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Reconstruction of an orthotropic thermal conductivity from nonlocal heat flux measurements

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Abstract

Raw materials are anisotropic and heterogeneous in nature, and recovering their conductivity is of utmost importance to the oil, aerospace and medical industries concerned with the identification of soils, reinforced fiber composites and organs. Due to the ill-posedness of the anisotropic inverse conductivity problem certain simplifications are required to make the model tracktable. Herein, we consider such a model reduction in which the conductivity tensor is orthotropic with the main diagonal components independent of one space variable. Then, the conductivity components can be taken outside the divergence operator and the inverse problem requires reconstructing one or two components of the orthotropic conductivity tensor of a two-dimensional rectangular conductor using initial and Dirichlet boundary conditions, as well as non-local heat flux over-specifications on two adjacent sides of the boundary. We prove the unique solvability of this inverse coefficient problem. Afterwards, numerical results indicate that accurate and stable solutions are obtained.

Keywords: Inverse problem; Orthotropic thermal conductivity; Two-dimensional heat equation; Nonlinear optimization.

1 Introduction

The reconstruction of coefficients in the parabolic heat equation, [3,10], has been the focus of attention in several fields, e.g. finance, groundwater flow, oil recovery, and heat transfer. In particular, the identification of coefficients in two-dimensional heat conduction problems has received significant attention from many researchers [5,6,11,15,20]. Most of these studies relate to isotropic materials. However, it has been found that factors such as manufacturing and curing processes have impact on the material properties of a structure, often introducing extra variations, including anisotropy, [7], which are difficult to measure directly. The estimation of thermal properties for multi-dimensional inhomogeneous and anisotropic media is quite limited in the literature, see e.g. [2,12]. Such a coefficient problem presents several difficulties because it is inverse, nonlinear and ill-posed.

At steady-state, the study on the determination of the diffusivity/conductivity of a layered and orthotropic medium has been addressed in [1,2]. At the same time, the general case concerning the identification of an anisotropic spacewise dependent conductivity in the elliptic Laplace-Beltrami equation was thoroughly investigated, [19]. However, in the time-dependent case the scenario has received limited attention from researchers. Here, we only highlight the nonlinear identification of a temperature-dependent orthotropic material, [18], the recovery of the leading coefficients of a heterogeneous orthotropic medium, [8,9,14], and the space-dependent anisotropic case addressed in [12].

In a recent paper, [8], the authors have investigated the recovery of the thermal conductivity coefficients $a(y,t) > 0$ and $b(x,t) > 0$ of an orthotropic rectangular conductor along with the temperature $u(x,y,t)$ in a two-dimensional problem given by the parabolic heat equation

$$\frac{\partial u}{\partial t}(x,y,t) = a(y,t)\frac{\partial^2 u}{\partial x^2}(x,y,t) + b(x,t)\frac{\partial^2 u}{\partial y^2}(x,y,t) + f(x,y,t),$$
$$(x,y,t) \in Q_T := (0,h) \times (0,\ell) \times (0,T), \tag{1}$$

where h, ℓ, T are given positive quantities and $f(x,y,t)$ is a given heat source, subject to the initial condition

$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in \overline{D} := [0,h] \times [0,\ell], \tag{2}$$

the Dirichlet boundary conditions

$$u(0,y,t) = \mu_{11}(y,t), \quad u(h,y,t) = \mu_{12}(y,t), \quad (y,t) \in [0,\ell] \times [0,T], \tag{3}$$

$$u(x,0,t) = \mu_{21}(x,t), \quad u(x,l,t) = \mu_{22}(x,t), \quad (x,t) \in [0,h] \times [0,T], \tag{4}$$

and the heat flux over-specifications

$$a(y,t)\frac{\partial u}{\partial x}(0,y,t) = \kappa_1(y,t), \quad (y,t) \in [0,\ell] \times [0,T], \tag{5}$$

$$b(x,t)\frac{\partial u}{\partial y}(x,0,t) = \kappa_2(x,t), \quad (x,t) \in [0,h] \times [0,T], \tag{6}$$

where φ, μ_{1i}, μ_{2i} for $i = 1,2$ are given functions satisfying compatibility conditions, and κ_1 and κ_2 are given heat flux measured data. In this paper, we generalise the local heat flux measurements (5) and (6) to the more general non-local over-specifications

$$a(y,t)\Big[\nu_{11}(y,t)\frac{\partial u}{\partial x}(0,y,t) + \nu_{12}(y,t)\frac{\partial u}{\partial x}(h,y,t)\Big] = \varkappa_1(y,t), \quad (y,t) \in [0,\ell] \times [0,T], \tag{7}$$

$$b(x,t)\Big[\nu_{21}(x,t)\frac{\partial u}{\partial y}(x,0,t) + \nu_{22}(x,t)\frac{\partial u}{\partial y}(x,\ell,t)\Big] = \varkappa_2(x,t), \quad (x,t) \in [0,h] \times [0,T], \tag{8}$$

where $\varkappa_1$ and $\varkappa_2$ are given functions and $(\nu_{i,j})_{i,j=1,2}$ are given coefficients. Of course, when $\nu_{11} = \nu_{21} = 1$ and $\nu_{12} = \nu_{22} = 0$, expressions (7) and (8) become (5) and (6), respectively. Expressions (7) and (8) are linear combinations of heat fluxes across the opposite sides of the rectangular heat conductor $D = (0,h) \times (0,\ell)$.

The organization of the paper is as follows. In Section 2, the existence and uniqueness of the solution $(a(y,t), b(x,t), u(x,y,t))$ of the inverse problem (1)–(4), (7) and (8) are proved. In Section 3, we briefly describe the explicit FDM used to discretise the direct problem. In Section 4, the numerical approach based on the minimization of the nonlinear least-squares objective function is introduced. Numerical results are presented and discussed in Section 5. Finally, conclusions are presented in Section 6.

2 Unique solvability of the inverse problem

Consider the following assumptions:

$(A1)$ $\varphi \in C^{2+\gamma}(\overline{D}), \mu_{1i} \in C^{2+\gamma,1+\gamma/2}([0,\ell]\times[0,T]), \mu_{2i} \in C^{2+\gamma,1+\gamma/2}([0,h]\times[0,T]), \nu_{1i} \in C^{\gamma,\gamma/2}([0,\ell]\times[0,T]), \nu_{2i} \in C^{\gamma,\gamma/2}([0,h]\times[0,T]), i\in\{1,2\},\ \varkappa_1 \in C^{\gamma,\gamma/2}([0,\ell]\times[0,T]),$ $\varkappa_2 \in C^{\gamma,\gamma/2}([0,h]\times[0,T]),\ f \in C^{\gamma,\gamma/2}(\overline{Q}_T),$ for some $\gamma\in(0,1)$;

$(A2)$ $\varphi_x(x,y)>0,\ \varphi_y(x,y)>0,\ (x,y)\in\overline{D};\ \varkappa_1(y,t)>0,\ \nu_{11}(y,t)+\nu_{12}(y,t)>0,\ (y,t)\in [0,\ell]\times[0,T],\ \varkappa_2(x,t)>0,\ \nu_{21}(x,t)+\nu_{22}(x,t)>0,\ (x,t)\in[0,h]\times[0,T]$;

$(A3)$ consistency conditions of the zero and the first orders hold.

In $(A1)$, $C^{k+\gamma,(k+\gamma)/2}$, for $k\in\{0,2\}$ and $\gamma\in(0,1)$, denotes the space of functions which are k-times continuously differentiable in space and $k/2$-times continuously differentiable in time, with the space partial derivatives of order k being Hölder continuous with exponent γ and the time partial derivative of order $k/2$ being Hölder continuous with exponent $\gamma/2$.

2.1 Local existence of solution

Theorem 1. *Suppose that the assumptions (A1)–(A3) hold. Then, for some $T_0\in(0,T]$ there exists a solution $(a(y,t),b(x,t),u(x,y,t))$ of the problem (1)–(4), (7) and (8) such that $0<a\in C^{\gamma,\gamma/2}([0,\ell]\times[0,T_0])$, $0<b\in C^{\gamma,\gamma/2}([0,h]\times[0,T_0])$ and $u\in C^{2+\gamma,1+\gamma/2}(\overline{Q}_{T_0})$.*

Proof. To prove the local existence of a solution to (1)–(4), (7) and (8) we are first going to reduce it to an equivalent, in a certain sense, operator equation with respect to (a,b) and afterwards apply the Schauder fixed point theorem.

To reduce the problem (1)–(4) to another problem with homogeneous initial and boundary conditions we denote

$$\begin{aligned}\psi(x,y,t) &:= \mu_{11}(y,t)-\mu_{11}(y,0)+\frac{x}{h}\Big(\mu_{12}(y,t)-\mu_{12}(y,0)-\mu_{11}(y,t)+\mu_{11}(y,0)\Big)\\ &+\mu_{21}(x,t)-\mu_{21}(x,0)-\Big[\mu_{11}(0,t)-\mu_{11}(0,0)+\frac{x}{h}\Big(\mu_{12}(0,t)-\mu_{12}(0,0)-\mu_{11}(0,t)\\ &+\mu_{11}(0,0)\Big)\Big]+\frac{y}{\ell}\Big[\mu_{22}(x,t)-\mu_{22}(x,0)-\mu_{11}(\ell,t)+\mu_{11}(\ell,0)-\frac{x}{h}\Big(\mu_{12}(\ell,t)-\mu_{12}(\ell,0)\\ &-\mu_{11}(\ell,t)+\mu_{11}(\ell,0)\Big)-\mu_{21}(x,t)+\mu_{21}(x,0)+\mu_{11}(0,t)-\mu_{11}(0,0)\\ &+\frac{x}{h}\Big(\mu_{12}(0,t)-\mu_{12}(0,0)-\mu_{11}(0,t)-\mu_{11}(0,0)\Big)\Big]\end{aligned}$$

and make the superposition

$$u(x,y,t)=v(x,y,t)+\varphi(x,y)+\psi(x,y,t). \tag{9}$$

For the function v we get the problem

$$\begin{aligned}v_t &= a(y,t)v_{xx}+b(x,t)v_{yy}+F(x,y,t)+a(y,t)(\varphi_{xx}(x,y)+\psi_{xx}(x,y,t))\\ &+b(x,t)(\varphi_{yy}(x,y)+\psi_{yy}(x,y,t)),\quad (x,y,t)\in Q_T,\end{aligned} \tag{10}$$

$$v(x,y,0)=0, \quad (x,y)\in\overline{D}, \tag{11}$$

$$v(0,y,t)=v(h,y,t)=0, \quad (y,t)\in[0,\ell]\times[0,T], \tag{12}$$

$$v(x,0,t)=v(x,\ell,t)=0, \quad (x,t)\in[0,h]\times[0,T], \tag{13}$$

where $F(x,y,t):=f(x,y,t)-\psi_t(x,y,t)$.

With the aid of the Green function $G(x,y,t;\xi,\eta,\tau)$ for the Dirichlet problem associated to the leading parabolic operator in (9), we have, [13],

$$v(x,y,t)=\int_0^t\iint_D G(x,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau)) +b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T, \tag{14}$$

and, using (9),

$$u(x,y,t)=\varphi(x,y)+\psi(x,y,t)+\int_0^t\iint_D G(x,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta) +\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau, \quad (x,y,t)\in\overline{Q}_T. \tag{15}$$

Finding from here the partial derivatives u_x, u_y and substituting them into (7) and (8) we get the following nonlinear system of equations for the determination of $a(y,t)$ and $b(x,t)$:

$$a(y,t)\Bigg(\nu_{11}(y,t)\Bigg(\varphi_x(0,y)+\psi_x(0,y,t)+\int_0^t\iint_D G_x(0,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) +a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg) +\nu_{12}(y,t)\Bigg(\varphi_x(h,y)+\psi_x(h,y,t)+\int_0^t\iint_D G_x(h,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) +a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg) =\varkappa_1(y,t), \quad (y,t)\in[0,\ell]\times[0,T], \tag{16}$$

$$b(x,t)\Bigg(\nu_{21}(x,t)\Bigg(\varphi_y(x,0)+\psi_y(x,0,t)+\int_0^t\iint_D G_y(x,0,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) +a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg) +\nu_{22}(x,t)\Bigg(\varphi_y(x,\ell)+\psi_y(x,\ell,t)+\int_0^t\iint_D G_y(x,\ell,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau) +a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg) =\varkappa_2(x,t), \quad (x,t)\in[0,h]\times[0,T]. \tag{17}$$

Using assumption $(A2)$ we can estimate from below the following expressions appearing in (16) and (17):

$$\nu_{11}(y,t)\varphi_x(0,y)+\nu_{12}(y,t)\varphi_x(h,y)\geq\left(\min_{\overline{D}}\varphi_x(x,y)\right)\left(\min_{[0,\ell]\times[0,T]}(\nu_{11}(y,t)+\nu_{12}(y,t))\right)=:M_1>0,$$

$$\nu_{21}(x,t)\varphi_y(x,0)+\nu_{22}(x,t)\varphi_y(x,l)\geq\left(\min_{\overline{D}}\varphi_y(x,y)\right)\left(\min_{[0,h]\times[0,T]}(\nu_{21}(x,t)+\nu_{22}(x,t))\right)=:M_2>0.$$

On the other hand, the rest of terms in (16) and (17) are equal to zero when $t=0$. Hence, there exists a number $T_0\in(0,T]$ such that

$$\Bigg|\nu_{11}(y,t)\Bigg(\psi_x(0,y,t)+\int_0^t\iint_D G_x(0,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)+\nu_{12}(y,t)\Bigg(\varphi_x(h,y)+\psi_x(h,y,t)+\int_0^t\iint_D G_x(h,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg|\leq\frac{M_1}{2},\quad (y,t)\in[0,\ell]\times[0,T_0],\quad (18)$$

$$\Bigg|\nu_{21}(x,t)\Bigg(\psi_y(x,0,t)+\int_0^t\iint_D G_y(x,0,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)+\nu_{22}(x,t)\Bigg(\varphi_y(x,\ell)+\psi_y(x,\ell,t)+\int_0^t\iint_D G_y(x,\ell,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg|\leq\frac{M_2}{2},\quad (x,t)\in[0,h]\times[0,T_0].\quad (19)$$

Now we can replace (16), (17) by the system

$$a(y,t)=\varkappa_1(y,t)\Bigg(\nu_{11}(y,t)\Bigg(\varphi_x(0,y)+\psi_x(0,y,t)+\int_0^t\iint_D G_x(0,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)+\nu_{12}(y,t)\Bigg(\varphi_x(h,y)+\psi_x(h,y,t)+\int_0^t\iint_D G_x(h,y,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta)+\psi_{\xi\xi}(\xi,\eta,\tau))+b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta)+\psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg)^{-1},\quad (y,t)\in[0,\ell]\times[0,T_0],\quad (20)$$

$$b(x,t) = \varkappa_2(x,t)\Bigg(\nu_{21}(x,t)\Bigg(\varphi_y(x,0) + \psi_y(x,0,t) + \int_0^t \iint_D G_y(x,0,t;\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)$$

$$+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)$$

$$+\nu_{22}(x,t)\Bigg(\varphi_y(x,\ell) + \psi_y(x,\ell,t) + \int_0^t \iint_D G_y(x,\ell;t,\xi,\eta,\tau)\Big[F(\xi,\eta,\tau)$$

$$+a(\eta,\tau)(\varphi_{\xi\xi}(\xi,\eta) + \psi_{\xi\xi}(\xi,\eta,\tau)) + b(\xi,\tau)(\varphi_{\eta\eta}(\xi,\eta) + \psi_{\eta\eta}(\xi,\eta,\tau))\Big]d\xi d\eta d\tau\Bigg)\Bigg)^{-1},$$

$$(x,t) \in [0,h] \times [0,T_0]. \quad (21)$$

With the aid of (18), (19) we find from (20), (21) the estimates

$$a(y,t) \le \frac{\max\limits_{[0,\ell]\times[0,T_0]} \varkappa_1(y,t)}{M_1/2} =: A_1, \quad a(y,t) \ge \frac{\max\limits_{[0,\ell]\times[0,T_0]} \varkappa_1(y,t)}{\max\limits_{\overline{D}} \varphi_x(x,y) + M_1/2} =: A_0 > 0,$$

$$(y,t) \in [0,\ell] \times [0,T_0], \quad (22)$$

$$b(x,t) \le \frac{\max\limits_{[0,h]\times[0,T_0]} \varkappa_2(x,t)}{M_2/2} =: B_1, \quad b(x,t) \ge \frac{\max\limits_{[0,h]\times[0,T_0]} \varkappa_2(x,t)}{\max\limits_{\overline{D}} \varphi_y(x,y) + M_2/2} =: B_0 > 0,$$

$$(x,t) \in [0,h] \times [0,T_0]. \quad (23)$$

Now, applying the Schauder fixed-point theorem we establish the existence of solution to the system of nonlinear equations (20) and (21). Denote $\mathcal{N} := \{(a,b) \in C([0,\ell] \times [0,T_0]) \times C([0,h] \times [0,T_0]) : A_0 \le a(y,t) \le A_1,\ B_0 \le b(x,t) \le B_1\}$, and represent the system (20) and (21) as an operator equation

$$\omega = P\omega, \quad \omega \in \mathcal{N}, \quad (24)$$

where $\omega := (a(y,t), b(x,t))$ and the operator P is defined by the right-hand sides of equations (20) and (21). Due to the construction of $\mathcal{N}$, the operator P maps $\mathcal{N}$ onto itself. The compactness of the operator P may be established analogously to [11]. Hence, there exists at least one solution $(a(y,t), b(x,t))$ of the system (20) and (21) in the space $\mathcal{N}$. Taking into account the assumption $(A1)$, it is easy to see that $(a,b) \in C^{\gamma,\gamma/2}([0,\ell] \times [0,T_0]) \times C^{\gamma,\gamma/2}([0,h] \times [0,T_0])$. Substituting a and b into equation (1) we find $u(x,y,t)$ as a solution of the direct problem (1)–(4) from the space $C^{2+\gamma,1+\gamma/2}(\overline{Q}_{T_0})$. The proof is complete.

2.2 Uniqueness of solution

Theorem 2. *Suppose that the assumption*
(A4) $\varkappa_1(y,t) \ne 0$, $(y,t) \in [0,\ell] \times [0,T]$, $\varkappa_2(x,t) \ne 0$, $(x,t) \in [0,h] \times [0,T]$,
is satisfied. Then, the solution $(a(y,t), b(x,t), u(x,y,t))$ *of the problem (1)–(4), (7) and (8) is unique in the space* $C^{\gamma,\gamma/2}([0,\ell] \times [0,T_0]) \times C^{\gamma,\gamma/2}([0,h] \times [0,T_0]) \times C^{2+\gamma,1+\gamma/2}(\overline{Q}_{T_0})$, *with* $a(y,t) > 0$, $(y,t) \in [0,\ell] \times [0,T_0]$, $b(x,t) > 0$, $(x,t) \in [0,h] \times [0,T_0]$.

Proof. Suppose that there exist two solutions $(a_k(y,t), b_k(x,t), u_k(x,y,t)), k \in \{1,2\}$, of the problem (1)–(4), (7) and (8) from the indicated class. Denote $a := a_1 - a_2,\ b := b_1 - b_2$ and $u := u_1 - u_2$. The triplet of functions $(a(y,t), b(x,t), u(x,y,t))$ is a solution to the problem

$$u_t = a_1(y,t)u_{xx} + b_1(x,t)u_{yy} + a(y,t)u_{2_{xx}}(x,y,t) + b(x,t)u_{2_{yy}}(x,y,t), \quad (x,y,t) \in Q_T, \tag{25}$$

$$u(x,y,0) = 0, \quad (x,y) \in \overline{D}, \tag{26}$$

$$u(0,y,t) = u(h,y,t) = 0, \quad (y,t) \in [0,\ell] \times [0,T], \tag{27}$$

$$u(x,0,t) = u(x,l,t) = 0, \quad (x,t) \in [0,h] \times [0,T], \tag{28}$$

$$a(y,t)(\nu_{11}(y,t)u_{1_x}(0,y,t) + \nu_{12}(y,t)u_{1_x}(h,y,t)) = -a_2(y,t)(\nu_{11}(y,t)u_x(0,y,t) + \nu_{12}(y,t)u_x(h,y,t)) \quad (y,t) \in [0,\ell] \times [0,T], \tag{29}$$

$$b(x,t)(\nu_{21}(x,t)u_{1_y}(x,0,t) + \nu_{22}(x,t)u_{1_y}(x,\ell,t)) = -b_2(x,t)(\nu_{21}(x,t)u_y(x,0,t) + \nu_{22}(x,t)u_y(x,\ell,t)), \quad (x,t) \in [0,h] \times [0,T]. \tag{30}$$

With the aid of the Green function $\tilde{G}(x,y,t;\xi,\eta,\tau)$ for the problem (25)-(28) we obtain

$$u(x,y,t) = \int_0^t \iint_D \tilde{G}(x,y,t;\xi,\eta,\tau)(a(\eta,\tau)u_{2_{\xi\xi}}(\xi,\eta,\tau) + b(\xi,\tau)u_{2_{\eta\eta}}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T. \tag{31}$$

Substituting (31) into (29) and (30) we obtain the system of Volterra-type integral equations

$$a(y,t) = -\frac{a_2(y,t)}{\nu_{11}(y,t)u_{1_x}(0,y,t) + \nu_{12}(y,t)u_{1_x}(h,y,t)} \int_0^t \iint_D (\nu_{11}(y,t)\tilde{G}_x(0,y,t;\xi,\eta,\tau) + \nu_{12}(y,t)\tilde{G}_x(h,y,t;\xi,\eta,\tau))(a(\eta,\tau)u_{2_{\xi\xi}}(\xi,\eta,\tau) + b(\xi,\tau)u_{2_{\eta\eta}}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (y,t) \in [0,\ell] \times [0,T], \tag{32}$$

$$b(x,t) = -\frac{b_2(x,t)}{\nu_{21}(x,t)u_{1_y}(x,0,t) + \nu_{22}(x,t)u_{1_y}(x,\ell,t)} \int_0^t \iint_D (\nu_{21}(x,t)\tilde{G}_y(x,0,t;\xi,\eta,\tau) + \nu_{22}(x,t)\tilde{G}_y(x,\ell,t;\xi,\eta,\tau))(a(\eta,\tau)u_{2_{\xi\xi}}(\xi,\eta,\tau) + b(\xi,\tau)u_{2_{\eta\eta}}(\xi,\eta,\tau))d\xi d\eta d\tau, \quad (x,t) \in [0,h] \times [0,T]. \tag{33}$$

Taking into account the equalities

$$\nu_{11}(y,t)u_{1_x}(0,y,t) + \nu_{12}(y,t)u_{1_x}(h,y,t) = \frac{\varkappa_1(y,t)}{a_1(y,t)} > 0,$$
$$\nu_{21}(x,t)u_{1_y}(x,0,t) + \nu_{22}(x,t)u_{1_y}(x,\ell,t) = \frac{\varkappa_2(x,t)}{b_1(x,t)} > 0,$$

and the homogeneity of the system (32) and (33), from the theory of Volterra integral equations of the second kind with integrable kernel, we conclude that $a(y,t) \equiv 0, (y,t) \in [0,\ell] \times [0,T]$ and $b(x,t) \equiv 0$, $(x,t) \in [0,h] \times [0,T]$. Then, $u(x,y,t) \equiv 0$, $(x,y,t) \in \overline{Q}_T$, and the proof is complete.

2.3 Statement of a simplified inverse problem

In this section, we give a statement of a simplified inverse problem obtained when the coefficient b is known and taken, for simplicity, to be unity. Then, equation (1) simplifies to

$$\frac{\partial u}{\partial t}(x,y,t) = a(y,t)\frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t) + f(x,y,t), \quad (x,y,t) \in Q_T. \tag{34}$$

The local existence and uniqueness of solution of the inverse problem (2)–(5), (34) were established in [11] and read as stated in the following two theorems.

Theorem 3. *Suppose that the following assumptions are satisfied:*

$(B1)$ $\varphi \in C^2(\overline{D}),\ \mu_{1i} \in C^{2,1}([0,\ell] \times [0,T]),\ \mu_{2i} \in C^{2,1}([0,h] \times [0,T]),\ i = 1,2,$
$\kappa_1 \in C^{\gamma,0}([0,\ell] \times [0,T]),\ f \in C^{1+\gamma,\gamma,0}(\overline{Q_T})$ *for some* $\gamma \in (0,1)$;

$(B2)$ $\varphi_x(x,y) > 0,\ (x,y) \in \overline{D},\ \mu_{11_t}(y,t) - \mu_{11_{yy}}(y,t) - f(0,y,t) \le 0,$
$\mu_{12_t}(y,t) - \mu_{12_{yy}}(y,t) - f(h,y,t) \ge 0,\ \kappa_1(y,t) > 0,\ (y,t) \in [0,\ell] \times [0,T],$
$\mu_{2i_x}(x,t) > 0,\ i = 1,2,\ (x,t) \in [0,h] \times [0,T],\ f_x(x,y,t) \ge 0,\ (x,y,t) \in \overline{Q_T};$

$(B3)$ *conditions of consistency of order zero [13] between the initial condition (2) and the Dirichlet boundary conditions (3) and (4) hold.*

Then, there exists $T_0 \in (0,T]$*, which is determined by the input data, such that the problem (2)–(5), (34) has a solution* $(a(y,t), u(x,y,t)) \in C^{\gamma,0}([0,\ell] \times [0,T_0]) \times C^{2,1}(\overline{Q_{T_0}})$*, with* $a(y,t) > 0$*,* $(y,t) \in [0,\ell] \times [0,T_0]$.

Theorem 4. *Suppose that the condition* $C^{\gamma,0}([0,\ell] \times [0,T]) \ni \kappa_1(y,t) \neq 0$*,* $(y,t) \in [0,\ell] \times [0,T]$*, is satisfied. Then, the inverse problem (2)–(5), (34) cannot have more than one solution in the class* $(a(y,t), u(x,y,t)) \in C^{\gamma,0}([0,\ell] \times [0,T]) \times C^{2,1}(\overline{Q_T})$*, with* $a(y,t) > 0$*,* $(y,t) \in [0,\ell] \times [0,T]$.

3 Numerical solution of the direct problem

In this section, we consider the direct initial boundary value problem (1)–(4), where $a(y,t)$, $b(x,t)$, $f(x,y,t)$, $\varphi(x,y)$ and μ_{ij}, $i,j = 1,2$, are known and the solution $u(x,y,t)$ is to be determined. To achieve this, we use the forward time central space (FTCS) finite-difference scheme which is conditionally stable.

We subdivide the solution domain Q_T into M_1, M_2 and N subintervals of equal step lengths Δx and Δy, and uniform time step Δt, where $\Delta x = h/M_1$, $\Delta y = \ell/M_2$ and $\Delta t = T/N$, for space and time, respectively. At the node (i,j,n), we denote $u_{i,j}^n := u(x_i, y_j, t_n)$, where $x_i = i\Delta x$, $y_j = j\Delta y$, $t_n = n\Delta t$, $a_{j,n} := a(y_j, t_n)$, $b_{i,n} := b(x_i, t_n)$ and $f_{i,j}^n := f(x_i, y_j, t_n)$ for $i = \overline{0, M_1}$, $j = \overline{0, M_2}$ and $n = \overline{0, N}$.

The simplest explicit difference scheme for equation (1) is given by

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = a_{j,n}\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{(\Delta x)^2}\right) + b_{i,n}\left(\frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2}\right) + f_{i,j}^n \tag{35}$$

for $i = \overline{1, M_1 - 1}$, $j = \overline{1, M_2 - 1}$ and $n = \overline{0, N}$. The initial and boundary conditions (2)–(4) give

$$u_{i,j}^0 = \varphi(x_i, y_j), \quad i = \overline{0, M_1}, \quad j = \overline{0, M_2}, \tag{36}$$

$$u_{0,j}^n = \mu_{11}(y_j, t_n), \quad u_{M_1,j}^n = \mu_{12}(y_j, t_n), \quad j = \overline{0, M_2}, \quad n = \overline{1, N}, \tag{37}$$

$$u_{i,0}^n = \mu_{21}(x_i, t_n), \quad u_{i,M_2}^n = \mu_{22}(x_i, t_n), \quad i = \overline{0, M_1}, \quad n = \overline{1, N}. \tag{38}$$

Let $\tilde{a}$ and $\tilde{b}$ be the maximum values of $a(y, t)$ and $b(x, t)$, respectively, then, the stability condition for the explicit FDM scheme (35) is [17],

$$\frac{\tilde{a}\Delta t}{(\Delta x)^2} + \frac{\tilde{b}\Delta t}{(\Delta y)^2} \leq \frac{1}{2}. \tag{39}$$

The combination of the heat fluxes (7) and (8) can be calculated using the second-order FDM approximations:

$$\varkappa_1(y_j, t_n) = a_{j,n}\Big(\nu_{11}(y_j, t_n)u_x(0, y_j, t_n) + \nu_{12}(y_j, t_n)u_x(h, y_j, t_n)\Big), \quad j = \overline{1, M_2 - 1}, \quad n = \overline{1, N}, \tag{40}$$

$$\varkappa_2(x_i, t_n) = b_{i,n}\Big(\nu_{21}(x_i, t_n)u_y(x_i, 0, t_n) + \nu_{22}(x_i, t_n)u_y(x_i, \ell, t_n)\Big), \quad i = \overline{1, M_1 - 1}, \quad n = \overline{1, N}, \tag{41}$$

where

$$u_x(0, y_j, t_n) = \frac{4u(x_1, y_j, t_n) - u(x_2, y_j, t_n) - 3\mu_{11}(y_j, t_n)}{2\Delta x}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{1, N}, \tag{42}$$

$$u_x(h, y_j, t_n) = \frac{4u(x_{M_1-1}, y_j, t_n) - u(x_{M_1-2}, y_j, t_n) - 3\mu_{12}(y_j, t_n)}{-2\Delta x}, \quad j = \overline{1, M_2 - 1}, \quad n = \overline{1, N}, \tag{43}$$

$$u_y(x_i, 0, t_n) = \frac{4u(x_i, y_1, t_n) - u(x_i, y_2, t_n) - 3\mu_{21}(x_i, t_n)}{2\Delta y}, \quad i = \overline{1, M_1 - 1}, \quad n = \overline{1, N}, \tag{44}$$

$$u_y(x_i, l, t_n) = \frac{4u(x_i, y_{M_2-1}, t_n) - u(x_i, y_{M_2-2}, t_n) - 3\mu_{22}(x_i, t_n)}{-2\Delta y}, \quad i = \overline{1, M_1 - 1}, \quad n = \overline{1, N}. \tag{45}$$

4 Numerical solution of the inverse problem

In this section, we aim to obtain stable reconstructions for the principal direction components $a(y,t) > 0$ and $b(x,t) > 0$ of the two-dimensional orthotropic rectangular medium together with the temperature $u(x,y,t)$ satisfying the equations (1)–(4), (7) and (8). One can remark that at initial time $t = 0$ the values $a(y,0)$ and $b(x,0)$ can be obtained from the non-local over-specifications (7) and (8) as

$$a(y,0) = \frac{\varkappa_1(y,0)}{\nu_{11}(y,0)\varphi_x(0,y) + \nu_{12}(y,0)\varphi_x(h,y)}, \tag{46}$$

$$b(x,0) = \frac{\varkappa_2(x,0)}{\nu_{21}(x,0)\varphi_y(x,0) + \nu_{22}(x,0)\varphi_y(x,\ell)}. \tag{47}$$

The inverse problem is solved based on the nonlinear minimization of the least-squares objective function

$$\begin{aligned} F(a,b) := &\left\| a(y,t)\Big(\nu_{11}(y,t)u_x(0,y,t) + \nu_{12}(y,t)u_x(h,y,t)\Big) - \varkappa_1(y,t)\right\|^2 \\ &+ \left\| b(x,t)\Big(\nu_{21}(x,t)u_y(x,0,t) + \nu_{22}(x,t)u_y(x,\ell,t)\Big) - \varkappa_2(x,t)\right\|^2, \end{aligned} \tag{48}$$

or, in discretised form

$$\begin{aligned} F(\underline{\mathbf{a}},\underline{\mathbf{b}}) = &\sum_{n=1}^{N}\sum_{j=0}^{M_2}\Big[a_{j,n}\Big(\nu_{11}(y_j,t_n)u_x(0,y_j,t_n) + \nu_{12}(y_j,t_n)u_x(h,y_j,t_n)\Big) - \varkappa_1(y_j,t_n)\Big]^2 \\ &+ \sum_{n=1}^{N}\sum_{i=0}^{M_1}\Big[b_{i,n}\Big(\nu_{21}(x_i,t_n)u_y(x_i,0,t_n) + \nu_{22}(x_i,t_n)u_y(x_i,\ell,t_n)\Big) - \varkappa_2(x_i,t_n)\Big]^2, \end{aligned} \tag{49}$$

where $u(x,y,t)$ solves (1)–(4) for given $\mathbf{a}$ and $\mathbf{b}$. The minimization of the objective functional (49), subject to the physical simple bound constraints $\underline{\mathbf{a}} > \underline{\mathbf{0}}$ and $\underline{\mathbf{b}} > \underline{\mathbf{0}}$ is accomplished using the MATLAB optimization toolbox routine *lsqnonlin*, which does not require supplying the gradient of the objective function, [16]. This routine attempts to find the minimum of a sum of squares by starting from an initial guesses. Furthermore, within *lsqnonlin*, we use the Trust Region Reflective (TRR) algorithm [4], which is based on the interior-reflective Newton method. Each iteration involves a large linear system of equations whose solution, based on a preconditioned conjugate gradient method, allows a regular and sufficiently smooth decrease of the objective functional (49). Since the MATLAB routine *lsqnonlin* accepts only a vector of unknowns we make the matrix a long vector by renumbering its components. Upper and lower bounds on the thermal conductivities a and b can be specified according to *a priori* information on these physical parameters.

In the numerical computation, we take the parameters of the routine *lsqnonlin*, as follows:

- Maximum number of iterations = $10^5\times$ (number of variables).
- Maximum number of objective function evaluations = $10^7\times$ (number of variables).

- Solution and objective function tolerances = 10^{-15}.

The inverse problem (1)–(4), (7) and (8) is solved subject to both exact and noisy measurements (7) and (8). The noisy data is numerically simulated as

$$\varkappa_1^{\epsilon 1}(y_j, t_n) = \varkappa_1(y_j, t_n) + \epsilon 1_{j,n}, \quad j = \overline{0, M_2}, \quad n = \overline{1, N} \tag{50}$$
$$\varkappa_2^{\epsilon 2}(x_i, t_n) = \varkappa_2(x_i, t_n) + \epsilon 2_{i,n}, \quad i = \overline{0, M_1}, \quad n = \overline{1, N}, \tag{51}$$

where $\epsilon 1_{j,n}$ and $\epsilon 2_{i,n}$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviations $\sigma 1$ and $\sigma 2$ given by

$$\sigma 1 = p \times \max_{(y,t)\in[0,\ell]\times[0,T]} |\varkappa_1(y_j, t_n)|, \quad \sigma 2 = p \times \max_{(y,t)\in[0,h]\times[0,T]} |\varkappa_2(x_i, t_n)|, \tag{52}$$

where p represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon 1} = (\epsilon 1_{j,n})_{j=\overline{0,M_2},n=\overline{1,N}}$ and $\underline{\epsilon 2} = (\epsilon 2_{i,n})_{i=\overline{0,M_1},n=\overline{1,N}}$, as follows:

$$\underline{\epsilon 1} = normrnd(0, \sigma 1, M_2, N), \quad \underline{\epsilon 2} = normrnd(0, \sigma 2, M_1, N). \tag{53}$$

In the case of noisy data (51), we replace $\varkappa_1(y_j, t_n)$ and $\varkappa_2(x_i, t_n)$ by $\varkappa_1^{\epsilon 1}(y_j, t_n)$ and $\varkappa_2^{\epsilon 2}(x_i, t_n)$, respectively, in (49).

5 Numerical results and discussion

In this section, we present numerical results for the reconstruction of the orthotropic thermal conductivity components $a(y, t)$, $b(x, t)$ and the temperature $u(x, y, t)$, in the case of exact and noisy data (50)-(53). To assess the accuracy of the numerical solution we employ the root mean square errors ($rmse$) defined by:

$$rmse(a) = \left[\frac{1}{N(M_2+1)} \sum_{n=1}^{N} \sum_{j=0}^{M_2} \left(a^{numerical}(y_j, t_n) - a^{exact}(y_j, t_n) \right)^2 \right]^{1/2}, \tag{54}$$
$$rmse(b) = \left[\frac{1}{N(M_1+1)} \sum_{n=1}^{N} \sum_{i=0}^{M_1} \left(b^{numerical}(x_i, t_n) - b^{exact}(x_i, t_n) \right)^2 \right]^{1/2}. \tag{55}$$

For simplicity, we take $h = \ell = T = 1$.

5.1 Example 1

Consider the inverse problem (1)–(4), (7) and (8) with unknown coefficients $a(y, t)$ and $b(x, t)$, and the input data φ, μ_{ij}, ν_{ij} and $\varkappa_i$, $i, j = \overline{1, 2}$:

$$\varphi(x, y) = u(x, y, 0) = -(-2 + x)^2 - (-2 + y)^2, \quad f(x, y, t) = 2 + \frac{3 + 2t + x + y}{100},$$
$$\mu_{11}(y, t) = u(0, y, t) = -4 + 2t - (-2 + y)^2, \quad \mu_{12}(y, t) = u(1, y, t) = -1 + 2t - (-2 + y)^2,$$
$$\mu_{21}(x, t) = u(x, 0, t) = -4 + 2t - (-2 + x)^2, \quad \mu_{22}(x, t) = u(x, 1, t) = -1 + 2t - (-2 + x)^2,$$
$$\nu_{11}(y, t) = \nu_{12}(y, t) = \nu_{21}(x, t) = \nu_{22}(x, t) = 1, \ \varkappa_1(t) = \frac{3(y + t + 1)}{100}, \ \varkappa_2(t) = \frac{3(2 + x + t)}{100}.$$

One can notice that the conditions of Theorem 2 are satisfied and therefore, the uniqueness of the solution is guaranteed. In fact, it can easily be checked by direct substitution that the analytical solution is given by

$$a(y,t) = \frac{y+t+1}{200}, \quad (y,t) \in [0,1] \times [0,1], \tag{56}$$

$$b(x,t) = \frac{2+x+t}{200}, \quad (x,t) \in [0,1] \times [0,1], \tag{57}$$

$$u(x,y,t) = -(x-2)^2 - (y-2)^2 + 2t, \quad (x,y,t) \in \overline{Q}_T. \tag{58}$$

We take $M_1 = M_2 = N = 10$ which together with the upper bound 1/40 for the unknown coefficients a and b ensure that the stability condition (39) is always satisfied at each iteration of the minimization process. We also take the lower bound 10^{-9} to impose the physical constraint that the thermal conductivity coefficients must be positive.

We start the investigation for simultaneously determining the unknown components a and b for exact $p = 0$ and noisy input $p = 10\%$ data. Figure 1 presents the objective function (49), as a function of the number of iterations. From this figure one can notice that a rapid monotonically decreasing convergence is achieved in 8-9 iterations, with the objective function reaching a very small value of $O(10^{-20})$.

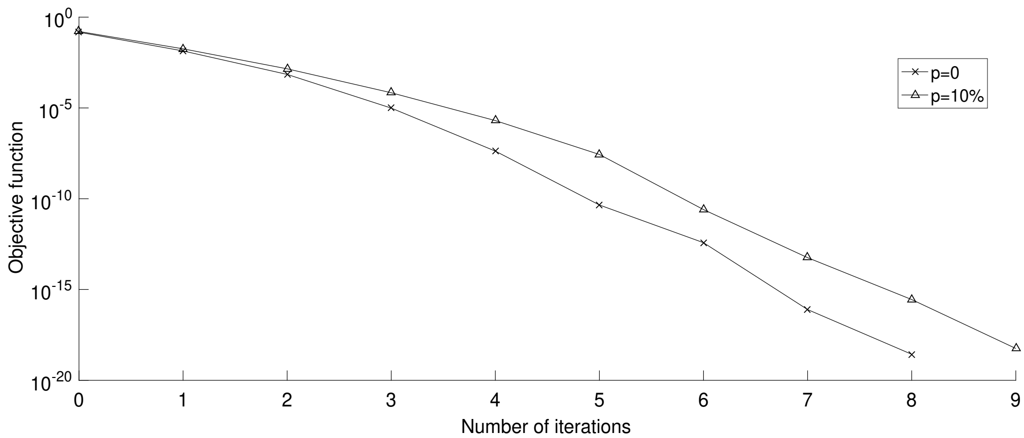

Figure 1: The objective function (49), for $p = 0$ and $p = 10\%$, for Example 1.

Figures 2 and 3 show the reconstructions of the orthotropic thermal conductivity components for $p = 0$ and $p = 10\%$, respectively. Table 1 shows more details of the numerical computation including the results for $p = 1\%$ noise. As expected, the numerically obtained results become more stable and accurate as the percentage of noise p decrease. From Figure 3 it can be seen that for the significant amount of noise $p = 10\%$, the numerical solution obtained by minimizing the nonlinear least-squares functional (49) becomes visibly oscillatory and unstable.

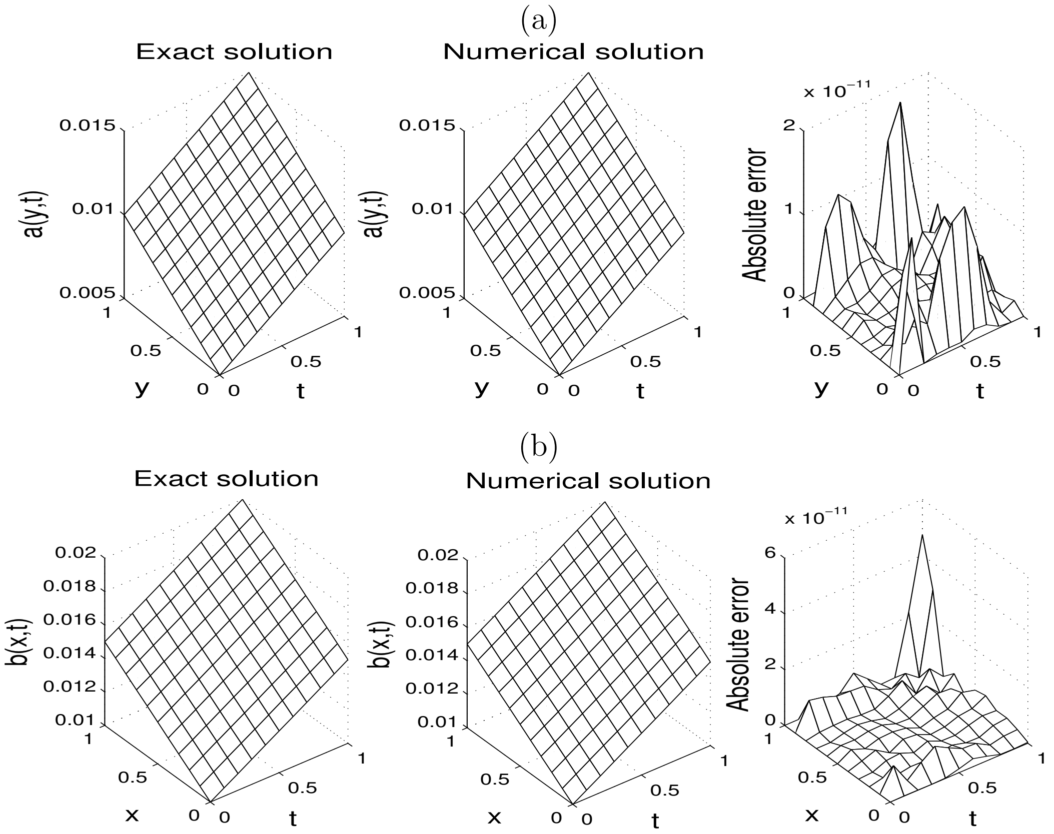

Figure 2: The exact ((56) and (57)) and numerical solutions for noise level $p = 0$ for (a) $a(y,t)$ and (b) $b(x,t)$, for Example 1. The absolute error between them is also included.

Table 1: The $rmse$ values (54) and (55) for various noise levels $p \in \{0, 1, 10\}\%$.

Example 1	$p = 0$	$p = 1\%$	$p = 10\%$
No. of iterations	8	8	9
Value of (49) at final iteration	2.7E-19	3.9E-19	5.6E-19
$rmse(a)$	4.7E-12	1.6E-4	1.6E-3
$rmse(b)$	6.6E-12	1.8E-4	1.9E-3
Computational time	21 min	23 min	25 min

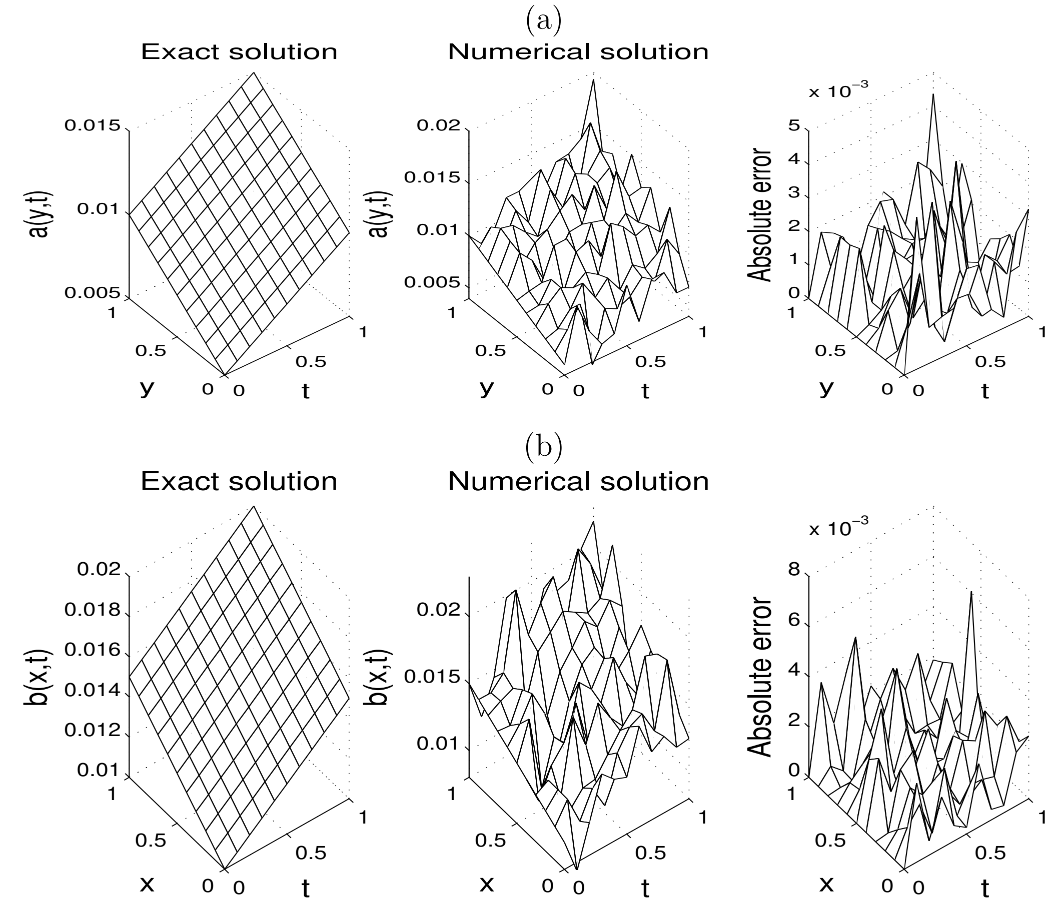

Figure 3: The exact ((56) and (57)) and numerical solutions for noise level $p = 10\%$ for (a) $a(y, t)$ and (b) $b(x, t)$, for Example 1. The absolute error between them is also included.

5.2 Example 2

In this example, we consider a simplification of the model described in subsection 2.3, obtained by taking one of the coefficients, say $b(x, t)$, known and, for simplicity, equal to unity. Consider the inverse problem (2)–(4) and (34) with unknown orthotropic thermal conductivity component $a(y, t)$ and solve this inverse problem with the input data φ, μ_{1i} and μ_{2i}, $i = 1, 2$, and κ_1 given by

$$\varphi(y, x) = u(x, y, 0) = x - y, \quad f(x, y, t) = \frac{1}{5}e^{t/5}(x - y),$$
$$\mu_{11}(y, t) = u(0, y, t) = -e^{t/5}y, \quad \mu_{12}(y, t) = u(1, y, t) = e^{t/5}(1 - y),$$
$$\mu_{21}(x, t) = u(x, 0, t) = e^{t/5}x, \quad \mu_{22}(x, t) = u(x, 1, t) = e^{t/5}(x - 1),$$
$$\kappa_1(y, t) = \frac{1}{100}e^{t/5}(1 + t + y). \tag{59}$$

Remark that the conditions of Theorem 3 and 4 are satisfied and therefore, the local existence and uniqueness of the solution are guaranteed. In fact, it can easily be checked by direct substitution that the analytical solutions $u(x, y, t)$ and $a(y, t)$ are given by

$$u(x, y, t) = e^{t/5}(x - y), \quad (x, y, t) \in \overline{Q_T}, \tag{60}$$

$$a(y, t) = \frac{y + t + 1}{100}, \quad (y, t) \in [0, 1] \times [0, 1]. \tag{61}$$

We investigate the inverse problem as we did in Example 1. We take $M_1 = M_2 = 5$ and $N = 60$, i.e. $\Delta x = \Delta y = 1/5$ and $\Delta t = 1/60$. We choose upper bound $UB = 0.2$ for a such that the stability condition (39) is always satisfied in the iterative process. Also, since a represents a positive physical quantity we take the lower bound for a to be a small positive number such as $LB = 10^{-4}$.

We start our investigation for reconstructing the unknown orthotropic thermal conductivity component $a(y,t)$ and the temperature $u(x,y,t)$ for exact and noisy measured input data (5), i.e., for the cases $p \in \{0,1,3,5\}\%$ of noise. The initial guess for $a(y,t)$ has been taken as

$$a^0(y,t) = a(y,0) = \frac{y+1}{100}, \quad y \in [0,1]. \tag{62}$$

Note that the value of $a(y,0)$ is available from (46). The objective function (49), as a function of the number of iterations, is plotted in Figure 4. From this figure, it can be seen that a monotonic decreasing convergence is achieved in about 10 to 11 iterations to reach a very low prescribed tolerance of $O(10^{-25})$. The numerically obtained results for $a(y,t)$ are illustrated in Figure 5 and summarised in Table 2. From this figure and table, it can be seen that as the percentage of noise p decreases the numerically obtained results becomes more stable and accurate.

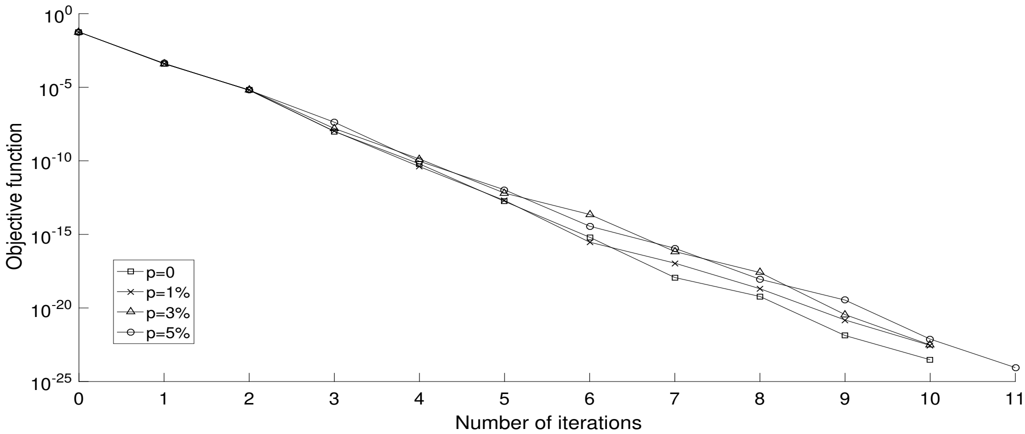

Figure 4: The objective function (49), as a function of the number of iterations, for various noise levels $p \in \{0,1,3,5\}\%$, for Example 2.

Table 2: The number of iterations, the value of the objective function (49) at final iteration, the $rmse(a)$ values (54) and the computational time, for various noise levels $p \in \{0,1,3,5\}\%$, for Example 2.

Numerical outputs	$p = 0$	$p = 1\%$	$p = 3\%$	$p = 5\%$
Number of iterations	10	10	10	11
Minimum value of (49)	3.0E-24	2.8E-23	3.0E-23	8.8E-25
$rmse(a)$	1.2E-6	3.4E-4	1.0E-3	1.7E-3
Computational time	37 mins	37 mins	37 mins	40 mins

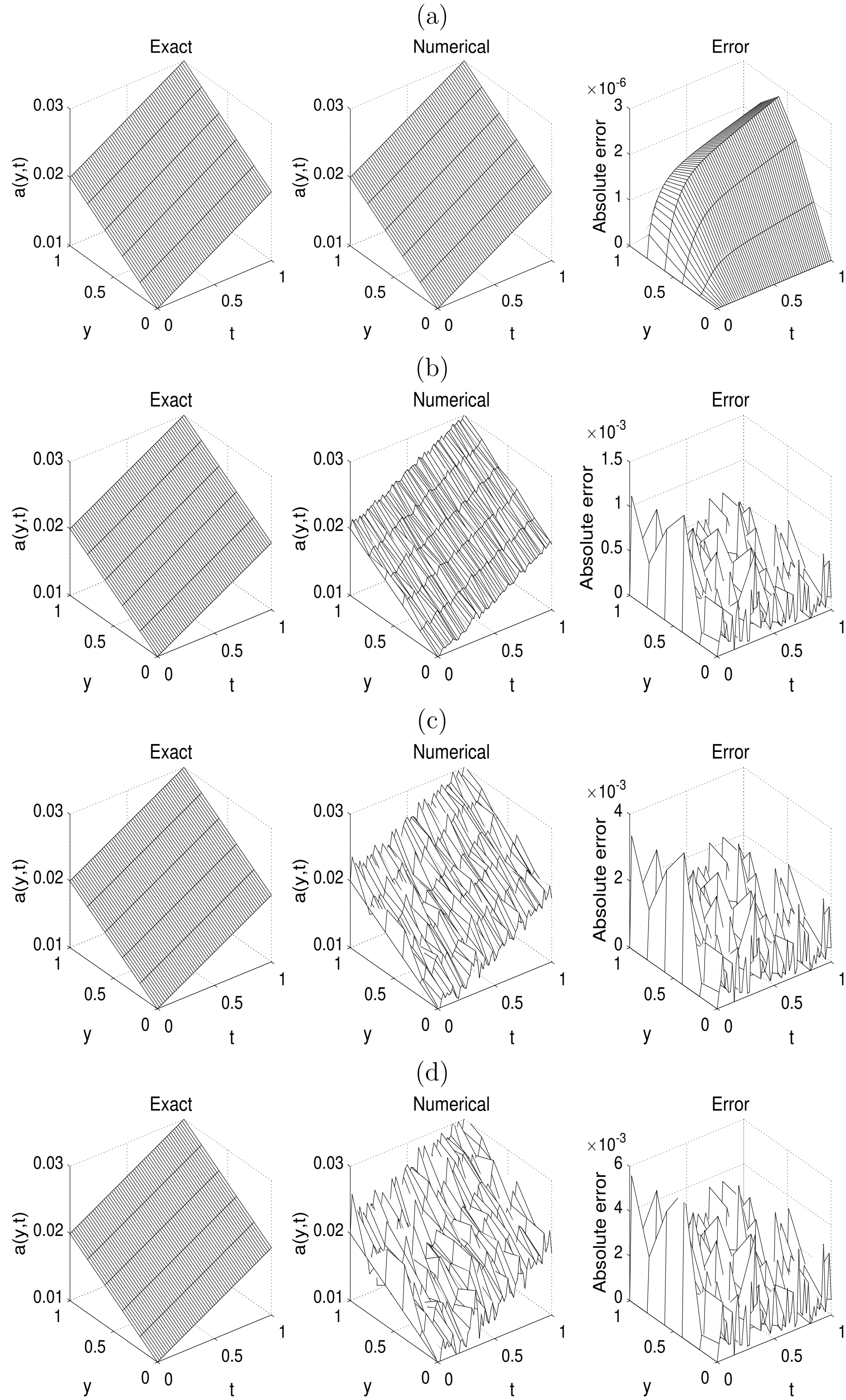

Figure 5: The exact (61) and numerical solutions for the orthotropic thermal conductivity component $a(y,t)$, for various noise levels: (a) $p = 0$, (b) $p = 1\%$, (c) $p = 3\%$ and (d) $p = 5\%$ noise, for Example 2. The absolute error between them is also included.

6 Conclusions

In this paper, the inverse problem involving the reconstruction of the orthotropic thermal conductivity components and the temperature in the two-dimensional parabolic heat equation (1) from the non-local heat flux over-specifications (7) and (8) has been investigated. Sufficient conditions which ensure the unique solvability of a local solution are provided and proved. The direct solver based on the FDM has been employed. The inverse problem solution based on a nonlinear least-squares minimization problem has been solved using the MATLAB optimisation toolbox routine *lsqnonlin*. As shown in Tables 1 and 2, the computational time is of the order of tens of minutes, which is reasonable bearing in mind that a nonlinear and ill-posed problem has been solved. Numerical results presented and discussed for both exact and noisy data show that accurate and stable solutions have been obtained. In principle, the analysis of this paper can be extended to three-dimensional problems; however, this non-trivial investigation is deferred to future work.

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