Onset of Floquet thermalisation

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In presence of interactions, a closed, homogeneous (disorder-free) many-body system is believed to generically heat up to an 'infinite temperature' ensemble when subjected to a periodic drive: in the spirit of the ergodicity hypothesis underpinning statistical mechanics, this happens as no energy or other conservation law prevents this. Here we present an interacting Ising chain driven by a field of time-dependent strength, where such heating onsets only below a threshold value of the drive amplitude, above which the system exhibits non-ergodic behaviour. The onset appears at strong, but not fast driving. This in particular puts it beyond the scope of high-frequency expansions. The onset location shifts, but it is robustly present, across wide variations of the model Hamiltonian such as driving frequency and protocol, as well as the initial state. The portion of nonergodic states in the Floquet spectrum, while thermodynamically subdominant, has a finite entropy. We find that the magnetisation as an emergent conserved quantity underpinning the freezing; indeed the freezing effect is readily observed, as initially magnetised states remain partially frozen up to infinite time. This result, which bears a family resemblance to the Kolmogorov-Arnold-Moser theorem for classical dynamical systems, could be a valuable ingredient for extending Floquet engineering to the interacting realm.

I. INTRODUCTION

Interacting many-body systems, by the ergodic hypothesis, generically thermalise, placing them in the purview of statistical mechanics and equilibrium thermodynamics¹. Our understanding of the corresponding situation for non-equilibrium systems is still in flux. For perhaps the simplest class of non-equilibrium systems, periodically driven (Floquet) systems, thermalisation physics at first pass looks maximally simple: removing time translation invariance destroys energy conservation, and hence the concept of temperature—which means thermalisation is to a featureless 'infinite-temperature' state^{2,3}.

Such Floquet systems have been predicted to be capable of sustaining new forms of spatiotemporal ordering when many-body localised as a result of strong quenched disorder⁴. The experimental search for such so-called discrete time crystals has been qualitatively more successful^{5,6} than may have been anticipated: the collection of systems appearing to exhibit such order now even includes a dense periodic array of nuclear spins initialised in a thermal state⁷.

All of this focuses the question on settings which permit long-lived correlations and order to persist despite the presence of periodic driving even in the absence of quenched disoroder. In periodically driven non-interacting systems, quantum heating can be suppressed^{8–10} and an extensive number of periodically conserved quantities identified¹¹. In turn, a prethermalisation regime has been identified¹² which resembles a frozen non-thermal state ⁸ which can be described by a periodic (generalized) Gibbs' ensemble¹¹. Tuning the drive parameters, and weakening the interactions, can substantially enhance the prethermalization period,

still expected to remain finite¹³. In fact, for disorder-free systems, a transient but exponentially long-lived regime exhibiting discrete time-crystalline phenomenology has already been identified¹⁴. These constitute lower bounds on the thermalisation timescales. For finite-size systems, an emergent integrability structure for strong drives has also been proposed as a way to avoid thermalisation¹⁵. There is further evidence indicating absence of heating at high drive frequencies in a variety of other settings^{16–24} and in specially designed models^{25,26}.

Here, we address the question whether there is an identifiable threshold for the ratio of driving and interaction strength, below which the system approaches a nontrivial steady state that depends on the drive and the initial state. We consider a spin chain subject to strong, but not fast driving, and use remanent infinite-time magnetisation of a initial magnetised state as measure of failure to Floquet-thermalise. As the driving is increased from low strength, where standard Floquet thermalisation is observed, we find a remarkably well-defined second regime, in which remanent magnetisation is present even in the infinite time limit. Its value is given by the Floquet diagonal ensemble average implied by the initial state. The location of this threshold moves, but its existence is stable to variations in state initialisation, driving strength, driving protocol, and driving frequency.

In all cases, however, we are able to identify an emergent approximately conserved quantity – in the case we discuss at length, the magnetisation itself – which becomes exactly conserved if the static part of the Hamiltonian is ignored. Thus, rather than an extensive set of integrals of motion, as is present in the case of the periodic Gibbs ensemble¹¹ and the Floquet many-body localised cases^{27–29}, all that appears to be needed to stop the system from heating up indefinitely is a single, ap-

proximately conserved quantity.

While our numerical investigation on systems up to L=14 spins naturally limits our capacity to extrapolate these results to the 'thermodynamic' limit, there are indications that this is not only a finite-size effect. Firstly, in plots of remanent magnetisation versus driving strength, we identify a crossing point for curves for different L separating the ergodic and the non-thermal regimes. Second, the set of Floquet eigenstates exhibiting memory, while accounting only for a vanishing fraction of the total Hilbert space, extrapolates to have a finite entropy in the thermodynamic limit. This means that such states can still be straightforwardly selected by an initial condition, not unlike initialising a static system in a low-temperature configuration.

In the following, we first we set the notation and provide a brief introduction to the Floquet concepts we have used. We then define our model and drive protocol. We characterise the ergodic and the non-thermal phases and the threshold between them using various measures, and demonstrate robustness to variations of drive patterns and system parameters. We close with an outlook and suggestions for further investigations. In particular, origin and nature of the sharp features in the memory as a function of driving strength merit further study.

II. MODEL

In this section, we introduce notation, model Hamiltonian, drive protocol, and observables to be studied.

A. Floquet basics

Let us decompose the time-dependent Hamiltonian H(t) into a static interacting Hamiltonian H_0 and a time-periodic drive $H_D(t)$ with $[H_0, H_D] \neq 0$:

$$H(t) = H_0 + H_D(t), \tag{1}$$

The time evolution operator evolving a state through a period from $t = \epsilon$ to $t = \epsilon + T$ ($0 \le \epsilon < T$) is $U(\epsilon)$. Since $U(\epsilon)$ is unitary, it can always be expressed in terms of a hermitian operator, the 'Floquet Hamiltonian' H_{eff} as

$$U(\epsilon) = e^{-iH_{eff}(\epsilon)T}. (2)$$

Formally,

$$\exp\left(-iH_{eff}\left(\epsilon\right)T\right) = \mathcal{T}\exp\left(-i\int_{\epsilon}^{\epsilon+T}dt\;H(t)\right),\quad(3)$$

where \mathcal{T} denotes time-ordering. Let $|\mu_i\rangle$ denote the *i*-th 'Floquet eigenstate' of H_{eff} corresponding to the 'Floquet eigenvalue' (also known as quasienergy) μ_i .

A sequence of stroboscopic observations at instants $t = \epsilon, \epsilon + T, \dots, \epsilon + nT$ (integer n) is identical to that

produced by the dynamics under the time-independent Hamiltonian H_{eff} . This applies for every ϵ , hence we get continuous family of stroboscopic series.

In the following, we are interested in long-time asymptotic behaviour, so that temporal variations within a driving period are of secondary importance. Hence, we arbitrarily pick $\epsilon=0$.

B. Infinite time limit: diagonal ensemble average

The nature of the asymptotic state under the drive can be understood as follows. Consider an initial state

$$|\psi(0)\rangle = \sum_{i} c_i |\mu_i\rangle$$

and the stroboscopic time series for an observable

$$\hat{\mathcal{O}} = \sum_{i,j} \mathcal{O}_{ij} |\mu_i\rangle \langle \mu_j|.$$

$$\langle \psi(nT+\epsilon)|\hat{\mathcal{O}}|\psi(nT+\epsilon)\rangle = \sum_{i,j} c_i c_j^* \mathcal{O}_{ij} e^{-i(\mu_i - \mu_j)(nT+\epsilon)}.$$
(4)

Like in the case of static Hamiltonians, under quite general and experimentally relevant conditions (see, e.g., Ref. 30), at long times $(n \to \infty)$ the off-diagonal $(i \neq j)$ terms 'average to zero' and the state of the system can hence be described by an effective "diagonal ensemble" (in the absence of synchronisation, e.g. for discrete time crystals, this is replaced by a block diagonal ensemble 31). This is captured by the mixed density matrix 32 $\hat{\rho}_{DE} = \sum_i |c_i|^2 |\mu_i\rangle\langle\mu_i|$.

Thus, the asymptotic properties of a periodically driven system are effectively given by a classical average (known as diagonal ensemble average or DEA) over the expectation values of the eigenstates of H_{eff} ,

$$\langle \hat{\mathcal{O}} \rangle (\text{DEA}) = \sum_{i} |c_{i}|^{2} \langle \mu_{i} | \hat{\mathcal{O}} | \mu_{i} \rangle .$$
 (5)

Hence it is sufficient to study the nature of the eigenstates and eigenvalues of H_{eff} , or equivalently of $U(\epsilon)$, in order to obtain the long-time behaviour.

C. Driving protocol

We consider L spins on a chain. We chose a binary drive protocol, which switches periodically between a pair of rectangular pulses. The time dependent Hamiltonian is

$$H(t) = H_0 + \operatorname{sgn}(\cos \omega t) H_D, \tag{6}$$

with the two components

$$H_{0} = -J \sum_{i} \sigma_{i}^{x} \sigma_{i+1}^{x} + \kappa \sum_{i} \sigma_{i}^{x} \sigma_{i+2}^{x} - h_{0}^{x} \sum_{i}^{L} \sigma_{i}^{x}$$
$$-h^{z} \sum_{i}^{L} \sigma_{i}^{z}; \tag{7}$$

$$H_D = -h_D^x \sum_{i}^{L} \sigma_i^x . (8)$$

The σ^{α} are Pauli matrices. We use periodic boundary condition, but tamper the system slightly by putting $J_{L,1} = 1.2J$ and $\kappa_{L-1,1} = 1.2\kappa$ to break translational invariance (and hence remove any remaining block-diagonal structure of the Hamiltonian). Here since we keep the interaction strengths constant during the drive, we use the drive amplitude itself as the tuning parameter.

In presence of the transverse field, the Hamiltonian H_0 is known to be ergodic due to the four-fermionic interaction terms arising from the next-nearest neighbour interactions under the spin to fermion mapping, and also due to the longitudinal field. We have explicitly verified that H_0 is ergodic for our case, see Suppl. Mat.).

We initialise the simulation in the time domain with different initial states. Unless otherwise stated, we use the default choice of the ground state of H(t = 0).

III. NUMERICAL RESULTS

The central quantity is the longitudinal magnetization

$$m^{x}(t) = \frac{1}{L} \sum_{i}^{L} \langle \psi(t) | \sigma_{i}^{x} | \psi(t) \rangle . \tag{9}$$

We monitor its real-time dynamics in a stroboscopic time series. We diagnose non-thermalisation/freezing via its long-time asymptotic behaviour, the remnant magnetisation, which we study as a function of various model parameters.

A. Onset of Floquet thermalisation

In the following, we provide numerical evidence that for a strong (but not fast) drive, the system fails to Floquet thermalise, instead retaining memory of its initially magnetised state. We then show that the onset of Floquet thermalisation occurs at a fairly well-defined threshold driving strength. For the results in the main text, we have chosen J=1, $\kappa=0.7$, $h^z=1.2$, and $h_0^x=0.02$.

The stroboscopic time series for the magnetisation m^x is shown in Fig. 1, left frame. Already at short times, three representative trajectories for different driving strengths show strikingly different behaviour. While for weak driving fields, the magnetisation disappears almost immediately, for stronger ones, the decay slows

down. Finally, for h_D^x beyond a threshold value, the decay is arrested: even at the longest times, a remnant magnetisation persists.

This remnant magnetisation agrees with the DEA of the magnetisation evaluated for the same system (see inset). Note that the nonvanishing DEA is already in itself a signature of the lack of Floquet thermalisation — in general, Floquet thermalised eigenstates individually show no non-trivial correlations.

In order to locate the onset, the DEA of m^x as a function of the drive amplitude h_D^x is plotted in Fig. 1, middle frame. A threshold for nonzero remnant magnetisation is observed, separating the ergodic $(m_{DE}^x \approx 0)$ from the nonergodic regime.

The lower inset shows freezing for an initial state with a reduced polarization in the x-direction. The black dotted line shows the initial value of m^x for the state, and the curve shows that for high enough h_D^x , the DEA of m^x almost coincides with it. In detail, this initial state is given as $|\psi_0\rangle = \sum_{i=1}^{2^L} c_i |i_x\rangle$, where $|i_x\rangle$ is the i-th eigenstate of the longitudinal field part (computational basis states in x-direction, or x-basis states), by choosing $Re[c_i]$ and $Im[c_i]$ from a uniform distribution between -1 and +1, multiplying them by $e^{\beta m_i^x}$, where $\beta > 0$ and m_x^i is the longitudinal magnetization of $|i_x\rangle$, and finally normalizing the state. This gives a 'generic' state with a bias towards positive longitudinal magnetization. For the plot in Fig. 1 (middle frame), we have chosen a random instance corresponding to $\beta = 1.75$. The right frame of Fig. 1 shows DEA of $1 - m^x$ on a doubly logarithmic log - log plot zoomed in around the threshold for better visibility.

B. Floquet eigenstates and an emergent conservation law

1. Localization and magnetization

We now turn to the properties of the Floquet eigenstates obtained by numerically diagonalizing the time evolution operator U(0), Eq. 2. We consider first their 'localisation' in Hilbert space, followed by their magnetisation content.

In order to investigate the localization properties of the Floquet states in the x-basis $\{|i_x\rangle\}$ we calculate the inverse participation ratio (IPR) in said basis defined as $IPR(|\mu_j\rangle) = \sum_{i=1}^{2^L} |\langle i_x | \mu_j \rangle|^4$. The left frame of Fig. 2 shows the IPR thus obtained, arranged in decreasing order. Indefinite heating corresponds to the states being delocalized in the eigenbasis of any local operator, which implies a uniformly small IPR given by the inverse dimension of Hilbert space, $1/D_H$. This is indeed what is observed for small drive fields. By contrast, for large drive fields, states appear which have an IPR close to 1, which indicates the presence of well-localised states, and hence the absence of Floquet thermalisation for the

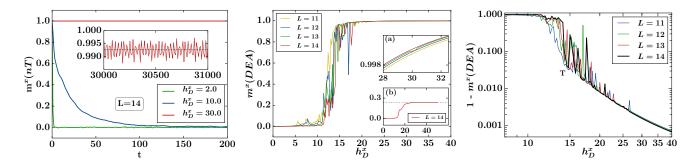


FIG. 1: Freezing and its onset threshold. Left frame: Stroboscopic time series of magnetisation $m^x(t)$ for different driving strengths showing initial state memory for strong driving. The inset zooms in on the long-time behaviour; the black horizontal line denotes the DEA of the magnetisation. Middle frame: Remnant magnetisation as a function of driving strength for different system sizes. The high-field regime (top inset) shows an increase of the remnant magnetisation with L. The bottom inset shows the DEA of m^x vs. drive amplitude for a 'generic' state (see the text for details) whose net initial magnetization is marked with the horizontal line, which remains almost unchanged for very strong drives. Right frame: Same data as middle frame on a doubly logarithmic plot for $1 - m^x(\text{DEA})$. The deviation away from almost complete thermalisation gets steeper and moves towards the right with increasing system size. The curves appear to accumulate from the left at a 'threshold point' (T) which itself appears to move little as the system size is increased from L = 11 to L = 14.

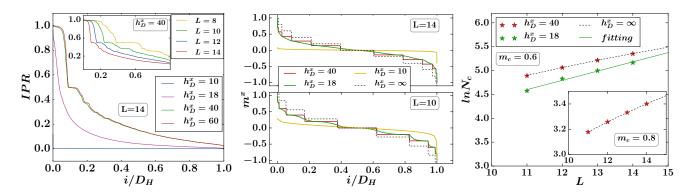


FIG. 2: Emergent conservation law for strong drives, as reflected in the Floquet eigenstates $|\mu_i\rangle$. Left frame: Values of the IPR in the x-basis, arranged in decreasing order. Unbounded heating requires these states to be delocalized in the eigenbasis of any local operator. This is the case for the drive with amplitude below the threshold $(h_D^x = 10,)$ but not above $(h_D^x = 18, 40, 60)$. The inset shows a decreasing IPR for different system-sizes for $h_D^x = 40$ due to the emergent conservation law evidenced in the middle frame: m^x for the Floquet eigenstates arranged in decreasing order, for different values of h_D^x . Black dotted lines $(h_D^x = \infty)$ show the values of m^x of the x-basis states (multiplied by a factor of 1.4 for visibility). For $h_D^x = 40$, clear step-like structures appear, indistinguishable from the steps of m^x for x-basis states for both system sizes L = 10, 14 (see Suppl. Mat. for finer details of L dependence of this matching). For a lower drive value $h_D^x = 18$, close to the threshold, the curve smoothes out, indicating weakening of the quasi-conservation, yet highly polarized Floquet states are still substantial in number. For still lower values (e.g. $h_D^x = 10$), the curve finally flattens. The pronounced asymmetry in the Floquet magnetizations for lower values of h_D^x is due to the small asymmetry in the drive. Right frame: The log of the number N_c of Floquet eigenstates with polarization above a given value m_c is shown to grows approximately exponentially with system size, corresponding to (a vanishing fraction of states but with a) finite entropy. For large h_D^x ($h_D^x = 40$), the numerical data points fall almost exactly on the analytically calculated (black dotted lines) corresponding to $h_D^x = \infty$ (see the matching of the step-like structures in the middle frame). For a lower value $h_D^x = 18$ a linear fit is done for the numerical data points.

corresponding part of the spectrum.

Complementary information can be gleaned by considering the correlations encoded in the non-ergodic states. The middle frame of Fig. 2 shows the magnetisation of different Floquet eigenstates, m_i^x , ordered according to their size. In the ergodic regime, these curves are featureless and m_i^x is uniformly tiny, showing a tendency to increase with increasing drive strength. Deep into the nonergodic regime, large values of m_i^x appear, which

together form plateaux. For the largest drives h_D^x , the plateaux correspond to essentially an integer number of spin flips, which indicates that the new basis is close to the computational basis in the x-direction mentioned above. As the drive is decreased, the plateaux give way to a smooth curve, which however still makes large excursion toward $m^x = \pm 1$ before assuming the featureless shape of the ergodic regime.

While the fraction of Floquet states with a magnetisa-

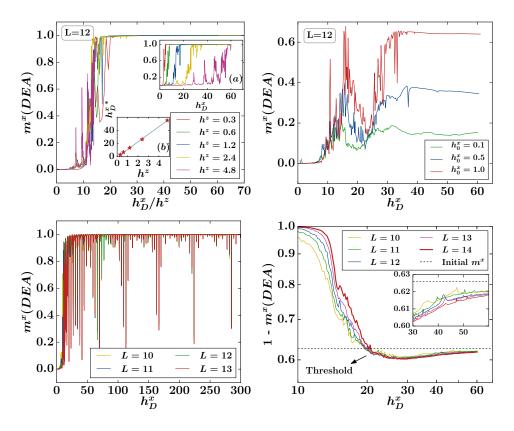


FIG. 3: Remnant magnetisation in various settings. Top left: Dependence of a DC the transverse field h^z which does not commute with the other, mutually commuting, terms of the model. h^z enhances thermalization (upper inset). The response approximately scales with h_D^x/h^z (main panel); in particular, the estimated threshold h_D^x is approximately proportional to h^z (lower inset). Top right: Robustness of freezing with respect to addition of DC field h_0^x . Bottom left: Freezing for uneven division of the total drive period. For $0 \le t < rT$, $h_D^x = +40$, while for $rT \le t < T$, $h_D^x = -40$, where r = 1/(Golden Ratio). Deep freezing minima persist to high driving strengths but show little size dependence. Bottom right: Behaviour for initial state chosen as the ground state of the non-integrable undriven part H_0 , with h^z and h_0^x chosen to created an initial state with a small positive polarization $m^x(0) \approx 0.37390$. For large h_D^x , freezing increases somewhat with L.

tion above a certain value is thermodynamically vanishing, their entropy is nonetheless finite, see Fig. 2, middle and right frames. This is analogous to the case of a finite-temperature ensemble of a magnet in a field, where a nonzero magnetisation arises as a thermodynamically vanishing fraction of magnetised states is preferentially populated, with their energy gain compensating for the entropy loss involved in concentrating the probability density on them. Here, the selection of the magnetised Floquet states arises via the state initialisation. It is interesting to note that in this 1D system there would be no magnetization at any finite temperature: the observation of a finite magnetization at finite energy density is purely a non-equilibrium effect.

2. Emergence of m^x as a local quasi-conserved quantity

We next address what we believe is the central feature underpinning the non-thermalisation, namely the existence of a conserved quantity in the drive Hamiltonian in isolation. In our example, this is the magnetisation in the x-direction, m^x , which persists as a quasi-conserved quantity even when the ratio of drive to static components of the Hamiltonian is finite.

The middle frame of Fig. 2 shows the value of m^x for the different Floquet eigenstates arranged by their size. For the strongest drives, the steps in this quantity are identical to the ones of the computational basis states in the x-basis, i.e. the steps simply reflect the number of spins flipped.

The static part of the Hamiltonian then mixes the states with the same value of m^x , which is reflected in the non-trivial distribution of the IPR of the Floquet states (left frame of Fig. 2). The growth of the size of each m^x sector (except for the fully polarised one) is in turn reflected in a decrease of the IPR.

For lower driving strengths, $h_D^x = 18$, the steps get washed out, but the range of m^x continues to span practically the full range in the interval between -1 and 1. This feature disappears below the threshold, $h_D^x = 10$, where the curve flattens substantially.

While the fraction of Floquet states with a non-zero magnetisation density vanishes with system size, these

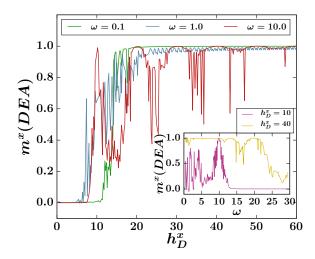


FIG. 4: Dependence of the remnant magnetisation on the drive frequency ω for L=14 initialised with the ground state of H(0). The basic morphology (in particular, the two regimes and the threshold) remains the same over two decades in frequency. The freezing decreases as ω is increased, with the threshold only varying slowly with ω . The inset shows m^x vs ω for $h_D^x=10$ (outside the frozen regime) and $h_D^x=40$, where the weakening of freezing with increasing ω is evident.

states nonetheless have nonzero entropy, Fig. 2 right panel, as is the case for magnetised states of a paramagnet generally.

The emergent quasi-conserved nature of m^x , along with the straightforward possibility of initialising the system in a magnetised state, acount for the main features of the results discussed in this work.

C. Robustness against variation of model and protocol parameters

We first address the existence of the onset for variants of the above model. We note that so far, no fine-tuning was necessary. The central demand was for the drive amplitude h_D^x to be the largest scale, while the other parameters of the Hamiltonian were chosen all to be in the same ballpark.

1. Role of non-commuting term

First, the location of the thermalisation threshold can be moved by varying the strength of the term in the static Hamiltonian H_0 which does not commute with the driving Hamiltonian H_D . Indeed, the top left frame of Fig. 3 shows that the threshold driving field is approximately proportional to the static transverse field strength h^z .

2. Drive shape and initial state

Also, we ask whether the 'symmetry' of having a vanishing mean drive of zero for symmetric pulse shapes about zero is an important ingredient. Fig 3, top right frame, shows that the freezing is quite robust to addition of a dc field of strength h_0^x . Indeed, the freezing actually grows with h_0^x .

Next, we consider a deviation of the drive protocol away from a time-symmetric switch in the sign of the driving term to one where more time is spent for one sign than the other (Fig. 3 bottom left frame). While the latter case has considerably more structure at high drives, in particular an apparently regular suppression of the remnant magnetisation even above the onset threshold, the former curve basically acts as a high-magnetisation envelope of the latter.

Further, we consider an initial state prepared as the ground-state of a many-body problem (rather than a more simply prepared polarised state). This displays (Fig. 3 bottom right frame) all the salient features observed with the simply polarized ground state in Fig. 1, right frame.

3. Drive frequency

What is particularly worth emphasizing is that the non-ergodic behaviour is *not* a high-frequency phenomenon. While such freezing also exists in the limit of a driving frequency in excess of the many-body bandwidth of the finite-size system, it is not even the case that the nonergodicity necessarily grows with frequency. This is illustrated in Fig. 4, where the remnant magnetisation is, if anything, more robust at small driving frequencies.

This is intriguing since at lower drive frequencies, Magnus-type high frequency expansions are divergent. Hence, this is an example of the breakdown of a Magnus expansion which is not associated with unbounded heating.

D. Finite-size behaviour

Our results indicate that absence of thermalization in this driven interacting system might persist even in the infinite-size limit. While there are some dips of the freezing strength in the nonergodic regime complicating a sharp identification of a threshold, the onset nonetheless appears to sharpen with increasing system-size. A closer view of the nonergodic regime, Fig. 1, middle frame top inset, shows smooth behaviour of the remnant magnetisation for the largest fields; this in fact grows with increasing system size. By contrast, for weak drives, the remnant magnetisation tends to decrease with system size. This results in a crossing pointas the curves for different system sizes of the deviation of the remnant magnetisation from its initial value, Fig. 1, right frame,

thus approximately cross at the threshold point. While it is hard entirely to rule out a slow drift to higher fields of the threshold with increasing system size, these observations suggest the possibility of a sharp transition at a finite threshold field in the thermodynamic limit.

Next, and most importantly, the step-structures in the m^x of the Floquet states are almost indistinguishable from that of the x-basis states for all system-sizes we investigated (Fig. 2, middle frame). This absence of system-size dependence indicates that at large values of h_D^x , the drive does not mix the x-basis states of different m^x values. A decrease in the fraction of Floquet states with $m^x > m_c$ with system-size is not because in larger systems the Floquet states are more delocalized between different magnetization sectors, but merely because the number of x-basis states in a given magnetization sector changes with the system-size. Delocalization between different magnetization sectors is suppressed strongly for all system-sizes at hand for h_D^x above the threshold. This is in keeping with the observations that on different types of initial states, Fig 1 and Fig.3, the freezing at the highest frequencies does not decrease with system size. and gives a further indication that our results are not merely finite-size effects.

IV. DISCUSSION

We have studied the onset of Floquet thermalisation in a driven interacting spin chain. We have found a fairly sharp threshold for the drive strength, above which Floquet thermalisation does not take place. The threshold value varies in different manners with parameters like pulse shape, drive frequency, or the (non-commuting) transverse field strength, but the freezing persists robustly under all these variations. The question of the existence of such a threshold is of fundamental importance, with a related issue appearing for classical dynamical systems, where the Kolmogorov-Arnold-Moser theorem deals with the onset of chaotic behaviour upon breaking of integrability.

An open question is the origin, and in particular the L-dependence, of the dips in the frozen component even beyond the threshold in the m^x vs h_D^x plots: the dips touching the x-axis correspond to points of thermalization. While their occurrence for certain discrete values of h_D^x has no significant consequence, if their number diverges with L, this may lead to a destruction of the frozen regime. For drives with pulse durations evenly placed about T/2, the dips disappear rapidly with increasing h_D^x . Such dips are, however, observed to persist even for very strong amplitudes for the case of drive with uneven division of the drive period (Fig. 3, bottom left). In this case, the total drive period is divided in two parts, T/GRand T(1-1/GR), where GR is the Golden ratio. While the depth of the dips seems to increase with L, their number and locations remain surprisingly independent of L, which points against their proliferation. Regarding an

extrapolation to the thermodynamic limit, we refer to our discussion at the end of the previous section.

Comparison of the magnetization and IPR of the Floquet states in the frozen regime allows one to conclude that the magnetization itself plays the role of a quasiconserved quantity, which becomes exactly conserved in the limit of infinitely strong driving. However, the emergence of only a single conserved quantity does not rule out non-trivial steady states, as can be gleaned from the structure of Floquet eigenstates in the frozen regime: these states have definite m^x values yet they are not fully localised in the x-basis. It is also interesting to note a single local conserved quantity like m^x does not preclude a non-local H_{eff} , yet is sufficient to result in a non-thermal Floquet spectrum.

While our driving term in isolation is integrable, it appears that the existence of a conserved quantity is all that is required for the existence of the frozen regime. A study of a non-integrable drive with an emergent conservation law is therefore an obvious item for future work.

This non-ergodicity is *not* a high-frequency phenomenon. Instead, it is particularly well-developed at lower driving frequencies, which a priori renders attempts to construct a Magnus-type high frequency expansion problematic. Instead, non-ergodicity is primarily associated with strong driving. Note that for the driving term in isolation, the instantaneous eigenvectors of the Hamiltonian are time-independent, while the instantaneous eigenvalues change; this suggests the development of a perturbation theory controlled by the instantaneous gap, rather than a high frequency. It would also be interesting to investigate the connections of this problem to the case of weakly driven interacting systems with approximate conservation laws³³.

The role of emergent conservation laws may in particular be important for experimental studies of driven manybody systems. Indeed, a first sighting of the physics we have analysed here has occurred in the context of an experiment of Floquet many-body localisation²², where the possibility of a finite threshold for delocalisation was also noted for the low-disorder limit. The main ingredient we have identified, an emergent conservation law, turns out to also have been present in that situation. Analogously, for the searches of time crystals taking place at present, it will be interesting to investigate if emergent conservation laws do, or can, play a role there as well.

Finally, while periodic driving is expected to heat a system and hence delocalize it, drive-induced destructive quantum interference can produce just the opposite effect. Competition of these might result in unexpected freezing behaviour, as has been observed in quantum counterpart of classically chaotic systems, namely, in the kicked rotators (see, e.g., ³⁴). Such a suppression of heating ³⁵ might not be impossible in a quantum many-body system where interactions lead to ergodicity. An absence of unbounded heating under periodic driving could be a step in that direction, and the availability of emergent approximate conservation laws may turn out to be a use-

ful ingredient for many-body Floquet engineering.

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V. SUPPLEMENTAL MATERIAL

A. Ergodicity of H_0

We first demonstrate the ergodicity of the undriven Hamiltonian H_0 for the parameter values we have used in the main text. This is displayed in Fig. 5.

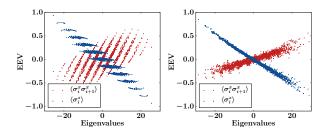


FIG. 5: Left frame shows expectation values of different local operators over the eigenstates (EEV) of the integrable transverse field Ising chain ($\kappa=h^x=0$) against the eigenvalues of those states (L=12). The non-smooth behaviour arises due to the existence of different non-mixing (block-diagonal) sectors. Right frame shows that introduction of the interaction and longitudinal field breaks the integrability and the EEV's become smooth, confirming that the undriven Hamiltonian H_0 is quite generic and satisfies the eigenstate thermalization hypothesis. The paremeters $\kappa=0.7$ and $h^x=1$ correspond to those in the main text.

B. Freezing and Thermalization via Level Statistics

Here we demonstrate the quasi-energy level repulsion and its absence for drive strengths above and below the threshold respectively. We plot the quasi-energy (folded to the first Brillouin zone) gap ratio defined by

$$r = \frac{\min\{\delta_n, \delta_{n-1}\}}{\max\{\delta_n, \delta_{n-1}\}},$$

where, $\delta_n = \mu_{n+1} - \mu_n$, μ_n being the *n*-th eigenvalue of H_{eff} , after folding them into the first Brillouin zone $[-\pi, \pi]$.

C. Quasi-energy Gap-ratio Statistics below and above the threshold:

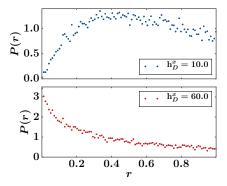


FIG. 6: Gap-ratio statistics r for $h_D^x = 10$ (below the threshold) and 60 (far above the threshold) respectively, showing presence and absence of quasi-energy level repulsion below and above the threshold respectively.

D. Threshold Phenomenon with Random Polarized Initial States

Fig. 7 shows the remanent magnetisation for a class of random polarised initial states.

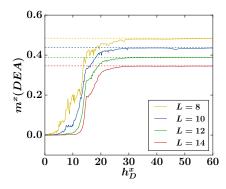


FIG. 7: Freezing for initially polarized but otherwise randomised states under longitudinal drive (for the same set of parameter as in the main text). These initial states are given as $|\psi(0)\rangle = \sum_{i=1}^{2^L} c_i |i_x\rangle$, where $|i_x\rangle$ is the i-th eigenstate in the computational basis states in x-direction, by choosing $Re[c_i]$ and $Im[c_i]$ from a uniform distribution between -1 and +1, multiplying them by $e^{\beta m_i^x}$, where $\beta>0$, where m_x^i is the longitudinal magnetization of $|i_x\rangle$, and finally normalizing the state. Results are shown for different L values, for random instances generated with $\beta=2.5$ (for which the initial magnetization decreases with L.) The system freezes for large values of the drive field h_D^x for all accessed system-sizes we could access, with the remanent magnetisation very close to the diagonal ensemble average (DEA) of the longitudinal magnetization m^x .

VI. L-DEPENDENCE OF THE FLOQUET STATE AVERAGE MAGNETIZATION

Fig. 2 (middle panel) of the main text shows that the step-structures in the m^x of the Floquet states are almost indistinguishable from that of the x-basis states for all L we investigated. Here we consider the following average quantity to show on a finer scale, that this difference systematically decreases with L. We order both the x-basis states and the Floquet states in order (decreasing, say) of their m^x values (eigenvalues and expectation values in respectively). Let $m_{h_{\mathcal{D}}}^{x}(i)$ and $m_{\infty}^{x}(i)$ denotes the m^x values of the *i*-th state thus ordered in respective basis. Now we compute the difference $|m_{\infty} - m_{h_{D}^{x}}|$, where $m_{\infty} = \frac{1}{2^L} \sum_i m_{\infty}^x(i)$ and $m_{h_D^x} = \frac{1}{2^L} \sum_i m_{h_D^x}^x(i)$. This provides a measure of the accuracy with which m^x is conserved (this difference vanishes if m^x is exactly conserved, since in that case each Floquet state corresponds to an exact eigenstate of m^x). In Fig. 8 we show that

this difference is tiny, and seems, if anything, to decrease with increasing L.

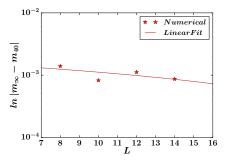


FIG. 8: L—dependence of the log of the deviation of the average of the magnetization of the Floquet states from the corresponding x—basis states