The Bondi-Sachs Formalism

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The Bondi-Sachs formalism of General Relativity is a metric-based treatment of the Einstein equations in which the coordinates are adapted to the null geodesics of the spacetime. It provided the first convincing evidence that gravitational radiation is a nonlinear effect of general relativity and that the emission of gravitational waves from an isolated system is accompanied by a mass loss from the system. The asymptotic behaviour of the Bondi-Sachs metric revealed the existence of the symmetry group at null infinity, the Bondi-Metzner-Sachs group, which turned out to be larger than the Poincare group.

Contents

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1

1 Introduction

In a seminal 1960 Nature article (Bondi, 1960), Hermann Bondi presented a new approach to the study gravitational waves in Einstein's theory of general relativity. It was based upon the outgoing null hypersurfaces along which the waves traveled. It was followed up in 1962 by a paper by Bondi, Metzner and van der Burg (Bondi et al., 1962), in which the details were given for axisymmetric spacetimes. In his autobiography (Bondi, 1990, page 79), Bondi remarked about this work: "The 1962 paper I regard as the best scientific work I have ever done, which is later in life than mathematicians supposedly peak". Soon after, Rainer Sachs (Sachs, 1962b) generalized this formalism to non-axisymmetric spacetimes and sorted out the asymptotic symmetries in the approach to infinity along the outgoing null hypersurfaces. The beautiful simplicity of the Bondi-Sachs formalism was that it only involved 6 metric quantities to describe a general spacetime. At this time, an independent attack on Einstein's equations based upon null hypersurfaces was underway by Ted Newman and Roger Penrose (Newman and Penrose, 1962, 2009). Whereas the fundamental quantity in the Bondi–Sachs formalism was the metric, the Newman-Penrose approach was based upon a null tetrad and its curvature components. Although the Newman-Penrose formalism involved many more variables it led to a more geometric treatment of gravitational radiation, which culminated in Penrose's (Penrose, 1963) description in terms of the conformal compactification of future null infinity, denoted by $\mathcal{I}^+$. It was clear that there were parallel results emerging from these two approaches but the two formalisms and notations were completely foreign. At meetings, Bondi would inquire of colleagues, including one of us (JW), "Are you you a qualified translator?". This article describes the Bondi-Sachs formalism and how it has evolved into a useful and important approach to the current understanding of gravitational waves.

Before 1960, it was known that linear perturbations h_{ab} of the Minkowski metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ obeyed the wave equation

$$\Big(- \frac{\partial^2}{\partial t^2} + \delta^{ij} \frac{\partial^2}{\partial y^i \partial y^j} \Big) h_{ab} = 0 \,, \tag{1}$$

where the standard Cartesian coordinates $y^i = (y^1, y^2, y^3)$ satisfy the harmonic coordinate condition to linear order. It was also known that these linear perturbations had coordinate (gauge) freedom which raised serious doubts about the physical properties of gravitational waves. The retarded time u and advanced time v,

$$u = t - r \;\;,\;\; v = t + r \;\;,\;\; r^2 = \delta_{ij} y^i y^j \,, \tag{2}$$

are characteristic hypersurfaces of the hyperbolic equations (1), i.e. hypersurfaces along which the wavefronts travel.

These characteristic hypersurfaces are also null hypersurfaces, i.e. their normals, $k_a = -\nabla_a u$ and $n_a = -\nabla_a v$ are null, $\eta^{ab}k_a k_b = \eta^{ab} n_a n_b = 0$. Note that it is a peculiar property of null hypersurfaces that their normal direction is also tangent to the hypersurface, i.e. $k^a = \eta^{ab} k_b$ is tangent to the $u = const$ hypersurfaces. The curves tangent to k^a are null geodesics, called null rays, and generate the $u = const$ outgoing null hypersurfaces. Bondi's ingenuity was to use such a family of outgoing null hypersurfaces to build spacetime coordinates for describing outgoing gravitational waves.

An analogous formalism based upon ingoing null hypersurfaces is also possible and finds applications in cosmology (Ellis et al., 1985) but is of less physical importance in the study of outgoing gravitational waves.

2 The Bondi–Sachs metric

The Bondi-Sachs coordinates $x^a = (u, r, x^A)$ are based on a family of outgoing null hypersurfaces $u = const$. Here $x^0 = u$ labels these hypersurfaces, x^A $(A, B, C, ... = 2, 3)$ label the null rays and $x^1 = r$ is a surface area coordinate. The coordinates x^A are constant along the null rays, $k^a \partial_a x^A = 0$. As a result, the metric satisfies

$$g^{uu} = g^{uA} = g_{rr} = g_{rA} = 0 \tag{3}$$

and only the coordinate r varies along the null rays. The choice that r be an areal coordinate implies that $\det[g_{AB}] = r^4 \mathfrak{q}$, where $\mathfrak{q}(x^A)$ is the determinant of the unit sphere metric q_{AB} associated with the angular coordinates x^A, e.g. $q_{AB} = \mathrm{diag}(1, \sin^2 \theta)$ for standard spherical coordinates $x^A = (\theta, \phi)$.

In the resulting $x^a = (u, r, x^A)$ coordinates, the metric takes the Bondi-Sachs form,

$$g_{ab}dx^a dx^b = -\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big), \tag{4}$$

where

$$g_{AB} = r^2 h_{AB} \qquad \text{with} \qquad \det[h_{AB}] = \mathfrak{q}(x^A). \tag{5}$$

This determinant condition implies $h^{AB}\partial_r h_{AB} = h^{AB}\partial_u h_{AB} = 0$, where $h^{AC}h_{CB} = \delta^A_B$. Hereafter D_A denotes the covariant derivative of the metric h_{AB}, with $D^A = h^{AB}D_B$. The corresponding non-zero contravariant

3

components of the metric (4) are

$$g^{ur} = -e^{2\beta} \ , \quad g^{rr} = \frac{V}{r}e^{-2\beta} \ , \quad g^{rA} = -U^A e^{-2\beta} \ , \quad g^{AB} = \frac{1}{r^2}h^{AB} \ . \quad (6)$$

A suitable representation of h_{AB} with two functions $\gamma(u,r,\theta,\phi)$ and $\delta(u,r,\theta,\phi)$ corresponding to the $+$ and $\times$ polarization of gravitational waves is (van der Burg, 1966; Winicour, 2013)

$$h_{AB}dx^A dx^B = \left(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right)\cosh(2\delta) + 2\sin\theta\sinh(2\delta)d\theta d\phi \ . \quad (7)$$

This differs from the original form of Sachs (Sachs, 1962b) by the transformation $\gamma \to (\gamma+\delta)/2$ and $\delta \to (\gamma-\delta)/2$, which gives a less natural description of gravitational waves in the weak field approximation. In the original axisymmetric Bondi metric (Bondi et al., 1962) with rotational symmetry in the ϕ-direction, $\delta = U^\phi = 0$ and $\gamma = \gamma(u,r,\theta)$, resulting in the metric

$$
\begin{aligned}
g_{ab}^{(B)}dx^a dx^b &= \left(-\frac{V}{r}e^{2\beta} + r^2 U e^{2\gamma}\right)du^2 - e^{2\beta}dudr - r^2 U e^{2\gamma}dud\theta \\
&\quad + r^2\left(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right) ,
\end{aligned} \quad (8)
$$

where $U \equiv U^\theta$. Note that the original Bondi metric also has the reflection symmetry $\phi \to -\phi$ so that it is not suitable for describing an axisymmetric rotating star.

In Bondi's original work, the areal coordinate r was called a luminosity distance but this terminology is misleading because of its different meaning in cosmology (Jordan et al., 1960, see Sec. 3.3). The areal coordinate r becomes singular when the expansion Θ of the null hypersurface vanishes, where (Sachs, 1961, 1962b)

$$\Theta = \nabla_a(e^{-2\beta}k^a) = \frac{2}{r}e^{-2\beta} \ , \quad k^a \partial_a = \partial_r. \quad (9)$$

In contrast, the standard radial coordinate along the null rays in the Newman-Penrose formalism (Newman and Penrose, 1962, 2009) is the affine parameter λ, which remains regular when $\Theta = 0$. The areal distance and affine parameter are related by $\partial_r\lambda = e^{2\beta}$. Thus the areal coordinate remains non-singular provided β remains finite. For a version of the Bondi-Sachs formalism based upon an affine parameter, see (Winicour, 2013).

2.1 The electromagnetic analogue

The electromagnetic field in Minkowski space with its two degrees of freedom propagating along null hypersurfaces provides a simple model to demonstrate the essential features and advantages of the Bondi–Sachs formalism (Tamburino and Winicour, 1966). Consider the Minkowski metric in outgoing null spherical coordinates (u, r, x^A) corresponding to the flat space version of the Bondi-Sachs metric,

$$\eta_{ab}dx^a dx^b = -du^2 - 2dr\,du + r^2 q_{AB}dx^A dx^B \ .$$

(10)

Assume that the charge-current sources of the electromagnetic field are enclosed by a 3-dimensional timelike worldtube Γ, with spherical cross-sections of radius $r = R$, such that the outgoing null cones N_u from the vertices $r = 0$ (Fig. 1) intersect Γ at proper time u in spacelike spheres S_u, which are coordinatized by x^A.
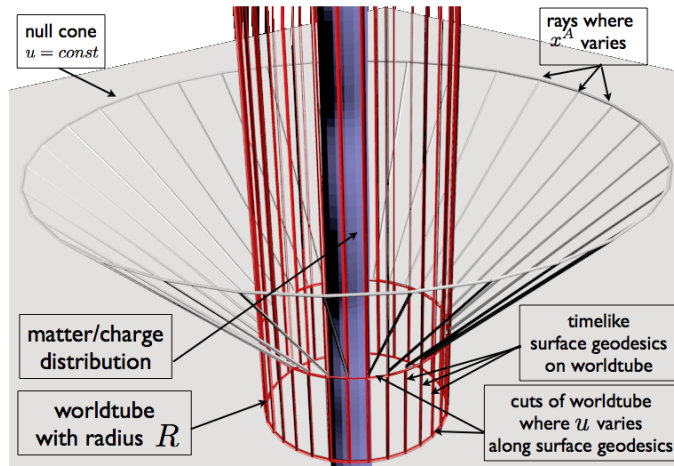

Figure 1: Illustration of Bondi-Sachs coordinates defined at a timelike worldtube surrounding a matter-charge distribution, along with an outgoing null cone.

The electromagnetic field F_{ab} is represented by a vector potential A_a,

5

$F_{ab} = \nabla_a A_b - \nabla_b A_a$, which has the gauge freedom

$$A_a \rightarrow A_a + \nabla_a \chi \ . \tag{11}$$

Choosing the gauge transformation

$$\chi(u, r, x^A) = -\int_R^r A_r dr' \tag{12}$$

leads to the null gauge $A_r = 0$, which is the analogue of the Bondi-Sachs coordinate condition (3), $g_{rr} = g_{rA} = 0$. The remaining gauge freedom $\chi(u, x^A)$ may be used to set either

$$A_u|_\Gamma = 0 \qquad \text{or} \qquad A_u|_{\mathcal{I}^+} = 0. \tag{13}$$

There remains the freedom $A_B \rightarrow A_B + \nabla_B \chi(x^C)$.

The vacuum Maxwell equations $M^b := \nabla_a F^{ab} = 0$ imply the identity

$$0 \equiv \nabla_b M^b = \partial_u M^u + \frac{1}{r^2} \partial_r(r^2 M^r) + \frac{1}{\sqrt{\mathsf{q}}} \partial_C(\sqrt{\mathsf{q}} M^C). \tag{14}$$

This leads to the following strategy. Designate as the main equations the components of Maxwell's equations $M^u = 0$ and $M^A = 0$, and designate $M^r = 0$ as the supplementary condition. Then if the main equations are satisfied (14) implies

$$0 = \partial_r(r^2 M^r) \ , \tag{15}$$

so that the supplementary condition is satisfied everywhere if it is satisfied at some specified value of r, e.g. on Γ or at $\mathcal{I}^+$.

The main equations separate into the

Hypersurface equation:
$$M^u = 0 \implies \partial_r(r^2 \partial_r A_u) = \partial_r(\eth_B A^B) \tag{16}$$

and the

Evolution equation:
$$M^A = 0 \implies \partial_r \partial_u A_B = \frac{1}{2}\partial_r^2 A_B - \frac{r^2}{2}\eth^C(\eth_B A_C - \eth_C A_B) + \frac{1}{2}\partial_r\eth_B A_u, \tag{17}$$

where hereafter $\eth_A$ denotes the covariant derivative with respect to the unit sphere metric q_{AB}, with $\eth^A = q^{AB}\eth_B$. The supplementary condition $M^r = 0$ takes the explicit form

$$\partial_u(r^2 \partial_r A_u) = \eth^B(\partial_r A_B - \partial_u A_B + \eth_B A_u). \tag{18}$$

A formal integration of the hypersurface equation yields

$$\partial_r A_u = \frac{Q(u, x^A) + \eth_B A^B}{r^2} + O(1/r^3) \ , \tag{19}$$

where $Q(u, x^A)$ enters as a function of integration. In the null gauge with $A_r = 0$, the radial component of the electric field corresponds to $E_r = F_{ru} = \partial_r A_u$. Thus, using the divergence theorem to eliminate $\eth_B A^B$, the total charge enclosed in a large sphere is

$$q(u) := \lim_{r \to \infty} \frac{1}{4\pi} \oint E_r r^2 \sin\theta d\theta d\phi = \frac{1}{4\pi} \oint Q(u, x^A) \sin\theta d\theta d\phi, \tag{20}$$

where $\oint$ indicates integration over the 2-sphere. This motivates calling $Q(u, x^A)$ the charge aspect. The integral of the supplementary condition (18) over a large sphere then gives the charge conservation law

$$\frac{dq(u)}{du} = 0. \tag{21}$$

The main equations give rise to a hierarchical integration scheme given the following combination of initial data on the initial null cone N_{u_0}, initial boundary data on the cross-section S_{u_0} of Γ and boundary data on Γ:

$$A_B\big|_{N_{u_0}}, \quad \partial_r A_u\big|_{S_{u_0}}, \quad \partial_u A_B\big|_{\Gamma}. \tag{22}$$

This leads to the following evolution algorithm:

1. In accord with (13), choose a gauge such that $A_u\big|_{\Gamma} = 0$.

2. Given the initial data $A_B\big|_{N_{u_0}}$ and $\partial_r A_u\big|_{S_{u_0}}$, the hypersurface equation (16) can be integrated along the null rays of N_{u_0} to determine A_u on the initial null cone N_{u_0}.

3. Given the initial boundary data $\partial_u A_B|_{S_{u_0}}$, the radial integration of the evolution equation (17) determines $\partial_u A_B$ on the initial null cone N_{u_0}.

4. (a) From $\partial_u A_B|_{N_{u_0}}$, A_B can be obtained in a finite difference approximation on the null cone $u = u_0 + \Delta u$.

   (b) From knowledge of $A_B|_{N_{u_0}}$ and $A_u|_{N_{u_0}}$, the the supplementary condition (18) determines $\partial_u \partial_r A_u\big|_{S_{u_0}}$ so that $\partial_r A_u|_{S_{u_0+\Delta u}}$ can also be obtained in a finite difference approximation.

5. This procedure can be iterated to determined a finite difference approximation for A_B and A_u on the null cone $u = u_0 + n\Delta u$.

An analogous algorithm for solving the Bondi-Sachs equations has been implemented as a convergent evolution code (see Sec. 5).

3   Einstein equations and their Bondi-Sachs solution

The Einstein equations, in geometric units $G = c = 1$, are

$$E_{ab} := R_{ab} - \frac{1}{2}g_{ab}R^c{}_c - 8\pi T_{ab} = 0 \ , \tag{23}$$

where R_{ab} is the Ricci tensor, $R^c{}_c$ its trace and T_{ab} the matter stress-energy tensor. Before expressing the Einstein equations in terms of the Bondi-Sachs metric variables (4), consider the consequence of the contracted Bianchi identity. Assuming the matter satisfies the local conservation condition $\nabla_b T^b{}_a = 0$, the Bianchi identities imply

$$0 = \nabla_b E_a^b = \frac{1}{\sqrt{-g}}\partial_b\Big(\sqrt{-g}E_a^b\Big) + \frac{1}{2}(\partial_a g^{bc})E_{bc} \ . \tag{24}$$

In analogy to the electromagnetic case, this leads to the designation of the components of Einstein's equations, consisting of

$$E_a^u = 0 \,, \quad E_{AB} - \frac{1}{2}g_{AB}g^{CD}E_{CD} = 0 \,, \tag{25}$$

as the main equations. Then if the main equations are satisfied, referring to the metric (4), $E_r^b = -e^{2\beta}E^{ub} = 0$ and the $a = r$ component of the conservation condition (24) reduces to $(\partial_r g^{AB})E_{AB} = -(2/r)g^{AB}E_{AB} = 0$ so that the component $g^{AB}E_{AB} = 0$ is trivially satisfied. Here we assume that the areal coordinate r is non-singular.

The retarded time u and angular components x^A of the conservation condition (24) now reduce to

$$\partial_r(r^2 e^{2\beta}E_u^r) = 0 \,, \quad \partial_r(r^2 e^{2\beta}E_A^r) = 0 \tag{26}$$

so that the E_u^r and E_A^r equations are satisfied everywhere if they are satisfied on a worldtube Γ or at $\mathcal{I}^+$. Furthermore, if the null foliation consists of non-singular null cones, they are automatically satisfied due to regularity conditions at the vertex $r = 0$. These equations were called supplementary

8

conditions by Bondi and Sachs. Evaluated at $\mathcal{I}^+$ they are related to the conservation of total energy and angular momentum. In particular, the equation $r^2 E^r_u|_{\mathcal{I}^+} = 0$ gives rise to the famous Bondi mass loss equation (see (57)).

The main Einstein equations separate further into the

$$\text{Hypersurface equations:} \quad E^u_a = 0 \tag{27}$$

and the

$$\text{Evolution equations:} \quad E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0. \tag{28}$$

In terms of the metric variables (4) the hypersurface equations consist of one first order radial differential equation determining β along the null rays,

$$E^u_r = 0 \;\; \Rightarrow \;\; \partial_r \beta = \frac{r}{16} h^{AC} h^{BD} (\partial_r h_{AB})(\partial_r h_{CD}) + 2\pi r T_{rr}\;, \tag{29}$$

two second order radial differential equations determining U^A,

$$E^u_A = 0 \;\; \Rightarrow \;\; \partial_r \left[r^4 e^{-2\beta} h_{AB}(\partial_r U^B) \right] = 2 r^4 \partial_r \left(\frac{1}{r^2} D_A \beta \right)$$
$$- r^2 h^{EF} D_E (\partial_r h_{AF}) + 16\pi r^2 T_{rA}\;, \tag{30}$$

and a radial equation to determine V,

$$E^u_u = 0 \;\; \Rightarrow \;\; 2 e^{-2\beta}(\partial_r V) = \mathcal{R} - 2 \left[D_A D^A \beta + (D_A \beta) D^A \beta \right]$$
$$+ \frac{e^{-2\beta}}{r^2} D_A \partial_r (r^4 U^A) - \frac{1}{2} r^4 e^{-4\beta} h_{AB}(\partial_r U^A)(\partial_r U^B)$$
$$+ 8\pi \left[h^{AB} T_{AB} - r^2 T^a{}_a \right]\;, \tag{31}$$

where D_A is the covariant derivative and $\mathcal{R}$ is the Ricci scalar with respect to the conformal 2-metric h_{AB}.

The evolution equations can be picked out by introducing a complex polarization dyad m^a satisfying $m^a \nabla_a u = 0$ which is tangent to the null hypersurfaces and points in the angular direction with components $m^a = (0, 0, m^A)$. Imposing the normalization $h^{AB} = \frac{1}{2}(m^A \bar{m}^B + m^B \bar{m}^A)$, with $m_A \bar{m}^A = 2$, $m_A = h_{AB} m^B$, and $m_A m^A = 0$ determines m^A up to the phase

freedom $m^A \to e^{i\eta}m^A$, which can be fixed by convention. The symmetric 2-tensor E_{AB} can then be expanded as

$$E_{AB} = \frac{1}{4}(E_{CD}m^C m^D)\bar{m}_A \bar{m}_B + \frac{1}{4}(E_{CD}\bar{m}^C \bar{m}^D)m_A m_B + \frac{1}{2}h_{AB}h^{CD}E_{CD},$$
(32)

where we have shown that $h^{CD}E_{CD} = 0$ is trivially satisfied. Consequently, the evolution equations reduce to the complex equation $m^A m^B E_{AB} = 0$, which takes the form (Winicour, 1983, 2012)

$$m^A m^B \Bigg\{ \quad r\partial_r[r(\partial_u h_{AB})] - \frac{1}{2}\partial_r[rV(\partial_r h_{AB})] - 2e^\beta D_A D_B e^\beta$$

$$+ \quad h_{CA}D_B[\partial_r(r^2 U^C)] - \frac{1}{2}r^4 e^{-2\beta}h_{AC}h_{BD}(\partial_r U^C)(\partial_r U^D)$$

$$+ \quad \frac{r^2}{2}(\partial_r h_{AB})(D_C U^C) + r^2 U^C D_C(\partial_r h_{AB})$$

$$- \quad r^2(\partial_r h_{AC})h_{BE}(D^C U^E - D^E U^C) - 8\pi e^{2\beta}T_{AB} \Bigg\} = 0. \text{ (33)}$$

It comprises a radial equation which determines the retarded time derivative of the two degrees of freedom in the conformal 2-metric h_{AB}.

As in the electromagnetic case, the main equations can be radially integrated in sequential order. In order to illustrate the hierarchical integration scheme we follow Bondi and Sachs by considering an asymptotic $1/r$ expansion of the solutions in an asymptotic inertial frame, with the matter sources confined to a compact region. This ansatz of a $1/r$-expansion of the metric leads to the peeling property of the Weyl tensor in the spin-coefficient approach (see (Newman and Penrose, 2009)). For a more general approach in which logarithmic terms enter the far field expansion and only a partial peeling property results, see (Winicour, 1985).

In the asymptotic inertial frame, often referred to as a Bondi frame, the metric approaches the Minkowski metric (10) at null infinity, so that

$$\lim_{r\to\infty} \beta = \lim_{r\to\infty} U^A = 0 \ , \quad \lim_{r\to\infty} \frac{V}{r} = 1 \ , \quad \lim_{r\to\infty} h_{AB} = q_{AB} \ . \qquad (34)$$

Later, in Sec. 4, we will justify these asymptotic conditions in terms of a Penrose compactification of $\mathcal{I}^+$.

For the purpose of integrating the main equations, we prescribe the following asymptotic data:

1. The conformal 2-metric h_{AB} on an initial null hypersurface N_0, $u = u_0$,

which has the asymptotic $1/r$ expansion

$$h_{AB} = q_{AB} + \frac{c_{AB}(u_0, x^E)}{r} + \frac{d_{AB}(u_0, x^E)}{r^2} + ...,\tag{35}$$

where the condition $h^{AC}h_{CB} = \delta^A{}_B$ implies

$$h^{AB} = q^{AB} - \frac{c^{AB}}{r} - \frac{d^{AB} - q^{AC}c^{BD}c_{CD}}{r^2} + ...\tag{36}$$

with $c^{AB} := q^{AD}q^{BE}c_{DE}$ and $d^{AB} := q^{AD}q^{BE}d_{DE}$. Furthermore, the derivative of the determinant condition $\det(h_{AB}) = \mathfrak{q}(x^C)$ requires

$$q^{AB}c_{AB} = 0 , \quad q^{AB}d_{AB} = \frac{1}{2}c^{AB}c_{AB} , \quad q^{AB}\partial_u c_{AB} = 0 ,$$
$$q^{AB}\partial_u d_{AB} - c^{AB}\partial_u c_{AB} = 0.\tag{37}$$

2. A function $M(u, x^A)$ at the initial time u_0,

$$M(u_0, x^A) := -\frac{1}{2}\lim_{r\to\infty}[V(u_0, r, x^C) - r] ,\tag{38}$$

   which is called the mass aspect.

3. A vector field $L^A(u, x^C)$ on the sphere at the initial time u_0,

$$L^A(u_0, x^C) := -\frac{1}{6}\lim_{r\to\infty} r^4\partial_r U^A(u_0, r, x^C),\tag{39}$$

   which is the angular momentum aspect.

4. The $1/r$ coefficient of the conformal 2-metric h_{AB} for retarded times $u \in [u_0, u_1]$, $u_1 > u_0$,

$$c_{AB}(u, x^C) := \lim_{r\to\infty} r(h_{AB} - q_{AB}) ,\tag{40}$$

   which describes the time dependence of the gravitational radiation.

In terms of a complex dyad $q^A = \lim_{r\to\infty} m^A$ on the unit sphere so that $q^{AB} = \frac{1}{2}(q^A\bar{q}^B + q^B\bar{q}^A)$, e.g. for the choice $q^A = (1, i/\sin\theta)$, the real and imaginary part of

$$\sigma = \frac{1}{2}q^A q^B c_{AB} = \frac{1}{2}\left(c_{\theta\theta} - \frac{c_{\phi\phi}}{\sin^2\theta}\right) + i\left(\frac{c_{\theta\phi}}{\sin\theta}\right)\tag{41}$$

11

correspond, respectively, to the $+$ and $\times$ polarization modes of the strain measured by a gravitational wave detector at large distance from the source. Traditionally, the radiative strain σ has also been called the shear because it measures the asymptotic shear of the outgoing null hypersurfaces in the sense of geometric optics,

$$\sigma = \lim_{r \to \infty} q^A q^B \nabla_A \nabla_B u \; . \tag{42}$$

The retarded time derivative $N_{AB} = \frac{1}{2} \partial_u c_{AB}(u, x^C)$, called the *news tensor*, is a gauge invariant field that determines the energy flux of the gravitational radiation. Relative to a choice of polarization dyad, the *Bondi news* function is

$$N = q^A q^B N_{AB} \; . \tag{43}$$

Note, in carrying out the $1/r$ expansion of the field equations the covariant derivative D_A corresponding to the metric h_{AB} is related to the covariant derivative $\eth_A$ corresponding to the unit sphere metric q_{AB} by

$$D_A V^B = \eth_A V^B + \mathcal{C}_{AE}^B V^E, \tag{44}$$

where

$$\mathcal{C}_{AE}^B = \frac{1}{2r} q^{BF} (\eth_A \, c_{FE} + \eth_E \, c_{FA} - \eth_F \, c_{AE}) + O(1/r^2). \tag{45}$$

Given the asymptotic gauge conditions (34) and the initial data (36), (38), (39), (40) on N_0, the formal integration of the main equations at large r proceeds in the following sequential order:

1. Integration of the β-hypersurface equation gives

$$\beta(u_0, r, x^A) = -\frac{1}{32} \frac{c^{AB} c_{AB}}{r^2} + O(r^{-3}) \; . \tag{46}$$

2. Insertion of the data (35) and the solution for β into the U^A hypersurface equation (30) yields

$$\partial_r \left[r^4 e^{-2\beta} h_{AB} (\partial_r U^B) \right] = \eth^E c_{AE} + \frac{S_A(u_0, x^C)}{r} + O(1/r^2) \tag{47}$$

where

$$S_A(u_0, x^C) = \eth^B (2 d_{AB} - q^{FG} c_{BG} c_{AF}). \tag{48}$$

As a result, unless $S_A = 0$, integration of (47) leads to a logarithmic $r^{-4} \ln r$ term in $\partial_r U^A$, which is ruled out by the assumption of an asymptotic $1/r$ expansion. This results in the condition

$$
\begin{aligned}
0 &= q^A S_A = q^A (q^B \bar{q}^E + q^E \bar{q}^B) \eth_E (d_{AB} - \frac{1}{2} q^{FG} c_{BG} c_{AF}) \\
&= q^A q^B \bar{q}^E \eth_E (d_{AB} - \frac{1}{2} q^F \bar{q}^G c_{BG} c_{AF}) \\
&= \bar{q}^E \eth_E (q^A q^B d_{AB}) ,
\end{aligned}
\tag{49}
$$

where the second line follows from the determinant condition (37) and the third line uses $\bar{q}^E \eth_E q^B \propto q^B$. It follows that

$$
q^A q^B d_{AB} = 0,
\tag{50}
$$

so that d_{AB} consists purely of a trace term dictated by the determinant condition (37). Hence, integrating (47) once and applying this constraint yields

$$
\partial_r U^A = \frac{\eth_B c^{AB}}{r^3} - \frac{6 L^A(u_0, x^B)}{r^4} + O(r^{-5}) \ .
\tag{51}
$$

3. Subsequent radial integration of $\partial_r U^A$ while using (36), (38), (39), (40) gives

$$
U^A(u_0, r, x^B) = -\frac{\eth_B c^{AB}}{2 r^2} + \frac{2 L^A}{r^3} + O(r^{-4}).
\tag{52}
$$

4. With the initial data (35) and initial values of β and U^A, the V-hypersurface equation (31) can be integrated to find the asymptotic solution
$$
V(u_0, r, x^A) = r - 2M(u_0, x^A) + O(r^{-1}) \ .
\tag{53}
$$

Here $M(u, x^A)$ is called the mass aspect since in the static, spherically symmetric case, where $h_{AB} = q_{AB}$, $\beta = U^A = 0$ and $M(u, xA) = m$, the metric (4) reduces to the Eddington-Finkelstein metric for a Schwarzschild mass m.

5. Insertion of the solutions for β, U^A and V into the evolution equation (33) yields to leading order that $q^A q^B \partial_u d_{AB} = 0$, consistent with the determinant condition (37).

6. With the asymptotic solution of the metric, the leading order coefficient of the E_u^r supplementary equation gives

$$2\partial_u M = -\frac{1}{2}\eth_A\eth_B N^{AB} - \frac{1}{4}N_{AB}N^{AB} \ . \tag{54}$$

Since N_{AB} is assumed known for $u_0 \leq u \leq u_1$, integration determines the mass aspect M in terms of its initial value $M(u_0, x^A)$.

7. The leading order coefficient of the E_A^r supplementary equation determines the time evolution of the angular momentum aspect L_A,

$$\begin{aligned} -3\partial_u L_A \ = \ & \eth_A M - \frac{1}{4}\eth^E\left(\eth_E\eth^F c_{AF} - \eth_A\eth^F c_{EF}\right) \\ & + \frac{1}{8}c_{EF}\eth_A N^{EF} - \frac{3}{8}N_{EF}\eth_A c^{EF} - N_{AB}\eth_E c^{BE} \\ & - c_{AB}\eth_E N^{BE} \ . \end{aligned} \tag{55}$$

The motivation for calling $L_A(u, x^A)$ the angular momentum aspect can be seen in the non-vacuum case where its controlling E_A^r supplementary equation is coupled to the angular momentum flux T_A^r of the matter field. Together with (54), (55) shows that the time evolution of L_A is entirely determined by N_{AB} for $u_0 \leq u \leq u_1$ and the initial values of L_A, M and c_{AB} at $u = u_0$.

This hierarchical integration procedure shows how the boundary conditions (34) and data (36), (38), (39), (40) uniquely determine a formal solution of the field equation in terms of the coefficients of an asymptotic $1/r$ expansion. In particular, the supplementary equations determine the time derivatives of M and L_A, whereas the hypersurface equations determine the higher order expansion coefficients. However, this formal solution cannot be cast as a well-posed evolution problem to determine the metric for $u > u_0$ because the necessary data, e.g. $c_{AB}(u, x^C)$, lies in the future of the initial hypersurface at u_0. Nevertheless, this formal solution led Bondi to the first clear understanding of mass loss due to gravitational radiation. It gives rise to the interpretation of the supplementary conditions as flux conservation laws for energy-momentum and angular momentum (Tamburino and Winicour, 1966; Goldberg, 1974).

The time-dependent *Bondi mass* $m(u)$ for an isolated system is

$$m(u) := \frac{1}{4\pi} \oint M(u, \theta, \phi)\sin\theta d\theta d\phi \ . \tag{56}$$

14

The integration of (54) over the sphere, using the definition of the news function (43), gives the famous Bondi mass loss formula

$$\frac{d}{du}m(u) = -\frac{1}{4\pi} \oint |N|^2 \sin\theta d\theta d\phi \ , \tag{57}$$

where the first term of (54) integrates out because of the divergence theorem. The positivity of the integrand in (57) shows that if a system emits gravitational waves, i.e. if there is news, then its Bondi mass must decrease. If there is no news, i.e. $N = 0$, the Bondi mass is constant.

Here (55) corrects the original equations Bondi and Sachs for the time evolution of the angular momentum aspect L_A. For the Bondi metric in which $\gamma(u, r, \theta) = c(u, \theta)/r + O(1/r^3)$, (55) becomes

$$-3\partial_u L_\theta = \partial_\theta M - \frac{3}{2}c\partial_u\partial_\theta c - \frac{7}{2}(\partial_u c)\partial_\theta c - 8c\partial_u c \cot\theta \ . \tag{58}$$

The asymptotic approach of Bondi and Sachs illustrates the key features of the metric based null cone formulation of general relativity. Nevertheless, assigning boundary data such as the news function N at large distances is non-physical as opposed to determining N by evolving an interior system (see Sec. 5). In particular, assignment of boundary data on a finite worldtube surrounding the source leads to gauge conditions in which the asymptotic Minkowski behavior (34) does not hold.

4   The Bondi-Metzner-Sachs (BMS) group

The asymptotic symmetries of the metric can be most clearly and elegantly described using a Penrose compactification of null infinity (Penrose, 1963). In that case the assumption of an asymptotic series expansion in $1/r$ becomes a smoothness condition at $\mathcal{I}^+$.

In Penrose's compactification of null infinity, $\mathcal{I}^+$ is the finite boundary of an unphysical space time containing the limiting end points of null geodesics in the physical space time. If g_{ab} is the metric of the physical space time and $\hat{g}_{ab}$ denotes the unphysical spacetime the two metrics are conformally related via $\hat{g}_{ab} = \Omega^2 g_{ab}$, where $\hat{g}_{ab}$ is smooth (at least C^3) and $\Omega = 0$ at $\mathcal{I}^+$. Asymptotic flatness requires that $\mathcal{I}^+$ has the topology $\mathbb{R} \times \mathbb{S}^2$ and that $\hat{\nabla}_a\Omega$ vanishes nowhere at $\mathcal{I}^+$. The conformal space and physical space Ricci tensors are related by

$$R_{ab} = \Omega^2 \hat{R}_{ab} + 2\Omega\hat{\nabla}_a\hat{\nabla}_b\Omega + \hat{g}_{ab}\left[\Omega\hat{\nabla}^c\hat{\nabla}_c\Omega - 3(\hat{\nabla}^c\Omega)\hat{\nabla}_c\Omega\right] \tag{59}$$

15

where $\hat{\nabla}_a$ is the covariant derivative with respect to $\hat{g}_{ab}$. Separating out the trace of (59), evaluation of the physical space vacuum Einstein equations $R_{ab} = 0$ at $\mathcal{I}^+$ implies

$$0 = [(\hat{\nabla}^c\Omega)\hat{\nabla}_c\Omega]_{\mathcal{I}^+} \tag{60a}$$

$$0 = \left[\hat{\nabla}_a\hat{\nabla}_b\Omega - \frac{1}{4}\hat{g}_{ab}\hat{\nabla}^c\hat{\nabla}_c\Omega\right]_{\mathcal{I}^+}. \tag{60b}$$

The first condition shows that $\mathcal{I}^+$ is a null hypersurface and the second assures the existence of a conformal transformation $\hat{\Omega}^{-2}\hat{g}_{ab} = \tilde{\Omega}^{-2}\tilde{g}_{ab}$ such that $\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\Omega}|_{\mathcal{I}^+} = 0$. Thus there is a set of preferred conformal factors $\tilde{\Omega}$ for which null infinity is a divergence-free ($\tilde{\nabla}^c\tilde{\nabla}_c\tilde{\Omega}|_{\mathcal{I}^+} = 0$) and shear-free ($\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\Omega}|_{\mathcal{I}^+} = 0$) null hypersurface.

A coordinate representation $\hat{x}^a = (u, \ell, x^A)$ of the compactified space can be associated with the Bondi–Sachs physical space coordinates in Sec. 2 by the transformation $\hat{x}^a = (u, \ell, x^A) = (u, 1/r, x^A)$. Here the inverse areal coordinate $\ell = 1/r$ also serves as a convenient choice of conformal factor $\Omega = \ell$. This gives rise to the conformal metric

$$\hat{g}_{ab}d\hat{x}^a d\hat{x}^b = \ell^3 V e^{2\beta}du^2 + 2e^{2\beta}dud\ell + h_{AB}\left(dx^A - U^A du\right)\left(dx^B - U^B du\right), \tag{61}$$

where $\det(h_{AB}) = \mathfrak{q}$. The leading coefficients are subject to the conformal space Einstein equations according to

$$h_{AB} = H_{AB}(u, x^C) + \ell c_{AB}(u, x^c) + O(\ell^2) \tag{62}$$

$$\beta = H(u, x^C) + O(\ell^2) \tag{63}$$

$$U^A = H^A(u, x^C) + 2\ell e^{2H}H^{AB}D_B H + O(\ell^2) \tag{64}$$

$$\ell^2 V = D_A H^A + \ell\left[\frac{1}{2}\mathcal{R} + D^A D_A e^{2H}\right] + O(\ell^2), \tag{65}$$

where here $\mathcal{R}$ is the Ricci scalar and D_A is the covariant derivative associated with H_{AB}.

In (61), H, H^A and H_{AB} have a general form which does not correspond to an asymptotic inertial frame. In order to introduce inertial coordinates consider the null vector $\hat{n}^a = \hat{g}^{ab}\hat{\nabla}_b\ell$ which is tangent to the null geodesics generating $\mathcal{I}^+$. In a general coordinate system, it has components at $\mathcal{I}^+$

$$\hat{n}^a|_{\mathcal{I}^+} = \left(e^{-2H}, 0, -e^{-2H}H^A\right) \tag{66}$$

arising from the contravariant metric components

$$\hat{g}^{ab}\Big|_{\mathcal{I}^+} = \begin{pmatrix} 0 & e^{-2H} & 0 \\ e^{-2H} & 0 & -H^A e^{-2H} \\ 0 & -H^A e^{-2H} & H^{AB} \end{pmatrix}. \tag{67}$$

Introduction of the inertial version of angular coordinates by requiring

$$\hat{n}^a \partial_a x^A |_{\mathcal{I}^+} = 0$$

results in $H^A = 0$. Next, introduction of the inertial version of a retarded time coordinate by requiring that u be an affine parameter along the generators of $\mathcal{I}^+$, with

$$\hat{n}^a \partial_a u \Big|_{\mathcal{I}^+} = 1,$$

results in $H = 0$. It also follows that ℓ is a preferred conformal factor so that the divergence free and shear free condition $\tilde{\nabla}_a \tilde{\nabla}_b \ell_{\mathcal{I}^+} = 0$ implies that $\partial_u H_{AB} = 0$. This allows a time independent conformal transformation $\ell \to \omega(x^C)\ell$ such that $H_{AB} \to q_{AB}$, so that the cross-sections of $\mathcal{I}^+$ have unit sphere geometry. In this process, the condition $H = 0$ can be retained by an affine change in u.

Thus it is possible to establish an inertial coordinate system $\hat{x}^a$ at $\mathcal{I}^+$, which justifies the Bondi-Sachs boundary conditions (34). In these inertial coordinates, the conformal metric has the asymptotic behavior

$$
\begin{align}
h_{AB} &= q_{AB}(u, x^C) + \ell c_{AB}(u, x^C) + O(\ell^2) \tag{68a} \\
\beta &= O(\ell^2) \tag{68b} \\
U^A &= 2L^A \ell^3 + O(\ell^4) \tag{68c} \\
\ell^3 V &= \ell^2 - 2M\ell^3 + O(\ell^4), \tag{68d}
\end{align}
$$

showing that the Bondi-Sachs variables c_{AB}, mass aspect M and angular momentum aspect L^A are the the leading order coefficients of a Taylor series at null infinity with respect to the preferred conformal factor ℓ.

The *BMS group* is the asymptotic isometry group of the Bondi-Sachs metric (4). In terms of the physical space metric, the infinitesimal generators ξ^a of the BMS group satisfy the asymptotic version of Killing's equation

$$\Omega^2 \mathcal{L}_\xi g_{ab}|_{\mathcal{I}^+} = -2\Omega^2 \nabla^{(a}\xi^{b)}|_{\mathcal{I}^+} = 0 \ , \tag{69}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ^a. In terms of the conformal space metric (68) with conformal factor $\Omega = \ell$, this implies

$$\left[\hat{\nabla}^{(a}\xi^{b)} - \ell^{-1}\hat{g}^{ab}\xi^c \partial_c \ell\right]_{\ell=0} = 0. \tag{70}$$

This immediately requires $\xi^c \partial_c \ell = 0$, i.e. the generator is tangent to $\mathcal{I}^+$ and $\ell^{-1}\xi^c \partial_c \ell|_{\ell=0} = \partial_\ell \xi^\ell|_{\ell=0}$. Then (70) takes the explicit form

$$\left[\hat{g}^{ac}\partial_c\xi^b + \hat{g}^{bc}\partial_c\xi^a - \xi^c\partial_c\hat{g}^{ab} - \hat{g}^{ab}\partial_\ell\xi^\ell\right]_{\ell=0} = 0 \ , \tag{71}$$

where (67) reduces in the inertial frame to

$$
\hat{g}^{ab}\Big|_{\ell=0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q^{AB} \end{pmatrix} \tag{72}
$$

Since only $\hat{g}^{ab}|_{\ell=0}$ enters (71), it is simple to analyze. This leads to the general solution

$$
\xi^a \partial_a|_{\ell=0} = \left[\alpha(x^C) + \frac{u}{2}\eth_B f^B(x^C) \right] \partial_u + f^A(x^C)\partial_A \tag{73}
$$

where $f^A(x^C)$ is a conformal killing vector of the unit sphere metric,

$$
\eth^{(A} f^{B)} - \frac{1}{2}q^{AB}\eth_C f^C = 0 \ . \tag{74}
$$

These constitute the generators of the BMS group.

The BMS symmetries with $f^A = 0$ are called *supertranslations*; and those with $\alpha = 0$ describe conformal transformations of the unit sphere, which are isomorphic to the orthochronous Lorentz transformations (Sachs, 1962a). The supertranslations form an infinite dimensional invariant subgroup of the BMS group. Of special importance, the supertranslations consisting of $l = 0$ and $l = 1$ spherical harmonics, e.g. $\alpha = a + a_x \sin\theta\cos\phi + a_y \sin\theta\sin\phi + a_z \cos\theta$, form an invariant 4-dimensional translation group consisting of time translations (a) and spatial translations (a_x, a_y, a_z). This allows an unambiguous definition of energy-momentum. However, because the Lorentz group is not an invariant subgroup of the BMS group there arises a supertranslation ambiguity in the definition of angular momentum. Only in special cases, such as stationary spacetimes, can a preferred Poincare group be singled out from the BMS group.

Consider the finite supertranslation, $\tilde{u} = u + \alpha(x^A) + O(\ell)$, with $\tilde{x}^A = x^A$, where the $O(\ell)$ term is required to maintain u as a null coordinate. Under this supertranslation, the radiation strain or asymptotic shear (42), i.e. $\sigma(u, x^C) = q^A q^B \nabla_A \nabla_B u|_{\ell=0}$, transforms according to

$$
\tilde{\sigma}(u, x^C) = q^A q^B \nabla_A \nabla_B \tilde{u}|_{\ell=0} = \sigma(u, x^C) + q^A q^B \eth_A \eth_B \alpha(x^C). \tag{75}
$$

This reveals the gauge freedom in the radiation stain under supertranslations. Note, because α is a real function, in the terminology of the Newman-Penrose spin-weight formalism (Newman and Penrose, 1962, 1966; Goldberg et al., 1967), this gauge freedom only affects the electric (or E-mode (Mädler and Winicour, 2016)) component of the shear.

5   The worldtube-null-cone formulation

In contrast to the Bondi-Sachs treatment in terms of a $1/r$ expansion at infinity, in the worldtube-null-cone formulation the boundary conditions for the hypersurface and evolution equations are provided on a timelike worldtube Γ with finite areal radius R and topology $\mathbb{R} \times \mathbb{S}^2$. This is similar to the electromagnetic analog discussed in Sec. (2.1). The worldtube data may be supplied by a solution of Einstein's equations interior to Γ, so that it satisfies the supplementary conditions on Γ. In the most important application, the worldtube data is obtained by matching to a numerical solution of Einstein's equations carried out by a Cauchy evolution of the interior. It is also possible to solve the supplementary conditions as a well-posed system on Γ if the interior solution is used to supply the necessary coefficients (Winicour, 2011).

Coordinates (u, x^A) on Γ have the same $2+1$ gauge freedom in the choice of lapse and shift as in a $3 + 1$ Cauchy problem. This produces a foliation of Γ into spherical cross-sections S_u. In one choice, corresponding to unit lapse and zero shift, u is the proper time along the null geodesics normal to some initial cross-section S_0 of Γ, with angular coordinates x^A constant along the geodesics. In the case of an interior numerical solution, the lapse and shift are coupled to the lapse and shift of the Cauchy evolution.

These coordinates are extended off the worldtube Γ by letting u label the family of outgoing null hypersurfaces N_u emanating from S_u and letting x^A label the null rays in N_u. A Bondi-Sachs coordinate system (u, r, x^A) is then completed by letting r be areal coordinate along the null rays, with $r = R$ on Γ, as depicted in Fig. 1. The resulting metric has the Bondi–Sachs form (4), which induces the $2 + 1$ metric intrinsic to Γ,

$$g_{ab}dx^a dx^b\big|_\Gamma = -\frac{V}{R}e^{2\beta}du^2 + R^2 h_{AB}(dx^A - U^A)(dx^B - U^B) \ , \qquad (76)$$

where $Ve^{2\beta}/R$ is the square of the lapse function and $(-U^A)$ is the shift.

The Einstein equations now reduce to the main hypersurface and evolution equations presented in Sec. 2, assuming that the worldtube data satisfy the supplementary conditions. As in the electromagnetic case, surface integrals of the supplementary equations (26) can be interpreted as conservation conditions on Γ, as described in (Tamburino and Winicour, 1966; Goldberg, 1974). The main equations can be solved with the prescription of the following mixed initial-boundary data:

- The areal radius R of Γ and $\partial_r U^A|_\Gamma$, as determined by matching to an interior solution.

- The conformal 2-metric $h_{AB}|_{N_0}$ on an entire initial null cone N_0 for $r > R$.

- The values of $\beta|_{S_0}$, $U^A|_{S_0}$, $\partial_r U^A|_{S_0}$ and $V|_{S_0}$ on the initial cross section S_0 of Γ.

- The retarded time derivative of the conformal 2-metric $\partial_u h_{AB}|_\Gamma$ on Γ for $u > u_0$.

Given this initial-boundary data, the hypersurface equations can be solved in the same hierarchical order as illustrated for the electromagnetic case in Sec. 3 and the evolution equation can be solved using a finite difference time-integrator. It has been verified in numerical testbeds, using either finite difference approximations (Bishop et al., 1996b, 1997) or spectral methods (Handmer and Szilágyi, 2015) for the spatial approximations, that this evolution algorithm is stable and converges to the analytic solution. However, proof of the well-posedness of the analytic initial-boundary problem for the above system remains an open issue.

A limiting case of the worldtube-null-cone problem arises when Γ collapses to a single world line traced out by the vertices of outgoing null cones. Here the metric variables are restricted by regularity conditions along the vertex worldline (Isaacson et al., 1983). For a geodesic worldline, the null coordinates can be based on a local Fermi normal coordinate system (Manasse and Misner, 1963), where u measures proper time along the worldline and labels the outgoing null cones. It has been shown for axially symmetric spacetimes (Mädler and Müller, 2013) that the regularity conditions on the metric in Fermi coordinates place very rigid constraints on the coefficients of the null data h_{AB} in a Taylor expansion in r about the vertices of the outgoing null cones. As a result, implementation of an evolution algorithm of the worldline-null-cone problem for the Bondi-Sachs equations is complicated and has been restricted to simple problems. Existence theorems have been established for a different formulation of the worldline-null-cone problem in terms of wave maps (Choquet-Bruhat et al., 2011) but this approach does not have a clear path toward numerical evolution.

6   Applications

By July 2016, the seminal works of Bondi, Sachs and their collaborators have together spawned more than 1500 citations on the Harvard ADS database [1]

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[1] http://adsabs.harvard.edu/abstract_service.html

(with more than 600 in the last 10 years), showing that the Bondi-Sachs formalism has found widespread applications. The main field of application of the Bondi-Sachs formalism is numerical relativity and an extensive overview is given in the *Living Review* article (Winicour, 2012). The BMS group has played an important role in defining the energy-momentum and angular momemtum of asymptotically flat spacetimes. For a historical account see (Goldberg, 2006).

Applications of the Bondi–Sachs formalism can be roughly grouped into the following sections, where a selective choice of references is given.

Numerical Relativity — Null cone evolution schemes

- axisymmetric simulations (Isaacson et al., 1983; Gómez et al., 1994; D'inverno and Vickers, 1996)

- Einstein-Scalar field evolutions (Gómez and Winicour, 1993; Barreto, 2014)

- spectral methods (de Oliveira and Rodrigues, 2011; Handmer and Szilágyi, 2015; Handmer et al., 2015, 2016)

- black hole physics (Bishop et al., 1996a; Papadopoulos, 2002; Husa et al., 2002; Poisson and Vlasov, 2010)

- relativistic stars (Linke et al., 2001; Siebel et al., 2002; Barreto et al., 2009)

Numerical Relativity — Waveform extraction

- Cauchy-characteristic extraction and conformal compactification (Bishop et al., 1996b, 1997; Babiuc et al., 2009)

- gauge invariant wave extraction with spectral methods (Handmer et al., 2015, 2016).

- extraction in physical space (Lehner and Moreschi, 2007; Nerozzi et al., 2006)

Cosmology

- reconstruction of the past light cone (Ellis et al., 1985)

- gravitational waves in cosmology (Bishop, 2016)

BMS group and gravitational memory

- BMS representation of emergy-momentum and angular momentum (Tamburino and Winicour, 1966; Geroch and Winicour, 1981; Ashtekar and Streubel, 1981; Dray and Streubel, 1984; Wald and Zoupas, 2000; Goldberg, 2006)

- BMS algebra in 3/4 dimensions and BMS/conformal field theory (CFT) correspondence (Barnich and Compère, 2007; Barnich and Troessaert, 2010a,b)

- soft theorems and the radiation memory effect (Strominger and Zhiboedov, 2016; Winicour, 2014; Mädler and Winicour, 2016)

- black hole information paradox (Hawking et al., 2016; Donnay et al., 2016)

Exact and Approximate Solutions

- Newtonian approximation (Winicour, 1983, 1984)

- linearized solutions and master equation approaches (Bishop et al., 1996b; Bishop, 2005; Mädler, 2013; Cedeño M. and de Araujo, 2016)

- boost-rotation symmetric solutions (Bičák et al., 1988; Bičák and Pravdová, 1998)

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