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Off-shell superconformal higher spin multiplets in four dimensions

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Abstract

We formulate off-shell $\mathcal{N} = 1$ superconformal higher spin multiplets in four spacetime dimensions and briefly discuss their coupling to conformal supergravity. As an example, we explicitly work out the coupling of the superconformal gravitino multiplet to conformal supergravity. The corresponding action is super-Weyl invariant for arbitrary supergravity backgrounds. However, it is gauge invariant only if the supersymmetric Bach tensor vanishes. This is similar to linearised conformal supergravity in curved background.

1 Introduction

The role of conformal field theories as cornerstones for the exploration of more general quantum field theories, which are connected to them via renormalization group flows, has been appreciated since a long time ago. Higher spin gauge theories [1, 2, 3, 4] have an even longer history and have attracted considerable interest recently. It is quite natural to combine the two symmetry principles and to study conformal higher spin theories [5]. A further symmetry which is compatible with conformal and higher spin symmetry is supersymmetry. This leads to superconformal higher spin theories, first advocated in [6], which are the main focus of this note. More specifically, we introduce off-shell $\mathcal{N} = 1$ superconformal higher spin multiplets in four dimensions and analyse in some detail the problem of lifting such supermultiplets to curved backgrounds. Our main technical tool, as far as the supersymmetry and supergravity aspects are concerned, is superspace and we refer to [7] for a thorough introduction to this formalism.

We first study superconformal higher spin theories in flat superspace. In Sect. 2 we review superconformal transformations and the important notion of superconformal primaries. In Sect. 3 we construct off-shell superconformal higher spin multiplets: starting from prepotentials and their transformation laws under higher spin gauge transformations and under superconformal transformations, we construct field strengths and invariant actions. The two cases of half-integer and integer superspin as well as the superconformal gravitino multiplet have to be treated separately. The component fields of these multiplets are totally symmetric traceless tensor and tensor-spinor fields. More general fields will be briefly discussed in the last part of Sect. 3. In Sect. 4 we couple the superconformal higher spin multiplets to conformal supergravity where the notion of superconformal transformations is replaced by that of super-Weyl transformations. While super-Weyl invariance is easy to achieve, gauge invariance requires non-minimal couplings. We explicitly discuss the gravitino multiplet, but defer the general case to the future. Sect. 5 contains concluding comments, including the explicit expressions for conserved higher spin current multiplets that correspond to the superconformal higher spin prepotentials. The main body of the paper is accompanied by two technical appendices. Appendix A contains those results concerning the Grimm-Wess-Zumino superspace geometry [8], which are important for understanding the supergravity part of this paper. Appendix B contains the essential information about the super-Weyl transformations [9].

There are different ways to describe $\mathcal{N} = 1$ conformal supergravity in superspace.[1]

[1]See [5] for a nice review of $\mathcal{N} = 1$ conformal supergravity and the complete list of references.

The simplest option is to make use of the superspace geometry of [8], which underlies the Wess-Zumino approach [10] to the old minimal formulation for $\mathcal{N}=1$ supergravity developed independently in [11]. Another option is to work with the $\mathsf{U}(1)$ superspace proposed by Howe [12]. Finally, one can make use of the so-called conformal superspace [13]. The three superspace approaches to $\mathcal{N}=1$ conformal supergravity are equivalent, although each of them has certain advantages and disadvantages (see [13] for a detailed discussion of the relationship between these formulations). In this paper we make use of the oldest and simplest superspace setting [8].

2 Superconformal transformations

In this section we briefly recall the structure of $\mathcal{N}=1$ superconformal transformations in Minkowski superspace $\mathbb{M}^{4|4}$, see [7] for more details. We denote by $z^A=(x^a,\theta^\alpha,\bar\theta_{\dot\alpha})$ the Cartesian coordinates for $\mathbb{M}^{4|4}$, and use the notation $D_A=(\partial_a,D_\alpha,\bar D^{\dot\alpha})$ for the superspace covariant derivatives.

Let $\xi=\xi^B D_B=\xi^b\partial_b+\xi^\beta D_\beta+\bar\xi_{\dot\beta}\bar D^{\dot\beta}$ be a real supervector field on $\mathbb{M}^{4|4}$. It is called conformal Killing if it obeys the equation

$$\Big[\xi+\frac{1}{2}K^{bc}[\xi]M_{bc}\,,D_A\Big]+\delta_{\sigma[\xi]}D_A=0~, \tag{2.1}$$

for some local Lorentz ($K^{bc}[\xi]$) and super-Weyl ($\sigma[\xi]$) parameters. The super-Weyl transformation of the covariant derivatives is defined in (B.1). Choosing $A=\alpha$ and $A=\dot\alpha$ in (2.1) implies that the spinor components of ξ^A as well as the parameters $K^{bc}[\xi]$ and $\sigma[\xi]$ are expressed in terms of the vector components of ξ^A:

$$\xi^\alpha=-\frac{\rm i}{8}\bar D_{\dot\alpha}\xi^{\dot\alpha\alpha}~,\qquad \bar D_{\dot\gamma}\xi^\alpha=0~, \tag{2.2a}$$
$$K_{\alpha\beta}[\xi]=D_{(\alpha}\xi_{\beta)}~,\qquad \bar D_{\dot\gamma}K_{\alpha\beta}[\xi]=0~, \tag{2.2b}$$
$$\sigma[\xi]=\frac{1}{3}(D_\alpha\xi^\alpha+2\bar D^{\dot\alpha}\bar\xi_{\dot\alpha})~,\qquad \bar D_{\dot\gamma}\sigma[\xi]=0~. \tag{2.2c}$$

The vector components of ξ^A obey the equations

$$D_{(\alpha}\xi_{\beta)\dot\beta}=0\quad\Longleftrightarrow\quad \bar D^{(\dot\alpha}\xi^{\dot\beta)\beta}=0~, \tag{2.3}$$

which imply

$$D^2\xi_{\beta\dot\beta}=0\quad\Longleftrightarrow\quad \bar D^2\xi^{\dot\beta\beta}=0~, \tag{2.4}$$

as well as the ordinary conformal Killing equation

$$\partial_a \xi_b + \partial_b \xi_a = \frac{1}{2}\eta_{ab}\partial_c \xi^c \ . \tag{2.5}$$

A useful corollary of (2.1) with $A = \alpha$ is

$$D_\gamma K^{\alpha\beta}[\xi] = \delta_\gamma^{(\alpha} D^{\beta)}\sigma[\xi] \quad \Longrightarrow \quad D^2\sigma[\xi] = 0 \ . \tag{2.6}$$

Another consequence of (2.1) is

$$\partial_a \sigma[\xi] = \partial_a \bar{\sigma}[\xi] \quad \Longrightarrow \quad \partial_a D_\beta \sigma[\xi] = 0 \ . \tag{2.7}$$

The most general conformal Killing supervector field proves to be

$$\begin{aligned} \xi_+^{\dot\alpha\alpha} &= a^{\dot\alpha\alpha} + \frac{1}{2}(\sigma+\bar\sigma)\, x_+^{\dot\alpha\alpha} + \bar K^{\dot\alpha}{}_{\dot\beta}\, x_+^{\dot\beta\alpha} + x_+^{\dot\alpha\beta} K_\beta{}^\alpha + x_+^{\dot\alpha\beta} b_{\beta\dot\beta} x_+^{\dot\beta\alpha} \\ &\quad + 4\mathrm{i}\, \bar\epsilon^{\dot\alpha}\theta^\alpha - 4 x_+^{\dot\alpha\beta}\eta_\beta\theta^\alpha \ , \end{aligned} \tag{2.8a}$$

$$\xi^\alpha = \epsilon^\alpha + \big(\bar\sigma - \frac{1}{2}\sigma\big)\theta^\alpha + \theta^\beta K_\beta{}^\alpha + + \theta^\beta b_{\beta\dot\beta} x_+^{\dot\beta\alpha} - \mathrm{i}\,\bar\eta_{\dot\beta} x_+^{\dot\beta\alpha} + 2\theta^2\eta^\alpha \ , \tag{2.8b}$$

where we have introduced the complex four-vector

$$\xi_+^a = \xi^a + \frac{\mathrm{i}}{8}\xi\sigma^a\bar\theta \ , \qquad \bar\xi^a = \xi^a \ , \tag{2.9}$$

along with the complex bosonic coordinates $x_+^a = x^a + \mathrm{i}\theta\sigma^a\bar\theta$ of the chiral subspace of $\mathbb{M}^{4|4}$. The constant bosonic parameters in (2.8) correspond to the spacetime translation ($a^{\dot\alpha\alpha}$), Lorentz transformation ($K_\beta{}^\alpha$, $\bar K^{\dot\alpha}{}_{\dot\beta}$), special conformal transformation ($b_{\alpha\dot\beta}$), and combined scale and R-symmetry transformations ($\sigma = \tau - \frac{2}{3}\mathrm{i}\varphi$). The constant fermionic parameters in (2.8) correspond to the Q-supersymmetry (ϵ^α) and S-supersymmetry (η_α) transformations. The constant parameters $K_{\alpha\beta}$ and σ are obtained from $K_{\alpha\beta}[\xi]$ and $\sigma[\xi]$, respectively, by setting $z^A = 0$.

A tensor superfield $\mathcal{T}$ (with its indices suppressed) is said to be superconformal primary of weight (p, q) if its superconformal transformation law is

$$\delta_\xi \mathcal{T} = \Big(\xi + \frac{1}{2}K^{bc}[\xi]M_{bc}\Big)\mathcal{T} + \Big(p\sigma[\xi] + q\bar\sigma[\xi]\Big)\mathcal{T} \ , \tag{2.10}$$

for some parameters p and q. The dimension of $\mathcal{T}$ is $(p+q)$, while $(p-q)$ is proportional to its R-symmetry charge. If $\mathcal{T}$ is superconformal primary and chiral, $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{T} = 0$, then $\mathcal{T}$ cannot possess dotted indices, i.e. $\bar M_{\dot\alpha\dot\beta}\mathcal{T} = 0$, and it must hold that $q = 0$. In the chiral case, it suffices to say that $\mathcal{T}$ is superconformal primary of dimension p.

Given a real scalar $\mathcal{L}$, which is superconformal primary of weight (1,1),

$$\delta_\xi \mathcal{L} = \xi \mathcal{L} + \big(\sigma[\xi] + \bar{\sigma}[\xi]\big)\mathcal{L} = \partial_a(\xi^a \mathcal{L}) - D_\alpha(\xi^\alpha \mathcal{L}) - \bar{D}^{\dot{\alpha}}(\bar{\xi}_{\dot{\alpha}} \mathcal{L}) \ , \tag{2.11}$$

the functional

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \mathcal{L} \tag{2.12}$$

is invariant under superconformal transformations. Given a chiral scalar $\mathcal{L}_{\rm c}$, which is superconformal primary of dimension +3,

$$\bar{D}_{\dot{\alpha}} \mathcal{L}_{\rm c} = 0 \ , \qquad \delta_\xi \mathcal{L}_{\rm c} = \xi \mathcal{L}_{\rm c} + 3\sigma[\xi]\mathcal{L}_{\rm c} = \partial_a(\xi^a \mathcal{L}_{\rm c}) - D_\alpha(\xi^\alpha \mathcal{L}_{\rm c}) \ , \tag{2.13}$$

the functional

$$S_{\rm c} = \int \mathrm{d}^4x \mathrm{d}^2\theta\, \mathcal{L}_{\rm c} \tag{2.14}$$

is invariant under superconformal transformations.

3 Off-shell superconformal multiplets in flat space

In this section we introduce off-shell superconformal higher spin multiplets. We first consider the half-integer and integer superspin cases, and then give some generalisations of the constructions proposed. Strictly speaking, the notion of superspin is defined only for super-Poincaré multiplets. The rationale for our use of this name in the superconformal framework is that our superconformal multiplets will be described solely in terms of the gauge prepotentials corresponding to the off-shell massless higher spin multiplets constructed in [14, 15]. Each of these massless multiplets also involves certain compensator superfields, in addition to the gauge prepotential.

3.1 Half-integer superspin

Let s be a positive integer. In the superspin-$(s+\frac{1}{2})$ case, the conformal prepotential $H_{\alpha(s)\dot{\alpha}(s)} \equiv H_{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_s}$ is a real superfield, which is symmetric in its undotted indices and, independently, in its dotted indices. The gauge transformation law of $H_{\alpha(s)\dot{\alpha}(s)}$ is

$$\delta H_{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_s} = \bar{D}_{(\dot{\alpha}_1}\Lambda_{\alpha_1\dots\alpha_s\dot{\alpha}_2\dots\dot{\alpha}_s)} - D_{(\alpha_1}\bar{\Lambda}_{\alpha_2\dots\alpha_s)\dot{\alpha}_1\dots\dot{\alpha}_s} \ , \tag{3.1}$$

with the gauge parameter $\Lambda_{\alpha(s)\dot{\alpha}(s-1)}$ being unconstrained.

In the $s = 1$ case, the transformation law (3.1) corresponds to linearised conformal supergravity [16]. The same transformation of $H_{\alpha\dot{\alpha}}$ occurs in all off-shell models for linearised $\mathcal{N} = 1$ supergravity, see [7] for a review. Such actions involve not only the gravitational superfield [16, 17, 18] $H_{\alpha\dot{\alpha}}$, but also certain compensators. For $s > 1$, the gauge transformation law (3.1) was introduced in [14] in the framework of the (two dually equivalent) off-shell formulations for the massless superspin-$(s+\frac{1}{2})$ multiplet. The massless actions of [14] involve not only the gauge prepotential $H_{\alpha(s)\dot{\alpha}(s)}$ but also certain compensators (see [7] for a pedagogical review).

The superconformal transformation law of $H_{\alpha(s)\dot{\alpha}(s)}$ is

$$\delta_\xi H_{\alpha(s)\dot{\alpha}(s)} = \Big(\xi + \frac{1}{2}K^{bc}[\xi]M_{bc}\Big)H_{\alpha(s)\dot{\alpha}(s)} - \frac{s}{2}\big(\sigma[\xi] + \bar{\sigma}[\xi]\big)H_{\alpha(s)\dot{\alpha}(s)} \ . \tag{3.2}$$

This transformation law is uniquely determined if one requires both the gauge superfield $H_{\alpha(s)\dot{\alpha}(s)}$ and the gauge parameter $\Lambda_{\alpha(s)\dot{\alpha}(s-1)}$ in (3.1) to be superconformal primary (see also [19]). It follows from (3.1) that the chiral symmetric spinor

$$\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} = -\frac{1}{4}\bar{D}^2\partial_{(\alpha_1}{}^{\dot{\beta}_1}\dots\partial_{\alpha_s}{}^{\dot{\beta}_s}D_{\alpha_{s+1}}H_{\alpha_{s+2}\dots\alpha_{2s+1})\dot{\beta}_1\dots\dot{\beta}_s} \tag{3.3}$$

is gauge invariant [14].[2] Our crucial observation is that $\mathcal{W}_{\alpha(2s+1)}$ is superconformal primary of dimension 3/2. We conclude that the gauge-invariant action

$$S_{s+\frac{1}{2}} = \int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{W}^{\alpha_1\dots\alpha_{2s+1}}\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} + \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{\mathcal{W}}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}}\bar{\mathcal{W}}^{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} \tag{3.4}$$

is superconformal. In the $s = 1$ case, it coincides with the action for linearised conformal supergravity [16]. One may check that

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathcal{W}^{\alpha_1\dots\alpha_{2s+1}}\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} = \int \mathrm{d}^4x\mathrm{d}^2\bar{\theta}\, \bar{\mathcal{W}}_{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}}\bar{\mathcal{W}}^{\dot{\alpha}_1\dots\dot{\alpha}_{2s+1}} \ . \tag{3.5}$$

We briefly comment on the component structure of the superconformal theory (3.4). The gauge parameter $\bar{D}_{(\dot{\alpha}_1}\Lambda_{\alpha_1\dots\alpha_s\dot{\alpha}_2\dots\dot{\alpha}_s)}$ in (3.1) may be represented as

$$\begin{aligned}\bar{D}_{(\dot{\alpha}_1}\Lambda_{\alpha(s)\dot{\alpha}_2\dots\dot{\alpha}_s)}(\theta,\bar{\theta}) = \mathrm{e}^{\mathrm{i}\mathcal{H}_0}\Big\{&g_{\alpha(s)\dot{\alpha}_1\dots\dot{\alpha}_s} + \mathrm{i}\bar{\theta}_{(\dot{\alpha}_1}\rho_{\alpha(s)\dot{\alpha}_2\dots\dot{\alpha}_s)} + \mathrm{i}\theta^\beta\chi_{\beta,\alpha(s)\dot{\alpha}_1\dots\dot{\alpha}_s} \\ &+\theta^2 v_{\alpha(s)\dot{\alpha}_1\dots\dot{\alpha}_s} + \theta^\beta\bar{\theta}_{(\dot{\alpha}_1}f_{\beta,\alpha(s)\dot{\alpha}_2\dots\dot{\alpha}_s)} + \theta^2\bar{\theta}_{(\dot{\alpha}_1}\omega_{\alpha(s)\dot{\alpha}_2\dots\dot{\alpha}_s)}\Big\} \ ,\end{aligned} \tag{3.6}$$

[2] The chiral superfield (3.3) is the only gauge-invariant field strength which remains non-vanishing on-shell in the supersymmetric higher spin theories introduced in [14]. In a model independent framework of superfield representations, field strengths of the form (3.3) appeared in [20].

where $\mathcal{H}_0 = \theta\sigma^a\bar\theta\partial_a$ and all component gauge parameters are complex. The parameters $\chi_{\beta,\alpha(s)\dot\alpha(s)}$ and $f_{\beta,\alpha(s)\dot\alpha(s-1)}$ transform in the representation $\mathbf{2}\otimes(\mathbf{2s+1})$ of $\mathsf{SL}(2,\mathbb{C})$ with respect to their undotted indices. It follows from (3.1) and (3.6) that a Wess-Zumino gauge may be chosen of the form

$$\begin{aligned} H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}h_{(\beta\alpha_1\dots\alpha_s)(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \bar\theta^2\theta^\beta\psi_{(\beta\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} \\ -\theta^2\bar\theta^{\dot\beta}\bar\psi_{\alpha_1\dots\alpha_s(\dot\beta\dot\alpha_1\dots\dot\alpha_s)} + \theta^2\bar\theta^2 h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s}\ , \end{aligned} \tag{3.7}$$

where the bosonic fields $h_{\alpha(s+1)\dot\alpha(s+1)}$ and $h_{\alpha(s)\dot\alpha(s)}$ are real. The residual gauge freedom is generated by

$$\begin{aligned} \bar D_{(\dot\alpha_1}\Lambda_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)}(\theta,\bar\theta) = \mathrm{e}^{\mathrm{i}\mathcal{H}_0}\Big\{ -\frac{\mathrm{i}}{2}\,\zeta_{\alpha(s)\dot\alpha_1\dots\dot\alpha_s} + \mathrm{i}\bar\theta_{(\dot\alpha_1}\rho_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)} - \mathrm{i}\theta_{(\alpha_1}\bar\rho_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} \\ +\frac{s}{s+1}\theta^\beta\bar\theta_{(\dot\alpha_1}\partial_{(\beta}{}^{\dot\gamma}\zeta_{\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}\dot\gamma} \\ -\frac{1}{2}\frac{s^2}{(s+1)^2}\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\partial^{\gamma\dot\gamma}\zeta_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)\dot\gamma} - 2\mathrm{i}\,\theta_{(\alpha_1}\bar\theta_{(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\alpha_s)} \\ -\frac{s}{s+1}\theta^2\bar\theta_{(\dot\alpha_1}\partial_{(\alpha_1}{}^{\dot\gamma}\bar\rho_{\alpha_2\dots\alpha_s)\gamma\dot\alpha_2\dots\dot\alpha_s)}\Big\}\ , \end{aligned} \tag{3.8}$$

where the bosonic parameters $\zeta_{\alpha(s)\dot\alpha(s)}$ and $\zeta_{\alpha(s-1)\dot\alpha(s-1)}$ are real. The residual gauge transformations are:

$$\delta h_{\alpha_1\dots\alpha_{s+1}\dot\alpha_1\dots\dot\alpha_{s+1}} = \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_{s+1})\dot\alpha_2\dots\dot\alpha_{s+1})}\ , \tag{3.9a}$$

$$\delta h_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \partial_{(\alpha_1(\dot\alpha_1}\zeta_{\alpha_2\dots\alpha_s)\dot\alpha_2\dots\dot\alpha_s)}\ , \tag{3.9b}$$

$$\delta\psi_{\alpha_1\dots\alpha_{s+1}\dot\alpha_1\dots\dot\alpha_s} = \partial_{(\alpha_1(\dot\alpha_1}\rho_{\alpha_2\dots\alpha_{s+1})\dot\alpha_2\dots\dot\alpha_s)}\ . \tag{3.9c}$$

These transformation laws correspond to conformal higher spin fields [5].

Reducing the actions (3.4) from superspace to components, for $s = 1, 2, \dots$, reproduces the conformal higher spin actions introduced by Fradkin and Tseytlin [5].

3.2 Integer superspin

In the superspin-s case, the superconformal multiplet is described in terms of an unconstrained prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)} \equiv \Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}}$ and its conjugate $\bar\Psi_{\alpha(s-1)\dot\alpha(s)}$. The prepotential is symmetric in its undotted indices and, independently, in its dotted indices. For $s > 1$ the gauge freedom is

$$\delta\Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}} = D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_{s-1})}\ , \tag{3.10}$$

with unconstrained gauge parameters $\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)}$ and $\zeta_{\alpha(s)\dot\alpha(s-2)}$. The choice $s=1$ will be considered in subsection 3.3.

As was shown in [15], the prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)}$ naturally originates within the so-called *longitudinal* formulation for the massless superspin-s multiplet, which also makes use of a real unconstrained compensator $H_{\alpha(s-1)\dot\alpha(s-1)}$. The prepotential $\Psi_{\alpha(s)\dot\alpha(s-1)}$ enters the action functional of [15] only via the *longitudinal linear* field strength $G_{\alpha(s)\dot\alpha(s)} := \bar D_{(\dot\alpha_1}\Psi_{\alpha(s)\dot\alpha_2\dots\dot\alpha_s)}$, which is manifestly invariant under the ζ-transformation (3.10). On the other hand, in the non-superconformal case the gauge parameter Λ is not arbitrary but instead has the form $\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)} = D^\beta L_{(\alpha_1\dots\alpha_{s-1}\beta)\dot\alpha(s-1)}$, with $L_{\alpha(s)\dot\alpha(s-1)}$ unconstrained. This is not critical since one may always make $\bar\Lambda_{\alpha(s-1)\dot\alpha(s-1)}$ unconstrained at the cost of introducing an additional compensator (in complete analogy with the massless gravitino case considered in [21] and reviewed in [7]). For the massless superspin-s multiplet, there exists another off-shell formulation which was constructed in [15] and called *transverse*. It is dual to the longitudinal one. It does not appear to be suitable to describe a superconformal multiplet.

The superconformal transformation law of $\Psi_{\alpha(s)\dot\alpha(s-1)}$ is postulated to be

$$\delta_\xi\Psi_{\alpha(s)\dot\alpha(s-1)} = \Big(\xi+\frac{1}{2}K^{bc}[\xi]M_{bc}\Big)\Psi_{\alpha(s)\dot\alpha(s-1)} - \frac{1}{2}\big(s\sigma[\xi]+(s-1)\bar\sigma[\xi]\big)\Psi_{\alpha(s)\dot\alpha(s-1)}~. \quad (3.11)$$

It follows from (3.10) that the following chiral descendants of the prepotentials

$$\mathcal{W}_{\alpha_1\dots\alpha_{2s}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_{s-1}}{}^{\dot\beta_{s-1}}D_{\alpha_s}\Psi_{\alpha_{s+1}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_{s-1}}~, \quad (3.12a)$$

$$\mathcal{Z}_{\alpha_1\dots\alpha_{2s}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_s}{}^{\dot\beta_s}D_{\alpha_{s+1}}\bar\Psi_{\alpha_{s+2}\dots\alpha_{2s})\dot\beta_1\dots\dot\beta_s} \quad (3.12b)$$

are gauge invariant.[3] As before, the crucial observation is that the field strengths $\mathcal{W}_{\alpha(2s)}$ and $\mathcal{Z}_{\alpha(2s)}$ are superconformal primaries of dimension 1 and 2, respectively. This allows us to construct a superconformal and gauge-invariant action

$$S_s = \mathrm{i}\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{W}^{\alpha_1\dots\alpha_{2s}}\mathcal{Z}_{\alpha_1\dots\alpha_{2s}} - \mathrm{i}\int \mathrm{d}^4x\mathrm{d}^2\bar\theta\,\bar{\mathcal{W}}_{\dot\alpha_1\dots\dot\alpha_{2s}}\bar{\mathcal{Z}}^{\dot\alpha_1\dots\dot\alpha_{2s}}~. \quad (3.13)$$

One checks that

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{W}^{\alpha_1\dots\alpha_{2s}}\mathcal{Z}_{\alpha_1\dots\alpha_{2s}} + \int \mathrm{d}^4x\mathrm{d}^2\bar\theta\,\bar{\mathcal{W}}_{\dot\alpha_1\dots\dot\alpha_{2s}}\bar{\mathcal{Z}}^{\dot\alpha_1\dots\dot\alpha_{2s}} = 0~. \quad (3.14)$$

We now comment on the component structure of (3.13). One may choose a Wess-Zumino gauge of the form

$$\Psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}}(\theta,\bar\theta) = \theta^\beta\bar\theta^{\dot\beta}\psi_{(\beta\alpha_1\dots\alpha_s)(\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1})} + \bar\theta^2\theta^\beta B_{(\beta\alpha_1\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_{s-1}}$$

[3]The field strength (3.12a) was introduced in [15].

$$-\theta^2\bar\theta^{\dot\beta}\boldsymbol{h}_{\alpha_1\dots\alpha_s(\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1})}+\theta^2\bar\theta^2\psi_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_{s-1}}\ , \tag{3.15}$$

where the bosonic fields $\boldsymbol{h}_{\alpha(s)\dot\alpha(s)}$ and $B_{\alpha(s+1)\dot\alpha(s-1)}$ are complex. In the Wess-Zumino gauge chosen, the bosonic fields $\boldsymbol{h}_{\alpha(s)\dot\alpha(s)}$ and $B_{\alpha(s+1)\dot\alpha(s-1)}$ and the fermionic fields $\psi_{\alpha(s+1)\dot\alpha(s)}$ and $\psi_{\alpha(s)\dot\alpha(s-1)}$ are defined modulo gauge freedom of the type (3.9).[4]

More specifically, the field $B_{\alpha(s+1)\dot\alpha(s-1)}$ belongs to a more general family of conformal fields than those described by the gauge transformation laws (3.9). The point is that one may consider conformal higher spin fields $\phi_{\alpha(m)\dot\alpha(n)}$, where m and n are integers such that $m>n>0$. Since $m\neq n$, the field $\phi_{\alpha(m)\dot\alpha(n)}$ is complex. Postulating the gauge transformation law

$$\delta\phi_{\alpha_1\dots\alpha_m\dot\alpha_1\dots\dot\alpha_n}=\partial_{(\alpha_1(\dot\alpha_1}\lambda_{\alpha_2\dots\alpha_m)\dot\alpha_2\dots\alpha_n)} \tag{3.16}$$

and requiring both the field $\phi_{\alpha(m)\dot\alpha(n)}$ and the gauge parameter $\lambda_{\alpha(m-1)\dot\alpha(n-1)}$ to be primary, the dimension of $\phi_{\alpha(m)\dot\alpha(n)}$ is fixed to be equal to $2-\frac{1}{2}(m+n)$. We can define two gauge-invariant field strengths

$$\hat C_{\alpha_1\dots\alpha_{m+n}}=\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_n}{}^{\dot\beta_n}\phi_{\alpha_{n+1}\dots\alpha_{m+n})\dot\beta_1\dots\dot\beta_n}\ , \tag{3.17a}$$

$$\check C_{\alpha_1\dots\alpha_{m+n}}=\partial_{(\alpha_1}{}^{\dot\beta_1}\dots\partial_{\alpha_m}{}^{\dot\beta_m}\bar\phi_{\alpha_{m+1}\dots\alpha_{m+n})\dot\beta_1\dots\dot\beta_m}\ . \tag{3.17b}$$

They are conformal primaries of dimension $2-\frac{1}{2}(n-m)$ and $2-\frac{1}{2}(m-n)$, respectively. In terms of those we can write a gauge-invariant conformal action

$$S=\mathrm{i}^{m+n}\int \mathrm{d}^4x\,\hat C^{\alpha_1\dots\alpha_{m+n}}\check C_{\alpha_1\dots\alpha_{m+n}}+\text{c.c.} \tag{3.18}$$

3.3 Superconformal gravitino multiplet

In the $s=1$ case, the gauge transformation law (3.10) has to be replaced with

$$\delta\Psi_\alpha=D_\alpha\bar\Lambda+\zeta_\alpha\ ,\qquad \bar D_{\dot\beta}\zeta_\alpha=0\ . \tag{3.19}$$

This gauge transformation was given in Ref. [21], which proposed the off-shell formulation for the massless gravitino multiplet in terms of the gauge spinor prepotential Ψ_α in conjunction with two compensators, an unconstrained real scalar and a chiral scalar.

The prepotential Ψ_α is required to be superconformal primary of weight $(-1,0)$, which is a special case of (3.11). The superconformal primary superfields (3.12) for $s=1$ are obviously invariant under the gauge transformations (3.19).

[4]The case $s=1$ is not considered here. Its special feature is that the field $B_{\alpha\beta}$ is not gauge.

3.4 Generalisations

Given two integers $m > n > 0$, we introduce a gauge prepotential $\Phi_{\alpha(m)\dot\alpha(n)} \equiv \Phi_{\alpha_1\dots\alpha_m\dot\alpha_1\dots\dot\alpha_n}$ and its conjugate $\bar\Phi_{\alpha(n)\dot\alpha(m)}$. The gauge transformation of $\Phi_{\alpha(m)\alpha(n)}$ is postulated to be

$$\delta\Phi_{\alpha_1\dots\alpha_m\dot\alpha_1\dots\dot\alpha_n} = D_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_m)\dot\alpha_1\dots\dot\alpha_n} + \bar D_{(\dot\alpha_1}\zeta_{\alpha_1\dots\alpha_m\dot\alpha_2\dots\dot\alpha_n)} \ , \tag{3.20}$$

with unconstrained gauge parameters $\bar\Lambda_{\alpha(m-1)\dot\alpha(n)}$ and $\zeta_{\alpha(m)\dot\alpha(n-1)}$. The superconformal transformation law of $\Phi_{\alpha(m)\dot\alpha(n)}$ is

$$\delta_\xi\Phi_{\alpha(m)\dot\alpha(n)} = \Big(\xi + \frac{1}{2}K^{bc}[\xi]M_{bc}\Big)\Phi_{\alpha(m)\dot\alpha(n)} - \frac{1}{2}\big(m\sigma[\xi] + n\bar\sigma[\xi]\big)\Phi_{\alpha(m)\dot\alpha(n)} \ . \tag{3.21}$$

Given Φ, we can define two gauge-invariant chiral field strengths

$$\mathbb{W}_{\alpha_1\dots\alpha_{m+n+1}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\cdots\partial_{\alpha_n}{}^{\dot\beta_n}D_{\alpha_{n+1}}\Phi_{\alpha_{n+2}\dots\alpha_{m+n+1})\dot\beta_1\dots\dot\beta_n} \ , \tag{3.22a}$$

$$\mathbb{Z}_{\alpha_1\dots\alpha_{m+n+1}} = -\frac{1}{4}\bar D^2\partial_{(\alpha_1}{}^{\dot\beta_1}\cdots\partial_{\alpha_m}{}^{\dot\beta_m}D_{\alpha_{m+1}}\bar\Phi_{\alpha_{m+2}\dots\alpha_{m+n+1})\dot\beta_1\dots\dot\beta_m} \ . \tag{3.22b}$$

which are superconformal primaries of dimension $\frac{1}{2}(3+n-m)$ and $\frac{1}{2}(3+m-n)$, respectively. Therefore, the following gauge-invariant action

$$S = \mathrm{i}^{m+n}\int \mathrm{d}^4x\mathrm{d}^2\theta\, \mathbb{W}^{\alpha_1\dots\alpha_{m+n+1}}\mathbb{Z}_{\alpha_1\dots\alpha_{m+n+1}} + \text{c.c.} \tag{3.23}$$

is superconformal.

4 Off-shell superconformal multiplets in supergravity

We now turn to exploring whether the superconformal higher spin multiplets introduced in the previous section may be consistently lifted to curved superspace backgrounds.

4.1 General considerations

Just as in the non-supersymmetric setting, where conformal invariance in Minkowski space is replaced by Weyl invariance, in a curved background geometry, superconformal invariance is replaced by super-Weyl invariance. In other words, super-Weyl invariance in curved superspace implies superconformal invariance in Minkowski superspace.

A tensor superfield $\mathcal{T}$ (with its indices suppressed) is said to be super-Weyl primary of weight (p,q) if its super-Weyl transformation law is

$$\delta_\sigma \mathcal{T} = \big(p\,\sigma + q\,\bar\sigma\big)\mathcal{T} ~, \tag{4.1}$$

for some parameters p and q. Similar to the rigid supersymmetric case (2.10), we will refer to $(p+q)$ as the dimension of $\mathcal{T}$. Given a covariantly chiral tensor superfield $\mathcal{T}$ defined on a general supergravity background, $\bar{\mathcal{D}}_{\dot\alpha}\mathcal{T} = 0$, it may carry only undotted indices, $\bar M_{\dot\alpha\dot\beta}\mathcal{T} = 0$, as a consequence of (A.3b). If $\mathcal{T}$ is covariantly chiral and super-Weyl primary, eq. (4.1), then $q=0$. An example is provided by the super-Weyl tensor $W_{\alpha\beta\gamma}$ with the transformation law (B.2c). In Appendix B we also collect the transformation properties of various other geometric quantities under super-Weyl transformations.

As reviewed in Appendix A, the curved superspace geometry of [8] does not possess torsion tensors of dimensions 1/2. This means that the gauge transformation (3.1) is uniquely extended to curved superspace as

$$\delta H_{\alpha_1\dots\alpha_s\dot\alpha_1\dots\dot\alpha_s} = \bar{\mathcal{D}}_{(\dot\alpha_1}\Lambda_{\alpha_1\dots\alpha_s\dot\alpha_2\dots\dot\alpha_s)} - \mathcal{D}_{(\alpha_1}\bar\Lambda_{\alpha_2\dots\alpha_s)\dot\alpha_1\dots\dot\alpha_s} ~. \tag{4.2}$$

It is compatible with the following super-Weyl transformation of the prepotential:

$$\delta_\sigma H_{\alpha(s)\dot\alpha(s)} = -\frac{s}{2}\big(\sigma+\bar\sigma\big)H_{\alpha(s)\dot\alpha(s)} ~. \tag{4.3}$$

The chiral field strength (3.3) may uniquely be lifted to curved superspace as a covariantly chiral superfield of the general form

$$\mathcal{W}_{\alpha(2s+1)} = -\frac{1}{4}(\bar{\mathcal{D}}^2-4R)\Big\{\mathcal{D}_{(\alpha_1}{}^{\dot\beta_1}\cdots\mathcal{D}_{\alpha_s}{}^{\dot\beta_s}\mathcal{D}_{\alpha_{s+1}}H_{\alpha_{s+2}\dots\alpha_{2s+1})\dot\beta_1\dots\dot\beta_s}+\dots\Big\} ~, \tag{4.4a}$$

$$\bar{\mathcal{D}}_{\dot\beta}\mathcal{W}_{\alpha(2s+1)} = 0 ~, \tag{4.4b}$$

with the super-Weyl transformation law

$$\delta_\sigma\mathcal{W}_{\alpha(2s+1)} = \frac{3}{2}\sigma\mathcal{W}_{\alpha(2s+1)} ~. \tag{4.5}$$

The ellipsis in (4.4a) stands for terms involving the super-Ricci tensor $G_{\alpha\dot\alpha}$ and its covariant derivatives. Such terms can always be found. A systematic construction is to start with conformal superspace [13], where $G_{\alpha\dot\alpha}$ appears as a connection, and then to implement the so-called de-gauging procedure in order to arrive at the ordinary curved superspace geometry of [8].[5] Details of the construction will be given elsewhere, but examples of the complete superfields for $s=1$ and $s=2$ are is given below in (4.7) and (4.9),

[5] In conformal superspace, the required primary chiral field strength $\mathcal{W}_{\alpha(2s+1)}$ has a minimal form $\mathcal{W}_{\alpha(2s+1)} = -\frac{1}{4}\bar\nabla^2\nabla_{(\alpha_1}{}^{\dot\beta_1}\cdots\nabla_{\alpha_s}{}^{\dot\beta_s}\nabla_{\alpha_{s+1}}H_{\alpha_{s+2}\dots\alpha_{2s+1})\dot\beta_1\dots\dot\beta_s}$, where $\nabla_A = (\nabla_a,\nabla_\alpha,\bar\nabla^{\dot\alpha})$ denotes the corresponding covariant derivatives [13].

respectively. Two important observations, which are crucial for this construction, are that the descendant $A_{\alpha(s+1)\dot\beta(s)} := \mathcal{D}_{(\alpha_{s+1}} H_{\alpha_1\dots\alpha_s)\dot\beta_1\dots\dot\beta_s}$ is super-Weyl primary and obeys the constraint $\mathcal{D}_{(\alpha_1} A_{\alpha_2\dots\alpha_{s+2})\dot\beta(s)} = 0$.

We may now consider a minimal extension of (3.4) to curved superspace given by

$$\int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,\mathcal{W}^{\alpha_1\dots\alpha_{2s+1}}\mathcal{W}_{\alpha_1\dots\alpha_{2s+1}} + \text{c.c.}\ , \tag{4.6}$$

where $\mathcal{E}$ is the chiral integration measure. It follows from (4.5) that this functional is super-Weyl invariant. However, for non-vanishing background super-Weyl tensor, $W_{\alpha\beta\gamma} \neq 0$, the field strength $\mathcal{W}_{\alpha(2s+1)}$ and, therefore, the action (4.6) are not gauge invariant. In general, the gauge variation $\delta_\Lambda \mathcal{W}_{\alpha(2s+1)}$ is proportional to the background super-Weyl tensor $W_{\alpha\beta\gamma}$, its conjugate $\bar W_{\dot\alpha\dot\beta\dot\gamma}$ and their covariant derivatives. For instance, in the $s=1$ case the variation $\delta_\Lambda \mathcal{W}_{\alpha(3)}$ is given by (4.8). The action (4.6) needs to be completed to include non-minimal terms which contain $W_{\alpha\beta\gamma}$, $\bar W_{\dot\alpha\dot\beta\dot\gamma}$ and their covariant derivatives. An example will be given in the following section where we discuss the gravitino supermultiplet.

As an example, we consider the simplest case, $s=1$, which corresponds to linearised conformal supergravity. The linearised super-Weyl tensor is

$$\mathcal{W}_{\alpha\beta\gamma} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big\{(\mathcal{D}_{(\alpha}{}^{\dot\gamma} + \mathrm{i}G_{(\alpha}{}^{\dot\gamma})\mathcal{D}_\beta H_{\gamma)\dot\gamma}\Big\}\ , \tag{4.7}$$

modulo normalisation. It varies homogeneously under the super-Weyl transformation, in accordance with (4.5). However, $\mathcal{W}_{\alpha\beta\gamma}$ is not invariant under the gauge transformation (4.2) with $s=1$. One may check that

$$\delta_\Lambda \mathcal{W}_{\alpha\beta\gamma} = \frac{\mathrm{i}}{2}(\bar{\mathcal{D}}^2 - 4R)\Big[\big(\mathcal{D}^\delta W_{\delta(\alpha\beta}\big)\Lambda_{\gamma)} - \mathcal{D}_{(\alpha}\big(W_{\beta\gamma)\delta}\Lambda^\delta\big)\Big]\ . \tag{4.8}$$

The important point is that each term in $\delta_\Lambda \mathcal{W}_{\alpha\beta\gamma}$ involves either the background super-Weyl tensor or its covariant derivative. The variation vanishes if the background superspace is conformally flat, $W_{\alpha\beta\gamma} = 0$. In this case the functional (4.6) is the required superconformal gauge-invariant action. Here 'superconformal' means that the action is invariant under arbitrary superconformal isometries of the background superspace.

As another example, we consider the next-to-leading case, $s=2$. The field strength $\mathcal{W}_{\alpha(5)}$ is uniquely determined to be

$$\begin{aligned}\mathcal{W}_{\alpha_1\dots\alpha_5} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big\{&\mathcal{D}_{(\alpha_1}{}^{\dot\beta_1}\mathcal{D}_{\alpha_2}{}^{\dot\beta_2} + 3\mathrm{i}G_{(\alpha_1}{}^{\dot\beta_1}\mathcal{D}_{\alpha_2}{}^{\dot\beta_2} - 2G_{(\alpha_1}{}^{\dot\beta_1}G_{\alpha_2}{}^{\dot\beta_2}\\ &-\frac{1}{4}([\mathcal{D}_{(\alpha_1}, \bar{\mathcal{D}}^{\dot\beta_1}]G_{\alpha_2}{}^{\dot\beta_2}) + \frac{3}{2}\mathrm{i}(\mathcal{D}_{(\alpha_1}{}^{\dot\beta_1}G_{\alpha_2}{}^{\dot\beta_2})\Big\}\mathcal{D}_{\alpha_3}H_{\alpha_4\alpha_5)\dot\beta_1\dot\beta_2}\ .\end{aligned} \tag{4.9}$$

It is a tedious exercise to check that $\mathcal{W}_{\alpha(5)}$ is superconformal primary.

In the case of anti-de Sitter superspace $\mathrm{AdS}^{4|4}$ [22, 23, 24] specified by

$$W_{\alpha\beta\gamma} = 0 \ , \qquad G_{\alpha\dot{\alpha}} = 0 \ , \qquad R \neq 0 \ , \tag{4.10}$$

the gauge-invariant chiral field strength $\mathcal{W}_{\alpha(2s+1)}$ was found in [25]. It is

$$\mathcal{W}_{\alpha(2s+1)} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_{(\alpha_1}{}^{\dot{\beta}_1} \cdots \mathcal{D}_{\alpha_s}{}^{\dot{\beta}_s}\mathcal{D}_{\alpha_{s+1}} H_{\alpha_{s+2}\dots\alpha_{2s+1})\dot{\beta}_1\dots\dot{\beta}_s} \ . \tag{4.11}$$

The curved-superspace extension of the gauge transformation (3.10) is

$$\delta\Psi_{\alpha_1\dots\alpha_s\dot{\alpha}_1\dots\dot{\alpha}_{s-1}} = \mathcal{D}_{(\alpha_1}\bar{\Lambda}_{\alpha_2\dots\alpha_s)\dot{\alpha}_1\dots\dot{\alpha}_{s-1}} + \bar{\mathcal{D}}_{(\dot{\alpha}_1}\zeta_{\alpha_1\dots\alpha_s\dot{\alpha}_2\dots\dot{\alpha}_{s-1})} \ . \tag{4.12}$$

It is compatible with the following super-Weyl transformation of the prepotential

$$\delta_\sigma\Psi_{\alpha(s)\dot{\alpha}(s-1)} = -\frac{1}{2}\big(s\sigma + (s-1)\bar{\sigma}\big)\Psi_{\alpha(s)\dot{\alpha}(s-1)} \ . \tag{4.13}$$

In the remainder of this section we specialise to the case of the superconformal gravitino multiplet.

4.2 Superconformal gravitino multiplet

The gravitino multiplet is characterised by the gauge freedom

$$\delta\Psi_\alpha = \mathcal{D}_\alpha\bar{\Lambda} + \zeta_\alpha \ , \qquad \bar{\mathcal{D}}_{\dot{\beta}}\zeta_\alpha = 0 \ , \tag{4.14}$$

and the super-Weyl transformation

$$\delta_\sigma\Psi_\alpha = -\frac{1}{2}\sigma\Psi_\alpha \ . \tag{4.15}$$

The following covariantly chiral field strengths

$$\mathcal{W}_{\alpha\beta} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_{(\alpha}\Psi_{\beta)} \ , \tag{4.16a}$$

$$\mathcal{Z}_{\alpha\beta} = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\Big[(\mathcal{D}_{(\alpha}{}^{\dot{\alpha}} + \mathrm{i}G_{(\alpha}{}^{\dot{\alpha}})\mathcal{D}_{\beta)}\bar{\Psi}_{\dot{\alpha}}\Big] \tag{4.16b}$$

are super-Weyl primary of dimension +1 and +2, respectively. These superfields are not invariant under the gauge transformations (4.14). One finds the following non-vanishing variations of $\mathcal{W}_{\alpha\beta}$ and $\mathcal{Z}_{\alpha\beta}$:

$$\delta_\zeta\mathcal{W}_{\alpha\beta} = 2W_{\alpha\beta\gamma}\zeta^\gamma \ , \tag{4.17a}$$

$$\delta_\Lambda \mathcal{Z}_{\alpha\beta} = \frac{\mathrm{i}}{2} W_{\alpha\beta\gamma} (\bar{\mathcal{D}}^2 - 4R) \mathcal{D}^\gamma \Lambda + \frac{\mathrm{i}}{2} (\bar{\mathcal{D}}^2 - 4R) \Big\{ \Lambda \mathcal{D}^\gamma W_{\alpha\beta\gamma} \Big\} ~. \tag{4.17b}$$

Consider the action

$$\begin{aligned} S_{\rm GM} &= \mathrm{i} \int \mathrm{d}^4x \mathrm{d}^2\theta \, \mathcal{E} \, \mathcal{W}^{\alpha\beta} \mathcal{Z}_{\alpha\beta} - 2\mathrm{i} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, E \, W^{\alpha\beta\gamma} \Psi_\alpha (\mathcal{D}_{\beta\dot{\beta}} + \mathrm{i} G_{\beta\dot{\beta}}) \mathcal{D}_\gamma \bar{\Psi}^{\dot{\beta}} \\ &\quad + \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, E \, (\mathcal{D}_\alpha W^{\alpha\beta\gamma}) (\bar{\mathcal{D}}_{\dot{\beta}} \Psi_\beta) \mathcal{D}_\gamma \bar{\Psi}^{\dot{\beta}} ~ + ~ \text{c.c.} \end{aligned} \tag{4.18}$$

Here $\mathcal{E}$ and E denote the chiral measure and the full superspace measure, respectively. The action $S_{\rm GM}$ is super-Weyl invariant,

$$\delta_\sigma S_{\rm GM} = 0 ~. \tag{4.19}$$

The second and third terms on the right of (4.18) are fixed by requiring $S_{\rm GM}$ to be invariant under the ζ-transformation (4.14),

$$\delta_\zeta S_{\rm GM} = 0 ~. \tag{4.20}$$

Finally, a lengthy calculation gives

$$\delta_\Lambda S_{\rm GM} = 2 \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, E \, B^{\alpha\dot{\alpha}} (\Psi_\alpha \bar{\mathcal{D}}_{\dot{\alpha}} \Lambda + \Lambda \bar{\mathcal{D}}_{\dot{\alpha}} \Psi_\alpha) + \text{c.c.} \tag{4.21}$$

Here $B^{\alpha\dot{\alpha}}$ denotes the $\mathcal{N}=1$ supersymmetric extension of the Bach tensor,

$$\begin{aligned} B^\alpha{}_{\dot{\alpha}} &= \mathrm{i} \mathcal{D}_{\beta\dot{\alpha}} \mathcal{D}_\gamma W^{\alpha\beta\gamma} + (\mathcal{D}_\beta G_{\gamma\dot{\alpha}}) W^{\alpha\beta\gamma} + G_{\beta\dot{\alpha}} \mathcal{D}_\gamma W^{\alpha\beta\gamma} \\ &= \mathrm{i} \mathcal{D}_{\alpha\dot{\beta}} \bar{\mathcal{D}}_{\dot{\gamma}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} - (\bar{\mathcal{D}}_{\dot{\beta}} G_{\alpha\dot{\gamma}}) \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} - G_{\alpha\dot{\beta}} \bar{\mathcal{D}}_{\dot{\gamma}} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} ~, \end{aligned} \tag{4.22}$$

with the super-Weyl transformation

$$\delta_\sigma B_{\alpha\dot{\alpha}} = \frac{3}{2} (\sigma + \bar{\sigma}) B_{\alpha\dot{\alpha}} ~. \tag{4.23}$$

One can rewrite $B_{\alpha\dot{\alpha}}$ is a manifestly real form [7, 26]

$$B_{\alpha\dot{\alpha}} = -\mathcal{D}^b \mathcal{D}_b G_{\alpha\dot{\alpha}} - W_\alpha{}^{\beta\gamma} \mathcal{D}_\beta G_{\gamma\dot{\alpha}} + \bar{W}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\beta}} G_{\alpha\dot{\gamma}} \tag{4.24}$$

$$\begin{aligned} &+ \frac{1}{4} \Big((\mathcal{D}^\beta R) \mathcal{D}_\beta + (\bar{\mathcal{D}}_{\dot{\beta}} \bar{R}) \bar{\mathcal{D}}^{\dot{\beta}} \Big) G_{\alpha\dot{\alpha}} - (\bar{\mathcal{D}}_{\dot{\alpha}} G^b) \mathcal{D}_\alpha G_b - 3 R \bar{R} G_{\alpha\dot{\alpha}} \\ &+ \frac{1}{8} G_{\alpha\dot{\alpha}} (\bar{\mathcal{D}}^2 \bar{R} + \mathcal{D}^2 R) + \frac{\mathrm{i}}{4} \mathcal{D}_{\alpha\dot{\alpha}} (\bar{\mathcal{D}}^2 \bar{R} - \mathcal{D}^2 R) ~. \end{aligned} \tag{4.25}$$

We recall that the super-Bach tensor may be introduced (see [7, 26] for the technical details) as a functional derivative of the conformal supergravity action [27, 28],

$$I_{\rm CSG} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathcal{E} \, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \text{c.c.} ~, \tag{4.26}$$

with respect to the gravitational superfield, specifically

$$\delta \int \mathrm{d}^4x\mathrm{d}^2\theta\,\mathcal{E}\,W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\Delta H^{\alpha\dot{\alpha}}B_{\alpha\dot{\alpha}}\ , \tag{4.27}$$

with $\Delta H^{\alpha\dot{\alpha}}$ the covariantised variation of the gravitational superfield defined in [29]. The super-Bach tensor obeys the conservation equation

$$\mathcal{D}^\alpha B_{\alpha\dot{\alpha}} = 0 \quad \Longleftrightarrow \quad \bar{\mathcal{D}}^{\dot{\alpha}} B_{\alpha\dot{\alpha}} = 0\ , \tag{4.28}$$

which expresses the gauge invariance of the conformal supergravity action.

It follows from (4.20) and (4.21) that the action (4.18) is gauge invariant if the background super-Bach tensor is equal to zero,

$$B_{\alpha\dot{\alpha}} = 0\ . \tag{4.29}$$

This holds, e.g., for all Einstein superspaces, which are characterised by

$$G_{\alpha\dot{\alpha}} = 0 \quad \Longrightarrow \quad R = \text{const}\ . \tag{4.30}$$

4.3 Linearised conformal supergravity

The condition (4.29) is also required to define an off-shell superconformal multiplet of superspin 3/2 in curved superspace. The point is that (4.29) is the equation of motion for conformal supergravity, since varying the action (4.26) with respect to the gravitational superfield gives[6]

$$\delta S_{\mathrm{CSG}} = 2 \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,E\,\Delta H^{\alpha\dot{\alpha}}B_{\alpha\dot{\alpha}}\ . \tag{4.31}$$

The gauge-invariant action for the superconformal superspin-$\frac{3}{2}$ multiplet in curved background is obtained by linearising the conformal supergravity action (4.26) around its arbitrary stationary point, $B_{\alpha\dot{\alpha}} = 0$. In accordance with [29] (see also [7] for a review), the linearised gauge transformation of the prepotential is given by (4.2) with $s = 1$. The linearised conformal supergravity action is automatically invariant under the gauge and super-Weyl transformations. Its explicit structure will be described elsewhere.

[6]The two terms in the right-hand side of (4.26) differ by a total derivative related to the Pontryagin invariant [7, 26].

5 Concluding comments

In this paper we constructed the off-shell $\mathcal{N}=1$ superconformal higher spin multiplets in four dimensions[7] and also sketched the general scheme of coupling such multiplets to conformal supergravity. Our work opens two new approaches to interacting conformal higher spin theories. Firstly, every conformal higher spin field may be embedded into an off-shell superconformal multiplet (the latter actually contains several bosonic and fermionic conformal fields). Instead of trying to couple the original conformal field to gravity, we can look for a consistent interaction of the superconformal multiplet with conformal supergravity. Since the gravitational field belongs to the conformal supergravity multiplet (also known as the Weyl multiplet), this will automatically lead to a consistent coupling of the component conformal fields to gravity.

The second avenue to explore is a superfield extension of the effective action approach to conformal higher spin fields advocated in [32, 33, 34]. One may start with a free massless chiral scalar superfield Φ, $\bar{D}_{\dot\alpha}\Phi=0$, and couple it to an infinite tower of background superconformal higher spin prepotentials $H_{\alpha(s)\dot\alpha(s)}$ (source superfields) by the rule

$$S[\Phi,\bar\Phi;H]=\int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\,\Big\{\Phi\bar\Phi+\sum_{s=1}^{\infty}H^{\alpha(s)\dot\alpha(s)}J_{\alpha(s)\dot\alpha(s)}\Big\}~,\quad \bar{J}_{\alpha(s)\dot\alpha(s)}=J_{\alpha(s)\dot\alpha(s)}~. \tag{5.1}$$

Here $J_{\alpha(s)\dot\alpha(s)}$ denotes a composite primary superfield, which describes a conserved current multiplet when Φ is on-shell. Then it is natural to consider the generating functional for correlation functions of these conserved higher spin supercurrents defined by

$$\mathrm{e}^{\mathrm{i}\Gamma[H]}=\int\mathcal{D}\Phi\mathcal{D}\bar\Phi\,\mathrm{e}^{\mathrm{i}S[\Phi,\bar\Phi;H]}~. \tag{5.2}$$

Similar to the non-supersymmetric analysis of [35], one may show that the action (5.1) has an exact non-Abelian gauge symmetry which reduces to (4.2) at lowest level in the superfields $H_{\alpha(s)\dot\alpha(s)}$, Φ and $\bar\Phi$. Here we restrict our discussion to giving the explicit expressions for $J_{\alpha(s)\dot\alpha(s)}$.

We turn to describing the structure of the conserved current multiplets $J_{\alpha(s)\dot\alpha(s)}$. In order for the source term

$$S^{(s+\frac{1}{2})}_{\text{source}}=\int\mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,H^{\alpha(s)\dot\alpha(s)}J_{\alpha(s)\dot\alpha(s)} \tag{5.3}$$

[7]In three dimensions, the off-shell superconformal higher spin multiplets have recently been described in [30] and [31] for the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases, respectively.

to be invariant under the superconformal transformations, the superfield $J_{\alpha(s)\dot\alpha(s)}$ must be superconformal primary of weight $(1+\frac{s}{2}, 1+\frac{s}{2})$. In order for $S^{(s+\frac12)}_{\rm source}$ to be invariant under the gauge transformations (3.1), $J_{\alpha(s)\dot\alpha(s)}$ must obey the conservation equations

$$D^\beta J_{\beta\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_s} = 0~, \qquad \bar D^{\dot\beta} J_{\alpha_1\dots\alpha_s\dot\beta\dot\alpha_1\dots\dot\alpha_{s-1}} = 0~. \tag{5.4}$$

The $s = 1$ case corresponds to the superconformal version [16] of the Ferrara-Zumino supercurrent [36]. Extension to the other cases $s > 1$ was given in [37] (building on [38]). The authors of [37] also postulated the prepotential $H_{\alpha(s)\dot\alpha(s)}$ as the source to generate the Noether coupling (5.3), as well as the gauge transformation (3.1) as the transformation of $H_{\alpha(s)\dot\alpha(s)}$ which leaves (5.3) invariant. However, no higher spin extensions of linearised conformal supergravity were given.

Consider a free on-shell massless chiral scalar Φ,

$$\bar D_{\dot\alpha}\Phi = 0~, \qquad D^2\Phi = 0~. \tag{5.5}$$

which is superconformal primary of dimension +1. By analogy with the construction of [39], the conserved current multiplets $J_{\alpha(s)\dot\alpha(s)}$, with $s = 1, 2, \dots,$ can be obtained as unique composites of Φ and $\bar\Phi$ of the form

$$\begin{aligned} J_{\alpha(s)\dot\alpha(s)} &= (2{\rm i})^{s-1}\sum_{k=0}^{s}(-1)^k\binom{s}{k} \\ &\quad\times\Big\{\binom{s}{k+1}\partial_{(\alpha_1(\dot\alpha_1}\dots\partial_{\alpha_k\dot\alpha_k}D_{\alpha_{k+1}}\Phi\,\bar D_{\dot\alpha_{k+1}}\partial_{\alpha_{k+2}\dot\alpha_{k+2}}\dots\partial_{\alpha_s)\dot\alpha_s)}\bar\Phi \\ &\qquad+2{\rm i}\binom{s}{k}\partial_{(\alpha_1(\dot\alpha_1}\dots\partial_{\alpha_k\dot\alpha_k}\Phi\,\partial_{\alpha_{k+1}\dot\alpha_{k+1}}\dots\partial_{\alpha_s)\dot\alpha_s)}\bar\Phi\Big\}~, \end{aligned} \tag{5.6}$$

where one should keep in mind that

$$\binom{s}{s+1} = 0~.$$

It is an instructive exercise to check that the conservation equations (5.4) are satisfied. Choosing $s = 1$ in (5.6) gives the well-known supercurrent [36]

$$J_{\alpha\dot\alpha} = D_\alpha\Phi\,\bar D_{\dot\alpha}\bar\Phi + 2{\rm i}(\Phi\,\partial_{\alpha\dot\alpha}\bar\Phi - \partial_{\alpha\dot\alpha}\Phi\,\bar\Phi)~. \tag{5.7}$$

The higher spin supercurrent (5.6) may be compared with the 3D $\mathcal{N} = 2$ result reported in [40].

Similar to the bosonic superfield prepotentials $H_{\alpha(s)\dot\alpha(s)}$, one may define conserved higher spin current supermultiplets associated with the fermionic superfield prepotentials $\Psi_{\alpha(s)\dot\alpha(s-1)}$. Consider a source term of the form

$$S^{(s)}_{\rm source} = \int {\rm d}^4x\, {\rm d}^2\theta\, {\rm d}^2\bar\theta\, \Psi^{\alpha(s)\dot\alpha(s-1)} J_{\alpha(s)\dot\alpha(s-1)} + {\rm c.c.} \tag{5.8}$$

In order for $S^{(s)}_{\rm source}$ to be invariant under the superconformal transformations, $J_{\alpha(s)\dot\alpha(s-1)}$ must be superconformal primary of weight $(1+\frac{s}{2}, \frac{1}{2}+\frac{s}{2})$. In order for $S^{(s)}_{\rm source}$ to be invariant under the gauge transformations (3.10), the superfield $J_{\alpha(s)\dot\alpha(s-1)}$ with $s>1$ must obey the conservation equations

$$D^\beta J_{\beta\alpha_1\dots\alpha_{s-1}\dot\alpha_1\dots\dot\alpha_{s-1}} = 0~, \qquad \bar D^{\dot\beta} J_{\alpha_1\dots\alpha_s\dot\beta\dot\alpha_1\dots\dot\alpha_{s-2}} = 0~. \tag{5.9}$$

In the $s=1$ case, the conservation equations are [41]

$$D^\beta J_\beta = 0~, \qquad \bar D^2 J_\alpha = 0~, \tag{5.10}$$

as a consequence of (3.19).

Conserved current multiplets $J_{\alpha(s)\dot\alpha(s-1)}$, with $s = 1, 2, \dots$, may be constructed from two free massless chiral superfields Φ_+ and Φ_-,

$$\bar D_{\dot\alpha} \Phi_\pm = 0~, \qquad D^2 \Phi_\pm = 0~. \tag{5.11}$$

One may check that the following composite

$$\begin{aligned} J_{\alpha(s)\dot\alpha(s-1)} = (2{\rm i})^{s-1} \sum_{k=0}^{s-1} \binom{s-1}{k} &\times \Big\{ \binom{s}{k+1} \partial_{(\alpha_1(\dot\alpha_1} \dots \partial_{\alpha_k\dot\alpha_k} D_{\alpha_s} \Phi_+\, \partial_{\alpha_{k+1}\dot\alpha_{k+1}} \dots \partial_{\alpha_{s-1})\dot\alpha_{s-1})} \Phi_- \\ &- \binom{s}{k} \partial_{(\alpha_1(\dot\alpha_1} \dots \partial_{\alpha_k\dot\alpha_k} \Phi_+\, \partial_{\alpha_{k+1}\dot\alpha_{k+1}} \dots \partial_{\alpha_{s-1}\dot\alpha_{s-1})} D_{\alpha_s)} \Phi_- \Big\} \end{aligned} \tag{5.12}$$

obeys the conservation equations (5.9) for $s>1$. One may also check that $J_{\alpha(s)\dot\alpha(s-1)}$ is antisymmetric with respect to the interchange $\Phi_+ \leftrightarrow \Phi_-$. Choosing $s=1$ in (5.12) gives the composite

$$J_\alpha = \Phi_- \overleftrightarrow{D}_\alpha \Phi_+~, \tag{5.13}$$

which obeys the conservation equations (5.10). The superfield J_α contains a conserved fermionic current $j_{\alpha\beta\dot\beta} = j_{\beta\alpha\dot\beta}$ that corresponds to the second supersymmetry current.[8] The conserved current multiplet (5.13) is obtained by reducing the $\mathcal{N} = 2$ supercurrent of a free massless hypermultiplet to $\mathcal{N} = 1$ superspace, see [41] for more details.

In the case of a free $\mathcal{N} = 2$ hypermultiplet described in terms of two $\mathcal{N} = 1$ chiral scalars Φ_+ and Φ_-, the action (5.1) should be replaced with

$$S[\Phi_\pm, \bar\Phi_\pm; H] = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar\theta \Big\{ \Phi_+\bar\Phi_+ + \Phi_-\bar\Phi_- + \sum_{s=1}^{\infty} H^{\alpha(s)\dot\alpha(s)} J_{\alpha(s)\dot\alpha(s)} + \sum_{s=1}^{\infty} \Big[\Psi^{\alpha(s)\dot\alpha(-1)} J_{\alpha(s)\dot\alpha(s-1)} + \text{c.c.} \Big] \Big\} . \tag{5.14}$$

The current superfields $J_{\alpha(s)\dot\alpha(s)}$ and $J_{\alpha(s)\dot\alpha(s-1)}$ are $\mathcal{N} = 1$ components of a conserved $\mathcal{N} = 2$ supermultiplet.

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A The Grimm-Wess-Zumino superspace geometry

In describing the Grimm-Wess-Zumino superspace geometry [8], we follow the notation and conventions of [7].[9] In particular, the coordinates of $\mathcal{N} = 1$ curved superspace

[8]This conserved current may be chosen as $j_{\alpha\beta\dot\beta} = \Big\{ \big[D_{(\beta}, \bar D_{\dot\beta}\big] J_{\alpha)} + \frac{2}{3} \mathrm{i}\partial_{(\beta\dot\beta} J_{\alpha)} \Big\}\Big|_{\theta=0}$. Also associated with J_α is the off-shell conserved current $v_{\alpha\beta\dot\beta} = \partial_{\beta\dot\beta} J_\alpha - 2\varepsilon_{\alpha\beta}\partial_{\gamma\dot\beta} J^\gamma$ such that $\partial^{\beta\dot\beta} v_{\alpha\beta\dot\beta} = 0$ for any J_α.

[9]These conventions are similar to those of Wess and Bagger [42]. To convert the notation of [7] to that of [42], one replaces $R \to 2R$, $G_{\alpha\dot\alpha} \to 2G_{\alpha\dot\alpha}$, and $W_{\alpha\beta\gamma} \to 2W_{\alpha\beta\gamma}$. In addition, the vector derivative has to be changed by the rule $\mathcal{D}_a \to \mathcal{D}_a + \frac{1}{4}\varepsilon_{abcd} G^b M^{cd}$, where G_a corresponds to [7]. Finally, the spinor Lorentz generators $(\sigma_{ab})_\alpha{}^\beta$ and $(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}$ used in [7] have an extra minus sign as compared with [42], specifically $\sigma_{ab} = -\frac{1}{4}(\sigma_a\tilde\sigma_b - \sigma_b\tilde\sigma_a)$ and $\tilde\sigma_{ab} = -\frac{1}{4}(\tilde\sigma_a\sigma_b - \tilde\sigma_b\sigma_a)$.

$\mathcal{M}^{4|4}$ are denoted $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$. The superspace geometry is described by covariant derivatives of the form

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A + \Omega_A \ . \tag{A.1}$$

Here E_A denotes the inverse vielbein, $E_A = E_A{}^M \partial_M$, and Ω_A the Lorentz connection,

$$\Omega_A = \frac{1}{2}\,\Omega_A{}^{bc} M_{bc} = \Omega_A{}^{\beta\gamma} M_{\beta\gamma} + \Omega_A{}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} \ , \tag{A.2}$$

with $M_{bc} = -M_{cb} \Leftrightarrow (M_{\beta\gamma} = M_{\gamma\beta}, \bar{M}_{\dot{\beta}\dot{\gamma}} = \bar{M}_{\dot{\gamma}\dot{\beta}})$ the Lorentz generators. The covariant derivatives obey the following anti-commutation relations:

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot{\alpha}} \ , \tag{A.3a}$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R} M_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 4R\bar{M}_{\dot{\alpha}\dot{\beta}} \ , \tag{A.3b}$$

$$\big[\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\big] = -\mathrm{i}\varepsilon_{\dot{\alpha}\dot{\beta}}\Big(R\,\mathcal{D}_\beta + G_\beta{}^{\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\gamma}} - (\bar{\mathcal{D}}^{\dot{\gamma}} G_\beta{}^{\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} + 2W_\beta{}^{\gamma\delta} M_{\gamma\delta}\Big) - \mathrm{i}(\mathcal{D}_\beta R)\bar{M}_{\dot{\alpha}\dot{\beta}} \ , \tag{A.3c}$$

$$\big[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}\big] = \mathrm{i}\varepsilon_{\alpha\beta}\Big(\bar{R}\,\bar{\mathcal{D}}_{\dot{\beta}} + G^\gamma{}_{\dot{\beta}}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot{\beta}})M_{\gamma\delta} + 2\bar{W}_{\dot{\beta}}{}^{\dot{\gamma}\dot{\delta}}\bar{M}_{\dot{\gamma}\dot{\delta}}\Big) + \mathrm{i}(\bar{\mathcal{D}}_{\dot{\beta}}\bar{R})M_{\alpha\beta} \ , \tag{A.3d}$$

$$\big[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}\big] = \varepsilon_{\dot{\alpha}\dot{\beta}}\psi_{\alpha\beta} + \varepsilon_{\alpha\beta}\psi_{\dot{\alpha}\dot{\beta}} \ , \tag{A.3e}$$

where

$$\psi_{\alpha\beta} := -\mathrm{i}G_{(\alpha}{}^{\dot{\gamma}}\mathcal{D}_{\beta)\dot{\gamma}} + \frac{1}{2}(\mathcal{D}_{(\alpha}R)\mathcal{D}_{\beta)} + \frac{1}{2}(\mathcal{D}_{(\alpha}G_{\beta)}{}^{\dot{\gamma}})\bar{\mathcal{D}}_{\dot{\gamma}} + W_{\alpha\beta}{}^\gamma\mathcal{D}_\gamma$$
$$+\frac{1}{4}\big((\bar{\mathcal{D}}^2 - 8R)\bar{R}\big)M_{\alpha\beta} + (\mathcal{D}_{(\alpha}W_{\beta)}{}^{\gamma\delta})M_{\gamma\delta} - \frac{1}{2}(\mathcal{D}_{(\alpha}\bar{\mathcal{D}}^{\dot{\gamma}}G_{\beta)}{}^{\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} \ , \tag{A.3f}$$

$$\psi_{\dot{\alpha}\dot{\beta}} := -\mathrm{i}G_{\gamma(\dot{\alpha}}\mathcal{D}^\gamma{}_{\dot{\beta})} - \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}\bar{R})\bar{\mathcal{D}}_{\dot{\beta})} - \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}G^\gamma{}_{\dot{\beta})})\mathcal{D}_\gamma - \bar{W}_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}\bar{\mathcal{D}}_{\dot{\gamma}}$$
$$+\frac{1}{4}\big((\mathcal{D}^2 - 8\bar{R})R\big)\bar{M}_{\dot{\alpha}\dot{\beta}} - (\bar{\mathcal{D}}_{(\dot{\alpha}}\bar{W}_{\dot{\beta})}{}^{\dot{\gamma}\dot{\delta}})\bar{M}_{\dot{\gamma}\dot{\delta}} + \frac{1}{2}(\bar{\mathcal{D}}_{(\dot{\alpha}}\mathcal{D}^\gamma G^\delta{}_{\dot{\beta})})M_{\gamma\delta} \ . \tag{A.3g}$$

The torsion tensors R, $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities

$$\bar{\mathcal{D}}_{\dot{\alpha}} R = 0 \ , \qquad \bar{\mathcal{D}}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \ , \tag{A.4a}$$

$$\bar{\mathcal{D}}^{\dot{\gamma}} G_{\alpha\dot{\gamma}} = \mathcal{D}_\alpha R \ , \tag{A.4b}$$

$$\mathcal{D}^\gamma W_{\alpha\beta\gamma} = \mathrm{i}\,\mathcal{D}_{(\alpha}{}^{\dot{\gamma}} G_{\beta)\dot{\gamma}} \ . \tag{A.4c}$$

A supergravity gauge transformation is defined to act on the covariant derivatives and any tensor superfield U (with its indices suppressed) by the rule

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] \ , \qquad \delta_{\mathcal{K}} U = \mathcal{K} U \ , \tag{A.5a}$$

where the gauge parameter $\mathcal{K}$ has the explicit form

$$\mathcal{K} = \xi^B \mathcal{D}_B + \frac{1}{2} K^{bc} M_{bc} = \xi^B \mathcal{D}_B + K^{\gamma\delta} M_{\gamma\delta} + \bar{K}^{\dot{\gamma}\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} = \bar{\mathcal{K}} \tag{A.5b}$$

and describes a general coordinate transformation generated by the supervector field $\xi = \xi^B E_B$ as well as a local Lorentz transformation generated by the antisymmetric tensor K^{bc}.

B Super-Weyl transformations

The algebra of covariant derivatives (A.3) preserves its functional form under super-Weyl transformations [9]

$$\delta_\sigma \mathcal{D}_\alpha = (\bar{\sigma} - \frac{1}{2}\sigma)\mathcal{D}_\alpha + (\mathcal{D}^\beta \sigma)\, M_{\alpha\beta} \ , \tag{B.1a}$$

$$\delta_\sigma \bar{\mathcal{D}}_{\dot{\alpha}} = (\sigma - \frac{1}{2}\bar{\sigma})\bar{\mathcal{D}}_{\dot{\alpha}} + (\bar{\mathcal{D}}^{\dot{\beta}} \bar{\sigma})\bar{M}_{\dot{\alpha}\dot{\beta}} \ , \tag{B.1b}$$

$$\begin{aligned}\delta_\sigma \mathcal{D}_{\alpha\dot{\alpha}} = {} & \frac{1}{2}(\sigma + \bar{\sigma})\mathcal{D}_{\alpha\dot{\alpha}} + \frac{\rm i}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\sigma})\mathcal{D}_\alpha + \frac{\rm i}{2}(\mathcal{D}_\alpha \sigma)\bar{\mathcal{D}}_{\dot{\alpha}} \\ & + (\mathcal{D}^\beta{}_{\dot{\alpha}}\sigma) M_{\alpha\beta} + (\mathcal{D}_\alpha{}^{\dot{\beta}}\bar{\sigma})\bar{M}_{\dot{\alpha}\dot{\beta}} \ ,\end{aligned} \tag{B.1c}$$

where σ is an arbitrary covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot{\alpha}}\sigma = 0$. The torsion tensors in (A.3) transform as follows:

$$\delta_\sigma R = 2\sigma R + \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar{\sigma} \ , \tag{B.2a}$$

$$\delta_\sigma G_{\alpha\dot{\alpha}} = \frac{1}{2}(\sigma + \bar{\sigma}) G_{\alpha\dot{\alpha}} + {\rm i}\mathcal{D}_{\alpha\dot{\alpha}}(\sigma - \bar{\sigma}) \ , \tag{B.2b}$$

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2}\sigma W_{\alpha\beta\gamma} \ . \tag{B.2c}$$

The local transformations (A.5) and (B.1) constitute the gauge freedom of conformal supergravity.

References

[1] C. Fronsdal, "Massless fields with integer spin," Phys. Rev. D **18**, 3624 (1978); J. Fang and C. Fronsdal, "Massless fields with half-integral spin," Phys. Rev. D **18**, 3630 (1978).

[2] C. Fronsdal, "Singletons and massless, integral-spin fields on de Sitter space," Phys. Rev. D **20**, 848 (1979); J. Fang and C. Fronsdal, "Massless, half-integer-spin fields in de Sitter space," Phys. Rev. D **22**, 1361 (1980).

[3] E. S. Fradkin and M. A. Vasiliev, "On the gravitational interaction of massless higher spin fields," Phys. Lett. B **189**, 89 (1987); "Cubic interaction in extended theories of massless higher spin fields," Nucl. Phys. B **291**, 141 (1987).

[4] M. A. Vasiliev, "Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions," Phys. Lett. B **243**, 378 (1990).

[5] E. S. Fradkin and A. A. Tseytlin, "Conformal supergravity," Phys. Rept. **119**, 233 (1985).

[6] E. S. Fradkin and V. Y. Linetsky, "Superconformal higher spin theory in the cubic approximation," Nucl. Phys. B **350**, 274 (1991).

[7] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, Bristol, 1995 (Revised Edition 1998).

[8] R. Grimm, J. Wess and B. Zumino, "Consistency checks on the superspace formulation of supergravity," Phys. Lett. B **73**, 415 (1978); "A complete solution of the Bianchi identities in superspace," Nucl. Phys. B **152**, 255 (1979).

[9] P. S. Howe and R. W. Tucker, "Scale invariance in superspace," Phys. Lett. B **80**, 138 (1978).

[10] J. Wess and B. Zumino, "Superfield Lagrangian for supergravity," Phys. Lett. B **74**, 51 (1978).

[11] K. S. Stelle and P. C. West, "Minimal auxiliary fields for supergravity," Phys. Lett. B **74**, 330 (1978); S. Ferrara and P. van Nieuwenhuizen, "The auxiliary fields of supergravity," Phys. Lett. B **74**, 333 (1978).

[12] P. S. Howe, "A superspace approach to extended conformal supergravity," Phys. Lett. B **100**, 389 (1981); "Supergravity in superspace," Nucl. Phys. B **199**, 309 (1982).

[13] D. Butter, "N=1 conformal superspace in four dimensions," Annals Phys. **325**, 1026 (2010) [arXiv:0906.4399 [hep-th]].

[14] S. M. Kuzenko, V. V. Postnikov and A. G. Sibiryakov, "Massless gauge superfields of higher half-integer superspins," JETP Lett. **57**, 534 (1993) [Pisma Zh. Eksp. Teor. Fiz. **57**, 521 (1993)].

[15] S. M. Kuzenko and A. G. Sibiryakov, "Massless gauge superfields of higher integer superspins," JETP Lett. **57**, 539 (1993) [Pisma Zh. Eksp. Teor. Fiz. **57**, 526 (1993)].

[16] S. Ferrara and B. Zumino, "Structure of conformal supergravity," Nucl. Phys. B **134**, 301 (1978).

[17] V. Ogievetsky and E. Sokatchev, "On vector superfield generated by supercurrent," Nucl. Phys. B **124**, 309 (1977).

[18] W. Siegel, "A derivation of the supercurrent superfield," Harvard preprint HUTP-77/A089 (December, 1977).

[19] J. H. Park, "Superconformal symmetry and correlation functions," Nucl. Phys. B **559**, 455 (1999) [hep-th/9903230].

[20] S. J. Gates Jr., M. T. Grisaru, M. Roček and W. Siegel, *Superspace, or One Thousand and One Lessons in Supersymmetry*, Benjamin/Cummings (Reading, MA), 1983, hep-th/0108200.

[21] S. J. Gates Jr. and W. Siegel, "(3/2, 1) superfield of O(2) supergravity," Nucl. Phys. **B164**, 484 (1980).

[22] B. W. Keck, "An alternative class of supersymmetries," J. Phys. A **8**, 1819 (1975).

[23] B. Zumino, "Nonlinear realization of supersymmetry in de Sitter space," Nucl. Phys. B **127**, 189 (1977).

[24] E. A. Ivanov and A. S. Sorin, "Superfield formulation of OSp(1,4) supersymmetry," J. Phys. A **13**, 1159 (1980).

[25] S. M. Kuzenko and A. G. Sibiryakov, "Free massless higher-superspin superfields on the anti-de Sitter superspace" Phys. Atom. Nucl. **57**, 1257 (1994) [Yad. Fiz. **57**, 1326 (1994)] [arXiv:1112.4612 [hep-th]].

[26] I. L. Buchbinder and S. M. Kuzenko, "Quantization of the classically equivalent theories in the superspace of simple supergravity and quantum equivalence," Nucl. Phys. B **308**, 162 (1988).

[27] W. Siegel, "Solution to constraints in Wess-Zumino supergravity formalism," Nucl. Phys. B **142**, 301 (1978).

[28] B. Zumino, "Supergravity and superspace," in *Recent Developments in Gravitation - Cargèse 1978*, M. Lévy and S. Deser (Eds.), N.Y., Plenum Press, 1979, pp. 405–459.

[29] M. T. Grisaru and W. Siegel, "Supergraphity (I). Background field formalism," Nucl. Phys. B **187**, 149 (1981); "Supergraphity (II). Manifestly covariant rules and higher loop finiteness," Nucl. Phys. B **201**, 292 (1982)

[30] S. M. Kuzenko, "Higher spin super-Cotton tensors and generalisations of the linearchiral duality in three dimensions," Phys. Lett. B **763**, 308 (2016) [arXiv:1606.08624 [hep-th]]; "Off-shell massive N=1 supermultiplets in three dimensions," Nucl. Phys. B **914**, 160 (2017) [arXiv:1609.06910 [hep-th]].

[31] S. M. Kuzenko and D. X. Ogburn, "Off-shell higher spin N=2 supermultiplets in three dimensions," Phys. Rev. D **94**, no. 10, 106010 (2016) [arXiv:1603.04668 [hep-th]].

[32] A. A. Tseytlin, "On limits of superstring in AdS(5) x S**5," Theor. Math. Phys. **133**, 1376 (2002) [Teor. Mat. Fiz. **133**, 69 (2002)] [hep-th/0201112].

[33] A. Y. Segal, "Conformal higher spin theory," Nucl. Phys. B **664**, 59 (2003) [hep-th/0207212].

[34] X. Bekaert, E. Joung and J. Mourad, "Effective action in a higher-spin background," JHEP **1102**, 048 (2011) [arXiv:1012.2103 [hep-th]].

[35] X. Bekaert, E. Joung and J. Mourad, "On higher spin interactions with matter," JHEP **0905**, 126 (2009) [arXiv:0903.3338 [hep-th]].

[36] S. Ferrara and B. Zumino, "Transformation properties of the supercurrent," Nucl. Phys. B **87**, 207 (1975).

[37] P. S. Howe, K. S. Stelle and P. K. Townsend, "Supercurrents," Nucl. Phys. B **192**, 332 (1981).

[38] W. Siegel, "On-shell O(N) supergravity in superspace," Nucl. Phys. B **177**, 325 (1981).

[39] N. S. Craigie, V. K. Dobrev and I. T. Todorov, "Conformally covariant composite operators in quantum chromodynamics," Annals Phys. **159**, 411 (1985).

[40] A. A. Nizami, T. Sharma and V. Umesh, "Superspace formulation and correlation functions of 3d superconformal field theories," JHEP **1407**, 022 (2014) [arXiv:1308.4778 [hep-th]].

[41] S. M. Kuzenko and S. Theisen, "Correlation functions of conserved currents in N = 2 superconformal theory," Class. Quant. Grav. **17**, 665 (2000) [hep-th/9907107].

[42] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, Princeton, 1983 (Second Edition 1992).