A criterion for detecting trivial elements of Burnside groups

Un critère pour détecter les éléments triviaux dans les groupes de Burnside

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Abstract

In this article we give a sufficient and necessary condition to determine whether or not an element of the free group induces a non-trivial element of the free Burnside group of sufficiently large odd exponent. This criterion can be stated without any knowledge about Burnside groups, in particular about the proof of its infiniteness. Therefore it provides a useful tool that we will use later to study outer automorphisms of Burnside groups. We also state an analogue result for periodic quotients of torsion-free hyperbolic groups.

Résumé

Dans cet article, on propose une condition nécessaire et suffisante pour déterminer si un élément du groupe libre induit ou non un élément trivial dans les groupes de Burnside libre d'exposants impairs suffisamment grands. Ce critère peut être énoncé sans aucun pré-requis sur les groupes de Burnside. En particulier il n'est pas nécessaire de comprendre pourquoi les groupes de Burnside sont infinis pour l'appliquer. Pour cette raison il fournit un outil effectif qui nous permettra plus tard d'étudier les automorphismes du groupe de Burnside. Nous donnons aussi un résultat analogue pour les quotients périodiques d'un groupe hyperbolique sans torsion.

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Introduction

Let n be an integer. A group G has exponent n if for all $g \in G$, $g^n = 1$. In 1902, W. Burnside asked whether a finitely generated group with finite exponent is necessarily finite or not [4]. To study this question, it is natural to look at the free Burnside group $\mathbf{B}_r(n) = \mathbf{F}_r/\mathbf{F}_r^n$ which is the quotient of the free goup of rank r, denoted by $\mathbf{F}_r$, by the subgroup $\mathbf{F}_r^n$ generated by all n-th powers. It is indeed the largest group of rank r and exponent n. Until the work of P.S. Novikov and S.I. Adian, it was only known that for some small exponents $\mathbf{B}_r(n)$ was finite ($n = 2$ [4], 3 [4, 16], 4 [24], 6 [14]). In 1968, they proved that for $r \geqslant 2$ and $n \geqslant 4381$ odd $\mathbf{B}_r(n)$ is infinite [20, 21, 19]. This result has been improved in many directions. A.Y. Ol'shanskiĭ [22] proposed an other proof of the Novikov-Adian theorem using graded diagramms. Moreover he extended the result to the periodic quotients of a hyperbolic group [23]. S.V. Ivanov [15] and I.G. Lysenok [17] solved the case of even exponents.

The crucial fact used by P.S. Novikov and S.I. Adian is the following result (see [2, Statement 1]). Let p be an integer and w a reduced word representing an element of $\mathbf{F}_r$. If w does not contain a subword of the form u^p, then w induces a non-trivial element of $\mathbf{B}_r(n)$ where n is an odd integer larger than $10000p$. The infiniteness of the Burnside groups follows then from the existence of infinite words without third-power (like Thue-Morse words [1]). Our goal is to improve this statement. Given a reduced word w of $\mathbf{F}_r$ we provide a sufficient and necessary condition to decide wether w represents a trivial element of $\mathbf{B}_r(n)$ or not.

Before describing the criterion we would like to motivate this work. We wish to investigate the outer automorphisms of Burnside groups. Since $\mathbf{F}_r^n$ is a characteristic subgroup of $\mathbf{F}_r$, the projection $\mathbf{F}_r \twoheadrightarrow \mathbf{B}_r(n)$ induces a map $\text{Out}\,(\mathbf{F}_r) \to \text{Out}\,(\mathbf{B}_r(n))$. This map is not onto. Nevertheless it provides numerous examples of automorphisms of the Burnside groups. For instance if n is an odd exponent large enough, the image of $\text{Out}\,(\mathbf{F}_r)$ in $\text{Out}\,(\mathbf{B}_r(n))$ contains free groups of arbitrary rank [7]. One important question is: which automorphisms of $\mathbf{F}_r$ induce automorphisms of infinite order of $\mathbf{B}_r(n)$? In [7] we provided a large class of automorphisms of $\mathbf{F}_r$ having this property. However we are looking for a sufficient and necessary condition to characterize them. To understand the difficulties that may appear, let us have a look at a simple example already studied by E.A. Cherepanov [5]. Let φ be the automorphism of $\mathbf{F}_2 = \mathbf{F}(a, b)$ defined by $\varphi(a) = ab$ and $\varphi(b) = a$. The idea is to compute the orbit of b under φ.

$$\begin{array}{lcllcl} \varphi^1(b) & = & a & \varphi^5(b) & = & abaababa \\ \varphi^2(b) & = & ab & \varphi^6(b) & = & abaababaabaab \\ \varphi^3(b) & = & aba & \varphi^7(b) & = & abaababaabaababaababa \\ \varphi^4(b) & = & abaab & \dots & & \end{array}$$

This sequence converges to a right-infinite word

$$\varphi^\infty(b) = abaababaabaababaababaabaababaabaab\dots$$

which does not contain a subword which is a fourth-power [18]. Using the criterion of P.S. Novikov and S.I. Adian, the $\varphi^k(b)$'s define pairwise distinct elements of $\mathbf{B}_r(n)$ for some large n. In particular φ induces an automorphism of infinite order of the Burnside groups of large exponents. For an arbitrary automorphism the situation becomes more complicated. Consider for instance the automorphism ψ of $\mathbf{F}_4 = \mathbf{F}(a, b, c, d)$ defined by $\psi(a) = a$, $\psi(b) = ba$, $\psi(c) = c^{-1}bcd$ and $\psi(d) = c$. As previously we compute the orbit of d under ψ.

$$\begin{array}{lcl} \psi^1(d) & = & c \\ \psi^2(d) & = & c^{-1}\mathbf{b}cd \\ \psi^3(d) & = & d^{-1}c^{-1}b^{-1}c\mathbf{ba}c^{-1}bcdc \\ \psi^4(d) & = & c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcd\mathbf{ba}^2d^{-1}c^{-1}b^{-1}cbac^{-1}bcdbcd \\ \psi^5(d) & = & d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcda^{-2}b^{-1}d^{-1}c^{-1}b^{-1}c\dots \\ & & bac^{-1}bcdc\mathbf{ba}^3c^{-1}d^{-1}c^{-1}b^{-1}ca^{-1}b^{-1}c^{-1}bcdba^2d^{-1}c^{-1}b^{-1}c\dots \\ & & bac^{-1}bcdcbac^{-1}bcdc \end{array}$$

Note that each time $\psi^k(d)$ contains a subword ba^m then $\psi^{k+1}(d)$ contains ba^{m+1}. Hence the $\psi^k(d)$'s contain arbitrary large powers of a. This cannot be avoided by choosing the orbit of an another element. The result of P.S. Novikov and S.I. Adian cannot tell us if the $\psi^k(d)$'s are pairwise distinct in $\mathbf{B}_r(n)$. Therefore, we need a more accurate criterion two distinguish two different elements of $\mathbf{B}_r(n)$. This question about automorphisms of $\mathbf{B}_r(n)$ is solved in [10].

To state our theorem we need to define elementary moves. Let ξ and n be two integers. A (ξ, n)-*elementary move* consists in replacing a reduced word of the form $pu^ms \in \mathbf{F}_r$ by the reduced representative of $pu^{m-n}s$, provided m is an integer larger than $n/2 - \xi$. Note that an elementary move may increase the length of the word.

Theorem. *There exist numbers ξ and n_0 such that for all odd integers $n \geqslant n_0$ we have the following property. Let w be a reduced word of $\mathbf{F}_r$. The element of $\mathbf{B}_r(n)$ defined by w is trivial if and only if there exists a finite sequence of (ξ, n)-elementary moves that sends w to the empty word.*

A.Y. Ol'shanksiĭ point us out that this theorem also follows from Lemma 5.5 of [22] when $m \geqslant n/3$. Moreover his method could be adapted to cover the case where $m \geqslant n/2 - \xi$. However in this paper we follow the construction given by T. Delzant and M. Gromov. In [12], they proposed an alternative proof of the

Novikov-Adian Theorem. Using a geometrical approach they built a sequence of hyperbolic groups $\mathbf{F}_r \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \dots$ whose direct limit is $\mathbf{B}_r(n)$. At each step the groups have - among others - the following properties.

- G_{k+1} is a small cancellation quotient of G_k
- The relations that define the the quotient $G_k \twoheadrightarrow G_{k+1}$ are n-th powers of elements of G_k.

Given a small cancellation group, one knows an algorithm solving the word problem. Consider for instance w a reduced word of $\mathbf{F}_r$ which is trivial in the first quotient G_1. According to the Greendlinger Lemma, w contains a subword which equals three fourth of a relation. In our situation, this means that w can be written $w = pu^m s$ where $m \geqslant 3n/4$. Applying an elementary move, we obtain a new word w' which represents $pu^{m-n}s$ and is shorter than the previous one. Moreover w' is still trivial in G_1. By iterating the process we get a sequence of elementary moves that sends w to the empty word.

For the Burnside groups the process is more tricky. Let w be a reduced word of $\mathbf{F}_r$ which is trivial in $\mathbf{B}_r(n)$. Since $\mathbf{B}_r(n)$ is the direct limit of the G_k's, there exists a step k such that w is trivial in G_{k+1} but not in G_k. Roughly speaking, the Greendlinger Lemma tells us that a geodesic word of G_k representing w contains three fourth of a relation, i.e. a subword of the form u^m with $m \geqslant 3n/4$. One would like to apply an elementary move. However there is no reason that u^m should be a subword of w in $\mathbf{F}_r$. Consider the following example. Let u and v be two reduced words of $\mathbf{F}_r$. Assume that u^n is trivial in G_1. Let $w = \left(u^l v\right)^q \left(u^{l-n} v\right)^{n-q}$. As an element of G_1, w represents $\left(u^l v\right)^n$ which contains an n-th power. Nevertheless this does not hold in $\mathbf{F}_r$. The fact is that the previous relations (here u^n) mess up the powers. However despite w does not contain a n-th power of $u^l v$, it contains a large power of u. Thus $n-q$ elementary moves send w to $\left(u^l v\right)^n$. We can now "read" the power of $u^l v$ directly in $\mathbf{F}_r$ and apply an elementary move to reduced the length of this last word. This example actually describes the general situation. Our main theorem is proved by induction on k using this kind of arguments. The technical difficulties come from the fact that to be rigorous we should formulate the ideas presented above in a hyperbolic framework, taking care of many parameters (hyperbolicity constants, small cancellation parameters,...).

Our study works in fact in a more general situation. Let (X, x_0) be a δ-hyperbolic, geodesic, pointed space and G a non-elementary, torsion-free group acting properly, co-compactly, by isometries on it. We provide indeed a sufficient and necessary condition to detect elements of G which are trivial in the quotient G/G^n. For this purpose we need to extend the definition of elementary moves to this context. Let v be a non-trivial isometry of G. Since G is torsion free, it fixes two points v^- and v^+ of ∂X, the boundary at infinity of X. We denote by Y_v the set of points of X which are 10δ-close to some bi-infinite geodesic joining v^- and v^+. This subset is quasi-isometric to a line. Moreover v roughly acts on it by translation of length $[v]$. A (ξ, n)-*elementary move* consists in replacing a point $y \in X$ by $v^{-n}y$ provided that we have in X

$$|[x_0, y] \cap Y_v| \geqslant [v^m], \text{ where } m \geqslant n/2 - \xi.$$

Here $|[x_0, y] \cap Y_v|$ is a quantity that measures the length of the part of the geodesic $[x_0, y]$ which is approximatively contained in Y_v.

Let us compare this definition with the previous one. Let X be the Cayley graph of $\mathbf{F}_r$ and x_0 the vertex representing 1. Let $g \in \mathbf{F}_r$. Assume that g can be written as a reduced word $g = pu^m s$. Then the geodesic $[x_0, gx_0]$, labeled by $pu^m s$, intersects the axis of $v = pup^{-1}$ along a path of length $[v^m]$. Moreover $v^{-n}g$ can be represented by the word $pu^{m-n}s$. The next theorem is a generalization for hyperbolic groups of the previous one. Not only does it tell that an element of G trivial in a periodic quotient G/G^n of G can be reduced to the trivial element using elementary moves but it also explain how to decide whether or not two element of G are the same in G/G^n using the same kind of elementary moves.

Theorem. *Let G be a non-elementary, torsion-free group acting freely, properly, co-compactly, by isometries on a proper, hyperbolic, geodesic, pointed space (X, x_0). There exist numbers ξ and n_0 such that for all odd integers $n \geqslant n_0$ we have the following property. Two elements g and g' of G induce the same element of G/G^n if and only if there are two finite sequences of (ξ, n)-elementary moves that respectively send gx_0 and $g'x_0$ to the same point.*

Outline of the article. In Section 1, we review some of the standard facts on hyperbolic geometry. Since the proofs in the rest of the article are already quite technical, we also tried to compile in this section all the results that only require hyperbolic geometry. Section 2 investigates the cone-off construction used by T. Delzant and M. Gromov, in [12]. In particular we compare at a large scale the relation between the geometry of the cone-off over a metric space and the one of its base. Section 3 is devoted to the study of small cancellation theory. Our goal is to understand how to lift figures from a small cancellation quotient $\bar{G} = G/K$ in the group G. For instance, let g be an element of G such that a geodesic of $\bar{G}$ representing the image of g contains a large power. Under which conditions g already contains a large power? If not, what kind of transformations could send g to an element containing a large power? In the last section we summarize all this results in an induction that will proves our main theorem.

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1 Hyperbolic spaces

Let X be a metric space. Given two points $x, x' \in X$, we denote by $|x - x'|_X$ (or simply $|x - x'|$) the distance between them. Although it may not be unique, we write $[x, x']$ for a geodesic joining x and x'. The Gromov's product of three points x, y and z of X is defined by

$$\langle x, y\rangle_z = \frac{1}{2}\Big(|x-z| + |y-z| - |y-z|\Big).$$

From now on, we assume that X is δ-hyperbolic, which means that for all $x, y, z, t \in X$

$$\langle x, z\rangle_t \geqslant \min\Big\{\langle x, y\rangle_t, \langle y, z\rangle_t\Big\} - \delta. \tag{1}$$

Equivalently, for all $x, y, z, t \in X$,

$$|x-y| + |z-t| \leqslant \Big\{|x-z| + |y-t|, |x-t| + |y-z|\Big\} + 2\delta. \tag{2}$$

It follows from the hyperbolicity assumption that the geodesic triangles of X are 4δ-thin (see [6, Chap. 1, Prop. 3.1]). More precisely for all $x, y, z \in X$, for all $(r, s) \in [x, y] \times [x, z]$, if $|x-r| = |x-s| \leqslant \langle y, z\rangle_x$ then $|r-s| \leqslant 4\delta$. The Gromov's product $\langle x, y\rangle_z$ can be interpreted as an estimate of the distance of z to $[x, y]$. We have indeed $\langle x, y\rangle_z \leqslant d(z, [x, y]) \leqslant \langle x, y\rangle_z + 4\delta$ (see [6, Chap. 3, Lemm. 2.7]). We denote by ∂X, the boundary at infinity of X (see [6, Chap.2] for the definition and the main properties).

1.1 Quasi-convex subsets

Let Y be a subset of X. We denote by $Y^{+\alpha}$ the α-neighbourhood of Y, i.e. the set of points $x \in X$ such that $d(x, Y) \leqslant \alpha$. A point y of Y is called an η-projection of x on Y if $|x-y| \leqslant d(x, Y) + \eta$. A 0-projection is simply called a projection.

Definition 1.1. *Let $\alpha \geqslant 0$. A subset Y of X is* α-quasi-convex *if for every $x \in X$ and $y, y' \in Y$, $d(x, Y) \leqslant \langle y, y'\rangle_x + \alpha$.*

Definition 1.2. *A subset Y of X is* strongly quasi-convex *if for all $y, y' \in Y$ there exist $z, z' \in Y$ and geodesics $[y, z]$, $[z, z']$, $[z', y']$ contained in Y such that $|y-z|, |y'-z'| \leqslant 10\delta$.*

Remark : Our definition of quasi-convex is slightly different from the one usually given in the literature (every geodesic joining two points of Y lies in the α-neighbourhood of Y). However an α-quasi-convex in the regular sense is $(\alpha + 4\delta)$-quasi-convex in our sense, and conversely. This definition has the advantage of working even in a length space which is not geodesic (see [9]). Moreover since we defined hyperbolicity using Gromov's products it is more convenient to work with. With this definition a geodesic is 4δ-quasi-convex. By hyperbolicity, a strongly quasi-convex subset is 6δ-quasi-convex.

Lemma 1.3 (compare [6, Chap. 10, Prop. 2.1]). *Let Y be an α-quasi-convex subset of X.*

- *Let $x \in X$ and $y \in Y$. If p is an η-projection of x on Y, then $\langle x, y\rangle_p \leqslant \alpha + \eta$.*
- *Let $x, x' \in X$. If p and p' are respectively η- and η'-projections of x and x' on Y then,*

$$|p-p'| \leqslant \max\Big\{\varepsilon, |x-x'| - |x-p| - |x'-p'| + 2\varepsilon\Big\},$$

where $\varepsilon = 2\alpha + \delta + \eta + \eta'$.

Lemma 1.4. *Let Y be an α-quasi-convex subset of X. Let x be a point of X and p an η-projection of x on Y. For every $x' \in X$, p is an ε-projection of x' on Y where $\varepsilon = \langle x, p \rangle_{x'} + 2\alpha + \delta + \eta$.*

Proof. Let $\eta' > 0$ and p' be an η'-projection of x' on Y. The previous lemma combined with the triangle inequality gives $|p - p'| \leqslant \varepsilon(\eta')$ where $\varepsilon(\eta') = \langle x, p \rangle_{x'} + 2\alpha + \delta + \eta + \eta'$. Therefore p is an $(\varepsilon(\eta') + \eta')$-projection of x' on Y. This property holds for every $\eta' > 0$ which gives the result. □

Definition 1.5. *Let Y and Z be two subsets of X we denote by $|Y \cap Z|$ the following quantity.*

$$|Y \cap Z| = \frac{1}{2} \sup_{\substack{y,y' \in Y \\ z,z' \in Z}} \left\{ 0, |y - y'| + |z - z'| - |y - z| - |y' - z'| \right\}.$$

Remark : It follows from the definition that $|Y \cap Z| \geqslant \operatorname{diam}(Y \cap Z)$. Actually, if Y and Z are respectively α- and β-quasi-convex subsets of X, $|Y \cap Z|$ roughly measures the intersection of Y and Z:

$$|Y \cap Z| \approx \operatorname{diam}\left(Y^{+\alpha+10\delta} \cap Z^{+\beta+10\delta}\right) + 10\delta.$$

However this notation has two advantages. First the definition does not involve the hyperbolicity constant δ nor the quasi-convexity parameters α and β. Moreover, given two points x and x' of X joined by a geodesic the triangle inequality yields $|[x, x'] \cap Y| = |\{x, x'\} \cap Y|$. Therefore $|[x, x'] \cap Y|$ does not depend on the choice of the geodesic but only on its endpoints. This is convenient since our space is not necessary uniquely geodesic.

Let Y and Z be two subsets of X. Applying the triangle inequality we obtain the followings.

(i) For all $A, B \geqslant 0$, $\left|Y^{+A} \cap Z^{+B}\right| \leqslant |Y \cap Z| + 2A + 2B$.

(ii) For all $x, x', z \in X$, $|[x, z] \cap Y| \leqslant |[x, x'] \cap Y| + \langle x, x' \rangle_z$.

Combining (ii) with the hyperbolicity condition (1) we obtain for all $x, x', z, z' \in X$,

$$|[z, z'] \cap Y| \leqslant |[x, x'] \cap Y| + \langle x, x' \rangle_z + \langle x, x' \rangle_{z'} + \delta. \tag{3}$$

Proposition 1.6. *Let Y be an α-quasi-convex subset of X. Let x and x' be two points of X. We assume that y and y' are respectively η- and η'-projections of x and x' on Y. Then $\left||[x, x'] \cap Y| - |y - y'|\right| \leqslant \varepsilon$, where $\varepsilon = 2\alpha + \delta + \eta + \eta'$.*

Proof. By projection on a quasi-convex we have,

$$\max\left\{ |x - x'| - |x - y| - |x' - y'| + 2\varepsilon, \varepsilon \right\} \geqslant |y - y'|,$$

where $\varepsilon = 2\alpha + \delta + \eta + \eta'$. Therefore

$$|[x, x'] \cap Y| \geqslant \frac{1}{2} \max\left\{ |x - x'| + |y - y'| - |x - y| - |x' - y'|, 0 \right\} \geqslant |y - y'| - \varepsilon.$$

On the other hand, y and y' being respective η- and η'-projections of x and x', the triangle inequality implies that for every $z, z' \in Y$

$$\begin{aligned}\frac{1}{2}\Big(|x-x'|+|z-z'|-|x-z|-|x'-z'|\Big) &\leqslant |y-y'|+\langle x,z\rangle_y+\langle x',z'\rangle_{y'}\\ &\leqslant |y-y'|+2\alpha+\eta+\eta'.\end{aligned}$$

This inequality holds for every $z, z' \in Y$ hence $|[x,x'] \cap Y| \leqslant |y-y'| + 2\alpha + \delta + \eta + \eta'$, which ends the proof. □

1.2 Quasi-geodesics

In this article, all the paths that we consider are continuous.

Definition 1.7. *Let $k \geqslant 1$, $l \geqslant 0$ and $L > 0$. Let J be an interval of* $\mathbf{R}$. *A path $\sigma : J \to X$ is*

- *a* (k,l)-quasi-geodesic *if for all $s, t \in J$,*

$$k^{-1}|s-t| - l \leqslant |\sigma(s)-\sigma(t)| \leqslant k|s-t| + l.$$

- *a* L-local (k,l)-quasi-geodesic *if its restriction to every close interval of diameter L is a (k,l)-quasi-geodesic.*
- *a* L-local geodesic *if it is a L-local $(1,0)$-quasi-geodesic.*

Remark : By abuse of notation, we often write σ for the image $\sigma(J)$ of σ in X.

Proposition 1.8 (Stability of quasi-geodesics)**.** *Let $l \geqslant 0$ and $k \geqslant 1$. There exist $L > 0$, $k' \geqslant k$ and $d \geqslant 0$ depending only on l and k (not on X nor δ) with the following property. The Hausdorff distance between two $L\delta$-local $(k, l\delta)$-quasi-geodesics joining the same endpoints (possibly in ∂X) is at most $d\delta$. Moreover every $L\delta$-local $(k, l\delta)$-quasi-geodesic is a (global) $(k', l\delta)$-quasi-geodesic.*

Proof. The case where $\delta = 1$ follows from [6, Chap. 4, Th. 1.4 and 3.1]. The general case is obtained by a rescaling argument. □

Corollary 1.9 (Stability of discrete quasi-geodesics)**.** *Let $l \geqslant 0$. There exist $L > 0$ and $d \geqslant 0$ depending only on l (not on X nor δ) with the following property. If $x_0, \dots, x_m$ is a sequence of points of X, such that for all $i \in \{0, \dots, m-2\}$, $|x_{i+1} - x_i| \geqslant L\delta$ and $\langle x_i, x_{i+2}\rangle_{x_{i+1}} \leqslant l\delta$. Then the Hausdorff distance between $[x_0, x_1] \cup \dots \cup [x_{m-1}, x_m]$ and $[x_0, x_m]$ is less than $d\delta$.*

If we only consider local geodesics, one can give simple quantitative estimations for the constants which appear in the stability of quasi-geodesics. They will be often used later.

Proposition 1.10. *Let $L > 32\delta$. The Hausdorff distance between two L-local geodesics joining the same endpoints of X (respectively $X \cup \partial X$) is at most 12δ (respectively 32δ). Moreover every L-local geodesic is a (global) $(k,0)$-quasi-geodesic with $k = \frac{L+24\delta}{L-24\delta}$.*

Proof. The case where the local geodesics join two points of X is done in [3, Chap. III.H, Th. 1.13]. The general case follows then as in [6, Chap. 3, Th. 3.1]. □

1.3 Isometries

In this section we assume that X is geodesic and proper i.e., every close ball is compact. Let g be an isometry of X. In order to measure its action on X, we define two translation lengths. By the *translation length* $[g]_X$ (or simply $[g]$) we mean

$$[g]_X = \inf_{x \in X} |gx - x| .$$

The *asymptotic translation length* $[g]^\infty_X$ (or simply $[g]^\infty$) is

$$[g]^\infty_X = \lim_{n \to +\infty} \frac{1}{n} |g^n x - x| .$$

These two lengths satisfy the following inequality $[g]^\infty \leqslant [g] \leqslant [g]^\infty + 16\delta$ (see [6, Chap. 10, Prop 6.4]). The *axis* A_g of g, defined as follows, is a 40δ-quasi-convex subset of X (see [12, Prop. 2.3.3]).

$$A_g = \Big\{ x \in X / \, |gx - x| \leqslant \max \left\{ [g], 40\delta \right\} \Big\}$$

The isometry g is *hyperbolic* if its asymptotic translation length is positive. In this case, g fixes exactly two points of ∂X denoted by g^- and g^+. The cylinder of g, denoted by Y_g, is defined to be the set of points of X which are 10δ-close to some geodesic joining g^- and g^+. It is a g-invariant, strongly quasi-convex subset of X.

Proposition 1.11 (see [7, Prop 2.3]). *Let g be a hyperbolic isometry of X. We denote by $[g^-, g^+]$ a geodesic joining the points of ∂X fixed by g. Then $[g^-, g^+]$ is contained in the 48δ-neighbourhood of A_g. In particular Y_g lies in the 58δ-neighbourhood of A_g.*

Let g be an isometry of X such that $[g] > 40\delta$. (In particular, g is hyperbolic.) Let x be a point of A_g. We consider a geodesic $N : J \to X$ between x and gx parametrized by arc length. We extend N in a g-invariant path $N : \mathbf{R} \to X$ in the following way: for all $t \in J$, for all $m \in \mathbf{Z}$, $N(t + m[g]) = g^m N(t)$. This is a $[g]$-local geodesic contained in A_g. We call such a path a *nerve* of g. It is a very convenient tool for the proofs. Indeed N is homeomorphic to a line on which g acts by translation of length $[g]$. Moreover the Hausdorff distance between N and Y_g is less than 42δ. Therefore one can replace Y_g by N with a little error. We summarize here some of its properties which follow from the stability of the local geodesics and the projection on a quasi-convex. In order to lighten the proofs we will later use these facts without any justification.

The nerve N is 16δ-quasi-convex. Given two points $u = N(s)$ and $v = N(t)$ of N, we denote by $(u, v)_N$ the path $N([s, t])$. The path N is injective thus this definition makes sense.

- Let x be a point of X and y its projection on N, for all $y' \in N$ and $z \in (y, y')_N$, $\langle x, y' \rangle_y \leqslant 16\delta$ and $\langle x, y' \rangle_z \leqslant 28\delta$.
- Let x, x' be two points of X and y, y' their respective projections on N. If $|y - y'| > 33\delta$ then for all $z \in (y, y')_N$, $\langle x, x' \rangle_y \leqslant 33\delta$ and $\langle x, x' \rangle_z \leqslant 45\delta$.

- For all $x, x' \in X$, we have $|d(x, N) - d(x, Y_g)| \leqslant 42\delta$. On the other hand, $\Big| |[x, x'] \cap N| - |[x, x'] \cap Y_g| \Big| \leqslant 84\delta$.

Lemma 1.12. *Let g be an isometry of X such that $[g] > 40\delta$. For all $x \in X$ we have*

$$\left| \langle gx, g^{-1}x \rangle_x - d(x, Y_g) \right| \leqslant 87\delta.$$

Proof. We denote by t the Gromov product $\langle gx, g^{-1}x \rangle_x$. Let N be a nerve of g and y a projection of x on N. By hyperbolicity we have

$$t - \langle gx, g^{-1}x \rangle_y \leqslant |x - y| \leqslant t + \max \left\{ \langle x, gx \rangle_y, \langle x, g^{-1}x \rangle_y \right\} + \delta.$$

However $[g] > 40\delta$ hence $|gy - g^{-1}y| > 33\delta$. Consequently $\langle x, gx \rangle_y \leqslant 33\delta$, $\langle x, g^{-1}x \rangle_y \leqslant 33\delta$ and $\langle g^{-1}x, gx \rangle_y \leqslant 45\delta$. It follows that $|t - |x - y|| \leqslant 45\delta$. However $|x - y|$ is exactly $d(x, N)$. Hence $|t - d(x, Y_g)| \leqslant 87\delta$. □

Lemma 1.13. *Let $a \geqslant 0$. Let g be an isometry of X such that $[g] > 40\delta$. Let x and x' be two points of X. We assume that $|[x, x'] \cap Y_g| > [g]/2 + a > 150\delta$. Then there exists $k \in \mathbf{Z}$ such that $|g^k x' - x| < |x' - x| - a + 183\delta$.*

Proof. Let N be a nerve of g. Its 42δ-neighbourhood contains Y_g, therefore $|[x, x'] \cap N| > [g]/2 + a - 84\delta$. We denote by y and y' respective projections of x and x' on N. Lemma 1.6 gives $|y' - y| > [g]/2 + a - 117\delta > 33\delta$. Combined with the projection on N we obtain

$$|x' - x| > |x' - y'| + \frac{1}{2}[g] + a + |y - x| - 183\delta$$

On the other hand g acts on N by translation of length $[g]$. Hence there exists $k \in \mathbf{Z}$ such that $|g^k y' - y| \leqslant [g]/2$. The triangle inequality yields

$$|g^k x' - x| \leqslant |x' - y'| + \frac{1}{2}[g] + |y - x| < |x' - x| - a + 183\delta,$$

which completes the proof. □

Lemma 1.14. *Let $a \geqslant 0$. Let g and h be two isometries of X such that $[g] > 40\delta$. We assume that*

$$\min \left\{ [h], |Y_h \cap Y_g| \right\} > \frac{1}{2}[g] + a > 324\delta.$$

Then, there exists $k \in \mathbf{Z}$ such that $[g^k h] < [h] - a + 357\delta$.

Proof. Let N be a nerve of h. Since Y_h lies in the 42δ-neighbourhood of N we have $|Y_g \cap N| > [g]/2 + a - 84\delta$. Hence there exist x and x' in Y_g such that $|[x, x'] \cap N| > [g]/2 + a - 84\delta$. We denote by $y = N(t)$ and $y' = N(t')$ respective projections of x and x' on N. Up to change the role of x and x' we can assume that $t' \geqslant t$. Recall that N is parametrized by arclength. Hence Lemma 1.6 gives

$$|t' - t| \geqslant |y' - y| > \frac{1}{2}[g] + a - 117\delta > 33\delta.$$

Let us set $s = [g]/2 + a - 117\delta$ and $z = N(t + s)$. The isometry h acts on N by translation of length $[h]$, thus $hy = N(t + [h])$. Note that $t \leqslant t + s \leqslant \min\{t', t + [h]\}$. Consequently $\langle y, hy\rangle_z \leqslant 12\delta$ and

$$|x - x'| \geqslant |x - y| + |y - z| + |z - x'| - 90\delta.$$

In particular $|Y_g \cap [y, z]| \geqslant |y - z| - 45\delta$. It follows that

$$|Y_g \cap [y, hy]| \geqslant |Y_g \cap [y, z]| - \langle y, hy\rangle_z \geqslant |y - z| - 57\delta \geqslant \frac{1}{2}[g] + a - 174\delta > 150\delta.$$

According to Lemma 1.13, there exists $k \in \mathbf{Z}$ such that $\left|g^k hy - y\right| < |hy - y| - a + 357\delta$. However y is a point of a nerve of h and thus of the axis of h. Consequently $\left[g^k h\right] \leqslant [h] - a + 357\delta$. □

The goal of the next two results is to describe a figure that will naturally arise in Part 3. Since the proof only requires some basic properties of hyperbolicity, we give it here. It will considerably lighten the proofs involving foldable configurations (see Sections 3.3-3.5). The constants a, b and c which appear in the following statements will be made precise in Part 3. They represent distances which are large in comparison to δ but small compared to $[g]$.

Proposition 1.15. *Let $a, b, c \geqslant 0$. Let g be an isometry of X such that $[g] > 2a + 2b + 2c + 612\delta$. Let x, y and z be three points of X. We assume that there exists a point $s \in X$ such that $|[s, y] \cap Y_g| \leqslant [g]/2 + a$ and $|x - s| \leqslant \langle y, z\rangle_x + b$. Let N be a nerve of g. We denote by p and q respective projections of y and z on N. Let r be a projection of x on $(p, q)_N$. If $|[y, z] \cap Y_g| \geqslant [g] - c$, then we have*

(i) $|p - q| \geqslant [g] - c - 117\delta$,
$|p - r| \leqslant [g]/2 + a + b + 144\delta$,
$|q - r| \geqslant [g]/2 - a - b - c - 261\delta$,

(ii) $\langle x, y\rangle_z \geqslant \langle x, y\rangle_r + |z - q| + |q - r| - 110\delta$.

Remark : The conditions on s have the following signification. By hyperbolicity, $[x, y]$ is contained in the 4δ-neighbourhood of $[x, z] \cup [z, y]$. The part of the geodesic $[x, y]$ which lies in the 4δ-neighbourhood of $[y, z]$ can not have a large overlap with the cylinder of g (see Figure 1). We could have chosen for s the point of $[x, y]$ such that $|x - s| = \langle y, z\rangle_x$ and asked that $|[s, y] \cap Y_g| \leqslant [g]/2 + a$. However in Part 3, we will need this more general assumption.

Proof. The 42δ-neighbourhood of N contains Y_g, thus $|[y, z] \cap N| \geqslant [g] - c - 84\delta$. Since p and q are respective projections of y and z on N, we get by Lemma 1.6 $|p - q| \geqslant [g] - c - 117\delta > 33\delta$. This proves the first inequality of Point (i).

Upper bound of $|p - r|$. We may assume that $|p - r| > 45\delta$. Hence $|p - q| \geqslant |p - r| - \langle p, q\rangle_r > 33\delta$. The points p and q are respective projections of y, z on N, thus $\langle y, z\rangle_x \leqslant |x - r| + \langle y, z\rangle_r \leqslant |x - r| + 45\delta$. Using our second assumption on s we obtain $|x - s| \leqslant |x - r| + b + 45\delta$. However, by hyperbolicity we have

$$\langle s, y\rangle_r \leqslant \max\left\{|x - s| - |x - r| + 2\langle x, y\rangle_r, \langle x, y\rangle_r\right\} + \delta$$

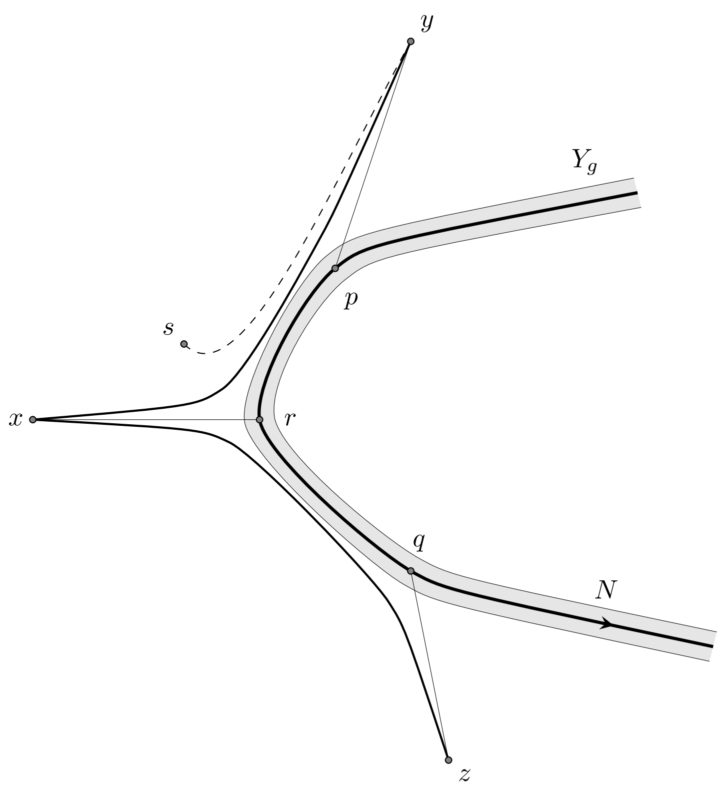


Figure 1: Signification of the point s

Since $\langle x, y\rangle_r \leqslant 33\delta$ we get $\langle s, y\rangle_r \leqslant b + 111\delta$. The point p is a projection of y on N. By Proposition 1.6 we have

$$|r-p| \leqslant |[r,y] \cap N| + 33\delta \leqslant |[s,y]\cap N| + \langle s, y\rangle_r + 33\delta \leqslant |[s,y]\cap Y_g| + b + 144\delta.$$

The second inequality of Point (i) follows then from the first assumption on s.

Lower bound of $|q-r|$. The third inequality of Point (i) follows by triangle inequality from the two previous ones.

Estimation of $\langle x, y\rangle_z$. As a consequence of Point (i), $|q-r| > 45\delta$, thus $\langle x, z\rangle_r \leqslant 33\delta$ and $\langle y, z\rangle_r \leqslant 45\delta$. However

$$\langle x, y\rangle_z = \langle x, y\rangle_r + |z-r| - \langle x, z\rangle_r - \langle y, z\rangle_r \geqslant \langle x, y\rangle_r + |z-r| - 78\delta.$$

Since q is a projection of z on N we have $|z-r| \geqslant |z-q| + |q-r| - 32\delta$, which combined with the previous inequality gives Point (ii). □

Proposition 1.16. *Let a, b and c be non-negative constants. Let g be an isometry of X such that $[g] > 2a + 4b + 2c + 830\delta$. Let x, y_1 and y_2 be three points of X. We assume that there exist two points $s_1, s_2 \in X$ such that for all $i \in \{1,2\}$, $|[s_i, y_i] \cap Y_g| \leqslant [g]/2 + a$ and $|x - s_i| \leqslant \langle y_1, y_2\rangle_x + b$. Let N be a* *nerve of g. We denote by r, q_1 and q_2 respective projections of x, y_1 and y_2 on N. If $|[y_1, y_2] \cap Y_g| \geqslant [g] - c$, then we have the followings*

(i) *r belongs to $(q_1, q_2)_N$,*

(ii) $|q_1 - q_2| \geqslant [g] - c - 117\delta$,
$[g]/2 - a - b - c - 261\delta \leqslant |r - q_i| \leqslant [g]/2 + a + b + 144\delta$,

(iii) $\langle x, y_i\rangle_r, \langle x, y_i\rangle_{q_i} \leqslant 33\delta$ *and* $\langle s_i, y_i\rangle_{q_i} \leqslant 34\delta$.

(iv) $\left|\langle y_1, y_2\rangle_x - |x - r|\right| \leqslant 45\delta$.

Remark : Intuitively, we have Figure 2 in mind. The goal of this proposition is to prove that this picture actually corresponds to the reality.

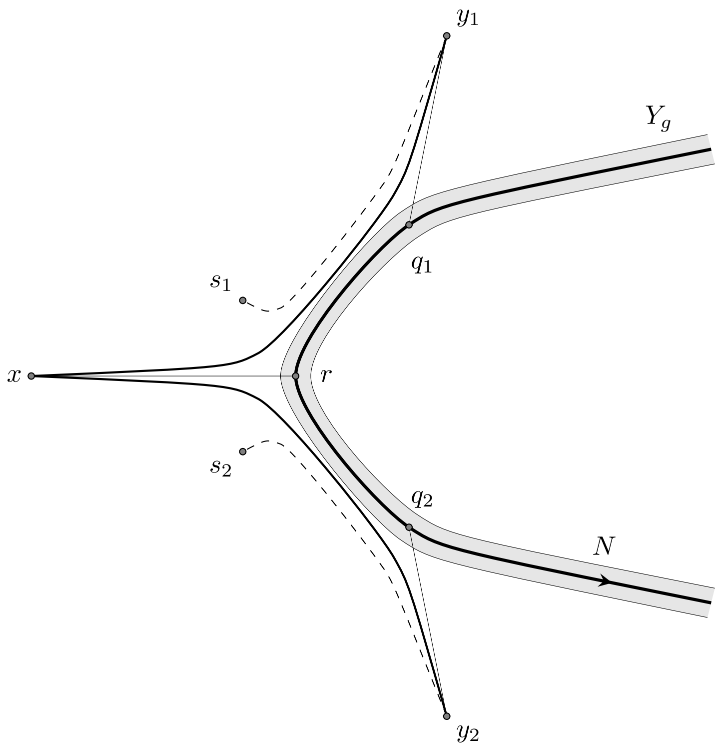


Figure 2: Positions of the points q_1, q_2 and r.

Proof. We prove Point (i) by contradiction. Assume that r does not belong to $(q_1, q_2)_N$. By symmetry we can assume that q_1 is a point of $(r, q_2)_N$. Let q be a point of $(q_1, q_2)_N$. Since r is a projection of x on N, $|x - q| \geqslant |x - r| + |r - q| - 32\delta$. However q_1 lies on N between r and q. Therefore we obtain $|x - q| \geqslant |x - q_1| - 44\delta$. Consequently q_1 is a 44δ-projection of x on $(q_1, q_2)_N$. By Proposition 1.3, the distance between q_1 and a projection t of x on $(q_1, q_2)_N$ is at most 154δ. Nevertheless Proposition 1.15 Point (i) gives $|q_1 - t| \geqslant [g]/2 - a - b - c - 261\delta$. Contradiction. Hence r belongs to $(q_1, q_2)_N$. Therefore, Point (ii) follows from Proposition 1.15.

The points r and q_i are respective projections of x and y_i on N. Thus $\langle x, y_i \rangle_r, \langle x, y_i \rangle_{q_i} \leqslant 33\delta$ and $\langle y_1, y_2 \rangle_r \leqslant 45\delta$, which proves in particular the first part of Point (iii). The hyperbolicity condition yields

$$\langle y_1, y_2 \rangle_x - \langle y_1, y_2 \rangle_r \leqslant |x - r| \leqslant \langle y_1, y_2 \rangle_x + \max \left\{ \langle x, y_1 \rangle_r, \langle x, y_2 \rangle_r \right\} + \delta$$

which leads to Point (iv). What is left to show is that $\langle s_i, y_i \rangle_{q_i} \leqslant 34\delta$. By hyperbolicity we have

$$\langle s_i, y_i \rangle_{q_i} \leqslant \max \left\{ |x - s_i| - |x - q_i| + 2 \langle x, y_i \rangle_{q_i}, \langle x, y_i \rangle_{q_i} \right\} + \delta.$$

However $\langle x, y_i \rangle_{q_i} \leqslant 33\delta$, thus it is sufficient to give an upper bound to $|x - s_i| - |x - q_i|$. Since r is a projection of x on N, one has $|x - q_i| \geqslant |x - r| + |r - q_i| - 32\delta$. However we already proved that $|x - r| \geqslant \langle y_1, y_2 \rangle_x - 45\delta \geqslant |x - s_i| - b - 45\delta$. Hence $|x - q_i| \geqslant |x - s_i| + |r - q_i| - b - 77\delta$. It follows then from (ii) that $|x - s_i| - |x - q_i| + 2 \langle x, y_i \rangle_{q_i} \leqslant \langle x, y_i \rangle_{q_i}$ which leads to the result. □

1.4 Hyperbolic groups

In this section X is still geodesic and proper. We consider a group G acting properly, co-compactly by isometries on X. It follows that every element of G is either *elliptic* (and has finite order) or *hyperbolic* (see [6, Chap. 9, Th. 3.4]). A subgroup of G is called *elementary* if it is virtually cyclic. Every non-elementary subgroup of G contains a copy of $\mathbf{F}_2$, the free group of rank 2 (see [13, Chap. 8, Th. 37]). Given a hyperbolic element g of G, the subgroup of G stabilizing $\{g^-, g^+\} \subset \partial X$ is elementary. In particular the normalizer of g is elementary (see [6, Chap. 10, Prop 7.1]).

Notation : If P is a subset of G, we denote by P^* the set of hyperbolic elements of P.

Definition 1.17. *Let P be a subset of G.*

- *The* injectivity radius *of P on X, denoted by $r_{inj}(P, X)$, is defined by $r_{inj}(P, X) = \inf_{\rho \in P^*} [\rho]^\infty_X$.*
- *The* maximal overlap *of P on X, denoted by $\Delta(P, X)$, is the quantity $\Delta(P, X) = \sup_{\rho \neq \rho' \in P^*} |Y_\rho \cap Y_{\rho'}|$.*

Definition 1.18. *The A invariant of G on X, denoted by $A(G, X)$, is the upper bound of $|A_g \cap A_h|$, where g and h are two elements of G which generate a non-elementary subgroup and whose translation lengths are smaller than 1000δ.*

Proposition 1.19 (see [12, Prop. 2.4.3], [9, Prop. 2.41])**.** *We assume that every elementary subgroup of G is cyclic. Let g and h be two elements of G such that $[g] \leqslant 1000\delta$. If the subgroup generated by g and h is non-elementary, then*

$$|A_g \cap A_h| \leqslant [h] + A(G, X) + 1000\delta.$$

Vocabulary : The group G satisfies the *small centralizers hypothesis* if G is non-elementary and every elementary subgroup of G is cyclic.

2 Cone-off over a metric space

In this section we focus on the cone-off over a metric space (see [12]). Let us fix a positive real number r_0. Its value will be made precise later. It should be thought as a very large scale parameter.

2.1 Cone over a metric space

We review the construction of a cone over a metric space. For more details see [3, Chap. I.5]. Let Y be metric space. The cone of radius r_0 over Y, denoted by $Z(Y, r_0)$ (or simply $Z(Y)$) is the quotient of $Y \times [0\,, r_0]$ by the equivalence relation which identifies all the points of the form $(y, 0)$, $y \in Y$. The equivalence class of $(y, 0)$ is the apex of the cone, denoted by v. We endow Y with a metric characterized as follows. Given any two points $x = (y, r)$ and $x' = (y', r')$ of $Z(Y)$,

$$\operatorname{ch}\left(|x - x'|\right) = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos\left(\min\left\{\pi, \frac{|y - y'|}{\operatorname{sh} r_0}\right\}\right).$$

In order to compare the cone $Z(Y)$ and its base Y we introduce two maps.

$$\begin{array}{rccc} \iota: & Y & \to & Z(Y) \\ & y & \to & (y, r_0) \end{array} \qquad\qquad \begin{array}{rccc} p: & Z(Y) \setminus \{v\} & \to & Y \\ & (y, r) & \to & y \end{array}$$

If y and y' are two points of Y, the distance between $\iota(y)$ and $\iota(y')$ is then given by $|\iota(y) - \iota(y')| = \mu\left(|y - y'|\right)$ where $\mu : \mathbf{R}^+ \to \mathbf{R}^+$ is defined in the following way: for all $t \in \mathbf{R}^+$,

$$\operatorname{ch}\left(\mu(t)\right) = \operatorname{ch}^2 r_0 - \operatorname{sh}^2 r_0 \cos\left(\min\left\{\pi, \frac{t}{\operatorname{sh} r_0}\right\}\right).$$

The function μ is non-decreasing, concave and subadditive. Moreover, for all $t \in \mathbf{R}^+$, $\mu(t) \leqslant t$ (see [8]). A coarse computation proves also that for all $t \in [0, \pi \operatorname{sh} r_0]$, $t \leqslant \pi \operatorname{sh}\left(\mu(t)/2\right)$. It follows from the concavity that for every $r, s, t \geqslant 0$

$$\mu(r + s) \leqslant \mu(r + t) + \mu(t + s) - \mu(t) \tag{4}$$

If Y is a length space, so is $Z(Y)$. More precisely, let $x = (y, r)$ and $x' = (y', r')$ be two points of $Z(Y)$. Let $\sigma : I \to Y$ be a rectifiable path between y and y'. If its length $L(\sigma)$ is strictly smaller than $\pi \operatorname{sh} r_0$, then there exists a rectifiable path $\tilde\sigma : I \to Z(Y) \setminus \{v\}$ between x and x' such that $p \circ \tilde\sigma = \sigma$ and whose length satisfies

$$\operatorname{ch}\left(L\left(\tilde\sigma\right)\right) \leqslant \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos\left(\frac{L(\sigma)}{\operatorname{sh} r_0}\right).$$

We now consider a group H acting properly, by isometries on Y. We denote by $\bar Y$ the quotient Y/H. For all $y \in Y$, we write $\bar y$ for the image of y in $\bar Y$. The space $\bar Y$ is endowed with a metric defined by $|\bar y - \bar y'| = \inf_{h \in H} |y - hy'|$. The action of H on Y can be extended to $Z(Y)$ by homogeneity: if $(y, r) \in Z(Y)$ and $h \in H$, then $h(y, r) = (hy, r)$. Hence H acts on $Z(Y)$ by isometries. If Y

is not compact, this action may not be proper. The stabilzer of v (i.e. H) may indeed be not finite. Nevertheless the formula $|\bar{x}-\bar{x}'| = \inf_{h\in H}|x-hx'|$ still defines a metric on $Z(Y)/H$. Moreover the spaces $Z(Y)/H$ and $Z(Y/H)$ are isometric (see [8]).

Lemma 2.1. *Let $l \geqslant 2\pi \operatorname{sh} r_0$. We assume that for every $h \in H\setminus\{1\}$, $[h] \geqslant l$. Let $x=(y,r)$ and $x'=(y',r')$ be two points of $Z(Y)$. If $|y-y'|_Y \leqslant l - \pi \operatorname{sh} r_0$ then $|\bar{x}-\bar{x}'| = |x-x'|$.*

Proof. Since $Z(Y/H)$ and $Z(Y)/H$ are isometric, the distance between $\bar{x}$ and $\bar{x}'$ in $Z(Y)/H$ is given by

$$\operatorname{ch}(|\bar{x}-\bar{x}'|) = \operatorname{ch} r \operatorname{ch} r' - \operatorname{sh} r \operatorname{sh} r' \cos\left(\min\left\{\pi, \frac{|\bar{y}-\bar{y}'|_{\bar{Y}}}{\operatorname{sh} r_0}\right\}\right).$$

If $|y-y'| < l/2$, then we have $|\bar{y}-\bar{y}'| = |y-y'|$. It follows that $|\bar{x}-\bar{x}'| = |x-x'|$. Assume now that $|y-y'| \geqslant l/2$. In particular $|y-y'| \geqslant \pi \operatorname{sh} r_0$. Thus $|x-x'| = r+r'$. On the other hand, using the triangle inequality, for all $h \in H\setminus\{1\}$, $|y-hy'| \geqslant l - |y-y'|$, thus $|\bar{y}-\bar{y}'| \geqslant \pi \operatorname{sh} r_0$. Consequently $|\bar{x}-\bar{x}'| = r+r' = |x-x'|$. □

2.2 Cone-off over a metric space

We give here a brief exposition of the construction of the cone-off over a metric space. For details and proofs we refer the reader to [8] and [9]. For the rest of this section X denotes a geodesic, δ-hyperbolic space and $Y=(Y_i)_{i\in I}$ a family of strongly quasi-convex subsets of X (see Definition 1.1).

Definition 2.2. *The maximal overlap between the Y_i's is measured by the quantity*

$$\Delta(Y) = \sup_{i\neq j} |Y_i \cap Y_j|\,.$$

For all $i \in I$ we define the following objects:

(i) Y_i is endowed with the length metric $|\,.\,|_{Y_i}$ induced by the restriction to Y_i of $|\,.\,|_X$. Since Y_i is strongly quasi-convex, for all $y, y' \in Y_i$ we have

$$|y-y'|_X \leqslant |y-y'|_{Y_i} \leqslant |y-y'|_X + 40\delta.$$

(ii) Z_i is the cone of radius r_0 over $(Y_i, |\,.\,|_{Y_i})$ and v_i its apex.

(iii) $\iota_i : Y_i \to Z_i$ and $p_i : Z(Y_i)\setminus\{v_i\} \to Y_i$ are the comparison maps defined in the previous section.

The *cone-off* of radius r_0 over X relatively to Y is the space obtained by attaching each cone Z_i on X along Y_i according to ι_i. We denote it by $\dot{X}(Y, r_0)$ or simply $\dot{X}$.

The next step is to define a metric on $\dot{X}$. Given x and x' two points of $\dot{X}$ we denote by $\|x-x'\|$ the minimal distance between two points of $X \sqcup \left(\bigsqcup_{i\in I} Z_i\right)$ whose images in $\dot{X}$ are respectively x and x'.

Remark : If x and x' are two points of the base X, $\|x - x'\|$ can be computed as follows:

$$\|x - x'\| = \min \left[|x - x'|_X , \inf \left\{ \mu \left(|x - x'|_{Y_i} \right) | i \in I,\ x, x' \in Y_i \right\} \right].$$

In particular,

$$\mu \left(|x - x'|_X \right) \leqslant \|x - x'\| \leqslant |x - x'|_X .$$

Moreover, if there is $i \in I$ such that $x, x' \in Y_i$ then $\|x - x'\| \leqslant \mu \left(|x - x'|_X \right) + 40\delta$.

Definition 2.3. *Let x and x' be two points of $\dot{X}$. A chain between x and x' is a finite sequence $C = (z_1, \dots, z_m)$ such that $z_1 = x$ and $z_m = x'$. Its length is $l(C) = \|z_1 - z_2\| + \dots + \|z_{m-1} - z_m\|$.*

Proposition 2.4. *Given x and x' in $\dot{X}$, the following formula defines a length metric on $\dot{X}$.*

$$|x - x'|_{\dot{X}} = \inf \left\{ l(C) | C \text{ chain between } x \text{ and } x' \right\}.$$

Note that given a chain between two points of X, one can always find an shorter chain joining the same extremities, whose points belong to X. (Just apply the triangle inequality in $X \sqcup \left(\bigsqcup_{i \in I} Z_i \right)$.) Therefore, in the rest of the section, we will only consider chains whose points lie in X.

Remark : In the rest of Section 2, we will work with two metric spaces : X and $\dot{X}$. Unless stated otherwise all distances, Gromov's products and geodesics are computed with the distance of X. To avoid any confusion the distance between two points x and x' in $\dot{X}$ will be written $|x - x'|_{\dot{X}}$.

Theorem 2.5 (see [9, Prop. 6.4] or [11, Coro. 5.27])**.** *There exist positive numbers δ_0, δ_1 and Δ_0 and r_1 which do not depend on X or Y with the following property. If $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$, then $\dot{X}(Y, r_0)$ is δ_1-hyperbolic.*

2.3 Shortening chains

Our goal is now to compare the geometry of $\dot{X}$ and X. In [12], T. Delzant and M. Gromov proved that the natural map $X \to \dot{X}$ restricted to any ball of radius 1000δ is a quasi-isometric embedding. For our purpose we need to compare X and $\dot{X}$ at a larger scale. In particular we have to take into account paths passing through the apices of $\dot{X}$.

Coarsly speaking we prove that the projection p preserves the shapes. For instance if x and x' are two points of X, the projection by p of a quasi-geodesic of $\dot{X}$ between them remains in the neighbourhood of any geodesic of X joining x and x' (see Proposition 2.12). To that end we proceed in two steps. Let x, y, z and t be four points of X. If $\langle x, t \rangle_y$ or $\langle x, t \rangle_z$ is large (compare to $\Delta(Y)$ and δ) we first explain how to shorten the chain $C = (x, y, z, t)$ (see Proposition 2.9). Then we combine this fact with the stability of discrete quasi-geodesics to show that the points of a chain between x and x' whose length approximates $|x - x'|_{\dot{X}}$ lie in the neighbourhood of $[x, x']$ (see Proposition 2.10).

Lemma 2.6. *Let $x, x' \in X$ and $p, p' \in [x, x']$. There exists a chain C joining p to p' whose length is at most $\|x - x'\| + 64\delta$.*

Proof. If $\|x - x'\| = |x - x'|$ then the chain $C = (p, p')$ works. Thus we can assume that there exists $i \in I$ such that $x, x' \in Y_i$. The subset Y_i being 6δ-quasi-convex, there are $q, q' \in Y_i$ such that $|p - q| \leqslant 6\delta$ and $|p' - q'| \leqslant 6\delta$. We choose for C the chain $C = (p, q, q', p')$. Its length is bounded above by $\mu\left(|q - q'|\right) + 52\delta$. However $|q - q'| \leqslant |x - x'| + 12\delta$. Consequently $l(C) \leqslant \mu\left(|x - x'|\right) + 64\delta \leqslant \|x - x'\| + 64\delta$. ☐

Lemma 2.7. *Let $x, y, z \in X$, $p \in [x, y]$ and $q \in [y, z]$. We assume that there is $i \in I$ such that $x, y \in Y_i$ but there is no $j \in I$ such that $x, y, z \in Y_j$. Then there exists a chain C joining p to z satisfying*

$$l(C) \leqslant 2\,|p - q| + \|y - z\| - |y - q| + \Delta(Y) + 64\delta.$$

Proof. We distinguish two cases. Assume first that there exists $j \in I$ such that $y, z \in Y_j$. According to our hypothesis we necessary have $i \neq j$. Therefore $|[x, y] \cap [y, z]| \leqslant |Y_i \cap Y_j| \leqslant \Delta(Y)$ i.e., $\langle x, z \rangle_y \leqslant \Delta(Y)$. It follows from the triangle inequality that

$$|y - q| \leqslant \langle x, z \rangle_y + |p - q| \leqslant |p - q| + \Delta(Y). \tag{5}$$

By Lemma 2.6, there exists a chain C_0 joining q to z whose length is at most $\|y - z\| + 64\delta$. We obtain C by adding p at the beginning of C_0. It satisfies $l(C) \leqslant |p - q| + \|y - z\| + 64\delta$. Combined with (5) we get the required inequality.

Assume now that $\|y - z\| = |y - z|$. Then $\|q - z\| \leqslant \|y - z\| - |y - q|$. We choose for C the chain $C = (p, q, z)$ which satisfies $l(C) \leqslant |p - q| + \|y - z\| - |y - q|$. ☐

Lemma 2.8. *Let $x, y, z, t \in X$. If there exists $i \in I$ such that $x, t \in Y_i$ then*

$$\|x - t\| \leqslant \|x - y\| + \|y - z\| + \|z - t\| - \mu\left(\max\left\{\langle x, t \rangle_y, \langle x, t \rangle_z\right\}\right) + 40\delta$$

Proof. Since x and t are in Y_i, $\|x - t\| \leqslant \mu\left(|x - t|\right) + 40\delta$. Applying (4) we get

$$\mu(|x - t|) \leqslant \mu(|x - y|) + \mu(|y - t|) - \mu(\langle x, t \rangle_y).$$

However by triangle inequality $\mu(|y - t|) \leqslant \mu(|y - z|) + \mu(|z - t|)$. Consequently $\|x - t\| \leqslant \|x - y\| + \|y - z\| + \|z - t\| - \mu(\langle x, t \rangle_y) + 40\delta$. By symmetry we have the same inequality with $\langle x, t \rangle_z$ instead of $\langle x, t \rangle_y$. ☐

Proposition 2.9. *Let $x, y, z, t \in X$. There exists a chain C joining x to t such that*

$$l(C) \leqslant \|x - y\| + \|y - z\| + \|z - t\| - \mu\left(\max\left\{\langle x, t \rangle_y, \langle x, t \rangle_z\right\}\right) + 2\Delta(Y) + 210\delta$$

Proof. If there is $i \in I$ such that $x, t \in Y_i$, Lemma 2.8 says that the chain $C = (x, t)$ works. Therefore, for now on we assume that there is no such $i \in I$. By hyperbolicity

$$|x - z| + |y - t| \leqslant \max\left\{|x - y| + |z - t|, |x - t| + |y - z|\right\} + 2\delta \tag{6}$$

Part 1: Assume first that the maximum is achieved by $|x-t|+|y-z|$. See Figure 3. In particular it follows that $\langle x,t\rangle_z \leqslant \langle y,t\rangle_z + \delta$ and $\langle x,t\rangle_y \leqslant \langle x,z\rangle_y + \delta$. Moreover $|y-z| \geqslant \langle x,t\rangle_y + \langle x,t\rangle_z - \delta$. We denote by p and q (respectively r and s) points of $[x,y]$ and $[y,z]$ (respectively $[t,z]$ and $[z,y]$) such that

$$|y-p| = |y-q| = \max\{0, \langle x,t\rangle_y - \delta\} \text{ and } |z-r| = |z-s| = \max\{0, \langle x,t\rangle_z - \delta\}.$$

By hyperbolicity $|p-q| \leqslant 4\delta$ and $|r-s| \leqslant 4\delta$. Furthermore $|y-z| \geqslant |y-q| + |s-z|$. We need to distinguish several cases depending on whether or not the points x, y, z and t lie in a quasi-convex Y_i. In each case we implicitly exclude the previous ones.

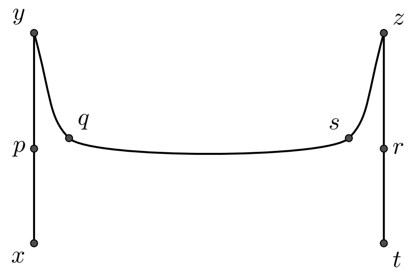


Figure 3: Shortening a four points chain - Part 1

Case 1.1: *There exist $i,j \in I$ such that $x,y,z \in Y_i$ and $y,z,t \in Y_j$.* According to our assumption at the beginning of the proof $i \neq j$. Since y and z belong to Y_i and Y_j they satisfy $|y-z| \leqslant |Y_i \cap Y_j| \leqslant \Delta(Y)$. Consequently $\langle x,t\rangle_y + \langle x,t\rangle_z \leqslant \Delta(Y) + \delta$. We choose the chain $C = (x,y,z,t)$. Thus

$$l(C) \leqslant \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + \Delta(Y) + \delta.$$

Case 1.2: *There exists $i \in I$ such that $x,y,z \in Y_i$.* The subset Y_i being 6δ–quasi-convex, there exists a point $s' \in Y_i$ such that $|s-s'| \leqslant 6\delta$. Hence $\|x-s'\| \leqslant \mu(|x-s|) + 46\delta$. Recall that q lies on $[y,z]$ between y and s. By (4) we get

$$\mu\left(|x-s|\right) \leqslant \mu\left(|x-p| + |q-s|\right) + 4\delta \leqslant \mu\left(|x-y|\right) + \mu\left(|y-s|\right) - \mu(\langle x,t\rangle_y) + 5\delta.$$

It follows that $\|x-s'\| \leqslant \|x-y\| + \|y-z\| - \mu(\langle x,t\rangle_y) + 51\delta$. On the other hand, by Lemma 2.7, there exists a chain C_0 joining s to t such that

$$l(C_0) \leqslant \|z-t\| - |z-r| + \Delta(Y) + 72\delta \leqslant \|z-t\| - \langle x,t\rangle_z + \Delta(Y) + 73\delta$$

We obtain C by adding x and s' at the beginning of C_0. Its length satisfies

$$l(C) \leqslant \|x-y\| + \|y-z\| + \|z-t\| - \mu(\langle x,t\rangle_y) - \langle x,t\rangle_z + \Delta(Y) + 130\delta.$$

Case 1.3: *There exists $i \in I$ such that $y,z,t \in Y_i$.* This case is just the symmetric of the previous one.

Case 1.4: *There exists $i \in I$ such that $y, z \in Y_i$.* By Lemma 2.6 there exists a chain C_0 joining q to s whose length is at most $\|y-z\| + 64\delta$. Applying Lemma 2.7, there is a chain C_- (respectively C_+) joining x to q (respectively s to t) such that

$$\begin{aligned} l(C_-) &\leqslant \|x-y\| - |y-p| + \Delta(Y) + 72\delta \leqslant \|x-y\| - \langle x,t\rangle_y + \Delta(Y) + 73\delta \\ l(C_+) &\leqslant \|z-t\| - |z-r| + \Delta(Y) + 72\delta \leqslant \|z-t\| - \langle x,t\rangle_z + \Delta(Y) + 73\delta \end{aligned}$$

Concatenating C_-, C_0 and C_+ we obtain a chain C such that

$$l(C) \leqslant \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 2\Delta(Y) + 210\delta.$$

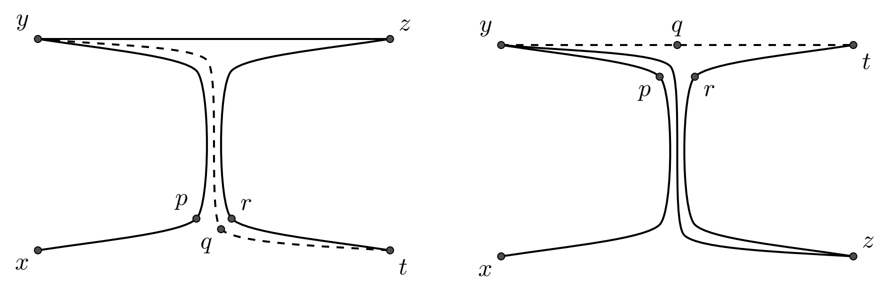
Case 1.5: *This is the last case of Part 1.* Negating the previous one there is no $i \in I$ such that $y, z \in Y_i$. In particular $\|y-z\| = |y-z|$. Hence

$$\|q-s\| \leqslant \|y-z\| - |y-q| - |z-s| \leqslant \|y-z\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 2\delta.$$

We put $C_0 = (q,s)$. According to Lemma 2.6 there is a chain C_- (respectively C_+) joining x to p (respectively r to t) whose length is at most $\|x-y\| + 64\delta$ (respectively $\|t-z\| + 64\delta$). Concatenating C_-, C_0 and C_+ we obtain a chain C such that

$$l(C) \leqslant \|x-y\| + \|y-z\| + \|z-t\| - \langle x,t\rangle_y - \langle x,t\rangle_z + 138\delta.$$

Part 2: Assume now that the maximum in (6) is achieved by $|x-y| + |z-t|$. See Figure 4. It follows that $\langle x,y\rangle_t \leqslant \langle y,z\rangle_t$. We assume that $\langle x,t\rangle_y \geqslant \langle x,t\rangle_z$ (the other case is symmetric). We denote by p and q the respective points of $[x,y]$ and $[t,y]$ such that $|y-p| = |y-q| = \langle x,t\rangle_y$. By hyperbolicity, $|p-q| \leqslant 4\delta$. On the other hand $|t-q| = \langle x,y\rangle_t \leqslant \langle y,z\rangle_t$. Consequently, if r is the point of $[z,t]$ such that $|t-r| = \langle x,y\rangle_t$ then $|q-r| \leqslant 4\delta$. Thus $|p-r| \leqslant 8\delta$. Moreover the triangle inequality leads to $\langle x,t\rangle_y \leqslant |z-y| + |z-t| - \langle x,y\rangle_t$ i.e., $\langle x,t\rangle_y \leqslant |y-z| + |z-r|$. According to Lemma 2.6 there exists a chain C_- (respectively C_+) joining x to p (respectively r to t) such that $l(C_-) \leqslant \|x-y\| + 64\delta$ (respectively $l(C_+) \leqslant \|z-t\| + 64\delta$). As previously we need to distinguish several cases.



(a) First configuration (b) Second configuration

Figure 4: Shortening a four points chain - Part 2

Case 2.1: *There exist $i, j \in I$ such that $x, y \in Y_i$ and $z, t \in Y_j$.* According to our assumption at the beginning of the proof $i \neq j$. In particular $|[x, y] \cap [z, t]| \leqslant |Y_i \cap Y_j| \leqslant \Delta(Y)$, thus $\langle x, t\rangle_y \leqslant |y - z| + \Delta(Y)$. It follows that $\mu(\langle x, t\rangle_y) \leqslant \|y - z\| + \Delta(Y)$. By contatenating C_- and C_+ we obtain a chain whose length satisfies

$$l(C) \leqslant \|x - y\| + \|y - z\| + \|z - t\| - \mu(\langle x, t\rangle_y) + \Delta(Y) + 136\delta$$

Case 2.2: *There exists $i \in I$ such that $x, y \in Y_i$.* In this case $\|z - t\| = |z - t|$, thus $\|r - t\| \leqslant \|z - t\| - |z - r|$. We obtain C by adding r and t at the end of C_-. This new chain satisfies.

$$l(C) \leqslant \|x - y\| + \|z - t\| - |z - r| + 72\delta$$

However we proved that $\langle x, y\rangle_t \leqslant |y - z| + |z - r|$. In particular $\mu(\langle x, y\rangle_t) \leqslant \|y - z\| + |z - r|$. Consequently

$$l(C) \leqslant \|x - y\| + \|y - z\| + \|z - t\| - \mu(\langle x, t\rangle_y) + 72\delta$$

Case 2.3: *This is the last case of Part 2.* In particular $\|x - y\| = |x - y|$. It follows that $\|x - p\| \leqslant \|x - y\| - |y - p|$ i.e., $\|x - p\| \leqslant \|x - y\| - \langle x, t\rangle_y$. We obtain C by adding x and p at the beginning of C_+. It satisfies

$$l(C) \leqslant \|x - y\| + \|z - t\| - \langle x, y\rangle_t + 72\delta$$

□

Proposition 2.10. *Let $\varepsilon > 0$. There exist positive numbers δ_0, Δ_0, r_1 and η which only depend on ε with the following property. Assume that $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$. Let $x, x' \in X$. Let C be a chain of points of X joining x to x'. If $l(C) \leqslant |x - x'|_{\dot{X}} + \eta$, then every point of C is contained in the ε-neighbourhood of $[x, x']$.*

Proof. We start by defining the constants δ_0, Δ_0, r_1 and η. Given r_0 the function μ defined in Section 2 satisfies

$$\forall t \in \mathbf{R}_+, \quad \mu(t) \geqslant t - \frac{1}{24}\left(1 + \frac{1}{\operatorname{sh}^2 r_0}\right) t^3$$

Thus there exist $r_1 \geqslant 0$ and $t_0 > 0$ with the following property. If $r_0 \geqslant r_1$ then for every $t \in [0, t_0]$, $\mu(t) \geqslant t/2$. We now fix $r_0 \geqslant r_1$. Since μ is increasing, for every $t \in \mathbf{R}_+$ if $\mu(t) < \mu(t_0)$ then $t \leqslant 2\mu(t)$. Let us put $l = 500$. The numbers L and d are given by the stability of discrete quasi-geodesics (Corollary 1.9). Without loss of generality, we can assume that $L > l$. We choose $\delta_0 > 0$, $\Delta_0 > 0$ and $\eta > 0$ such that

(i) $2\Delta_0 + 24\delta_0^3 (L + l)^3 + 210\delta_0 + \eta < \mu(t_0)$,

(ii) $4\Delta_0 + 48\delta_0^3 (L + l)^3 + 421\delta_0 + 2\eta \leqslant l\delta_0$

(iii) $\delta_0 (d + 3L + 3l) \leqslant \varepsilon$

From now on we assume that $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$. In particular X is δ_0-hyperbolic. Let $x, x' \in X$ and $C = (z_0, \dots, z_n)$ be a chain of points of X joining x to x' such that $l(C) \leqslant |x - x'|_{\dot{X}} + \eta$. Note that for every $i \leqslant j$, the length of the subchain $(z_i, z_{i+1}, \dots, z_{j-1}, z_j)$ is at most $|z_j - z_i|_{\dot{X}} + \eta$.

We now extracts a subchain of C. To that end we proceed in two steps. First we define a subchain $C_1 = (z_{i_0}, \dots z_{i_m})$ of C as explained in [8, Section 3.2].

- Put $i_0 = 0$.
- Assume that i_k is defined. If $|z_{i_k+1} - z_{i_k}| > 2\delta_0 (L+l)$ then $i_{k+1} = i_k + 1$, otherwise i_{k+1} is the largest integer $i \in \{i_k + 1, \dots, n\}$ such that $|z_i - z_{i_k}| \leqslant 2\delta_0 (L+l)$.

By construction, for all $k \in \{0, \dots, m-2\}$ either $\left|z_{i_{k+2}} - z_{i_{k+1}}\right| > \delta_0 (L+l)$ or $\left|z_{i_{k+1}} - z_{i_k}\right| > \delta_0 (L+l)$. Moreover every point of C is $2\delta_0 (L+l)$-close to a point of C_1.

Claim 1. For every $k, k' \in \{0, \dots, m\}$ the length of the subchain $(z_{i_k}, \dots, z_{i_{k'}})$ of C_1 is bounded above by $\left|z_{i_k} - z_{i_{k'}}\right|_{\dot X} + 8\delta_0^3 (L+l)^3 |k - k'| + \eta$. (See [8, Lemma 3.2.3]).

We now build the chain $C_2 = (x_0, y_0, x_1, y_1, \dots y_{p-1}, x_p)$ as follows.

- Put $x_0 = z_{i_0}$.
- Assume that $x_j = z_{i_k}$ is already defined. If $\left|z_{i_{k+1}} - z_{i_k}\right| > \delta_0 (L+l)$ we put $y_j = x_j$ and $x_{j+1} = z_{i_{k+1}}$, otherwise we chose $y_j = z_{i_{k+1}}$ and $x_{j+1} = z_{i_{k+2}}$. (If $z_{i_{k+1}}$ is already the last point of C_1 i.e., if $k+1 = m$ we chose $x_{j+1} = z_{i_{k+1}}$.)

In this way for all $j \in \{0, \dots, p-2\}$, $|x_{j+1} - y_j| > \delta_0 (L+l)$. Moreover, every point of C is $3\delta_0 (L+l)$-close to a point of $\{x_0, x_1, \dots, x_p\}$.

Claim 2. For all $j \in \{0, \dots, p-1\}$, we have $\langle x_j, x_{j+1} \rangle_{y_j} \leqslant l\delta_0$. Let $j \in \{0, \dots, p-1\}$. According to Claim 1, we have

$$\|x_j - y_j\| + \|y_j - x_{j+1}\| \leqslant |x_{j+1} - x_j|_{\dot X} + 16\delta_0^3 (L+l)^3 + \eta.$$

On the other hand applying Proposition 2.9 with the points x_j, y_j ,y_j and x_{j+1} we obtain a chain joining x_j to x_{j+1} whose length is at most $\|x_j - y_j\| + \|y_j - x_{j+1}\| - \mu(\langle x_j, x_{j+1} \rangle_{y_j}) + 2\Delta(Y) + 210\delta$. Hence

$$\mu(\langle x_j, x_{j+1} \rangle_{y_j}) \leqslant 2\Delta_0 + 16\delta_0^3 (L+l)^3 + 210\delta_0 + \eta < \mu(t_0)$$

It follows from the definitions of t_0, δ_0, Δ_0 and η that

$$\langle x_j, x_{j+1} \rangle_{y_j} \leqslant 4\Delta_0 + 32\delta_0^3 (L+l)^3 + 420\delta_0 + 2\eta \leqslant l\delta_0.$$

Claim 3. For all $j \in \{0, \dots, p-2\}$, we have $\langle x_j, x_{j+2} \rangle_{x_{j+1}} \leqslant l\delta_0$. Let $j \in \{0, \dots, p-2\}$. Applying to Claim 1, we have

$$\|y_j - x_{j+1}\| + \|x_{j+1} - y_{j+1}\| + \|y_{j+1} - x_{j+2}\| \leqslant |x_{j+2} - y_j|_{\dot X} + 24\delta_0^3 (L+l)^3 + \eta.$$

On the other hand according to Proposition 2.9 applied to the points y_j, x_{j+1}, y_{j+1} and x_{j+2} there exists a chain joining y_j to x_{j+2} whose length is at most

$\|y_j - x_{j+1}\| + \|x_{j+1} - y_{j+1}\| + \|y_{j+1} - x_{j+2}\| - \mu(\langle y_j, x_{j+2}\rangle_{x_{j+1}}) + 2\Delta(Y) + 210\delta$. Using the same argument as in Claim 2, we obtain that

$$\langle y_j, x_{j+2}\rangle_{x_{j+1}} \leqslant 4\Delta_0 + 48\delta_0^3 (L+l)^3 + 420\delta_0 + 2\eta \leqslant (l-1)\delta_0.$$

By hyperbolicity we get

$$\min\left\{\langle y_j, x_j\rangle_{x_{j+1}}, \langle x_j, x_{j+2}\rangle_{x_{j+1}}\right\} \leqslant \langle y_j, x_{j+2}\rangle_{x_{j+1}} + \delta_0 \leqslant l\delta_0$$

However using Claim 2,

$$\langle y_j, x_j\rangle_{x_{j+1}} = |x_{j+1} - y_j| - \langle x_j, x_{j+1}\rangle_{y_j} > \delta_0 (L+l) - l\delta_0 > l\delta_0.$$

Consequently $\langle x_j, x_{j+2}\rangle_{x_{j+1}} \leqslant l\delta_0$.

Claim 4. For all $j \in \{0, \dots, p-2\}$ we have $|x_{j+1} - x_j| > L\delta_0$. The triangle inequality combined with Claim 2 gives

$$|x_{j+1} - x_j| \geqslant |x_{j+1} - y_j| - \langle x_j, x_{j+1}\rangle_{y_j} > \delta_0 (L+l) - l\delta_0.$$

Claims 3 and 4 exactly say that $x_0, x_1, \dots, x_p$ satisfies the assumptions of the stability of discrete quasi-geodesics (Proposition 1.9). Therefore for every $j \in \{0, \dots, p\}$, x_j lies in the $d\delta_0$-neigbourhood of $[x_0, x_p]$ i.e., $[x, x']$. Nevertheless we noticed that every point of C is $3\delta_0 (L+l)$-close to some x_j. Thus the distance between any point of C and $[x, x']$ is a most $\delta_0 (d + 3L + 3l) \leqslant \varepsilon$. □

2.4 Paths in a cone-off

In this section, X is still a geodesic, δ-hyperbolic space and $Y = (Y_i)_{i \in I}$ a family of strongly quasi-convex subsets of X. We denote by $\dot{X}$ the cone-off $\dot{X}(Y, r_0)$.

Lemma 2.11. *Let x and x' be two points of X. For all $\eta > 0$, there exists a path $\sigma : J \to \dot{X}$ between them whose length $L(\sigma)$ is smaller than $\|x - x'\| + \eta$ and for all $t \in J$, if $\sigma(t)$ is not the apex of a cone Z_i then $p \circ \sigma(t)$ belongs to the 65δ-neighbourhood of $[x, x']$.*

Proof. If $\|x - x'\| = |x - x'|_X$ the geodesic of X joining x to x' works. Therefore we can assume that $\|x - x'\| \neq |x - x'|_X$. Let $\varepsilon > 0$. By definition of $\| \, . \, \|$, there exists $i \in I$ such that $x, x' \in Y_i$ and $|x - x'|_{Z_i} < \|x - x'\| + \varepsilon$. We distinguish two cases.

Case 1: If $|x - x'|_{Y_i} \geqslant \pi \operatorname{sh} r_0$, then $|x - x'|_{Z_i} = 2r_0$. We chose for $\sigma : J \to Z_i$ the geodesic of Z_i $[x, v_i] \cup [v_i, x']$. (Recall that v_i is the apex of the cone Z_i.) Its length (as a path of Z_i) is $2r_0$. Moreover for all $t \in J$, if $\sigma(t) \neq v_i$, then $p \circ \sigma(t) \in \{x, x'\}$.

Case 2: If $|x - x'|_{Y_i} < \pi \operatorname{sh} r_0$. The space $\left(Y_i, | \, . \, |_{Y_i}\right)$ is a length space. Thus there exists a path $\sigma_Y : J \to Y_i$ parametrized by arc length between x and x' whose length is less than $\min\{|x - x'|_{Y_i} + \varepsilon, \pi \operatorname{sh} r_0\}$. Hence there exists a path

$\sigma : J \to Z_i \setminus \{v_i\}$ between x and x' such that $p_i \circ \sigma = \sigma_Y$ and its length $L(\sigma)$ (as a path of Z_i) satisfies

$$L(\sigma) \leqslant \mu\left(L\left(\sigma_Y\right)\right) \leqslant \mu\left(|x - x'|_{Y_i} + \varepsilon\right) \leqslant \|x - x'\| + 2\varepsilon$$

However Y_i is strongly quasi-convex. It follows that for all $y, y' \in Y_i$, $|y - y'|_X \leqslant |y - y'|_{Y_i} \leqslant |y - y'|_X + 40\delta$. Consequently, as a path of X, σ_Y is a $(1, \varepsilon + 40\delta)$-quasi-geodesic. In particular $\sigma_Y(J)$ lies in the $\left(\frac{3}{2}\varepsilon + 64\delta\right)$-neighbourhood of $[x, x']$.

Hence we have build a path $\sigma : J \to Z_i$, whose length (as a path of Z_i) is smaller than $\|x - x'\| + 2\varepsilon$ and such that for all $t \in J$, if $\sigma(t) \neq v_i$, $p \circ \sigma(t)$ belongs to the $\left(\frac{3}{2}\varepsilon + 64\delta\right)$-neighbourhood of $[x, x']$. However the map $Z_i \to \dot{X}$ is 1-lipschitz. It follows that the length of σ as a path of $\dot{X}$ is also smaller than $\|x - x'\| + 2\varepsilon$. By choosing ε small enough we obtain the announced result. □

Proposition 2.12. *Let $\varepsilon > 0$. There exist positive constants δ_0, Δ_0 and r_1 which only depend on ε having the following property. Assume that $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$. Let x and x' be two points of $X \subset \dot{X}(Y, r_0)$. For all $\eta > 0$, there exists a $(1, \eta)$-quasi-geodesic $\sigma : J \to \dot{X}$ joining x and x' such that for all $t \in J$, if $\sigma(t)$ is not an apex of $\dot{X}$, $p \circ \sigma(t)$ belongs to the ε-neighbourhood of $[x, x']$.*

Proof. By Proposition 2.10, there exist positive constants δ_0, Δ_0, r_1 and η_0 which only depend on ε satisfying the following property. Assume that $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$. Let x and x' be two points of X and C a chain of X between them. If $l(C) \leqslant |x - x'|_{\dot{X}} + \eta_0$, then every point of C belongs to the $\varepsilon/2$-neigbourhood of $[x, x']_X$. By replacing δ_0 by a smaller constant if necessary, we may also assume that $71\delta_0 \leqslant \varepsilon/2$.

Consider now $\eta \in (0, \eta_0)$ and x and x' two points of X. By definition of $|x - x'|_{\dot{X}}$, there exists a chain $C = (z_0, \dots, z_m)$ of X between x and x' such that $l(C) \leqslant |x - x'|_{\dot{X}} + \eta/2$. By Proposition 2.10, every z_j belongs to the $\varepsilon/2$-neighbourhood of $[x, x']$. Let $k \in \{0, \dots, m - 1\}$. Applying Lemma 2.11, there exists a rectifiable path $\sigma_k : J_k \to \dot{X}$ joining z_k and z_{k+1} whose length is smaller than $\|z_k - z_{k+1}\| + \eta/2m$ and such that for all $t \in J_k$, if $\sigma_k(t)$ is not an apex of $\dot{X}$, $p \circ \sigma_k(t)$ belongs to the 65δ-neighbourhood of $[z_k, z_{k+1}]$. In particular the distance of $p \circ \sigma_k(t)$ to $[x, x']$ is less than $\varepsilon/2 + 71\delta \leqslant \varepsilon$. We now choose for σ the concatenation of the σ_k's. Its length is smaller than $l(C) + \eta/2 \leqslant |x - x'|_{\dot{X}} + \eta$. We reparametrize σ by arc length, hence σ is a $(1, \eta)$-quasi-geodesic. Moreover it satisfies the announced property. □

Proposition 2.13. *There exist positive constants δ_0, δ_1, Δ_0 and r_1 which do not depend on X or Y having the following property. Assume that $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$. For every $x, y, z \in X$ we have*

$$\mu\left(\langle y, z\rangle_x\right) \leqslant \frac{1}{2}\left(|y - x|_{\dot{X}} + |z - x|_{\dot{X}} - |y - z|_{\dot{X}}\right) + r_0 + 14\delta_1.$$

Proof. The constant δ_1, δ_0, Δ_0 and r_1 are given by Proposition 2.5. We fix ε_1 such that $\mu(\varepsilon_1) = \delta_1$. According to Proposition 2.12, by decreasing (respectively increasing) if necessary δ_0, Δ_0 (respectively r_1) the following hold. Assume that $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$ and $\Delta(Y) \leqslant \Delta_0$ then

(i) $\dot{X}$ is δ_1-hyperbolic,

(ii) for every $x, x' \in X$, for every $\eta > 0$ there is a $(1,\eta)$-quasi-geodesic $\sigma : J \to \dot{X}$ joining x and x' such that for all $t \in J$, if $\sigma(t)$ is not an apex of $\dot{X}$, $p \circ \sigma(t)$ belongs to the ε_1-neighbourhood of $[x, x']$.

Let x, y and z be three points of $X \subset \dot{X}$. In all this section we kept the notation $\langle x, y\rangle_z$ for the Gromov product computed with the distance of X. Exceptionally we will denote the Gromov product of these three points computed in $\dot{X}$ by

$$\langle x, y\rangle_z^{\dot{X}} = \frac{1}{2}\left(|z-x|_{\dot{X}} + |z-y|_{\dot{X}} - |x-z|_{\dot{X}}\right)$$

Let $\eta > 0$. There exists a $(1,\eta)$-quasi-geodesic $\gamma : [0\,, a] \to \dot{X}$ joining y to z and satisfying (ii). Let us put

$$t = \min\left\{\langle x, z\rangle_y^{\dot{X}}\,, a - \langle x, y\rangle_z^{\dot{X}}\right\}$$

Note that the definition of $\gamma(t)$ is symmetric in y and z: using the reverse parametrization for the quasi-geodesic γ would lead to the same point. The point $\gamma(t)$ is not necessary in X. However the diameter of the cones that were attached to buid $\dot{X}$ is at most $2r_0$. The path γ being a continuous $(1,\eta)$-quasi-geodesic, there exists $s \in [0\,, a]$ such that $|s-t| \leqslant r_0 + \eta$ and $\gamma(s) \in X$. The points y and z playing a symmetric role, we can assume without loss of generality that $s \leqslant t$.

We consider now a $(1,\eta)$-quasi-geodesic $\sigma : [0\,, b] \to \dot{X}$ joining y to x, satisfying (ii) and put $r = \min\{s, b\}$. Since σ is $(1,\eta)$-quasi-geodesic we have $|x - \sigma(r)|_{\dot{X}} \leqslant |x-y|_{\dot{X}} - r + 2\eta$ which leads to

$$|x - \sigma(r)|_{\dot{X}} \leqslant \langle y, z\rangle_x^{\dot{X}} + r_0 + 5\eta \tag{7}$$

Moreover by hyperbolicity of $\dot{X}$, $|\sigma(r) - \gamma(r)|_{\dot{X}} \leqslant 4\delta_1 + 5\eta$. In particular $\sigma(r)$ belongs to the $(4\delta_1 + 5\eta)$-neighbourhood of X in $\dot{X}$. Hence $|p\circ\sigma(r) - \gamma(r)|_{\dot{X}} \leqslant 8\delta_1 + 10\eta$. It follows that $|p\circ\sigma(r) - \gamma(r)| \leqslant \varepsilon$, where $\mu(\varepsilon) = 8\delta_1 + 10\eta$. Nevertheless $p\circ\sigma(r)$ and $\gamma(r)$ respectively lie in the ε_1-neighbourhood of $[y, x]$ and $[y, z]$. By triangle inequality

$$|p\circ\sigma(r) - y| \leqslant \langle x, z\rangle_y + \langle x, y\rangle_{p\circ\sigma(r)} + \langle y, z\rangle_{\gamma(r)} + |p\circ\sigma(r) - \gamma(r)|\,.$$

Consequently $|p\circ\sigma(r) - y| \leqslant \langle x, z\rangle_y + \varepsilon + 2\varepsilon_1$, and

$$\langle y, z\rangle_x \leqslant |x-y| - |y - p\circ\sigma(r)| + \varepsilon + 2\varepsilon_1 \leqslant |x - p\circ\sigma(r)| + \varepsilon + 2\varepsilon_1$$

Applying μ to this inequality we get $\mu(\langle y, z\rangle_x) \leqslant |x - p\circ\sigma(r)|_{\dot{X}} + 10\delta_1 + 10\eta$ which combined with (7) gives

$$\mu\left(\langle y, z\rangle_x\right) \leqslant \langle y, z\rangle_x^{\dot{X}} + r_0 + 14\delta_1 + 20\eta.$$

This inequality holds for every $\eta > 0$ which completes the proof. □

3 Small cancellation theory

In this section we will be concerned with the small cancellation theory. We expose the geometrical point of view developed by T. Delzant and M. Gromov in [12] and used in Section 4 to prove the main theorem.

3.1 General framework

We require X to be a proper, geodesic, δ-hyperbolic space and G a group acting properly, co-compactly, by isometries on X. We assume that G satisfies the small centralizers hypothesis (see Section 1.4).

Let P be a set of hyperbolic elements of G. We assume that P is the union of a finite number of conjugacy classes. We denote by K the (normal) subgroup of G generated by P. Our goal is to study the quotient $\bar{G} = G/K$. The small cancellation parameters $\Delta(P, X)$ and $r_{inj}(P, X)$ (see Definition 1.17), respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory. We are interested in situations where the ratios $\delta / r_{inj}(P, X)$ and $\Delta(P, X)/r_{inj}(P, X)$ are very small. To that end, we build a space $\bar{X}$ with an action of $\bar{G}$. We only recall the main steps of this construction. This approach has been studied in [12], [11] and [7]. We follow here [9].

Fix $r_0 > 0$. Its value will be made precise in Theorem 3.1. We consider the family of strongly quasi-convex subsets $Y = (Y_\rho)_{\rho \in P}$. The cone-off of radius r_0 over X relatively to Y is denoted by $\dot{X}$. We extend by homogeneity the action of G on X in an action of G on $\dot{X}$. Given a point $x = (y, r)$ of C_ρ and g an element of G, gx is the point of $C_{g\rho g^{-1}} = gC_\rho$ defined by $gx = (gy, r)$. The group G acts by isometries on $\dot{X}$ (see [8, Lemma 4.3.1]). The space $\bar{X}$ is the quotient of $\dot{X}$ by K.

Theorem 3.1 (Small cancellation theorem, see [12, Th. 5.5.2] or [9, Prop. 6.7])**.** *There exist positive numbers δ_0, δ_1, Δ_0 and r_1 which do not depend on X or P with the following property. If $r_0 \geqslant r_1$, $\delta \leqslant \delta_0$, $\Delta(P, X) \leqslant \Delta_0$ and $r_{inj}(P, X) \geqslant \pi \operatorname{sh} r_0$, then $\bar{X}$ is proper, geodesic and $\bar{\delta}$-hyperbolic, with $\bar{\delta} \leqslant \delta_1$. Moreover $\bar{G}$ acts properly, co-compactly, by isometries on it.*

Note that the constants δ_0, δ_1, Δ_0 and r_1 in Theorem 3.1 are a priori different from the ones of Theorem 2.5 or Propositions 2.12 and 2.13. However by decreasing (respectively increasing) if necessary δ_0, Δ_0 (respectively δ_1, r_1) we can always assume that they work for the three results. Similarly we can require that $r_1 \geqslant 10^{100}\delta_1$ and $\delta_0, \Delta_0 < 10^{-5}\delta_1$. We now fix them once for all. By Proposition 1.8, we can find constants $r_0 \geqslant r_1$ and $k_S \geqslant 1$ having the following property. Let $\eta \in (0, \delta_1)$. If σ is a $\frac{1}{100}r_0$-local $(1, \eta)$-quasi-geodesic in a δ_1-hyperbolic space then it is a (k_S, η)-quasi-geodesic and lies in the $\frac{1}{500}r_0$-neighbourhood of every geodesic joining its endpoints. Using Theorems 2.5 and 3.1, Propositions 2.12 and 2.13 we obtain that if $\delta \leqslant \delta_0$, $\Delta(P, X) \leqslant \Delta_0$ and $r_{inj}(P, X) \geqslant 500\pi \operatorname{sh} r_0$, then the followings hold.

(i) (Theorem 2.5) The cone-off $\dot{X}$ is δ_1-hyperbolic.

(ii) (Theorem 3.1) The space $\bar{X}$ is proper, geodesic and $\bar{\delta}$-hyperbolic, with $\bar{\delta} \leqslant \delta_1$. Moreover $\bar{G}$ acts properly, co-compactly, by isometries on it.

(iii) (Proposition 2.13) For all $x, y, z \in X$,

$$\mu\left(\langle y, z\rangle_x\right) \leqslant \frac{1}{2}\left(|y-x|_{\dot{X}} + |z-x|_{\dot{X}} - |y-z|_{\dot{X}}\right) + r_0 + 14\delta_1.$$

(iv) (Proposition 2.12) For all $x, x' \in X$, for all $\eta > 0$, there exists a $(1,\eta)$-quasi-geodesic $\sigma : J \to \dot{X}$ between x and x' such that for all $t \in J$, if $\sigma(t)$ is not an apex of $\dot{X}$, then $p \circ \sigma(t)$ lies in the $\pi \operatorname{sh} r_0$-neighbourhood of $[x, x']$.

Remark : The parameters δ_0, Δ_0, δ_1 and r_0 are certainly not chosen in an optimal way. What only matters is their orders of magnitude recalled below.

$$\max\{\delta_0, \Delta_0\} \ll \delta_1 \ll r_0 \ll \pi \operatorname{sh} r_0.$$

An other important point to remember is the following. The constants δ_0, Δ_0 and $\pi \operatorname{sh} r_0$ are used to describe the geometry of X whereas δ_1 and r_0 refers to the one of $\dot{X}$ or $\bar{X}$.

Notations :

- Given g is an element of G we write $\bar{g}$ for the image of g by the canonical projection $\pi : G \twoheadrightarrow \bar{G}$.
- We will denote by $\bar{x}$ the image of a point x of X by the natural map $\nu : X \to \dot{X} \to \bar{X}$.
- Unless otherwise stated all distances, diameters, Gromov's products, etc will be compute with the distance of X or $\bar{X}$ (but not of $\dot{X}$).

3.2 A Greendlinger Lemma

Lemma 3.2 (see [12, Prop. 5.6.1] or [9, Prop. 3.15]). *Let x be a point of $\dot{X}$ such that $d(x, X) \leqslant \frac{r_0}{2}$. The map $\dot{X} \to \bar{X}$ induces an isometry from $B\left(x, \frac{1}{50} r_0\right)$ onto its image.*

Proposition 3.3. *Let x and x' be two points of X. We assume that for all $\rho \in P$, $|[x, x'] \cap Y_\rho| \leqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. Then for all $\eta > 0$ there exists a $(1,\eta)$-quasi-geodesic $\sigma : J \to \dot{X}$ between x and x', such that the path $\bar{\sigma} : J \to \dot{X} \to \bar{X}$ is a $\frac{1}{100} r_0$-local $(1,\eta)$-quasi-geodesic of $\bar{X}$.*

Proof. Let $\eta \in \left(0, \frac{1}{100} r_0\right)$. Applying Proposition 2.12 there exists a $(1,\eta)$-quasi-geodesic $\sigma : J \to \dot{X}$ between x and x' such that for all $t \in J$, if $\sigma(t)$ is not an apex of $\dot{X}$, then $p \circ \sigma(t)$ lies in the $\pi \operatorname{sh} r_0$-neighbourhood of $[x, x']$. Let $s, t \in J$ such that $|s - t| \leqslant \frac{1}{100} r_0$. Since σ is a $(1,\eta)$-quasi-geodesic, $|\sigma(s) - \sigma(t)|_{\dot{X}} \leqslant |s - t| + \eta < \frac{1}{50} r_0$. We now distinguish two cases.

- Assume that $d(\sigma(s), X) \leqslant \frac{1}{2} r_0$. By Lemma 3.2, the map $\dot{X} \to \bar{X}$ restricted to the ball of center $\sigma(s)$ and radius $\frac{1}{50} r_0$ preserves the distances. Hence $|\bar{\sigma}(s) - \bar{\sigma}(t)|_{\bar{X}} = |\sigma(s) - \sigma(t)|_{\dot{X}}$.

- Assume that $d\,(\sigma(s), X) > \frac{1}{2}r_0$. There exists $\rho \in P$ such that $\sigma(s)$ and $\sigma(t)$ are two points of the same cone C_ρ. If $\sigma(s)$ or $\sigma(t)$ is the apex of the cone then $|\bar\sigma(s) - \bar\sigma(t)|_{\bar X} = |\sigma(s) - \sigma(t)|_{\dot X}$, otherwise $p \circ \sigma(s)$ and $p \circ \sigma(t)$ belong to Y_ρ and the $\pi \operatorname{sh} r_0$-neghbourhood of $[x, x']$. Thus

$$|p \circ \sigma(s) - p \circ \sigma(t)|_{Y_\rho} \leqslant |[x, x'] \cap Y_\rho| + 2\pi \operatorname{sh} r_0 + 40\delta \leqslant [\rho]_{Y_\rho} - \pi \operatorname{sh} r_0$$

It follows from Lemma 2.1, that $|\bar\sigma(s) - \bar\sigma(t)|_{\bar X} = |\sigma(s) - \sigma(t)|_{\dot X}$.

Thus for all $s, t \in J$, if $|s - t| \leqslant \frac{1}{100}r_0$, $|\bar\sigma(s) - \bar\sigma(t)|_{\bar X} = |\sigma(s) - \sigma(t)|_{\dot X}$. Since σ is a $(1, \eta)$-quasi-geodesic, $\bar\sigma$ is a $\frac{1}{100}r_0$-local $(1, \eta)$-quasi-geodesic. ☐

Theorem 3.4 (Greendlinger's Lemma)**.** *Let x be a point of X. Let g be an element of $G \setminus \{1\}$. If g belongs to K, then there exists $\rho \in P$ such that $|[x, gx] \cap Y_\rho| > [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$.*

Proof. We prove the theorem by contradiction. Assume that for all $\rho \in P$, $|[x, gx] \cap Y_\rho| \leqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. Let $\eta \in (0, \delta_1)$. Applying Proposition 3.3, there exists a $(1, \eta)$-quasi-geodesic $\sigma : [a\,, b] \to \dot X$ between x and gx, such that the path $\bar\sigma : [a\,, b] \to \dot X \to \bar X$ is a $\frac{1}{100}r_0$-local $(1, \eta)$-quasi-geodesic of $\bar X$. In particular $\bar\sigma$ is a (k_S, η)-quasi-geodesic (see Proposition 1.8). Hence, $|gx - x|_{\dot X} \leqslant k_S\,|\bar g \bar x - \bar x| + 3\eta = 3\eta$. This inequality holds for all $\eta > 0$. It implies $gx = x$. However K acts freely on X (see [12, Prop. 5.6.2]), thus $g = 1$. Contradiction. ☐

Proposition 3.5 (Preserving shape Lemma)**.** *Let x, y and z be three points of X such that for all $\rho \in P$,*

$$\max\Big\{|[x, y] \cap Y_\rho|\,, |[x, z] \cap Y_\rho|\Big\} \leqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta.$$

If $\langle \bar y, \bar z\rangle_{\bar x} \leqslant \frac{1}{250}r_0$, then $\langle y, z\rangle_x \leqslant \pi \operatorname{sh} r_0$

Proof. As we wrote before, we keep the notation $\langle y, z\rangle_x$ for the Gromov product computed with the distance of X. Therefore we denote by t the same product computed with the distance of $\dot X$.

$$t = \frac{1}{2}\left(|y - x|_{\dot X} + |z - x|_{\dot X} - |y - z|_{\dot X}\right)$$

By Proposition 2.13, we have $\mu(\langle y, z\rangle_x) \leqslant t + r_0 + 14\delta_1$. The goal is now to compare t and $\langle \bar y, \bar z\rangle_{\bar x}$. We can assume that $\min\{|x - y|_{\dot X}\,, |x - z|_{\dot X}\} > \langle \bar y, \bar z\rangle_{\bar x} + \frac{1}{250}r_0 + 2\delta_1$. Otherwise we would have $t \leqslant \langle \bar y, \bar z\rangle_{\bar x} + \frac{1}{250}r_0 + 2\delta_1$.

Let $\eta \in (0, \delta_1)$ such that $\min\{|x - y|_{\dot X}\,, |x - z|_{\dot X}\} > \langle \bar y, \bar z\rangle_{\bar x} + \frac{1}{250}r_0 + 2\delta_1 + 5\eta$. According to Proposition 3.3, there exists a $(1, \eta)$-quasi-geodesic $\sigma : [0\,, a] \to \dot X$ between x and y whose image $\bar\sigma : J \to \dot X \to \bar X$ in $\bar X$ is a $\frac{1}{100}r_0$-local $(1, \eta)$-quasi-geodesic. In particular $\bar\sigma$ lies in the $\frac{1}{500}r_0$-neighbourhood of $[\bar x, \bar y]$. We also construct a path $\gamma : [0\,, b] \to \dot X$ between x and z having the same properties. Let $s \in [0, \min\{a, b\}]$ such that $s > \langle \bar y, \bar z\rangle_{\bar x} + \frac{1}{250}r_0 + 2\delta_1 + 4\eta$. Without loss of generality we can also require that $s \leqslant \frac{1}{100}r_0$. Let us denote by p and q the points $\sigma(s)$ and $\gamma(s)$. By hyperbolicity of $\dot X$ we have

$$|p - q|_{\dot X} \leqslant \max\Big\{\Big||x - p|_{\dot X} - |x - q|_{\dot X}\Big| + 3\eta, |x - p|_{\dot X} + |x - q|_{\dot X} - 2t\Big\} + 4\delta_1$$

which leads to

$$|p-q|_{\dot{X}} \leqslant \max\{3\eta, 2s-2t\} + 4\delta_1 + 2\eta \tag{8}$$

Our next step is to give a lower bound for $|p-q|_{\dot{X}}$. Recall that $s \leqslant \frac{1}{100}r_0$. Thus p and q are contained in the ball of $\dot{X}$ of center x and radius $\frac{1}{50}r_0$. However the map $\dot{X} \to \bar{X}$ restricted to this ball is an isometry, hence $|p-q|_{\dot{X}} = |\bar{p}-\bar{q}|$ and $|\bar{x}-\bar{p}| + |\bar{x}-\bar{q}| \geqslant 2s - 2\eta$. By triangle inequality that

$$|\bar{p}-\bar{q}| \geqslant |\bar{x}-\bar{p}| + |\bar{x}-\bar{q}| - 2\langle \bar{y}, \bar{z}\rangle_{\bar{x}} - 2\langle \bar{x}, \bar{y}\rangle_{\bar{p}} - 2\langle \bar{x}, \bar{z}\rangle_{\bar{q}}$$

Since $\bar{p}$ (respectively $\bar{q}$) lies in the $\frac{1}{500}r_0$-neighbourhood of $[\bar{x}, \bar{y}]$ (respectively $[\bar{x}, \bar{z}]$) we get

$$|p-q|_{\dot{X}} = |\bar{p}-\bar{q}| \geqslant 2s - 2\langle \bar{y}, \bar{z}\rangle_{\bar{x}} - \frac{1}{125}r_0 - 2\eta > 4\delta_1 + 5\eta$$

It follows then from (8) that $t \leqslant s + 2\delta_1 + \eta$. This inequality holds for every sufficiently small η and for every $s > \langle \bar{y}, \bar{z}\rangle_{\bar{x}} + \frac{1}{250}r_0 + 2\delta_1 + 4\eta$ thus $t \leqslant \langle \bar{y}, \bar{z}\rangle_{\bar{x}} + \frac{1}{250}r_0 + 4\delta_1$.

We proved that $\mu(\langle y, z\rangle_x) \leqslant \langle \bar{y}, \bar{z}\rangle_{\bar{x}} + \frac{251}{250}r_0 + 18\delta_1 < 2r_0$. The conclusion follows from the estimate of the function μ (see Section 2.1). □

3.3 P-close points

Definition 3.6. *Two points x and x' of X are* P-close *if for all $\rho \in P$, $|[x, x'] \cap Y_\rho| \leqslant [\rho]/2 + 6\pi \operatorname{sh} r_0$.*

Remark : There is a very simple way to get P-close points. Let x and x' be two points of X. Let $u \in K$. If $|x - ux'| \leqslant \inf_{v\in K} |x - vx'| + \delta$, then x and ux' are P-close. Indeed, if it was not the case, according to Lemma 1.13 one could reduce the distance between x and ux'.

Proposition 3.7. *Let $\alpha \geqslant 0$ Let x and x' be two P-close points of X. Let $y \in X$ such that for all $u \in K$, $\langle x, x'\rangle_y < \langle x, x'\rangle_{uy} + 2\alpha$. Then for all $\rho \in P$, $|[x, y] \cap Y_\rho| \leqslant [\rho] - c$ where $c = 122\pi \operatorname{sh} r_0 - \alpha - 252\delta$.*

Proof. We prove this result by contradiction. Assume that there exists $\rho \in P$ such that $|[x, y] \cap Y_\rho| > [\rho] - c$. Let N be a nerve of ρ. We denote by p and q respective projections of x and y on N. Let r be a projection of x' on $(p, q)_N$. Recall that $[\rho] \geqslant 500\pi \operatorname{sh} r_0$. It follows from Proposition 1.15, that

(i) $|p-q| \geqslant [\rho] - c - 117\delta$,
$|q-r| \geqslant [\rho]/2 - c - 6\pi \operatorname{sh} r_0 - 261\delta$,

(ii) $\langle x, x'\rangle_y \geqslant \langle x, x'\rangle_r + |y-q| + |q-r| - 110\delta$.

The isometry ρ acts on N by translation of length $[\rho]$. Therefore there exists $\varepsilon \in \{\pm 1\}$, such that p and $\rho^\varepsilon q$ belong to the same component of $N \setminus \{q\}$. We want to compare $\langle x, x'\rangle_y$ and $\langle x, x'\rangle_{\rho^\varepsilon y}$. To that end, we distinguish two cases depending on the relative positions of p, r, and $\rho^\varepsilon q$ on N.

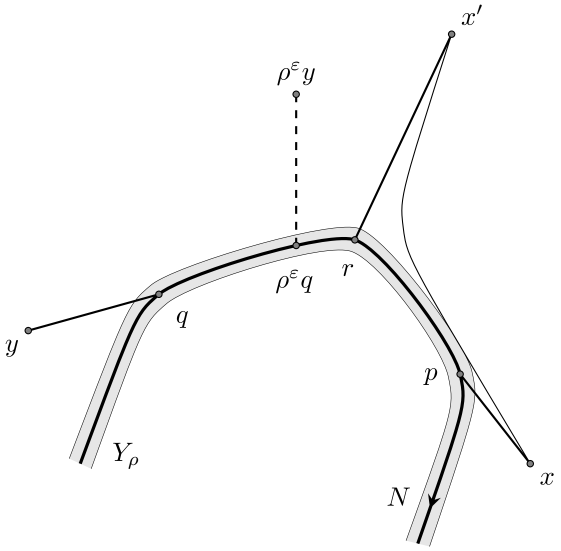
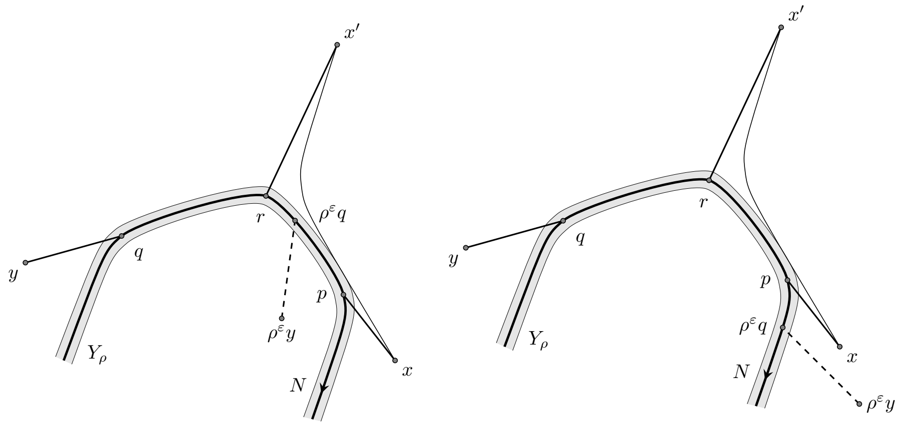


Figure 5: Case 1

Case 1. Assume that $\rho^\varepsilon q$ belongs to $(q,r)_N$ (see Fig. 5). Since q is a projection of y on N we have $|y-r| \geqslant |\rho^\varepsilon y - r| + [\rho] - 56\delta$. Combined with the lower bound of $\langle x, x'\rangle_y$ given by (ii), we get

$$\langle x, x'\rangle_y \geqslant \langle x, x'\rangle_r + |\rho^\varepsilon y - r| + [\rho] - 166\delta \geqslant \langle x, x'\rangle_{\rho^\varepsilon y} + [\rho] - 166\delta.$$



(a) $\rho^\varepsilon q$ between r and p (b) p between r and $\rho^\varepsilon q$

Figure 6: Case 2

Case 2. Assume now that $\rho^\varepsilon q$ does not belong to $(q,r)_N$. We claim that $\langle x, r\rangle_{\rho^\varepsilon q} \leqslant c + 133\delta$. If $\rho^\varepsilon q$ lies on N between r and p (see Fig. 6(a)) it follows from the definition of N. If not (see Fig. 6(b)) N being a $[\rho]$-local geodesic we have

$$\langle x, r\rangle_{\rho^\varepsilon q} \leqslant |p - \rho^\varepsilon q| + \langle x, r\rangle_p = [\rho] - |p - q| + \langle x, r\rangle_p .$$

The point p is a projection of x on N, thus $\langle x, r\rangle_p \leqslant 16\delta$. Moreover by (i) $[\rho] - |p - q| \leqslant c + 117\delta$, which completes the proof of our claim. Applying the triangle inequality we get $\langle x, x'\rangle_{\rho^\varepsilon q} \leqslant \langle x, x'\rangle_r + \langle x, r\rangle_{\rho^\varepsilon q} \leqslant \langle x, x'\rangle_r + c + 133\delta$. Combined with (i) and (ii) it gives

$$\langle x, x'\rangle_y \geqslant \langle x, x'\rangle_{\rho^\varepsilon q} + |\rho^\varepsilon q - \rho^\varepsilon y| + \frac{1}{2}[\rho] - 2c - 6\pi \operatorname{sh} r_0 - 504\delta.$$

In both cases $\langle x, x'\rangle_{\rho^\varepsilon y} \leqslant \langle x, x'\rangle_y + 2c - 244\pi \operatorname{sh} r_0 + 504\delta \leqslant \langle x, x'\rangle_y + 2\alpha$, which contradicts our assumption on y. □

3.4 P-reduced isometries

Definition 3.8. *Let g be an element of G. The isometry g is P-reduced if its image $\bar g$ in $\bar G$ is hyperbolic and for all $\rho \in P$, $|Y_g \cap Y_\rho| \leqslant [\rho]/2 + \pi \operatorname{sh} r_0$.*

Remark : Since P is invariant under conjugation, all conjugates of a P-reduced isometry are also P-reduced.

The next proposition explains how to construct P-reduced elements of G. To that end we need to assume that the elements of P are proper powers of small isometries.

Proposition 3.9. *Let $n \in \mathbf{N}^*$. We assume that*

(i) for all $\rho \in P$, there exists $r \in G$ such that $[r] \leqslant 1000\delta$ and $\rho = r^n$,

(ii) $A(G, X) \leqslant \pi \operatorname{sh} r_0 - 1590\delta$

Let $g \in G$, such that its image $\bar g$ in $\bar G$ is hyperbolic. Then, there exists $u \in K$ such that ug is P-reduced.

Proof. We choose $u \in K$ such that for all $v \in K$, $[ug] \leqslant [vg] + \delta$. Since $\bar g = \bar u \bar g$ is a hyperbolic element of $\bar G$, so is ug in G. We suppose now that the isometry ug is not P-reduced. There is $\rho \in P$, such that $|Y_{ug} \cap Y_\rho| > [\rho]/2 + \pi \operatorname{sh} r_0$. By assumption, there exists $r \in G$ such that $[r] \leqslant 1000\delta$ and $\rho = r^n$. From Proposition 1.11, $Y_\rho = Y_r$ (respectively Y_{ug}) lies in the 58δ-neighbourhood of A_r and (respectively A_{ug}). Hence $|A_{ug} \cap A_r| > [\rho]/2 + \pi \operatorname{sh} r_0 - 232\delta$. Note that ug and r do not generate an elementary subgroup. The group G satisfies indeed the small centralizers hypothesis. If it was the case, $\bar g$ would have finite order as $\bar r$, which contradicts the fact that $\bar g$ is hyperbolic. Thus Proposition 1.19 leads to $[ug] > [\rho]/2 - A(G, X) + \pi \operatorname{sh} r_0 - 1232\delta$. It follows from our assumptions and Lemma 1.14 that there exists $m \in \mathbf{Z}$ such that $[\rho^m ug] < [ug] + A(G, X) - \pi \operatorname{sh} r_0 + 1589\delta$. However $\rho^m u$ belongs to K. This last inequality contradicts the definition of u. Consequently ug is P-reduced. □

Lemma 3.10. *Let g be a P-reduced element of G. Let x and x' be two points of X. For all $\rho \in P$ we have*

$$|[x, x'] \cap Y_\rho| \leqslant \frac{1}{2}[\rho] + d(x, Y_g) + d(x', Y_g) + \pi \operatorname{sh} r_0 + \delta.$$

In particular, if $d(x, Y_g) + d(x', Y_g) \leqslant 5\pi \operatorname{sh} r_0 - \delta$*, then* x *and* x*' are* P*-closed.*

Proof. Let ρ be an element of P. Let y and y' be respective projections of x and x' on Y_g. One knows by (3) that

$$|[x,x'] \cap Y_\rho| \leqslant |[y,y'] \cap Y_\rho| + \langle y,y' \rangle_x + \langle y,y' \rangle_{x'} + \delta.$$

However g is P-reduced, therefore $|[y,y'] \cap Y_\rho| \leqslant |Y_g \cap Y_\rho| \leqslant [g]/2 + \pi \operatorname{sh} r_0$. On the other hand, $\langle y,y' \rangle_x \leqslant |x-y| = d(x,Y_g)$. Similarly $\langle y,y' \rangle_{x'} \leqslant d(x',Y_g)$. □

Proposition 3.11. *Let* $\alpha \geqslant 0$*. Let* g *be a* P*-reduced element of* G*. Let* x *be a point of* X *such that for all* $u \in K$*,* $d(x,Y_g) \leqslant d(ux,Y_g) + 2\alpha$*. Then, there exists* k_0 *such that for all* $k \geqslant k_0$*, for all* $\rho \in P$*,* $\left|\left[x, g^k x\right] \cap Y_\rho\right| \leqslant [\rho] - c$ *where* $c = 122\pi \operatorname{sh} r_0 - \alpha - 288\delta$*.*

Proof. Let y be a projection of x on Y_g. The family P only contains a finite number of conjugacy classes. Since g is hyperbolic, there exists k_0 such that for all $k \geqslant k_0$, for all $\rho \in P$, $\left|y - g^k y\right| > [\rho]/2 + \pi \operatorname{sh} r_0 + 53\delta$. Assume now that our proposition is false i.e., there exists $k \geqslant k_0$ and $\rho \in P$ such that $\left|\left[x, g^k x\right] \cap Y_\rho\right| > [\rho] - c$. The point y is a projection of x on Y_g, thus $\left\langle y, g^k y\right\rangle_x \leqslant d(x, Y_g)$. Moreover Y_g is 6δ-quasi-convex. It follows from our assumption on x that that for all $u \in K$, $\left\langle y, g^k y\right\rangle_x \leqslant \left\langle y, g^k y\right\rangle_{ux} + 2\alpha + 6\delta$. On the other hand, g is P-reduced. By Lemma 3.10, y and $g^k y$ are P-close. According to Proposition 3.7 $\left|\left[x, g^k y\right] \cap Y_\rho\right| \leqslant [\rho] - c'$, where $c' = 122\pi \operatorname{sh} r_0 - \alpha - 255\delta$. The same inequality holds if one replaces $\left[x, g^k y\right]$ by $\left[y, g^k x\right]$. We now denote by p and q respective projections of x and $g^k x$ on Y_ρ. According to Lemma 1.6

$$|p-q| \geqslant \left|\left[x, g^k x\right] \cap Y_\rho\right| - 13\delta > [\rho] - c - 13\delta. \tag{9}$$

Claim. y is a 20δ-projection of p on Y_g. Thanks to Lemma 1.4 it is sufficient to show that $\langle x,y \rangle_p \leqslant 7\delta$. Assume that this statement is false. Let $z \in Y_\rho$. By hyperbolicity we have

$$\min\left\{\langle x,y \rangle_p, \langle y,z \rangle_p\right\} \leqslant \langle x,z \rangle_p + \delta \leqslant 7\delta.$$

Thus for every $z \in Y_\rho$, $\langle y,z \rangle_p \leqslant 7\delta$. In particular p is a 7δ-projection of y on Y_ρ. Using Lemma 1.6 we obtain that $|p-q| \leqslant \left|\left[y, g^k x\right] \cap Y_\rho\right| + 20\delta. \leqslant [\rho] - c' + 20\delta$, which contradicts (9).

In the the same way, we prove that $g^k y$ is a 20δ-projection of q on Y_g. It follows then from Lemma 1.6 that

$$\left|y - g^k y\right| \leqslant |[p,q] \cap Y_g| + 53\delta \leqslant |Y_\rho \cap Y_g| + 53\delta.$$

By assumption g is P-reduced. Consequently, $\left|y - g^k y\right| \leqslant [\rho]/2 + \pi \operatorname{sh} r_0 + 53\delta$, which contradicts our assumption on k. Thus the proposition is true. □

3.5 Foldable configurations

In this section, we are interested in the following situation. Let x, p and q be three points of X such that x and p (respectively x and q) are P-close. We assume that p and q have the same image $\bar{p} = \bar{q}$ in $\bar{X}$, but are distinct as points of X. We would like to understand the reason why $p \neq q$ in X and which transformation could move p closer to q.

The idea is roughly the following. Since $\bar{p} = \bar{q}$, there exists $g \in K \setminus \{1\}$ such that $q = gp$. By the Greendlinger Lemma (Proposition 3.4), there exists $\rho \in P$, such that

$$|[p,q] \cap Y_\rho| > [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta.$$

However x and p (respectively x and q) are P-closed. Hence, half of the overlap between Y_ρ and $[p,q]$ is covered by $[x,p]$ and the other half by $[x,q]$ (see Fig. 7). Using ρ we translate the point p. In particular there exists $\varepsilon \in \{\pm 1\}$ such that

$$\langle \rho^\varepsilon p, q \rangle_x \geqslant \langle p, q \rangle_x + \frac{1}{2}[\rho] - 9\pi \operatorname{sh} r_0 - 40\delta.$$

By iterating the process, we increase at each step $\langle p, q \rangle_x$ (which is bounded

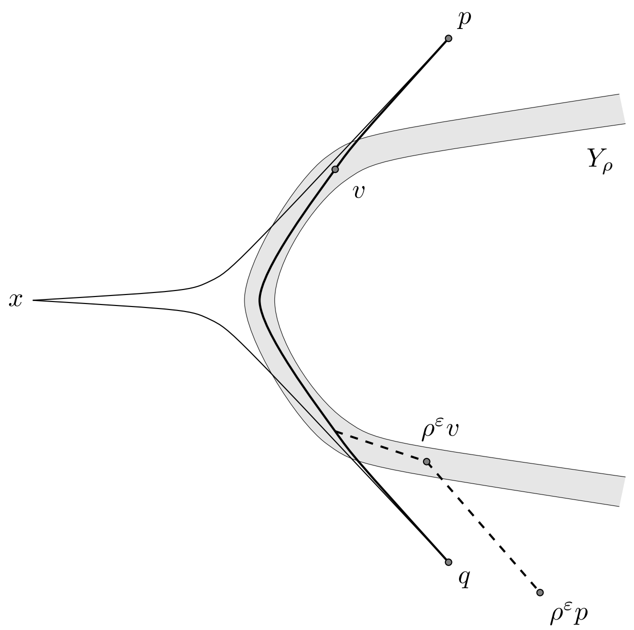


Figure 7: Folding a geodesic.

above by $|x-q|$) until $p = q$. To that end we need the points x and $\rho^\varepsilon p$ to be P-close, which is unfortunately not exactly the case: we might approximatively have

$$|[x, \rho^\varepsilon p] \cap Y_\rho| \simeq \frac{1}{2}[\rho] + 9\pi \operatorname{sh} r_0 + 40\delta$$

The definition of *foldable configuration* gives a set of conditions on x, p and q which are sufficient to detail the previous discussion and which will be still satisfied by x, $\rho^\varepsilon p$ and q.

Definition 3.12 (Foldable configuration). *Let x, p, q and y be four points of X. We say that the configuration (x,p,q,y) is* foldable *if there exist $s, t \in X$ satisfying the following conditions (see Fig. 8).*

(C1) s and p are P-close and $|x-s| \leqslant \langle p,q \rangle_x + 4\pi \operatorname{sh} r_0$,

(C2) t and q are P-close and $|x-t| \leqslant \langle p,q \rangle_x + 4\pi \operatorname{sh} r_0$.

(C3) s and y are P-close and $\langle s,y\rangle_p = 0$.

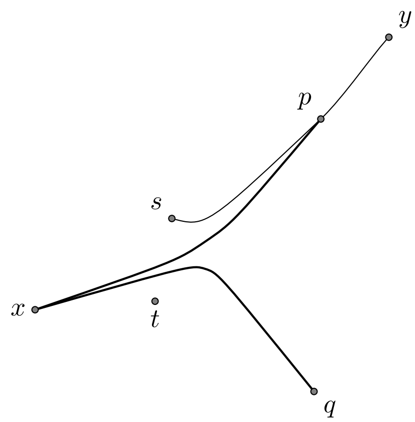


Figure 8: Definition of a foldable configuration.

Remark : This framework is a little more general than the one presented above. It naturally arises when folding a geodesic $[x,y]$ on the axe of a relation (see Proposition 3.17) The reason to keep track of the point y will appear in Proposition 4.12.

Proposition 3.13. *Let* (x,p,q,y) *be a foldable configuration such that* $\bar{p} = \bar{q}$ *but* $p \neq q$. *There exist* $\rho \in P$ *and* $\varepsilon \in \{\pm 1\}$ *satisfying the followings.*

(i) $|[x,y] \cap Y_\rho| \geqslant [\rho]/2 - 13\pi \operatorname{sh} r_0 - 419\delta$,

(ii) $\langle \rho^\varepsilon p, q\rangle_x \geqslant \langle p,q\rangle_x + [\rho]/2 - 13\pi \operatorname{sh} r_0 - 424\delta$,

(iii) the configuration $(x, \rho^\varepsilon p, q, \rho^\varepsilon y)$ *is foldable.*

(iv) $\langle x,y\rangle_p \leqslant \delta$ *and* $\langle x, \rho^\varepsilon y\rangle_{\rho^\varepsilon p} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$

Proof. The points s and t are the one given by the definition of a foldable configuration. We assumed that $\bar{p} = \bar{q}$ but $p \neq q$. By Greendlinger's Lemma there exists $\rho \in P$ such that $|[p,q] \cap Y_\rho| \geqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. We denote by N a nerve of ρ and by u, v, w and z respective projections of x, p, q and y on N. According to Proposition 1.16, u lies on N between v and w (see Fig. 9). Moreover we have

(a) $|v-w| \geqslant [\rho] - 3\pi \operatorname{sh} r_0 - 157\delta$,

(b) $[\rho]/2 - 13\pi \operatorname{sh} r_0 - 301\delta \leqslant |u-v| \leqslant [\rho]/2 + 10\pi \operatorname{sh} r_0 + 144\delta$,

(c) $[\rho]/2 - 13\pi \operatorname{sh} r_0 - 301\delta \leqslant |u-w| \leqslant [\rho]/2 + 10\pi \operatorname{sh} r_0 + 144\delta$,

(d) $||x-u| - \langle p,q\rangle_x| \leqslant 45\delta$,

(e) $\langle s,p\rangle_v \leqslant 34\delta$,

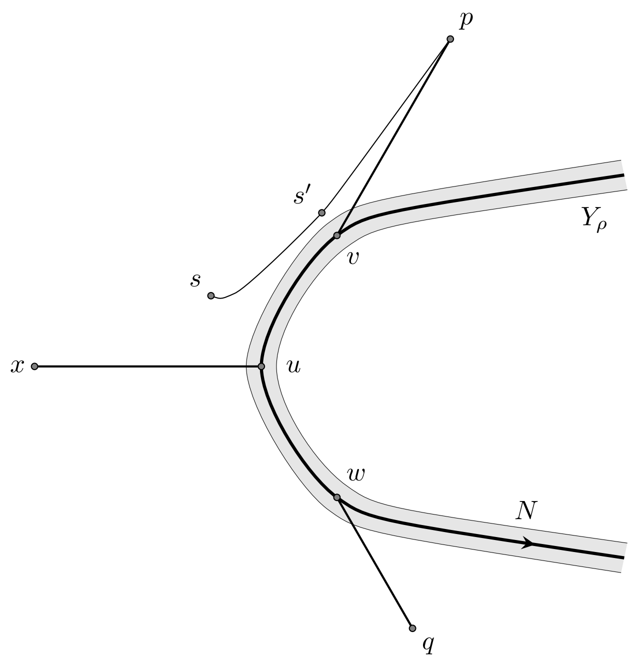


Figure 9: Positions of the points u, v, w and s'

On the configuration (x, p, q, y)**.** The points u and v are respective projections of x and p on N, thus $|x - p| \geqslant |x - u| + |u - v| - 66\delta$. Combined with Points (b) and (d), we get

$$|x - p| > \langle p, q\rangle_x + 4\pi \operatorname{sh} r_0 + \delta \geqslant |x - s| + \delta. \tag{10}$$

By hyperbolicity, $\min\left\{\langle x, y\rangle_p, |x - p| - |x - s|\right\} \leqslant \langle s, y\rangle_p + \delta \leqslant \delta$. According to (10) we necessary have $\langle x, y\rangle_p \leqslant \delta$, which proves the first part of Point (iv). The nerve N is contained in the 42δ-neighbourhood of Y_ρ. Applying Proposition 1.6 with (b) we get

$$|[x, y] \cap Y_\rho| \geqslant |[x, p] \cap Y_\rho| - \langle x, y\rangle_p \geqslant [\rho]/2 - 13\pi \operatorname{sh} r_0 - 419\delta,$$

which corresponds to Point (iv).

Claim 1. $|u - z| \leqslant [\rho]/2 + 10\pi \operatorname{sh} r_0 + 231\delta$. By hyperbolicity, we have

$$\langle s, y\rangle_u \leqslant \max\left\{|x - s| - |x - u| + 2\langle x, y\rangle_u, \langle x, y\rangle_u\right\} + \delta.$$

By (d) we know that $|x - s| \leqslant \langle p, q\rangle_x + 4\pi \operatorname{sh} r_0 \leqslant |x - u| + 4\pi \operatorname{sh} r_0 + 45\delta$. On the other hand the triangle inequality leads to $\langle x, y\rangle_u \leqslant \langle x, y\rangle_p + \langle x, p\rangle_u \leqslant 34\delta$. It follows that $\langle s, y\rangle_u \leqslant 4\pi \operatorname{sh} r_0 + 114\delta$. However z is a projection of y on N. The points s and y being P-close Proposition 1.6 yields

$$|u - z| \leqslant |[u, y] \cap Y_\rho| + 117\delta \leqslant |[s, y] \cap Y_\rho| + \langle s, y\rangle_u + 117\delta \leqslant \frac{1}{2}[\rho] + 10\pi \operatorname{sh} r_0 + 231\delta.$$

Claim 2. $\langle z, y\rangle_p \leqslant 23\pi \operatorname{sh} r_0 + 566\delta$. By triangle inequality, $\langle z, y\rangle_p \leqslant \langle x, y\rangle_p + \langle x, p\rangle_v + |v - z|$. The Gromov products on the left hand side of the inequality are small ($\langle x, y\rangle_p \leqslant \delta$ and $\langle x, p\rangle_v \leqslant 33\delta$) therefore it is sufficient to find an upper bound for $|v - z|$. In particular we can assume that $|v - z| > 79\delta$. Note that, since $\langle x, y\rangle_p \leqslant \delta$ the points z and u cannot belong to the same component of $N \setminus \{v\}$. In other words v lies between u and z. It follows from Claim 1 and Point (b) that $|v - z| = |u - z| - |u - v| \leqslant 23\pi \operatorname{sh} r_0 + 532\delta$.

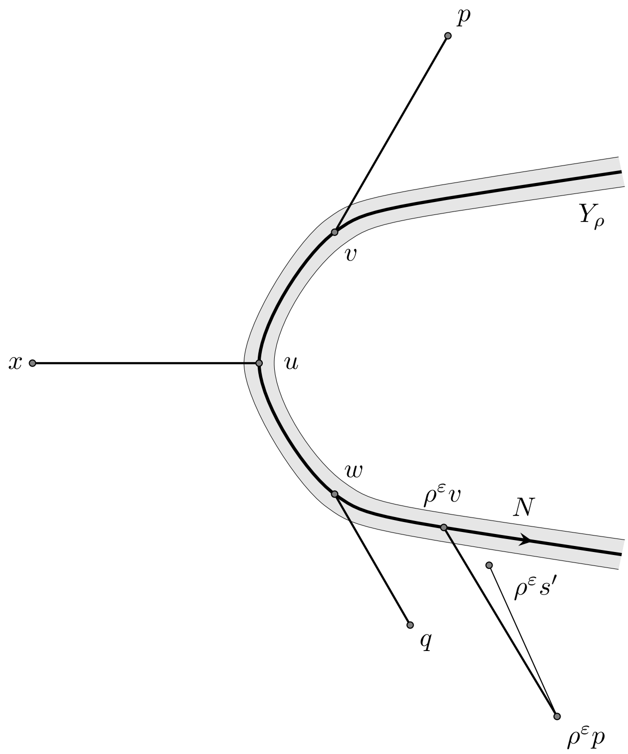


Figure 10: Positions of the point $\rho^\varepsilon v$, $\rho^\varepsilon s'$ and $\rho^\varepsilon p$

Translation by ρ. The isometry ρ acts by translation on N. Therefore there exists $\varepsilon \in \{\pm 1\}$ such that $\rho^\varepsilon v$ and w belong to the same component of $N \setminus \{v\}$ (see Fig. 10).

Claim 3. $|x - \rho^\varepsilon v| \leqslant \langle \rho^\varepsilon p, q\rangle_x + 3\pi \operatorname{sh} r_0 + 203\delta$. Note that $|u - v| \leqslant [\rho] \leqslant |\rho^\varepsilon v - v|$. Thus u lies on N between v and $\rho^\varepsilon v$. Since N is a $[\rho]$-local geodesic, $|\rho^\varepsilon v - u| = [\rho] - |u - v| \geqslant [\rho]/2 - 10\pi \operatorname{sh} r_0 - 144\delta$. We now distinguish two cases. If $\rho^\varepsilon v$ lies on N between u and w. Then $\langle x, q\rangle_{\rho^\varepsilon v} \leqslant 45\delta$ and $\langle x, \rho^\varepsilon p\rangle_{\rho^\varepsilon v} \leqslant 33\delta$. By hyperbolicity we obtain

$$|x - \rho^\varepsilon v| \leqslant \langle \rho^\varepsilon p, q\rangle_x + \max\left\{\langle x, \rho^\varepsilon p\rangle_{\rho^\varepsilon v}, \langle x, q\rangle_{\rho^\varepsilon v}\right\} + \delta \leqslant \langle \rho^\varepsilon p, q\rangle_x + 46\delta.$$

Assume now that w lies on N between u and $\rho^\varepsilon v$. As previously we show that $|x - w| \leqslant \langle \rho^\varepsilon p, q\rangle_x + 46\delta$. On the other hand N is a $[\rho]$-local geodesic, thus using

Point (a), $|w - \rho^\varepsilon v| = [\rho] - |v - w| \leqslant 3\pi \operatorname{sh} r_0 + 157\delta$. It follows from the triangle inequality that $|x - \rho^\varepsilon v| \leqslant |x - w| + |w - \rho^\varepsilon v| \leqslant \langle \rho^\varepsilon p, q\rangle_x + 3\pi \operatorname{sh} r_0 + 203\delta$, which completes the proof of our claim.

Combined with Point (d), we get in particular

$$\langle \rho^\varepsilon p, q\rangle_x \geqslant |x-u| + |u - \rho^\varepsilon v| - 3\pi \operatorname{sh} r_0 - 235\delta \geqslant \langle p, q\rangle_x + \frac{1}{2}[\rho] - 13\pi \operatorname{sh} r_0 - 424\delta,$$

which is exactly Point (ii). We now prove that $(x, \rho^\varepsilon p, q, \rho^\varepsilon y)$ is foldable. Note that the point t already satisfies the condition (C2). Let us denote by s' a projection of v on $[s,p]$. Since s and p are P-close, so are s' and p and thus $\rho^\varepsilon s'$ and $\rho^\varepsilon p$. On the other hand, by Point (e), $|v - s'| \leqslant 38\delta$. Using Claim 3 we obtain $|x - \rho^\varepsilon s'| \leqslant |x - \rho^\varepsilon v| + |v - s'| \leqslant \langle \rho^\varepsilon p, q\rangle_x + 3\pi \operatorname{sh} r_0 + 241\delta$. Consequently $\rho^\varepsilon s'$ satisfies the condition (C1). Since $\langle s, y\rangle_p = 0$ there exists a geodesic joining s to y which extends the geodesic between s and p containing s'. In particular $\langle \rho^\varepsilon s', \rho^\varepsilon y\rangle_{\rho^\varepsilon p} = \langle s', y\rangle_p = 0$. The points s and y being P-close, so are s' and y and thus $\rho^\varepsilon s'$ and $\rho^\varepsilon y$. Thus (C3) is also fulfilled and $(x, \rho^\varepsilon p, q, \rho^\varepsilon y)$ is foldable.

In only remains to prove that $\langle x, \rho^\varepsilon y\rangle_{\rho^\varepsilon p} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$. The isometry ρ acts on N by translation of length $[\rho]$. Moreover by Claim 1, $|u - z| \leqslant [\rho]/2 + 10\pi \operatorname{sh} r_0 + 231\delta$. Thus $|u - \rho^\varepsilon z| \geqslant [\rho]/2 - 10\pi \operatorname{sh} r_0 - 231\delta$. In particular $\langle x, \rho^\varepsilon y\rangle_{\rho^\varepsilon z} \leqslant 33\delta$. The triangle inequality and Claim 2 lead to $\langle x, \rho^\varepsilon y\rangle_{\rho^\varepsilon p} \leqslant \langle x, \rho^\varepsilon y\rangle_{\rho^\varepsilon z} + \langle z, y\rangle_p \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$, which completes the proof of Point (iv) and of the proposition. □

3.6 Lifting figures of $\bar X$ in X

In this section we try to find the best way to lift in X a figure of $\bar X$. Lemma 3.14 (respectively Lemma 3.15) explains how to lift a point of $\bar X$ which is close to a geodesic (respectively the cylinder of an isometry) with a point of X having a similar property. In Proposition 3.17 we are interested in the following situation. Let x and y be two P-close points of X and g a P-reduced isometry of G. We assume that $[\bar x, \bar y]$ and $Y_{\bar g}$ have a large overlap in $\bar X$ (for instance larger than $\left[\bar g^k\right]$ with $k \gg 1$) and would like to "lift" this overlap. By replacing if necessary g by a conjugate of g we may translate Y_g such that $[x,y]$ and Y_g have more or less a non-empty intersection. However there is no reason that this overlap should be as large in X as in $\bar X$. We face the same kind of problem exposed at the beginning of Section 3.5. Nevertheless, lifting the endpoints of $[\bar x, \bar y] \cap Y_{\bar g}$, one can build a foldable configuration. In the same way as explained in Section 3.5, we will use this configuration in Section 4 in order to translate y by elements of P and fold the geodesic $[x,y]$ onto Y_g.

Lemma 3.14. *Let x and x' be two P-close points of X. Let $y \in X$ such that for all $u \in K$, $\langle x, x'\rangle_y \leqslant \langle x, x'\rangle_{uy} + 2\delta$. Moreover we assume that $\langle \bar x, \bar x'\rangle_{\bar y} \leqslant \frac{1}{250} r_0$, Then $\langle x, x'\rangle_y \leqslant \pi \operatorname{sh} r_0$.*

Proof. The points x and x' are P-close. Hence by Proposition 3.7, for all $\rho \in P$, $|[x,y] \cap Y_\rho|$ and $|[x',y] \cap Y_\rho|$ are smaller than $[\rho] - 122\pi \operatorname{sh} r_0 + 253\delta$. The result follows then from Proposition 3.5. □

Lemma 3.15. *Let g be a P-reduced element of G. Let $x \in X$ such that for all $u \in K$, $d(x, Y_g) \leqslant d(ux, Y_g) + 2\delta$. We assume also that $d(\bar{x}, Y_{\bar{g}}) \leqslant \frac{1}{250} r_0 - 87\bar{\delta}$. Then $d(x, Y_g) \leqslant \pi \operatorname{sh} r_0 + 87\delta$.*

Proof. By Proposition 3.11, there exists $k_0 \in \mathbf{N}$ such that for all $k \geqslant k_0$, for all $\rho \in P$, $\left|\left[x, g^k x\right] \cap Y_\rho\right| \leqslant [\rho] - 122\pi \operatorname{sh} r_0 + 289\delta$. However $\bar{g}$ is a hyperbolic isometry. Therefore, there exists $k \geqslant k_0$ such that $\left[g^k\right] > 40\delta$ and $\left[\bar{g}^k\right] > 40\bar{\delta}$. It follows from Lemma 1.12 that the distance from x to Y_g is approximatively given by $\left\langle g^{-k}x, g^k x\right\rangle_x$. The same works for $\bar{x}$ and $Y_{\bar{g}}$. More precisely,

$$\left\langle \bar{g}^{-k}\bar{x}, \bar{g}^k \bar{x}\right\rangle_{\bar{x}} \leqslant d(\bar{x}, Y_{\bar{g}}) + 87\bar{\delta} \leqslant \frac{1}{250} r_0.$$

Applying Proposition 3.5 we get

$$d(x, Y_g) \leqslant \left\langle g^{-k}x, g^k x\right\rangle_x + 87\delta \leqslant \pi \operatorname{sh} r_0 + 87\delta,$$

which completes the proof of the lemma. □

Proposition 3.16. *Let $k \in \mathbf{N}$. Let $L \geqslant 2r_0$. Let g be a P-reduced element of G such that $\left[\bar{g}^k\right] > 40\bar{\delta}$. Let p and q be two points of X satisfying the followings*

(i) $d(\bar{p}, Y_{\bar{g}}), d(\bar{q}, Y_{\bar{g}}) \leqslant \frac{1}{250} r_0 - 87\bar{\delta}$,

(ii) for all $u \in K$, $d(p, Y_g) \leqslant d(up, Y_g) + 2\delta$ *and* $d(q, Y_g) \leqslant d(uq, Y_g) + 2\delta$,

(iii) $|\bar{p} - \bar{q}| \geqslant \left[\bar{g}^k\right] + L$.

Then $|p - q| \geqslant \left[g^k\right] + L - 3\pi \operatorname{sh} r_0$.

Proof. Let $\bar{N}$ be a nerve of $\bar{g}^k$ (in $\bar{X}$). We denote by $\bar{r}$ and $\bar{s}$ respective projections of $\bar{p}$ and $\bar{q}$ on $\bar{N}$. The isometry $\bar{g}^k$ acts on $\bar{N}$ by translation of length $\left[\bar{g}^k\right]$. By replacing if necessary g by g^{-1}, we can assume that $\bar{s}$ and $\bar{g}^k\bar{r}$ belong to the same component of $\bar{N} \setminus \{\bar{r}\}$. Since $\left[\bar{g}^k\right] > 40\bar{\delta}$, $Y_{\bar{g}}$ is contained in the $42\bar{\delta}$-neighbourhood of $\bar{N}$. In particular $|\bar{p} - \bar{r}| \leqslant \frac{1}{250} r_0 - 45\bar{\delta}$ and $|\bar{q} - \bar{s}| \leqslant \frac{1}{250} r_0 - 45\bar{\delta}$. It follows from the triangle inequality that $|\bar{r} - \bar{s}| \geqslant |\bar{p} - \bar{q}| - \frac{1}{125} r_0 + 90\bar{\delta} \geqslant \left[\bar{g}^k\right]$. However $\bar{N}$ is a $\left[\bar{g}^k\right]$-local geodesic, thus $\bar{g}^k\bar{r}$ necessary belongs to $(\bar{r}, \bar{s})_{\bar{N}}$ and $\langle \bar{p}, \bar{q}\rangle_{\bar{g}^k\bar{r}} \leqslant 45\bar{\delta}$. Hence $\langle \bar{p}, \bar{q}\rangle_{\bar{g}^k\bar{p}} \leqslant \langle \bar{p}, \bar{q}\rangle_{\bar{g}^k\bar{r}} + |\bar{r} - \bar{p}| \leqslant \frac{1}{250} r_0$. According to Point (ii) p and q are the respective lifts of $\bar{p}$ and $\bar{q}$ which are the "closest" to Y_g. Hence by Lemma 3.15, p and q belongs to the $(\pi \operatorname{sh} r_0 + 87\delta)$-neighbourhood of Y_g. It follows from Lemma 3.10 that for all $\rho \in P$, $\left|\left[p, g^k p\right] \cap Y_\rho\right|$ and $\left|\left[q, g^k p\right] \cap Y_\rho\right|$ are bounded above by $[\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. Consequently by Proposition 3.5 $\langle p, q\rangle_{g^k p} \leqslant \pi \operatorname{sh} r_0$. In particular

$$|p - q| \geqslant \left|p - g^k p\right| + \left|g^k p - q\right| - 2\pi \operatorname{sh} r_0 \geqslant \left[g^k\right] + \left|g^k p - q\right| - 2\pi \operatorname{sh} r_0. \quad (11)$$

However the map $X \to \bar{X}$ shorten the distances, thus

$$\left|g^k p - q\right| \geqslant \left|\bar{g}^k \bar{p} - \bar{q}\right| \geqslant |\bar{p} - \bar{q}| - \left|\bar{g}^k\bar{p} - \bar{p}\right| \geqslant |\bar{p} - \bar{q}| - \left[\bar{g}^k\right] - 2|\bar{p} - \bar{r}|$$

Using Point (iii) we deduced that $\left|g^k p - q\right| \geqslant L - \pi \operatorname{sh} r_0$, which together with (11) leads to the result. □

Proposition 3.17. *Let x and y be two P-close points of X. Let g be a P-reduced element of G. Let $k \in \mathbf{N}$ such that $\left[\bar{g}^k\right] > 40\bar{\delta}$. Let $L \geqslant 6r_0 + 13\bar{\delta}$ such that $|[\bar{x},\bar{y}] \cap Y_{\bar{g}}| \geqslant \left[\bar{g}^k\right] + L$. There exists three points $r,p,q \in X$ and $v \in K$ satisfying the following properties*

(i) $\bar{p} = \bar{q}$.

(ii) $d\left(r, vY_g\right) \leqslant \pi \operatorname{sh} r_0 + 87\delta$, $d\left(q, vY_g\right) \leqslant 2\pi \operatorname{sh} r_0 + 91\delta$, $\langle x, q\rangle_r \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$ *and* $\langle x, y\rangle_p \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$, ,

(iii) $|r - q| \geqslant \left[g^k\right] + L - 5\pi \operatorname{sh} r_0 - 4\delta$.

(iv) The configuration (x,p,q,y) is foldable.

Proof. Let us denote by $\bar{a}$ and $\bar{b}$ respective projections of $\bar{x}$ and $\bar{y}$ on $Y_{\bar{g}} \subset \bar{X}$. By Proposition 1.6, $\left|\bar{a} - \bar{b}\right| \geqslant \left[\bar{g}^k\right] + L - 13\bar{\delta}$. Recall that $\bar{X}$ is obtained by attaching cones on X/K. Hence $\bar{a}$ and $\bar{b}$ may not belong to $\nu(X)$, the image of X in $\bar{X}$. However these cones have diameter $2r_0$. Thus there exists two points $\bar{r}$ and $\bar{z}$ in $\left[\bar{a},\bar{b}\right] \cap \nu(X)$, such that $|\bar{a} - \bar{r}|, \left|\bar{b} - \bar{z}\right| \leqslant 2r_0$. In particular $|\bar{r} - \bar{z}| \geqslant \left[\bar{g}^k\right] + L - 4r_0 - 13\bar{\delta}$. Since $Y_{\bar{g}}$ is $6\bar{\delta}$-quasi-convex, $\bar{r}$ and $\bar{z}$ are in the $6\bar{\delta}$-neighbourhood of $Y_{\bar{g}}$. Moreover, $\langle \bar{x}, \bar{y}\rangle_{\bar{r}}, \langle \bar{x}, \bar{y}\rangle_{\bar{z}} \leqslant 13\bar{\delta}$ and $\langle \bar{x}, \bar{z}\rangle_{\bar{r}} \leqslant 6\bar{\delta}$. The next step of the proof consists in lifting this figure in X. First we define lifts of $\bar{r}$ and $\bar{z}$ which are as close as possible from $[x,y]$. Let $r, z \in X$ be respective pre-images of $\bar{r}$ and $\bar{z}$ such that for all $u \in K$, we have in X $\langle x, y\rangle_r \leqslant \langle x, y\rangle_{ur} + 2\delta$ and $\langle x, y\rangle_z \leqslant \langle x, y\rangle_{uz} + 2\delta$. Since x and y are P-close, Lemma 3.14 leads to $\langle x, y\rangle_r, \langle x, y\rangle_z \leqslant \pi \operatorname{sh} r_0$. In particular there is a point p on $[x,y]$ such that $|p - z| \leqslant \pi \operatorname{sh} r_0 + 4\delta$ and $\langle x, y\rangle_p \leqslant \langle x, y\rangle_z + |p - z| \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$.

We now chose a conjugate of g whose axes in X approximatively passes through r. To that end, we fix $v \in K$ such that for all $u \in K$, we have $d\left(r, vY_g\right) \leqslant d\left(ur, vY_g\right) + 2\delta$. By assumption g is P-reduced. Hence vY_g is the cylinder of vgv^{-1} which is P-reduced as well. By Lemma 3.15, $d\left(r, vY_g\right) \leqslant \pi \operatorname{sh} r_0 + 87\delta$. We chose for z a lift of $\bar{z}$ which was close to $[x,y]$. Unfortunately z is not necessarily in the neighbourhood of vY_g. That is why we have to introduce a second pre-image of $\bar{z}$. Let $w \in K$ such that for all $u \in K$, $d\left(wz, vY_g\right) \leqslant d\left(uwz, vY_g\right) + 2\delta$. By Lemma 3.15, $d\left(wz, vY_g\right) \leqslant \pi \operatorname{sh} r_0 + 87\delta$. We finally put $q = wp$. In particular $d(q, vY_g) \leqslant 2\pi \operatorname{sh} r_0 + 91\delta$. Moreover $\bar{p} = \bar{q}$, which proves Point (i).

By construction $\langle x, y\rangle_r \leqslant \pi \operatorname{sh} r_0$. However x and y are P-close. Hence for all $\rho \in P$, $|[x,r] \cap Y_\rho| \leqslant |[x,y] \cap Y_\rho| + \langle x, y\rangle_r \leqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. On the other hand, $d(r, vY_g)$ and $d(wz, vY_g)$ are bounded above by $\pi \operatorname{sh} r_0 + 87\delta$. The isometry vgv^{-1} being P-reduced, Lemma 3.10 implies that for all $\rho \in P$, $|[r, wz] \cap Y_\rho| \leqslant [\rho] - 3\pi \operatorname{sh} r_0 - 40\delta$. Since $\langle \bar{x}, \bar{z}\rangle_{\bar{r}} \leqslant 6\bar{\delta}$, applying Lemma 3.5 we get $\langle x, q\rangle_r \leqslant \langle x, wz\rangle_r + |p - z| \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$, which completes the proof of Point (ii). (In the same way, we can prove that $\langle x, p\rangle_r \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$.) Note that vgv^{-1}, r and wz satisfy the assumptions of Proposition 3.16. Therefore $|r - wz| \geqslant \left[g^k\right] + L - 4\pi \operatorname{sh} r_0$. Thus $|r - q| \geqslant \left[g^k\right] + L - 5\pi \operatorname{sh} r_0 - 4\delta$, which gives Point (iii).

It only remains to prove that (x,p,q,y) is foldable. In the definition of foldable configuration we choose $s=x$. Since x and y are P-close and p lies on a geodesic between them, Assumption (C1) is fulfilled. So is the condition (C3). We choose for t the point r. We proved that $d(r,vY_g) \leqslant \pi \operatorname{sh} r_0 + 87\delta$ and $d(q,vY_g) \leqslant 2\pi \operatorname{sh} r_0 + 91\delta$. Moreover vgv^{-1} is P-reduced. By Lemma 3.10, r and q are P-close. On the other hand $\langle x,q\rangle_r \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$ and $\langle x,p\rangle_r \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$. Therefore by hyperbolicity $|x-r| \leqslant \langle p,q\rangle_x + 2\pi \operatorname{sh} r_0 + 5\delta$. Thus Condition (C2) holds. □

4 Burnside groups

4.1 General framework

This section is dedicated to the proof of our main theorem. Let (X,x_0) be a geodesic, proper, hyperbolic pointed space. Let G be a non-elementary, torsion-free group acting freely, properly, co-compactly, by isometries on X.

In order to study the quotient G/G^n, T. Delzant and M. Gromov provides in [12] a sequence of appropriate hyperbolic groups (G_k) whose direct limit is G/G^n. We recall here the main steps of this construction as it is exposed in [9].

The constants δ_1, r_0, δ_0 and Δ_0 are the one given at the end of Section 3.1. The rescaling parameter λ_n is defined by

$$\lambda_n = \frac{\pi \operatorname{sh} r_0}{5\sqrt{n r_0 \delta_1}}.$$

The integer n_0 is chosen large enough in such a way that λ_{n_0} satisfy a set of inequalities[1]. For our purpose, we also require that $\lambda_{n_0}^{-1} \geqslant 500$. We build by induction two sequences (X_k) and (G_k) as follows.

Initialization. Among other things, we can assume, by rescaling X if necessary, that X is δ-hyperbolic, with $\delta \leqslant \delta_0$ and $A(G,X) \leqslant \Delta_0/2$. Up to increase n_0, we may also require that $r_{inj}(G,X) \geqslant 20\sqrt{r_0\delta_1/n_0}$. We fix now ξ such that

$$40(\xi-1)\sqrt{r_0\delta_1/n_0} \geqslant 30\pi \operatorname{sh} r_0$$

and an odd integer $n \geqslant \max\{100, n_0, 2\varepsilon+1\}$ satisfying

$$\frac{500\pi \operatorname{sh} r_0}{n} \leqslant 20\sqrt{r_0\delta_1/n_0}$$

We put $X_0 = X$ and $G_0 = G$. For simplicity of notation we write λ instead of λ_{n_0}.

[1] In this article, the exact statement of the inequalities it should satisfy is not important. There are chosen in such a way that one can iterate the small cancellation process explained below. The conditions to fulfill coarsely say that $\lambda_n\delta_1 \ll \min\{\delta_0, \Delta_0\}$. For more details see [9].

Heredity. We assume that X_k and G_k are built and satisfy (among others) the following assumptions.

(i) The metric space X_k is geodesic, proper and δ-hyper-bolic, with $\delta \leqslant \delta_0$.

(ii) The group G_k acts properly, co-compactly by isometries on X_k and satisfies the small centralizers hypothesis (i.e. it is non-elementary and all its elementary subgroups are cyclic).

(iii) $A(G_k, X_k) \leqslant \Delta_0/2$.

(iv) $r_{inj}(G_k, X_k) \geqslant 20\sqrt{r_0\delta_1/n_0} \geqslant 500\frac{\pi \operatorname{sh} r_0}{n}$. In particular, the injectivity radius of G_k satisfies $2(\xi - 1)r_{inj}(G_k, X_k) \geqslant 30\pi \operatorname{sh} r_0$.

We denote by R_k the set of elements $g \in G_k$ such that g is hyperbolic, not a proper power and $[g]_{X_k} \leqslant 1000\delta$. There exists a subset R_k^0 of R_k stable under conjugation such that R_k is the disjoint union of R_k^0 and the set of all inverses of R_k^0. We define P_k by $P_k = \left\{g^n, g \in R_k^0\right\}$. This set satisfies the hypothesis of the small cancellation theorem (Theorem 3.1), i.e. $\Delta(P_k, X_k) \leqslant \Delta_0$ and $r_{inj}(P_k, X_k) \geqslant 500\pi \operatorname{sh} r_0$. Let G_{k+1} be the quotient $G_k/ \ll P_k \gg$. The space $\bar{X}_k$ is the one constructed from X_k by small cancellation (see Section 3). It is $\bar{\delta}$-hyperbolic, with $\bar{\delta} \leqslant \delta_1$. We define X_{k+1} as the rescaled space $\lambda \bar{X}_k$. Using the conditions satisfied by λ, one can prove that X_{k+1} and G_{k+1} satisfy also the assumptions (i)–(iv). Moreover the canonical map $\nu_k : X_k \rightarrow X_{k+1}$ has the following property: for all $x, x' \in X_k$, $|\nu_k(x) - \nu_k(x')|_{X_{k+1}} \leqslant \lambda |x - x'|_{X_k}$.

The sequence (G_k) constructed in this way approximates the Burnside group G/G^n in the sense that $\varinjlim G_k = G/G^n$.

Notations :

(i) For all $k \in \mathbf{N}$ the kernel of the projection $G \twoheadrightarrow G_k$ is denoted by K_k. In particular, for all $k \in \mathbf{N}$, $K_k \lhd K_{k+1}$.

(ii) Let x be a point of X (respectively g be an element of G). For simplicity of notation, we still write x (respectively g) for its image by the natural map $X \rightarrow X_k$ (respectively $G \twoheadrightarrow G_k$).

4.2 Close points, reduced elements of rank k

Remark : From now on, unless otherwise stated, all the metric objects (distances, diameters, Gromov's products) are measured with the distance of X_k (and never with the one of $\bar{X}_k$).

Definition 4.1. *Let $k \in \mathbf{N}$. Two points x and x' of X are close of rank k if for all $j < k$, for all $\rho \in P_j$, $|[x, x'] \cap Y_\rho| \leqslant [\rho]/2 + 6\pi \operatorname{sh} r_0$ in the space X_j.*

Definition 4.2. *Let $k \in \mathbf{N}$. An element g of G is reduced of rank k if g is hyperbolic as element of G_k and for all $j < k$, $|Y_g \cap Y_\rho| \leqslant [\rho]/2 + \pi \operatorname{sh} r_0$ in the space X_j.*

Remark : Note that *being close* (respectively *reduced*) *of rank 0* is an empty condition. Any two points of X are close of rank 0. Any hyperbolic element of G is reduced of rank 0.

Proposition 4.3. *Let $k \in \mathbf{N}$. Let $g \in G$. If g is hyperbolic in G_k then there exists $u \in K_k$ such that ug is reduced of rank k.*

Proof. The proof is by induction on k. Since every hyperbolic element of G is reduced of rank 0, the proposition is true for $k = 0$. Assume now that the proposition holds for $k \in \mathbf{N}$. Let $g \in G$ such that g is hyperbolic in G_{k+1}. By Proposition 3.9 there exists $u \in K_{k+1}$ such that ug is P_k-reduced, i.e. for all $\rho \in P_k$, $|Y_{ug} \cap Y_\rho| \leqslant [\rho]/2 + \pi \operatorname{sh} r_0$ in the space X_k. Note that $g = ug$ in G_{k+1}. Thus ug is hyperbolic in G_{k+1} and therefore in G_k. We apply the induction hypothesis on ug: there exists $v \in K_k$ such that vug is reduced of rank k. However $vug = ug$ in G_k. Hence for all $j \leqslant k$, for all $\rho \in P_j$, $|Y_{vug} \cap Y_\rho| \leqslant [\rho]/2 + \pi \operatorname{sh} r_0$ in the space X_j, which means that vug is reduced of rank $k+1$. Moreover, since $K_k \lhd K_{k+1}$, $vu \in K_{k+1}$. Consequently the proposition holds for $k+1$. □

4.3 Elementary moves in X

Recall that x_0 is a base point of X.

Definition 4.4. *Let y and z be two points of X.*

- *We say that z is the image of y by a (ξ, n)-*elementary move *(or simply an* elementary move*), if there exist $g \in G$ such that*

  (i) $|[x_0, y] \cap Y_g| \geqslant [g^m]$ in the space X, with $m \geqslant n/2 - \xi$.

  (ii) $z = g^{-n} y$ in X.

- *We say that z is the image of y by a* sequence of elementary moves*, and we write $y \to z$, if there exists a finite sequence of points of X, $y = y_0, y_1, \dots, y_l = z$ such that for all $j \in \{0, \dots, l-1\}$, y_{j+1} is the image of y_j by an elementary move.*

Our theorems are consequences of the following one

Theorem 4.5. *Let y be a point of X. An element $g \in G$ belongs to G^n if and only if there exist two sequences of elementary moves which respectively send y and gy to the same point.*

Remark : Assume that there are two sequences of elementary moves which respectively send y and gy to the same point. By definition this common point can be written $uy = vgy$ where u and v belong to G^n. Since G acts freely on X it directly follows that g belongs to G^n. What we need to prove is the other direction. To that end we first show the following induction proposition.

Proposition 4.6. *Let $k \in \mathbf{N}$.*

(A) Let $y \in X$. There exists $u \in K_k$ such that x_0 and uy are close of rank k and uy is the image of y by a sequence of elementary moves.

(B) Let $y, z \in X$ such that x_0 and y (respectively x_0 and z) are close of rank k. If $y = z$ in X_k, then z is the image of y by a sequence of elementary moves.

(C) Let $y \in X$ such that x_0 and y are close of rank k. Let g be an element of G which is reduced of rank k. We assume that there exists an integer $m \geqslant n/2 - \xi$ such that

$$|[x_0, y] \cap Y_g| \geqslant [g^m] + \pi \operatorname{sh} r_0 \text{ in } X_k.$$

Then there exist $u, v \in K_k$ such that uy is the image of y by a sequence of elementary moves and

$$|[x_0, uy] \cap vY_g| \geqslant [g^m] + \pi \operatorname{sh} r_0 \text{ in } X.$$

Proof. The rest of this section is devoted to the proof of this proposition. The proof is by induction of k. If $k = 0$, all the conclusions are already contained in the hypothesis (take $u = v = 1$). Hence the proposition is true for $k = 0$. Assume now that the proposition holds for $k \in \mathbf{N}$.

Lemma 4.7. *Let $y \in X$ such that x_0 and y are close of rank k but not close of rank $k+1$. There exists $u \in K_{k+1}$ such that*

(i) x_0 and uy are close of rank k,

(ii) uy is the image of y by a sequence of elementary moves,

(iii) $|x_0 - uy|_{X_k} < |x_0 - y|_{X_k} - 6\pi \operatorname{sh} r_0 + 183\delta$.

Proof. By assumption, there exists $r \in R_k^0$ such that

$$|[x_0, y] \cap Y_r| > \frac{1}{2}[r^n] + 6\pi \operatorname{sh} r_0 \text{ in } X_k.$$

Applying Lemma 1.13, there exists $\kappa \in \mathbf{Z}$ such that $|x_0 - r^{\kappa n} y|_{X_k} < |x_0 - y|_{X_k} - 6\pi \operatorname{sh} r_0 + 183\delta$. However r is hyperbolic in G_k. By Proposition 4.3, there exists $s \in G$ which is reduced of rank k such that $s = r$ in G_k. In particular s^n belongs to K_{k+1} and $|[x_0, y] \cap Y_s| > [s^n]/2 + 6\pi \operatorname{sh} r_0$ in X_k. We put $m = \lfloor n/2 - \xi \rfloor + 1$. Recall that $(\xi - 1) r_{inj}(G_k, X_k) \geqslant 30\pi \operatorname{sh} r_0$. It follows that

$$[s^n]_{X_k} \geqslant 2[s^m]^\infty_{X_k} + 2(\xi - 1)[s]^\infty_{X_k} \geqslant 2[s^m]_{X_k} + 30\pi \operatorname{sh} r_0 - 32\delta.$$

Consequently we have in X_k, $|[x_0, y] \cap Y_s| > [s^m] + \pi \operatorname{sh} r_0$, with $m \geqslant n/2 - \xi$. By construction x_0 and y are close of rank k and s is reduced of rank k. Applying the induction hypothesis (Prop. 4.6(C)), there exist $u, v \in K_k$ such that uy is the image of y by a sequence of elementary moves and

$$|[x_0, uy] \cap vY_s| \geqslant [s^m] + \pi \operatorname{sh} r_0 \geqslant \left[vs^m v^{-1}\right] \text{ in } X.$$

Therefore $(vs^{\kappa n} v^{-1}) uy$ is the image of uy by an elementary move. However, by induction hypothesis (Prop. 4.6(A)), there exists $w \in K_k$ such that x_0 and $w(vs^{\kappa n} v^{-1}) uy$ are close of rank k and $w(vs^{\kappa n} v^{-1}) uy$ is the image of $(vs^{\kappa n} v^{-1}) uy$ by a sequence of elementary moves.

Let us now summarize. Using a finite number of elementary moves, we have done the following transformations:

$$y \to uy \to (vs^{\kappa n}v^{-1})uy \to w(vs^{\kappa n}v^{-1})uy.$$

On the other hand $u, v, w \in K_k$ and $s^n \in K_{k+1}$. Thus $w(vs^{\kappa n}v^{-1})u$ belongs to K_{k+1} and $w(vs^{\kappa n}v^{-1})u = s^{\kappa n} = r^{\kappa n}$ in G_k. Hence

$$\left|x_0 - w(vs^{\kappa n}v^{-1})uy\right|_{X_k} = |x_0 - r^{\kappa n}y|_{X_k} < |x_0 - y|_{X_k} - 6\pi \operatorname{sh} r_0 + 183\delta.$$

which concludes the proof of the lemma. □

Lemma 4.8. *Let $y \in X$. There exists $u \in K_{k+1}$ such that x_0 and uy are close of rank $k+1$ and uy is the image of y by a sequence of elementary moves.*

Remark : This lemma proves Prop. 4.6(A) for $k+1$.

Proof. Let $\mathcal{U}$ be the set of elements of $u \in K_{k+1}$ such that x_0 and uy are close of rank k and uy is the image of y by a sequence of elementary moves. According to the induction hypothesis (Prop. 4.6(A)), $\mathcal{U}$ is non-empty (more precisely $\mathcal{U} \cap K_k \neq \emptyset$). Hence we can choose $u \in \mathcal{U}$ such that for all $u' \in \mathcal{U}$, $|x_0 - uy|_{X_k} \leqslant |x_0 - u'y|_{X_k} + \delta$. We claim that x_0 and uy are close of rank $k+1$. On the contrary, suppose that this assertion is false. By construction of $\mathcal{U}$, x_0 and uy are close of rank k. By Lemma 4.7, there exists v in K_{k+1} such that vu belongs to $\mathcal{U}$ and $|x_0 - vuy|_{X_k} < |x_0 - uy|_{X_k} - 6\pi \operatorname{sh} r_0 + 183\delta$, which contradicts the definition of u. □

Lemma 4.9. *Let $y \in X$ such that x_0 and y are close of rank k. Let $p, q \in X_k$ such that the configuration (x_0, p, q, y) is foldable in X_k. We assume that p and q are equal in X_{k+1} but not in X_k. There exists $u \in K_{k+1}$ such that*

(i) x_0 and uy are close of rank k,

(ii) uy is the image of y by a sequence of elementary moves,

(iii) $\langle up, q\rangle_{x_0} \geqslant \langle p, q\rangle_{x_0} + 237\pi \operatorname{sh} r_0 - 424\delta$ in X_k,

(iv) the configuration (x_0, up, q, uy) is foldable and

$$\langle x_0, uy\rangle_{up} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta.$$

Proof. Let us apply Proposition 3.13 in X_k with (x_0, p, q, y). There exist $r \in R_k^0$ and $\varepsilon \in \{\pm 1\}$ satisfying the followings.

- $|[x_0, y] \cap Y_r| \geqslant [r^n]/2 - 13\pi \operatorname{sh} r_0 - 419\delta$.
- $\langle r^{\varepsilon n}p, q\rangle_{x_0} \geqslant \langle p, q\rangle_{x_0} + [r^n]/2 - 13\pi \operatorname{sh} r_0 - 424\delta$.
- The configuration $(x_0, r^{\varepsilon n}p, q, r^{\varepsilon n}y)$ is foldable. Furthermore

$$\langle x_0, r^{\varepsilon n}y\rangle_{r^{\varepsilon n}p} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta.$$

However r is hyperbolic in G_k. By Proposition 4.3, there exists $s \in G$ which is reduced of rank k such that $s = r$ in G_k. In particular s^n belongs to K_{k+1}. Moreover, we have $|[x_0, y] \cap Y_s| \geqslant [s^n]/2 - 13\pi \operatorname{sh} r_0 - 419\delta$ in X_k. We put $m = \lfloor n/2 - \xi \rfloor + 1$. Just as in Lemma 4.7, we have $[s^n]_{X_k} \geqslant 2[s^m]_{X_k} + 30\pi \operatorname{sh} r_0 - 32\delta$. Consequently we get in X_k, $|[x_0, y] \cap Y_s| > [s^m] + \pi \operatorname{sh} r_0$, with $m \geqslant n/2 - \xi$. By construction x_0 and y are close of rank k and s is reduced of rank k. Applying the induction hypothesis (Prop. 4.6(C)), there exist $u, v \in K_k$ such that uy is the image of y by a sequence of elementary moves and

$$|[x_0, uy] \cap vY_s| \geqslant [s^m] + \pi \operatorname{sh} r_0 \;\geqslant \left[vs^m v^{-1}\right] \text{ in } X.$$

Therefore $(vs^{\varepsilon n}v^{-1})uy$ is the image of uy by an elementary move. By induction hypothesis (Prop. 4.6(A)), there exists $w \in K_k$ such that x_0 and $w(vs^{\varepsilon n}v^{-1})uy$ are close of rank k and $w(vs^{\varepsilon n}v^{-1})uy$ is the image of $(vs^{\varepsilon n}v^{-1})uy$ by a sequence of elementary moves.

Let us now summarize. Using a finite number of elementary moves, we have done the following transformations:

$$y \to uy \to (vs^{\varepsilon n}v^{-1})uy \to w(vs^{\varepsilon n}v^{-1})uy.$$

On the other hand $u, v, w \in K_k$ and $s^n \in K_{k+1}$. Thus $w(vs^{\varepsilon n}v^{-1})u$ belongs to K_{k+1} and $w(vs^{\varepsilon n}v^{-1})u = s^{\varepsilon n} = r^{\varepsilon n}$ in G_k. Consequently the configuration $(x_0, w(vs^{\varepsilon n}v^{-1})up, q, w(vs^{\varepsilon n}v^{-1})uy)$ is foldable (in X_k) and

$$\left\langle x_0, w(vs^{\varepsilon n}v^{-1})uy \right\rangle_{w(vs^{\varepsilon n}v^{-1})up} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta.$$

□

Lemma 4.10. *Let $y \in X$ such that x_0 and y are close of rank k. Let $p, q \in X_k$ such that the configuration (x_0, p, q, y) is foldable in X_k and $\langle x_0, y\rangle_p \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$. We assume that p and q are equal in X_{k+1}. There exists $u \in K_{k+1}$ such that*

(i) x_0 and uy are close of rank k,

(ii) uy is the image of y by a sequence of elementary moves,

(iii) in X_k, $up = q$ and $\langle x_0, uy\rangle_q \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$.

Proof. Let us denote by $\mathcal{U}$ the set of elements $u \in K_{k+1}$ such that,

- x_0 and uy are close of rank k,
- uy is the image of y by a sequence of elementary moves,
- in X_k, the configuration (x_0, up, q, uy) is foldable, furthermore

$$\langle x_0, uy\rangle_{up} \leqslant 23\pi \operatorname{sh} r_0 + 599\delta.$$

The set $\mathcal{U}$ is non empty ($1 \in \mathcal{U}$). On the other hand, for all $u \in \mathcal{U}$, $\langle up, q\rangle_{x_0}$ is bounded above by $|q - x_0|_{X_k}$ in X_k. Hence we can choose $u \in \mathcal{U}$ such that for all $u' \in \mathcal{U}$, $\langle up, q\rangle_{x_0} \geqslant \langle u'p, q\rangle_{x_0} - \delta$ in X_k. We claim that $up = q$. On the contrary, suppose that this assertion is false. By definition of $\mathcal{U}$, the configuration (x_0, up, q, uy) is foldable in X_k. Therefore applying Lemma 4.9, there exists $v \in K_{k+1}$ such that vu belongs to $\mathcal{U}$ and $\langle vup, q\rangle_{x_0} \geqslant \langle up, q\rangle_{x_0} + 237\pi \operatorname{sh} r_0 - 424\delta$ in X_k, which contradicts the definition of u. Consequently $up = q$ in X_k. It follows from the definition of $\mathcal{U}$ that $\langle x_0, uy\rangle_q \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$ in X_k. □

Lemma 4.11. *Let $y, z \in X$ such that x_0 and y (respectively x_0 and z) are close of rank $k+1$. If $y = z$ in X_{k+1} then z is the image of y by a sequence of elementary moves.*

Remark : This lemma proves Prop. 4.6(B) for $k+1$.

Proof. By assumption x_0 and y are close of rank k. Moreover x_0 and y (respectively x_0 and z) are P_k-close in X_k. Thus the configuration (x_0, y, z, y) is foldable in X_k (take $s = t = x_0$ in Definition 3.12) and $\langle x_0, y\rangle_y = 0$. Applying Lemma 4.10, there exists $u \in K_{k+1}$ such that uy is the image of y by a sequence of elementary moves, $uy = z$ in X_k and x_0 and uy are close of rank k. By assumption, x_0 and z are also close of rank k. According to the induction hypothesis (Prop. 4.6(B)), z is the image of uy by a sequence of elementary moves. Hence z is the image of y by a sequence of elementary moves. □

Lemma 4.12. *Let $y \in X$ such that x_0 and y are close of rank $k+1$. Let $g \in G$ which is reduced of rank $k+1$. We assume that there exists an integer $m \geqslant n/2 - \xi$ such that*

$$|[x_0, y] \cap Y_g| \geqslant [g^m] + \pi \operatorname{sh} r_0, \text{ in } X_{k+1}.$$

Then there exist $u, v \in K_{k+1}$ such that uy is the image of y by a sequence of elementary moves and

$$|[x_0, uy] \cap vY_g| \geqslant [g^m] + \pi \operatorname{sh} r_0 \text{ in } X.$$

Remark : This lemma proves Prop. 4.6(C) for $k+1$.

Proof. Exceptionally we begin the proof by working in $\bar{X}_k = \lambda^{-1} X_{k+1}$ (instead of X_{k+1}). Written in $\bar{X}_k$, our assumption says that $|[x_0, y] \cap Y_g| \geqslant [g^m] + \lambda^{-1}\pi \operatorname{sh} r_0$. According to Proposition 3.17, there exist $r, p, q \in X_k$ and $v \in K_{k+1}$ satisfying the following

(i) $d(r, vY_g) \leqslant \pi \operatorname{sh} r_0 + 87\delta$, $d(q, vY_g) \leqslant 2\pi \operatorname{sh} r_0 + 91\delta$, $\langle x_0, y\rangle_p \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$ and $\langle x_0, q\rangle_r \leqslant 2\pi \operatorname{sh} r_0 + 4\delta$ in X_k,

(ii) $|r - q|_{X_k} \geqslant [g^m]_{X_k} + \pi \operatorname{sh} r_0 \left(\lambda^{-1} - 5\right) - 4\delta$.

(iii) $\bar{p} = \bar{q}$ in $\bar{X}_k$ and thus in X_{k+1}. Moreover the configuration (x_0, p, q, y) is foldable in X_k.

Applying Lemma 4.10, there exists $u \in K_{k+1}$ such that

- x_0 and uy are close of rank k,
- uy is the image of y by a sequence of elementary moves,
- in X_k, $up = q$ and $\langle x_0, uy\rangle_q \leqslant 23\pi \operatorname{sh} r_0 + 599\delta$.

In X_k we have

$$|[x_0, uy] \cap vY_g| \geqslant |[r,q] \cap vY_g| - \langle x_0, uy \rangle_q - \langle x_0, q \rangle_r .$$

On the other hand $d(r, vY_g) \leqslant \pi \operatorname{sh} r_0 + 87\delta$ and $d(q, vY_g) \leqslant 2\pi \operatorname{sh} r_0 + 91\delta$, thus

$$|[r,q] \cap vY_g| \geqslant |r-q| - 4\pi \operatorname{sh} r_0 - 182\delta \geqslant [g^m]_{X_k} + \pi \operatorname{sh} r_0 \left(\lambda^{-1} - 9\right) - 186\delta.$$

It follows that in X_k, $|[x_0, uy] \cap vY_g| \geqslant [g^m] + \pi \operatorname{sh} r_0$. (Recall that $\lambda^{-1} \geqslant 500$.) According to the induction hypothesis (Prop. 4.6(C)) there exist $u', v' \in K_k$ such that $u'uy$ is the image of uy by a sequence of elementary moves and $|[x_0, u'uy] \cap v'vY_g| \geqslant [g^m] + \pi \operatorname{sh} r_0$ in X. In particular $u'u, v'v \in K_{k+1}$ and $u'uy$ is the image of y by a sequence of elementary moves, which ends the proof of the lemma. □

Lemmas 4.8, 4.11 and 4.12 proves that Proposition 4.6 holds for $k+1$. □

Proof of Theorem 4.5. Let $g \in G$ such that its image in G/G^n is trivial. By construction the direct limit of the sequence (G_k) is G/G^n. There exists $k \in \mathbf{N}$ such that g is trivial in G_k. In particular, $y = gy$ in X_k. By Proposition 4.6(A), there exist $u, v \in K_k$ such that x_0 and uy (respectively x_0 and vgy) are close of rank k. Moreover uy (respectively vgy) is the image of y (respectively gy) be a sequence of elementary moves. However u and v belong to K_k, thus $uy = vgy$ in X_k. Applying Proposition 4.6(B), vgy is the image of uy by a sequence of elementary moves. □

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