

# Vertex functions in QCD - preparation for beyond two loops

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March, 2019.

**Abstract.** We summarize the algorithm the determine the two loop off-shell 3-point vertex functions of QCD before outlining the steps required to extend the results to three and higher loops.

# 1 Introduction.

In our current generation of high energy particle accelerators involving hadron collisions a major source of background radiation derived from the strong sector. As this is governed by Quantum Chromodynamics (QCD) as the basic quantum field theory in order to quantify the background effects one has to carry out high loop order computations. There has been remarkable activity and progress in this direction since around the turn of the Millennium. The primary focus has been with the evaluation of on-shell  $n$ -point gluonic and fermionic amplitudes to several loop orders both analytically and numerically. Indeed such results have been crucial in ensuring that the Higgs particle was observed at CERN's LHC. However having information on the off-shell Green's functions such as the 3-point vertices of QCD are also important for theory and experiment as well. For instance, various articles in this direction have appeared over the years. A non-exhaustive set of references for the status of 3- and 4-point functions at various external momenta configurations is [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. There are various theoretical reasons for having such off-shell Green's functions. One is that knowing, say, the two loop off-shell vertex functions then higher loop  $n$ -point on-shell amplitudes could be modelled numerically. This would be in the interim absence of the technology to compute them fully explicitly. Such an approach is not uncommon. Equally in solving QCD beyond the perturbative limit analytically to probe deep infrared properties using the Schwinger-Dyson formalism, approximations have to be made in order to solve the infinite tower of Green's functions. Until recent years the validity of such approximations could not be fully quantified. However with explicit perturbative results, for instance, such error analyses have been possible. For instance one approximation in solving 2- and 3-point Schwinger-Dyson equations is to neglect the summed graphs deriving from the quartic gluon vertex. Work in this direction over a period of time, [13, 14, 15, 16, 17], has been checked that such a step does not affect final results by more than a few percent. Equally the Schwinger-Dyson method has been applied to finding the behaviour of the vertex functions. While similar approximations have been made such analyses have to be consistent with explicit perturbative results where *no* approximation is made at a particular loop order to drop a subset of contributing graphs. As an aside lattice gauge theory calculations of vertex functions equally have to match on to perturbative results. Therefore in light of these different areas of activity there is a clear need to compute QCD  $n$ -point and specifically vertex functions off-shell as well as on-shell. For the former, which is the focus of this article, we will review the status of the two loop evaluation of the 3-point vertices as well as outline the algorithm to extend this to higher loop order. While the discussion will be technical by nature, we will pool together all the necessary ingredients for the goal to be obtained at three loops.

While it is not immediately obvious it is the case that the route to achieve this will involve higher level mathematics extracted from an algebraic geometry approach. Indeed this also lies at the heart of on-shell amplitude computations. This technology has revolutionized the programme of loop calculations. An example of this can be seen in the results for two loop off-shell vertex results of [18] where harmonic polylogarithms based on a specific type of polynomials known as cyclotomic, [19], appeared. One corollary of such results is the possibility of effecting renormalization schemes other than the canonical  $\overline{\text{MS}}$  one which is universally accepted as the default scheme. Although it is the scheme with which one can carry out very high loop order calculations it is not a kinematic one and retains no data within the  $\beta$ -function, for instance, of the information on the subtraction point. In [20] the momentum subtraction scheme, denoted by MOM, was introduced and the  $R$ -ratio studied [21]. Extending [20] to the next order in [22] produced the three loop renormalization group functions. This allowed for studies of physical quantities at a loop order where scheme effects were apparent, [23]. One consequence is that choosing alternative renormalization schemes could lead to a different way of estimating

theory errors in measurements. In other words similar to an experiment estimating in different schemes the average of the result could be a more sound way of assessing truncation errors as an alternative to using values at different scales. With fully off-shell vertex functions, for instance, this ideas can be extended beyond the symmetric point subtraction of the MOM case to have a region bounding the central value.

The article is organized as follows. The method used to evaluate 3-point off-shell vertex functions is discussed in the next section with reference to the triple gluon vertex. The forms the basis for higher loop computations with the algorithm being outlined in Section 3. Concluding remarks are made in Section 4.

## 2 Current status

At the outset it is worth reviewing aspects of the history of QCD vertex evaluation. By this we mean that our focus will be on cases where there is *no* nullification of an extenal momentum. This is important since in the computation of the QCD  $\beta$ -function to very high loop order the extraction of the  $\overline{\text{MS}}$  coupling constant renormalization constant can be facilitated by setting the momentum of one of the fields of the Green's function to zero. This is a mathematical shortcut since the ultraviolet divergence is not contaminated by any infrared ones. By contrast this infrared safe procedure does not produce the *correct* finite part of the vertex functions. So it is not an appropriate method for gaining insight into any aspect of the kinematic properties of the vertex functions themselves. To be more concrete in the discussion we will focus on the triple gluon vertex function of Figure 1 which represents

$$\left\langle A_\mu^a(p_1)A_\nu^b(p_2)A_\sigma^c(-p_1 - p_2) \right\rangle = f^{abc}\Sigma_{\mu\nu\sigma}^{\text{ggg}}(p_1, p_2) = f^{abc} \sum_{k=1}^{14} \mathcal{P}_{(k)\mu\nu\sigma}^{\text{ggg}}(p_1, p_2) \Sigma_{(k)}^{\text{ggg}}(p_1, p_2) . \quad (2.1)$$

The momenta  $p_i$  satisfy energy-momentum conservation

$$\sum_{n=1}^3 p_n = 0 \quad (2.2)$$

and the underlying Lorentz invariants which the 3-point functions depend on are expressed in terms of two dimensionless variables,  $x$  and  $y$ , and one mass scale  $\mu$  which are defined by

$$x = \frac{p_1^2}{p_3^2} , \quad y = \frac{p_2^2}{p_3^2} , \quad p_3^2 = -\mu^2 \quad (2.3)$$

and we assume that none of  $p_i^2$  vanishes. In (2.1) we have decomposed the vertex into its 14 scalar amplitudes  $\Sigma_{(k)}^{\text{ggg}}(p_1, p_2)$  with respect to a basis of Lorentz tensors  $\mathcal{P}_{(k)\mu\nu\sigma}^{\text{ggg}}(p_1, p_2)$ . With this structure in mind for the other two 3-point vertices the full one loop vertex functions were studied in [20] in the early years following the discovery of asymptotic freedom.

Two important main early papers which stand out are [20, 24]. The former focussed on the vertex functions at the fully symmetric subtraction point defined by  $x = y = 1$  and introduced the kinematic renormalization scheme known as MOM for momentum subtraction. Unlike the  $\overline{\text{MS}}$  scheme the renormalization is carried out at this specific symmetric point and the finite part of the vertex functions absorbed into the renormalization constants. Therefore the  $\beta$ -functions contain kinematic data. The motivation of [20] was to study if the convergence of the perturbative series could be improved in this new scheme. The other article [24] was a

systematic study of each fully off-shell 3-point vertex with a view to writing each in terms of amplitudes dictated by external gluons being transverse. As such it has served as the vertex function convention where Schwinger-Dyson techniques are used to approximate other Green's functions. Consequently there have been a large number of one loop studies of the three 3-point vertices for different external momentum configurations as noted earlier. In some cases these studies have been at two loops but for the most part one or more external gluon legs were on-shell and quarks have been massless except in the case of [3, 7]. In the main the evaluation has been by standard quantum field theory techniques via Feynman graphs. However modern string inspired methods have been used [11, 12] for off-shell one loop vertex functions. The case where a gluon for example is on-shell has to be treated separately from the configuration introduced in (2.3) due to potential infrared singularities in taking the on-shell limit from the fully off-shell results.

Studies of the vertex functions for the special cases where one or more external line is on-shell has direct applications to experimental set-ups. One of the reasons why these were computed was in the main that the calculational tools for the off-shell case were not developed until much later. Several main components were necessary for this with the main breakthrough arriving in the form of the Laporta algorithm [25]. This is a procedure of relating scalar Feynman integrals of a particular  $n$ -point function at a specified loop order to core or master integrals of  $r$ -point functions with  $r \leq n$  and the same loop order. The connection between integrals being made via integration by parts. Then starting with the most complicated integral the relations derived from the integration by parts could be solved algebraically. While clearly such a set of large equations contains a degree of redundancy the whole process can be encoded for a computer to handle this and several packages are publicly available to achieve this, [26, 27, 28, 29, 30, 31, 32]. The second breakthrough was necessary to complete this task which was the determination of the master integrals. For 3-point functions these had to be constructed by specialized methods [33, 34, 35, 36] to two loops as integration by parts had been exhausted by the Laporta algorithm. To have a flavour of the resultant mathematical structure the one loop master integral of Figure 2 is for instance given by, [33, 34, 35],

$$I_1(x, y) = -\frac{1}{\mu^2} \left[ \Phi_1(x, y) + \Psi_1(x, y)\epsilon + \left[ \frac{\zeta(2)}{2}\Phi_1(x, y) + \chi_1(x, y) \right] \epsilon^2 + O(\epsilon^3) \right] \quad (2.4)$$

in  $d = 4 - 2\epsilon$  dimensions where  $\zeta(z)$  is the Riemann zeta function. Here the functions are related to polylogarithms  $\text{Li}_n(z)$ . For instance

$$\Phi_1(x, y) = \frac{1}{\lambda} \left[ 2\text{Li}_2(-\rho x) + 2\text{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right) \ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) + \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{3} \right]. \quad (2.5)$$

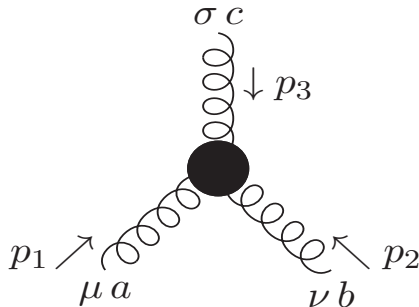


Figure 1: Triple gluon vertex function.

with

$$\rho(x, y) = \frac{2}{[1 - x - y + \lambda(x, y)]} \quad , \quad \lambda(x, y) = \sqrt{[1 - 2x - 2y + x^2 - 2xy + y^2]} \quad (2.6)$$

with the other functions of (2.4) given in [33, 34, 35] too. While the  $O(\epsilon)$  terms may not at first sight appear to be necessary they are required for various reasons. One is that at higher loops these one loop expressions are multiplied by the counterterms. So when a pole in  $\epsilon$  multiplies a term which is  $O(\epsilon)$  then that will contribute to the finite part of the vertex function at the next loop order. Accordingly one needs the master integrals to at least  $O(\epsilon^2)$  at one loop for a three loop evaluation. We have indicated at least since it could be the case that in the reduction using the Laporta algorithm a spurious pole in  $\epsilon$  arises. This is not an uncommon occurrence but the latest Laporta algorithm packages have now tools to circumvent this possibility. These technical issues aside the full off-shell 3-point QCD vertex functions are now known to two loops with the more details provided in [18].

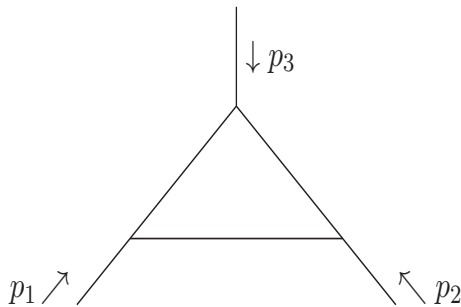


Figure 2: One loop 3-point master integral  $I_1(x, y)$ .

### 3 Three loop strategy

One of the reasons for detailing the formalism to carry out the two loop computations is that it points the way for higher loop corrections. With that basis we outline the next parts of the jigsaw to construct the three loop extension of [18]. First we assume the procedure of the general algorithm for the Green's functions is applied to obtain the three loop scalar amplitudes as illustrated in (2.1). From these the large set of scalar Feynman integrals are assembled which have to be reduced to the master integrals. The Laporta algorithm can in principle be applied in the three loop case using one of the latest packages which have the built-in improvements such as the refined algebraic reduction of the KIRA package, [32]. However to speed the integration by parts procedure it is not inconceivable that a faster algorithm could be developed. For instance for many years the MINCER package served the multiloop community well for three loop massless 2-point graphs in four dimensions, [37, 38]. It implemented the star-triangle rule to produce an efficient code to evaluate even the heaviest fully gluonic three loop graphs. With the need for more precision experimentally the four loop FORCER package, [39, 40], has superseded MINCER in the journey to hit the latest precision benchmark. Each have been encoded in the symbolic manipulation language FORM, [41, 42]. With the increase in loop order the evaluation time for a Green's function increases. However the FORCER algorithm implements a *new* integration rule to handle an internal topology which has no three loop antecedents and hence is a purely four loop feature. We have mentioned this since FORCER like MINCER applies only to 2-point functions. However the same new rule should be applicable or adaptable to three loop 3-point

functions since such a configuration emerges when one slices the vertex off a 2-point function where that vertex contains one of the external legs. The remaining graph would retain the internal topology of the 2-point four loop case. Therefore an adaptation of the new feature of FORCER could in principle be transferred to the 3-point case to provide an efficient alternative to the application of the Laporta algorithm for massless 3-point functions.

While such technology is already in effect in situ the main obstacle to the full implementation of a three loop evaluation is the determination of the required three loop master integrals. In recent years this field has advanced with progress having been made with understanding the mathematical properties of high order Feynman integrals. Examples of such articles include [43, 44] which provide novel procedures to compute Feynman graphs. The background to this is that there are a wide range of tools to evaluate a graph. One is to introduce the Schwinger parameter representation of each propagator and convert the  $L$ -loop  $d$ -dimensional spacetime integral into an integral over Schwinger parameters. The resulting integral has a large number of these parameter integrations to carry out and there is no guarantee that this can be achieved analytically. This is to be preferred over a numerical approach as the latter, if a Monte Carlo approach is used, could require a sizeable amount of computer resources to get a reasonable accuracy. In certain instances an analytic evaluation is possible and in essence uses algebraic geometry to produce an integration strategy. Such higher mathematics is relevant since the integrand contains polynomials of the parameters which represent higher dimensional geometries. Established mathematical theorems are then effected which determine which parameter integration order is to be used with the guiding principle being linear reducibility. By this we mean that after each parameter integration the polynomial degree reduces but the key to achieve this is to have the polynomial factor off a smaller polynomial involving only factors linear in the next variable to be integrated. It is this linearity which is key as it allows one to use the machinery of hyperlogarithms to carry out the integration over that Schwinger parameter. What was not immediately evident is if this procedure could be iterated without obstruction and that when it terminates the value of the integral is found. It has now been shown that if an integral is linearly reducible, [45, 46], in the above sense there is at least one choice of integration order which allows the integral to be determined. While this is in essence the general current position it is known that to three loops the 3-point vertex integrals are all linearly reducible. So in principle the required master integrals can be determined.

The actual practicalities of this have yet to be carried out. However several packages are available to assist with this task. For instance the converting a scalar Feynman integral into Schwinger parameter representation via the underlying graphs polynomials is now a standard feature of integration packages such as in HYPERINT, [47]. This package is appropriate for an analytic determination since any evaluation can be written in various hyperlogarithm representations. It has features which allows one to find the order of integration over the parameter variables to ensure that there is no obstruction to the linear reducibility. In principle one can expand to several orders in the  $\epsilon$  expansion in  $d = 4 - 2\epsilon$  dimensions. However for terms beyond the leading few the parameter integration can become tedious especially for high loop order. Therefore a more appropriate strategy would one where only the first term of the  $\epsilon$  expansion of a master integral was required which would then require the Laporta reduction to be constrained to producing a basis of masters which is finite. There is a caveat with this due to the fact that one is using dimensional regularization which means that the reduction produces factors of rational polynomials in  $d$ . Such functions can include poles in  $(d-4)$  which are termed spurious poles. This is in the sense that while they correspond to a divergence it is not necessarily one due to the divergence of an actual graph. There are now ways to circumvent this which work hand in hand with another property of the beauty of computing in  $d$ -dimensions. This was analysed in depth in [48, 49] where it was shown that  $d$ -dimensional integrals can be related to

the corresponding topology in  $(d+2)$ -dimensions plus a sum of others which have the same core topology but with propagators missing. Such higher dimensional integrals can be incorporated in the Laporta reduction process and has been implemented in the REDUZE package, [26, 27]. The advantage is that with the increase in dimensionality in the higher dimensional integral, it is not as ultraviolet divergent as its lower dimensional counterpart. Thereby in principle one reduces the evaluation of the more difficult master integrals to finite higher dimensional ones which should therefore be more accessible to the HYPERINT package.

In summarizing the algorithm to extend the two loop QCD off-shell vertex functions it is worth noting that for the triple gluon vertex there will be 2382 three loop graphs to evaluate and 63992 at four loops. For both the other 3-point vertices the numbers of graphs in each case are the same and are 688 and 17311 respectively at three and four loops and the evaluation of even just the three loop vertex functions will require a substantial amount of work and computing time. This would especially be the case at four loops without access to appropriate computers to build the necessary databases of integral relations. In the interim there is a potential alternative to gain some insight into or estimate of the three loop contributions. In the period between the early work of Celmaster and Gonsalves, [20], and its extension to the next order in [22] a method was developed in [9] where the vertex functions were computed at the fully symmetric point numerically at two loops in QCD. The approach was to apply a large momentum expansion of the vertex functions to very high order. This produced a set of 2-point integrals which were evaluated using MINCER, [37, 38]. Provided enough terms were computed the approximate value of the contributing graphs could be accurately estimated numerically. The stability and accuracy of the expansion could be checked by choosing different external momenta to play the role of the large momentum. What was remarkable when the analytic two loop expressions came available in [22], was how accurate the large momentum expansion values were. The only major difference was for a colour group Casimir coefficient in one three loop MOM  $\beta$ -function which turned out to be of the order 0.01. The numerical coefficient was small and the expansion needed to a higher accuracy than was computationally available at the time of [9]. With the advances in symbolic manipulation such as the provision of the FORCER programme which is significantly more efficient than MINCER such an interim numerical evaluation of the vertex functions would at least give information on the magnitude of the next order corrections. As a corollary it would provide the four loop MOM  $\beta$ -functions numerically.

## 4 Discussion

To recap we have reviewed recent results in the determination of the 3-point vertex functions of QCD at two loops. We have for the most part concentrated on the off-shell case to achieve this could not have been possible without the earlier work on different external momentum configurations. While the two loop off-shell results followed a long time after the one loop case, the main reason for this was lack of the computational technology. The last decade has seen a revolution in this direction with the Laporta algorithm, [25], as well as a systematic way of computing master integrals from high level mathematics. Consequently the road to achieve the extension to three loops is in principle possible. One useful corollary of such a computation would be the extension of the renormalization group functions to four loops in kinematic schemes such as MOM. To go to higher orders beyond three this depends on whether the linear reducibility of four loop masters can be established. One case which we have not touched on is that of the 4-point functions. The technology to compute the full off-shell amplitudes is already available. However the current situation is that the relevant two loop off-shell masters have not been computed. Moreover it has not been established if they are linearly reducible so that the

hyperlogarithm approach can be applied. This at present appears to be an open question for future work. Finally, in either the 3- or 4-point cases the massive quarks would have to be included in the results. At two loops the 3-point masters with one mass scale and off-shell momentum configuration are not known.

**Acknowledgements.** This work was supported by a DFG Mercator Fellowship.

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