SYMPLECTIC RESOLUTIONS OF QUIVER VARIETIES AND CHARACTER VARIETIES

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Abstract. In this article, we consider Nakajima quiver varieties from the point of view of symplectic algebraic geometry. We prove that they are all symplectic singularities in the sense of Beauville and completely classify which admit symplectic resolutions. Moreover we show that the smooth locus coincides with the locus of θ-canonically polystable points, generalizing a result of Le Bruyn, and we describe the Namikawa Weyl group. An interesting consequence of our results is that not all symplectic resolutions of quiver varieties appear to come from variation of GIT.

We apply this to the G-character variety of a compact Riemann surface of genus $g > 0$, when G is $\mathrm{SL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{C})$. We show that these varieties are symplectic singularities and classify when they admit symplectic resolutions: they do when $g = 1$ or $(g, n) = (2, 2)$ (assuming $n \geq 2$). This is analogous to the case of a quiver with one vertex, g arrows, and dimension vector (n).

We note that our results show that existence of proper and projective symplectic resolutions are equivalent for the varieties in question. This does not seem to be known in general.

Dedicated, with admiration and thanks,
to Victor Ginzburg, on the occasion of his 60th Birthday.

Table of Contents

1. Introduction

Nakajima's quiver varieties [45], [47], have become ubiquitous throughout representation theory. For instance, they play a key role in the categorification of representations of Kac-Moody Lie

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algebras and the corresponding theory of canonical bases. They provide étale-local models of singularities appearing in many important moduli spaces, together with, in most cases, a canonical symplectic resolution given by varying the stability parameter. They give explicit constructions via symplectic or hyperkähler reduction of some important moduli spaces, including resolutions of du Val singularities, Hilbert schemes of points on them, and Uhlenbeck and Gieseker instanton moduli spaces.

Surprisingly, there seems to be no explicit criterion in the literature for when a quiver variety admits a symplectic resolution; often, in applications, suitable sufficient conditions for their existence are provided, but they do not appear always to be necessary. The main purpose of this article is to give such an explicit criterion. Following arguments of Kaledin, Lehn and Sorger (who consider the surprisingly similar case of moduli spaces of semistable sheaves on a $K3$ or abelian surface), our classification ultimately relies upon a result of Drezet on the local factoriality of certain GIT quotients.

Our classification begins by generalizing Crawley-Boevey's decomposition theorem [13] of affine quiver varieties into products of such varieties, which we will call *indecomposable*, to the non-affine case; i.e., to quiver varieties with nonzero stability condition (Theorem 1.4). Along the way, we also generalize Le Bruyn's theorem, [37, Theorem 3.2], which computes the smooth locus of these varieties, again from the affine to nonaffine setting (Theorem 1.13).

Then, our main result, Theorem 1.6, states that those quiver varieties admitting resolutions are exactly those whose indecomposable factors, as above, are one of the following types of varieties. Here, Σ-indivisibility is a condition slightly weaker than indivisibility, which we will recall below; see Section 2 for precise definitions.

(a) Varieties whose dimension vectors are Σ-indivisible roots;
(b) Symmetric powers of deformations or partial resolutions of du Val singularities ($\mathbb{C}^2/\Gamma$ for $\Gamma < \mathrm{SL}_2(\mathbb{C})$);
(c) Varieties whose dimension vector are twice a root whose Cartan pairing with itself is -2 (i.e., the variety has dimension ten).

The last type is perhaps surprising: it is closely related to O'Grady's examples [38]. In this case, one cannot fully resolve or smoothly deform via a quiver variety, but after maximally smoothing in this way, the remaining singularities are étale-equivalent to the product of $V = \mathbb{C}^4$ with the locus of square-zero matrices in $\mathfrak{sp}(V)$ (which O'Grady considers, see also [38]), and the latter locus is resolved by the cotangent bundle of the Lagrangian Grassmannian of V. In fact, the resolution can be constructed by blowing up the singular locus (once), as for the O'Grady examples.

In the case of type (a), one can resolve or deform by varying the quiver (GIT) parameters, whereas in the case of type (b), one cannot resolve in this way, but the variety is well-known to be isomorphic to another quiver variety (whose quiver is obtained by adding an additional vertex, usually called a framing, and arrows from it to the other vertices), which does admit a resolution via varying the parameters. Moreover, in this case, if the stability parameter is chosen to lie in the appropriate chamber, then the resulting resolution is a punctual Hilbert scheme of the minimal resolution of the original (deformed) du Val singularity.

1.1. **Symplectic resolutions.** In order to state precisely our main results, we will require some notation, which we will restate in more detail in Section 2. Let $Q = (Q_0, Q_1)$ be a quiver with finitely many vertices and arrows. We fix a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, deformation parameter $\lambda \in \mathbb{C}^{Q_0}$, and stability parameter $\theta \in \mathbb{Z}^{Q_0}$, such that $\lambda \cdot \alpha = \theta \cdot \alpha = 0$. Unless otherwise stated, we make the following assumption throughout the paper:

$$\textit{If } \theta \neq 0 \textit{ then } \lambda \in \mathbb{R}^{Q_0}. \tag{1}$$

Nakajima associated to this data the (generally singular) variety, called a "quiver variety." We briefly recall the definition; see Section 2 for more details. Let $\operatorname{Rep}(Q, \alpha)$ be the vector space of representations of Q of dimension α. The group $G(\alpha) := \prod_{i \in Q_0} GL_{\alpha_i}(\mathbb{C})$ acts on $\operatorname{Rep}(Q, \alpha)$; write $\mathfrak{g}(\alpha) = \operatorname{Lie} G(\alpha)$. Then $G(\alpha)$ acts on $T^* \operatorname{Rep}(Q, \alpha) \cong \operatorname{Rep}(\overline{Q}, \alpha)$ with a moment map $\mu : T^* \operatorname{Rep}(Q, \alpha) \to \mathfrak{g}(\alpha)^* \cong \mathfrak{g}(\alpha)$ (here $\overline{Q}$ is the doubled quiver). To $\lambda \in \mathbb{C}^{Q_0}$ we can associate $(\lambda \operatorname{Id}_i)_{i \in Q_0} \in \mathfrak{g}(\alpha)$. Let $\mu^{-1}(\lambda)^\theta \subseteq \mu^{-1}(\lambda)$ be the θ-semistable locus; this is the locus corresponding to representations of $\overline{Q}$ such that the dimension vector β of every subrepresentation satisfies $\theta \cdot \beta \leq 0$. Then Nakajima defined the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ as:

$$\mathfrak{M}_\lambda(\alpha, \theta) := \mu^{-1}(\lambda)^\theta /\!\!/ G(\alpha).$$

It does not seem to be known whether $\mathfrak{M}_\lambda(\alpha, \theta)$, equipped with its natural scheme structure, is reduced (though we expect it is the case). Therefore, following Crawley-Boevey [14], we will consider *throughout the paper* all quiver varieties as *reduced schemes.*

Remark 1.1. The construction in [45, 47] is apparently more general, depending on an additional dimension vector, called the framing. However, as observed by Crawley-Boevey [12], every framed variety can be identified with an unframed one. In more detail, for the variety as in [45, 47] with framing $\beta \in \mathbb{N}^{Q_0}$, it is observed in [12, Section 1] that the resulting variety can alternatively be constructed by replacing Q by the new quiver $(Q_0 \cup \{\infty\}, \widetilde{Q_1})$, where $\widetilde{Q_1}$ consists of Q_1 together with, for every $i \in Q_0$, β_i new arrows from ∞ to i; then Nakajima's β-framed variety is the same as $\mathfrak{M}_{(\lambda,0)}((\alpha, 1), (\theta, -\alpha \cdot \theta))$. Thus, for the purposes of the questions addressed in this article, it is sufficient to consider the unframed varieties.

Let $R^+_{\lambda,\theta}$ denote those positive roots of Q that pair to zero with both λ and θ. If $\alpha \notin \mathbb{N}R^+_{\lambda,\theta}$ then $\mathfrak{M}_\lambda(\alpha, \theta) = \emptyset$, therefore we assume $\alpha \in \mathbb{N}R^+_{\lambda,\theta}$. Recall that a normal variety X is said to be a symplectic singularity if there exists an (algebraic) symplectic 2-form ω on the smooth locus of X such that $\pi^*\omega$ extends to a regular 2-form on the whole of Y, for any resolution of singularities $\pi : Y \to X$. We say that π is a symplectic resolution if $\pi^*\omega$ extends to a non-degenerate 2-form on

Y. Note that *a symplectic resolution does not always exist,* and when it does exist, it is not always unique.

Theorem 1.2. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is an irreducible symplectic singularity.*

This theorem is important because symplectic singularities have become important in representation theory: on the one hand they include many of the most important examples (aside from quiver varieties, they include linear quotient singularities, nilpotent cones, orbit closures, Slodowy slices, hypertoric varieties, and so on), and on the other hand they exhibit important properties, at least in the conical case, such as the existence of a nice universal family of deformations [34, 50, 49] and of quantizations [4, 8, 40].

From both the representation theoretic and the geometric point of view, it is important to know when the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a symplectic resolution. In this article, we address this question, giving a complete answer. The first step is to reduce to the case where α is a root for which there exists a θ-stable point in $\mu^{-1}(\lambda)$. This is done via the *canonical decomposition* of α, as described by Crawley-Boevey; it is analogous to Kac's canonical decomposition. In this article, the term canonical decomposition will only refer to the former, which we now recall. Associated to λ, θ is a combinatorially defined set $\Sigma_{\lambda,\theta} \subset R^+_{\lambda,\theta}$; see Section 2 below. Then α admits a canonical decomposition

$$\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)} \tag{2}$$

with $\sigma^{(i)} \in \Sigma_{\lambda,\theta}$ pairwise distinct, such that any other decomposition of α into a sum of roots belonging to $\Sigma_{\lambda,\theta}$ is a refinement of the decomposition (2). Generalizing [12, Theorem 1.2.], Proposition 3.18 implies

Theorem 1.3. *There exists a θ-stable representation of the deformed preprojective algebra $\Pi^\lambda(Q)$ of dimension α if and only if $\alpha \in \Sigma_{\lambda,\theta}$.*

Crawley-Boevey's Decomposition Theorem [13], which we will show holds in somewhat greater generality, then implies that the canonical decomposition gives a decomposition of the quiver variety as a product of varieties for each of the summands (the first statement of the next theorem). We show that the question of existence of symplectic resolutions of $\mathfrak{M}_\lambda(\alpha, \theta)$ can be reduced to the analogous question for each factor:

Theorem 1.4. *With respect to the canonical decomposition* (2)*:*

(a) The symplectic variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is isomorphic to $S^{n_1}\mathfrak{M}_\lambda(\sigma^{(1)}, \theta) \times \cdots \times S^{n_k}\mathfrak{M}_\lambda(\sigma^{(k)}, \theta)$.

(b) $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a symplectic resolution if and only if each $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a symplectic resolution.

(c) The existence of proper and projective symplectic resolutions is equivalent for $\mathfrak{M}_\lambda(\alpha, \theta)$.

Here S^nX denotes the nth symmetric product of X.

Remark 1.5. Most of the literature deals with projective rather than proper resolutions. However, there are interesting examples of proper symplectic resolutions that are not projective. For example, in [2] such examples are constructed admitting Hamiltonian torus actions of maximal dimension (this condition is called hypertoric there, which generalizes the usual definition of hypertoric variety). It seems to be an interesting question if, whenever a proper symplectic resolution exists, also a projective symplectic resolution exists. More generally, it seems reasonable to ask whether, if a proper symplectic resolution exists, then every proper $\mathbb{Q}$-factorial terminalization is symplectic; if we restrict to projective resolutions and terminalizations, then the proof of [50, Theorem 5.5] shows that this holds at least when the singularity is conical with homogeneous generic symplectic form.

To finish the classification, it suffices to describe the case $\alpha \in \Sigma_{\lambda,\theta}$. As we will recall below, α is called Σ-indivisible if there does not exist $n > 1$ such that $\frac{1}{n}\alpha \in \Sigma_{\lambda,\theta}$.

Let $p(\alpha) := 1 - \frac{1}{2}(\alpha, \alpha)$ where $(-, -)$ is the Cartan pairing associated to the undirected graph underlying the quiver, i.e., $(e_i, e_j) = 2 - |\{a \in Q_1 : a : i \to j \text{ or } a : j \to i\}|$, for elementary vectors e_i, e_j. As we will show below (in Corollary 3.23), $2p(\alpha) = \dim \mathfrak{M}_\lambda(\alpha, \theta)$. Finally, as we will recall in Section 2, elements $\alpha \in \Sigma_{\lambda,\theta}$ are divided into real roots (when $p(\alpha) = 0$) and imaginary roots (when $p(\alpha) > 0$). The case $p(\alpha) = 1$ is particularly important and called *isotropic*, since it means $(\alpha, \alpha) = 0$. When $p(\alpha) > 0$ we say that α is *anisotropic*.

Our main theorem is then:

Theorem 1.6. *Let $\alpha \in \Sigma_{\lambda,\theta}$. Then $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a symplectic resolution if and only if α is Σ-indivisible or α is twice a root $\beta \in \Sigma_{\lambda,\theta}$ with $p(\beta) = 2$.*

If $\alpha \in \Sigma_{\lambda,\theta}$ is Σ-indivisible and anisotropic, then a projective symplectic resolution of $\mathfrak{M}_\lambda(\alpha, \theta)$ is given by moving θ to a generic stability parameter. However, this fails if α is twice a root $\beta \in \Sigma_{\lambda,\theta}$ with $p(\beta) = 2$. It seems unlikely that $\mathfrak{M}_\lambda(\alpha, \theta)$ can be resolved by another quiver variety in this case. Instead, we show that the 10-dimensional symplectic singularity $\mathfrak{M}_\lambda(\alpha, \theta)$ can be resolved by blowing up the singular locus. We will need the partial ordering $\geq$ on stability conditions, where $\theta' \geq \theta$ if every θ'-semistable representation is θ-semistable; see Section 2.4 below.

Theorem 1.7. *Let $\alpha \in \Sigma_{\lambda,\theta}$, and suppose α is twice a root $\beta \in \Sigma_{\lambda,\theta}$ with $p(\beta) = 2$. Let θ' be a generic stability parameter such that $\theta' \geq \theta$. If $\widetilde{\mathfrak{M}}_\lambda(\alpha, \theta')$ is the blowup of $\mathfrak{M}_\lambda(\alpha, \theta')$ along the reduced singular locus, then the canonical morphism $\pi : \widetilde{\mathfrak{M}}_\lambda(\alpha, \theta') \to \mathfrak{M}_\lambda(\alpha, \theta)$ is a projective symplectic resolution of singularities.*

1.2. **Factoriality of quiver varieties.**

The real difficulty in the proof of Theorem 1.6 is in showing that if $\alpha \in \Sigma_{\lambda,\theta}$ is Σ-divisible and anisotropic,

$$\left(\gcd(\alpha), p\left(\gcd(\alpha)^{-1}\alpha\right)\right) \neq (2, 2), \tag{3}$$

then $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a proper symplectic resolution. Based upon a result of Drezet [19], who considered instead the moduli space of semistable sheaves on a rational surface, we show in

Corollary 6.9 the following result. Recall that a variety is (locally) factorial if all of its local rings are unique factorization domains.

Theorem 1.8. *Assume that* $\alpha \in \Sigma_{\lambda,\theta}$ *is an anisotropic root satisfying condition* (3)*, and that* θ *is generic. Then the quiver variety* $\mathfrak{M}_\lambda(\alpha, \theta)$ *is factorial.*

Observe that we did not require α to be Σ-divisible, although if it is Σ-indivisible we already noted that $\mathfrak{M}_\lambda(\alpha, \theta)$ is smooth for generic θ. On the other hand, in the Σ-divisible case, we will see that, for θ generic, the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ has terminal singularities, using that, by [48], this is equivalent to having singularities in codimension at least four. Therefore, by a well-known fact, the above theorem implies that it cannot admit a proper symplectic resolution.

In fact, we prove in Corollary 6.9 a more precise statement than Theorem 1.8 which does not require that θ be generic. By the argument given in the proof of Theorem 6.14, we see that the corollary implies that this statement holds for open subsets of $\mathfrak{M}_\lambda(\alpha, \theta)$. Therefore we conclude the following strengthening of the nonexistence direction of Theorem 1.6:

Corollary 1.9. *Assume that* $\alpha \in \Sigma_{\lambda,\theta}$ *is* Σ*-divisible, it satisfies condition* (3)*, and* θ *is generic. Under the assumptions of Theorem 1.8, if* $U \subseteq \mathfrak{M}_\lambda(\alpha, \theta)$ *is any singular Zariski open subset, then* U *does not admit a symplectic resolution.*

In fact, by Corollary 6.9 below, we can drop in Corollary 1.9 the assumption that θ is generic, at the price of replacing $\mathfrak{M}_\lambda(\alpha, \theta)$ by a certain canonical open set: the locus of direct sums of stable representations of dimension vector proportional to α.

In particular, in many cases, there are open subsets $U \subseteq \mathfrak{M}_\lambda(\alpha, \theta)$ which formally locally admit symplectic resolutions everywhere, but do not admit one globally. For example, if $\alpha = 2\beta$ for some $\beta \in \Sigma_{\lambda,\theta}$ with $p(\beta) \geq 3$ (cf. the definition of p above Theorem 1.6), then we can let U be the locus of representations which are either stable or decompose as $X = Y \oplus Y'$ for Y, Y' *nonisomorphic* θ-stable representations of dimension vectors equal to β.

There is one quiver variety in particular that captures the "unresolvable" singularities of $\mathfrak{M}_\lambda(\alpha, \theta)$. This variety, which we denote $\mathfrak{X}(g, n)$ with $g, n \in \mathbb{N}$, has been studied in the works of Lehn, Kaledin and Sorger. Concretely,

$$\mathfrak{X}(g,n) := \left\{ (X_1, Y_1, \ldots, X_g, Y_g) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \;\middle|\; \sum_{i=1}^{d} [X_i, Y_i] = 0 \right\} /\!\!/ \mathrm{GL}(n, \mathbb{C}),$$

Viewed as a special case of Corollary 6.9, we see that $\mathfrak{X}(g, n)$ does not admit a proper symplectic resolution if $g, n \geq 2$ and $(g, n) \neq (2, 2)$.

When $g = 1$, the Hilbert scheme of n points in the plane provides a symplectic resolution of $\mathfrak{X}(g, n) \simeq S^n\mathbb{C}^2$; see [23, Theorem 1.2.1, Lemma 2.8.3]. When $n = 1$, one has $\mathfrak{X}(g, n) \simeq \mathbb{C}^{2g}$.

Remark 1.10. It is interesting to note that [12, Theorem 1.1] implies that the moment map

$$(X_1, Y_1, \ldots, X_g, Y_g) \mapsto \sum_{i=1}^{g} [X_i, Y_i]$$

is flat when $g > 1$, in contrast to the case $g = 1$, which is easily seen not to be flat.

Remark 1.11. Generalizing the Geiseker moduli spaces that arise from framings of the Jordan quiver, it seems likely that the framed versions of $\mathfrak{X}(g,n)$, which are smooth for generic stability parameters, should have interesting combinatorial and representation theoretic properties.

Remark 1.12. One does not need the full strength of Theorem 1.8 to prove that $\mathfrak{M}_\lambda(\alpha,\theta)$ does not admit a symplectic resolution: it suffices to show that a formal neighborhood of some point does not admit a symplectic resolution. This reduces the problem to the one-vertex case, i.e., to $\mathfrak{X}(g,n)$. However, the techniques (following [33]) do not actually simplify in this case. Moreover, this would not be enough to imply Corollary 1.9.

1.3. **Smooth versus canonically polystable points.** In order to decide when the variety $\mathfrak{M}_\lambda(\alpha,\theta)$ is smooth, we describe the smooth locus in terms of θ-stable representations. Write the canonical decomposition $n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ of $\alpha \in \mathbb{N}R^+_{\lambda,\theta}$ as $\beta^{(1)} + \cdots + \beta^{(\ell)}$, where a given $\beta \in \Sigma_{\lambda,\theta}$ may appear multiple times. Recall that a representation is said to be θ-polystable if it is a direct sum of θ-stable representations. We say that a representation x is *θ-canonically polystable* if $x = x_1 \oplus \cdots \oplus x_\ell$ where each x_i is θ-stable, $\dim x_i = \beta^{(i)}$ and $x_i \not\simeq x_j$ for $i \neq j$, unless $\beta^{(i)} = \beta^{(j)}$ is a real root, i.e., $p(\beta^{(i)}) = 0$. Observe that the notion of θ-canonical polystability reduces to θ-stability precisely in the case that $\alpha \in \Sigma_{\lambda,\theta}$. In general, the set of points of $\mathfrak{M}_\lambda(\alpha,\theta)$ which are the image of θ-canonically polystable representations is a dense open subset. When $\theta = 0$, the result below is due to Le Bruyn [37, Theorem 3.2] (whose arguments we generalize).

Theorem 1.13. *A point $x \in \mathfrak{M}_\lambda(\alpha,\theta)$ belongs to the smooth locus if and only if it is θ-canonically polystable.*

Remark 1.14. Theorem 1.13 confirms the expectation stated after Lemma 4.4 of [27].

An element $\sigma \in \Sigma_{\lambda,\theta}$ is said to be *minimal* if there are no $\beta^{(1)}, \ldots, \beta^{(r)} \in \Sigma_{\lambda,\theta}$, with $r \geq 2$, such that $\sigma = \beta^{(1)} + \cdots + \beta^{(r)}$.

Corollary 1.15. *The variety $\mathfrak{M}_\lambda(\alpha,\theta)$ is smooth if, and only if, in the canonical decomposition $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ of α, each $\sigma^{(i)}$ is minimal, and the multiplicity n_i is one if $\sigma^{(i)}$ is isotropic.*

1.4. **Namikawa's Weyl group.** When both λ and θ are zero, $\mathfrak{M}_0(\alpha,0)$ is an affine conic symplectic singularity. Associated to $\mathfrak{M}_0(\alpha,0)$ is *Namikawa's Weyl group* W [49], a finite reflection group. In order to compute W, one needs to describe the codimension two symplectic leaves of $\mathfrak{M}_0(\alpha,0)$. More generally, we consider the codimension two leaves in a general quiver variety $\mathfrak{M}_\lambda(\alpha,\theta)$. It is enough by Crawley–Boevey's canonical decomposition to consider the case $\alpha \in \Sigma_{\lambda,\theta}$. We show that the codimension two symplectic leaves are parameterized by *isotropic decompositions* of α.

Definition 1.16. The decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(s)} + m_1\gamma^{(1)} + \cdots m_t\gamma^{(t)}$ is said to be an isotropic decomposition if

(a) $\beta^{(i)}, \gamma^{(j)} \in \Sigma_{\lambda,\theta}$.
(b) The $\beta^{(i)}$ are imaginary roots.
(c) The $\gamma^{(i)}$ are *pairwise distinct* real roots.
(d) If $\overline{Q}''$ is the quiver with $s+t$ vertices without loops and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows from vertex i to vertex $j \neq i$, where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \ldots, \beta^{(s)}, \gamma^{(1)}, \ldots, \gamma^{(t)}\}$, then Q'' is an affine Dynkin quiver.
(e) The dimension vector $(1, \ldots, 1, m_1, \ldots, m_t)$ of Q'' (where there are s ones) equals δ, the minimal imaginary root.

Remark 1.17. In fact, as we will show in Lemma 7.2 below, in an isotropic decomposition of $\alpha \in \Sigma_{\lambda,\theta}$, all of the anisotropic $\beta^{(i)}$ are pairwise distinct. This may help in finding these decompositions.

However, the isotropic $\beta^{(i)}$ need not be distinct. As an example, when Q is the quiver with two vertices $1, 2$ and two arrows, one loop at 1 and one arrow from 1 to 2, then we can take $\alpha = (4, 2)$, $\beta^{(1)} = (1, 0) = \beta^{(2)} = \beta^{(3)} = \beta^{(4)}$, and $\gamma^{(1)} = (0, 1)$. Then $p(\alpha) = 5$ and $\alpha \in \Sigma_{0,0}$, and the quiver $\overline{Q}''$ is of affine D_4 type with central vertex corresponding to $\gamma^{(1)}$ and external vertices corresponding to the $\beta^{(i)}$. This example is also interesting since $\alpha \in \Sigma_{0,0}$ is divisible, but $\frac{1}{2}\alpha \notin \Sigma_{0,0}$, so it is not Σ-divisible.

Given an isotropic decomposition with affine Dynkin quiver Q'', let Q''_f be the finite part, which is a Dynkin diagram.

Theorem 1.18. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be imaginary. Then the codimension two strata of $\mathfrak{M}_\lambda(\alpha, \theta)$ are in bijection with the isotropic decompositions of α. The singularity along each such stratum is étale-equivalent to the du Val singularity of the type A_n, D_n, E_n corresponding to Q''_f.*

As a consequence, for $\lambda = 0 = \theta$, by [49, Theorem 1.1] the Namikawa Weyl group is a product over all isotropic decompositions B of a group W_B. This group W_B is either the Weyl group of the corresponding Dynkin diagram Q''_f, or else the centralizer therein of an automorphism of this diagram, corresponding to the monodromy around the fiber over a point of the stratum under a crepant resolution of the complement of the codimension > 2 strata.

1.5. **Character varieties.** The methods we use seem to be applicable to many other situations. Indeed, as we have noted previously, they were first developed by Kaledin-Lehn-Soerger in the context of semistable sheaves on a $K3$ or abelian surface. Any situation where the symplectic singularity is constructed as a Hamiltonian reduction with respect to a reductive group of type A

is amenable to this sort of analysis. One such situation, which is of crucial importance throughout geometry, topology, and group theory, is that of character varieties of a Riemannian surface.

Let Σ be a compact Riemannian surface of genus $g > 0$ and π its fundamental group. The SL-*character variety* of Σ is the affine quotient

$$\mathcal{Y}(g,n) := \mathrm{Hom}(\pi, \mathrm{SL}(n,\mathbb{C}))/\!\!/\mathrm{SL}(n,\mathbb{C}).$$

Similarly, the GL-character variety is

$$\mathcal{X}(g,n) = \mathrm{Hom}(\pi, \mathrm{GL}(n,\mathbb{C}))/\!\!/\mathrm{GL}(n,\mathbb{C}).$$

If $g > 1$ then $\dim \mathcal{Y}(g,n) = 2(g-1)(n^2-1)$, and when $g = 1$, it has dimension $2(n-1)$. On the other hand $\dim \mathcal{X}(g,n) = \dim \mathcal{Y}(g,n) + 2g$. We do not consider the case where Σ has punctures (in this case it is natural to impose conditions on the monodromy about the punctures, and we will address this in future work).

Theorem 1.19. *The varieties $\mathcal{X}(g,n)$ and $\mathcal{Y}(g,n)$ are irreducible symplectic singularities.*

The same arguments, using Drezet's Theorem, that we have used to proof Theorem 1.8 are also applicable to the symplectic singularities of these varieties. We show that:

Theorem 1.20. *Assume that $g > 1$ and $(g,n) \neq (2,2)$. The varieties $\mathcal{X}(g,n)$ and $\mathcal{Y}(g,n)$ are factorial with terminal singularities.*

Arguing as in the proof of Theorem 6.14, Theorem 1.20 implies:

Corollary 1.21. *Assume $g > 1$ and $(g,n) \neq (2,2)$. Then the symplectic singularities $\mathcal{X}(g,n)$ and $\mathcal{Y}(g,n)$ do not admit proper symplectic resolutions. The same holds for any singular open subset.*

Remark 1.22. Parallel to Remark 1.12, we can give an alternative proof of the first statement of Corollary 1.21 using formal localization, reducing to the quiver variety case. The formal neighborhood of the identity of $\mathcal{X}(g,n)$ is well-known to identify with the formal neighborhood of $(0,\ldots,0)$ in the quotient

$$\left\{ (X_1, Y_1, \ldots, X_g, Y_g) \in \mathfrak{gl}(n,\mathbb{C}) \;\middle|\; \sum_{i=1}^{d} [X_i, Y_i] = 0 \right\} /\!\!/\mathrm{SL}(n,\mathbb{C}).$$

This is nothing but the formal neighborhood of zero of the quiver variety $\mathfrak{M}_{(n)}(0,0)$ for the quiver Q with one vertex and g arrows. However, we cannot directly conclude Theorem 1.20 using formal localization, and neither the stronger last statement of Corollary 1.21.

Remark 1.23. Similarly to the discussion after Corollary 1.9, one can obtain singular open subsets $U \subseteq \mathcal{Y}(g,n)$ in the case $g > 1$ and $(g,n) \neq (2,2)$ for which the formal neighborhood of every point does admit a resolution, even though the entire U does not admit one by the corollary. Indeed, by Remark 8.8, one example is analogous to the one given after Corollary 1.9: for $n = 2$ and $g \geq 3$, and U the complement of the locus of representations of the form $Y^{\oplus 2}$ for Y one-dimensional (and hence irreducible).

Once again, the case of a genus two Riemann surface and 2-dimensional representations of π i.e. $(g,n) = (2,2)$, is special. In this case $\mathcal{Y}(2,2)$ does not have terminal singularities. Moreover, by work of Lehn and Sorger [38], $\mathcal{Y}(2,2)$ does admit a symplectic resolution. It is constructed exactly as in the quiver $(2,2)$ case:

Theorem 1.24. *The blowups $\widetilde{\mathcal{X}}(2,2) \to \mathcal{X}(2,2)$ and $\widetilde{\mathcal{Y}}(2,2) \to \mathcal{Y}(2,2)$ along the singular loci are projective symplectic resolutions of singularities.*

Remark 1.25. When $g = 1$, the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ provides a projective symplectic resolution of $\mathcal{X}(g,n)$, and the barycentric Hilbert scheme provides a projective symplectic resolution of $\mathcal{Y}(g,n)$.

We first prove these results for $\mathcal{X}(g,n)$, and then in section 8.6, we deduce the results for $\mathcal{Y}(g,n)$ from these. Similar techniques are applicable to Hitchin's moduli spaces of semistable Higgs bundles over smooth projective curves: see [55].

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1.7. **Proof of the main results.** The proof of the theorems and corollaries stated in the introduction can be found in the following sections.

Theorem 1.2	:	Section 6.3
Theorem 1.4	:	Section 6.4
Theorem 1.6	:	Section 6.4
Theorem 1.7	:	Section 5.2
Theorem 1.8	:	Section 6.2
Corollary 1.9	:	Section 6.4
Theorem 1.13	:	Section 4.2
Corollary 1.15	:	Section 4.3
Theorem 1.18	:	Section 7.1
Theorem 1.19	:	Section 8.6
Theorem 1.20	:	Section 8.6
Corollary 1.21	:	Section 8.6
Theorem 1.24	:	Section 8.6

Throughout, a variety will mean a reduced, quasi-projective scheme of finite type over $\mathbb{C}$. If X is a (quasi-projective) variety equipped with the action of a reductive algebraic group G, then $X/\!\!/G$ will denote the good quotient (when it exists). In this case, let $\xi : X \to X/\!\!/G$ denote the quotient map. Then each fibre $\xi^{-1}(x)$ contains a unique closed G-orbit. Following Luna, this closed orbit is denoted $T(x)$.

2. Quiver varieties

In this section we fix notation.

2.1. **Notation.** Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$. We work over $\mathbb{C}$ throughout. All quivers considered will have a finite number of vertices and arrows. We *allow* Q to have loops at vertices. Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 denotes the set of vertices and Q_1 denotes the set of arrows. Given $a \in Q_1$, let $a_s, a_t \in Q_0$ be the source and target, so $a : a_s \to a_t$. For a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, $\mathrm{Rep}(Q, \alpha) := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{a_s}}, \mathbb{C}^{\alpha_{a_t}})$ denotes the vector space of representations of Q of dimension α. The group $G(\alpha) := \prod_{i \in Q_0} GL_{\alpha_i}(\mathbb{C})$ acts on $\mathrm{Rep}(Q, \alpha)$; write $\mathfrak{g}(\alpha) = \mathrm{Lie}\, G(\alpha)$. The torus $\mathbb{C}^\times$ in $G(\alpha)$ of diagonal matrices acts trivially on $\mathrm{Rep}(Q, \alpha)$. Thus, the action factors through $PG(\alpha) := G(\alpha)/\mathbb{C}^\times$.

Let $\overline{Q}$ be the doubled quiver so that there is a natural identification $T^* \mathrm{Rep}(Q, \alpha) = \mathrm{Rep}(\overline{Q}, \alpha)$. The group $G(\alpha)$ acts symplectically on $\mathrm{Rep}(\overline{Q}, \alpha)$ and the corresponding moment map is $\mu : \mathrm{Rep}(\overline{Q}, \alpha) \to \mathfrak{g}(\alpha)$, where we have identified $\mathfrak{g}(\alpha)$ with its dual using the trace form. An element $\lambda \in \mathbb{C}^{Q_0}$ is identified with the tuple of scalar matrices $(\lambda_i \mathrm{Id}_{V_i})_{i \in Q_0} \in \mathfrak{g}(\alpha)$. The affine quotient $\mu^{-1}(\lambda)/\!/G(\alpha)$ parameterizes semi-simple representations of the *deformed preprojective algebra* $\Pi^\lambda(Q) := \mathbb{C}\overline{Q}/(\sum_{a \in Q_1}(aa^* - a^*a) - \sum_{i \in Q_0} \lambda_i p_i)$, where p_i is the length-zero path at the vertex i. See [12] for details.

If M is a finite dimensional $\Pi^\lambda(Q)$-module, then $\dim M$ will always denote the dimension *vector* of M, and not just its total dimension.

2.2. **Root systems.** The coordinate vector at vertex i is denoted e_i. The set $\mathbb{N}^{Q_0}$ of dimension vectors is partially ordered by $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i and we say that $\alpha > \beta$ if $\alpha \geq \beta$ with $\alpha \neq \beta$. Following [14, Section 8], the vector α is called *sincere* if $\alpha_i > 0$ for all i. The Ringel form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

Let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ denote the corresponding Euler form and set $p(\alpha) = 1 - \langle \alpha, \alpha \rangle$. The fundamental region $\mathcal{F}(Q)$ is the set of $0 \neq \alpha \in \mathbb{N}^{Q_0}$ with connected support and with $(\alpha, e_i) \leq 0$ for all i.

If i is a loopfree vertex, so $p(e_i) = 0$, there is a reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by $s_i \alpha = \alpha - (\alpha, e_i) e_i$. There is also the dual reflection, $r_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}, (r_i \lambda)_j = \lambda_j - (e_i, e_j) \lambda_i$. The real roots (respectively imaginary roots) are the elements of $\mathbb{Z}^{Q_0}$ which can be obtained from the coordinate vector at a loopfree vertex (respectively $\pm$ an element of the fundamental region) by applying some sequence of reflections at loopfree vertices. Let R^+ denote the set of positive roots. Recall that a root β is *isotropic* imaginary if $p(\beta) = 1$ (i.e., $(\beta, \beta) = 0$) and *anisotropic* imaginary if $p(\beta) > 1$. Abusing terminology slightly, we will simply say that a root α is (a) real if $p(\alpha) = 0$, (b) isotropic if $p(\alpha) = 1$, and (c) anisotropic if $p(\alpha) > 1$.

2.3. **The canonical decomposition.** In this section we recall the *canonical decomposition* defined by Crawley-Boevey (not to be confused with Kac's canonical decomposition). Fix $\lambda \in \mathbb{C}^{Q_0}$ and $\theta \in \mathbb{Z}^{Q_0}$. Then $R^+_{\lambda,\theta} := \{\alpha \in R^+ \mid \lambda \cdot \alpha = \theta \cdot \alpha = 0\}$. Following [12], we define

$$\Sigma_{\lambda,\theta} = \left\{ \alpha \in R^+_{\lambda,\theta} \;\middle|\; p(\alpha) > \sum_{i=1}^{r} p\left(\beta^{(i)}\right) \text{ for any decomposition } \alpha = \beta^{(1)} + \cdots + \beta^{(r)} \text{ with } r \geq 2,\ \beta^{(i)} \in R^+_{\lambda,\theta} \right\}.$$

Example 1. Suppose that $\lambda = 0 = \theta$ and $\alpha \in \Sigma_{\lambda,\theta}$ is real, i.e., $p(\alpha) = 0$. Then α is a coordinate vector. Indeed, if not, by definition there is a vertex $i \in Q_0$ such that $\alpha = s_i \alpha + k e_i$ with $k \geq 1$. Then $0 = p(\alpha) = p(s_i \alpha) + k p(e_i)$ contradicts the fact that $\alpha \in \Sigma_{0,0}$.

Example 2. Again suppose that $\lambda = 0 = \theta$, and now assume that $\alpha \in \Sigma_{\lambda,\theta}$ is isotropic i.e., $p(\alpha) = 1$. Then as observed in the proof of [13, Proposition 1.2.(2)], α is supported on an affine Dynkin subquiver and there is the minimal imaginary root. We repeat the argument for the reader's convenience. First, α is Σ-indivisible, since $\alpha = k\beta$ would imply $p(\alpha) < kp(\beta)$, and as β is also a root, this contradicts the assumption $\alpha \in \Sigma_{0,0}$. Next, α is in the fundamental region, since otherwise $\alpha = s_i \alpha + k e_i$ for some $i \in Q_0$ and $k \geq 1$, which implies $1 = p(\alpha) = p(s_i \alpha) + k p(e_i)$, again contradicting the assumption that $\alpha \in \Sigma_{0,0}$. Now the support of α is connected. Letting

Q' be its supporting quiver (i.e., the result of discarding all vertices not in the support and all incident arrows), we obtain a connected quiver for which α is in the kernel of the Cartan pairing. By [30, Lemma 1.9.(d)], Q' is affine (ADE) Dynkin and α is an imaginary root. Since it is also Σ-indivisible, it is the minimal imaginary root δ of Q'.

Choosing a parameter $\lambda' \in \mathbb{C}^{Q_0}$ such that $R^+_{\lambda,\theta} = R^+_{\lambda'}$, [13, Theorem 1.1] implies that[1]

Proposition 2.1. *Let $\alpha \in \mathbb{N}R^+_{\lambda,\theta}$. Then α admits a unique decomposition $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ as a sum of element $\sigma^{(i)} \in \Sigma_{\lambda,\theta}$ such that any other decomposition of α as a sum of elements from $\Sigma_{\lambda,\theta}$ is a refinement of this decomposition.*

As is apparent from the results stated in the introduction, indivisible roots in $\Sigma_{\lambda,\theta}$ play an important role in this paper. However, being an indivisible root and belonging to $\Sigma_{\lambda,\theta}$ is stronger than being indivisible in $\Sigma_{\lambda,\theta}$; recall that a root β that is indivisible in $\Sigma_{\lambda,\theta}$ is said to be Σ-indivisible.

Theorem 2.2. *If $\alpha \in \Sigma_{\lambda,\theta}$ is imaginary, with $\alpha = m\beta$ for some indivisible root β, then one of the following hold:*

(a) α is isotropic and $m = 1$,
(b) β is anisotropic and $\beta \in \Sigma_{\lambda,\theta}$; or
(c) β is anisotropic, $\beta \notin \Sigma_{\lambda,\theta}$ and $m > 1$ can be chosen arbitrarily.

Proof. For simplicity, choose once again $\lambda' \in \mathbb{C}^{Q_0}$ such that $R^+_{\lambda,\theta} = R^+_{\lambda'}$ and let $\mathcal{F}_{\lambda'}$ be the "relative fundamental domain", as defined in [12, §7]. Then Theorem 2.2 follows from [12, Theorem 8.1] provided that $\alpha \in \mathcal{F}_{\lambda,\theta}$.

If α is not in $\mathcal{F}_{\lambda'}$ then, by definition, there is a sequence of admissible reflections (whose product is w say) mapping α to $w(\alpha) \in \mathcal{F}_{w(\lambda')}$ (where $w(\lambda')$ uses the action of dual reflections rather than reflections). Moreover, by [12, Lemma 5.2], $w(\alpha)$ also belongs to $\Sigma_{w(\lambda')}$. Thus, it suffices to note that if trichotomy of the theorem holds for $w(\alpha)$, then it also holds for the root α. □

Notice that Theorem 2.2 says that if β is an indivisible anisotropic root such that some multiple of β belongs to $\Sigma_{\lambda,\theta}$, then every *proper* multiple of β belongs to $\Sigma_{\lambda,\theta}$. However, in some cases β itself need not belong to $\Sigma_{\lambda,\theta}$.

2.4. **Stability.** Let $\theta \in \mathbb{Z}^{Q_0}$ be a stability condition. Given a representation M of $\overline{Q}$ (e.g., a module over $\Pi^\lambda(Q)$), let $\theta(M) := \theta \cdot \dim M$. Note that a representation M of $\Pi^\lambda(Q)$ is the same as a point in the zero fiber $\mu^{-1}(\lambda)$. Recall that a $\Pi^\lambda(Q)$-representation M (hence also a point in $\mu^{-1}(\lambda)$) such that $\theta(M) = 0$, is said to be θ-stable, respectively θ-semistable, if $\theta(M') < 0$, respectively $\theta(M') \leq 0$, for all proper nonzero subrepresentations M' of M. A representation M is said to be θ-polystable if $M = M_1 \oplus \cdots \oplus M_k$ with $\theta(M_i) = 0$, such that each M_i is θ-stable. The set

[1] We don't have to choose such a λ', since the arguments of [13] can be simply generalized to the context of the pair (θ, λ).

of θ-semistable points in $\mu^{-1}(\lambda)$ is denoted $\mu^{-1}(\lambda)^\theta$. We define a partial order on $\mathbb{Z}^{Q_0}$ by setting $\theta' \geq \theta$ if M θ'-semistable implies that M is θ-semistable, i.e.,

$$\theta' \geq \theta \quad \iff \quad \mu^{-1}(\lambda)^{\theta'} \subset \mu^{-1}(\lambda)^\theta.$$

The space $\mathrm{Rep}(\overline{Q}, \alpha)$ has a natural Poisson structure. Since the action of $G(\alpha)$ on $\mathrm{Rep}(\overline{Q}, \alpha)$ is Hamiltonian,

$$\mathfrak{M}_\lambda(\alpha, \theta) = \mu^{-1}(\lambda)^\theta /\!\!/ G(\alpha) := \mathrm{Proj} \bigoplus_{k \geq 0} \mathbb{C}\left[\mu^{-1}(\lambda)\right]^{k\theta}$$

is a Poisson variety.

Lemma 2.3. *If $\theta' \geq \theta$, then there is a projective Poisson morphism $\mathfrak{M}_\lambda(\alpha, \theta') \to \mathfrak{M}_\lambda(\alpha, \theta)$.*

Proof. By definition, we have a $G(\alpha)$-equivariant embedding $\mu^{-1}(\lambda)^{\theta'} \hookrightarrow \mu^{-1}(\lambda)^\theta$. This induces a morphism

$$\mathfrak{M}_\lambda(\alpha, \theta') = \mu^{-1}(\lambda)^{\theta'} /\!\!/ G(\alpha) \longrightarrow \mu^{-1}(\lambda)^\theta /\!\!/ G(\alpha) = \mathfrak{M}_\lambda(\alpha, \theta),$$

between geometric quotients. We need to show that this morphism is projective. This is local on $\mathfrak{M}_\lambda(\alpha, \theta)$. Therefore we may choose $n \gg 0$ and a $n\theta$-semi-invariant f and consider the open subsets $U \cap \mu^{-1}(\lambda)^{\theta'}$ and $U \cap \mu^{-1}(\lambda)^\theta$, where $U = (f \neq 0) \subset \mathrm{Rep}(\overline{Q}, \alpha)$. Then $\left(U \cap \mu^{-1}(\lambda)^\theta\right) /\!\!/ G(\alpha) = \mathrm{Spec}\, \mathbb{C}\left[U \cap \mu^{-1}(\lambda)\right]^{G(\alpha)}$ is an open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$ and

$$\left(U \cap \mu^{-1}(\lambda)^{\theta'}\right) /\!\!/ G(\alpha) = \mathrm{Proj} \bigoplus_{k \geq 0} \mathbb{C}\left[U \cap \mu^{-1}(\lambda)\right]^{k\theta'}$$

such that $\left(U \cap \mu^{-1}(\lambda)^{\theta'}\right) /\!\!/ G(\alpha) \to \left(U \cap \mu^{-1}(\lambda)\right) /\!\!/ G(\alpha)$ is the projective morphism

$$\mathrm{Proj} \bigoplus_{k \geq 0} \mathbb{C}\left[U \cap \mu^{-1}(\lambda)\right]^{k\theta'} \longrightarrow \mathrm{Spec}\, \mathbb{C}\left[U \cap \mu^{-1}(\lambda)\right]^{G(\alpha)}.$$

It is clear that this morphism is Poisson. $\square$

It follows from the proof of Lemma 2.3 that if $\theta'' \geq \theta' \geq \theta$ then the projective morphism $\mathfrak{M}_\lambda(\alpha, \theta'') \to \mathfrak{M}_\lambda(\alpha, \theta)$ factors through $\mathfrak{M}_\lambda(\alpha, \theta')$.

We will frequently use the fact that for each point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$, there is a unique closed $G(\alpha)$-orbit in the fibre over x of the quotient map $\xi : \mu^{-1}(\lambda)^\theta \to \mathfrak{M}_\lambda(\alpha, \theta)$. Recall that this closed orbit is denoted $T(x)$.

3. Canonical Decompositions of the Quiver Variety

In this section we recall the canonical decomposition of quiver varieties described in [13], and show that it holds in slightly greater generality than stated there.

3.1. **A stratification.** Let $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ be a closed point and $y \in T(x)$. Recall the following basic fact:

Proposition 3.1. [35, Proposition 3.2 (i)] *A point of a closed $G(\alpha)$-orbit in $\mu^{-1}(\lambda)^\theta$ is a θ-polystable representation.*

In more detail, [35, Proposition 3.2 (ii)] states that two points of $\mu^{-1}(\lambda)^\theta$ determine the same point of $\mathfrak{M}_\lambda(\alpha, \theta)$ if and only if the corresponding representations admit filtrations whose associated graded subquotients are isomorphic θ-polystable representations.

Therefore y decomposes into a direct sum $y_1^{e_1} \oplus \cdots \oplus y_k^{e_k}$ of θ-stable representations, with multiplicity. Let $\beta^{(i)} = \dim y_i$. The point x is said to have *representation type* $\tau = (e_1, \beta^{(1)}; \ldots ; e_k, \beta^{(k)})$. Associated to this is the stabilizer group $G_\tau = G(\alpha)_y$, which is independent of the choice of y up to conjugation in $G(\alpha)$. Even though $\mu^{-1}(\lambda)^\theta$ is not generally affine, the fact that a nonzero morphism between θ-stable representations is an isomorphism implies:

Lemma 3.2. *The group G_τ is reductive.*

In fact, it is isomorphic to $\prod_{i=1}^k GL_{e_i}(\mathbb{C})$. We denote the conjugacy class of a closed subgroup H of $G(\alpha)$ by (H). Given a reductive subgroup H of $G(\alpha)$, let $\mathfrak{M}_\lambda(\alpha, \theta)_{(H)}$ denote the set of points x such that the stabilizer of any $y \in T(x)$ belongs to (H). We order the conjugacy classes of reductive subgroups of $G(\alpha)$ by $(H) \leq (L)$ if and only if L is conjugate to a subgroup of H.

3.2. **Étale local structure.** In this section, we recall the étale local structure of $\mathfrak{M}_\lambda(\alpha, \theta)$, as described in [14, Section 4]. Since it is assumed in *op. cit.* that $\theta = 0$, we provide some details to ensure the results are still applicable in this more general setting. Let $x, y, y_1, \ldots, y_k, \beta^{(1)}, \ldots, \beta^{(k)}$, and τ be as in Section 3.1. Let Q' be the quiver with k vertices whose double has $2p(\beta^{(i)})$ loops at vertex i and $-(\beta^{(i)}, \beta^{(j)})$ arrows from vertex i to j. The k-tuple $\mathbf{e} = (e_1, \ldots, e_k)$ defines a dimension vector for the quiver Q'.

If X and Y are Poisson varieties, then we say that there is a étale Poisson isomorphism between a neighborhood of $x \in X$ and $y \in Y$ if there exists a Poisson variety Z and Poisson morphisms $Y \xleftarrow{\psi} Z \xrightarrow{\phi} X$ and $z \in Z$ such that $\phi(z) = x$, $\psi(z) = y$ and both ϕ and ψ are étale at z.

Theorem 3.3. *There is an étale Poisson isomorphism between a neighborhood of 0 in $\mu_{Q'}^{-1}(0) /\!\!/ G(\mathbf{e})$ and a neighborhood of $x \in \mathfrak{M}_\lambda(\alpha, \theta)$.*

The proof of Theorem 3.3 is given in section 3.3 below. By taking the completion $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x$ of $\mathfrak{M}_\lambda(\alpha, \theta)$ at x and the completion $\widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$ of $\mathfrak{M}_0(\mathbf{e}, 0)$ at 0, the formal analogue of Theorem 3.3 is:

Corollary 3.4. *There is an isomorphism of formal Poisson schemes $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x \simeq \widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$.*

Remark 3.5. An easy calculation shows that $p(\alpha) = p(\mathbf{e})$. It can also be deduced from the fact that $\dim \widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x = \dim \widehat{\mathfrak{M}}_0(\mathbf{e}, 0)_0$. This fact will be useful later.

The following result is an important consequence of the proof of Theorem 3.3.

Proposition 3.6. *The strata $\mathfrak{M}_\lambda(\alpha,\theta)_\tau := \mathfrak{M}_\lambda(\alpha,\theta)_{(G_\tau)}$ define a finite stratification of $\mathfrak{M}_\lambda(\alpha,\theta)$ into locally closed subsets such that*

$$\mathfrak{M}_\lambda(\alpha,\theta)_{(H)} \subset \overline{\mathfrak{M}_\lambda(\alpha,\theta)_{(L)}} \quad \Leftrightarrow \quad (H) \le (L).$$

Moreover, the connected components of the strata are precisely the symplectic leaves of $\mathfrak{M}_\lambda(\alpha,\theta)$, with respect to its natural Poisson bracket.

Proof. It is well-known that the stratification of $\mathrm{Rep}(Q,\alpha)^\theta /\!\!/ G(\alpha)$ by stabilizer type is finite, with smooth locally closed strata. Therefore the stratification $\{\mathfrak{M}_\lambda(\alpha,\theta)_\tau\}$ of $\mathfrak{M}_\lambda(\alpha,\theta)$ is finite with locally closed strata. Thus it suffices to show that (a) each stratum is smooth, and (b) the Poisson structure is non-degenerate on each stratum. In fact, (b) implies (a), and both statements are implied by Theorem 3.15. $\square$

We will show in Corollary 3.24 that each stratum $\mathfrak{M}_\lambda(\alpha,\theta)_\tau$ is connected.

3.3. **The proof of Theorem 3.3.** Fix $M = \mathrm{Rep}(\overline{Q},\alpha)$ and $G = G(\alpha)$. Recall that M has a canonical G-invariant symplectic form ω. Since $y \in M^\theta$, there exists some $n > 0$ and $n\theta$-semi-invariant function γ such that $\gamma(y) \neq 0$. We fix such a γ, and let $M_\gamma \subset M^\theta$ be the *affine* open subset of M defined by the non-vanishing of γ. Let $H := G(\alpha)_y$ be the stabilizer of y in $G(\alpha)$ and $\mathfrak{h}$ the Lie algebra of H. Since $\mathfrak{h}$ is reductive we can fix a $\mathfrak{h}$-stable complement L to $\mathfrak{h}$ in $\mathfrak{g}$. By [14, Lemma 4.1], the H-submodule $\mathfrak{g}\cdot y \subset M$ is isotropic, and by [14, Corollary 2.3], there exists a coisotropic H-module complement C to $\mathfrak{g}\cdot y$ in M. Let $W = (\mathfrak{g}\cdot y)^\perp \cap C$. The composition of $\mu : M \to \mathfrak{g}^*$ with the restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$ is denoted μ_H. Notice that μ_H is simply the moment map for the action of H on M. The restriction of μ_H to W is denoted $\hat{\mu}$. There is a natural identification of W with $\mathrm{Rep}(\overline{Q}',\mathbf{e})$ such that $\hat{\mu} = \mu_{Q'}$.

Lemma 3.7. *The group H is isomorphic to $G(\mathbf{e})$ and $\theta|_H$ is the trivial character.*

Proof. The isomorphism $H \simeq G(\mathbf{e})$ follows from the fact that $\mathrm{Hom}_{\Pi^\lambda(Q)}(M_1,M_2) = 0$ if M_1 and M_2 are non-isomorphic θ-stable representations and $\mathrm{End}_{\Pi^\lambda(Q)}(M_i,M_i) = \mathbb{C}$. Under this identification,

$$\theta|_{G(\mathbf{e})} = (\theta\cdot\beta^{(1)},\dots,\theta\cdot\beta^{(k)}) = (0,\dots,0) \in \mathbb{Z}^{Q_0'}$$

is the trivial stability condition. $\square$

As in [14], define $\nu : C \to L^*$ by

$$\nu(c)(l) = \omega(c, l\cdot y) + \omega(c, l\cdot c) + \omega(y, l\cdot c).$$

Theorem 3.3 follows from the following more precise result.

Theorem 3.8. *There exists a G-saturated affine open set $V \subset M^\theta$, and H-saturated affine open sets $Z \subset C$ and $U \subset \mathrm{Rep}(\overline{Q}',\mathbf{e})$ such that*

(a) *there are étale Poisson morphisms*

$$\phi : G \times_H Z \to V, \quad \psi : Z \cap \nu^{-1}(0) \to V;$$

(b) *the morphisms ϕ and ψ induce étale Poisson maps*

$$(Z \cap \nu^{-1}(0) \cap \mu_H^{-1}(0))/\!\!/H \to (U \cap \hat{\mu}^{-1}(0))/\!\!/H,$$
$$(\phi^*\mu)^{-1}(\lambda)^\theta /\!\!/G \to (V \cap \mu^{-1}(\lambda))/\!\!/G.$$

(c) *There is an isomorphism of Poisson varieties,*

$$\Phi : (Z \cap \nu^{-1}(0) \cap \mu_H^{-1}(0))/\!\!/H \xrightarrow{\sim} (\phi^*\mu)^{-1}(\lambda)^\theta /\!\!/G.$$

If we assume that $y \in \mu^{-1}(\lambda)$ then for $k \in \mathfrak{h}$ and $l \in L$,

$$\begin{aligned}\mu(y+c)(k+l) &= \lambda(k+l) + \nu(c)(l) + \mu_H(c)(k) + \omega(y, k \cdot c) - \omega(k \cdot y, c) \\ &= \lambda(k+l) + \nu(c)(l) + \mu_H(c)(k)\end{aligned}$$

because $k \cdot y = 0$. We define $\delta : C \to \mathbb{C}$ by $\delta(c) = \gamma(c+y)$. Then δ is H-invariant. We let C_δ denote the non-vanishing locus of δ. Then

$$\{c \in C \mid c + y \in M_\gamma \cap \mu^{-1}(\lambda)\} = C_\delta \cap \mu_H^{-1}(0) \cap \nu^{-1}(0). \tag{4}$$

Let $X = G \times_H C_\delta$. Since $M = C \oplus \mathfrak{g} \cdot y$, the map $\phi : X \to M_\gamma$, $\phi(g, c) = g \cdot (c+y)$ is étale at $(1, 0)$. We recall that a G-morphism $\phi : X \to Y$ is said to be *excellent* if

(a) ϕ is étale.
(b) The induced map $\phi/G : X/\!\!/G \to Y/\!\!/G$ is étale.
(c) The morphism $X \to Y \times_{Y/\!\!/G} X/\!\!/G$ is an isomorphism.

Lemma 3.9. *There exists an affine, H-saturated open neighbourhood Z of 0 in C_δ, such that ϕ restricts to an excellent Poisson morphism*

$$\phi : G \times_H Z \to V := \mathrm{Im}\ \phi \subset M_\gamma,$$

inducing a étale Poisson morphism

$$(\phi^*\mu)^{-1}(\lambda)/\!\!/G \to (\mu^{-1}(\lambda) \cap V)/\!\!/G.$$

Proof. This is a direct consequence of Luna's Fundamental Lemma [41], together with the fact that every G-saturated affine open subset of X is of the form $G \times_H Z$ for some H-saturated open subset of C_δ. Since $\phi : G \times_H Z \to V$ is excellent, the form $\phi^*\omega$ on X is symplectic, with moment map $\phi^*\mu$. In particular, [29, Lemma 3.7] says that the corresponding étale morphism of Hamiltonian reductions $(\phi^*\mu)^{-1}(\lambda)/\!\!/G \to (\mu^{-1}(\lambda) \cap V)/\!\!/G$ is Poisson. □

Proposition 3.10. *There exist H-saturated open subsets Z of $\nu^{-1}(0)$ and U of W such that the morphism*

$$(\mu_H^{-1}(0) \cap Z)/\!\!/H \to (\hat{\mu}^{-1}(0) \cap U)/\!\!/H$$

is Poisson and étale.

Proof. Let $\hat{\omega} = \omega|_W$. As in [14, Lemma 4.3], $\hat{\omega}$ is a H-invariant symplectic form on W, with corresponding moment map $\hat{\mu}$. Write $p : C \to W$ for the projection map along $C^\perp$ and $\overline{p} : \nu^{-1}(0) \to W$ for the restriction of p to $\nu^{-1}(0)$. We claim that $\overline{p}^*\hat{\omega} = \omega|_{\nu^{-1}(0)}$ and $\overline{p}^*\hat{\mu} = \mu_H|_{\nu^{-1}(0)}$. This follows, by definition, from $p^*\hat{\omega} = \omega|_C$ and $p^*\hat{\mu} = \mu_H|_C$. The latter two equalities can be check by a direct computation.

By [14, Lemma 4.5], the map ν is smooth at 0 and $\omega|_{\nu^{-1}(0)}$ is non-degenerate at 0 with moment map $\mu_H|_{\nu^{-1}(0)}$. Moreover, *loc. cit.* shows that the kernel of $d_0\nu$ is W, thus $d_0 p : T_0\nu^{-1}(0) \to T_0 W$ is the identity map. This implies that $\overline{p} : \nu^{-1}(0) \to W$ is étale at 0. Applying Luna's Fundamental Lemma once again, we deduce that there are H-saturated affine open subset $Z \subset \nu^{-1}(0)$ and $U = \overline{p}(Z)$ such that $\overline{p} : Z \to U$ and $\overline{p}/H : Z/\!\!/H \to U/\!\!/H$ are étale. Since $\overline{p}^*\hat{\mu} = \mu_H|_{\nu^{-1}(0)}$, pulling back $\overline{p}/H$ along the closed embedding $\hat{\mu}^{-1}(0)/\!\!/H \to W/\!\!/H$ gives an étale morphism $(Z \cap \mu_H^{-1}(0))/\!\!/H \to (U \cap \hat{\mu}^{-1}(0))/\!\!/H$.

Shrinking Z if necessary, we may assume that $\overline{p}^*\hat{\omega} = \omega|_{\nu^{-1}(0)}$ is non-degenerate on Z. Since $\overline{p}^*\hat{\mu} = \mu_H|_{\nu^{-1}(0)}$, it follows from [29, Lemma 3.7] that the map $(Z \cap \mu_H^{-1}(0))/\!\!/H \to (U \cap \hat{\mu}^{-1}(0))/\!\!/H$ is Poisson. □

The H-equivariant closed embedding $j : \nu^{-1}(0) \cap C_\delta \hookrightarrow G \times_H C_\delta$ given by $j(c) = (1, c)$ induces an isomorphism

$$\Psi : (\mu_H^{-1}(0) \cap \nu^{-1}(0) \cap Z)/\!\!/H \xrightarrow{\sim} (\phi^*\mu)^{-1}(\lambda)/\!\!/G. \tag{5}$$

We will show later that this isomorphism is Poisson. Let $M_{(H)}$ be the set of points m in M^θ such that

(a) $G \cdot m$ is closed in M^θ; and
(b) G_m is conjugate to H.

If V is a G-module, then V_G denotes the complement to V^G.

Lemma 3.11. *The set $M_{(H)}$ is a smooth locally closed subset of M^θ with*

$$T_m M_{(H)} = T_m M^H \oplus (\mathfrak{g}/\mathfrak{h})_H, \quad \forall\, m \in \left(M_{(H)}\right)^H. \tag{6}$$

Proof. To show that $M_{(H)}$ is locally closed, it suffices to prove that, for each $m \in M_{(H)}$, there exists some G-stable affine open neighbourhood U of m in M^θ such that $U \cap M_{(H)}$ is closed in U. By a result of Richardson, [53, Proposition 3.3], the fact that all stabilizers are connected implies that there is a G-stable open set U such that the stabilizer G_u of each $u \in U$ is conjugate to a subgroup of H. In particular, we see that if $n = \dim H$ then $\dim G_u < n$ for all $u \in U \smallsetminus M_{(H)}$. Therefore,

$U \cap M_{(H)} = \{u \in U \mid \dim G_u \geq n\}$. This is closed by [7, Lemma 2.2]. It will follow that $M_{(H)}$ is smooth if we can prove identity (6), since M^H is smooth by [57, Corollary 6.5].

In order to prove identity (6), we apply Luna's slice theorem [41]. There exists an excellent map $\phi : G \times_H S \to U$, where S is a slice to the G-orbit at m. Then, $\phi^{-1}(M_{(H)}) = G \times_H S_{(H)} = G/H \times S^H$. Thus,

$$d_{(1,m)}\phi : T_{(1,m)}\left(G/H \times S^H\right) \xrightarrow{\sim} T_p M_{(H)}$$

has image $T_m S^H \oplus \mathfrak{g} \cdot m$ in $T_m M$, hence $T_m M_{(H)} = T_m S^H \oplus \mathfrak{g} \cdot m$. Since $S^H \subset M^H \subset M_{(H)}$, we have $T_m S^H \subset T_m M^H \subset T_m M_{(H)}$ and hence

$$T_m M^H = (T_m M)^H = (T_p M_{(H)})^H = T_m S^H \oplus (\mathfrak{g} \cdot m)^H.$$

Thus,

$$T_p M_{(H)} = T_m S^H \oplus (\mathfrak{g} \cdot m)^H \oplus (\mathfrak{g} \cdot m)_H = T_m M^H \oplus (\mathfrak{g}/\mathfrak{h})_H$$

as required. □

Lemma 3.12. *The variety $\mu^{-1}(\lambda)^\theta \cap M_{(H)}$ is smooth, with*

$$T_y\left(\mu^{-1}(\lambda)^\theta \cap M_{(H)}\right) = (M^H \cap (\mathfrak{g} \cdot y)^\perp) \oplus (\mathfrak{g} \cdot y)_H,$$

for all $y \in \mu^{-1}(\lambda)^\theta \cap (M_{(H)})^H$.

Proof. Note that every point of $\mu^{-1}(\lambda)^\theta \cap M_{(H)}$ is conjugate by G to some point in $\mu^{-1}(\lambda)^\theta \cap (M_{(H)})^H$. By Lemma 3.11, we have

$$\begin{aligned} T_y\left(\mu^{-1}(\lambda)^\theta \cap M_{(H)}\right) &= T_y M_{(H)} \cap \operatorname{Ker} d_y\mu, \\ &= (M^H \oplus \mathfrak{g} \cdot y) \cap (\mathfrak{g} \cdot y)^\perp \\ &= (M^H \cap (\mathfrak{g} \cdot y)^\perp) \oplus (\mathfrak{g} \cdot y)_H \end{aligned}$$

since $\mathfrak{g} \cdot y \subset (\mathfrak{g} \cdot y)^\perp$ is isotropic. Therefore, we just need to show that the dimension of $\mu^{-1}(\lambda)^\theta \cap M_{(H)}$, as a reduced variety, is also equal to $\dim((M^H \cap (\mathfrak{g} \cdot y)^\perp) \oplus (\mathfrak{g} \cdot y)_H)$. We have

$$(\phi^*\mu)^{-1}(\lambda) \cap (G \times_H C)_{(H)} = G \times_H (\nu^{-1}(0) \cap \mu_H^{-1}(0))_{(H)}.$$

Set-theoretically, this equals $G/H \times \nu^{-1}(0)^H$ (which is smooth) and there is an étale map from this space to $G/H \times W^H$. Thus, we just need to show that

$$\dim((M^H \cap (\mathfrak{g} \cdot y)^\perp) \oplus (\mathfrak{g} \cdot y)_H) = \dim G/H \times \nu^{-1}(0)^H.$$

If $M = C \oplus (\mathfrak{g} \cdot y)$, then $M^H = C^H \oplus (\mathfrak{g} \cdot y)^H$. The fact that $W = C \cap (\mathfrak{g} \cdot y)^\perp$ implies that

$$C^H \cap (\mathfrak{g} \cdot y)^\perp = \left(C \cap (\mathfrak{g} \cdot y)^\perp\right)^H = W^H.$$

Thus,

$$\begin{aligned}\dim((M^H \cap (\mathfrak{g}\cdot y)^\perp) \oplus (\mathfrak{g}\cdot y)_H) &= \dim C^H \cap (\mathfrak{g}\cdot y)^\perp + \dim(\mathfrak{g}\cdot y)^H + \dim(\mathfrak{g}\cdot y)_H \\ &= \dim W^H + \dim(\mathfrak{g}\cdot y) \\ &= \dim G/H \times \nu^{-1}(0)^H\end{aligned}$$

as required. □

Theorem 3.13. *There exists a unique symplectic form ω_H on $\mathfrak{M}_\lambda(\alpha,\theta)_{(H)}$ such that*

$$\pi^*\omega_H = \omega|_{\mu^{-1}(\lambda)^\theta \cap M_{(H)}},$$

where $\pi : \mu^{-1}(\lambda)^\theta \cap M_{(H)} \to \mathfrak{M}_\lambda(\alpha,\theta)_{(H)}$ is the quotient map.

Proof. For brevity, let $Y = \mu^{-1}(\lambda)^\theta \cap M_{(H)} \cap V$, where V is the affine open set of Lemma 3.9, and set $\mathfrak{M} = \mathfrak{M}_\lambda(\alpha,\theta)$. Abusing notation, we will also write $\mathfrak{M} \cap V$ for the affine open subset $(V \cap \mu^{-1}(\lambda))/\!\!/G$ of $\mathfrak{M}$. We claim that we have a commutative diagram of linear maps

$$\begin{array}{ccc} W^H \oplus \mathfrak{g}/\mathfrak{h} & \xrightarrow{\sim} & T_y Y \\ \downarrow & & \downarrow{\scriptstyle d_y\pi} \\ W^H & \xrightarrow{\sim} & T_y\mathfrak{M}_{(H)}, \end{array}$$

where the vertical map on the left is just projection.

Since ϕ is excellent, we have an identification

$$\phi^{-1}(Y) = G/H \times (\nu^{-1}(0)^H \cap U),$$

which means that the diagram

$$\begin{array}{ccc} G/H \times \nu^{-1}(0)^H & \xrightarrow{\phi} & Y \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\pi} \\ \left(G/H \times \nu^{-1}(0)^H\right)/\!\!/G & \xrightarrow{\phi/G} & \mathfrak{M}_{(H)} \end{array}$$

commutes, with ϕ and ϕ/G being étale. Under the identification $T_0\nu^{-1}(0)^H = W^H$, the differential map $d\eta : T_0\nu^{-1}(0)^H \oplus \mathfrak{g}/\mathfrak{h} \to T_0\nu^{-1}(0)^H$ is the projection map $W^H \oplus \mathfrak{g}/\mathfrak{h} \to W^H$, as required.

We deduce that π is a smooth morphism on Y. Hence $\pi^* : \Omega^2_{(\mathfrak{M}\cap V)_{(H)}} \to \Omega^2_Y$ is an embedding, with image $\left(\Omega^2_Y\right)^G$. Thus, there is a unique (closed) 2-form ω_H on $(\mathfrak{M} \cap V)_{(H)}$, whose pull-back along π equals $\omega|_Y$.

Finally, to prove that ω_H is symplectic it suffices to prove that the radical of $\omega|_Y$ at m equals $\mathfrak{g}/\mathfrak{h}$. Clearly the latter is contained in the former. Since $T_mY = W^H \oplus (\mathfrak{g}/\mathfrak{h})$, it suffices to show that $\omega|_{W^H}$ is non-degenerate. Recall that $\hat{\omega} = \omega|_W$ is non-degenerate. Then W^H is a symplectic subspace since $\hat{\omega}$ is H-invariant. □

Next, we show that the symplectic forms ω_H come from the Poisson structure on $\mu^{-1}(\lambda)^\theta/\!/G$.

Lemma 3.14. *For each $f \in \mathbb{C}[V]^G$, the Hamiltonian vector field ζ_f is tangent to $M_{(H)}$.*

Proof. By Lemma 3.11, $M_{(H)}$ is smooth, therefore it suffices to show that $(\zeta_f)_y \in T_y M_{(H)}$ for all $y \in (M_{(H)})^H$. Recall from Lemma 3.11 that $T_y M_{(H)} = M^H \oplus (\mathfrak{g}/\mathfrak{h})_H$. The canonical map $\mathrm{Der}(V) \to T_y M_{(H)}$ is H-equivariant. Since $\{-,-\}$ is G-invariant, and $f \in \mathbb{C}[V]^G$, the Hamiltonian vector field ζ_f belongs to $\mathrm{Der}(V)^G \subset \mathrm{Der}(V)^H$. Hence $(\zeta_f)_y \in (T_y M)^H = M^H \subset T_y M_{(H)}$, as required. □

Theorem 3.15. *The space $\mathfrak{M}_\lambda(\alpha,\theta)_{(H)}$ is a locally closed Poisson subvariety, such that the restriction $\{-,-\}|_{\mathfrak{M}_\lambda(\alpha,\theta)_{(H)}}$ of the Poisson bracket on $\mathfrak{M}_\lambda(\alpha,\theta)$ equals the Poisson structure induced by ω_H. In particular, it is non-degenerate.*

Proof. Again, let $\mathfrak{M} = \mathfrak{M}_\lambda(\alpha,\theta)$. First we show that it is a Poisson subvariety. It suffices to show that each Hamiltonian vector field $\zeta_{\bar{f}}$ on $\mathfrak{M} \cap V$ is tangent to $(\mathfrak{M} \cap V)_{(H)}$. Let $f \in \mathbb{C}[V]^G$ be a lift of $\bar{f}$. Then, by Lemma 3.14, ζ_f is tangent to $M_{(H)} \cap \mu^{-1}(\lambda)^\theta$. By definition of Hamiltonian reduction, ζ_f is also tangent to $V \cap \mu^{-1}(\lambda)^\theta$. Therefore, it descends to the vector field $\zeta_{\bar{f}}$ on $\mathfrak{M}$, which is tangent to $(M_{(H)} \cap V \cap \mu^{-1}(\lambda)^\theta)/\!\!/G$. But, by Theorem 3.13,

$$(M_{(H)} \cap V \cap \mu^{-1}(\lambda)^\theta)/\!\!/G = (\mathfrak{M} \cap V)_{(H)},$$

as required.

Next, we show that the two Poisson structures agree. Once again, we let $Y = V \cap M_{(H)} \cap \mu^{-1}(\lambda)^\theta$, and let $\pi : Y \to (V \cap \mathfrak{M})_{(H)}$ be the quotient map.

Choose a function $\bar{f}$ defined on $(V \cap \mathfrak{M})_{(H)}$ and denote by the same symbol an arbitrary lift to $V \cap \mathfrak{M}$. Since the form ω_H is non-degenerate on $(V \cap \mathfrak{M})_{(H)}$ there exists a Hamiltonian vector field $\zeta'_{\bar{f}}$ on $(V \cap \mathfrak{M})_{(H)}$ satisfying the defining equation $\omega_H(\zeta'_{\bar{f}}, \eta) = -\eta(\bar{f})$ for all vector fields η. The non-degeneracy of ω_H implies that it suffices to prove that $\omega_H(\zeta_{\bar{f}}, \eta) = \omega_H(\zeta'_{\bar{f}}, \eta)$ for all η, since $\zeta_{\bar{f}} = \zeta'_{\bar{f}}$ implies that $\{\bar{f}, g\} = \{\bar{f}, g\}'$ for all functions g on $(V \cap \mathfrak{M})_{(H)}$. Thus, we must show that $\omega_H(\zeta_{\bar{f}}, \eta) = -\eta(\bar{f})$.

Since the quotient map $\pi : Y \to (V \cap \mathfrak{M})_{(H)}$ is smooth, we can choose a lift of η. In fact, if we ask that the lift be G-invariant, it is unique, and so we will denote it by η too. If f is a lift of $\bar{f}$ to $\mathbb{C}[V]^G$, then ζ_f is tangent to Y, and $\zeta_f|_Y$ is a lift of $\zeta_{\bar{f}}$. Therefore,

$$\omega_H(\zeta_{\bar{f}}, \eta) = \pi^* \omega_H(\zeta_f|_Y, \eta) = \omega|_Y(\zeta_f|_Y, \eta).$$

Finally, if we choose an arbitrary lift η' of η to V, then

$$\omega|_Y(\zeta_f|_Y, \eta) = \omega(\zeta_f, \eta')|_Y = -\eta'(f)|_Y = -\eta(f|_Y) = -\eta(\bar{f}).$$

□

Finally, we complete the proof of Theorem 3.8.

Proof of Theorem 3.8. All claims, except for the final one, follow from Lemma 3.9 and Proposition 3.10. Thus, it suffices to note that the isomorphism Ψ of (5) is Poisson. Choose a generic point n in

$$(Z \cap \nu^{-1}(0) \cap \mu_H^{-1}(0))/\!\!/H \simeq (\phi^*\mu)^{-1}(\lambda)^\theta /\!\!/G.$$

Then there exits some $K \subset H$ such that $n \in ((Z \cap \nu^{-1}(0) \cap \mu_H^{-1}(0))/\!\!/H)_{(K)}$. Both Poisson structures on this open stratum are non-degenerate. Therefore, it suffices to show that the corresponding symplectic 2-forms agree via Ψ. Recall that the symplectic form on $((Z \cap \nu^{-1}(0) \cap \mu_H^{-1}(0))/\!\!/H)_{(K)}$ is the unique form such that its pull-back to $Z_{(K)} \cap \nu^{-1}(0) \cap \mu_H^{-1}(0)$ agrees with $\omega|_{Z_{(K)} \cap \nu^{-1}(0) \cap \mu_H^{-1}(0)}$. Similarly, the symplectic form on $((\phi^*\mu)^{-1}(\lambda)^\theta /\!\!/G)_{(K)}$ is the unique symplectic form whose pull-back to $D := (G \times_H Z_{(K)}) \cap (\phi^*\mu)^{-1}(\lambda)^\theta$ equals $(\phi^*\omega)|_D$. Therefore, since the map Ψ is induced by the closed embedding j, it suffices to show that

$$j^*((\phi^*\omega)|_D) = \omega|_{Z_{(K)} \cap \nu^{-1}(0) \cap \mu_H^{-1}(0)}.$$

But this follows from the fact that

$$D = \phi^{-1}(V_{(K)} \cap \mu^{-1}(\lambda)^\theta), \quad j^{-1}(D) = Z_{(K)} \cap \nu^{-1}(0) \cap \mu_H^{-1}(0),$$

and $\phi \circ j$ is the map $c \mapsto c + m$, so that $j^*\phi^*\omega = \omega|_{\nu^{-1}(0) \cap C_\delta}$, since ω is invariant under translation. □

3.4. **Hyperkähler twisting.** Let $\alpha = m_1\nu^{(1)} + \cdots + m_t\nu^{(t)}$ be the canonical decomposition of α with respect to Σ_λ. It is shown in [13] that

Theorem 3.16. [13] *There is an isomorphism of varieties* $\prod_i S^{m_i}\left(\mathfrak{M}_\lambda(\nu^{(i)}, 0)\right) \simeq \mathfrak{M}_\lambda(\alpha, 0)$.

Moreover, if $\nu^{(i)}$ is real then $S^{m_i}\left(\mathfrak{M}_{\nu^{(i)}}(\lambda, 0)\right) = \{\mathrm{pt}\}$ and if $\nu^{(i)}$ is anisotropic then $m_i = 1$. We now adapt Crawley-Boevey's result to the case where $\theta \neq 0$:

Theorem 3.17. *Let $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ be the canonical decomposition of α with respect to $\Sigma_{\lambda,\theta}$. Then, there is an isomorphism of* ***Poisson*** *varieties*

$$\phi : \prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)\right) \xrightarrow{\sim} \mathfrak{M}_\lambda(\alpha, \theta).$$

The proof of Theorem 3.17 is given at the end of section 3.5. In order to deduce Theorem 3.17 from [13, Theorem 1.1], we use hyperkähler twists. By our main assumption (1), $\lambda \in \mathbb{R}^{Q_0}$.

Proposition 3.18. *Let $\nu = -\lambda - \mathbf{i}\theta$ and consider $\mathfrak{M}_\lambda(\alpha, \theta)$, $\mathfrak{M}_\nu(\alpha, 0)$ as complex analytic spaces. Hyperkähler twisting defines a homeomorphism of stratified spaces*

$$\Psi : \mathfrak{M}_\lambda(\alpha, \theta) \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0),$$

i.e. Ψ restricts to a homeomorphism $\mathfrak{M}_\lambda(\alpha, \theta)_{(H)} \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0)_{(H)}$ for all classes (H). In particular, the homeomorphism maps stable representations to stable (= simple) representations.

Proof. We follow the setup described in the proof of [11, Lemma 3]. We have moment maps

$$\mu_{\mathbb{C}}(x) = \sum_{a \in Q_1} [x_a, x_{a^*}], \quad \mu_{\mathbb{R}}(x) = \frac{\sqrt{-1}}{2} \sum_{a \in Q_1} [x_a, x_a^\dagger] + [x_{a^*}, x_{a^*}^\dagger].$$

As shown in [35, Corollary 6.2], the Kempf-Ness Theorem says that the embedding $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\lambda)$ induces a bijection

$$\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha) \xrightarrow{\sim} \mathfrak{M}_\lambda(\alpha, \theta). \tag{7}$$

Since the embedding is clearly continuous and the topology on the quotients $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(i\theta)/U(\alpha)$ and $\mathfrak{M}_\lambda(\alpha, \theta)$ is the quotient topology (for the latter space, see [52, Corollary 1.6 and Remark 1.7]), the bijection (7) is continuous.

Define a stratification $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha)$ analogous to the stratification of $\mathfrak{M}_\lambda(\alpha, \theta)$ described in section 3.1. Let $y \in \mathfrak{M}_\lambda(\alpha, \theta)$, and $x = x_1^{e_1} \oplus \cdots \oplus x_k^{e_k} \in T(y)$ a θ-polystable lift in $\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(i\theta)$ (which exists by Proposition 3.1). Then Lemma 3.7 says that $G_x = G(\mathbf{e})$ and [35, Proposition 6.5] implies that $U(\alpha)_x = U(\mathbf{e})$. Hence $G(\alpha)_x = U(\alpha)_x^{\mathbb{C}}$. Therefore the homeomorphism (7) restricts to a bijection

$$(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)/U(\alpha))_{(K)} \to \mathfrak{M}_\lambda(\alpha, \theta)_{(K^{\mathbb{C}})}$$

for each (K).

Let the quaternions $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ act on $\mathrm{Rep}(\overline{Q}, \alpha)$ by extending the usual complex structure so that $\mathbf{j} \cdot (x_a, x_{a^*}) = (-x_{a^*}^\dagger, x_a^\dagger)$. In general,

$$(z_1 + z_2\mathbf{j}) \cdot (x_a, x_{a^*}) = (z_1 x_a - z_2 x_{a^*}^\dagger, z_1 x_{a^*} + z_2 x_a^\dagger).$$

This action commutes with the action of $U(\alpha)$ and satisfies

$$\mu_{\mathbb{R}}(z \cdot x) = (||z_1||^2 - ||z_2||^2)\mu_{\mathbb{R}}(x) - \mathbf{i} z_1 \overline{z}_2 \mu_{\mathbb{C}}(x) - \mathbf{i} z_2 \overline{z}_1 \mu_{\mathbb{C}}(x)^\dagger, \tag{8}$$

$$\mu_{\mathbb{C}}(z \cdot x) = z_1^2 \mu_{\mathbb{C}}(x) - z_2^2 \mu_{\mathbb{C}}(x)^\dagger - 2\mathbf{i} z_1 z_2 \mu_{\mathbb{R}}(x), \quad \forall\ z \in \mathbb{H}. \tag{9}$$

Let $h = (\mathbf{i} - \mathbf{j})/\sqrt{2}$. Then multiplication by h defines a homeomorphism

$$\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta) \xrightarrow{\sim} \mu_{\mathbb{C}}^{-1}(-\lambda - \mathbf{i}\theta) \cap \mu_{\mathbb{R}}^{-1}(0)$$

Since multiplication by h commutes with the action of $U(\alpha)$, this homeomorphism descends to a homeomorphism

$$\left(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(\mathbf{i}\theta)\right)/U(\alpha) \xrightarrow{\sim} \left(\mu_{\mathbb{C}}^{-1}(-\lambda - \mathbf{i}\theta) \cap \mu_{\mathbb{R}}^{-1}(0)\right)/U(\alpha)$$

which preserves the stratification by stabilizer type.

Thus, the map Ψ is the composition of three homeomorphisms, each of which preserves the stratification. □

Remark 3.19. Our general assumption that $\lambda \in \mathbb{R}^{Q_0}$ if $\theta \neq 0$ is required in the proof of Proposition 3.18 to ensure that multiplication by h lands in $\mu_{\mathbb{R}}^{-1}(0)$. Equation (8) implies that it would suffice to assume more generally that there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $z\lambda \in \mathbb{R}^{Q_0}$. It is natural to expect that Theorem 3.17 holds with out the assumption $\lambda \in \mathbb{R}^{Q_0}$.

Remark 3.20. Using the notion of smooth structures on stratified symplectic spaces, as defined in [54], one can presumably strengthen Proposition 3.18 to the statement that there is a diffeomorphism of stratified symplectic spaces $\mathfrak{M}_\lambda(\alpha, \theta) \xrightarrow{\sim} \mathfrak{M}_\nu(\alpha, 0)$.

Proposition 3.21. *The variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible and normal.*

Proof. We begin by showing that the variety $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected. Proposition 3.18 implies that $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected if and only if $\mathfrak{M}_\nu(\alpha, 0)$ is connected. The latter is known to be connected by [13, Corollary 1.4].

Next, we show that $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible. Since $\mathfrak{M}_\lambda(\alpha, \theta)$ is connected, it suffices to show that, for each $\mathbb{C}$-point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$, the local ring $\mathcal{O}_{\mathfrak{M}_\lambda(\alpha,\theta),x}$ is a domain. This ring embeds into the formal neighborhood of x in $\mathfrak{M}_\lambda(\alpha, \theta)$. By Corollary 3.4, the formal neighborhood of x in $\mathfrak{M}_\lambda(\alpha, \theta)$ is isomorphic to the formal neighborhood of 0 in $\mathfrak{M}_0(e, 0)$. By [13, Corollary 1.4], this is a domain. Finally, normality is an etalé local property, [44, Remark 2.24 and Proposition 3.17]. Therefore, as in the previous paragraph this follows from Theorem 3.3 and [14, Theorem 1.1]. ∎

3.5. **The proof of Theorem 3.17.** Recall that $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$ is the canonical decomposition of α in $R^+_{\lambda,\theta}$. The map ϕ is defined as follows. Let $H(\alpha)$ be the product $G(\sigma^{(1)})^{n_1} \times \cdots \times G(\sigma^{(k)})^{n_k}$, thought of as a subgroup of $G(\alpha)$. There is a natural $H(\alpha)$-equivariant inclusion $\prod_i T^* \operatorname{Rep}(Q, \sigma^{(i)})^{n_i} \hookrightarrow T^* \operatorname{Rep}(Q, \alpha)$. This is an inclusion of symplectic vector spaces. Since the moment map for the action of $H(\alpha)$ on $T^* \operatorname{Rep}(Q, \alpha)$ is the composition of the moment map for $G(\alpha)$ followed by projection from the Lie algebra of $G(\alpha)$ to the Lie algebra of $H(\alpha)$, the above inclusion restricts to an inclusion $\prod_i (\mu_{\sigma^{(i)}}^{-1}(\lambda)^\theta)^{n_i} \hookrightarrow \mu_\alpha^{-1}(\lambda)^\theta$, inducing a map of GIT quotients

$$\prod_i \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)^{n_i} \to \mathfrak{M}_\lambda(\alpha, \theta).$$

This map, which sends a tuple of representations $(M_{i,j})$ to the direct sum $\bigoplus_{i,j} M_{i,j}$ clearly factors through $\prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)\right)$. It is this map that we call ϕ.

Passing to the analytic topology, Proposition 3.18 implies that we get a commutative diagram

$$\begin{array}{ccc} \prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)\right) & \longrightarrow & \mathfrak{M}_\lambda(\alpha, \theta) \\ \downarrow & & \downarrow \\ \prod_i S^{n_i}\left(\mathfrak{M}_\lambda(-\sigma^{(i)} - \mathbf{i}\theta, 0)\right) & \longrightarrow & \mathfrak{M}_{-\lambda-\mathbf{i}\theta}(\alpha, 0). \end{array} \tag{10}$$

where both vertical arrows are homeomorphisms and the bottom horizontal arrow is an isomorphism by Theorem 3.16. Therefore, we conclude that ϕ is bijective. Since we are working over the complex numbers, and we have shown in Proposition 3.21 that $\mathfrak{M}_\lambda(\alpha,\theta)$ is normal, we conclude by Zariski's main theorem that ϕ is an isomorphism.

As a consequence, we can compute the dimension of $\mathfrak{M}_\lambda(\alpha,\theta)$, which in the case $\theta \neq 0$ is [12, Corollary 1.4]. We begin with the following basic lemma:

Lemma 3.22. *If $\alpha \in \Sigma_{\lambda,\theta}$, then $\dim \mathfrak{M}_\lambda(\alpha,\theta) = 2p(\alpha)$. Moreover $\dim \mu^{-1}(\lambda)^\theta \geq \alpha\cdot\alpha + 2p(\alpha) - 1$.*

Proof. We note that Proposition 3.18, together with the results of [12], imply that there exists a θ-stable representation of $\Pi^\lambda(Q)$ of dimension α if and only if $\alpha \in \Sigma_{\lambda,\theta}$. Let U be the subset of $\mathfrak{M}_\lambda(\alpha,\theta)$ consisting of θ-stable representations. Since α is assumed to be in $\Sigma_{\lambda,\theta}$, Proposition 3.21 implies that U is a dense open subset of $\mathfrak{M}_\lambda(\alpha,\theta)$. Let V be the open subset of $\mathrm{Rep}(\overline{Q},\alpha)$ consisting of θ-stable representations. Then U is the image of $\mu^{-1}(\lambda) \cap V$ under the quotient map and hence V is non-empty. The group $PG(\alpha)$ acts freely on V and μ is smooth when restricted to V. Thus,

$$\dim U = \dim \mathrm{Rep}(\overline{Q},\alpha) - 2(\dim G(\alpha) - 1) = 2p(\alpha),$$

as required. For the second statement, observe that $\dim(V \cap \mu^{-1}(\lambda)) = \dim U + \dim PG(\alpha)$ since $PG(\alpha)$ acts freely on V. □

Then we immediately conclude

Corollary 3.23. *For $\alpha \in R^+_{\lambda,\theta}$ with canonical decomposition $\alpha = n_1\sigma^{(1)} + \cdots + n_k\sigma^{(k)}$, the variety $\mathfrak{M}_\lambda(\alpha,\theta)$ has dimension $2\sum_{i=1}^k n_i p(\sigma^{(i)})$.*

Finally, we need to check that the morphism ϕ is Poisson. Since both varieties are normal by Proposition 3.21, it suffices to show that ϕ induces an isomorphism of smooth symplectic varieties between the open leaf of $\mathfrak{M}_\lambda(\alpha,\theta)$ and the open leaf of $\prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)},\theta)\right)$. By Proposition 3.6, the symplectic leaves of $\mathfrak{M}_\lambda(\alpha,\theta)$ are connected components of the strata given by stabilizer type. The explicit description of ϕ given at the start of this section shows that ϕ restricts to an isomorphism between strata. In particular, ϕ restricts to an isomorphism between the open leaves.

The symplectic structure on the open leaf of $\mathfrak{M}_\lambda(\alpha,\theta)$ comes from the symplectic structure on $T^*\mathrm{Rep}(Q,\alpha)$. More specifically, the non-degenerate closed form on the latter space restricts to a degenerate $G(\alpha)$-equivariant two-form on $\mu^{-1}(\lambda)^\theta$. Hence it descends to a closed two-form on $\mathfrak{M}_\lambda(\alpha,\theta)$. The restriction of this two-form to the open leaf is non-degenerate. The two-form on the open leaf of $\prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)},\theta)\right)$ is defined similarly. Now the point is that under the embedding $\prod_i(\mu^{-1}_{\sigma^{(i)}}(\lambda)^\theta)^{n_i} \hookrightarrow \mu^{-1}_\alpha(\lambda)^\theta$, the $H(\alpha)$-equivariant closed two-form on $\prod_i(\mu^{-1}_{\sigma^{(i)}}(\lambda)^\theta)^{n_i}$ is simply the pull-back of the $G(\alpha)$-equivariant closed two-form on $\mu^{-1}_\alpha(\lambda)^\theta$. This implies that the two-form on the open leaf of $\prod_i S^{n_i}\left(\mathfrak{M}_\lambda(\sigma^{(i)},\theta)\right)$ is the pull-back, under ϕ, of the symplectic two-form on the open leaf of $\mathfrak{M}_\lambda(\alpha,\theta)$.

Using Proposition 3.21, we can now show that each stratum $\mathfrak{M}_\lambda(\alpha,\theta)_\tau$ is connected.

Corollary 3.24. *Each stratum $\mathfrak{M}_\lambda(\alpha,\theta)_\tau$ of $\mathfrak{M}_\lambda(\alpha,\theta)$ is connected.*

Proof. Writing $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$, we can repeat the construction of ϕ given above (even though τ is not the canonical decomposition of α) to get a morphism

$$\phi : \prod_i S^{e_i} \mathfrak{M}_\lambda(\beta^{(i)}, \theta) \to \mathfrak{M}_\lambda(\alpha, \theta).$$

The stratum $\mathfrak{M}_\lambda(\alpha,\theta)_\tau$ is contained in the image of ϕ and $\phi^{-1}(\mathfrak{M}_\lambda(\alpha,\theta)_\tau)$ is dense in the domain of ϕ. Since the domain is irreducible by Proposition 3.21, we deduce that $\mathfrak{M}_\lambda(\alpha,\theta)_\tau$ is irreducible. □

3.6. **Flatness of the moment map.** We need an additional result which follows from [14]. Let $\xi : \mu^{-1}(\lambda)^\theta \to \mathfrak{M}_\lambda(\alpha,\theta)$ be the quotient map.

Theorem 3.25. [14, Corollary 6.4] *For $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$ a representation type,*

$$\dim \xi^{-1}(\mathfrak{M}_\lambda(\alpha,\theta)_\tau) \le \alpha \cdot \alpha - 1 + p(\alpha) + \sum_{i=1}^{k} p(\beta^{(i)}). \tag{11}$$

Proof. The proof follows verbatim as in [14, Corollary 6.4], substituting θ-stable representations for simple representations. Alternatively, [14, Corollary 6.4] as written together with Theorem 3.3 yields the result. □

Proposition 3.26. *The moment map $\mu : \mathrm{Rep}(\overline{Q}, \alpha)^\theta \to \mathbb{R}^{Q_0}$ is flat over the open subset $B_{\alpha,\theta} := \{\lambda \in \mathbb{R}^{Q_0} \mid \alpha \in \Sigma_{\lambda,\theta}\} \subseteq \mathbb{R}^{Q_0}$, with all fibers of dimension $\alpha\cdot\alpha + p(\alpha) - 1$. In particular, if $\alpha \in \Sigma_{\lambda,\theta}$, then the variety $\mu^{-1}(\lambda)^\theta$ is a complete intersection in the open subset $\mu^{-1}(B_{\alpha,\theta})^\theta$.*

Proof. By Lemma 3.22 and Theorem 3.25, all of the fibers $\mu^{-1}(\lambda)$ for $\lambda \in B_{\alpha,\theta}$ have the same dimension, $\alpha\cdot\alpha + p(\alpha) - 1$. Then, since $B_{\alpha,\theta}$ is smooth, and $\mu^{-1}(B_{\alpha,\theta})$ is open (hence smooth and therefore Cohen-Macaulay), it follows that the moment map is flat as stated, and therefore that every fiber is a complete intersection. □

4. Smooth vs. stable points

As usual, choose a deformation parameter $\lambda \in \mathbb{R}^{Q_0}$, a stability parameter $\theta \in \mathbb{Z}^{Q_0}$, and a dimension vector $\alpha \in \mathbb{N}R^+_{\lambda,\theta}$. The main goal of this section is to prove Theorem 1.13, which says that $x \in \mathfrak{M}_\lambda(\alpha,\theta)$ is θ-canonically polystable if and only if it is in the smooth locus of $\mathfrak{M}_\lambda(\alpha,\theta)$.

4.1. **Isotropic roots.** In this section, we briefly consider quiver varieties associated to isotropic roots. The subgroup of $\mathrm{GL}(\mathbb{Z}^{Q_0})$ generated by the reflection at loop free vertices is denoted $W(Q)$.

Lemma 4.1. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be an isotropic root. Then there exists $w \in W(Q)$ such that $\delta = w\alpha$ is in the fundamental domain, $Q' = \mathrm{Supp}\ \delta$ is an affine Dynkin quiver, $\delta|_{Q'}$ is the minimal imaginary root and $\mathfrak{M}_\lambda(\alpha,\theta) \simeq \mathfrak{M}_{w\lambda}(\delta, w\theta)$.*

Proof. As the name implies, the fundamental domain $\mathcal{F}(Q)$ is a fundamental domain for the action of the reflection group $W(Q)$ of Q on the set of imaginary roots. Therefore there exists w such that $w\alpha \in \mathcal{F}(Q)$. The fact that Q' is affine Dynkin and $\delta|_{Q'}$ is the minimal imaginary root follows from [30, Lemma 1.9 (d)].

Thus, we show that $\delta \in \Sigma_{w\lambda,w\theta}$ and $\mathfrak{M}_\lambda(\alpha,\theta) \simeq \mathfrak{M}_{w\lambda}(\delta,w\theta)$. The Lusztig-Maffei-Nakajima reflection isomorphisms of quiver varieties (see in particular [42, Theorem 26]) shows that if either λ_i or θ_i is non-zero (equivalently, as explained in example 1, if $e_i \notin \Sigma_{\lambda,\theta}$) then $\mathfrak{M}_\lambda(\alpha,\theta) \simeq \mathfrak{M}_{s_i\lambda}(s_i\alpha, s_i\theta)$. It is easily checked that if $e_i \in \Sigma_{\lambda,\theta}$ then $(\alpha, e_i) \le 0$ (otherwise $\alpha = (\alpha - e_i) + e_i$ with $p(\alpha) = p(\alpha - e_i)$). Moreover, the fact that s_i permutes the set $R^+ \smallsetminus \{e_i\}$ implies that $s_i\alpha \in \Sigma_{s_i\lambda,s_i\theta}$ if $e_i \notin \Sigma_{\lambda,\theta}$. Hence, we need to show that $w = s_{i_r} \cdots s_{i_1}$ can be chosen so that $(s_{i_l} \cdots s_{i_1}\lambda)_{i_{l+1}} \neq 0$ or $(s_{i_l} \cdots s_{i_1}\theta)_{i_{l+1}} \neq 0$ for all $l = 1, \dots, r-1$. Recall that every positive root $\beta = \sum_{i\in Q_0} k_i e_i$ has height $\mathrm{ht}(\beta) := \sum_{i\in Q_0} k_i \ge 1$. As in the proof of [10, Proposition 16.10], the key thing to note is that δ is specified by the fact that it is the unique element of minimal height in the orbit $W(Q)\cdot\alpha$. Thus, if $\alpha \notin \mathcal{F}(Q)$, then there exists $i \in Q_0$ such that $(\alpha, e_i) > 0$. This implies that $e_i \notin \Sigma_{\lambda,\theta}$ and $\mathrm{ht}(s_i\alpha) < \mathrm{ht}(\alpha)$. Since every element in the orbit $W(Q)\cdot\alpha$ is a positive root (and hence has positive height) this cannot continue forever, and the result follows. □

In particular, we note that Lemma 4.1 implies that if $\alpha \in \Sigma_{\lambda,\theta}$ is an isotropic root, then $\mathfrak{M}_\lambda(\alpha,\theta)$ is the partial resolution of a partial deformation of a Kleinian singularity. Moreover, the type of the Kleinian singularity is specified by the support of $w\alpha \in \mathcal{F}$.

4.2. **The proof of Theorem 1.13.** The proof of Theorem 1.13 follows closely the arguments given in [37, Theorem 3.2]. We provide the necessary details that show that the arguments of *loc. cit.* are valid in our setting. First, notice that, under the isomorphism of Theorem 3.17, the open subset of θ-canonically polystable points in $\mathfrak{M}_\lambda(\alpha,\theta)$ is the product of the θ-canonically polystable points in the spaces $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$. Therefore it suffices to show that the set of θ-canonically polystable points in $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ is precisely the smooth locus. If $\sigma^{(i)}$ is real then $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ is a point. If $\sigma^{(i)}$ is an isotropic root then by Lemma 4.1, $\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ is a partial resolution of a du Val singularity. In particular, it is a 2-dimensional (quasi-projective) variety. This implies that the smooth locus of $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ equals

$$S^{n_i,\circ}\,\mathfrak{M}_\lambda(\sigma^{(i)},\theta)_{\mathrm{sm}} := \left\{ \sum_{j=1}^{n_i} p_j \;\middle|\; p_j \in \mathfrak{M}_\lambda(\sigma^{(i)},\theta)_{\mathrm{sm}},\ p_j \neq p_k \text{ for } j \neq k \right\}.$$

On the other hand, the set of θ-canonically polystable points in $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ equals $S^{n_i,\circ}U$, where $U \subset \mathfrak{M}_\lambda(\sigma^{(i)},\theta)$ is the set of θ-canonically polystable points. Therefore, in this case it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)},\theta)_{\mathrm{sm}}$ equals U. Finally, in the case where $\sigma^{(i)}$ is an anisotropic root, we have $n_i = 1$.

Thus, we are reduced to considering the situation where $\alpha \in \Sigma_{\lambda,\theta}$ is an imaginary root. In this case, a point x is θ-canonically polystable if and only if it is θ-stable. As in the proof of Corollary 3.23, it is clear from the definition of $\mathfrak{M}_\lambda(\alpha,\theta)$ that the set of θ-stable points is contained in the smooth locus. Therefore it suffices to show that if x is not θ-stable then it is a singular point. As in section 3.2, let x be the image of a θ-polystable representation $y = y_1^{e_1} \oplus \cdots \oplus y_\ell^{e_\ell}$ (with the y_i θ-stable). Let $\beta^{(i)} = \dim y_i$. Let Q' be the quiver with ℓ vertices whose double has $2p(\beta^{(i)})$ loops at vertex i and $-(\beta^{(i)}, \beta^{(j)})$ arrows between vertex i and j. The ℓ-tuple $\mathbf{e} = (e_1, \ldots, e_\ell)$ defines a dimension vector for the quiver Q'. By Theorem 3.3, it suffices to show that 0 is contained in the singular locus of $\mathfrak{M}_0(\mathbf{e}, 0)$.

In order to proceed, we require [36, Proposition 1.1], stated in our generality. The proof is identical to the proof given in *loc. cit.*, this time using Theorem 3.3.

Proposition 4.2. *Assume that $\alpha \in \Sigma_{\lambda,\theta}$ and let x be a geometric point of $\mathfrak{M}_\lambda(\alpha, \theta)$, of representation type $\tau = (e_1, \beta_1; \ldots; e_k, \beta_k)$. Then $\mathbf{e}$ is the dimension vector of a simple $\Pi^0(Q')$-module i.e. $\mathbf{e} \in \Sigma_0(Q')$.*

Returning to the proof of Theorem 1.13, with Proposition 4.2 in hand, the argument given in the proof of [37, Theorem 3.2] goes through *verbatim.* This completes the proof of Theorem 1.13.

4.3. **The proof of Corollary 1.15.** By Theorem 1.13, $\mathfrak{M}_\lambda(\alpha, \theta)$ is smooth if and only if every point is θ-canonically polystable. As in the reduction argument given at the start of the proof of Theorem 1.13, this means that n_i must be 1 when $\sigma^{(i)}$ is an isotropic root. Moreover, it is clear that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ consists only of θ-stable points if and only if $\sigma^{(i)}$ is minimal.

5. The (2, 2) case

In this section we will prove Theorem 1.7. First we restrict to the one vertex case.

5.1. **The variety $\mathfrak{X}(2,2)$.** Recall that $\mathfrak{X}(g, n)$ denotes the quiver variety

$$\left\{ (X_1, Y_1, \ldots, X_g, Y_g) \in \operatorname{End}_{\mathbb{C}}(\mathbb{C}^n) \;\middle|\; \sum_{i=1}^{d} [X_i, Y_i] = 0 \right\} /\!\!/ \mathrm{GL}(n, \mathbb{C}).$$

We note that $\mathfrak{X}(g, n)$ is an irreducible, normal affine variety of dimension $2(n^2(g-1)+1)$.

Set $(g, n) = (2, 2)$, so $\dim \mathfrak{X}(g, n) = 10$. We recall results of Kaledin-Lehn [32], see also [38], which explain that $\mathfrak{X}(2,2)$ admits a projective symplectic resolution.

Let $W = \mathfrak{sl}_2$ and (V, ω) a 4-dimensional symplectic vector space. Let κ denote the Killing form on W. Then $\kappa \otimes \omega$ is a symplectic form on $W \otimes V$. We identify $\mathfrak{sp}(V)^*$ with $\mathfrak{sp}(V)$ via its Killing form. There is an action of PGL(2) on W by conjugation and hence on $W \otimes V$. This action is Hamiltonian and commutes with the natural action of $\mathrm{Sp}(V)$ on $W \otimes V$. The moment map for the action of PGL(2) is given by

$$\mu\left(\sum_i A_i \otimes v_i\right) = \sum_{i,j} A_i A_j \omega(v_i, v_j)$$
$$= \sum_{i<j} [A_i, A_j] \omega(v_i, v_j).$$

The moment map for the action of $\mathrm{Sp}(V)$ is given by $\sum_i A_i \otimes v_i \mapsto \nu(\sum_i A_i \otimes v_i)$, where

$$\nu\left(\sum_i A_i \otimes v_i\right)(u) = \sum_{i,j} \kappa(A_i, A_j)\omega(v_i, u)v_j.$$

Since the actions of $\mathrm{PGL}(2)$ and $\mathrm{Sp}(V)$ on $\mu^{-1}(0)$ commute, the map ν descends to a map $\mu^{-1}(0)/\!\!/\mathrm{PGL}(2) \to \mathfrak{sp}(V)$, which we also denote by ν. Let $\mathcal{N}_2^2 \subset \mathfrak{sp}(V)$ be the set $\{B \mid B^2 = 0,\ \mathrm{rk}B = 2\}$. The set $\mathcal{N}_2^2$ is a 6-dimensional adjoint $\mathrm{Sp}(V)$-orbit. Its closure $\mathcal{N} := \overline{\mathcal{N}}_2^2 = \mathcal{N}_2^2 \cup \mathcal{N}_1^2 \cup \{0\}$ consists of three $\mathrm{Sp}(V)$-orbits and one can check that $\overline{\mathcal{N}}_1^2 \simeq \mathbb{C}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{C}^4$ with weights $(-1,-1,-1,-1)$. The following result is proven in [32].

Theorem 5.1. [32] *The map ν defines an isomorphism $\mu^{-1}(0)/\!\!/\mathrm{PGL}(2) \xrightarrow{\sim} \mathcal{N}$ of Poisson varieties. In particular, $\mu^{-1}(0)/\!\!/\mathrm{PGL}(2)$ is a symplectic singularity.*

Taking trace of the matrices $(X_1, X_2, Y_1, Y_2) \in \mathfrak{X}(2,2)$ defines an isomorphism of symplectic singularities $\mathfrak{X}(2,2) \simeq \mu^{-1}(0)/\!\!/\mathrm{PGL}(2) \times \mathbb{C}^4$, where $\mathbb{C}^4$ is given the usual symplectic structure. Thus, $\mathfrak{X}(2,2) \simeq \mathcal{N} \times \mathbb{C}^4$.

5.2. **Proof of Theorem 1.7.** We now prove Theorem 1.7, following the arguments of [32, Remark 4.6]; see also [38].

Since α is anisotropic, 2α is also an anisotropic root. Choose a generic stability parameter $\theta' \geq \theta$ with $\theta' \cdot \beta \neq 0$ for all nonzero $\beta \leq 2\alpha$, $\beta \neq \alpha$. Then the projective Poisson morphism $\mathfrak{M}_\lambda(2\alpha, \theta') \to \mathfrak{M}_\lambda(2\alpha, \theta)$ of Lemma 2.3 is a partial projective resolution. The proof of Lemma 6.12 below shows that if $Y \to \mathfrak{M}_\lambda(2\alpha, \theta')$ is a projective symplectic resolution then so is the composite $Y \to \mathfrak{M}_\lambda(2\alpha, \theta)$ i.e. it is enough to show that we can resolve $\mathfrak{M}_\lambda(2\alpha, \theta')$ symplectically. Fix $X = \mathfrak{M}_\lambda(2\alpha, \theta')$. Then $X = X_2 \sqcup X_1 \sqcup X_0$, where, by Theorem 1.13, X_0 is the smooth locus consisting of θ'-stable representations, X_1 parameterizes representations $M = M_1 \oplus M_2$ with $\dim M_1 = \dim M_2 = \alpha$, $M_1 \not\simeq M_2$ are θ'-stable representations and X_2 consists of all points M^2, with $\dim M = \alpha$. By Proposition 3.6, X_2 and $X_2 \cup X_1$ are closed in X.

Let $\widetilde{X}$ denote the blowup of X the along the sheaf of ideals of the reduced singular locus $X_1 \sqcup X_0$. The corollary will follow from the following claim: $\widetilde{X} \to X$ is a projective symplectic resolution.

Clearly, $\widetilde{X} \to X$ is a projective birational morphism, therefore we just need to show that $\widetilde{X}$ is smooth and the symplectic 2-form on X_0 extends to a symplectic 2-form on $\widetilde{X}$. We check this in a neighborhood of $x \in X_2$ and of $y \in X_1$. First consider $x \in X_2$. Replacing X by some affine open neighborhood of x, Theorem 3.3 says that there is an affine Z with

$$X \xleftarrow{\ \pi\ } Z \xrightarrow{\ \rho\ } \mathfrak{X}(2,2)$$

where π and ρ are étale. Let $\widetilde{\mathfrak{X}}(2,2) \to \mathfrak{X}(2,2)$, resp. $\widetilde{Z} \to Z$, denote the blowup along the reduced singular locus. Then

$$\widetilde{Z} \simeq \widetilde{X} \times_X Z \simeq \widetilde{\mathfrak{X}}(2,2) \times_{\mathfrak{X}(2,2)} Z. \tag{12}$$

As noted in [32, Remark 4.6], $\widetilde{\mathfrak{X}}(2,2) \to \mathfrak{X}(2,2)$ is a projective symplectic resolution. Now Lemma 5.2 below and (12) imply that $\widetilde{X} \to X$ is a projective symplectic resolution.

For $y \in X_1$, Theorem 3.3 shows that there is an étale equivalence between a neighborhood of y and a neighborhood of the origin in a certain quiver variety, independent of the choice of $y \in X_1$. In particular such a neighborhood is also étale equivalent to a neighborhood of a point of X_1 inside the neighborhood of $x \in X_2$ used above, so the result follows from the previous statement. (One can also compute explicitly: the quiver needed is the one with two vertices, one arrow in each direction between the two vertices, and also two loops at each vertex, so the quiver variety is isomorphic to $\mathbb{C}^8 \times \mathbb{C}^2/\mathbb{Z}_2$, which is an A_1 singularity and hence blowing up the reduced ideal sheaf of the singular locus gives a projective symplectic resolution).

It remains to prove the following standard lemma:

Lemma 5.2. *Let X be a symplectic singularity and $\pi : \widetilde{X} \to X$ a proper morphism. Then π is a symplectic resolution if and only if it is so after a surjective étale base change i.e. being a symplectic resolution is an étale local property.*

Notice that we are not making the (false) claim that X admits a symplectic resolution if and only if it does so étale locally.

Proof. Passing to the generic points of $\widetilde{X}$ and X, the fact that a surjective étale morphism is faithfully flat implies that π is birational if and only if it is so after base change. Therefore it suffices to check that the extension ω' of the pullback $\pi^*\omega$ is non-degenerate. If $b : Z \to X$ is a surjective étale morphism, then so too is $\widetilde{b} : \widetilde{Z} = \widetilde{X} \times_X Z \to \widetilde{X}$. The form ω' will be non-degenerate if and only if $\widetilde{b}^*\omega'$ is non-degenerate. □

6. Factoriality of quiver varieties

In this section, which is the technical heart of the paper, we consider the case of a Σ-divisible anisotropic root. Fix $\alpha \in \Sigma_{\lambda,\theta}$ to be a Σ-indivisible anisotropic root, and let $n \geq 2$ such that such that $(p(\alpha), n) \neq (2,2)$. We prove the key result, Corollary 6.9, which says that if θ is generic then $\mathfrak{M}_\lambda(n\alpha, \theta)$ is a factorial variety.

6.1. A *weighted partition* ν of n is a sequence $(\ell_1, \nu_1; \ldots; \ell_k, \nu_k)$, where $\nu_1 \geq \nu_2 \geq \cdots$ and $\sum_{i=1}^k \ell_i \nu_i = n$. Recall from Proposition 3.6 that the quiver variety $\mathfrak{M}_\lambda(\alpha, \theta)$ has a finite stratification by representation type. Given a weighted partition ν of n we can associate naturally a representation type of $n\alpha$:

$$\nu\alpha := (\ell_1, \nu_1\alpha; \ldots; \ell_k, \nu_k\alpha). \tag{13}$$

Lemma 6.1. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be a Σ-indivisible anisotropic root. Let $n \geq 2$.*

1. *The set $\Sigma_{\lambda,\theta}$ contains $\{m\alpha \mid m \geq 1\}$.*
2. *We have the formula*

$$\dim \mathfrak{M}_\lambda(n\alpha,\theta)_{\nu\alpha} = 2\left(k + (p(\alpha)-1)\sum_{i=1}^{k} \nu_i^2\right).$$

3. *For $(p(\alpha), n) \neq (2,2)$, we have $\dim \mathfrak{M}_\lambda(n\alpha,\theta) - \dim \mathfrak{M}_\lambda(n\alpha,\theta)_{\nu\alpha} \geq 4$ for all $\nu \neq (1,n)$.*
4. *For $(p(\alpha), n) \neq (2,2)$ and $\nu \neq (1,n)$, we furthermore have*

$$\dim \mathfrak{M}_\lambda(n\alpha,\theta) - \dim \mathfrak{M}_\lambda(n\alpha,\theta)_{\nu\alpha} \geq 8$$

   unless one of the following holds:
   (i) $(p(\alpha), n) = (2,3)$ *and* $\nu = (1,2;1,1)$*; or*
   (ii) $(p(\alpha), n) = (3,2)$ *and* $\nu = (1,1;1,1)$.
5. *In the case that $\theta \cdot \beta \neq 0$ for all $\beta \leq n\alpha$ not a multiple of α, all strata of $\mathfrak{M}_\lambda(n\alpha,\theta)$ are of the form $\mathfrak{M}_\lambda(n\alpha,\theta)_\nu$, and they are parameterized by weighted partitions of n. (It suffices to make the weaker assumption that $\Sigma_{\lambda,\theta} \cap \{\beta \mid \beta \leq n\alpha\} \subseteq \{m\alpha \mid m \leq n\}$.)*

We remark that we will only use part (4) later.

Proof. The first claim follows from [13, Proposition 1.2 (3)].

Next consider the second claim. Set $d := p(\alpha)$. Then $p(n\alpha) = n^2(d-1)+1$. We have a finite surjective map $\prod_i \mathfrak{M}_\lambda(\nu_i\alpha,\theta) \to \mathfrak{M}_\lambda(n\alpha,\theta)_{\nu\alpha}$, so the dimension formula follows from Corollary 3.23.

For the third part, notice that

$$\begin{aligned}\dim \mathfrak{M}_\lambda(n\alpha,\theta) - \dim \mathfrak{M}_\lambda(n\alpha,\theta)_\nu &= 2(n^2(d-1)+1) - 2\sum_{i=1}^{k}(\nu_i^2(d-1)+1) \\ &= 2(d-1)\sum_{i,j=1}^{k}(\ell_i\ell_i - \delta_{i,j})\nu_i\nu_j - 2(k-1). \qquad (14)\end{aligned}$$

Since $\sum_{i,j=1}^{k}(\ell_i\ell_j - \delta_{i,j})\nu_i\nu_j - (k-1) \geq 1$, we clearly have $\dim \mathfrak{M}_\lambda(n\alpha,\theta) - \dim \mathfrak{M}_\lambda(n\alpha,\theta)_\nu \geq 4$ when $d > 2$. When $d = 2$, a simple computation shows that $\dim \mathfrak{M}_\lambda(n\alpha,\theta) - \dim \mathfrak{M}_\lambda(n\alpha,\theta)_\nu = 2$ if and only if $n = 2$ and $\nu = (1,1;1,1)$.

For the fourth part, we use again (14), noticing the following points: the RHS of (14) is increasing in d; the RHS is increased if we replace (ℓ_i, n_i) by $(\ell_i - 1, n_i); (1, n_i)$; the RHS is increased if we replace $(1,a)$ and $(1,b)$ by $(1, a+b)$ (when $a + b < n$); and for $a > b > 1$, the RHS is increased if we replace $(1,a)$ and $(1,b)$ by $(1,a+1)$ and $(1,b-1)$. Since it suffices to prove the inequality after performing operations that increase the RHS, the result follows once we observe that the inequality holds in the following cases: (i) $\nu = (1, n-1; 1, 1)$ whenever $n \geq 4$ as well as $(1,1;1,1;1,1)$; (ii) for $\nu = (1,1;1,1)$ whenever $p(\alpha) \geq 4$, as well as $\nu = (2,1)$ for $p(\alpha) = 3$.

For the final claim, observe that each stratum of $\mathfrak{M}_\lambda(n\alpha, \theta)$ consists of representations of the form $x = x_1^{\oplus \ell_1} \oplus \cdots \oplus x_k^{\oplus \ell_k}$, where the x_i are pairwise non-isomorphic θ-stable representations of fixed dimension vectors $\alpha_i \in \Sigma_{\lambda,\theta}$. Under the assumptions given, each α_i must be a multiple of α. Therefore the representation type is of the form $\nu\alpha$ for some weighted partition ν of n. □

In particular, Lemma 6.1 describes the stratification of $\mathfrak{X}(g, n)$. Since $p(\alpha) > 1$, there exist infinitely many non-isomorphic θ-stable $\Pi^\lambda(Q)$-modules of dimension α. Therefore, for all representation types $\nu\alpha = (\ell_1, \nu_1\alpha; \ldots; \ell_k, \nu_k\alpha)$ with $\sum_i \ell_i \nu_i = n$, the stratum $\mathfrak{M}_\lambda(n\alpha, \theta)_{\nu\alpha}$ is non-empty. Let U be the union of all strata of "type $\nu\alpha$".

Lemma 6.2. *The subset U is open in $\mathfrak{M}_\lambda(n\alpha, \theta)$.*

Proof. Since the stratum of representation type $\rho = (n, \alpha)$ is contained in the closure of all the other strata of type $\nu\alpha$, it suffices to show that there is no stratum $\beta = (e_1, \beta^{(1)}; \ldots; e_l, \beta^{(l)})$ of any other type such that $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho \subset \overline{\mathfrak{M}_\lambda(n\alpha, \theta)}_\beta$. Assume otherwise. If $G_\rho \simeq GL_n(\mathbb{C})$ is the stabilizer of some $x \in \mathfrak{M}_\lambda(n\alpha, \theta)_\rho$, then the Hilbert-Mumford criterion implies that there exists some $y \in \mathfrak{M}_\lambda(n\alpha, \theta)_\beta$ whose stabilizer G_β is contained in G_ρ. Let V_i be the $n\alpha_i$-dimensional vector space at the vertex i on which $G(n\alpha)$ acts. Then, for each $g \in G(n\alpha)$ and $u \in \mathbb{C}^\times$, the u-eigenspace of g is the direct sum over the u-eigenspaces $g|_{V_i}$. In particular, it has a well-defined dimension vector. Now the elements g of G_ρ all have the property that the dimension vector of the u-eigenspace of g is of the form $r\alpha$ for some $r \in \mathbb{Z}_{\geq 0}$. On the other hand, since β is not "of type $\nu\alpha$", there is some i such that $e_i\beta^{(i)} \neq r\alpha$ for any r. Take $u \neq 1$ and $g \in G_\beta$ that rescales the summand of y of dimension $e_i\beta^{(i)}$ by u and is the identity on all other summands. Then the u-eigenspace of g has dimension vector $e_i\beta^{(i)}$ which implies that $G_\beta \not\subset G_\rho$ - a contradiction. Thus, U is open. □

The open subset of $\mu^{-1}(\lambda)^\theta$ consisting of stable representations is denoted $\mu^{-1}(\lambda)^\theta_s$, and its image in $\mathfrak{M}_\lambda(n\alpha, \theta)$ is denoted $\mathfrak{M}_\lambda(n\alpha, \theta)^s$. Note that $\mathfrak{M}_\lambda(n\alpha, \theta)^s$ is an open subset of U, and the quotient map $\mu^{-1}(\lambda)^\theta_s \to \mathfrak{M}_\lambda(n\alpha, \theta)^s$ is a principal $PG(n\alpha)$-bundle.

6.2. Factoriality of $\mathfrak{M}_\lambda(n\alpha, \theta)$. A closed point $x \in X$ is said to be factorial if the local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. We say that X is factorial if X is factorial at every closed point. If $\xi : \mu^{-1}(\lambda)^\theta \to \mathfrak{M}_\lambda(n\alpha, \theta)$ is the quotient map, then let $V = \xi^{-1}(U)$, where U is the open subset of Lemma 6.2. We will need the following result from [14]:

Theorem 6.3. [14, Theorem 6.3, Corollary 6.4] *Consider a stratum Z in $\mathfrak{M}_\lambda(\beta, \theta)$ of representation type $(k_1, \beta^{(1)}; \ldots; k_r, \beta^{(r)})$. Then for all $z \in Z$, $\xi^{-1}(z) \subseteq \mu^{-1}(\lambda)^\theta$ has dimension at most $\beta \cdot \beta - 1 + p(\beta) - \sum_t p(\beta^{(t)})$, so the dimension of $\xi^{-1}(Z)$ is at most $\beta \cdot \beta - 1 + p(\beta) + \sum_t p(\beta^{(t)})$.*

We note that in [14], this is stated and proved for λ and θ equal to zero, but the proof and result extends verbatim to the general case (replacing simple modules by θ-stable modules). In the case that $\beta \in \Sigma_{\lambda,\theta}$, applying Proposition 3.26 immediately yields

Corollary 6.4. *The codimension of* $\xi^{-1}(Z)$ *is at least* $\frac{1}{2}\operatorname{codim}(Z) = p(\beta) - \sum_t p(\beta^{(t)})$.

Proposition 6.5. V *is a local complete intersection, factorial and normal.*

Proof of Proposition 6.5. Since $\alpha \in \Sigma_{\lambda,\theta}$, Proposition 3.26 implies that $\mu^{-1}(\lambda)^\theta$ and hence V is a local complete intersection of dimension $n^2\alpha \cdot \alpha - 1 + 2p(n\alpha)$, and the stable locus is open and dense.

The main step is the second assertion. For this we will show that V is smooth outside a subset of codimension four, i.e., it satisfies the R_3 property. For any $G(n\alpha)$-stable subset X of $\mu^{-1}(\lambda)^\theta$, we write X_{free} for the subset of all points where $PG(n\alpha)$ acts freely. The assertion will follow from estimating the codimension of the complement to V_{free} in V. Note that the free locus is the same as the locus of representations whose endomorphism algebra has dimension one, i.e., the "bricks". We represent $\mu^{-1}(\lambda)^\theta$ as the union of preimages of the (finitely many) strata, and consider over each such preimage the non-free locus.

If the preimage of the stratum has codimension at least four, it can be ignored. Thus, we just need to show that the complement to $\xi^{-1}(Z)_{\text{free}}$ has codimension at least four for those strata Z with $\operatorname{codim} \overline{\xi^{-1}(Z)} \leq 3$. Since we are explicitly excluding the case $(p(\alpha), n) = (2, 2)$, Corollary 6.4, together with Lemma 6.1 (4), imply that we are reduced to considering the cases $(p(\alpha), n) = (2, 3)$ and $\nu = (1, 2; 1, 1)$, or $(p(\alpha), n) = (3, 2)$ and $\nu = (1, 1; 1, 1)$.

Observe first that if Z is a stratum, then the polystable part of the preimage $\xi^{-1}(Z)$ has codimension (in $\mu^{-1}(\lambda)^\theta$) at least the codimension of Z itself (in $\mathfrak{M}_\lambda(n\alpha, \theta)$), since the fiber over a polystable representation M has dimension $\alpha \cdot \alpha - \dim \operatorname{End}(M)$, which is maximized when M is stable. Thus if Z has codimension at least four (which is the case for us), then we can ignore the polystable part of $\xi^{-1}(Z)$.

Next, if we consider a stratum Z of type $(1, a; 1, b)$, note that every representation in this stratum is either polystable or an indecomposable extension of two non-isomorphic representations. The latter type is a brick, since there is a unique stable quotient and a unique stable subrepresentation and the two are nonisomorphic. Therefore applying the previous paragraph together with Lemma 6.1 (3) shows that we can ignore $\xi^{-1}(Z)$ (the non-free locus has overall codimension at least four). This proves the final assertion.

Since μ is regular on the locus where $PG(n\alpha)$ acts freely, $\mu^{-1}(\lambda)^\theta_{\text{free}}$ lies in the smooth locus of V. We conclude from the last assertion of the proposition that the singular locus of V has codimension at least 4 (i.e., property R_3 holds). Since V is a local complete intersection, and hence Cohen-Macaulay, it satisfies Serre's condition S_2, so it is normal.

Finally, it follows from a theorem of Grothendieck, [33, Theorem 3.12], that since V is a complete intersection and satisfies R_3, it is factorial. □

The result that allows us to descend factoriality from V to the quotient U is the following theorem by Drezet. Since the version given in [19] concerns the moduli space of semistable sheaves on a smooth surface, we provide full details to ensure the arguments are applicable in our situation. Let G be a connected reductive group.

Lemma 6.6. *Let V be a factorial normal affine G-variety and $V_s \subset V$ a dense open subset of V, whose complement has codimension at least two in V. Then every G-equivariant line bundle on V_s extends to a G-equivariant line bundle on V.*

Proof. The fact that V is normal and factorial implies that

$$\mathrm{Pic}(V) = \mathrm{Div}(V) = \mathrm{Div}(V_s) = \mathrm{Pic}(V_s).$$

Hence if L_0 is a G-equivariant line bundle on V_s, forgetting the equivariant structure, there is an extension L to V. To show that the extension L has a G-equivariant structure, one repeats the argument of [20, Lemme 5.2], which uses the fact that the codimension of $V \smallsetminus V_s$ is at least two. $\square$

Theorem 6.7 ([19], Theorem A)**.** *Let V be a factorial, normal G-variety, with good quotient $\xi : V \to U := V /\!\!/ G$. Assume that there exists an open subset $U_s \subset U$ such that*

(a) the complement to U_s has codimension at least two in U,
(b) $V_s := \xi^{-1}(U_s) \to U_s$ is a principal G-bundle; and
(c) the complement to V_s has codimension at least two in V.

Let $x \in U$ and $y \in T(x)$ a lift in V (so that $G \cdot y$ is closed in V). The following are equivalent:

(i) The local ring $\mathcal{O}_{U,x}$ is a unique factorization domain.
(ii) For every line bundle M_0 on U_s, there exists an open subset $U_0 \subset U$ containing both x and U_s such that M_0 extends to a line bundle M on U_0.
(iii) For every G-equivariant line bundle L on V, the stabilizer of y acts trivially on the fiber L_y.

Proof. Recall that $\mathcal{O}_{U,x}$ is a unique factorization domain if and only if every height one prime is principal. Geometrically, this means that for every hypersurface Y of U, the sheaf of ideals $\mathcal{I}_Y$ is free at x.

(i) implies (ii). It suffices to assume that $M_0 = \mathcal{I}_Y$, where Y is a hypersurface in U_s. If $\overline{Y}$ is the closure of Y in U, then $M = \mathcal{I}_{\overline{Y} \cap U_0}$ is the required extension.

(ii) implies (i). Let Y be a hypersurface in U. We wish to show that $\mathcal{I}_Y$ is free at x. Let M be the extension of $\mathcal{I}_Y|_{U_s}$ to U_0. The line bundle M corresponds to a Cartier divisor D on U_0; $M = \mathcal{O}_{U_0}(D)$. Then,

$$\mathcal{I}_Y|_{U_s} = \mathcal{O}_{U_s}(D \cap U_s),$$

and the divisors Y and $-D \cap U_s$ are linearly equivalent. Since, by assumption, the codimension of the complement to U_s in U has codimension at least two and U is normal, $\overline{Y} \simeq -D$. Hence $M = \mathcal{I}_{\overline{Y}}$ is free at x.

(ii) implies (iii). Suppose that L is a G-equivariant line bundle on V. Since G acts freely on V_s, the restriction $L|_{V_s}$ descends to the line bundle $M_0 = (L|_{V_s})/G$ on U_s. Let M be the extension of

M_0 to U_0. Then the G-equivariant line bundle $\xi^* M$ agrees with L on V_s. This implies, as in the previous paragraph, that $\xi^* M = L$ on $\xi^{-1}(U_0)$. In particular, since $y \in \xi^{-1}(U_0)$, the stabilizer of y acts trivially on L_y.

(iii) implies (ii). Let M_0 be a line bundle on U_s. By Lemma 6.6, $\xi^* M_0$ extends to a G-equivariant line bundle L on V. Recall by definition of lift that $G \cdot y$ is closed in V. Therefore Lemma 6.8 below says that there is an affine open neighborhood U' of x such that $G_{y'}$ acts trivially on $L_{y'}$ for all $y' \in \xi^{-1}(U')$ such that $G \cdot y'$ is closed in V. Let $U_0 = U' \cup U_s$. Then, by descent [19, Theorem 1.1], there exists a line bundle M on U_0 such that $\xi^* M \simeq L$. In particular, M extends M_0. □

Let Y be a variety admitting an algebraic action of a reductive group G. Assume that there exists a good quotient $\xi : Y \to X = Y /\!\!/ G$. The following result, which says that the descent locus of an equivariant line bundle is open, is presumably well-known, but we were unable to find it in the literature.

Lemma 6.8. *Let L be a G-equivariant line bundle on Y and $y \in Y$ a closed point such that the orbit $\mathcal{O} = G \cdot y$ is closed and the stabilizer G_y of y acts trivially on the fiber L_y. Then there exists an affine open neighborhood U of $\xi(y)$ such that the stabilizer $G_{y'}$ acts trivially on $L_{y'}$ for all $y' \in \xi^{-1}(U)$ such that $G \cdot y'$ is closed.*

Proof. The proof of the lemma can be easily deduced from the proof of [20, Theorem 2.3]. It is shown there that one can find a G-invariant section $s' : \mathcal{O} \to L|_{\mathcal{O}}$, which trivializes $L|_{\mathcal{O}}$. As explained in *loc. cit.*, the fact that $\mathcal{O}$ is closed in Y implies that one can lift s' to a G-invariant section $s \in \Gamma(\xi^{-1}(U'), L)$, where U' is some affine open neighborhood of $\xi(y)$. Let W be the (non-empty) open subset of $\xi^{-1}(U')$ consisting of all points y' such that $s(y') \neq 0$ i.e. s trivializes L over W. Then it suffices to show that there is some affine neighborhood U of $\xi(y)$ such that $\xi^{-1}(U) \subset W$. Again, following [20, Theorem 2.3], the sets $\xi^{-1}(U') \smallsetminus W$ and $\mathcal{O}$ are G-stable closed subsets of $\xi^{-1}(U')$. Therefore the fact that ξ is a good quotient implies that $\xi(\xi^{-1}(U') \smallsetminus W)$ and $\xi(\mathcal{O}) = \{\xi(y)\}$ are closed, disjoint subsets of U'. Thus, there exists an affine neighborhood U of $\xi(y)$ such that $U \cap \xi(\xi^{-1}(U') \smallsetminus W) = \emptyset$, as required. □

Corollary 6.9. *Assume that $(p(\alpha), n) \neq (2, 2)$. Then the variety U is factorial.*

Proof. Let $G = PG(\alpha)$ and $U_s = \mathfrak{M}_\lambda(n\alpha, \theta)_s$. Proposition 6.5 implies that V is normal and factorial. This implies that U is normal. Moreover, by Lemma 6.1, the codimension of the complement to U_s in U has codimension at least two. Thus, assumptions (a) and (b) of Theorem 6.7 are satisfied. By Lemma 6.1 (3) and Corollary 6.4, the complement to $\mu^{-1}(\lambda)^\theta_s$ in V has codimension at least 4. In particular, assumption (c) of Theorem 6.7 is also satisfied.

Next, recall from the proof of Lemma 6.2, the stratum of type $\rho = (n, \alpha)$ is contained in the closure of all other strata in U. If y is a lift in $\mu^{-1}(\lambda)^\theta$ of a point of $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho$ then y corresponds to a representation $M_0^{\oplus n}$, where $M_0 \in \mathfrak{M}_\lambda(\alpha, \theta)$ is a stable $\Pi^\lambda(Q)$-module. Therefore $PG(n\alpha)_y =$

PGL_n has no non-trivial characters. In particular, $PG(n\alpha)_y$ will act trivially on L_y for any $PG(n\alpha)$-equivariant line bundle on V. Hence, we deduce from Theorem 6.7 that $\mathfrak{M}_\lambda(n\alpha, \theta)$ is factorial at every point of $\mathfrak{M}_\lambda(n\alpha, \theta)_\rho$.

Now consider an arbitrary stratum $\mathfrak{M}_\lambda(n\alpha, \theta)_{\nu\alpha}$ in U. If $\mathfrak{M}_\lambda(n\alpha, \theta)$ is factorial at one point of the stratum then it will be factorial at every point in the stratum (for a rigorous proof of this fact, repeat the argument given in the proof of [33, Theorem 5.3]). On the other hand, a theorem of Boissière, Gabber and Serman [6] says that the subset of factorial points of U is an open subset. Since this open subset is a union of strata and contains the unique closed stratum, it must be the whole of U. $\square$

Remark 6.10. Notice that if θ is generic then $U = \mathfrak{M}_\lambda(n\alpha, \theta)$. Hence Corollary 6.9 says that $\mathfrak{M}_\lambda(n\alpha, \theta)$ is a factorial variety. This is precisely the statement of Theorem 1.8.

6.3. The proof of Theorem 1.2. By Proposition 3.21, we know that $\mathfrak{M}_\lambda(\alpha, \theta)$ is irreducible and normal. Therefore, it suffices to show that it admits symplectic singularities. Since the isomorphism of Theorem 3.17 is Poisson, it suffices to show that the varieties $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admit symplectic singularities. If $\sigma^{(i)}$ is real there is nothing to check.

Lemma 6.11. *Let X be a smooth irreducible Poisson variety and Y a smooth symplectic variety. If $\pi : Y \to X$ is a birational, surjective Poisson morphism, then it is an isomorphism.*

Proof. Since the morphism is birational, there is a dense open subset $U \subset X$ over which it is an isomorphism. By [28, I, Corollary 6.12], the complement of U has codimension at least two. On the other hand, since X is smooth, the locus where the Poisson structure on X is degenerate has codimension one. Therefore, X is symplectic too. This implies that $d_y\pi$ is an isomorphism for all $y \in Y$. Thus, by Zariski's Main Theorem, π is an isomorphism. $\square$

Lemma 6.12. *Let X be a normal irreducible Poisson variety and assume that $\pi : Y \to X$ is a proper birational Poisson morphism from a variety Y with symplectic singularities. Then X has symplectic singularities.*

Proof. Let $\rho : Z \to Y$ be a resolution of singularities. If ω' is the symplectic 2-form on the smooth locus of Y then $\rho^*\omega'$ extends to a regular form on Z. Let $Y' = \pi^{-1}(X_{\mathrm{sm}})$. Then, since $\pi : Y' \to X_{\mathrm{sm}}$ is proper and birational and X_{sm} is irreducible, π is surjective. Lemma 6.11 implies that it is an isomorphism. In particular, there is a symplectic 2-form on X_{sm} such that the Poisson structure on X_{sm} is non-degenerate and induced from ω. Moreover, $\pi^*\omega = \omega'$. Thus, $(\pi \circ \rho)^*\omega = \rho^*\omega'$ extends to a regular form, and hence X has symplectic singularities. $\square$

Remark 6.13. One can drop the assumption in Lemma 6.12 that X is Poisson and π is a Poisson morphism; since $R^0\pi_*\mathcal{O}_Y = \mathcal{O}_X$ it naturally inherits a Poisson structure making π Poisson.

If $\sigma^{(i)}$ is a Σ-indivisible anisotropic root then $n_i = 1$ and choosing a generic stability parameter $\theta' \geq \theta$ defines a projective, Poisson resolution $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta') \to \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ with $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ a smooth symplectic variety; see [14, Section 8]. Similarly, if $\sigma^{(i)}$ is isotropic imaginary then it is well-know that one can frame the quiver so that there exists a projective, Poisson resolution of singularities from a quiver variety that is a smooth symplectic variety. Thus, Lemma 6.12 implies that $S^{n_i}\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits symplectic singularities in these two cases.

Therefore we may assume that there exists a Σ-indivisible anisotropic root β such that $\alpha = n\beta$, for some $n > 1$. Let $g = p(\beta)$. If $(g, n) = (2, 2)$, then Theorem 1.7 and Lemma 6.12 imply that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ has symplectic singularities. Therefore, it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ has symplectic singularities when $(g, n) \neq (2, 2)$. Again, choose a generic stability parameter $\theta' \geq \theta$. Then $\pi : \mathfrak{M}_\lambda(\sigma^{(i)}, \theta') \to \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is projective and Poisson by Lemma 2.3. Moreover, since both $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ and $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ are irreducible by Proposition 3.21, and a generic element of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ is θ-stable, the map π is birational. Thus, by Lemma 6.12, it suffices to show that $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ admits symplectic singularities. This follows from Flenner's Theorem [22], once we show that the singular locus of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ has codimension at least four. By Theorem 1.13, the singular locus of $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$ is the union of all strata except the open stratum. By Lemma 6.1 (3) each of these strata has codimension at least 4 in $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta')$.

6.4. The proof of Theorems 1.4 and 1.6. We begin by considering the case of a Σ-divisible anisotropic root. Recall that a normal variety X with $\mathbb{Q}$-Cartier canonical divisor K_X is said to have *terminal singularities* if $K_Y = f^*(K_X) + \sum_i a_i E_i$ with $a_i > 0$, where $f : Y \to X$ is any resolution of singularities, and the sum is over all exceptional divisors of f.

Theorem 6.14. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be a Σ-indivisible anisotropic root and fix $n \geq 2$ such that $(n, p(\alpha))$ does not equal $(2, 2)$. Then the symplectic singularity $\mathfrak{M}_\lambda(n\alpha, \theta)$ does not admit a proper symplectic resolution.*

Proof. If $\mathfrak{M}_\lambda(n\alpha, \theta)$ admits a proper symplectic resolution then so too by restriction does the open subset U of Lemma 6.2. Recall from Theorem 1.13 that the singular locus of U is the complement of the open stratum. By Lemma 6.1, this has codimension at least four in U. Therefore, since U has symplectic singularities by Theorem 1.2, [48] says that U has terminal singularities. This implies that if $f : Y \to U$ is a proper symplectic resolution then the exceptional locus of f has codimension at least two in Y. On the other hand, we have shown in Corollary 6.9 that U is factorial. This implies by van der Waerden purity, see [17, Section 1.40], that the exceptional locus of f is a divisor. This is a contradiction. □

The proof also shows, more generally, that any singular open subset of U does not admit a symplectic resolution. If θ is generic, then $U = \mathfrak{M}_\lambda(n\alpha, \theta)$. This implies Corollary 1.9 (as well as the stronger result discussed afterwards).

Corollary 6.15. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be Σ-indivisible and $n \geq 1$. Then $\mathfrak{M}_\lambda(n\alpha, \theta)$ admits a projective symplectic resolution if one of the following conditions hold:*

(o) α is a real root ($p(\alpha) = 0$);

(i) $n = 1$;

(ii) $p(\alpha) = 1$; or

(iii) $(n, p(\alpha)) = (2, 2)$.

If none of these conditions hold, then $\mathfrak{M}_\lambda(n\alpha, \theta)$ does not admit a proper symplectic resolution. In particular, existence of projective and proper symplectic resolutions is equivalent for $\mathfrak{M}_\lambda(n\alpha, \theta)$.

Proof. In case (o), $\mathfrak{M}_\lambda(n\alpha, \theta)$ is a point, so there is nothing to show. In case (i), let D be the open subset of $\{\theta' \in \mathbb{Q}^{Q_0} \mid \theta'(\alpha) = 0\}$ consisting of all stability conditions vanishing on α, but not on any other $\beta < \alpha$ with $\beta \in \Sigma_{\lambda,\theta}$. Since α is Σ-indivisible, the set D is non-empty. Its closure is the whole space, thus there exists a connected component C such that $\theta \in \overline{C}$. Choose $\theta' \in C$ (rescaling if necessary, we may assume that $\theta' \in \mathbb{Z}^{Q_0}$). Then, just as shown in [14, Section 8], the morphism $\mathfrak{M}_\lambda(\alpha, \theta') \to \mathfrak{M}_\lambda(\alpha, \theta)$ is a projective symplectic resolution. For case (ii), first note that case (i) implies that $X := \mathfrak{M}_\lambda(\alpha, \theta') \to \mathfrak{M}_\lambda(\alpha, \theta)$ is a projective symplectic resolution of (du Val) singularities for some $\theta' \geq \theta$. In particular, X is a smooth symplectic surface. Next, $\mathfrak{M}_\lambda(n\alpha, \theta) \cong S^n \mathfrak{M}_\lambda(\alpha, \theta)$ by Theorem 3.16 because the canonical decomposition of $n\alpha$ is $\alpha + \cdots + \alpha$. We therefore obtain a partial resolution $S^n X \to \mathfrak{M}_\lambda(n\alpha, \theta)$. Now recall that the natural map $\operatorname{Hilb}^n \mathfrak{M}_\lambda(\alpha, \theta') \to S^n \mathfrak{M}_\lambda(\alpha, \theta')$ is a projective symplectic resolution; see [46, Theorem 1.8, Theorem 1.10]. Finally in case (iii) the resolution is given in Theorem 1.7. If none of the conditions hold, then α is an anisotropic root and the non-existence of a proper symplectic resolution is a consequence of Theorem 6.14. □

We now proceed to the proof of Theorem 1.4. The isomorphism of Theorem 1.4 follows directly from Theorem 3.17. Therefore, it suffices to show that $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a symplectic resolution if and only if each $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a symplectic resolution.

First note that if $\sigma^{(i)}$ is a real root or an isotropic root, then $\sigma^{(i)}$ is Σ-indivisible (for the latter property, a Σ-divisible isotropic root is not in $\Sigma_{\lambda,\theta}$). In these cases, as recalled in Corollary 6.15, $S^{n_i} \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution, as does $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$.

On the other hand, if $\sigma^{(i)}$ is an anisotropic root, then $n_i = 1$, since every multiple of an anisotropic root $\sigma \in \Sigma_{\lambda,\theta}$ also belongs to $\Sigma_{\lambda,\theta}$. Moreover, by Corollary 6.15, $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution if $\sigma^{(i)}$ is Σ-indivisible or if $\sigma^{(i)}$ is twice a root $\beta \in \Sigma_{\lambda,\theta}$ satisfying $p(\beta) = 2$, and otherwise it admits no proper symplectic resolution.

Therefore, if $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a proper symplectic resolution for all i, it follows that each $S^{n_i} \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ admits a projective symplectic resolution, and hence so does $\mathfrak{M}_\lambda(\alpha, \theta)$.

If, on the other hand, some $\mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$ did not admit a projective symplectic resolution, then for some i, $\sigma^{(i)}$ is a Σ-divisible anisotropic root which is not twice a root $\beta \in \Sigma_{\lambda,\theta}$ satisfying $p(\beta) = 2$. In this case, we show that some open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a proper symplectic resolution, which completes the proof. Take $x = (x_1, \ldots, x_k) \in \prod_{j=1}^k S^{n_j}\mathfrak{M}_\lambda(\sigma^{(j)}, \theta) = \mathfrak{M}_\lambda(\alpha, \theta)$ such that x_j is in the smooth locus of $S^{n_j}\mathfrak{M}_\lambda(\sigma^{(j)}, \theta)$ for $j \neq i$ and $x_i \in U \subset \mathfrak{M}_\lambda(\sigma^{(i)}, \theta)$. Then, as in the proof of Theorem 6.14, $\mathfrak{M}_\lambda(\alpha, \theta)$ is factorial at x and has terminal singularities, and hence cannot admit a proper symplectic resolution. This completes the proof of Theorem 1.4.

Notice that Theorem 1.6 also follows from the above argument.

6.5. Formal resolutions. Let α be an anisotropic root. Though it might not be obvious from Corollary 1.9, the nature of the obstructions to the existence of a symplectic resolution of $\mathfrak{M}_\lambda(\alpha, \theta)$ are quite subtle. We have shown that Zariski locally no resolution exists if α is Σ-divisible. But then one can ask if a resolution exists étale locally, or in the formal neighborhood of a point? In this section we give a precise answer to this question.

Definition 6.16. The closed point $x \in \mathfrak{M}_\lambda(\alpha, \theta)$ is said to be *formally resolvable* if the formal neighborhood $\widehat{\mathfrak{M}}_\lambda(\alpha, \theta)_x$ of x in $\mathfrak{M}_\lambda(\alpha, \theta)$ admits a projective symplectic resolution.

Lemma 6.17. *If $0 \in \mathfrak{M}_0(\alpha, 0)$ is formally resolvable, then $\mathfrak{M}_0(\alpha, 0)$ also admits a projective symplectic resolution, and conversely.*

Proof. Let $\mathbb{C}^\times$ act on $\mathrm{Rep}(\overline{Q}, \alpha)$ by dilations. Then the moment map μ is homogeneous of degree two and the action of $G(\alpha)$ commutes with the action of $\mathbb{C}^\times$. This implies that $\mathbb{C}[\mathfrak{M}_\lambda(\alpha, \theta)]$ is an $\mathbb{N}$-graded, connected algebra. Note also that the Poisson bracket on $\mathbb{C}[\mathfrak{M}_\lambda(\alpha, \theta)]$ has degree -2. The lemma follows from standard arguments; see [24, Proposition 5.2], [31, Theorem 1.4], and the references therein. The idea is that: 1) The induced action of $\mathbb{C}^\times$ on $\widehat{\mathfrak{M}}_0(\alpha, 0)_0$ lifts to the resolution. 2) The $\mathbb{C}^\times$-action allows one to globalize the resolution of the formal neighborhood of 0 to a resolution of the whole of $\mathfrak{M}_0(\alpha, 0)$. For the converse statement, we restrict a symplectic resolution of $\mathfrak{M}_0(\alpha, 0)$ to the formal neighborhood of zero. □

Remark 6.18. The same result holds if we define formally resolvable using proper resolutions instead of projective resolutions. Then by Theorem 1.4, it follows that the two definitions of formally resolvable are actually equivalent (since we are only considering quiver varieties).

By Corollary 3.4, if one point in a stratum $\mathfrak{M}_\lambda(\alpha, \theta)_\tau \subset \mathfrak{M}_\lambda(\alpha, \theta)$ is formally resolvable, then so too is every other point in the stratum. If $\tau = (e_1, \beta^{(1)}; \ldots; e_k, \beta^{(k)})$, then define the greatest common divisor $\gcd(\tau)$ of τ to be the greatest common divisor of the e_i. If the greatest common divisor of τ is k, then each point in $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ corresponds to a representation of the form $Y^{\oplus k}$ for some θ-polystable representation Y. Let $U_{\mathrm{fr}} \subset \mathfrak{M}_\lambda(\alpha, \theta)$ be the union of all strata $\mathfrak{M}_\lambda(\alpha, \theta)_\tau$ such that $\gcd(\tau) = 1$.

Lemma 6.19. *Let $\alpha \in \Sigma_{\lambda,\theta}$. Then U_{fr} is a dense open subset of $\mathfrak{M}_\lambda(\alpha, \theta)$.*

Proof. The set U_{fr} is dense because it contains the open stratum $\mathfrak{M}_\lambda(\alpha, \theta)_{(1,\alpha)}$, consisting of stable representations. We will show that the complement to U_{fr} is closed in $\mathfrak{M}_\lambda(\alpha, \theta)$. It suffices to show that if the greatest common divisor of ρ is greater than one and $\mathfrak{M}_\lambda(\alpha,\theta)_\tau \subset \overline{\mathfrak{M}_\lambda(\alpha,\theta)_\rho}$ then $\gcd(\tau) > 1$ too. The argument is similar to the proof of Lemma 6.2. Let $x \in \mathfrak{M}_\lambda(\alpha,\theta)_\rho$ and $G_\rho \subset G(\alpha)$ its stabilizer. By Proposition 3.6, there exists $x' \in \mathfrak{M}_\lambda(\alpha,\theta)_\tau$ such that its stabilizer G_τ contains G_ρ. Let $\gcd(\rho) = k$, so that x corresponds to a representation $Y \otimes V$ for some θ-polystable representation Y, and k-dimensional vector space V. Notice that $\alpha = k \dim Y$. Then $GL(V)$ is a subgroup of G_ρ, and hence of G_τ too. An elementary argument shows that this implies that x' corresponds to a representation $Y' \otimes V$ for some θ-polystable representation Y'. Thus, $\gcd(\tau) > 1$. In fact, we have shown that if $\mathfrak{M}_\lambda(\alpha,\theta)_\tau \subset \overline{\mathfrak{M}_\lambda(\alpha,\theta)_\rho}$, then $\gcd(\rho)$ divides $\gcd(\tau)$. Thus, U_{fr} is open in $\mathfrak{M}_\lambda(\alpha,\theta)$. □

Theorem 6.20. *Let $\alpha \in \Sigma_{\lambda,\theta}$ be an anisotropic root, $\alpha = n\beta$ for some Σ-indivisible root and some $n > 1$. Assume that $(n, p(\beta)) \neq (2,2)$. Then a point x is formally resolvable if and only if $x \in U_{fr}$.*

Proof. Let $x \in \mathfrak{M}_\lambda(\alpha,\theta)$ have representation type $\tau = (e_1, \beta^{(1)}; \dots; e_k, \beta^{(k)})$, where $m := \gcd(\tau)$. By Corollary 3.4, $\widehat{\mathfrak{M}_\lambda}(\alpha,\theta)_x \simeq \widehat{\mathfrak{M}_0}(\mathbf{e},0)_0$ and hence Lemma 6.17 says that x is formally resolvable if and only if $\mathfrak{M}_0(\mathbf{e},0)$ admits a projective symplectic resolution. By definition, the greatest common divisor of $\mathbf{e}$ is m. Proposition 4.2 says that $\mathbf{e}$ belongs to $\Sigma_{0,0}$ for the quiver underlying $\mathfrak{M}_0(\mathbf{e},0)$. Moreover, by remark 3.5, we have $p(\alpha) = p(\mathbf{e})$ which implies that $\mathbf{e} = m\mathbf{f}$ with both $\mathbf{e}$ and $\mathbf{f}$ anisotropic. If $m = 1$ then Corollary 6.15 (i) implies that x is formally resolvable. Thus, we just need to show that if $m > 1$ then x is not formally resolvable.

First, we show:

$$(m, p(\mathbf{f})) = (2,2) \quad \Leftrightarrow \quad (n, p(\beta)) = (2,2). \tag{15}$$

Assume that the left hand side of (15) holds. Then $p(\alpha) = n^2(p(\beta)-1)+1$ implies that

$$n^2(p(\beta)-1)+1 = p(\alpha) = p(\mathbf{e}) = 5$$

and hence $n^2(p(\beta)-1) = 4$. But $p(\beta) > 1$ since β is anisotropic, and 2 divides n. Thus, $n = 2$ and $p(\beta) = 2$. Conversely, assume that the right hand side of (15) holds. Then $5 = p(\alpha) = p(\mathbf{e})$ implies that $m^2(p(\mathbf{f})-1) = 4$. Since $\mathbf{f}$ is anisotropic and we have assumed that $m > 1$, we deduce that $(m, p(\mathbf{f})) = (2,2)$.

Notice that we have assumed in the statement of the theorem that $(n, p(\beta)) \neq (2,2)$. Thus, $(m, p(\mathbf{f}))$ cannot equal $(2,2)$. By Theorem 2.2, either β is indivisible, or there exists a prime p such that $\beta = p\gamma$, for some indivisible anisotropic root γ that does not belong to $\Sigma_{\lambda,\theta}$ and prime p. We consider the two cases separately. Assume that β is indivisible. Then m divides n and Proposition 4.2 says that $\mathbf{f}$ belongs to $\Sigma_{0,0}$ because $m^{-1}\alpha$ belongs to $\Sigma_{\lambda,\theta}$. Then Theorem 6.14 says that $\mathfrak{M}_0(\mathbf{e},0)$ does not admit a symplectic resolution because $m > 1$. Assume now that β is not indivisible. Let q be a prime dividing m. Then q divides np since

$$np\gamma = \alpha = m \sum_i \mathbf{f}_i \beta^{(i)}.$$

Moreover, $npq^{-1} > 1$ since $n > 1$. Therefore, Theorem 2.2 implies that $q^{-1}\alpha$ belongs to $\Sigma_{\lambda,\theta}$. By Proposition 4.2, this means that $q^{-1}\mathbf{e}$ belongs to $\Sigma_{0,0}$ and hence Theorem 6.14 says that $\mathfrak{M}_0(\mathbf{e}, 0)$ does not admit a symplectic resolution. $\square$

In the case where $\alpha \in \Sigma_{\lambda,\theta}$ equals 2β for some anisotropic root $\beta \in \Sigma_{\lambda,\theta}$ with $p(\beta) = 2$, every point in $\mathfrak{M}_\lambda(\alpha, \theta)$ is formally resolvable. Similarly, if α is Σ-indivisible anistropic then every point in $\mathfrak{M}_\lambda(\alpha, \theta)$ is formally resolvable.

Remark 6.21. If $U_{\mathrm{fr}} \subsetneq \mathfrak{M}_\lambda(\alpha, \theta)$ then Corollary 1.9 implies that any open subset of U_{fr} not contained in the smooth locus of $\mathfrak{M}_\lambda(\alpha, \theta)$ does not admit a symplectic resolution i.e. the singular locus of U_{fr} consists of points that cannot be resolved Zariski locally, but do admit a resolution in a formal neighborhood (in fact étale locally).

7. Namikawa's Weyl group

In the paper [49], Namikawa defined a finite group W associated to any conic affine symplectic singularity X such that the symplectic form on X has weight $\ell > 0$ with respect to the torus action. The group W acts as a reflection group on $H^2(Y, \mathbb{R})$, where $Y \to X$ is any $\mathbb{Q}$-factorial terminalization of X, whose existence is guaranteed by the minimal model programme. The group W plays a key role in the birational geometry of X; see [51] and [3].

One computes W as follows: let $\mathcal{L}$ be a codimension 2 leaf of X and $x \in \mathcal{L}$. Then the formal neighborhood of x in X is isomorphic to the formal neighborhood of 0 in $\mathbb{C}^{2(n-1)} \times \mathbb{C}^2/\Gamma$, where $2n = \dim X$ and $\Gamma \subset SL_2(\mathbb{C})$ is a finite group; see [50, Lemma 1.3]. Associated to Γ, via the McKay correspondence, is a Weyl group $W_{\mathcal{L}}$ of type A, D or E. The fundamental group $\pi_1(\mathcal{L})$ acts on $W_{\mathcal{L}}$ via Dynkin automorphisms. Let $W'_{\mathcal{L}}$ denote the centralizer of $\pi_1(\mathcal{L})$ in $W_{\mathcal{L}}$. Then

$$W := \prod_{\mathcal{L}} W'_{\mathcal{L}}.$$

Thus, in order to compute W, it is essential to classify the codimension 2 leaves of X, and describe $\pi_1(\mathcal{L})$. This is the goal of this section.

7.1. The proof of Theorem 1.18.

We assume throughout that $\alpha \in \Sigma_{\lambda,\theta}$, hence it is a root. Therefore the support of α on the quiver is connected. We can assume, up to replacing the quiver by the subquiver whose vertices are the support of α, and whose arrows are the ones with endpoints in the support, that α is sincere. Then, the quiver is connected. We may assume that α is imaginary, otherwise the statement is vacuous.

Our goal is to compute the codimension two leaves of $\mathfrak{M}_\lambda(\alpha, \theta)$, proving Theorem 1.18. Recall from Definition 1.16 that $\alpha = \beta^{(1)} + \cdots + \beta^{(s)} + m_1\gamma^{(1)} + \cdots m_t\gamma^{(t)}$ is an isotropic decomposition if

(a) $\beta^{(i)}, \gamma^{(j)} \in \Sigma_{\lambda,\theta}$.
(b) The $\beta^{(i)}$ are imaginary roots.
(c) The $\gamma^{(i)}$ are *pairwise distinct* real roots.

(d) If $\overline{Q}''$ is the quiver with $s+t$ vertices without loops and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows from vertex i to vertex $j \neq i$, where $\alpha^{(i)}, \alpha^{(j)} \in \{\beta^{(1)}, \ldots, \beta^{(s)}, \gamma^{(1)}, \ldots, \gamma^{(t)}\}$, then Q'' is an affine Dynkin quiver.

(e) The dimension vector $(1, \ldots, 1, m_1, \ldots, m_t)$ of Q'' (where there are s ones) equals δ, the minimal imaginary root.

To prove this, first let us consider a general stratum τ of $\mathfrak{M}_\lambda(\alpha, \theta)$,

$$\tau = \left(n_1, \beta^{(1)}; \ldots; n_s, \beta^{(s)}; m_1, \gamma^{(1)}; \ldots; m_t, \gamma^{(t)}\right), \quad \beta^{(i)} \text{ are imaginary, and } \gamma^{(i)} \text{ are real.} \tag{16}$$

Since there is only one θ-stable representation of dimension equal to each real root in $\Sigma_{\lambda,\theta}$, it follows that the $\gamma^{(i)}$ are all distinct.

Let us now set $\alpha^{(i)} := \beta^{(i)}$ for $1 \leq i \leq s$ and $\alpha^{(i)} = \gamma^{(i-s)}$ for $s+1 \leq i \leq s+t$. Let $k_i := n_i$ for $1 \leq i \leq s$ and $k_i = m_{i-s}$ for $s+1 \leq i \leq s+t$; let $\mathbf{k} = (k_1, \ldots, k_{s+t})$. By Theorem 3.3, at a point of this stratum, $\mathfrak{M}_\lambda(\alpha, \theta)$ is étale-equivalent to $\mathfrak{M}_{Q'}(0, \mathbf{k})_0$, where the notation means we use the quiver Q' instead of Q. Recall that $\overline{Q}'$ is the quiver with $s+t$ vertices, $2p(\alpha^{(i)})$ loops at the ith vertex and $-(\alpha^{(i)}, \alpha^{(j)})$ arrows between i and j.

Note that Q'' is obtained from Q' by discarding all loops at vertices. We will prove that, in the case that the stratum has codimension two, $\mathfrak{M}_{Q''}(0, \mathbf{k})_0$ étale-locally describes a transverse slice to the stratum.

Lemma 7.1. *Suppose that τ is as in* (16) *and moreover $n_i = 1$ for all i. Then at every point of the stratum there is an étale-local transverse slice isomorphic to a neighborhood of zero in $\mathfrak{M}_{Q''}(0, \mathbf{k})$.*

Proof. A neighborhood of a point of the stratum is étale-equivalent to a neighborhood of zero in $\mathfrak{M}_{Q''}(0, \mathbf{k})$. Inside the latter, the stratum containing zero consists of the representations which are a direct sum of simple representations, one at each vertex. At the vertices $1, \ldots, s$, this representation has dimension one; at the other vertices there are no loops and hence the simple representations are the standard ones. The stratum has dimension $2\sum_{i=1}^{s} p(\beta^{(i)})$, where $p(\beta^{(i)})$ equals the number of loops at the vertex i. A transverse slice is thus given by the representations which assign zero to all of the loops, which obviously identifies with $\mathfrak{M}_{Q''}(0, \mathbf{k})$. □

Lemma 7.2. *Suppose that τ has codimension two. Then $n_i = 1$ for all i. Moreover, the anisotropic $\beta^{(i)}$ are pairwise distinct.*

Proof. The codimension two condition can be written as:

$$1 = p(\alpha) - \sum_i p\left(\beta^{(i)}\right) - \sum_i p\left(\gamma^{(i)}\right) = p(\alpha) - \sum_i p\left(\beta^{(i)}\right), \tag{17}$$

since $\dim \mathfrak{M}_\lambda(\alpha, \theta)_\tau = 2\sum_i p\left(\beta^{(i)}\right) + 2\sum_i p\left(\gamma^{(i)}\right)$. Note that the $n_i\beta^{(i)}$ are themselves imaginary roots. Let $I_{\text{ani}} \subseteq I := \{1, \ldots, s\}$ be the set of indices such that $\beta^{(i)}$ is anisotropic, and let $I_{\text{iso}} := I \setminus I_{\text{ani}}$; so $p(\beta^{(i)}) = 1$ if and only if $i \in I_{\text{iso}}$. Note the identity

$$p(m\alpha) = m^2(p(\alpha) - 1) + 1.$$

Since $\alpha \in \Sigma_{\lambda,\theta}$,

$$p(\alpha) \geq \sum_{i \in I_{\mathrm{ani}}} p\left(n_i \beta^{(i)}\right) + \sum_{i \in I_{\mathrm{iso}}} n_i p\left(\beta^{(i)}\right) + \sum_i p\left(\gamma^{(i)}\right) = \sum_{i \in I_{\mathrm{ani}}} (n_i^2 p\left(\beta^{(i)}\right) + (1 - n_i^2)) + \sum_{i \in I_{\mathrm{iso}}} n_i,$$

with equality holding only if $s = 1$ and $\beta^{(1)}$ is anisotropic. Therefore, the RHS of (17) is greater than or equal to

$$\sum_{i \in I_{\mathrm{ani}}} (n_i^2 - 1)(p\left(\beta^{(i)}\right) - 1) + \sum_{i \in I_{\mathrm{iso}}} (n_i - 1),$$

again with equality only if $s = 1$ and $\beta^{(1)}$ is anisotropic. Therefore, if $n_i > 1$ for any i, then the RHS of (17) is strictly greater than one, a contradiction.

To see that the anisotropic $\beta^{(i)}$ are all distinct, suppose not. Group the ones that are not distinct together: let I' be the index set so that $\beta^{(i)}, i \in I$ gives all of the distinct roots once each, and let $\ell_i := |\{j : \beta_j = \beta'_i\}|$. Then, we obtain the inequality

$$p(\alpha) - \sum_i p\left(\beta^{(i)}\right) \geq p(\alpha) - \sum_{i \in I'} \ell_i p\left(\beta^{(i)}\right) > \sum_{i \in I} (\ell_i^2 - \ell_i)(p\left(\beta^{(i)}\right) - 1),$$

which is greater than one if any $\ell_i > 1$ with $i \in I_{\mathrm{ani}}$. Thus $n_i = 1$ for all i and the anisotropic $\beta^{(i)}$ are pairwise distinct. $\square$

We can now proceed with the proof of the theorem:

Proof of Theorem 1.18. Consider a general stratum τ as in (16). By Lemmas 7.1 and 7.2, we know that τ has codimension two if and only if $n_i = 1$ for all i and $\dim \mathfrak{M}_{Q''}(0, \mathbf{k}) = 2$. The latter is certainly true if τ is given by an isotropic decomposition. Moreover, in this case $\mathfrak{M}_{Q''}(0, \mathbf{k})$ is a du Val singularity of type given by the quiver Q''.

It remains only to show that, if τ has codimension two, then Q'' is affine Dynkin (ADE), with $\mathbf{k}$ the minimal imaginary root. Consider the canonical decomposition of $\mathbf{k}$, say $\mathbf{k} = \mathbf{k}^{(1)} + \cdots + \mathbf{k}^{(r)}$. Then $\mathfrak{M}_{Q''}(0, \mathbf{k}) \cong \prod_{i=1}^r \mathfrak{M}_{Q''}(0, \mathbf{k}^{(i)})$. The dimension of the latter is $2 \sum_{i=1}^r p(\mathbf{k}^{(i)})$. Hence exactly one of the $\mathbf{k}^{(i)}$ is isotropic, i.e., $p(\mathbf{k}^{(i_0)}) = 1$ for some i_0, and the others are real, i.e., $p(\mathbf{k}^{(i)}) = 0$ for $i \neq i_0$. Let $\mathbf{k}' := \mathbf{k}^{(i_0)}$. Since $\mathbf{k}' \in \Sigma_{0,0}(Q'')$, it follows that it is the minimal imaginary root of some affine Dynkin subquiver of Q'' (by the argument of the proof of [13, Proposition 1.2.(2)]).

We claim that $\mathbf{k} = \mathbf{k}'$. Given this, since every component of $\mathbf{k}$ is nonzero, Q'' is indeed affine Dynkin, which completes the proof.

To prove the claim, write $\mathbf{k}' = (k'_1, \ldots, k'_{s+t})$ with $k'_i \leq k_i$ for all i. Let $\alpha' := \sum_{i=1}^{s+t} k'_i \alpha^{(i)}$. Then Lemmas 7.1 and 7.2 applied to α' show also that the stratum τ' corresponding to $\mathbf{k}'$ in $\mathfrak{M}_\lambda(\alpha', \theta)$ has codimension two. That is:

$$2 \sum_{i : k'_i \neq 0} p(\alpha^{(i)}) = \dim \mathfrak{M}_\lambda(\alpha', \theta) - 2. \tag{18}$$

By Lemma 7.3 below, the RHS of (18) is at most $2p(\alpha')-2$. Now adding $\sum_{i:k'_i=0} p(\alpha^{(i)})$ to both sides, we obtain:

$$\sum_i p(\alpha^{(i)}) \leq p(\alpha') + \sum_{i:k'_i=0} p(\alpha^{(i)}) - 1. \tag{19}$$

The LHS of (19) equals $p(\alpha)-1$ by assumption. Therefore we obtain:

$$p(\alpha) \leq p(\alpha') + \sum_{i:k'_i=0} p(\alpha^{(i)}). \tag{20}$$

Now, replace α' and each of the $\alpha^{(i)}$ in the RHS of (20) by their canonical decompositions and let $\eta_1, \ldots, \eta_q$ be the resulting elements of $\Sigma_{\lambda,\theta}$ with multiplicity. By Lemma 7.3 again, we obtain that $p(\alpha) \leq \sum_{i=1}^q p(\eta_i)$. Since $\alpha \in \Sigma_{\lambda,\theta}$, this can only happen if $q=1$, i.e., $\alpha = \alpha' = \eta_1$. This is true if and only if $\mathbf{k} = \mathbf{k}'$. ☐

Lemma 7.3. *Suppose $\alpha \in \mathbb{N}R^+_{\lambda,\theta}$ has canonical decomposition $\alpha = \sum_i n_i \sigma^{(i)}$ with respect to λ and θ. Then $p(\alpha) \leq \sum_i n_i p(\sigma^{(i)})$.*

Proof. Let λ' be such that $R^+_{\lambda'} = R^+_{\lambda,\theta}$. As $\alpha \in \mathbb{N}R^+_{\lambda,\theta} = \mathbb{N}R^+_{\lambda'}$, we know that $\mu_\alpha^{-1}(\lambda')$ is nonempty. The latter is a fiber of a map $\bigoplus_{a\in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{t(a)}}, \mathbb{C}^{\alpha_{h(a)}}) \to \mathfrak{pg}(\alpha)$, where $\mathfrak{pg}(\alpha)$ is the Lie algebra of $PG(\alpha)$. All of the irreducible components of the latter must have dimension at least $\sum_{a\in\overline{Q_1}} \alpha_{t(a)}\alpha_{h(a)} - \sum_{i\in Q_0} \alpha_i^2 + 1 = \alpha\cdot\alpha - 2\langle\alpha,\alpha\rangle + 1 = \alpha\cdot\alpha + 2p(\alpha) - 1$.

On the other hand, by [12, Theorem 4.4], $\dim \mu_\alpha^{-1}(\lambda') = \alpha\cdot\alpha - \langle\alpha,\alpha\rangle + m = \alpha\cdot\alpha + p(\alpha) + (m-1)$ where m is the maximum value of $\sum_i p(\alpha^{(i)})$ with $\alpha^{(i)} \in R^+_{\lambda'}$ and $\alpha = \sum \alpha^{(i)}$; as remarked at the top of page 3 in [13], we have $m = \sum_i n_i p(\sigma^{(i)})$ (it is a direct consequence of [13, Theorem 1.1] which we discussed before Theorem 3.17).[2] We conclude that $\alpha\cdot\alpha + p(\alpha) + (m-1) \geq \alpha\cdot\alpha + 2p(\alpha) - 1$. Therefore, $m \geq p(\alpha)$, as desired. ☐

Remark 7.4. The lemma can be strengthened to prove: for any decomposition $\alpha = \sum_j \alpha^{(j)}$ with $\alpha^{(j)} \in \mathbb{N}R^+_{\lambda,\theta}$, we have $\sum_j p(\alpha^{(j)}) \leq \sum_i n_i p(\sigma^{(i)})$. This generalizes an observation on [13, p. 3] (dealing with the case where the $\alpha^{(j)}$ are roots). To prove this, for arbitrary $\alpha^{(j)}$, we can apply the lemma to each of the $\alpha^{(j)}$, and then we get that $\sum_j p(\alpha^{(j)}) \leq \sum_j p(\beta^{(j)})$ for some roots $\beta^{(j)} \in R^+_{\lambda,\theta}$ with $\alpha = \beta^{(j)}$; then we are back in the case of roots so that $\sum_j n_i p(\sigma^{(i)}) \geq \sum_j p(\beta^{(j)})$.

Remark 7.5. The arguments of [12, 13] can be generalized to the context of the pair (λ,θ), which as we pointed out in §2.3 would eliminate the need of picking a λ' as in the proof of the lemma above.

[2]Another interpretation of these facts is that $\mu_\alpha^{-1}(\lambda')$ has some irreducible component of maximum dimension whose generic element is semisimple with the canonical decomposition. The same fact can be deduced for $\mu_\alpha^{-1}(\lambda)^\theta$, replacing semisimple by canonically polystable.

8. Character varieties

Recall from section 1.5 that Σ is a compact Riemannian surface of genus $g > 0$ and π is its fundamental group. We have defined the character varieties

$$\mathcal{Y}(g,n) = \mathrm{Hom}(\pi, \mathrm{SL}(n,\mathbb{C}))/\!\!/\mathrm{SL}(n,\mathbb{C}), \quad \mathcal{X}(g,n) = \mathrm{Hom}(\pi, \mathrm{GL}(n,\mathbb{C}))/\!\!/\mathrm{GL}(n,\mathbb{C}).$$

These are affine varieties. Except in the last subsection, we will only consider $\mathcal{X}(g,n)$. Then, in section 8.6 we deduce the corresponding results for $\mathcal{Y}(g,n)$. We begin by recalling the basic properties of the affine varieties $\mathrm{Hom}(\pi, \mathrm{GL})$ and $\mathcal{X}(g,n)$. Based on results of Li [39], as explained in Theorem 2.1 of [21],

Theorem 8.1. *(1) Both* $\mathrm{Hom}(\pi, \mathrm{GL})$ *and* $\mathcal{X}(g,n)$ *are reduced and irreducible.*

(2) $\mathrm{Hom}(\pi, \mathrm{GL})$ *is a complete intersection in* GL^{2g}.

(3) The generic points of $\mathrm{Hom}(\pi, \mathrm{GL})$ *and* $\mathcal{X}(g,n)$ *correspond to irreducible representations of the fundamental group* π.

As shown originally by Goldman [25], the varieties $\mathcal{X}(g,n)$ and $\mathcal{Y}(g,n)$ have a natural Poisson structure. This Poisson structure becomes clear in the realization of these spaces as quasi-Hamiltonian reductions; see [1], where it is shown that the symplectic structure defined by Goldman on the smooth locus of $\mathcal{X}(g,n)$ agrees with the Poisson structure of $\mathcal{X}(g,n)$ as a quasi-Hamiltonian reduction. In particular, if $C_{(1,n)}$ denotes the dense open subset of $\mathcal{X}(g,n)$ parameterizing simple representations of π, then it is shown in [1] that the Poisson structure on $C_{(1,n)}$ is non-degenerate. It will be useful for us to reinterpret the quasi-Hamiltonian reduction as a moduli space of semi-simple representations of the multiplicative preprojective algebra. Let Q be the quiver with a single vertex and g loops, labeled $a_1, \ldots, a_g$. Let a_i^* denote the loop dual to a_i in the doubled quiver $\overline{Q}$. Associated to Q is the multiplicative preprojective algebra $\Lambda(Q)$, as defined in [16]. Namely, $\mathbb{C}\overline{Q} \to \Lambda(Q)$ is the universal homomorphism such that each $1 + a_i a_i^*$ and $1 + a_i^* a_i$ is invertible and

$$\prod_{i=1}^{g} (1 + a_i a_i^*)(1 + a_i^* a_i)^{-1} = 1.$$

Here the product is ordered. Following [15], let $\Lambda(Q)'$ denote the universal localization of $\Lambda(Q)$, where each a_i is also required to be invertible. Let $(T^* \mathrm{Rep}(Q,n))^\circ$ denote the space of all n-dimensional representations (A_i, A_i^*) of $\mathbb{C}\overline{Q}$ such that $1 + A_i A_i^*, 1 + A_i^* A_i$ and A_i are invertible for all i. It is an open, $\mathrm{GL}(n,\mathbb{C})$-stable affine subset of $T^* \mathrm{Rep}(Q,n)$. The action of $\mathrm{GL}(n,\mathbb{C})$ on $(T^* \mathrm{Rep}(Q,n))^\circ$ is quasi-Hamiltonian, with multiplicative moment map

$$\Psi : \mathrm{Rep}(\Lambda(Q)', n) \to \mathrm{GL}, \quad (A_i, A_i^*) \mapsto \prod_{i=1}^{g} (1 + A_i A_i^*)(1 + A_i^* A_i)^{-1}.$$

As noted in Proposition 2 of [15], the category $\Lambda(Q)'$-mod of finite dimensional $\Lambda(Q)'$-modules is equivalent to π-mod, in such a way that we have a GL-equivariant identification

$$\Psi^{-1}(1) \xrightarrow{\sim} \mathrm{Hom}(\pi, \mathrm{GL}), \quad (A_i, A_i^*) \mapsto (A_i, B_i) = (A_i, A_i^{-1} + A_i^*).$$

Hence, we have an identification of Poisson varieties

$$\Psi^{-1}(1)/\!\!/\mathrm{GL} = \mathcal{X}(g,n).$$

See [56] for further details.

8.1. Symplectic singularities. The space $\mathcal{X}(g,n)$ has a stratification by representation type, which is also the stratification by stabilizer type; see [43, Theorem 5.4]. As in section 6.1, a *weighted partition* ν of n is a sequence $(\ell_1,\nu_1;\dots;\ell_k,\nu_k)$, where $\nu_1 \ge \nu_2 \ge \cdots$ and $\sum_{i=1}^k \ell_i\nu_i = n$.

Lemma 8.2. *Assume $n, g > 1$.*

(1) The strata C_ν of $\mathcal{X}(g,n)$ are labeled by weighted partitions of n such that

$$\dim C_\nu = 2\left(k + (g-1)\sum_{i=1}^k \nu_i^2\right).$$

(2) If $(g,n) \neq (2,2)$, then $\dim \mathcal{X}(g,n) - \dim C_\nu \ge 4$ for all $\nu \neq (1,n)$.

(3) If $(g,n) \neq (2,2)$ and $\nu \neq (1,n)$, then $\dim \mathcal{X}(g,n) - \dim C_\nu \ge 8$ unless either

- (i) *$(g,n) = (2,3)$ and $\nu = (1,2;1,1)$; or*
- (ii) *$(g,n) = (3,2)$ and $\nu = (1,1;1,1)$.*

Proof. By Theorem 8.1, the set of points $C_{(1,n)}$ in $\mathcal{X}(g,n)$ parameterizing irreducible representations of π is a dense open subset contained in the smooth locus. Therefore $\dim C_{(1,n)} = 2(1+n^2(g-1))$. An arbitrary semi-simple representation of π of dimension n has the form $x = x_1^{\oplus \ell_1} \oplus \cdots \oplus x_k^{\oplus \ell_k}$, where the x_i are pairwise non-isomorphic irreducible π-modules of dimension ν_i and $n = \sum_{i=1}^k \ell_i\nu_i$. Thus, the representation type strata correspond to weighted partitions of n. Let C_ν denote the locally closed subvariety of all such representations. If we write the multiset $\{\{\nu_1,\dots,\nu_k\}\}$ as $\{\{m_1\cdot\nu_1,\dots,m_r\cdot\nu_r\}\}$, with $\nu_i \neq \nu_j$, then

$$C_\nu \simeq S^{m_1,\circ}C_{(1,\nu_1)} \times \cdots \times S^{m_r,\circ}C_{(1,\nu_r)},$$

where $S^{n,\circ}X$ is the open subset of S^nX consisting of n pairwise distinct points. Thus,

$$\dim C_\nu = \sum_{i=1}^r 2(1+\nu_i^2(g-1))m_i = 2\left(k + (g-1)\sum_{i=1}^k \nu_i^2\right).$$

The second and third claims are identical to Lemma 6.1, parts (3) and (4). □

Proposition 8.3. *The variety $\mathcal{X}(g,n)$ is normal.*

Proof. The case $g = 1$ follows from Proposition 8.13 below. The case $n = 1$ is trivial since $\mathcal{X}(1,g) \cong (\mathbb{C}^\times)^{2g}$.

Assume $g, n > 1$. We consider the case $(g,n) = (2,2)$ separately below. Notice that since $\mathrm{Hom}(\pi,\mathrm{GL})$ is a complete intersection, it is Cohen-Macaulay. Thus, it satisfies Serre's condition (S_2). Let $C_{(1,n)} \subset \mathcal{X}(g,n)$ be the open subset of points corresponding to irreducible π-modules. It is contained in the smooth locus of $\mathcal{X}(g,n)$, and hence is normal. Let Z denote its complement. By

Lemma 8.2 (2), Z has codimension at least four in $\mathcal{X}(g,n)$ when $(g,n) \neq (2,2)$. By [16, Corollary 7.3], if $\xi : \mathrm{Hom}(\pi, \mathrm{GL}) \to \mathcal{X}(g,n)$ is the quotient map, then

$$\dim \mathrm{Hom}(\pi, \mathrm{GL}) - \dim \xi^{-1}(Z) \geq \frac{1}{2} \min_{\nu \neq (1,n)} \left(\dim \mathcal{X}(g,n) - \dim C_\nu\right). \tag{21}$$

Thus, Z has codimension at least two in $\mathrm{Hom}(\pi, \mathrm{GL})$, implying that (R_1) holds too. We deduce that $\mathrm{Hom}(\pi, \mathrm{GL})$, and hence $\mathcal{X}(g,n)$ too, is normal.

Finally, we consider the case where $(g,n) = (2,2)$. As noted in Theorem 8.1, $\mathrm{Hom}(\pi, \mathrm{GL})$ is a complete intersection and hence Cohen-Macaulay. Thus, it satisfies (S_2). We claim that the locus $\mathrm{Hom}(\pi, \mathrm{GL})_{\mathrm{free}}$ on which PGL acts freely has complement having codimension at least two. Since this open set is contained in the smooth locus, this will imply that $\mathrm{Hom}(\pi, \mathrm{GL})$ satisfies (R_1). Therefore, by Serre's criterion, $\mathrm{Hom}(\pi, \mathrm{GL})$ will be normal. It will then follow that $\mathcal{X}(2,2) = \mathrm{Hom}(\pi, \mathrm{GL})/\!\!/\mathrm{GL}$ is normal too.

It remains only to prove the claim. Inside $\mathcal{X}(2,2)$, there are three strata: the open stratum consisting of the image of simple representations in $\mathrm{Hom}(\pi, \mathrm{GL})$, the codimension two stratum corresponding to the semisimple representations of the form $V \oplus W$ for nonisomorphic one-dimensional representations V, W, and the four-dimensional (codimension six) stratum corresponding to semisimple representations isomorphic to $V^{\oplus 2}$ for some V. The preimage of the open stratum is smooth in $\mathrm{Hom}(\pi, \mathrm{GL})$, since it consists entirely of simple representations. By inequality (21), the preimage of the codimension-six stratum has codimension at least three. This stratum is therefore irrelevant for the claim. The preimage of the codimension-two stratum has codimension at least one by (21), or because $\mathrm{Hom}(\pi, \mathrm{GL})$ is irreducible (Theorem 8.1.1)). So we only have to show that this stratum, call it Z, has open dense intersection with $\mathrm{Hom}(\pi, \mathrm{GL})_{\mathrm{free}}$.

Let $Z^{\mathrm{ss}} \subseteq Z$ be the semisimple locus, consisting of the representations which are decomposable into nonisomorphic one-dimensional representations. As observed in the proof of Proposition 6.5, the codimension of Z^{ss} in $\mathrm{Hom}(\pi, \mathrm{GL})_{\mathrm{free}}$ must be at least (in fact, must exceed) the codimension, two, of its image $\xi(Z^{\mathrm{ss}}) = \xi(Z)$. Explicitly, for every $z \in Z^{\mathrm{ss}}$, the endomorphism space $\mathrm{End}_\pi(z)$ has dimension two. So the PGL-orbit of z has dimension at most $\dim \mathrm{PGL} - 1$. We obtain that $\mathrm{codim}_{\mathrm{Hom}(\pi,\mathrm{GL})} Z^{\mathrm{ss}} \geq 1 + \mathrm{codim}_{\mathcal{X}(2,2)} \xi(Z^{\mathrm{ss}}) = 3$.

Thus we only have to show that every non-semisimple point of Z has a free PGL-orbit. But these points consist of extensions of two non-isomorphic simple representations. So their endomorphism algebra is indeed one-dimensional (the same argument was used in the proof of Proposition 6.5). □

Proposition 8.4. *The Poisson variety $\mathcal{X}(g,n)$ is a symplectic singularity.*

Proof. When $g = 1$ the claim follows from Proposition 8.13. The case $(g,n) = (2,2)$ is dealt with in Corollary 8.12 below. The case $n = 1$ is trivial.

We assume $g, n > 1$ and $(g,n) \neq (2,2)$. We have shown in Proposition 8.3 that the irreducible variety $\mathcal{X}(g,n)$ is normal. By Theorem 8.1, the Poisson structure on the dense open subset $C_{(1,n)}$ of $\mathcal{X}(g,n)$ is non-degenerate. This implies that the Poisson structure on the whole of the smooth locus is non-degenerate since the complement to $C_{(1,n)}$ in $\mathfrak{X}(g,n)$ has codimension at least four. Therefore, since the singular locus of $\mathcal{X}(g,n)$ must also have codimension at least four, it follows from Flenner's Theorem [22] that $\mathcal{X}(g,n)$ has symplectic singularities. $\square$

As for quiver varieties, the symplectic leaves of the character variety $\mathcal{X}(g,n)$ are precisely the stabilizer type strata. Since this result is not needed elsewhere, we only sketch the proof.

Proposition 8.5. *The symplectic leaves of* $\mathcal{X}(g,n)$ *are the stabilizer type strata* C_ν.

Proof. Since $\mathcal{X}(g,n)$ has symplectic singularities, it has only finitely many leaves. The stratification by stabilizer type is a finite stratification by smooth, connected locally closed subvarieties (which can be deduced from the corresponding statement, Proposition 3.6, for quiver varieties by using Theorem 8.6 below). Therefore it suffices to show that the Hamiltonian vector fields on $\mathcal{X}(g,n)$ are all tangent to the strata. This follows from Lemma 3.14, suitably adapted. $\square$

The same statement can be shown to hold for $\mathcal{Y}(g,n)$ using Lemma 8.17 below.

8.2. Passage to the normal cone. In order to study the singularities of $\mathcal{X}(g,n)$, we describe the normal cone to a closed GL-orbit in $\mathrm{Hom}(\pi,\mathrm{GL})$. Let ϕ be a point whose GL-orbit is closed, and denote by V the corresponding n-dimensional representation of π. Composing ϕ with the adjoint action of GL on $\mathfrak{gl}(V)$, the space $\mathfrak{gl}(V)$ is a π-module. Since Σ is a $K(\pi,1)$-space, we have natural identifications

$$\mathrm{Ext}^i_\pi(V,V)=\mathrm{Ext}^i_\pi(\mathbb{C},\mathfrak{gl}(V))=H^i(\pi,\mathfrak{gl}(V))=H^i(\Sigma,\mathcal{V}\otimes\mathcal{V}^\vee),$$

where $\mathcal{V}$ is the local system on Σ corresponding to the π-module V; see page 59 and Proposition 2.2 of [9]. Cup product in cohomology, followed by the Lie bracket $[-,-]:\mathfrak{gl}(V)\times\mathfrak{gl}(V)\to\mathfrak{gl}(V)$, defines the *Kuranishi* map

$$\kappa:\mathrm{Ext}^1_\pi(V,V)=\mathrm{Ext}^1_\pi(\mathbb{C},\mathfrak{gl}(V))\longrightarrow\mathrm{Ext}^2_\pi(\mathbb{C},\mathfrak{gl}(V)\otimes\mathfrak{gl}(V))\longrightarrow\mathrm{Ext}^2_\pi(\mathbb{C},\mathfrak{gl}(V))=\mathrm{Ext}^2_\pi(V,V),$$

given by $\varphi\mapsto[\varphi\cup\varphi]$. As shown in [26, Section 4], if $C_V(\pi)$ denotes the tangent cone to V in $\mathrm{Hom}(\pi,\mathrm{GL})$, and $N_V(\pi)$ its image in $T_V\,\mathrm{Hom}(\pi,\mathrm{GL})/T_V\,\mathrm{GL}\cdot V$, then there is a $\mathrm{Stab}_{\mathrm{GL}}(V)$-equivariant isomorphism

$$N_V(\pi)\simeq\kappa^{-1}(0)\subset\mathrm{Ext}^1_\pi(V,V).$$

As explained in [25], the space $\mathrm{Ext}^1_\pi(V,V)$ has a natural symplectic structure, such that the action of $\mathrm{Stab}_{\mathrm{GL}}(V)$ on $\mathrm{Ext}^1_\pi(V,V)$ is Hamiltonian. Decompose the semi-simple representation V as $\bigoplus_{i=1}^k V_i\otimes W_i$, where the V_i are pairwise non-isomorphic simple π-modules. Let $\overline{Q}$ be the quiver with k vertices and $\dim\mathrm{Ext}^1_\pi(V_i,V_j)$ arrows between vertex i and j. Let α be the dimension vector for $\overline{Q}$ given by $\alpha_i=\dim W_i$.

Theorem 8.6. *(1) There is a natural identification* $\mathrm{Stab}_{\mathrm{GL}}(V)=G(\alpha)$.

(2) The quiver $\overline{Q}$ *is the double of some quiver* Q.

(3) *We have a $G(\alpha)$-equivariant identification $\mathrm{Ext}^1_\pi(V,V) \xrightarrow{\sim} \mathrm{Rep}(\overline{Q},\alpha)$ of symplectic vector spaces and a $G(\alpha)$-equivariant identification $\mathrm{Ext}^2_\pi(V,V) \xrightarrow{\sim} \mathfrak{g}(\alpha)$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ext}^1_\pi(V,V) & \xrightarrow{\sim} & \mathrm{Rep}(\overline{Q},\alpha) \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \mu} \\ \mathrm{Ext}^2_\pi(V,V) & \xrightarrow{\sim} & \mathfrak{g}(\alpha) \end{array}$$

Proof. The first claim follows directly from the decomposition $\bigoplus_{i=1}^k V_i \otimes W_i$, since $\mathrm{Stab}_{\mathrm{GL}}(V)$ only acts on the W-tensorand.

If $\mathcal{V}_i$ denotes the irreducible local system on Σ corresponding to V_i, then we have natural identifications $\mathrm{Ext}^1_\pi(V_i,V_j) = H^1(\Sigma, \mathcal{V}_i \otimes \mathcal{V}_j^\vee)$ and $\mathrm{Ext}^2_\pi(V_i,V_j) = H^2(\Sigma, \mathcal{V}_i \otimes \mathcal{V}_j^\vee)$ imply by Poincaré-Verdier duality (see [18, Corollary 3.3.12]) that

- $\mathrm{Ext}^2_\pi(V_i,V_i) \simeq \mathbb{C}$ and $\mathrm{Ext}^2_\pi(V_i,V_j) = 0$ for $i \neq j$.
- The cup product defines a non-degenerate pairing

$$\langle -,- \rangle : \mathrm{Ext}^1_\pi(V_i,V_j) \times \mathrm{Ext}^1_\pi(V_j,V_i) \to \mathbb{C}.$$

The existence of the non-degenerate pairing implies that $\dim \mathrm{Ext}^1_\pi(V_i,V_j) = \dim \mathrm{Ext}^1_\pi(V_j,V_i)$ when $i \neq j$. Moreover, each $\mathrm{Ext}^1_\pi(V_i,V_i)$ is a symplectic vector space [25], and hence $\dim \mathrm{Ext}^1_\pi(V_i,V_i)$ is even. Thus, $\overline{Q}$ is the double of some quiver Q, confirming (2).

Finally, we have $G(\alpha)$-equivariant identifications

$$\mathrm{Ext}^1_\pi(V,V) = \bigoplus_{i,j} \mathrm{Ext}^1_\pi(V_i,V_j) \otimes \mathrm{Hom}(W_i,W_j) = \mathrm{Rep}(\overline{Q},\alpha)$$

and

$$\mathrm{Ext}^2_\pi(V,V) = \bigoplus_{i,j} \mathrm{Ext}^2_\pi(V_i,V_j) \otimes \mathrm{Hom}(W_i,W_j) = \bigoplus_{i=1}^k \mathrm{End}(W_i),$$

since $\mathrm{Ext}^2_\pi(V_i,V_j) = 0$ for $i \neq j$. Now view the quadratic map $\kappa : \mathrm{Ext}^1_\pi(V,V) \to \mathrm{Ext}^2_\pi(V,V)$ as a linear one, $\mathrm{Ext}^1_\pi(V,V) \otimes \mathrm{Ext}^1_\pi(V,V) \to \mathrm{Ext}^2_\pi(V,V)$. This map can be written as $a \otimes b \mapsto a \circ b - b \circ a$, where the map $\circ$ is the usual composition,

$$\circ : \mathrm{Ext}^1_\pi(V_i \otimes W_i, V_j \otimes W_j) \times \mathrm{Ext}^1_\pi(V_j \otimes W_j, V_k \otimes W_k) \to \mathrm{Ext}^2_\pi(V_i,V_k) \otimes \mathrm{Hom}(W_i,W_k),$$

which is only nonzero when $i = k$. For $i = k$, if we write $\mathrm{Ext}^1_\pi(V_i \otimes W_i, V_j \otimes W_j) = \mathrm{Ext}^1_\pi(V_i,V_j) \otimes \mathrm{Hom}(W_i,W_j)$, the above map becomes the tensor product of the symplectic pairing between $\mathrm{Ext}^1_\pi(V_i,V_j)$ and $\mathrm{Ext}^1_\pi(V_j,V_i)$ and the composition on Hom spaces. Thus, linearly extending to V, we obtain the usual moment map μ, as required. □

Luna's slice theorem implies:

Corollary 8.7. *The tangent cone to $[V] \in \mathfrak{X}(g,n)$ is isomorphic to $\mathfrak{M}_0(\alpha,0)$ for the quiver Q and dimension vector α described above.*

Remark 8.8. In fact, by [5, Theorem 6.3, Theorem 6.6],[3] the whole formal neighborhood of $[V] \in \mathfrak{X}(g,n)$ is isomorphic to the formal neighborhood of 0 in $\mathfrak{M}_0(\alpha, 0)$, since the group algebra $\mathbb{C}[\pi]$ is a two-dimensional Calabi–Yau algebra. The argument given there also begins the same way as above, but we included details for the benefit of the reader.

Lemma 8.9. *The singular locus of $\mathfrak{X}(g,n)$ is the closed subset consisting of non-simple representations. Its irreducible components are labeled by integers $1 \le n' \le n/2$.*

Proof. The proof is identical to the proof of [33, Proposition 6.1]. Theorem 8.6 implies that if the point $x \in \mathfrak{X}(g,n)$ corresponds to a simple representation V, then x is smooth. For each $1 \le n' \le n/2$, let $\varphi(n') : \mathfrak{X}(n', g) \times \mathfrak{X}(n-n', g) \to \mathfrak{X}(g,n)$ denote the map $([V_1],[V_2]) \mapsto [V_1 \oplus V_2]$. It is a finite morphism. Clearly, every semi-simple, but not simple, π-module of dimension n lies in the image of some $\varphi(n')$. Also, $\mathrm{Im}\ \varphi(n') \cap \mathrm{Im}\ \varphi(n'')$ is a proper subset of $\mathrm{Im}\ \varphi(n')$ for all $n' \neq n''$ since a generic point of $\mathrm{Im}\ \varphi(n')$ is the direct sum of exactly two simple modules. Therefore the $\mathrm{Im}\ \varphi(n')$ are precisely the irreducible components of the complement to the open subset of simple representations. Thus, it suffices to show that the generic point of $\mathrm{Im}\ \varphi(n')$ is singular in $\mathfrak{X}(g,n)$. Such a generic point is $[V_1 \oplus V_2]$, where V_1 and V_2 are simple π-modules of dimension n' and $n-n'$ respectively.

It suffices to show that the tangent cone at this point is singular. By Corollary 8.7, the tangent cone is isomorphic to $0 \in \mathfrak{M}_0(\alpha, 0)$ for some quiver Q and dimension vector α. In this case, we get the quiver Q with $\frac{1}{2} \dim \mathrm{Ext}^1_\pi(V_1, V_1)$ loops at vertex 1, $\frac{1}{2} \dim \mathrm{Ext}^1_\pi(V_2, V_2)$ loops at vertex 2 and $\dim \mathrm{Ext}^1_\pi(V_1, V_2)$ arrows from vertex 1 to vertex 2. The dimension vector is $\alpha = (1,1)$. The space $\mathfrak{M}_0(\alpha, 0)$ is singular if and only if $\dim \mathrm{Ext}^1_\pi(V_1, V_2) > 1$ (removing the loops, which do not contribute to the singularities, $\mathfrak{M}_0(\alpha, 0)$ is isomorphic to the closure of the minimal nilpotent orbit in $\mathfrak{gl}_n$, where $n = \dim \mathrm{Ext}^1_\pi(V_1, V_2)$). Since V_1 and V_2 are not isomorphic, [16, Theorem 1.6] implies that

$$\dim \mathrm{Ext}^1_\pi(V_1, V_2) = (2g-2)n'(n-n') > 1$$

as required. ☐

Remark 8.10. We note that [16, Theorem 1.6] allows one to easily compute the Euler characteristic of local systems on compact Riemann surfaces. For instance, it implies that if $\mathcal{L}$ is an irreducible local system on Σ then

$$\chi(\mathcal{L}) = (2g-2)\ \mathrm{rk}(\mathcal{L}).$$

Presumably, this is well-known to experts.

8.3. The case $(g,n) = (2,2)$.

The case $(g,n) = (2,2)$ can be thought of as a "local model" for the moduli space M_{2v} of semistable shaves with Mukai vector $2v$ on an abelian or $K3$ surface, where v is primitive, such that $\langle v, v \rangle = 2$. Therefore we are able to apply directly the results of Lehn

[3] Thanks to Raf Bocklandt for pointing out these results.

and Sorger [38] in this case. Lemma 8.2 (1) says that $\mathcal{X}(2,2)$ has three strata, $C_{(1,2)}$ consisting of simple representations E, $C_{(1,1;1,1)}$ consisting of semi-simple representations $E = F_1 \oplus F_2$, where F_1 and F_2 are a pair of non-isomorphic one-dimensional representations of π, and $C_{(2,1)}$ the stratum of semi-simple representations $E = F^{\oplus 2}$, where F is a one-dimensional representation. By Corollary 8.9, the singular locus of $\mathcal{X}(2,2)$ equals $\overline{C}_{(1,1;1,1)} = C_{(1,1;1,1)} \sqcup C_{(2,1)}$.

Theorem 8.11 (Lehn-Sorger, [38])**.** *The blowup $\sigma : \widetilde{\mathcal{X}}(2,2) \to \mathcal{X}(2,2)$ along the reduced ideal defining the singular locus of $\mathcal{X}(2,2)$ defines a semi-small resolution of singularities.*

Proof. We sketch the proof, based on the results in [38]. Fix a point $E \in C_{(1,1;1,1)}$ and $E' \in C_{(2,1)}$. Theorem 8.6 says that the tangent cone $C_E(\mathcal{X}(2,2))$ is isomorphic to $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ and the tangent cone $C_{E'}(\mathcal{X}(2,2))$ is isomorphic to $\mathbb{C}^4 \times \mathcal{N}$, where $\mathcal{N}$ is the orbit closure in $\mathfrak{sp}(4)$ defined in section 5.1. The proof of [38, Théorème 4.5] goes through word for word in this situation (one has to check that Propositions A.1 and A.2 of the appendix to *op. cit.* hold in this setting), and we deduce that there are isomorphisms of analytic germs

$$(\mathcal{X}(2,2), E) \simeq (\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2), 0), \quad (\mathcal{X}(2,2), E') \simeq (\mathbb{C}^4 \times \mathcal{N}, 0).$$

(The first isomorphism follows from [50, Lemma 1.3]). Clearly, blowing up $\mathbb{C}^8 \times (\mathbb{C}^2/\mathbb{Z}_2)$ along the singular locus gives a semi-small resolution of singularities. The key result [32, Remark 5.4], see also [38, Théorème 2.1], says that blowing up along the reduced ideal defining the singular locus in $\mathbb{C}^4 \times \mathcal{N}$ also produces a semi-small resolution of singularities. ☐

Corollary 8.12. *The blowup $\widetilde{\mathcal{X}}(2,2)$ of $\mathcal{X}(2,2)$ along the reduced ideal defining the singular locus of $\mathcal{X}(2,2)$ is a smooth symplectic variety and $\mathfrak{X}(2,2)$ has symplectic singularities.*

Proof. Let $\sigma : \widetilde{\mathcal{X}}(2,2) \to \mathcal{X}(2,2)$ denote the blowup map. The singularities of $\mathcal{X}(2,2)$ in a an analytic neighborhood of a point in $C_{(1,1;1,1)}$ are equivalent to an A_1 singularity. Therefore the pullback $\sigma^*\omega$ of the symplectic 2-form ω on the smooth locus of $\mathcal{X}(2,2)$ extends to a symplectic 2-form on $\sigma^{-1}(U)$, where U is the open set $C_{(1,2)} \cup C_{(1,1;1,1)}$. Since σ is semi-small, $\sigma^{-1}(C_{(2,1)})$ has codimension at least 3 in $\widetilde{\mathcal{X}}(2,2)$. Therefore, $\sigma^*\omega$ extends to a symplectic 2-form on the whole of $\widetilde{\mathcal{X}}(2,2)$. Since we have shown in Proposition 8.3 that $\mathcal{X}(2,2)$ is normal, Lemma 6.12 implies that $\mathcal{X}(2,2)$ has symplectic singularities. ☐

8.4. The genus one case. Let G be either GL or SL and $\mathbb{T}$ a maximal torus in G. The following is well-known. It can be deduced from the corresponding statement for the commuting variety in $\mathfrak{g} \times \mathfrak{g}$; see [23, Sections 2.7 and 2.8]

Proposition 8.13. *Fix $g = 1$. As symplectic singularities, the* G*-character variety of* Σ *is isomorphic to* $(\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$.

Unlike the case $g > 1$, it is not clear whether $\mathrm{Hom}(\pi, \mathrm{G})$ is reduced, but it is shown in [23] that the corresponding G-character variety is reduced. In the case $\mathrm{G} = \mathrm{GL}$, the Hilbert-Chow morphism defines a symplectic resolution $\pi : \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times) \to (\mathbb{T} \times \mathbb{T})/\mathfrak{S}_n$. Similarly, the the preimage $\mathrm{Hilb}^n_0(\mathbb{C}^\times \times \mathbb{C}^\times) \subset \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ of $\mathcal{Y}(n,1) \subset \mathcal{X}(n,1)$ under π defines a symplectic resolution of $\mathcal{Y}(n,1)$; for want of a better name, we call $\mathrm{Hilb}^n_0(\mathbb{C}^\times \times \mathbb{C}^\times)$ the *barycentric Hilbert scheme.* Notice that the case $n = 1$ is trivial since $\mathcal{X}(1,1) = \mathbb{C}^\times \times \mathbb{C}^\times$ with its standard symplectic structure.

8.5. Factoriality. We begin with the following analogue of Theorem 6.3:

Theorem 8.14. [16, Theorem 7.2, Corollary 7.3] *Consider a stratum Z in $\mathcal{X}(g,n)$ of representation type $(k_1, \beta^{(i_1)}; \dots; k_r, \beta^{(i_r)})$. Then for all $z \in Z$, the fibre $\xi^{-1}(z) \subseteq \mathrm{Hom}(\pi, \mathrm{GL})$ has dimension at most $\beta \cdot \beta - 1 + p(\beta) - \sum_t p(\beta^{(t)})$, so the dimension of $\xi^{-1}(Z)$ is at most $\beta \cdot \beta - 1 + p(\beta) + \sum_t p(\beta^{(t)})$.*

Recall that $\xi : \mathrm{Hom}(\pi, \mathrm{GL}) \to \mathcal{X}(g,n)$ is the quotient map. The action of GL on $\mathrm{Hom}(\pi, \mathrm{GL})$ factors through PGL. The open subset of $\mathrm{Hom}(\pi, \mathrm{GL})$ where PGL acts freely is denoted $\mathrm{Hom}(\pi, \mathrm{GL})_{\mathrm{free}}$.

Lemma 8.15. *Assume that $g, n > 1$ and $(g,n) \neq (2,2)$. The variety $\mathrm{Hom}(\pi, \mathrm{GL})$ is normal and factorial. Moreover, the complement to $\mathrm{Hom}(\pi, \mathrm{GL})_{\mathrm{free}}$ in $\mathrm{Hom}(\pi, \mathrm{GL})$ has codimension at least four.*

Proof. As noted previously, $\mathrm{Hom}(\pi, \mathrm{GL})$ is a complete intersection and hence Cohen-Macaulay. Thus, it satisfies (S_2). By a theorem of Grothendieck, [33, Theorem 3.12], in order to show that $\mathrm{Hom}(\pi, \mathrm{GL})$ is factorial, it suffices to check that it satisfies (R_3) too. But this follows from the proof of Proposition 6.5: the same arguments hold now substituting Lemma 8.2 for Lemma 6.1 and Theorem 8.14 for Theorem 6.3. □

Next, we prove the part of Theorem 1.20 dealing with $\mathcal{X}(g,n)$:

Proof of Theorem 1.20 for $\mathcal{X}(g,n)$. Let $\mathcal{X}(g,n)_s$ denote the dense open subset consisting of simple representations and $\mathrm{Hom}(\pi, \mathrm{GL})_s$ its preimage in $\mathrm{Hom}(\pi, \mathrm{GL})$. Then $\xi : \mathrm{Hom}(\pi, \mathrm{GL}) \to \mathcal{X}(g,n)_s$ is a principal PGL-bundle. Moreover, by Lemma 8.2 (2), the complement to $\mathcal{X}(g,n)_s$ has codimension at least 4 in $\mathcal{X}(g,n)$. Therefore we may apply the results of Theorem 6.7 to $\mathcal{X}(g,n)$.

The stratum C_ρ of type $\rho = (n,1)$ is contained in the closure of all other strata in $\mathcal{X}(g,n)$ (this can be proven by induction on n using the morphisms $\varphi(n')$ defined in the proof of Lemma 8.9). If $y \in T(x)$ is a lift in $\mathrm{Hom}(\pi, \mathrm{GL})$ of a point x of C_ρ then y corresponds to the representation $\mathbb{C}^{\oplus n}$, where $\mathbb{C}$ denotes here the trivial π-module. Therefore $\mathrm{PGL}_y = \mathrm{PGL}$ has no non-trivial characters. In particular, PGL_y will act trivially on L_y for any PGL-equivariant line bundle on $\mathrm{Hom}(\pi, \mathrm{GL})$. Hence, we deduce from Theorem 6.7 that $\mathcal{X}(g,n)$ is factorial at every point of C_ρ.

Now consider an arbitrary stratum C_τ in $\mathcal{X}(g,n)$. If $\mathcal{X}(g,n)$ is factorial at one point of the stratum then it will be factorial at every point in the stratum. On the other hand, the main result of [6] says that the subset of factorial points of $\mathcal{X}(g,n)$ is an open subset. Since this open subset is a union of strata and contains the unique closed stratum, it must be the whole of $\mathcal{X}(g,n)$. □

Arguing as in the proof of Theorem 6.14, Theorem 1.20 implies Corollary 1.21 for $\mathcal{X}(g,n)$.

Remark 8.16. A similar analysis can be done in order to classify which moduli spaces of semi-simple representations of an arbitrary multiplicative deformed preprojective algebra admit symplectic resolutions. Details will appear elsewhere.

8.6. The SL-character variety. Recall that $\mathcal{Y}(g,n)$ is the character variety associated to the compact Riemann surface Σ, of genus g, with values in $\mathrm{SL}(n,\mathbb{C})$. Let $T \simeq (\mathbb{C}^\times)^{2g}$ denote the $2g$-torus.

Lemma 8.17. *The character variety $\mathcal{X}(g,n)$ is an étale locally trivial fiber bundle over T with fiber $\mathcal{Y}(g,n)$.*

Proof. Let $\varrho : \mathrm{Hom}(\pi, \mathrm{GL}) \to T$ be the map sending (A_i, B_i) to $(\det(A_i), \det(B_i))$. This map is GL-equivariant, where the action on T is trivial. Moreover, it fits into a commutative diagram of GL-varieties

$$\begin{array}{ccc} \mathrm{Hom}(\pi,\mathrm{SL}) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathrm{Hom}(\pi,\mathrm{GL}) \\ & {\scriptstyle \mathrm{pr}} \searrow \quad \swarrow {\scriptstyle \varrho} & \\ & T & \end{array} \tag{22}$$

where $\mathbb{Z}_n^{2g}$ acts freely on T, and the map $\mathrm{Hom}(\pi, \mathrm{SL}) \times T \to \mathrm{Hom}(\pi, \mathrm{GL})$ sends $((A_i, B_i), (t_i, s_i))$ to $(t_i A_i, s_i B_i)$. Therefore it descends to a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}(g,n) \times_{\mathbb{Z}_n^{2g}} T & \xrightarrow{\sim} & \mathcal{X}(g,n) \\ & {\scriptstyle \mathrm{pr}} \searrow \quad \swarrow {\scriptstyle \varrho} & \\ & T & \end{array} \tag{23}$$

where $\mathbb{Z}_n^{2g}$ acts freely on $\mathcal{Y}(g,n) \times T$. □

Proof of Theorem 1.24. It remains to prove the result for the SL_2 case. Let I denote the reduced ideal in $\mathbb{C}[\mathcal{Y}(2,2)]$ defining the singular locus. Since the singular locus is stable under the action of $\mathbb{Z}_n^{2g}$ so too is I. Therefore, the action of $\mathbb{Z}_n^{2g}$ lifts to the blowup $\widetilde{\mathcal{Y}}(2,2)$ making $\sigma : \widetilde{\mathcal{Y}}(2,2) \to \mathcal{Y}(2,2)$ equivariant. Theorem 8.11, together with the fact that

$$\widetilde{\mathcal{X}}(2,2) \simeq \widetilde{\mathcal{Y}}(2,2) \times_{\mathbb{Z}_n^{2g}} T,$$

implies that $\widetilde{\mathcal{Y}}(2,2)$ is smooth. Moreover, the fact that $\widetilde{\mathcal{X}}(2,2) \to \mathcal{X}(2,2)$ is semi-small implies that $\sigma : \widetilde{\mathcal{Y}}(2,2) \to \mathcal{Y}(2,2)$ is semi-small. The argument that this implies that σ is a symplectic resolution is identical to the first part of the proof of Corollary 8.12. □

Proof of Theorem 1.19. It remains to prove this result in the SL_n case. Proposition 8.13 implies that Theorem 1.19 holds when $g = 1$. When $(g,n) = (2,2)$, Theorem 1.24 and Lemma 6.12 imply that $\mathcal{Y}(2,2)$ has symplectic singularities. When $n = 1$ then $\mathcal{Y}(g,1)$ is a point. Therefore we assume that $g, n > 1$ and $(g,n) \neq (2,2)$.

We begin by showing that $\mathcal{Y}(g,n)$ is normal. Lemma 8.17 implies that $\mathcal{Y}(g,n)$ is an irreducible variety of dimension $2(g-1)(n^2-1)$ since $\dim \mathcal{X}(g,n) = 2n^2(g-1)+2$. If $\mathcal{Y}(g,n)$ were not normal, then $\mathcal{Y}(g,n) \times T$ would also not be normal. But the fact that $\mathcal{Y}(g,n) \times_{\mathbb{Z}_n^{2g}} T \simeq \mathcal{X}(g,n)$ is normal, and the map $\mathcal{Y}(g,n) \times T \to \mathcal{X}(g,n)$ is étale, implies by [44, Proposition 3.17] that $\mathcal{Y}(g,n) \times T$ is normal. Thus, $\mathcal{Y}(g,n)$ is normal. The identification $\mathcal{Y}(g,n) \times_{\mathbb{Z}_n^{2g}} T \simeq \mathcal{X}(g,n)$ of Lemma 8.17 is Poisson, where we equip $\mathcal{Y}(g,n) \times T$ with the product Poisson structure. We deduce that the Poisson structure on the smooth locus of $\mathcal{Y}(g,n)$ is non-degenerate, and the singular locus of $\mathcal{Y}(g,n)$ has codimension at least 4 when $(g,n) \neq (2,2)$. Therefore, repeating the proof of Proposition 8.4, we deduce that $\mathcal{Y}(g,n)$ has symplectic singularities. □

Proof of Theorem 1.20. It remains to prove this statement in the SL_n case. Recall that we have assumed that $g > 1$ and $(g,n) \neq (2,2)$.

As noted in the proof of Theorem 1.19, $\mathcal{Y}(g,n)$ is a symplectic singularity whose singular locus has codimension at least 4. Therefore $\mathcal{Y}(g,n)$ has terminal singularities. To show that $\mathcal{Y}(g,n)$ is factorial, one simply repeats word for word the arguments of section 8.5, but with GL replaced by SL throughout, and using diagrams (22) and (23) to deduce the required dimension inequalities. □

Proof of Corollary 1.21. This follows because, as recalled in Section 6.4, any normal factorial variety with terminal singularities cannot admit a proper crepant resolution, and hence not a proper symplectic resolution. □

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