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Quantum discord in photon added Glauber coherent states of GHZ-type

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Abstract

We investigate the influence of photon excitations on quantum correlations in tripartite Glauber coherent states of Greenberger-Horne-Zeilinger type. The pairwise correlations are measured by means of the entropy-based quantum discord. We also analyze the monogamy property of quantum discord in this class of tripartite states in terms of the strength of Glauber coherent states and the photon excitation order.

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1 Introduction

In the context of information processing and transmission, several theoretical and experimental results confirm the advantages of quantum protocols compared to their classical counterparts (see for instance [1, 2, 3]). Quantum technology exploiting the intriguing phenomena of quantum world, such as entanglement, offers secure ways for communication [4, 5] and potentially powerful algorithms in quantum computation [6]. Originally, quantum information processing focused on discrete (finite-dimensional) entangled states like the polarizations of a photon or discrete levels of an atom. But, the extension from discrete to continuous variables has been also proven beneficial in coding and manipulating efficiently quantum information. Coherent states, which constitute the prototypical instance of continuous-variables states, are expected to play a central role in this context. They are appealing for their mathematical elegance (continuity and over-completion property) and closeness to classical physical states (minimization of Heisenberg uncertainty relation). Implementing a logical qubit encoding by treating entangled coherent states as qubits in a two dimensional Hilbert space has been shown a promising strategy in performing successfully various quantum tasks such as quantum teleportation [7, 8], quantum computation [9, 10, 11], entanglement purification [12] and errors correction [13]. In view of these potential applications, a special attention was paid, during the last years, to the identification, characterization and quantification of quantum correlations in bipartite coherent states systems (see for instance the papers [14, 15, 16] and references therein). The bipartite treatment was extended to superpositions of multimode coherent states [17, 18, 19, 20, 21] which exhibit multipartite entanglement. One may quote for instance entanglement properties in GHZ (Greenberger-Horne-Zeilinger), W (Werner) states discussed in [22, 23] and entangled coherent state extensions of cluster qubits investigated in [24, 25, 26]. To quantify quantum correlations beyond entanglement in coherent states systems, measures such as quantum discord [27, 28] and its geometric variant [29] were used. Explicit results were derived for quantum discord [30, 31, 32, 33, 34, 35, 36, 37] and geometric quantum discord [38, 39, 40, 41] for some special sets of coherent states.

On the other hand, decoherence is a crucial process to understand the emergence of classicality in quantum systems. It describes the inevitable degradation of quantum correlations due to experimental and environmental noise. Various decoherence models were investigated and in particular the phenomenon of entanglement sudden death was considered in a number of distinct contexts (see for instance [42] and reference therein). For optical qubits based on coherent states, the influence of the environment, is mainly due to energy loss or photon absorption. The photon loss or equivalently amplitude damping in a noisy environment can be modeled by assuming that some of field energy and information is lost after transmission through a beam splitter [36, 43]. Interestingly, it has been shown that a beam spitting device with a coherent in the first input and a number state in the second input generates photon-added coherent states [44]. Henceforth, understanding the influence of photon excitations might be useful to develop the adequate strategies in improving the performance of noise reduction in quantum processing protocols involving coherent states. In this sense, some authors considered the concurrence as measure of the entanglement in bipartite and tripartite photon added coherent states [23, 45].

In this work, we derive the analytical expression of pairwise quantum discord in a three modes system initially prepared in a tripartite Glauber coherent state of GHZ-type. In particular, we shall consider the influence of photon excitation of a single mode on the dynamics of pairwise quantum correlations. Mathematically, this process is represented by the action of a suitable creation operator on the state of the first subsystem. Another important issue in photon added GHZ-type coherent states concerns the distribution of quantum discord between the different parts of the whole system. In fact, we study the shareability of quantum correlations which obeys a restrictive inequality termed in the literature as the monogamy property [46] (see also [47, 48, 49, 50, 51, 52]).

This paper is organized as follows. In section 2, basic definitions and equations related to photon added coherent states are presented. We also consider the quantum correlations as measured by the entanglement of formation in quasi-Bell states. In particular, we introduce an encoding map to pass from continuous variables (coherent states) to discrete variables (logical quantum bits). Along the same line of reasoning, this qubit encoding is extended, in section 3, to tripartite photon added coherent states of GHZ-type. The pairwise quantum discord quantifying the amount of quantum correlations existing in the system is analytically derived. In section 4, we study the monogamy property of quantum discord. Numerical illustrations of the monogamy inequality are presented in some special cases. Concluding remarks close this paper.

2 Entanglement in photon added quasi-Bell states

2.1 Photon added coherent states and qubit mapping

The basic objects in this work are the Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ where α is a complex number which determines the coherent amplitude of the electromagnetic field. Mathematically, a single-mode quantized radiation field is represented by the harmonic oscillator algebra spanned by the creation a^+ and annihilation a^- operators. The process of adding m photons to coherent states of type $|\alpha\rangle$ and $|-\alpha\rangle$ is usually represented by the action of the operator $(a^+)^m$ (m is a non negative integer) [53]. Several experimental as well theoretical studies were devoted to the generation and nonclassical properties of photon-added coherent states [54] (for a recent review see [55]). Explicitly, m successive actions of creation operator a^+ on the Glauber coherent states

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{1}$$

lead to the un-normalized states

$$||\alpha, m\rangle = \left(a^{+}\right)^{m} \mid \alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \sqrt{(n+m)!} \mid n+m\rangle. \tag{2}$$

The vectors $|n\rangle$ denote the Fock-Hilbert states of the harmonic oscillator. The normalized m-photon added coherent states are defined by

$$|\alpha, m\rangle = \frac{(a^{+})^{m}|\alpha\rangle}{\sqrt{\langle\alpha|(a^{-})^{m}(a^{+})^{m}|\alpha\rangle}}, \tag{3}$$

where the quantity

$$\langle\alpha|(a^{-})^{m}(a^{+})^{m}|\alpha\rangle = m! L_m(-|\alpha|^2), \tag{4}$$

involves the Laguerre polynomial of order m defined by

$$L_m(x) = \sum_{n=0}^{m} \frac{(-1)^n m! x^n}{(n!)^2 (m-n)!}. \tag{5}$$

Photon added coherent states interpolate between electromagnetic field coherent states (quasi-classical states) and Fock states $|n\rangle$ (purely quantum states). Furthermore, they exhibit non-classical features such as squeezing, negativity of Wigner distribution and sub Poissonian statistics [55]. Their experimental generation using parametric down conversion in a nonlinear crystal was reported in [54]. Photon-coherent states $|\alpha, m\rangle$ and $|-\alpha, m\rangle$, of the same amplitude and phases differing by π, are not orthogonal to each other. Indeed using the expression

$$\langle-\alpha|(a^{-})^{m}(a^{+})^{m}|\alpha\rangle = e^{-2|\alpha|^2} m! L_m(|\alpha|^2), \tag{6}$$

it is simply verified that the overlap between the two states is

$$\langle-\alpha, m|\alpha, m\rangle = e^{-2|\alpha|^2} \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{7}$$

Considering the nonorthogonality property (7), the identification of photon added coherent states $|\alpha, m\rangle$ and $|-\alpha, m\rangle$ as basis of a logical qubit is only possible for $|\alpha|$ large ($|\alpha| \geq 2$). Alternatively, the Schrödinger cat states, the even and odd coherent states, can be used to encode a qubit. Indeed, based on the encoding scheme proposed in [11], we introduce a two dimensional basis spanned by the orthogonal qubits $|+, m\rangle$ and $|-, m\rangle$ defined by

$$|\pm, m\rangle = \frac{1}{\sqrt{2 \pm 2\kappa_m e^{-2|\alpha|^2}}} \left(|\alpha, m\rangle \pm |-\alpha, m\rangle\right) \tag{8}$$

where

$$\kappa_m \equiv \kappa_m(|\alpha|^2) := \frac{L_m(|\alpha|^2)}{L_m(-|\alpha|^2)}. \tag{9}$$

Clearly, for $m = 0$, one has $\kappa_0 = 1$ and the logical qubits (8) reduce to

$$|\pm\rangle = \frac{1}{\sqrt{2 \pm 2e^{-2|\alpha|^2}}} \left(|\alpha\rangle \pm |-\alpha\rangle\right), \tag{10}$$

which coincide with even and odd Glauber coherent states providing the qubit encoding scheme introduced in [11]. This qubit encoding is important in dealing with quantum correlation in photon added coherent states and to investigate the influence of the photon excitations processes. To illustrate this, we shall first consider the entanglement in quasi-Bell states which are very interesting in quantum optics and serve as valuable resource for quantum teleportation and many other quantum computing operations. The quasi-Bell states

$$|\mathrm{B}_k(\alpha)\rangle = \mathcal{N}_k(\alpha)\ \big[|\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}| - \alpha\rangle \otimes | - \alpha\rangle\big], \tag{11}$$

with $k = 0 \pmod 2$ (resp. $k = 1 \pmod 2$) stands for even (resp. odd) quasi-Bell states and the normalization factor $\mathcal{N}_k(\alpha)$ is

$$\mathcal{N}_k^{-2}(\alpha) = 2 + 2e^{-4|\alpha|^2} \cos k\pi. \tag{12}$$

By repeated actions of the creation operator on the first mode, the resulting excited quasi-Bell states are

$$||\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha)\ \Big[\big[(a^+)^m \otimes \mathbb{I}\big]|\alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}\ \big[(a^+)^m \otimes \mathbb{I}\big]| - \alpha\rangle \otimes | - \alpha\rangle\Big] \tag{13}$$

are un-normalized ($\mathbb{I}$ stands for the unity operator). Using the norm of the vectors $||\mathrm{B}_k(\alpha, m)\rangle$ given by

$$\langle \mathrm{B}_k(\alpha, m)||\mathrm{B}_k(\alpha, m)\rangle = m!\ \frac{L_m(-|\alpha|^2) + e^{-4|\alpha|^2} L_m(|\alpha|^2)\ \cos k\pi}{1 + e^{-4|\alpha|^2} \cos k\pi}, \tag{14}$$

we introduce the normalized photon-added quasi-Bell states as

$$|\mathrm{B}_k(\alpha, m)\rangle = \frac{||\mathrm{B}_k(\alpha, m)\rangle}{\sqrt{\langle \mathrm{B}_k(\alpha, m)||\mathrm{B}_k(\alpha, m)\rangle}}. \tag{15}$$

They can be rewritten as

$$|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m)\ \big[|m, \alpha\rangle \otimes |\alpha\rangle + e^{ik\pi}\ |m, -\alpha\rangle \otimes | - \alpha\rangle\big], \tag{16}$$

in terms of the normalized photon added coherent state (3). The normalization factor in (16) is

$$\mathcal{N}_k^{-2}(\alpha, m) = 2 + 2\kappa_m e^{-4|\alpha|^2} \cos k\pi, \tag{17}$$

which reduces for $m = 0$ to (12) and the quasi-Bell states (11) are recovered.

2.2 Dynamics of the entanglement of formation under photon excitation

Using the qubit mapping (8) for the first mode and (10) for the second, the bipartite state (16) is converted in the two qubit state

$$|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m) \sum_{i=\pm} \sum_{j=\pm} C_{ij}\ |i, m\rangle \otimes |j\rangle, \tag{18}$$

where the vectors $|i, m\rangle$ (resp. $|j\rangle$) are defined by (8) (resp. (10)) and the expansion coefficients are given by

$$C_{++} = c_m^+ c^+(1 + e^{ik\pi}), \quad C_{-+} = c^+ c_m^-(1 - e^{ik\pi}), \quad C_{+-} = c_m^+ c^-(1 - e^{ik\pi}), \quad C_{--} = c^- c_m^-(1 + e^{ik\pi}),$$

with

$$c_m^{\pm} = \sqrt{\frac{1 \pm \kappa_m e^{-2|\alpha|^2}}{2}} \qquad c^{\pm} = \sqrt{\frac{1 \pm e^{-2|\alpha|^2}}{2}}.$$

In a pure bipartite system, the quantum discord coincides with entanglement of formation (see for instance [30, 31, 32]). Thus, to discuss the effect of the photon excitations of quasi-Bell states (16), we quantify the quantum correlations by means of the entanglement of formation. We recall that for ρ_{12} the density matrix for a pair of qubits 1 and 2 which may be pure or mixed, the entanglement of formation is defined by [56]

$$E(\rho_{12}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{12})|^2}), \tag{19}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. The concurrence $C(\rho_{12})$ is given by

$$C(\rho_{12}) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \tag{20}$$

for $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ the square roots of the eigenvalues of the "spin-flipped" density matrix $\varrho_{12} \equiv \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12}^{\star}(\sigma_y \otimes \sigma_y)$ where the star stands for complex conjugation and σ_y is the usual Pauli matrix. In the state (16), it easy to check that the concurrence (20) gives

$$C_{12} = 2\mathcal{N}_k^2(\alpha, m)|C_{++}C_{--} - C_{+-}C_{-+}|, \tag{21}$$

which rewrites explicitly as

$$C_{12} = \frac{\sqrt{1 - e^{-4|\alpha|^2}}\sqrt{1 - \kappa_m^2 e^{-4|\alpha|^2}}}{1 + \kappa_m e^{-4|\alpha|^2}\cos k\pi} \tag{22}$$

in terms of the coherent states amplitude $|\alpha|$ and the excitation order m. This result coincides with one obtained in [45] using a different qubit encoding. It follows that entanglement of formation is

$$E_{12} \;=\; H\left[\frac{1}{2} + \frac{e^{-2|\alpha|^2}(1 + \kappa_m \cos k\pi)}{2 + 2\kappa_m e^{-4|\alpha|^2}\cos k\pi}\right]. \tag{23}$$

For $m = 0$, one has

$$C_{12} = \frac{1 - e^{-4|\alpha|^2}}{1 + e^{-4|\alpha|^2}\cos k\pi}. \tag{24}$$

To illustrate the influence of the photon excitation on the quantum correlation between the modes of the quasi-Bell state (11), we report in the figures 1 and 2 the behavior of the entanglement of formation E_{12} (23) versus Glauber coherent states amplitude $|\alpha|^2$ and the overlap $p = \langle\alpha| - \alpha\rangle = e^{-2|\alpha|^2}$ for different values of m. We note that for $|\alpha|$ large ($|\alpha|^2 \geq 1.5$), the entanglement of formation tends to unit independently of the number of added photons m. Indeed, from equation (23), one gets $E_{12} = 1$ for $|\alpha| \longrightarrow \infty$. Note that, in this limit, the Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ tends to orthogonality and an orthogonal basis can be constructed such that $|\mathbf{0}\rangle \equiv |\alpha\rangle$ and $|\mathbf{1}\rangle \equiv |-\alpha\rangle$. Thus, in the strong regime $|\alpha| \longrightarrow \infty$, the quasi-Bell states (11) become maximally entangled

$$|\mathrm{B}_k(\infty)\rangle = \frac{1}{\sqrt{2}}\left[|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + e^{ik\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle\right].$$

Subsequently, maximally entangled quasi-Bell states are robust against any photon addition process. Another interesting limiting situation concerns quasi-Bell states with smaller values of α (weak regime). For $\alpha \longrightarrow 0$, the symmetric ($k = 0$ (mod 2))-quasi-Bell state (11) reduces to the separable state $|0\rangle \otimes |0\rangle$ and by adding m photons it becomes $|m\rangle \otimes |0\rangle$. No quantum correlation is created by the photon excitation ($E_{12} = 0$). This result can be also obtained from (23) for $|\alpha| \longrightarrow 0$. As depicted in the figure 2, the situation is completely different for anti-symmetric quasi-Bell states ($k = 1$ (mod 2)) (11). For α approaching zero, the entanglement of formation decreases as the photon excitation number m increases. For $|\alpha| \longrightarrow 0$, the Laguerre polynomial (5) can be approximated by $L_m(|\alpha|^2) \simeq 1 - m|\alpha|^2$ and the quantity (9) writes

$$\kappa_m(|\alpha|^2) \simeq 1 - 2m|\alpha|^2. \tag{25}$$

Reporting (25) in (23), one gets

$$E_{12}(\mathrm{B}_1(0, m)) \simeq H\left(\frac{m+1}{m+2}\right). \tag{26}$$

It is interesting to note that in the situation when $|\alpha| \longrightarrow 0$, the anti-symmetric quasi-Bell states (11) reduce to the maximally entangled two qubit state of W-type

$$|\mathrm{B}_1(0)\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle\right] \tag{27}$$

where $|0\rangle$ and $|1\rangle$ denote the Fock number states. The action of the operator $(a^+)^m$ on the state $|\mathrm{B}_1(0)\rangle$ gives

$$|\mathrm{B}_1(0, m)\rangle = \frac{1}{\sqrt{m+2}} \left[|m\rangle \otimes |1\rangle + \sqrt{m+1}|m+1\rangle \otimes |0\rangle\right].$$

In this case, the concurrence is

$$C_{12}(\mathrm{B}_1(0, m)) = 2\, \frac{\sqrt{m+1}}{m+2}, \tag{28}$$

which agrees with the result (26). Clearly, adding photons to maximally entangled states of W-type (27) diminishes the amount of pairwise quantum correlations. For the intermediate regime, corresponding to $|\alpha|^2$ ranging between 0 and 1.5, the entanglement of formation increases as the Glauber coherent state amplitude α increases. We note that adding photon process induces a quick activation of the creation of quantum correlation for the symmetric quasi-Bell states (figure 1). Similarly, the results presented in figure 2 show that increasing the amplitude of anti-symmetric quasi-Bell states tends to compensate the quantum correlation loss due to photon excitations in states of W type.

3 Pairwise quantum correlations in excited quasi-GHZ coherent states

3.1 Photon added quasi-GHZ coherent states

The analysis and results derived in the previous section are useful in investigating pairwise quantum correlations in tripartite states involving Glauber coherent states. In this respect, we consider the quasi-GHZ coherent states defined by

$$|\mathrm{GHZ}_k(\alpha)\rangle = \mathcal{C}_k(\alpha)(|\alpha, \alpha, \alpha\rangle + e^{ik\pi}| - \alpha, -\alpha, -\alpha\rangle). \tag{29}$$

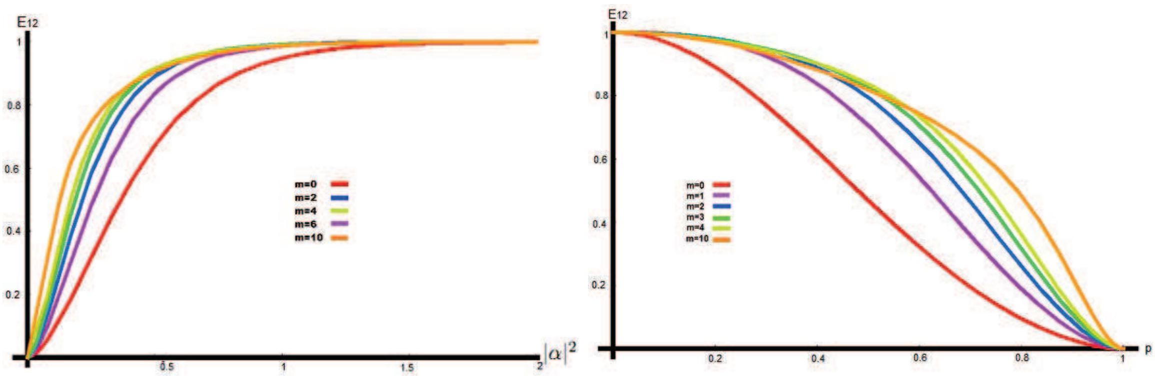

Figure 1. The entanglement of formation E_{12} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 0$ and different values of photon excitation number m.

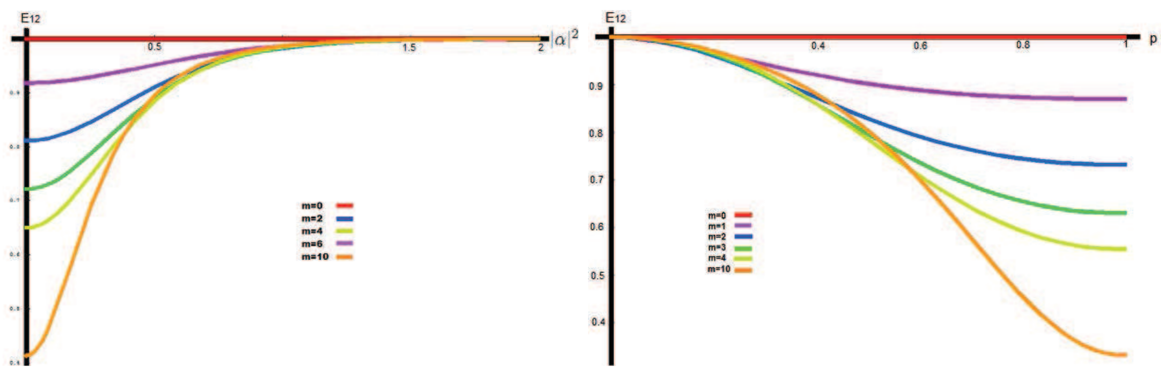

Figure 2. The entanglement of formation E_{12} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 1$ and different values of photon excitation number m.

where the normalization constant $\mathcal{C}_k$ is given by

$$\mathcal{C}_k^{-2}(\alpha) = 2 + 2e^{-6|\alpha|^2} \cos k\pi. \tag{30}$$

The excitation of the first mode by adding m photon leads to the tripartite state

$$||\mathrm{GHZ}_k(\alpha, m)\rangle = ((a^+)^m \otimes \mathbb{I} \otimes \mathbb{I})\ |\mathrm{GHZ}_k(\alpha)\rangle, \tag{31}$$

from which we introduce the normalized photon added quasi-GHZ coherent states as

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \frac{||\mathrm{GHZ}_k(\alpha, m)\rangle}{\sqrt{\langle \mathrm{GHZ}_k(\alpha, m)||\mathrm{GHZ}_k(\alpha, m)\rangle}}. \tag{32}$$

Using the expressions (4) and (6), the vector (32) rewrites as

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \mathcal{C}_k(\alpha, m)(|m, \alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle, + e^{ik\pi}|m, -\alpha\rangle \otimes |-\alpha\rangle \otimes |-\alpha\rangle). \tag{33}$$

where the normalization factor is

$$\mathcal{C}_k^{-2}(\alpha, m) = 2 + 2\kappa_m e^{-6|\alpha|^2} \cos k\pi. \tag{34}$$

For $m = 0$, the state $|\mathrm{GHZ}_k(\alpha, m)\rangle$ (33) reduces to $|\mathrm{GHZ}_k(\alpha)\rangle$ (29). It is also important to note that for $|\alpha|$ large, the overlap between Glauber coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ approaches zero and then they are quasi-orthogonal. In this case, the state $|\mathrm{GHZ}_k(\alpha)\rangle$ (29) reduces to the usual GHZ three qubit state

$$|\mathrm{GHZ}_k(\infty)\rangle = \frac{1}{\sqrt{3}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + e^{ik\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle), \tag{35}$$

where $|\mathbf{0}\rangle \equiv |\alpha\rangle$ and $|\mathbf{1}\rangle \equiv |-\alpha\rangle$.

In investigating the pairwise quantum discord in a tripartite system $1-2-3$ prepared in the state $|\mathrm{GHZ}_k(\alpha, m)\rangle$, one needs the reduced density matrices describing the two qubit subsystems $1-2$, $2-3$ and $1-3$. Since only the first mode is affected by the photon excitations, it is simply seen that the reduced density matrices $\rho_{12} = \mathrm{Tr}_3\rho_{123}$ and $\rho_{13} = \mathrm{Tr}_2\rho_{123}$ are identical. The pure three mode density matrix ρ_{123} is given

$$\rho_{123} = |\mathrm{GHZ}_k(\alpha, m)\rangle\langle\mathrm{GHZ}_k(\alpha, m)|. \tag{36}$$

After some algebra, the reduced density matrices ρ_{12} and ρ_{13} can be written as

$$\rho_{12} = \rho_{13} = \frac{\mathcal{C}_k^2(\alpha, m)}{\mathcal{N}_k^2(\alpha, m)}\left[\left(\frac{1+e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha, m)\rangle\langle\mathrm{B}_k(\alpha, m)| + \left(\frac{1-e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha, m)\rangle\langle\mathrm{B}_k(\alpha, m)|Z\right] \tag{37}$$

in terms of photon added quasi-Bell states (16). The operator Z is the third Pauli generator defined by

$$Z|\mathrm{B}_k(\alpha, m)\rangle = \mathcal{N}_k(\alpha, m)[|m, \alpha) \otimes |\alpha\rangle - e^{ik\pi}|m, -\alpha) \otimes |-\alpha\rangle].$$

Similarly, by tracing out the first mode, the reduced matrix density ρ_{23} takes the form

$$\rho_{23} = \frac{\mathcal{C}_k^2(\alpha, m)}{\mathcal{N}_k^2(\alpha, 0)}\left[\left(\frac{1+\kappa_m e^{-2|\alpha|^2}}{2}\right)|\mathrm{B}_k(\alpha, 0)\rangle\langle\mathrm{B}_k(\alpha, 0)| + \left(\frac{1-\kappa_m e^{-2|\alpha|^2}}{2}\right)Z|\mathrm{B}_k(\alpha, 0)\rangle\langle\mathrm{B}_k(\alpha, 0)|Z\right]. \tag{38}$$

To derive the pairwise correlation between the components of the subsystems $1-2$, $2-3$ and $1-3$, we assume that the information is encoded in even and odd Glauber coherent states (Schrödinger cat states). In this sense, we introduce for the first mode the following qubit mapping

$$|m, \pm\alpha\rangle = \sqrt{\frac{1+\kappa_m e^{-2|\alpha|^2}}{2}}\,|0\rangle_1 \pm \sqrt{\frac{1-\kappa_m e^{-2|\alpha|^2}}{2}}\,|1\rangle_1. \tag{39}$$

This coincides with the encoding scheme (8) introduced in the previous section to study the entanglement in quasi-Bell states. For the second and third modes, we consider the qubits defined by

$$|\pm\alpha\rangle = \sqrt{\frac{1+e^{-2|\alpha|^2}}{2}}\,|0\rangle_i \pm \sqrt{\frac{1-e^{-2|\alpha|^2}}{2}}\,|1\rangle_i, \qquad i = 2, 3. \tag{40}$$

Substituting (39) and (40) in (37) (resp. (38)), one can express the density matrix ρ_{12} (resp. ρ_{23}) in the two qubit basis $\{|0\rangle_1 \otimes |0\rangle_2, |0\rangle_1 \otimes |1\rangle_2, |1\rangle_1 \otimes |0\rangle_2, |1\rangle_1 \otimes |1\rangle_2\}$ (resp. $\{|0\rangle_2 \otimes |0\rangle_3, |0\rangle_2 \otimes |1\rangle_3, |1\rangle_2 \otimes |0\rangle_3, |1\rangle_2 \otimes |1\rangle_3\}$). The resulting density matrices have non-vanishing entries only along the diagonal and the anti-diagonal.

3.2 Bipartite measures of quantum discord

The state $|\mathrm{GHZ}_k(\alpha, m)\rangle$ (33) has rank two reduced density matrices (37) and (38). For these two qubit states, the Koashi-Winter relation which provides the connection between the quantum discord and the entanglement of formation, can be exploited to obtain the relevant pairwise quantum correlations. It is important to note that for two qubit states with rank larger than two, the derivation of quantum discord involves optimization procedures that are in general complicated to achieve analytically.

The total correlation in the subsystem $1-2$ comprising the optical modes 1 and 2, is quantified by the mutual information

$$I_{12} = S_1 + S_2 - S_{12}, \tag{41}$$

where S_{12} is the von Neumann entropy of the quantum state ρ_{12} (37) ($S(\rho) = -\mathrm{Tr}\,\rho\ln\rho$) and S_1 (resp. S_2) is the entropy of the reduced state $\rho_1 = \mathrm{Tr}_2(\rho_{12})$ (resp. $\rho_2 = \mathrm{Tr}_1(\rho_{12})$) of the mode 1(resp. 2). The mutual information I_{12} contains both quantum and classical correlations. The classical correlations C_{12} can be determined by a local measurement optimization procedure. To remove the measurement dependence, a maximization over all possible measurements is performed and the classical correlation writes

$$C_{12} = S_2 - \widetilde{S}_{\min}, \tag{42}$$

where $\widetilde{S}_{\min}$ denotes the minimal value of the conditional entropy [57, 58] (for more details, see the recent review [59]). Thus, the quantum discord, defined as the difference between total correlation I_{12} and classical correlation C_{12} [57, 58], writes

$$D_{12} = I_{12} - C_{12} = S_1 + \widetilde{S}_{\min} - S_{12}. \tag{43}$$

The main difficulty in deriving the analytical expression of bipartite quantum discord (43), in arbitrary mixed state, arises in the minimization process of conditional entropy. This explains why the explicit expressions of quantum discord were obtained only for few exceptional two-qubit quantum states, especially ones of rank two. One may quote for instance the results obtained in [31, 32] (see also [36, 37, 41]). Since the density matrix ρ_{12} (37) is of rank two, the derivation of the analytical expression of $\widetilde{S}_{\min}$ in equation (42), can be performed by purifying the density matrix ρ_{12} and making use of Koashi-Winter relation [60] (see also [33]). This relation establishes the connection between the classical correlation of a bipartite state ρ_{12} and the entanglement of formation E_{23} of its complement ρ_{23} in the pure state ρ_{123} (36). The minimal value of the conditional entropy coincides with the entanglement of formation of ρ_{23} [60]:

$$\widetilde{S}_{\min} = E_{23}. \tag{44}$$

The Koaschi-Winter relation and the purification procedure provide us with a computable expression of quantum discord in the bipartite state ρ_{12}

$$D_{12} = S_1 - S_{12} + E_{23} \tag{45}$$

when the measurement is performed on the subsystem 1. The von Neumann entropy of the reduced density $\rho_1 = \mathrm{Tr}_2\rho_{12}$ is

$$S_1 = H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-2|\alpha|^2})(1+e^{-4|\alpha|^2}\ \cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg), \tag{46}$$

and the entropy of the bipartite density ρ_{12} is explicitly given by

$$S_{12} = H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)(1+e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg). \tag{47}$$

It is important to note that the entanglement of formation measuring the entanglement of the subsystem 2 with the ancillary qubit, required in the purification process to minimize the conditional entropy, is exactly the entanglement of formation measuring the degree of intricacy between the optical modes 2 and 3. Using the definition of Wootters concurrence (20), one gets

$$C_{23} = \kappa_m e^{-2|\alpha|^2}\ \frac{(1-e^{-4|\alpha|^2})^2}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi} \tag{48}$$

and subsequently the corresponding entanglement of formation writes

$$E_{23} = H\bigg(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{\kappa_m^2 e^{-4|\alpha|^2}(1-e^{-4|\alpha|^2})^2}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\bigg). \tag{49}$$

Reporting (46), (47) and (49) in (45), the quantum discord in the state ρ_{12} is explicitly given by

$$\begin{aligned} D_{12} &= H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-2|\alpha|^2})(1+e^{-4|\alpha|^2}\ \cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg) \\ &- H\bigg(\frac{1}{2}\frac{(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)(1+e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg) \\ &+ H\bigg(\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{\kappa_m^2 e^{-4|\alpha|^2}(1-e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\bigg), \end{aligned} \tag{50}$$

The pairwise quantum discord existing in the mixed states ρ_{23} (38) can be computed along the same procedure. As result, when the measurement is performed on the subsystem 2, the quantum discord is

$$D_{23} = S_2 - S_{23} + E_{13}. \tag{51}$$

The von Neumann entropy of the reduced density $\rho_2 = \mathrm{Tr}_1\rho_{12}$ is

$$S_2 = H\bigg(\frac{1}{2}\frac{(1+e^{-2|\alpha|^2})(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg), \tag{52}$$

and the entropy of the bipartite density ρ_{23} is

$$S_{23} = H\bigg(\frac{1}{2}\frac{(1+e^{-4|\alpha|^2}\cos k\pi)(1+\kappa_m e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\bigg). \tag{53}$$

In purifying the state ρ_{23} to derive the minimal amount of conditional entropy, it is simple to show here also that the entanglement of formation measuring the entanglement of the mode 3 with an ancillary qubit, is exactly the entanglement of formation measuring the degree of intricacy between the optical modes 1 and 3. From (20), the concurrence between the modes 1 and 3 takes the following form

$$C_{13} = e^{-2|\alpha|^2} \frac{\sqrt{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}, \tag{54}$$

from which one gets

$$E_{13} = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{e^{-4|\alpha|^2}(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\right). \tag{55}$$

Finally, the expression of quantum discord in the state ρ_{23} is

$$\begin{aligned} D_{23} &= H\left(\frac{1}{2}\frac{(1+e^{-2|\alpha|^2})(1+\kappa_m e^{-4|\alpha|^2}\cos k\pi)}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\ &- H\left(\frac{1}{2}\frac{(1+e^{-4|\alpha|^2}\cos k\pi)(1+\kappa_m e^{-2|\alpha|^2})}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right) \\ &+ H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{e^{-4|\alpha|^2}(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-4|\alpha|^2})}{(1+\kappa_m e^{-6|\alpha|^2}\cos k\pi)^2}}\right). \end{aligned} \tag{56}$$

3.3 Some special cases

In order to analyze the influence of the photon excitation number m on the bipartite quantum discord D_{12} (50) and D_{23} (56), we first give the figures 3 and 4 representing respectively D_{12} and D_{23} for symmetric states ($k = 0$).

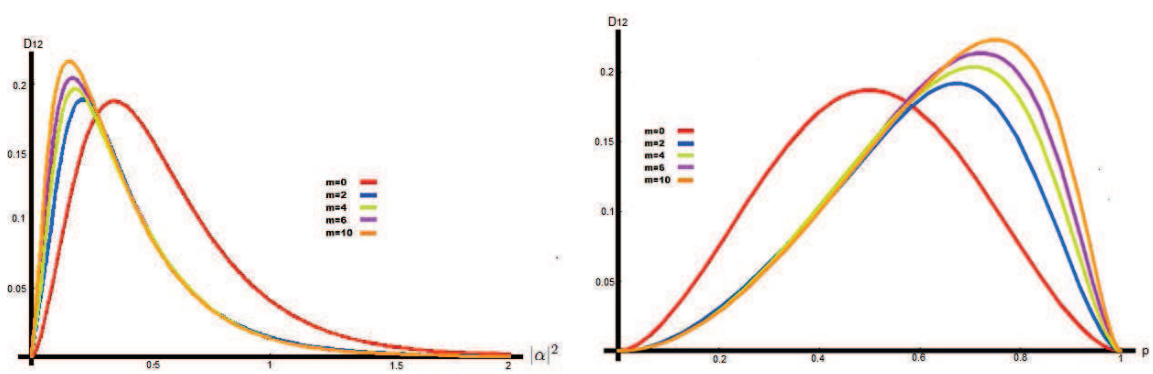

Figure 3. The quantum discord D_{12} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 0$ and different values of photon excitation number m.

We can see from figure 3 that the quantum discord $D_{12}(|\alpha|^2)$ between the optical modes 1 and 2, in the symmetric case ($k = 0$), exhibits peaks which move to the left-hand when the photon excitation number m increases. It must be noticed that the height of peaks, $D_{12}^{\max}(m)$, increases with increasing the number of added photons. We observe also that on the left-hand side of the peak (weak regime), the quantum discord D_{12} rises rapidly with increasing the optical strength $|\alpha|^2$. This indicates that the photon excitation of Glauber coherent states, in the weak regime, induces an activation of the correlations between the modes 1 and 2. In the strong regime ($|\alpha|$ large), the quantum discord tends

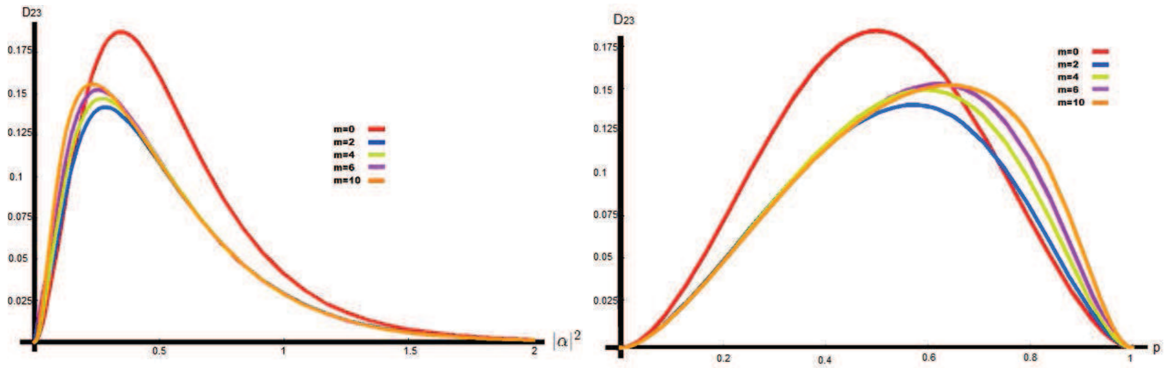

Figure 4. The quantum discord D_{23} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 0$ and different values of photon excitation number m.

to zero quickly as m increases. The behavior of $D_{23}(|\alpha|^2)$ in symmetric quasi-GHZ coherent states ($k = 0$), depicted in figure 4, shows that the maximal amount of quantum discord $D_{23}^{\max}(m)$ is obtained for $m = 0$ and $|\alpha|^2 \sim 0.5$. In contrast with D_{12}, $D_{23}^{\max}(m)$ decreases as m increases (figure 5). Thus, the increase of the quantum discord D_{12} is accompanied by a decrease of D_{23} when the photon excitation number m increases. Remark also that for symmetric states ($k = 0$), the photon excitation does not affect the amount of pairwise quantum correlations D_{12} and D_{23} in the limiting situations $|\alpha| \longrightarrow 0$ and $|\alpha| \longrightarrow \infty$. This is no longer valid for antisymmetric states $k = 1$ especially for $|\alpha|$ approaching zero (see figures 5 and 6). Indeed, the quantum discord D_{12} and D_{23} decreases for $\alpha \longrightarrow 0$ as the GHZ-like coherent states become more excited.

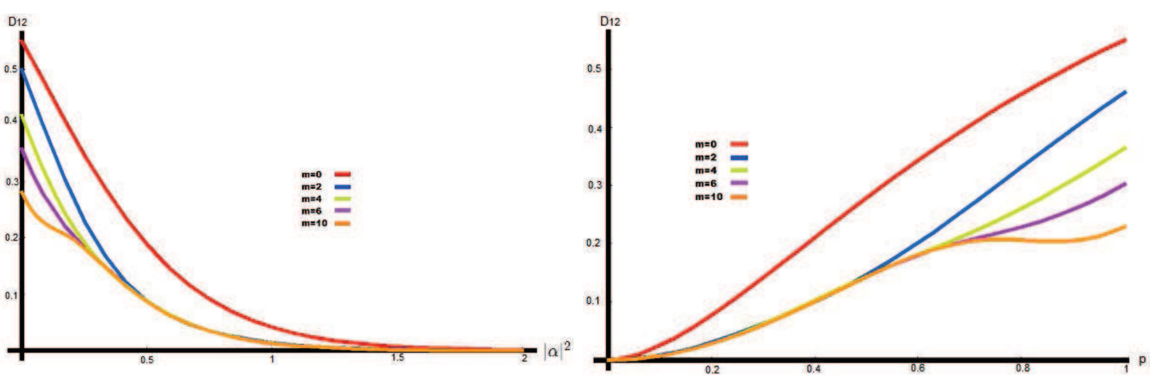

Figure 5. The quantum discord D_{12} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 1$ and different values of photon excitation number m.

The behavior of quantum discord D_{12} and D_{23} in anti-symmetric states ($k = 1$), when $|\alpha|$ approaches zero, can be confirmed analytically. In fact, using (25), one shows that for $|\alpha| \longrightarrow 0$ the quantum discord D_{12} (50) and D_{23} (56) are given by

$$D_{12} = D_{13} = H\left(\frac{2}{m+3}\right) - H\left(\frac{m+2}{m+3}\right) + H\left(\frac{1}{2} + \frac{1}{2}\frac{\sqrt{(m+1)(m+5)}}{m+3}\right), \tag{57}$$

and

$$D_{23} = H\left(\frac{m+2}{m+3}\right) - H\left(\frac{2}{m+3}\right) + H\left(\frac{1}{2} + \frac{1}{2}\frac{\sqrt{m^2+2m+5}}{m+3}\right). \tag{58}$$

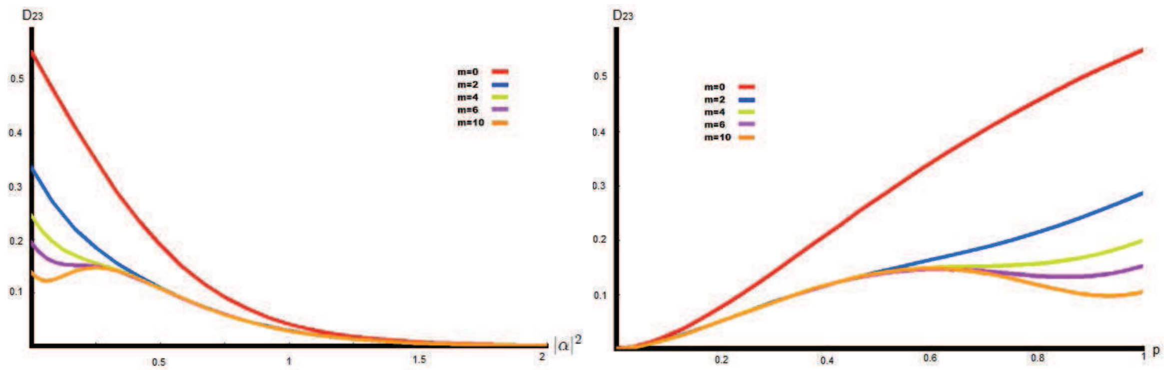

Figure 6. The quantum discord D_{23} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 1$ and different values of photon excitation number m.

It is interesting to note that the antisymmetric photon added GHZ-type coherent states $|\text{GHZ}_1(\alpha, m)\rangle$ (33) reduces, for $|\alpha| \longrightarrow 0$, to

$$|\text{GHZ}_1(0, m)\rangle = \frac{1}{\sqrt{m+3}}(\sqrt{m+1}|m+1, 0, 0\rangle + |m, 1, 0\rangle + |m, 0, 1\rangle) \tag{59}$$

which coincides with the usual three qubit W states for $m = 0$ [61]. The state (59) is expressed in the Fock-Hilbert basis. Hence, according to the results plotted in figures 5 and 6, one concludes that photon excitations diminish the pairwise quantum correlations existing in three qubit states of W type.

4 Monogamy of quantum discord in a three-qubit entangled state of GHZ-type

Having investigated the pairwise quantum discord in the state $|\text{GHZ}_k(\alpha, m)\rangle$ (33), we shall consider the distribution of quantum correlations among its three optical modes. It is well established that in a multi-qubit quantum system, the monogamy property imposes restrictive constraints for the qubits to share freely quantum correlations. Now, it is well established that, unlike the square of concurrence and the squashed entanglement, the quantum discord does not follow the monogamy relation. In this section, we investigate the influence of photon excitation number m on monogamy relation of quantum entropy-based quantum discord in tripartite state of type $|\text{GHZ}_k(\alpha, m)\rangle$ (33).

The entropy-based quantum discord, in the three modes states $\text{GHZ}_k(\alpha, m)$, is monogamous if, and only if, the quantum discord deficit defined by

$$\Delta_{123} = \Delta_{123}(m, |\alpha|^2) = D_{1|23} - D_{12} - D_{13}, \tag{60}$$

is positive. This condition reflects that the monogamy property is satisfied when the quantum discord $D_{1|23}$ between the first mode and the modes 2-3 (viewed as a single subsystem) exceeds the sum of pairwise quantum discord D_{12} and D_{13}. We recall that the concept of monogamy was originally introduced by Coffman, Kundu and Wootters in 2001 [46] in analyzing the distribution of entanglement in a tripartite qubit system. After, several works considered the monogamy of other quantum correlations quantifiers. In this section, we shall determine the conditions under which the quantum discord satisfies the monogamy property and a special attention will be devoted to the influence of photon excitations of Glauber coherent states. For this end, one has to determine the pairwise quantum discord $D_{1|23}$ in the pure state $|\mathrm{GHZ}_k(\alpha, m)\rangle$ (33). In pure states the quantum discord coincides with the entanglement of formation. Hence, to compute the entanglement between qubit (1) and the joint qubits (23), we introduce the orthogonal basis $\{|0\rangle_1, |1\rangle_1\}$ defined by

$$|0\rangle_1 = \frac{|\alpha, m\rangle + |-\alpha, m\rangle}{\sqrt{2(1+\kappa_m e^{-2|\alpha|^2})}}, \qquad |1\rangle_1 = \frac{|\alpha, m\rangle - |-\alpha, m\rangle}{\sqrt{2(1-\kappa_m e^{-|\alpha|^2})}}, \tag{61}$$

for the first subsystem. For the modes (23), grouped into a single subsystem, we introduce the orthogonal basis $\{|0\rangle_{23}, |1\rangle_{23}\}$ given by

$$|0\rangle_{23} = \frac{|\alpha, \alpha\rangle + |-\alpha, -\alpha\rangle}{\sqrt{2(1+e^{-4|\alpha|^2})}} \qquad |1\rangle_{23} = \frac{|\alpha, \alpha\rangle - |-\alpha, -\alpha\rangle}{\sqrt{2(1-e^{-4|\alpha|^2})}}. \tag{62}$$

Inserting (61) and (62) in $|\mathrm{GHZ}_k(\alpha, m)\rangle$, we get the expression of the pure state $|\mathrm{GHZ}_k(\alpha, m)\rangle$ in the basis $\{|0\rangle_1 \otimes |0\rangle_{23}, |0\rangle_1 \otimes |1\rangle_{23}, |1\rangle_1 \otimes |0\rangle_{23}, |1\rangle_1 \otimes |1\rangle_{23}\}$. Explicitly, it is given by

$$|\mathrm{GHZ}_k(\alpha, m)\rangle = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_1 \otimes |\beta\rangle_{23} \tag{63}$$

where the coefficients $C_{\alpha,\beta}$ are

$$C_{0,0} = \mathcal{C}_k(\alpha, m)(1+e^{ik\pi})c_1^+ c_{23}^+, \qquad C_{0,1} = \mathcal{C}_k(\alpha, m)(1-e^{ik\pi})c_1^+ c_{23}^-,$$

$$C_{1,0} = \mathcal{C}_k(\alpha, m)(1-e^{ik\pi})c_{23}^+ c_1^-, \qquad C_{1,1} = \mathcal{C}_k(\alpha, m)(1+e^{ik\pi})c_1^- c_{23}^-,$$

in terms of the quantities

$$c_1^{\pm} = \sqrt{\frac{1 \pm \kappa_m e^{-2|\alpha|^2}}{2}}, \qquad c_{23}^{\pm} = \sqrt{\frac{1 \pm e^{-4|\alpha|^2}}{2}}.$$

The concurrence between the two logical qubits 1 and 23 is given by

$$C_{1|23} = \frac{\sqrt{(1-\kappa_m^2 e^{-4|\alpha|^2})(1-e^{-8|\alpha|^2})}}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}, \tag{64}$$

from which we obtain

$$D_{1|23} = E_{1|23} = H\left(\frac{1}{2} + \frac{1}{2}\frac{\kappa_m e^{-2|\alpha|^2} + e^{-4|\alpha|^2}\cos k\pi}{1+\kappa_m e^{-6|\alpha|^2}\cos k\pi}\right). \tag{65}$$

Inserting $D_{1|23}$ (65) and $D_{12} = D_{13}$ (45) in (60), one gets the explicit expression of the quantum discord deficit Δ_{123}. The corresponding behavior as function of $|\alpha|^2$ (and $p = e^{-2|\alpha|^2}$) for various values of photon excitation order m is displayed in the figures 7 and 8.

It can be inferred that the photon excitation of symmetric quasi GHZ-coherent states ($k = 0$) does not affect the monogamy property of quantum discord. The quantum discord deficit Δ_{123} is always

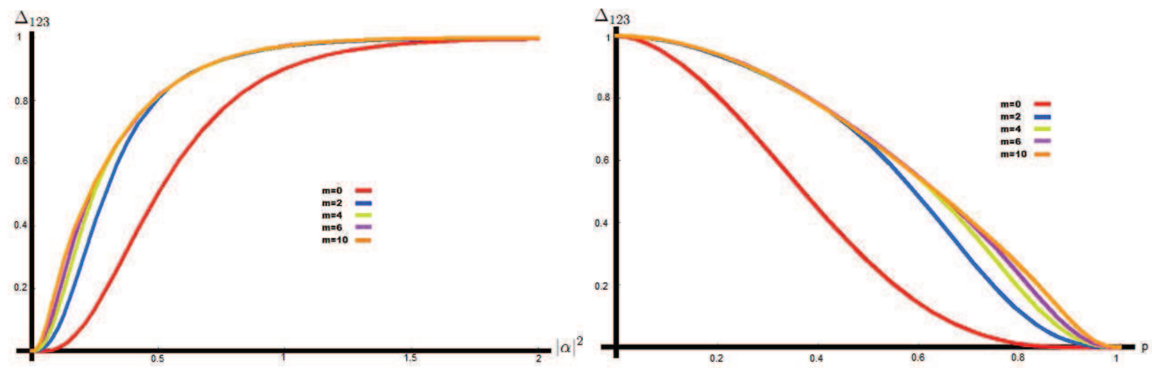

Figure 7. The quantum discord deficit Δ_{123} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 0$ and different values of photon excitation number m.

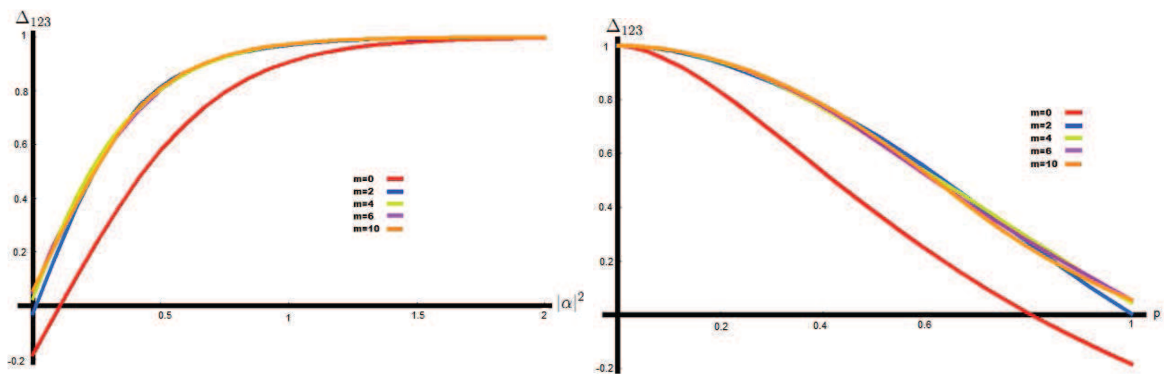

Figure 8. The quantum discord deficit Δ_{123} versus $|\alpha|^2$ and $p = e^{-2|\alpha|^2}$ for $k = 1$ and different values of photon excitation number m.

positive. The situation is slightly different for antisymmetric quasi GHZ-coherent states ($k = 1$). In absence of photon excitation ($m = 0$), the quantum discord violates the monogamy inequality for $|\alpha|^2 < 0.1075$ (or equivalently $p > 0.806$). Remarkably, this violation tends to disappear when photons are added and the quantum discord becomes monogamous. For symmetric as well antisymmetric quasi GHZ-coherent states comprising $m \geq 2$ added photons, Δ_{123} is almost identical in particular for high values of $|\alpha|$ ($|\alpha|^2 \geq 1.5$). Remark that for $|\alpha|$ large the photon added three mode coherent states $|\mathrm{GHZ}_k(\alpha, m)\rangle$ tend to the usual Greenberger-Horne-Zeilinger three qubit states (35). This indicates that, in this case, photon addition process does not affect the distribution of the quantum correlations. Another special limiting situation concerns Glauber states with amplitude approaching zero. For symmetric states $|\mathrm{GHZ}_0(\alpha = 0, m)$, it is easy to verify from the equations (45) and (65) that $\Delta_{123} = 0$ for any photon excitation order m. For the antisymmetric states $|\mathrm{GHZ}_1(\alpha = 0, m)$, which coincide with three qubit states of W-type (59), the monogamy discord deficit increases as m increases (see figure 8). This result can be recovered analytically. Indeed, for $k = 1$ and $|\alpha| \longrightarrow 0$, one shows that

$$D_{1|23} \longrightarrow H\left(\frac{2}{m+3}\right), \tag{66}$$

and using the result (58) one has

$$\Delta_{123} \longrightarrow 2H\left(\frac{m+2}{m+3}\right) - 2H\left(\frac{1}{2} + \frac{1}{2}\frac{\sqrt{(m+1)(m+5)}}{m+3}\right) - H\left(\frac{2}{m+3}\right). \tag{67}$$

The behavior Δ_{123} near the point $\alpha = 0$, plotted in the figure 8, reflects that the photon addition tends to increase the quantum deficit Δ_{123} and subsequently to reduce the violation of monogamy relation in states of W-type.

5 Concluding remarks

In multipartite quantum systems, the monogamy is probably one of the most important relation which imposes severe restriction on the structure of entanglement distributed among many parties. In this context, the main interest of this paper was the monogamy property of quantum discord in three qubit systems where the information is encoded in even and odd Glauber coherent states. In particular, we investigated the influence of photon excitations on the shareability of quantum discord between the three optical modes of a quantum of GHZ-type. We derived the quantum discord deficit by evaluating analytically the pairwise correlations in terms of the photon excitation number and the optical strength of Glauber coherent states. The symmetric quasi-GHZ coherent states follow the monogamy property for any photon excitation order. We have also shown that the photon excitation of antisymmetric quasi-GHZ coherent states reduces the violation of the monogamy property especially in states involving Glauber coherent states with small amplitudes.

Finally, the investigation of the influence of photon excitations on the monogamy of quantum correlations in the states of GHZ-type using geometric based quantifiers such as Hilbert-Schmidt norm or trace distance would be interesting. On the other hand, another significant issue which deserves to be examined concerns the dynamics of quantum discord under the effect of subtracting photons on the pairwise correlations in multipartite coherent states.

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