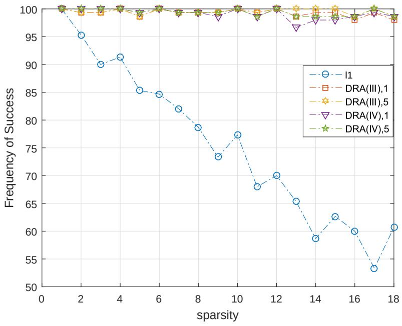


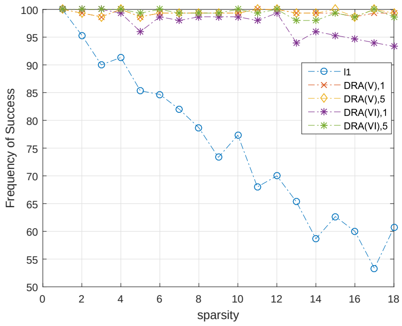
Figure 5.10: Comparison of the performance of the dual re-weighted algorithms and primal re-weighted algorithms using updated ε in the case of $A \in R^{50\times 200}$, $B = -I$ and $b = 0$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 40. All re-weighted algorithms are performed five iterations for each example.

We pick the algorithms with the best performance in Figures 5.9 (i)-(iv) and present them in Figure 5.10. Note that all the re-weighted algorithms in Figure 5.10 perform much better than ℓ_1-minimization. In addition, the performance of DRA(VI) almost the same as the performance of ARCTAN and CWB with fixed $\varepsilon = 0.1$.

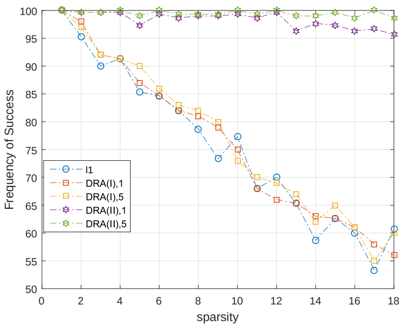
5.4 Monotone sparse model



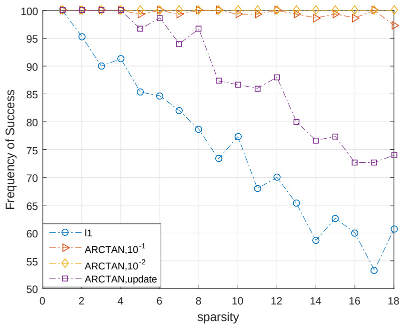
(i) DRA(III)) and DRA(IV)



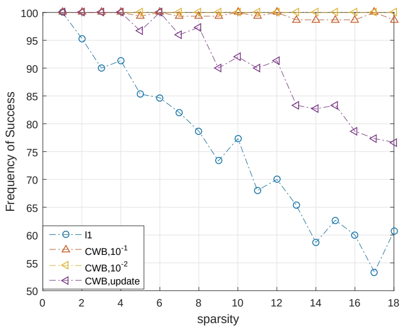
(ii) DRA(V) and DRA(VI)



(iii) DRA(I) and DRA(II)



(iv) ARCTAN with $\varepsilon = 0.1$ and updating rule (4.20)



(v) CWB with $\varepsilon = 0.1$ and updating rule (4.20)

Figure 5.11: (i)-(iii) Comparison of the performance of the dual re-weighted algorithms with one iteration and five iterations. (iv)-(v) Comparison of the performance of primal re-weighted algorithms with five iterations. All of experiments are implemented in the case that $A \in R^{50\times 200}$ and B and b are given as (1.4). Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 30.

The success rate of locating the sparse vectors by ℓ_1-minimization in this case is lower than that in the above case when the sparsity level is low (for example, from 1 to 6). As demonstrated in Figures (i), (ii) and (iii), the dual re-weighted ℓ_1-algorithms have an enormous improvement of the performance of finding the sparse vectors in T. For example, by using five iterations, the success rate of sparse recovery of DRA(III), DRA(IV), DRA(V) and DRA(VI) is nearly 100% from sparsity $k = 1$ to $k = 18$. Figure 5.11 (iv) and (v) show an interesting phenomena. First, CWB or ARCTAN with update rule (4.20) of ε performs worse than the one with a fixed ε. Moreover, we notice that the primal re-weighted algorithms with relatively smaller ε have a better performance than those with larger ε, which is different from the above cases. Note that the monotonic constraint $x_1 \geqslant x_2 \geqslant ... \geqslant x_n$ forces the algorithms to find the solutions in such a sparsity structure. Thus, a smaller ε might make the weights be updated effectively to keep the sparsity structure of the solution so that the algorithms might effectively find a sparser solution.

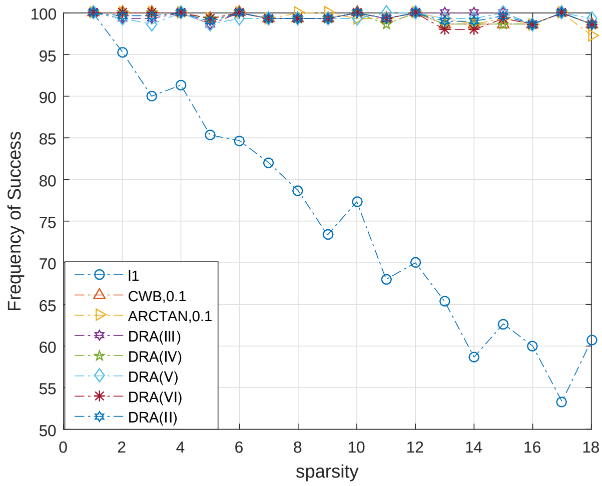
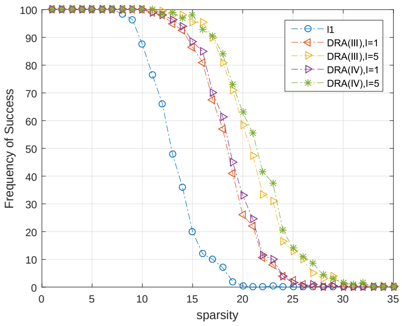


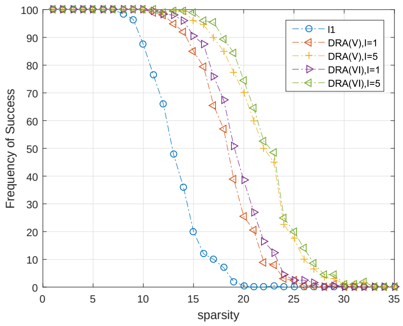
Figure 5.12: Comparison of the performance of the dual re-weighted algorithms and primal re-weighted algorithms using updated ε in the case of random Gaussian matrices $A \in R^{50\times 200}$ and $B = 0$ and $b = 0$. Each algorithm is tested by using 200 randomly generated examples for each sparsity level from 1 to 30. All re-weighted algorithms are performed five iterations for each example.

As usual, we pick the best algorithms in Figures 5.11 (i)-(v). Notice that the performance of all 'best' re-weighted algorithms is similar. It is interesting to note that the dual and primal re-weighted ℓ_1-algorithms not only outperform ℓ_1-minimization, but also have nearly 100% of success rate of locating the sparse vectors from the sparsity level 1 to 18 although this case admits larger noisy level than the others.

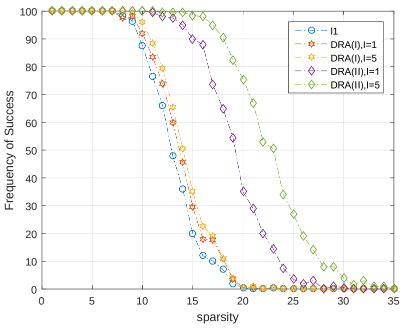
5.5 $B \in R^{15\times 200}$



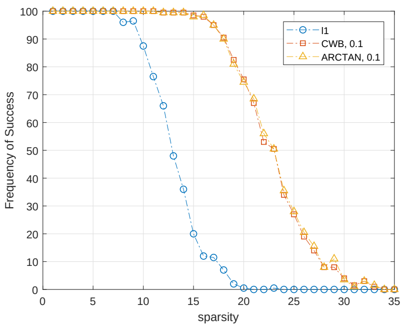
(i) DRA(III) and DRA(IV)



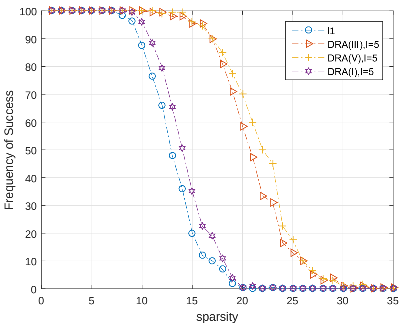
(ii) DRA(V) and DRA(VI)



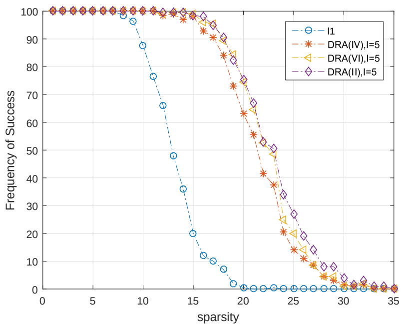
(iii) DRA(I) and DRA(II)



(iv) CWB and ARCTAN with $\varepsilon = 0.1$



(v) Algorithms with updating rule (4.29)



(vi) Algorithms with updating rule (4.30)

Figure 5.13: (i)-(iii) Comparison of the performance of the dual re-weighted algorithms with one iteration and five iterations. (iv) Comparison of the performance of unified re-weighted algorithms with $\varepsilon = 0.1$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 35. The size of matrices in these examples are $A \in R^{50\times 200}$ and $B \in R^{15\times 200}$. (v)-(vi) Comparison of the performance of dual re-weighted algorithms with $\mathcal{W}$ (4.29) or (4.30)

As shown in the above Figure (i)-(iv), the result in the case that $B \in R^{15\times 200}$ is nearly the same as in the case $B = -I$ and $b = 0$. As shown in (v), for the dual algorithms using the updating rule (4.29) for $\mathcal{W}$, by executing five iterations, DRA(V) performs better than others. We compare the performance of the dual algorithms with updating rule (4.30) and find that DRA(II) and DRA(VI) perform slightly better than DRA(IV).

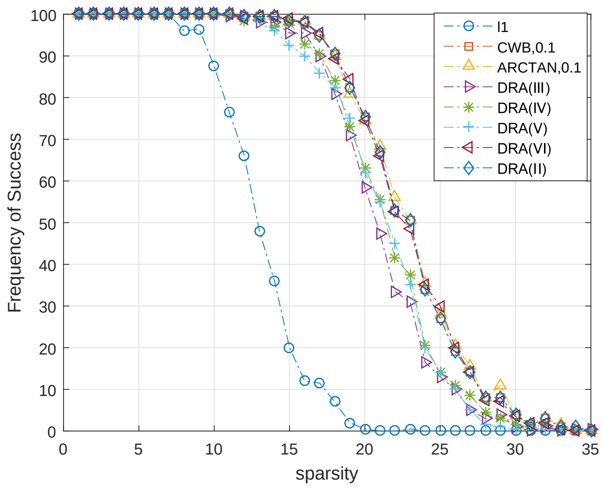
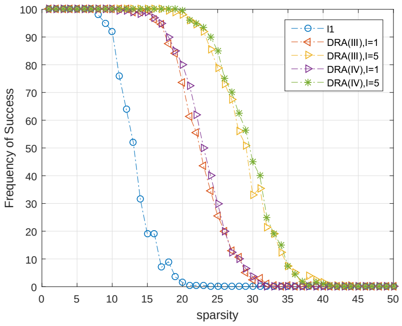


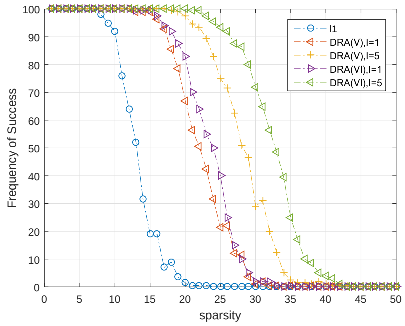
Figure 5.14: Comparison of the performance of the dual re-weighted algorithms and primal re-weighted algorithms in the case of $A \in R^{50\times 200}$ and $B \in R^{15\times 200}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 35. All re-weighted algorithms are performed five iterations for each example.

The results shown in Figure 5.14 demonstrate that CWB, ARCTAN, DRA(II) and DRA(VI) perform slightly better than DRA(III)), DRA(IV) and DRA(V) while the difference is not remarkable.

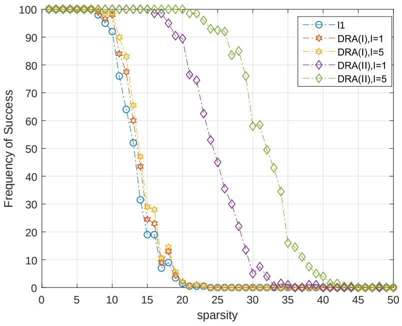
5.6 $B \in R^{50\times 200}$



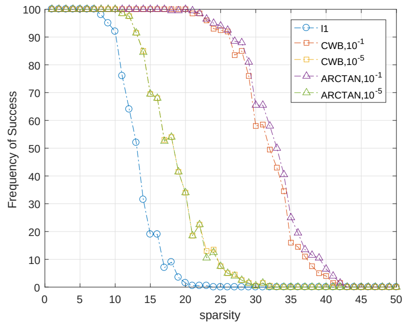
(i) DRA(III) and DRA(IV)



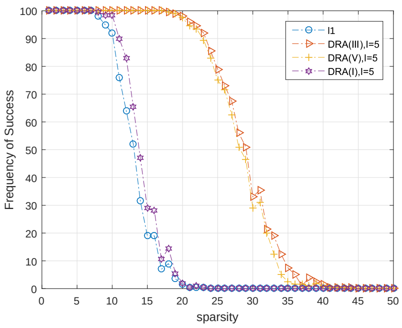
(ii) DRA(V) and DRA(VI)



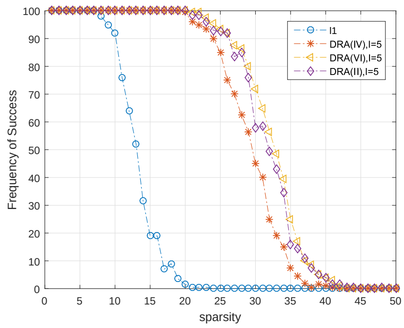
(iii) DRA(I) and DRA(II)



(iv) CWB and ARCTAN with $\varepsilon = 10^{-1}, 10^{-5}$



(v) Algorithms with updating rule (4.29)



(vi) Algorithms with updating rule (4.30)

Figure 5.15: (i)-(iii) Comparison of the performance of the dual re-weighted algorithms with one iteration and five iterations. (iv) Comparison of the performance of primal re-weighted algorithms with $\varepsilon = 0.1$ and 10^{-5}. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 45. The size of matrices in these examples are $A \in R^{50\times 200}$ and $B \in R^{50\times 200}$. (v)-(vi) Comparison of the performance of dual re-weighted algorithms with (4.29) or (4.30)

Similar to the results in the above sections, the performance of DRA(I) resembles the performance of DRA(I) regardless of one iteration or five iterations being executed, which is shown in Figure 5.15 (iii). We compare the re-weighted ℓ_1-algorithms with updating rule (4.29) ((4.30)), which are shown in (v) ((vi)). For the dual algorithms using the updating rule (4.29), when executing 5 iterations, Figure (5.15) (v) shows that DRA(III) and DRA(V) performs much better than DRA(I). For the dual algorithms using the updating rule (4.30), when executing 5 iterations, Figure (5.15) (vi) indicates that the success rates of finding the sparse vectors in T by DRA(II) and DRA(VI) are very similar. The other behaviours are similar to the case of $B = -I$ and $b = 0$ or $B \in R^{15\times 200}$.

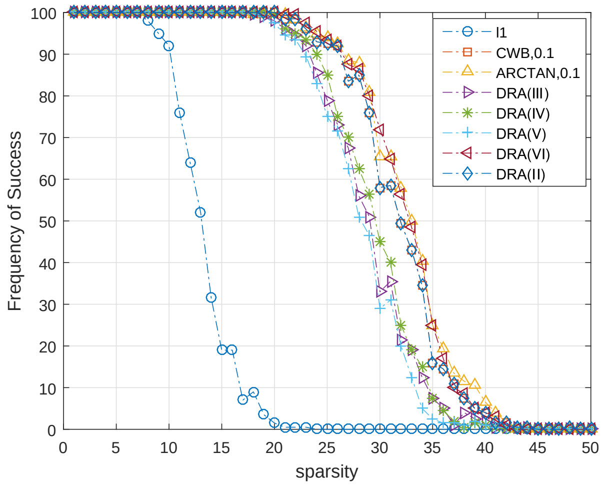
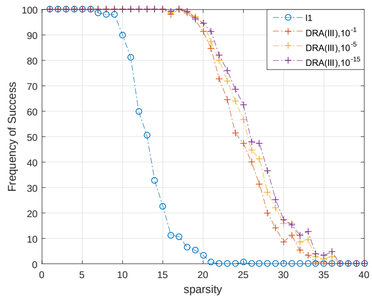


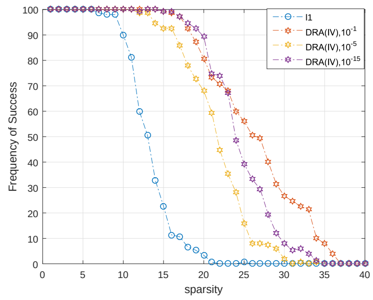
Figure 5.16: Comparison of the performance of the dual re-weighted algorithms and unified re-weighted algorithms using updated ε in the case of $A \in R^{50\times 200}$ and $B \in R^{50\times 200}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 45. All re-weighted algorithms perform five iterations for each example.

We still put all the best algorithms in each Figure 5.15 (i)-(iv) into Figure 5.16. It reveals that although the performance of ARCTAN and DRA(VI) is slightly better than that of DRA(II) and CWB, these four algorithms can compete to each other in successfully finding sparse vectors at high sparsity level in many situations.

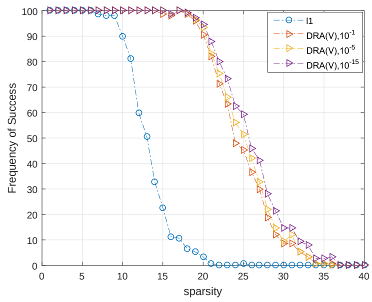
5.7 The influence of ε



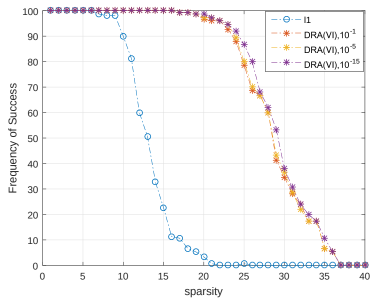
(i) DRA(III)



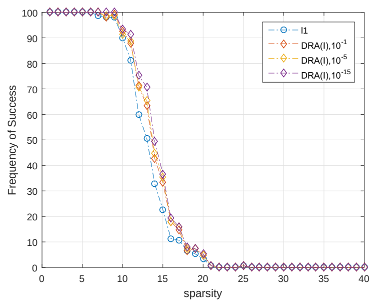
(ii) DRA(IV)



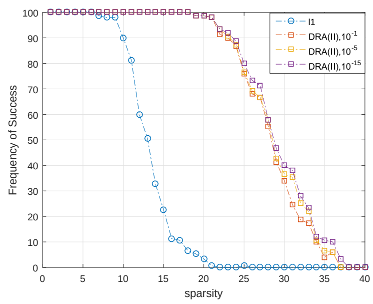
(iii) DRA(V)



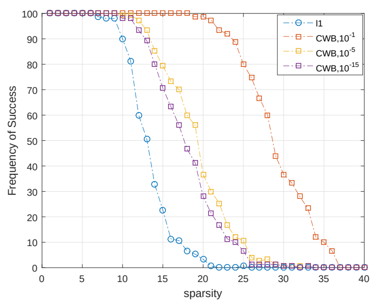
(iv) DRA(VI)



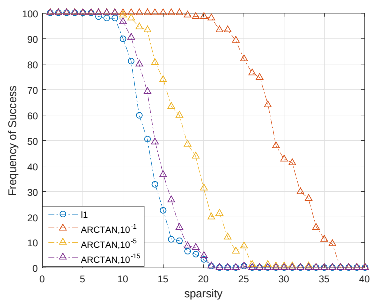
(v) DRA(I)



(vi) DRA(II)



(vii) CWB



(viii) ARCTAN

Figure 5.17: Comparison of the performance of dual re-weighted algorithms with different ε in the case of $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 40, and performed five iterations for each example.

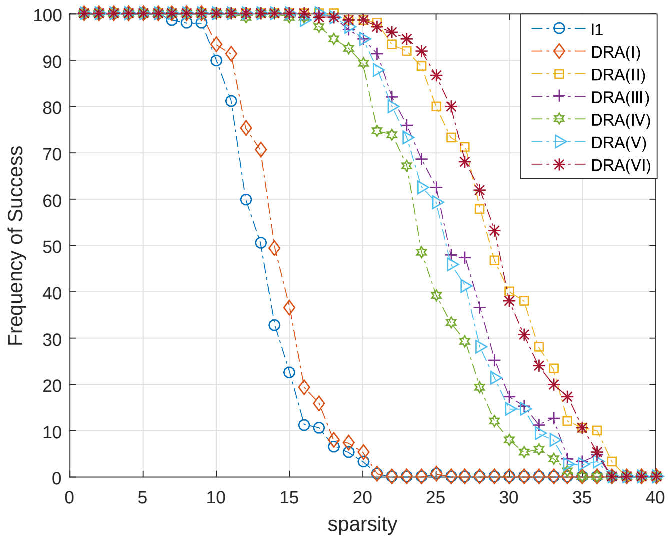


Figure 5.18: Comparison of the performance of dual re-weighted algorithms with $\varepsilon = 10^{-15}$ in the case of $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 40, and performed five iterations for each example.

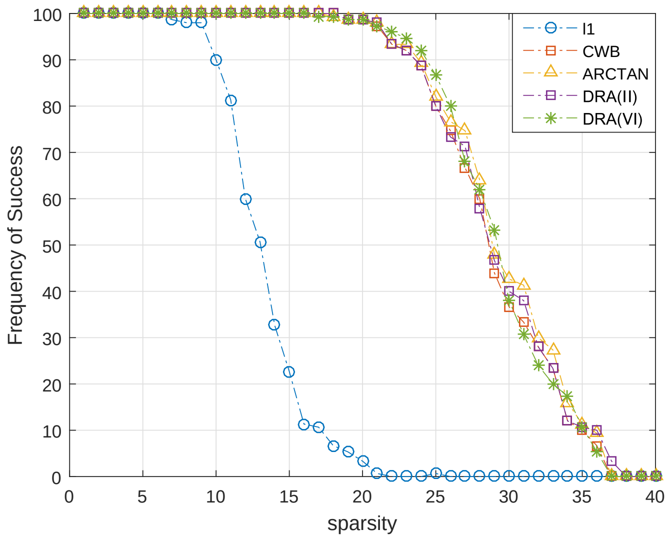


Figure 5.19: Comparison of the performance of dual re-weighted algorithms with $\varepsilon = 10^{-15}$ and primal re-weighted algorithms with $\epsilon = 10^{-1}$ in the case of $A \in R^{50\times 200}$ and $B \in R^{40\times 200}$. Each algorithm is tested by using 200 randomly generated examples with Gaussian matrices for each sparsity level from 1 to 40, and performed five iterations for each example.

In this section, we compare how the parameter ε of merit functions affect the performance of locating the sparse vectors in T by dual re-weighted ℓ_1-algorithms. The numerical results for each primal and dual algorithm with different ε are shown in Figure 5.17. It indicates that the performance of all our dual algorithms is relatively insensitive to the choice of small ε compared to the primal re-weighted algorithms.

The performance of the dual re-weighted algorithms with $\varepsilon = 10^{-15}$ are summarized into Figure 5.18, and we conclude that by executing 5 iterations, DRA(II) and DRA(VI) perform better than the others in this case.

In Figure 5.19, we compare the performance of the 'best' dual algorithms in Figure 5.18 and the 'best' algorithm in Figure 5.17 (vii) and (viii). It can be seen that these algorithms perform similarly in finding the sparse vectors in T, and they outperform ℓ_1-minimization. It worth nothing that we did not use any iterative scheme for ε in our dual re-weighted ℓ_1-algorithms, which can be studied in the near future.

Chapter 6

Stability of ℓ_1-minimization Methods under Restricted Range Space Property

6.1 Introduction

We have studied some algorithms for the general sparsity model (1.1), and carried out numerical experiments for these algorithms in previous chapters. In this chapter, we establish a stability result for the ℓ_1-minimization method for locating the solution of the following ℓ_0-minimization model:

$$\begin{aligned} \min_{x \in R^n} \quad & \|x\|_0 \\ \text{s.t.} \quad & a_1 \|y - Ax\|_2 + a_2 \left\|U^T(Ax - y)\right\|_\infty + a_3 \left\|U^T(Ax - y)\right\|_1 \leq \epsilon, \\ & Bx \leq b. \end{aligned} \tag{6.1}$$

where ϵ is a given parameter, and $A \in R^{m \times n}$ $(m \ll n)$ and $U \in R^{m \times h}$ are two matrices with full row rank, and a_1, a_2, a_3 are three given parameters satisfying $a_i \in [0, 1]$ and $\sum_{i=1}^{3} a_i = 1$. The above model covers several important special cases depending on the special choices of a_i's, B and b. For example, the ℓ_0-minimization models with only one of the following constraints are clearly the special cases of (6.1):

(C1). $y = Ax$;

(C2). $\|y - Ax\|_2 \leqslant \epsilon$;

(C3). $\left\|U^T(Ax-y)\right\|_1 \leqslant \epsilon$;

(C4). $\left\|U^T(Ax-y)\right\|_\infty \leqslant \epsilon$.

Constraints (C1) or (C2) appears in standard basis pursuit (1.19) and quadratically constrained basis pursuit (1.16) [26,34,35,43]. $(C4)$ reduces to the Dantzig Selector constraint in (1.18) [19, 23, 43] when $U = A$. The general ℓ_0-minimization model (1.1) is still a special case of (6.2). Cleraly, the nonnegative sparsity model (1.7) [20, 21, 43] and monotonic sparsity model (1.5) (isotonic regression [82]) are also the special cases of (6.1). Moreover, the model (6.1) can be used to deal with sparsity recovery problems with certain structure which is one of the important aspects in compressive sensing.

The ℓ_1-minimization associated with (6.1) can be stated as follows:

$$\begin{aligned} \min_{x\in R^n} \quad & \|x\|_1 \\ \text{s.t.} \quad & a_1\|y-Ax\|_2 + a_2\left\|U^T(Ax-y)\right\|_\infty + a_3\left\|U^T(Ax-y)\right\|_1 \le \epsilon, \\ & Bx \le b. \end{aligned} \tag{6.2}$$

The stability of (6.2) concerns about how close the solution of (6.2) to that of the original problem (6.1). We use the classic Hoffman Theorem to develop a stability result for the model (6.2) under the assumption of restricted weak range space property (RSP) of order k (which will be introduced in next section). Our result generalizes several stability theorems established by Zhao et al. [94, 95, 98].

This chapter is organized as follows. In Section 6.2, we recall some basic definitions about range space property (RSP) and introduce restricted weak RSP of order k. Then we review Hoffman Theorem. An approximation of the solution set of (6.2) and its related problem will be discussed in Sections 6.3 and 6.4, respectively. Finally, in Section 6.5, we show the main stability results of this chapter.

6.2 Basic facts and restricted weak RSP

6.2.1 Hoffman theorem

To develop a stability theory for basis pursuits, Zhao et al. [94, 95] utilized the classic Hoffman theorem [53, 73] associated with the following polyhedral set:

$$\mathcal{Q} = \{x \in R^n : M_1 x \leq p,\ M_2 x = q\}. \tag{6.3}$$

Lemma 42 (Hoffman Theorem)**.** [53,73] Given two matrices $M_1 \in R^{m\times n}$ and $M_2 \in R^{l\times n}$ and the set $\mathcal{Q}$ in (6.3), for any vector $x \in R^n$, there exists a vector $x^* \in \mathcal{Q}$ satisfying

$$\|x - x^*\|_2 \leq \sigma(M_1, M_2) \left\| \begin{bmatrix} (M_1 x - p)^+ \\ M_2 x - q \end{bmatrix} \right\|_1,$$

where $\sigma(M_1, M_2)$ is a constant determined by M_1 and M_2. In addition, Rosenbloom [75] obtain the same result.

6.2.2 Restricted weak RSP of order k

The range space property (RSP) of order k was first introduced by Zhao in [92,93], which is described as follows:

Definition 43 (RSP of order k)**.** [93], [100], [95], [94] Given a matrix $A \in R^{m\times n}$, if for any two disjoint sets J_1, J_2 such that $J_1, J_2 \subseteq \{1, ..., n\}$ and $|J_1| + |J_2| \leq k$, there is a vector $\eta \in \mathcal{R}(A^T)$ such that

$$\begin{cases} \eta_i = 1, & i \in J_1; \\ \eta_i = -1, & i \in J_2; \\ |\eta_i| < 1, & i \in \bar{J}_1 \cap \bar{J}_2, \end{cases},$$

then A^T is said to satisfy the RSP of order k.

Closely related to NSP of order k and RSP of order $2k$, the RSP of order k is one of the important matrix properties in compressive sensing. It originates from the uniqueness condition of the solution of basis pursuit for noiseless situations (ℓ_1-minimization with linear constraints $y = Ax$). More specifically, it has been shown in [43, 44, 48, 69, 93] that for any given vector $\bar{x}$, $\bar{x}$ is the unique least ℓ_1-norm solution of the system $Ax = A\bar{x}$ if and only if the following two conditions are satisfied:

- the matrix $A_{\text{supp}(\bar{x})}$ has full column rank;
- there exists a vector $\eta \in \mathcal{R}(A^T)$ such that

$$\eta_i = 1 \text{ for } \bar{x}_i > 0; \; \eta_i = -1 \text{ for } \bar{x}_i < 0; \; |\eta_i| < 1 \text{ for } \bar{x}_i = 0.$$

In fact, Zhao [92, 93] has shown that any k sparse vector can be exactly recovered by the basis pursuit if and only if A^T satisfies the RSP of order k. So NSP of order k is a sufficient and necessary condition for the uniform recovery of sparse signals via basis pursuit. This means NSP of order k is equivalent to RSP of order k.

Zhao et al. [95] used a relaxed form of RSP of order k (i.e., weak RSP of order k) to develop the stability theorem for compressive sensing algorithms. The weak RSP of order k can be immediately obtained from the RSP of order k by changing $|\eta_i| < 1$ to $|\eta_i| \leq 1$ for $i \in \bar{J}_1 \cap \bar{J}_2$, which is stated as follows:

Definition 44 (weak RSP of order k)**.** *Given a matrix $A \in R^{m\times n}$, A^T is said to satisfy the weak* RSP *order k if for any two disjoint sets J_1, J_2 such that $J_1, J_2 \subseteq \{1, ..., n\}$ and $|J_1| + |J_2| \leq k$, there exists a vector $\eta \in \mathcal{R}(A^T)$ such that*

$$\begin{cases} \eta_i = 1, & i \in J_1; \\ \eta_i = -1, & i \in J_2; \\ |\eta_i| \leq 1, & i \in \bar{J}_1 \cap \bar{J}_2. \end{cases}$$

In [94, 95], Zhao et al. used Hoffman theorem to show that the ℓ_1-minimization with linear constraints such that basis pursuit [26] and Dantzig Selector [23], are stable in sparse recovery under the weak RSP order k. Moreover, the weak RSP order k is also the necessary condition for the basis pursuit and Dantzig Selector to be stable in sparse recovery. Different from the standard ℓ_1-minimization, the model (6.2) has more complex structures. To investigate the stability of (6.2), we need to extend the notion of RSP of order k to the so-called restricted weak RSP of order k, which is stated as follows:

Definition 45 (Restricted weak RSP of order k)**.** *Given matrices $A \in R^{m\times n}$ and $B \in R^{l\times n}$, (A^T, B^T) is said to satisfy the restricted weak* RSP *of order k if for any two disjoint sets J_1, J_2 such that $J_1, J_2 \subseteq \{1, ..., n\}$ and $|J_1| + |J_2| \leq k$, there exists a vector $\eta \in \mathcal{R}(A^T, B^T)$ such that $\eta = (A^T, B^T)\begin{pmatrix}\nu\\ u\end{pmatrix}$ where $\nu \in R^m$, $u \in R^l_-$ and*

$$\begin{cases} \eta_i = 1, & i \in J_1; \\ \eta_i = -1, & i \in J_2; \\ |\eta_i| \leq 1, & i \in \bar{J}_1 \cap \bar{J}_2. \end{cases}$$

It is worth mentioning that a generalized version of RSP of order k is also used to study the exact sign recovery in 1-bit compressive sensing in [100].

6.2.3 Polytope approximation of unit ball $\mathcal{B}$

The unit ℓ_2-ball is defined as $\mathcal{B} = \{s \in R^m : \|s\|_2 \leqslant 1\}$. The unit ball $\mathcal{B}$ can also be seen as the intersection of the half spaces as follows:

$$\mathcal{B} = \bigcap_{\|a\|_2=1} \{s \in R^m : a^T s \leqslant 1\}. \tag{6.4}$$

The unit ball $\mathcal{B}$ appears in (6.2) since the ℓ_2-norm $\|y - Ax\|_2$ appears in the constraints of (6.2). To use Hoffman Theorem to establish stability result for (6.2), we need to approximate the unit ball $\mathcal{B}$ by a certain polyhedral set in order to make the constraint linear. To this need, we use Lemma 47 (Zhao et al. [95]) which is based on Dudley approximation of the unit ball to create a polytope as an approximation of $\mathcal{B}$. After that, under the restricted weak RSP of order k, we prove that the model (6.2) is stable in the sense that

$$\|x - x^*\| \leq C_1\sigma_k(x)_1 + C_2\epsilon,$$

where x^* is the found solution by (6.2), and x denotes a solution of (6.1) or x denotes the signal to recover in CS, and C_1, C_2 are two constants and $\sigma_k(x)_1$ is the best k term approximation of x.

By introducing the slack variables μ, s, ξ and v, the ℓ_1-model (6.2) can be rewritten as the following model:

$$\begin{aligned} \min_{(x,\mu,s,\xi,v)} \quad & \|x\|_1 \\ \text{s.t.} \quad & a_1 s + a_2\xi + a_3 e^T v \leqslant \epsilon,\ Bx \leqslant b, \\ & \mu \in s\mathcal{B},\ \mu = y - Ax,\ (s,\xi,v) \geq 0, \\ & \left\|U^T(Ax-y)\right\|_\infty \leq \xi,\ \left|U^T(Ax-y)\right| \leq v. \end{aligned} \tag{6.5}$$

Before we discuss the stability of (6.2), let us specify the polytope approximation of the unit ball $\mathcal{B}$ first. Through taking K half spaces in (6.4), the unit ball can be approximated by the polytope

$$P_K = \bigcap_{\|a^i\|_2=1, 1\leqslant i\leqslant K} \{s \in R^m : (a^i)^T s \leqslant 1\}. \tag{6.6}$$

Dudley [36] established the following lemma to ensure that P_K can approximate $\mathcal{B}$ to a certain level of accuracy. Recall the Hausdorff metric of two sets $M_1, M_2 \subseteq R^m$:

$$\delta^{\mathcal{H}}(M_1, M_2) = \max\left\{ \sup_{x\in M_1} \inf_{z\in M_2} \|x - z\|_2,\ \sup_{z\in M_2} \inf_{x\in M_1} \|x - z\|_2 \right\}. \tag{6.7}$$

Lemma 46. (Dudley, [36]) There exists a constant τ such that for every integer number $K > m$, there exists a polytope P_K of the form (6.6) containing $\mathcal{B}$ and satisfying

$$\delta^{\mathcal{H}}(\mathcal{B}, P_K) \leq \frac{\tau}{K^{2/(m-1)}}, \tag{6.8}$$

where $\delta^{\mathcal{H}}(\mathcal{B}, P_K)$ is the Hausdorff metric of $\mathcal{B}$ and P_K.

For the convenience of later analysis, we include the following $2m$ half spaces into P_K in (6.6):

$$\pm\beta_i^T s \leq 1, i = 1, ..., m, \tag{6.9}$$

where β_i is the ith column of the $m \times m$ identity matrix. These extra half spaces can further shrink the original P_K ($K \geq 2m$), so (6.8) remains valid. Therefore, without loss of generality, we assume that P_K always include the half spaces in (6.9).

6.3 Approximation of the solution set of (6.5)

Denote the set C by

$$C = \{(x, s, \xi, v) : a_1 s + a_2 \xi + a_3 e^T v \leq \epsilon,\ Bx \leq b,\ \left\|U^T(Ax - y)\right\|_\infty \leq \xi,$$
$$\left|U^T(Ax - y)\right| \leq v,\ (s, \xi, v) \geq 0\},$$

and hence the solution set of (6.5) can be represented as

$$\Omega^* = \{(x, \mu, \xi, s, v) : \|x\|_1 \leq \theta^*,\ \mu \in s\mathcal{B},\ \mu = y - Ax,\ (x, s, \xi, v) \in C\}, \tag{6.10}$$

where θ^* is the optimal value of (6.5) or (6.2). In what follows, we will create a sequence of polytopes, denoted by $\{\widetilde{P_V}\}$. By replacing $\mathcal{B}$ in (6.10) with $\widetilde{P_V}$, we can get the relaxation (approximation) of Ω^*, denoted by $\Omega_{\widetilde{P_V}}$, i.e.,

$$\Omega_{\widetilde{P_V}} = \{(x, \mu, \xi, s, v) : \|x\|_1 \leq \theta^*,\ \mu \in s\widetilde{P_V},\ \mu = y - Ax,\ (x, s, \xi, v) \in C\}. \tag{6.11}$$

The sequence $\{\widetilde{P_V}\}$ is chosen such that the Hausdorff distance between $\Omega_{\widetilde{P_V}}$ and Ω^* tends to 0 as V tends to ∞.

In fact, we consider the polytopes $P_K, K > 2m$ in Lemma 46, which is an approximation of $\mathcal{B}$ when K is sufficiently large. Let $\{P_K\}_{K>2m}$ be a sequence of such polytopes. Let M_{P_K} be the matrix whose columns are $a^i \in R^m,\ i = 1, ..., K$, which define the half spaces in P_K, i.e.,

$$M_{P_K} = [a^1, ..., a^K],$$

and hence

$$P_K = \{s \in R^m : (M_{P_K})^T s \leq e^K\},$$

where $e^K \in R^K$ is the vector of ones. We now consider the sequence of such polytopes $\{\widetilde{P_V}\}_{V>2m}$ such that

$$\widetilde{P_V} = \bigcap_{2m<K\leq V} P_K. \tag{6.12}$$

Clearly, $\widetilde{P_V}$ is still a polytope consisting of a finite number of half spaces with the form: $(a^i)^T s \leq 1, \|a^i\|_2 = 1, i = 1, ..., \widetilde{K}$, where $\widetilde{K}$ is an integer number. Let $M_{\widetilde{P_V}}$ be the matrix whose columns are such vectors a^i. Thus $\widetilde{P_V}$ can be written as

$$\widetilde{P_V} = \{s \in R^m : (M_{\widetilde{P_V}})^T s \leq e^{\widetilde{K}}\}, \tag{6.13}$$

where $e^{\widetilde{K}}$ is the vector of ones in $R^{\widetilde{K}}$. The following result was established by Zhao, Jiang and Luo [94, 95].

Lemma 47. [94, 95] Let $\{P_K\}_{K>m}$ be the sequence of polytopes given in (6.8) in Lemma 46, and let $\{\widetilde{P_I}\}_{I>m}$ be the sequence of polytopes such that $\widetilde{P_I} = \bigcap_{m<K\leq I} P_K$. For any $s \in R^m$ with $\|s\|_2 = 1$, there is a column of $M_{\widetilde{P_I}}$, denoted by a^i, satisfying

$$\left\|s - a^i\right\|_2 \leq \sqrt{\frac{2\tau}{I^{2/(m-1)} + \tau}},$$

where τ is given in Lemma 46.

By using the above lemma, it is not very difficult to prove the following lemma.

Lemma 48. *Let the sequence $\{P_K\}_{K>m}$ given as Lemma 47, and let $\{\widetilde{P_V}\}_{V>m}$ be the sequence of polytopes such that $\widetilde{P_V} = \bigcap_{m<K\leq V} P_K$. The corresponding sets Ω^* and $\Omega_{\widetilde{P_V}}$ are*

defined as (6.10) *and* (6.11)*, respectively. Then the following property is satisfied:*

$$\delta^{\mathcal{H}}(\Omega^*, \Omega_{\widetilde{P_V}}) \to 0 \ \ as \ \ V \to \infty. \tag{6.14}$$

The proof is omitted since it is similar to the proof of Lemma 5.3 in [95]. Let ε' be any fixed small number. Thus from the above observations, there is an integer number $V_0 > 2m$ such that

$$\delta^{\mathcal{H}}(\Omega^*, \Omega_{\widetilde{P_{V_0}}}) \leq \varepsilon'. \tag{6.15}$$

In the remainder of the chapter, we use a polytope $\widetilde{P_{V_0}}$ satisfying (6.15) to replace the unit ℓ_2-ball $\mathcal{B}$ to obtain the approximation model of (6.2).

6.4 Approximation of (6.2)

Let N be the number of half spaces $((a^i)^T s \leq 1, \|a^i\|_2 = 1)$ which determine $\widetilde{P_{V_0}}$, and let $M_{\widetilde{P_{V_0}}}$ be a matrix with the columns $a^i, 1 \leq i \leq N$, i.e., $M_{\widetilde{P_{V_0}}} = [a^1, ..., a^N]$. By replacing $\mathcal{B}$ by $M_{\widetilde{P_{V_0}}}$, we obtain the following approximation of the optimal value θ^* of (6.2):

$$\begin{aligned} \theta^*_{\widetilde{P_{V_0}}} &:= \min_{(x,\mu,\xi,s,v)} \{\|x\|_1 : \ \mu \in s\widetilde{P_{V_0}}, \ \mu = y - Ax, \ (x, s, \xi, v) \in C\} \\ &= \min_{(x,\xi,s,v)} \{\|x\|_1 : \ (M_{\widetilde{P_{V_0}}})^T (y - Ax) \leq se^N, \ (x, s, \xi, v) \in C\}, \end{aligned} \tag{6.16}$$

and the associated problem of (6.2) can be written as

$$\begin{aligned} \min_{(x,s,\xi,v)} \quad & \|x\|_1 \\ \text{s.t.} \quad & a_1 s + a_2 \xi + a_3 e^T v \leq \epsilon, \ Bx \leq b, \\ & (M_{\widetilde{P_{V_0}}})^T (y - Ax) \leq se^N, \ (s, \xi, v) \geq 0, \\ & \left\|U^T(Ax - y)\right\|_\infty \leq \xi, \left|U^T(Ax - y)\right| \leq v, \end{aligned} \tag{6.17}$$

where e^N is the vector of ones in R^N. Clearly, (6.17) is an approximation of (6.2). The optimal solution set of (6.17) is

$$\begin{aligned}\Omega^*_{\widetilde{P_{V_0}}} &= \{x \in R^n : \ \|x\|_1 \le \theta^*_{\widetilde{P_{V_0}}}, \ \mu \in s\widetilde{P_{V_0}}, \ \mu = y - Ax, \ (x, s, \xi, v) \in C\} \\ &= \{x \in R^n : \ \|x\|_1 \le \theta^*_{\widetilde{P_{V_0}}}, \ (M_{\widetilde{P_{V_0}}})^T (y - Ax) \le se^N, \ (x, s, \xi, v) \in C\}\end{aligned} \tag{6.18}$$

Note that $\mathcal{B} \subseteq \widetilde{P_{V_0}}$ implies that $\theta^* \ge \theta^*_{\widetilde{P_{V_0}}}$. So we can see that $\Omega^*_{\widetilde{P_{V_0}}} \subseteq \Omega_{P_{V_0}}$. Let $\Omega_{\widetilde{P_{V_0}}}$ be the set defined as (6.11) by replacing $\widetilde{P_V}$ with $\widetilde{P_{V_0}}$. By the definition, we know that $\Omega^* \subseteq \Omega_{\widetilde{P_{V_0}}}$. In the next section, we develop the main stability result for the model (6.2) and we will prove it by using the following result established by Zhao [94, 98].

Lemma 49. [94,98] Let $\pi_S(s)$ be the projection of s into S. Let the three convex compact sets T_1, T_2 and T_3 satisfy that $T_1 \subseteq T_2$ and $T_3 \subseteq T_2$, then we have the following properties: for any $x \in R^n$ and any $z \in T_3$,

$$\|x - \pi_{T_1}(x)\|_2 \le \delta^{\mathcal{H}}(T_1, T_2) + 2\,\|x - z\|_2\,; \tag{6.19}$$

Before we analyse the stability for (6.2), we also need to define some constants. Let

$$C = \begin{bmatrix} A \\ B \end{bmatrix} \tag{6.20}$$

be the matrix with full row rank. Then given three positive numbers $c, d, e \in [1, \infty]$, we define two constants $\Upsilon(d, e)$ and $\vartheta(c)$ as follows:

$$\Upsilon(d, e) = \max_{\mho \subseteq \{1, \ldots, h\}, |\mho| = m} \left\|U_{\mho}^{-1}\right\|_{e \to d} \left\|(CC^T)^{-1}C\right\|_{\infty \to e}, \tag{6.21}$$

$$\vartheta(c) = \left\|(CC^T)^{-1}C\right\|_{\infty \to c}. \tag{6.22}$$

Specifically, we will use the following values for the analysis of the stability of (6.2):

$$\Upsilon(1,1) = \max_{\mho\subseteq\{1,\ldots,h\},|\mho|=m} \left\|U_{\mho}^{-1}\right\|_{1\to 1} \left\|(CC^T)^{-1}C\right\|_{\infty\to 1}, \ \vartheta(1) = \left\|(CC^T)^{-1}C\right\|_{\infty\to 1}, \tag{6.23}$$

$$\Upsilon(\infty,\infty) = \max_{\mho\subseteq\{1,\ldots,h\},|\mho|=m} \left\|U_{\mho}^{-1}\right\|_{\infty\to\infty} \left\|(CC^T)^{-1}C\right\|_{\infty\to\infty}. \tag{6.24}$$

6.5 Main stability theorem

Introducing a variable t yields the equivalent form of (6.17):

$$\begin{aligned} \min_{(x,t,s,\xi,v)} \quad & e^T t \\ \text{s.t.} \quad & a_1 s + a_2\xi + a_3 e^T v \le \epsilon, \ Bx \le b, \ |x| \le t, \\ & (M_{\widetilde{P_{V_0}}})^T (y - Ax) \le s e^N, \ (t,s,\xi,v) \ge 0, \\ & \left\|U^T(Ax-y)\right\|_\infty \le \xi, \left|U^T(Ax-y)\right| \le v. \end{aligned} \tag{6.25}$$

The solution set of (6.25) is given as (6.18). Note that the above optimization is equivalent to a linear programming. In fact, the constraint $\left\|U^T(Ax-y)\right\|_\infty \le \xi$ can be rewritten as the following constraint:

$$\left|U^T(Ax-y)\right| \le \xi e.$$

Thus the model (6.25) can be rewritten explicitly as a linear programming problem as follows:

$$\begin{aligned} \min_{(x,t,s,\xi,v)} \quad & e^T t \\ \text{s.t.} \quad & x + t \ge 0, \ \ -x + t \ge 0, \\ & -a_1 s - a_2\xi - a_3 e^T v \ge -\epsilon, M_{\widetilde{P_{V_0}}}^T Ax + e^N s \ge M_{\widetilde{P_{V_0}}}^T y, \\ & U^T Ax + \xi e \ge U^T y, -U^T Ax + \xi e \ge -U^T y, \\ & U^T Ax + v \ge U^T y, -U^T Ax + v \ge -U^T y, \\ & -Bx \ge -b, \ (t,s,\xi,v) \ge 0. \end{aligned} \tag{6.26}$$

Now the dual problem of (6.26) is given as follows:

$$
\begin{aligned}
\max_{w} \quad & -\epsilon w_3 + y^T M_{\widetilde{P_{V_0}}} w_4 + y^T U(w_5 - w_6 + w_7 - w_8) - b^T w_9 \\
\text{s.t.} \quad & w_1 - w_2 + A^T M_{\widetilde{P_{V_0}}} w_4 + A^T U(w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \\
& w_1 + w_2 \leq e, \\
& -a_1 w_3 + (e^N)^T w_4 \leq 0, \\
& -a_2 w_3 + e^T(w_5 + w_6) \leq 0, \\
& -a_3 e w_3 + w_7 + w_8 \leq 0, \\
& w_1, w_2 \in R^n_+, \ w_3 \in R_+, \ w_4 \in R^N_+, \ , w_{5-8} \in R^h_+, \ w_9 \in R^l_+.
\end{aligned} \tag{6.27}
$$

The optimality condition of linear programmings yields the following proposition:

Proposition 50. *A vector x^* is an optimal solution of* (6.17) *if and only if there exist vectors $(x^*, t^*, s^*, \xi^*, v^*, w^*) \in \Pi$, where Π is the set given as*

$$
\begin{aligned}
\Pi = \Big\{ (x, t, s, \xi, v, w) : \quad & -x - t \leq 0, \ x - t \leq 0, \ a_1 s + a_2 \xi + a_3 e^T v \leq \epsilon, \\
& -M^T_{\widetilde{P_{V_0}}} Ax - e^N s \leq -M^T_{\widetilde{P_{V_0}}} y, \ Bx \leq b, \\
& -U^T Ax - \xi e \leq -U^T y, \ U^T Ax - \xi e \leq U^T y, \\
& -U^T Ax - v \leq -U^T y, \ U^T Ax - v \leq U^T y, \\
& w_1 - w_2 + A^T M_{\widetilde{P_{V_0}}} w_4 + A^T U(w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \\
& w_1 + w_2 \leq e, \ - a_1 w_3 + (e^N)^T w_4 \leq 0, \ (t, s, \xi, v, w) \geq 0, \\
& -a_2 w_3 + e^T(w_5 + w_6) \leq 0, \ - a_3 e w_3 + w_7 + w_8 \leq 0, \\
& e^T t = -\epsilon w_3 + y^T M_{\widetilde{P_{V_0}}} w_4 + y^T U(w_5 - w_6 + w_7 - w_8) - b^T w_9 \Big\}.
\end{aligned} \tag{6.28}
$$

Clearly, $|x^*| = t^*$ must hold for every $(x^*, t^*, s^*, \xi^*, v^*, w^*) \in \Pi$. Note that the set Π can be rewritten as the form (6.3) by setting the matrices M_1' and M_2' and the vectors p'

and q' as follows:

$$M_1' = \begin{bmatrix} D^1 & 0 \\ 0 & D^2 \\ D^3 & 0 \\ 0 & \widetilde{I} \end{bmatrix} \tag{6.29}$$

$$M_2' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I & -I & 0 & A^T M_{\widetilde{P_{V_0}}} & A^T U & -A^T U & A^T U & -A^T U & -B^T \\ 0 & e^T & 0 & 0 & 0 & 0 & 0 & \epsilon & -y^T M_{\widetilde{P_{V_0}}} & -y^T U & y^T U & -y^T U & y^T U & b^T \end{bmatrix} \tag{6.30}$$

$$p' = \begin{bmatrix} 0 & 0 & \epsilon & -M^T_{\widetilde{P_{V_0}}} y & b & -U^T y & U^T y & -U^T y & U^T y & e & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, q' = 0, \tag{6.31}$$

where 0's in (6.29), (6.30) and (6.31) are zero matrix with suitable sizes and the matrices D^1, D^2 and D^3 and $\widetilde{I}$ have the following forms:

$$D^1 = \begin{bmatrix} -I & -I & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 e^T \\ -M^T_{\widetilde{P_{V_0}}} A & 0 & -e^N & 0 & 0 \\ B & 0 & 0 & 0 & 0 \\ -U^T A & 0 & 0 & -e & 0 \\ U^T A & 0 & 0 & -e & 0 \\ -U^T A & 0 & 0 & 0 & -I \\ U^T A & 0 & 0 & 0 & -I \end{bmatrix}, \quad \widetilde{I} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^N & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I^h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^h & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^l \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I^h \end{bmatrix}, \quad D^2 = \begin{bmatrix} I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & (e^N)^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & e^T & e^T & 0 & 0 & 0 \\ 0 & 0 & -a_3 e & 0 & 0 & 0 & I & I & 0 \end{bmatrix},$$

where I, I^N, I^h and I^l are the identity matrices with the size of $n \times n$, $N \times N$, $h \times h$ and $l \times l$, respectively. Then we can apply Lemmas 42 and 49 to prove the following theorem, which is the main result for the model (6.2). The idea of this proof follows that of Zhao, Jiang and Luo [95].

Theorem 51. *Let the problem data $(U, A, B, \epsilon, a, b, y)$ be given, where $y \in R^m$ and $b \in R^l$ are two vectors, a_i's are three nonnegative numbers satisfying $a_i \geq 0$ and $\sum_{i=1}^3 a_i = 1$, and $A \in R^{m\times n}$, $B \in R^{l\times n}$, $C \in R^{(m+l)\times n}$ given in (6.20) as well as $U \in R^{m\times h}$ are full-row-rank matrices. Let $\widetilde{P_{V_0}}$ be given as the polytope in (6.12) satisfying (6.15). If (A^T, B^T) satisfies restricted weak RSP of order k, then for any $x \in R^n$, there is an optimal solution x^* of (6.2) satisfying the following bound:*

$$\begin{aligned} \|x - x^*\|_2 \leq \; & \varepsilon' + 2\sigma(M_1', M_2')\Big\{ \|(Bx-b)^+\|_1 + \epsilon\hat{\Upsilon} + 2\sigma_k(x)_1 + \|Bx-b\|_{c'}\vartheta(c) + \\ & \Big(a_1\|y - Ax\|_2 + a_2\left\|U^T(Ax-y)\right\|_\infty + a_3\left\|U^T(Ax-y)\right\|_1 - \Upsilon\Big)^+ + \\ & \left\|U^T(Ax-y)\right\|_{d'}\Upsilon(d,e)\Big\}. \end{aligned} \tag{6.32}$$

where $\sigma(M_1', M_2')$ is the constant determined by (6.29) and (6.30), $\hat{\Upsilon} = (\max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}$ is a constant, $\Upsilon(e,d)$ is the constant given in (6.21) and $\vartheta(c)$ is the constant given in (6.22), e, d, c, d', c' are five given positive numbers (allowing to be ∞) satisfying

$$\frac{1}{c} + \frac{1}{c'} = 1, \; \frac{1}{d} + \frac{1}{d'} = 1, \; e, d, c, d', c' \in [1, +\infty]. \tag{6.33}$$

$\Upsilon(1,1)$, $\Upsilon(\infty,\infty)$ and $\vartheta(1)$ are the constants given in (6.23) and (6.24). If x is a feasible

solution of (6.2)*, then there is an optimal solution* x^* *of* (6.2) *such that the following inequality holds:*

$$\|x - x^*\|_2 \leq \varepsilon' + 2\sigma(M_1', M_2')\left\{\epsilon\hat{\Upsilon} + 2\sigma_k(x)_1 + \left\|U^T(Ax - y)\right\|_{d'}\Upsilon(d, e) + \|Bx - b\|_{c'}\vartheta(c)\right\}. \tag{6.34}$$

Proof. Let x be any vector in R^n and $\widetilde{P_{V_0}}$ be the polytope given as (6.12) which can approximate the unit ℓ_2-ball with error bound in (6.15). Clearly, it can be represented as a finite number of half spaces

$$(a^i)^T z \leq 1, \ \left\|a^i\right\|_2 = 1, \ i = 1, ..., N \text{ for some N}.$$

Let $M_{\widetilde{P_{V_0}}}$ be the matrix with columns a^i such that

$$M_{\widetilde{P_{V_0}}} = [a^1, ..., a^N].$$

Now we construct vectors (t, s, ξ, v, w) satisfying some properties. We first let (t, s, ξ, v) satisfy that

$$t = |x|, \ s = \left\|(M_{\widetilde{P_{V_0}}})^T(y - Ax)\right\|_\infty, \ \xi = \left\|U^T(y - Ax)\right\|_\infty, \ v = \left|U^T(y - Ax)\right|. \tag{6.35}$$

With such a choice of (t, s, ξ, ν), we have

$$\begin{aligned}&(-x - t)^+ = 0, \ (x - t)^+ = 0, \ (M_{\widetilde{P_{V_0}}}^T(y - Ax) - e^N s)^+ = 0, \ (U^T(y - Ax) - \xi e)^+ = 0,\\ &(-U^T(y - Ax) - \xi e)^+ = 0, \ (U^T(y - Ax) - v)^+ = 0, \ (-U^T(y - Ax) - v)^+ = 0.\end{aligned} \tag{6.36}$$

Let S be the support set of k largest components of x, and S_1 and S_2 be the sets such that

$$S_1 = \{i : \ x_i > 0, \ i \in S\}, \ S_2 = \{i : \ x_i < 0, \ i \in S\}.$$

Clearly, $|S_1 \bigcup S_2| = |S| = |S_1| + |S_2| \leq k$. Let S_3 be the complementary set of S. Clearly, S_1, S_2 and S_3 are disjoint. By the restricted weak RSP of order k, there exists a vector $\eta \in R(A^T, B^T)$ such that $\eta = A^T\nu^* + B^T h^*$ for some $\nu^* \in R^m$ and $h^* \in R^l_-$ satisfying

$$\eta_i = 1 \text{ for } i \in S_1;\ \eta_i = -1 \text{ for } i \in S_2;\ \ |\eta_i| \leq 1 \text{ for } i \in S_3. \tag{6.37}$$

Now we construct a feasible solution $w = (w_1, ..., w_9)$ to the dual problem (6.27).

Constructing (w_1, w_2): Set w_1 and w_2 as follows:

$$\begin{cases} (w_1)_i = 0,\ (w_2)_i = 1, & i \in S_1; \\ (w_1)_i = 1,\ (w_2)_i = 0, & i \in S_2; \\ (w_1)_i = \frac{1-\eta_i}{2},\ (w_2)_i = \frac{1+\eta_i}{2}, & i \in S_3. \end{cases} \tag{6.38}$$

Such w_1 and w_2 satisfy that

$$w_1 + w_2 \leq e,\ w_2 - w_1 = \eta,\ w_1, w_2 \geq 0. \tag{6.39}$$

Constructing (w_5-w_8): Note that U is a matrix with full row rank. Thus there must exist an invertible $m \times m$ matrix of U, denoted by $U_{\mho}$, where $\mho \subseteq \{1, ..., h\}$ with $|\mho| = m$. Denote the complementary set of $\mho$ by $\bar{\mho} = \{1, ..., h\} \setminus \mho$. Then we construct a vector $g \in R^h$ with

$$g_{\mho} = U_{\mho}^{-1}\nu^*,\ g_{\bar{\mho}} = 0.$$

Clearly, it implies

$$Ug = \nu^*. \tag{6.40}$$

Let g^+ (g^-) be a vector obtained by keeping the components with positive (negative) value and setting the remaining components to 0 in g. By using the vector g, $w_5 - w_9$ can be constructed as follows:

$$w_5 = a_2 g_+,\ w_6 = -a_2 g_-,\ w_7 = a_3 g_+,\ w_8 = -a_3 g_-, \tag{6.41}$$

which implies

$$w_5 - w_6 + w_7 - w_8 = (a_2 + a_3)g, \; w_5, w_6, w_7, w_8 \geq 0. \tag{6.42}$$

Constructing (w_4): Without loss of generality, we suppose that the first m columns in $M_{\widetilde{P_{V_0}}}$ are $\beta_i, \; i = 1, ..., m$, and $-\beta_i, \; i = 1, ..., m$ are the second m columns of $M_{\widetilde{P_{V_0}}}$. The components of w^4 can be assigned as follows:

$$\begin{cases} (w_4)_i = a_1\nu_i^*, & \text{if} \;\; \nu_i^* > 0, \; i = 1, ..., m; \\ (w_4)_{i+m} = -a_1\nu_i^*, & \text{if} \;\; \nu_i^* < 0, \; i = 1, ..., m; \\ 0, & \text{otherwise.} \end{cases} \tag{6.43}$$

From this choice of w_4, we can see that

$$M_{\widetilde{P_{V_0}}} w_4 = a_1\nu^*, \;\; \|w_4\|_1 = a_1 \|\nu^*\|_1 \text{ and } w_4 \geq 0. \tag{6.44}$$

Constructing (w_3): Let $w_3 = \max\{\|\nu^*\|_1, \|g\|_1, \|g\|_\infty\}$. Such a choice of w_3 together with the choice of w_4-w_8 implies

$$\begin{cases} (-a_1w_3 + (e^N)^T w_4)^+ & \leq \;\; (-a_1 \|\nu^*\|_1 + (e^N)^T w_4)^+ = 0, \\ (-a_2w_3 + e^T(w_5 + w_6))^+ & \leq \;\; (-a_2 \|g\|_1 + a_2 \|g\|_1)^+ = 0, \\ (-a_3ew_3 + w_7 + w_8)^+ & \leq \;\; (-a_3e \|g\|_\infty + a_3g)^+ = 0. \end{cases} \tag{6.45}$$

Constructing (w_9): Let $w_9 = -h^*$. Clearly, $w_9 \geq 0$ due to $h^* \leq 0$.

With the above choice of w and (6.39), (6.42), (6.44) and (6.45), we deduce that

$$\begin{cases} w_1 - w_2 + A^T M_{\widetilde{P_{V_0}}} w_4 + A^T U(w_5 - w_6 + w_7 - w_8) - B^T w_9 = 0, \\ (w_1 + w_2 - e)^+ = 0, \; (-a_1w_3 + (e^N)^T w_4)^+ = 0, \\ (-a_2w_3 + e^T(w_5 + w_6))^+ = 0, \; (-a_3ew_3 + w_7 + w_8)^+ = 0, \\ t^- = 0, \; s^- = 0, \; \xi^- = 0, \; v^- = 0, \; w^- = 0. \end{cases} \tag{6.46}$$

Note that Π given as (6.28) can be represented as $\mathcal{Q}$ in terms of (M_1', M_2', p', q') given in

(6.29) and (6.30). Thus for the vector (x, t, s, ξ, ν, w) constructed above, we can apply Lemma 42 to get the following results: For the vector (x, t, s, ξ, ν, w), there exists a vector $(\hat{x}, \hat{t}, \hat{s}, \hat{\xi}, \hat{v}, \hat{w}) \in \Pi$ such that

$$\left\| \begin{bmatrix} x \\ t \\ s \\ \xi \\ v \\ w \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{t} \\ \hat{s} \\ \hat{\xi} \\ \hat{v} \\ \hat{w} \end{bmatrix} \right\|_2 \leq \sigma(M_1', M_2') \left\| \begin{bmatrix} (-x-t)^+ \\ (x-t)^+ \\ (a_1 s + a_2 \xi + a_3 e^T v - \epsilon)^+ \\ (M_{\widetilde{P_{V_0}}}^T (y - Ax) - e^N s)^+ \\ (Bx - b)^+ \\ (U^T(y - Ax) - \xi e)^+ \\ (-U^T(y - Ax) - \xi e)^+ \\ (U^T(y - Ax) - v)^+ \\ (-U^T(y - Ax) - v)^+ \\ (w_1 + w_2 - e)^+ \\ (-a_1 w_3 + (e^N)^T w_4)^+ \\ (-a_2 w_3 + e^T(w_5 + w_6))^+ \\ (-a_3 e w_3 + w_7 + w_8)^+ \\ \{w_1 - w_2 + A^T M_{\widetilde{P_{V_0}}} w_4 + \\ A^T U(w_5 - w_6 + w_7 - w_8) - B^T w_9\} \\ \{e^T t + \epsilon w_3 - y^T M_{\widetilde{P_{V_0}}} w_4 - \\ y^T U(w_5 - w_6 + w_7 - w_8) + b^T w_9\} \\ (t^-, s^-, \xi^-, v^-, w^-) \end{bmatrix} \right\|_1 \tag{6.47}$$

where $\sigma(M_1', M_2')$ is a constant depending only on M_1' and M_2'' given in (6.29) and (6.30). Since the vector (x, t, s, ξ, v, w) satisfies (6.46) and (6.36), the inequality (6.47) can be simplified to

$$\left\| \begin{bmatrix} x \\ t \\ s \\ \xi \\ v \\ w \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{t} \\ \hat{s} \\ \hat{\xi} \\ \hat{v} \\ \hat{w} \end{bmatrix} \right\|_2 \leq \sigma(M_1', M_2') \left\| \begin{bmatrix} (a_1 s + a_2\xi + a_3 e^T v - \epsilon)^+ \\ (Bx-b)^+ \\ \{e^T t + \epsilon w_3 - y^T M_{\widetilde{P_{V_0}}} w_4 - \\ y^T U(w_5 - w_6 + w_7 - w_8) + b^T w_9\} \end{bmatrix} \right\|_1$$

$$= \left|(a_1 s + a_2\xi + a_3 e^T v - \epsilon)^+\right| + \|(Bx-b)^+\|_1 + \left|e^T t + \epsilon w_3 - y^T M_{\widetilde{P_{V_0}}} w_4 - y^T U(w_5 - w_6 + w_7 - w_8) + b^T w_9\right| \tag{6.48}$$

Note that $|(a^i)^T(Ax-y))| \leq \|a^i\|_2 \|y-Ax\|_2$ for all $i \in \{1, ..., N\}$, which imply that $\max_{1\leq i\leq N} \left|(M_{\widetilde{P_{V_0}}})^T(Ax-y)\right|_i \leq \|y-Ax\|_2$. The value of s in (6.35) implies that

$$s \leq \|y-Ax\|_2 .$$

Therefore the following inequality holds:

$$(a_1 s + a_2\xi + a_3 e^T v - \epsilon)^+ \leq (a_1 \|y-Ax\|_2 + a_2 \left\|U^T(y-Ax)\right\|_\infty + a_3 \left\|U^T(y-Ax)\right\|_1 - \epsilon)^+. \tag{6.49}$$

Due to (6.40), (6.42) and (6.44), we have

$$\left|e^T t + \epsilon w_3 - y^T M_{\widetilde{P_{V_0}}} w_4 - y^T U(w_5 - w_6 + w_7 - w_8) + b^T w_9\right| = \left|e^T t + \epsilon w_3 - y^T M g - b^T h^*\right|$$
$$= \left|e^T t + \epsilon w_3 - x^T A^T \nu^* + (U^T(Ax-y))^T g + (Bx-b)^T h^* - x^T B^T h^*\right|.$$

The fact $A^T\nu^* + B^T h^* = \eta$ and the triangle inequality imply that

$$\left|e^T t + \epsilon w_3 - y^T \nu^* - b^T h^*\right| \leq \left|e^T t - x^T\eta\right| + \epsilon\,|w_3| + \left|(U^T(Ax-y))^T g\right| + \left|(Bx-b)^T h^*\right|. \tag{6.50}$$

Now we deal with the right-hand side of the above inequality. First, By using the index sets S and S_3, we have

$$\left|e^T t - x^T \eta\right| = \left|e_S^T t_S + e_{S_3}^T t_{S_3} - x_S^T \eta_S - x_{S_3}^T \eta_{S_3}\right|. \tag{6.51}$$

It follows from $t = |x|$ and (6.37) that

$$\begin{aligned}
\left|e_S^T t_S + e_{S_3}^T t_{S_3} - x_S^T \eta_S - x_{S_3}^T \eta_{S_3}\right| &= \left|e_{S_3}^T t_{S_3} - x_{S_3}^T \eta_{S_3}\right| \le \left|e_{S_3}^T t_{S_3}\right| + \left|x_{S_3}^T \eta_{S_3}\right| \\
&= \|x_{S_3}\|_1 + \left|x_{S_3}^T\right| |\eta_{S_3}| \le \|x_{S_3}\|_1 + \left|x_{S_3}^T\right| e \\
&= 2\|x_{S_3}\|_1 .
\end{aligned}$$

We recall the error of best k term approximation $\sigma_k(x)_1 = \inf_z\{\|x-z\|_1 : \ \|z\|_0 \le k\}$. Then we obtain

$$\left|e^T t - x^T \eta\right| \le 2\|x_{S_3}\|_1 = 2\sigma_k(x)_1. \tag{6.52}$$

By using the restricted weak RSP of order k, we have

$$\|\nu^*\|_1 \le \left\| \begin{bmatrix} \nu^* \\ h^* \end{bmatrix} \right\|_1 \le \left\|(CC^T)^{-1} C\eta\right\|_1 \le \left\|(CC^T)^{-1}C\right\|_{\infty\to 1} \|\eta\|_\infty \le \vartheta(1),$$

where $C \in R^{(m+l)\times n}$ is a matrix given in (6.20) and $\vartheta(1)$ is defined in (6.24). Moreover, we have

$$\begin{aligned}
\|g\|_1 &= \left\|U_\mho^{-1}\nu^*\right\|_1 \le \left\|U_\mho^{-1}\right\|_{1\to 1} \|\nu^*\|_1 \\
&\le \left\|U_\mho^{-1}\right\|_{1\to 1} \left\|(CC^T)^{-1}C\eta\right\|_1 \\
&\le \left\|U_\mho^{-1}\right\|_{1\to 1} \left\|(CC^T)^{-1}C\right\|_{\infty\to 1} \|\eta\|_\infty \\
&\le \left\|U_\mho^{-1}\right\|_{1\to 1} \left\|(CC^T)^{-1}C\right\|_{\infty\to 1} .
\end{aligned}$$

Recall that $\Upsilon(1,1)$ is given in (6.23). Then $\|g\|_1 \le \Upsilon(1,1)$. Similarly, $\|g\|_\infty \le \Upsilon(\infty,\infty)$

can be obtained. Due to $w_3 = \max\{\|\nu^*\|_1, \|g\|_1, \|g\|_\infty\}$, we have

$$\epsilon\,|w_3| \le \epsilon(\max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}). \tag{6.53}$$

Let $c, d, e \in [1, +\infty]$ be three given positive numbers and d, d' be two given numbers satisfying (6.33). For the term $\left|(U^T(Ax-y))^T g\right|$ in (6.50), by using Hölder inequalities, we have

$$\begin{aligned}
\left|(U^T(Ax-y))^T g\right| &\le \left\|U^T(Ax-y)\right\|_{d'} \|g\|_d \\
&= \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}\nu^*\right\|_d \\
&\le \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}\right\|_{e\to d} \|\nu^*\|_e \\
&\le \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}\right\|_{e\to d} \left\|(CC^T)^{-1}C\right\|_{\infty\to e}.
\end{aligned} \tag{6.54}$$

Let $\Upsilon(d, e)$ be given as (6.21), i.e.,

$$\Upsilon(d,e) = \max_{\mho\subseteq\{1,\dots,h\},|\mho|=m} \left\|U_{\mho}^{-1}\right\|_{e\to d} \left\|(CC^T)^{-1}C\right\|_{\infty\to e}.$$

Thus we have

$$\left|(U^T(Ax-y))^T g\right| \le \Upsilon(d,e) \left\|U^T(Ax-y)\right\|_{d'}. \tag{6.55}$$

Similarly, the following inequalities holds

$$\begin{aligned}
\left|(Bx-b)^T h^*\right| &\le \|Bx-b\|_{c'} \|h^*\|_c \\
&\le \|Bx-b\|_{c'} \left\|(CC^T)^{-1}C\right\|_{\infty\to c} \|\eta\|_\infty \\
&\le \|Bx-b\|_{c'} \left\|(CC^T)^{-1}C\right\|_{\infty\to c}.
\end{aligned} \tag{6.56}$$

Let $\vartheta(c)$ be given as (6.22), i.e.,

$$\vartheta(c) = \left\|(CC^T)^{-1}C\right\|_{\infty\to c}.$$

Thus for any given c,

$$\left|(Bx-b)^T h^*\right| \le \vartheta(c)\left\|Bx-b\right\|_{c'}. \tag{6.57}$$

Due to (6.52), (6.53), (6.55) and (6.57), the inequality (6.50) is reduced to

$$\begin{aligned}\left|e^T t+\epsilon w_3-y^T\nu^*-b^T h^*\right| \;\le\; & \epsilon(\max\{\Upsilon(1,1),\Upsilon(\infty,\infty),\vartheta(1)\})+2\sigma_k(x)_1 \\ & +\left\|U^T(Ax-y)\right\|_{d'}\Upsilon(e,d)+\left\|Bx-b\right\|_{c'}\vartheta(c).\end{aligned} \tag{6.58}$$

Note that $\|x-\hat{x}\|_2 \le \left\|(x,t,s,\xi,\nu,w)-(\hat{x},\hat{t},\hat{s},\hat{\xi},\hat{\nu},\hat{w})\right\|_2$. Thus it follows from (6.49), (6.57) and (6.48) that

$$\begin{aligned}\|x-\hat{x}\|_2 \;\le\; & \sigma(M_1',M_2')\Big\{\,\|(Bx-b)^+\|_1+\epsilon(\max\{\Upsilon(1,1),\Upsilon(\infty,\infty),\vartheta(1)\})+ \\ & (a_1\|y-Ax\|_2+a_2\left\|U^T(Ax-y)\right\|_\infty+a_3\left\|U^T(Ax-y)\right\|_1-\epsilon)^+ + \\ & 2\sigma_k(x)_1+\left\|U^T(Ax-y)\right\|_{d'}\Upsilon(e,d)+\left\|Bx-b\right\|_{c'}\vartheta(c)\Big\}.\end{aligned} \tag{6.59}$$

We recall the three sets Ω^*, $\Omega_{\widetilde{P_{V_0}}}$ and $\Omega^*_{\widetilde{P_{V_0}}}$, where Ω^* and $\Omega^*_{\widetilde{P_{V_0}}}$ are the solution sets of (6.2) and (6.17), given as (6.10) and (6.18), respectively, and $\Omega_{\widetilde{P_{V_0}}}$ is given as (6.11) with $V=V_0$. Clearly, $x^*\in\Omega^*$ and $\hat{x}\in\Omega^*_{\widetilde{P_{V_0}}}$. Let x^* denote the projection of x onto Ω^*, that is,

$$x^*=\pi_{\Omega^*}(x).$$

Note that the three sets are compact convex sets, and $\Omega^*\subseteq\Omega_{\widetilde{P_{V_0}}}$ and $\Omega^*_{\widetilde{P_{V_0}}}\subseteq\Omega_{\widetilde{P_{V_0}}}$. Then by applying Lemma 49 with $T_1=\Omega^*$, $T_2=\Omega_{\widetilde{P_{V_0}}}$ and $T_3=\Omega^*_{\widetilde{P_{V_0}}}$, (6.19) implies that

$$\|x-\pi_{T_1}(x)\|_2=\|x-x^*\|_2\le\delta^{\mathcal{H}}(\Omega^*,\Omega_{\widetilde{P_{V_0}}})+2\|x-\hat{x}\|_2.$$

Since P_{V_0} satisfies (6.15), it implies that

$$\|x-x^*\|_2\le\varepsilon'+2\|x-\hat{x}\|_2.$$

Let $\hat{\Upsilon}=\max\{\Upsilon(1,1),\Upsilon(\infty,\infty),\vartheta(1)\}$. Combination of the above inequality and (6.59) yields the desired results (6.32). If x is the feasible solution of (6.2), then $\|(Bx-b)^+\|_1 = 0$ and

$$(a_1 \|y - Ax\|_2 + a_2 \left\|U^T(Ax-y)\right\|_\infty + a_3 \left\|U^T(Ax-y)\right\|_1 - \epsilon)^+ = 0,$$

and thus the desired error bound (6.34) is obtained. □

Before closing this chapter, we make some remarks for Theorem 51.

Remark 52. *By setting different values of a_1, a_2 and a_3, (6.2) can be reduced to several special cases, and the corresponding stability results for these special cases, can be obtained from (6.32) and (6.34) immediately. Note that if any of a_1, a_2 and a_3 becomes zero, the constant $\hat{\Upsilon} = \max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}$ in (6.32) and (6.34) will be simplified as well. For example, if $a_1 = 0$, the constant $\hat{\Upsilon}$ is reduced to $\max\{\Upsilon(1,1), \Upsilon(\infty,\infty)\}$. Thus we use the following table to show the form of the constant $\hat{\Upsilon}$ for different choices a_1, a_2 and a_3.*

Table 6.1: The constant $\hat{\Upsilon}$

a_i	$\hat{\Upsilon}$
$a_1 + a_2 = 0$	$\Upsilon(\infty,\infty)$
$a_1 + a_3 = 0$	$\Upsilon(1,1)$
$a_2 + a_3 = 0$	$\vartheta(1)$
$a_1 = 0$	$\max\{\Upsilon(1,1), \Upsilon(\infty,\infty)\}$
$a_2 = 0$	$\max\{\Upsilon(\infty,\infty), \vartheta(1)\}$
$a_3 = 0$	$\max\{\Upsilon(1,1), \vartheta(1)\}$
$a_1, a_2, a_3 \neq 0$	$\max\{\Upsilon(1,1), \Upsilon(\infty,\infty), \vartheta(1)\}$

Note that for any case with $a_1 = 0$, we have $\Omega^* = \Omega_{\widetilde{P_{V_0}}} = \Omega^*_{\widetilde{P_{V_0}}}$ so that $\hat{x} = x^*$ where $\hat{x} \in \Omega^*_{\widetilde{P_{V_0}}}$ and $x^* \in \Omega^*$. Thus instead of using Lemma (49), the stability results can be immediately obtained from (6.59).

Remark 53. *If the matrix B does not appear in (6.2), then (6.2) is reduced to the model*

$$\begin{aligned} &\min_{x\in R^n} \quad \|x\|_1 \\ &s.t. \quad a_1 \|y-Ax\|_2 + a_2 \left\|U^T(Ax-y)\right\|_\infty + a_3 \left\|U^T(Ax-y)\right\|_1 \le \epsilon. \end{aligned} \tag{6.60}$$

In this case, the restricted weak RSP *of order k is reduced to weak* RSP *of order k, which means $A^T\nu^* = \eta$. In fact, the upper bound of $\left|(U^T(Ax-y))^T g\right|$ in (6.54) can be improved to*

$$\begin{aligned} \left|(U^T(Ax-y))^T g\right| &\le \left\|U^T(Ax-y)\right\|_{d'} \|g\|_d \\ &= \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}\nu^*\right\|_d \\ &\le \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}(AA^T)^{-1}A\eta\right\|_d \\ &\le \left\|U^T(Ax-y)\right\|_{d'} \left\|U_{\mho}^{-1}(AA^T)^{-1}A\right\|_{\infty\to d}. \end{aligned} \tag{6.61}$$

Then in order to obtain a tighter bound, $\Upsilon(e,d)$ can be replaced by

$$\Upsilon'(d) = \max_{\mho\subseteq\{1,\dots,h\},|\mho|=m} \left\|U_{\mho}^{-1}(AA^T)^{-1}A\right\|_{\infty\to d}.$$

Then we have $\left|(U^T(Ax-y))^T g\right| \le \left\|U^T(Ax-y)\right\|_{d'} \Upsilon'(d)$. Similarly, the constants $\Upsilon_{1,1}$ and $\Upsilon_{\infty,\infty}$ are replaced by $\Upsilon'(1)$ and $\Upsilon'(\infty)$, respectively. Clearly, in this case, $\vartheta(c) = \left\|(AA^T)^{-1}A\right\|_{\infty\to c}$. Let $\hat{\Upsilon}' = \max\{\Upsilon'(1), \Upsilon'(\infty), \vartheta(1)\}$. Then the bound (6.34) is reduced to the following one:

$$\|x-x^*\|_2 \le \varepsilon' + 2\sigma(M_1', M_2'')\left\{\epsilon\hat{\Upsilon}' + 2\sigma_k(x)_1 + \left\|U^T(Ax-y)\right\|_{d'} \Upsilon'(d)\right\}. \tag{6.62}$$

Similarly, we list the constants $\hat{\Upsilon}'$ for different choices of a_i, $i = 1,2,3$.

Table 6.2: The constant $\hat{\Upsilon}'$

a_i	$\hat{\Upsilon}'$
$a_1 + a_2 = 0$	$\Upsilon'(\infty)$
$a_1 + a_3 = 0$	$\Upsilon'(1)$
$a_2 + a_3 = 0$	$\vartheta(1)$
$a_1 = 0$	$\max\{\Upsilon'(1), \Upsilon'(\infty)\}$
$a_2 = 0$	$\max\{\Upsilon'(\infty), \vartheta(1)\}$
$a_3 = 0$	$\max\{\Upsilon'(1), \vartheta(1)\}$
$a_1, a_2, a_3 \neq 0$	$\max\{\Upsilon'(1), \Upsilon'(\infty), \vartheta(1)\}$

Note that when $a_1 = 0$, $\hat{\Upsilon}' = \Upsilon'(1)$ since $\left\|U_{\mho}^{-1}(AA^T)^{-1}A\right\|_{\infty\to 1} \geq \left\|U_{\mho}^{-1}(AA^T)^{-1}A\right\|_{\infty\to\infty}$. Setting $d = 1$ yields

$$\|x - x^*\|_2 \leq \sigma(M_1', M_2'')\left\{\epsilon\Upsilon'(1) + 2\sigma_k(x)_1 + \left\|U^T(Ax - y)\right\|_\infty \Upsilon'(1)\right\}, \tag{6.63}$$

which is identical to the bound for the ℓ_1-minimization established by Zhao and Li [98] (see also in Zhao [94]),

$$\min\{\|x\|_1 : a_2 \left\|U^T(Ax - y)\right\|_\infty + a_3 \left\|U^T(Ax - y)\right\|_1 \leq \epsilon\}.$$

In particular, when $\hat{x}$ is the solution to ℓ_0-minimization, it must be feasible to ℓ_1-minimization. So (6.32) *and* (6.63) *are exactly the error bound between $\hat{x}$ and the solution of ℓ_0-minimization x^*.*

Our stability result is only determined by the problem data itself and can be applied to many sparsity models. Different from the stability results obtained by NSP of order k or RSP of order k, Theorem 51 is proven under the so-called restricted weak RSP of order k.

Chapter 7

Conclusions and Future Work

7.1 Conclusions

Optimization models with sparsity lie at the heart of many practical applications such as compressive sensing, 1-bit compressive sensing, statistical and machine learning, etc. Although these models are often difficult to solve in general, certain efficient algorithms are still urgently needed. In the meantime, the practical algorithms are also required to be stable in finding the sparse solutions of these models. In this thesis, we have studied the general ℓ_0-minimization models (1.1) and (6.1). We have mainly discussed the nonuniqueness properties of the solutions to (1.1) and the existence of an optimal weight guaranteeing the problem (1.1) can be solved by weighted ℓ_1-minimization. The reformulation of (1.1) as a bilevel optimization was established under the assumption of strict complementarity. Based on this analysis, the primal and dual re-weighted ℓ_1-algorithms were developed for the model (1.1) and some numerical experiments were carried out to illustrate the efficiency of these methods. Moreover, a stability result was established for the model (6.2). The main results of this thesis are described in more detail as follows.

Properties of the solutions of (1.1): Some basic properties of the solutions of (1.1) are developed in Chapter 2, such as the necessary conditions for a point being the sparsest point in T given in (1.2). Since the model (1.1) is more complex than the basis pursuit (1.8), it seems difficult to guarantee the uniqueness of the solutions of (1.1) under some standard conditions such as the NSP or RIP condition. So we have shown the nonuniqueness of the sparsest solutions of (1.1) under some mild conditions. We also discussed the boundedness of the solution set of (1.1) under certain conditions. Based on this, a lower bound for the absolute entries of the solutions to (1.1) can be guaranteed when the solution set of (1.1) is bounded.

Existence of optimal weights and reformulation of (1.1): Due to the complexity of the model (1.1), it is difficult to ensure it being exactly solved by the ℓ_1- or weighted ℓ_1-minimization model. In Chapter 3, we introduce the concept of optimal weights which, in theory, guarantee the solution of the weighted ℓ_1-problem is the sparsest point in the feasible set of the sparse optimization problem. We have shown in Theorems 19 and 20 that the existence of an optimal weight for the weighted ℓ_1-minimization problem associated with (1.1). We further discussed some fundamental properties of the weighted ℓ_1-problem (3.5) and its dual problem (3.20) such as strong duality and strictly complementary condition. We show that under the well-known Slater condition, strong duality for (3.5) and (3.20) is satisfied. Moreover, strict complementarity is proven to be guaranteed under Assumption 32. Finally, we have shown that the ℓ_0-problem (1.1) can be formulated as a bilevel programming problem under Assumption 34. This fact provides a basis for the development of algorithms in Chapter 4.

Primal and dual re-weighted algorithms: The main contribution of this thesis is to develop two new types of re-weighted algorithms for the problem (1.1). In Chapter 4, a family of merit functions is introduced, including a new merit function (4.9). We apply the first order approximation (linearization) to the model (3.5) to develop the primal re-weighted ℓ_1-algorithm. We test the numerical behaviour of the primal algorithm PRA with this new merit function and other merit functions. Our simulations indicate that PRA with this new merit function is very promising compared with other tested merit functions. We also demonstrate the influence of the parameter ε of merit functions, and our experiments indicate that a sufficiently small ε might lead to a low success rate of finding the sparsest point in T.

Next, we develop a framework for the dual re-weighted ℓ_1-algorithms. We have introduced two new ideas to relax the bilevel programming problem (3.37), which, together with the idea of Zhao and Luo [99], lead to three relaxation models (4.25), (4.32) and (4.38). As a result, we develop several specific algorithms: DWA(I), DWA(II) and DRA(I)-DRA(VI). It is worth stressing that the new dual re-weighted ℓ_1-algorithms provide a competitive and alternative computational approach for solving the ℓ_0-problems.

Numerical experiments: In Chapter 5, we have carried out a number of simulations in order to choose a good candidate for parameters and merit functions in our dual algorithms. We have also performed experiments on the noisy sparse model (1.9), monotonic sparse model (1.5) and nonnegative sparse model (1.7), in which the matrix B has special structures. We also consider the cases where the matrix B is a random Gaussian matrix. Our numerical results show that the performance of DWA(I) and DWA(II) are almost identical to that of ℓ_1-method (2.24). The dual re-weighted ℓ_1-algorithms, except for DRA(I), remarkably outperform the ℓ_1-minimization and the dual re-weighted ℓ_1-algorithms DRA(II), DRA(IV) and DRA(VI) always have a better performance of finding the sparse vector in T among these dual algorithms. Finally, we have carried out experiments to demonstrate that the dual algorithms are insensitive to the choice of parameter ε provided that ε is relatively small.

Stability theory: In Chapter 6, we have discussed the stability issue of the ℓ_1-minimization method for a class of sparse optimization problems. To establish our results, we first introduced the restricted weak RSP of order k which is one of the mildest assumption governing the stability of sparsity-seeking algorithms. By using the classic Hoffman theorem and Lemma 49, we have shown that under the restricted weak RSP of order k, the ℓ_1-minimization method (6.2) is stable (see Theorem 51 for details). Our results can apply to a wide range of situations, including the problem with simple ℓ_1, ℓ_2 and ℓ_∞ constraints. Also several existing stability results based on the RSP conditions can be re-obtained from our general results.

7.2 Future work

In our view, the study of dual re-weighted ℓ_1-algorithms remains incomplete, as for example, some other relaxation versions of the bilevel model (3.37) and a different choice of $\mathcal{W}$ or a different constraint that reflects the relation of the dual objective $-\lambda_1\epsilon - \lambda_2^T b + \lambda_3^T y$ and $\|\lambda_6\|_0$ can still be developed. Next, it is necessary to analyse the convergence of the dual re-weighted ℓ_1-algorithms. It seems such an analysis can be possibly made under a certain assumption such as restricted isometry property (RIP) or restricted weak RSP of transposed matrices.

Secondly, it would be interesting to develop re-weighted ℓ_1-algorithms via the second order approximation of merit function for sparsity. By using the second order approximation, (4.14) can be extended to

$$\Psi_\varepsilon(t) \leqslant \Psi_\varepsilon(t^k) + \nabla\Psi_\varepsilon^T(t^k)(t - t^k) + \frac{L}{2}\|t - t^k\|_2^2 \tag{7.1}$$

when Ψ_ε is a continuously differentiable function with a Lipschitz continuous gradient (the Lipschitz constant is L) (see more details in Descent Lemma [6]). How to determine L or to choose the merit functions satisfying (7.1) is also a future work.

Last but not least, the main stability result (6.34) in Chapter 6 has indicated that an error bound can be measured by using the parameters d, e and c. How to choose such proper parameters to obtain a tighter error bound would be an interesting research topic.

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