Arithmetic Notation…*now in 3D!*

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Abstract

When people reason formally, they often make use of special notations—algebra and arithmetic are familiar examples. These notations are often treated as mere shorthand—a concise way of referring to meaningful mathematical concepts. Other authors have argued that people treat notations as pictures—literal diagrams of an imagined set of objects (Dörfler, 2003; Landy & Goldstone, 2009). If notations depict objects that exist in space, then it makes sense to wonder how they are arranged not just in the two visible dimensions, but in depth. In four experiments, we find a consistent pattern: properties that increase mathematical precedence also tend to make objects appear closer in space. This alignment of formal pressures and informal pressures suggests that perceived depth may play a role in supporting computational reasoning processes. Although our primary focus is documenting the existence of depth illusions in notations, we also evaluate several sources of information that might guide depth judgments: availability of an object for computational actions, formal syntactic structure, relative symbol salience and voluntary attention shifts. We consider relationships between these nonexclusive possible sources of information in guiding how people judge depth in mathematics.

Keywords: Mathematical cognition, embodied cognition, depth perception

Introduction

Special notations are ubiquitous markers of mathematical thinking. Often, these notations are treated as mere conventional patterns, which serve as the target of rule-learning systems such as generic production systems (e.g., (Koedinger, Alibali, & Nathan, 2008; Anderson, 2005; Koedinger & MacLaren, 2002). From this perspective, the exact format and layout of the expression doesn't much affect how reasoning happens—what makes learning difficult is the rules, not the layout. However, growing evidence suggests a different account. It seems formal operations—from reasoning to logic to simple mathematics—are not always computed through the action of an abstract reasoning system, but instead make use of perceptual-motor systems typically involved in real-world perception and action.

In the case of mathematics in particular, reasoning that on the surface appears to require formal operations can be simplified if reasoners treat notations as pictures of a physical scene (Dörfler, 2003; Landy & Goldstone, 2009). For instance, algebraic syntax has a hierarchical structure partially described by the order of operations. In any equation, operations bind in the following order: parentheses, exponents, multiplications and divisions, additions and subtractions. This apparently arbitrary system appears to require explicit memorization, but in fact can be computed using basic perceptual-motor mechanisms such as grouping (Landy & Goldstone, 2007) and automatic attentional biases (Landy, Jones, & Goldstone, 2008). On this account, computing the answer to a math problem involves taking physical actions to transform abstract forms.

If notations really are fundamentally abstract, then their implied physical structure is entirely given by their surface form: there is no sense in which any of the symbols are 'close' or 'far away.' However, if people indeed often reason by treating symbols as pictures of objects in space, then these objects must be laid out in some three-dimensional arrangement. Thus, it is at least possible that different symbols would be seen as closer or further away than others. In the next section, we outline some reasons to expect such differences.

Reasons to Expect Illusions of Differential Depth

Actual equations and expressions are of course purely two-dimensional; thus actual depth experience should not directly inform perceived depth judgments with mathematical forms. Furthermore, accounts that treat notation as basically abstract predict no particular differences in apparent depth of different symbols. However, several factors might affect the perceived depth of symbols seen as objects that exist in space, which can be acted on in particular kinds of ways.

One clear prediction is that symbols that afford action appear proximal relative to those that do not. Several studies have found that depth perception can be affected by the action capabilities of the observer (Linkenauger, Witt, Stefanucci, Bakdash, & Proffitt, 2009; Witt, Linkenauger, Bakdash, & Proffitt, 2008; Witt & Proffitt, 2005; Witt, Proffitt, & Epstein, 2005). Therefore, if solving a mathematical equation requires actions on the part of the solver, high precedence terms—those most available for actions—should generally seem most proximal in arithmetic expressions. Put simply, years of experience acting first on multiplications in expressions like $3 + 4 \times 5 = 23$ will lead the multiplication to appear closer than the addition.

We hypothesized, more generally, that terms and operation signs that were most immediately available for

action would be seen as more proximal. In some cases, general syntactic factors do not align with action. That is, unlike in the stimuli in Figure 1, in some cases low-precedence operations afford more immediate action than high-precedence operations. This issue is taken up again in Experiment 3.

Another reason to expect systematic biases in perceived depth of arithmetic signs comes from the salience structure of those signs. As mentioned above, typical multiplication signs (the dot and the cross) are more salient and readily attended than addition signs. Salience and attention present conflicting pressures in the paradigm employed here. Voluntary attention has been shown to influence the perceived depth of ambiguous figures (Kawabata, 1986). In the case of arithmetic expressions, attention shifts systematically from high-precedence operations to low-precedence over the course of problem solution. Since participants made depth judgments after solving the problem, attention would most recently have been primarily allocated to additions, potentially causing addition signs to seem closer immediately after computation.

Salience also affects perception of figure and ground such that highly salient parts tend to be interpreted as parts of figures (Hoffman & Singh, 1997). Generally speaking, this may imply that salient objects will tend to be seen as proximal (though see Huang & Pashler 2009). The higher salience of multiplications should then also cause them to be perceived as proximal.

In three of the experiments reported below, participants were asked to solve simple mathematical problems, superimposed over two views of a baseball (see Figure 1). After solving the problem, they judged the relative distance of the two baseballs. If the relative availability of computational action affects perceived depth, the baseball associated with the high-precedence sign (the multiplication sign) should appear closer than the baseball under the low-precedence sign. The baseballs were used to ground the participants' judgments and make clear the nature of the task; our assumption is that judged distance of the baseball reflects primarily the relative perceived depth of the symbol superimposed on it.

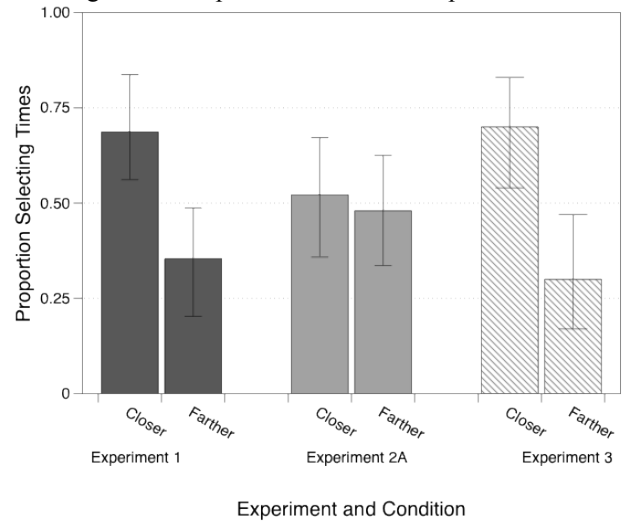Figure 1: Sample stimuli used in Experiment 1.

Figure 2: Results of Experiments. Errors represent 95% confidence intervals on proportions.

Experiment 1

Method

Participants Forty-eight students from the University of Richmond received partial course credit for participation. All participants had normal or corrected to normal vision.

Materials Participants viewed simple two-operation arithmetic expressions superimposed over images of a baseball (see Figure 1). The same image that appeared under the plus sign appeared under the multiplication sign, rotated 90°. Instructions informed participants that one baseball was close, and approaching, while the other baseball was farther away, and receding.

Procedure Participants were asked to solve the arithmetic problem. After doing so, they were asked to decide the relative distance of the two baseballs. Half of the participants were asked to circle the baseball that was closer; the other half to circle the baseball that was farther. Each participant responded to two expressions; one was in the format $a + b \times c$, the other $a \times b + c$. The order of presentation of the problems was counterbalanced across participants, as was which image orientation appeared on the left (thus image orientation and operation sign were independent). The task was performed as part of a distracter, between two phases of an unrelated experiment. Several other short problems appeared in between the two baseball judgments.

Results On each trial, the participant circled either the multiplication sign or the addition sign (see Figure 2). These choices were analyzed using two repeated measures logistic regressions: one included just an intercept, while the other also included judgment type (closer vs. farther) as a between-participants factor. Including judgment type

significantly improved the quality of the fit over the baseline model, ($\chi^2(1) = 10.2$, p<.01).

Discussion The primary purpose of Experiment 1 was to evaluate whether there was consistency in how depth would be judged in a simple arithmetic expression. The results demonstrated that there is. In a simple two operation problem, multiplications are seen as closer than additions. In simple arithmetic processes, the higher precedence operation is perceived as more proximal to the reasoner than is the lower precedence operation.

Though predicted by the idea that multiplication is more available than addition in this expression, these results could be produced by any combination of an effect of the higher salience of cross signs over plus signs, the syntactic order of operations, and the relative availability of multiplication in this expression. The prediction that voluntary attention toward the plus sign at the end of solving the problem would dominate proximity judgment was not borne out.

Experiment 2a

In this experiment, we tested whether the bias in perceived depth revealed in Experiment 1 was due to particular salience differences between the addition and multiplication sign rather than order of operations. In Experiment 2a, we used the same stimuli and design as in Experiment 1, except that we used parentheses to make the plus sign the first operation rather than the multiplication. If the result in Experiment 1 was due simply to perceptual differences between the multiplication and addition signs, then we should expect to replicate Experiment 1. However, if the result is due to order of operations, adding the parentheses should result in either a null effect or the opposite effect.

Method

Participants Forty-eight students from the University of Virginia received partial course credit for participation. All participants had normal or corrected to normal vision.

Materials Packets were created using the same stimuli as in Experiment 1, except the stimuli were modified so that the two numbers adjacent to the plus sign were enclosed by parentheses.

Procedure The procedure was the same as in Experiment 1 except that rather than serving as a distracter in an unrelated task, these judgments served as the primary experiment. In between the two trials, participants completed a distracter task which involved solving a maze.

Results and Discussion Participant choices were analyzed using a repeated measures logistic regression (see Figure 2).

Including judgment type did not improve the fit by a likelihood ratio test over the null model ($\chi^2(1) = .12$, p~.68).

Unlike Experiment 1, there was no sign of a consistent relationship between perceived relative depth and operation sign in simple expressions with parentheses. One plausible interpretation of the disparity is that the higher precedence of multiplication sign interacts with a pressure to see parenthesized terms as closer. There are (at least) two plausible reasons why this would occur: first, as hypothesized, terms that can be computed early may appear closer than they otherwise would. Second, visual factors intrinsic to parentheses may make them things inside parentheses appear differentially closer.

Consider Figure 3. In the top part, the circle on the left appears to be in front of (and consequently partially occluding) the illusory oval induced by the curved lines. This, in turn, causes it to appear closer than the circle on the right. In a similar manner, it may well be that the parentheses create a (relatively weak) illusory oval. If the symbols are interpreted by the visual system as being in front of that oval, then they may be perceived as being closer than the symbols not inside the parentheses.

Experiment 2b

One difference between Experiment 1 and 2a is that the populations differed, one being drawn from University of Richmond students and the other from the University of Virginia. Therefore, to eliminate the possibility that the null results in Experiment 2a were due to differences between the student populations, a second experiment was run at the University of Virginia to ensure that similar perceptual effects as in Experiment 1 could be
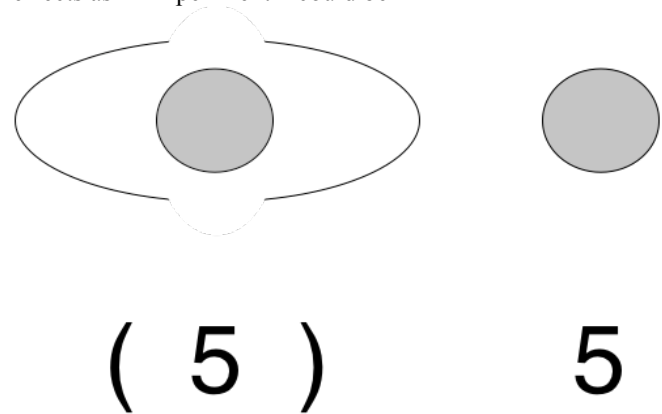

Figure 3: Visual factors could influence the relative perceived proximity of terms inside parentheses. Just as the circle on the left appears to be closer than the circle on the right, so apparent occlusion may cause the 5 on the left to be in closer than the 5 on the right.
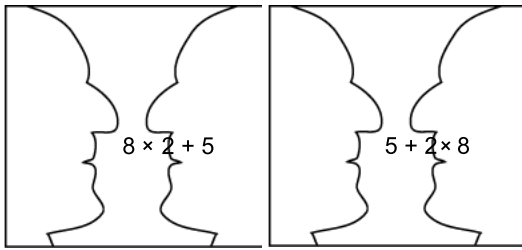
Figure 4. Example Stimuli from Experiment 2b.

found in this population. In addition, this experiment offers a more indirect judgment of relative proximity.

Method

Participants Twenty-two students from the University of Virginia received partial course credit for participation. All participants had normal or corrected to normal vision.

Stimuli and Apparatus Mathematical equations were superimposed onto an illustration of the vase-face illusion so that either the multiplication sign was on the vase and the plus on the face or vice versa (see Figure 4). Instructions printed above the picture directed participants to solve the equation and then indicate whether or not they saw a vase or faces. The equation was either on the right side of the illustration, as in Figure 4, or on the left.

Packets were created which consisted of two face-vase trials. A maze was inserted in between the vase-face trials, which acted as a distracter task. Half of the packets contained vase-face illusions in which equations were located on the right; the other half had the equations located on the left. Within each packet, one illusion had the multiplication sign on the vase, and the other had the multiplication sign on the face. Order was counter-balanced across packets.

Procedure Participants were given a packet and told to complete it in full. They were asked to follow the instructions written on each sheet, and not to look at the subsequent sheets until they had completed their task on the current sheet.

Results and Discussion Arbitrarily, each trial was coded as positive if the participant chose the faces as the foreground. A significantly better fit was found for a logistical regression including position of the multiplication sign as a factor, than for the null model ($\chi^2(1) = 5.1$, p<0.05). Overall, when the multiplication sign was over the vases, participants chose the face interpretation less often ($M=.5$, $CI=0.28$-0.71) than when the multiplication sign was over the faces ($M=.82$, $CI=0.60$-0.95).

These results show that the effect in Experiment 1 can generalize across populations and across tasks. This finding is also a less direct manipulation of perceived depth and is less likely to be affected by demand characteristics. Whether the face or vase is seen in the foreground is indicative of the depth relationship between the vase and the face. Therefore, participants determined which figure they saw instead of directly specifying the depth relationship between the mathematical operators. Interestingly, order of operations influenced the figure-ground and therefore, the depth relationship in an ambiguous figure illusion.

Prior results have shown that fixations (Gibson & Peterson, 1994) and exogenous cues (Vecera, Flevaris, & Filapek, 2004) guide figure-ground segmentation, as long as the cues appear inside figures (as was done here). Thus, these results could result from the greater salience of multiplication signs. However, once again, voluntary attention shifts are unlikely to account for these effects, as voluntary attention is most likely directed toward the addition at the end of computation (when participants were instructed to make their judgments).

The first two experiments indicated that syntactic precedence, available formal actions (computations), and perceived proximity tend to go together. Experiment 3 distinguishes between syntactic precedence and action structure by repeating the structure of Experiments 1 and 2A in the context of an algebraic rather than an arithmetic task. Because linear equations are solved starting from the lowest, rather than the highest precedence operations, in Experiment 3 the two theories make opposite predictions. If availability of formal actions guides perceived depth, then lower precedence items should be seen as proximal in Experiment 3; if syntactic precedence predicts or guides depth, then high precedence operations should appear proximal, as in Experiments 1 and 2B.

Experiment 3: Linear Equations

Method

Participants Twenty students from the University of Virginia received partial course credit for participation. All participants had normal or corrected to normal vision.

Materials Packets were created using a format identical to Experiment 2A. Similar stimuli were used, except the stimuli were modified from an arithmetic computation to the solution of a linear equation (see Figure 5). Participants were instructed to solve the equation before deciding which ball was closer (further).
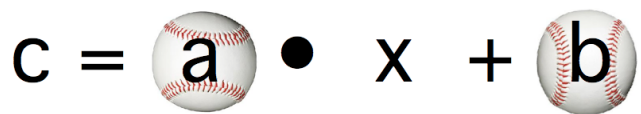

Figure 5. Example Stimulus from Experiment 3.

Two other changes were made to the stimuli: first, the dot notation was used for multiplication in place of the cross. This sign is much more typical in algebraic contexts, and is less likely to be confused for a variable than the cross. Furthermore, crosses and dots have similar salience advantages in mathematical contexts (Landy et al., 2008). Second, the baseballs were moved from behind the operations to behind the operands. This was done because, in an equation solution context, the operands are more relevant units of action. That is, one solves the problem illustrated by cancelling the *b*, and then the *a*. Thus, the effect of perceived action was predicted to be strongest in this configuration.

Procedure The procedure was identical to Experiment 2A.

Results and Discussion Participant choices were analyzed using a repeated measures logistic regression (see Figure 2). Including judgment type significantly improved the fit over the baseline model ($\chi^2(1) = 6.0$, p<0.05). Participants were more likely to judge the baseball under the multiplicative term as closer than the baseball under the additive term.

The results of Experiment 3 contradicted our original hypothesis that perceived depth would align with available actions, instead supporting the idea that syntactically central symbols appear to be closer than syntactically peripheral symbols in mathematical expressions.

The results of Experiment 3 are to some degree compatible with the interpretation that low-level visual features guide perceived proximity. In this case, perceived depth of the baseball images would be affected by the terms adjacent to the judged baseballs, rather than those directly behind it. In a pre-cuing task, Baylis and Driver (1995) reported that exogenous cues to attention did not affect figure-ground segregation (which tends to align with depth perception in most cases, though see Huang & Pashler, 2009), when the cue appeared outside the area in which the figure appeared (see also Vecera et al., 2004). Nonetheless, in this case, it is possible that the highly salient dot adjacent to the baseball increases its apparent proximity. It is also possible that the multiplicative terms and the multiplication sign are visually grouped and therefore that the salience of one part of the group (the multiplication sign) causes the entire group to appear closer.

General Discussion

Although mathematical notation is a formal language, and is inherently two dimensional, readers of these notations come to quite consistent judgments about the relative proximity of terms in formal expressions. Three experiments demonstrated that factors that determine formal precedence (operation sign and parentheses) also systematically influence perceived depth.

Variations in perceived depth aligned in our stimuli with formal precedence. The current results do not distinguish between the possibilities that syntactic precedence directly affects perceived depth, and that the low-level visual

features of typical mathematical notation determine apparent depth. Although future work should distinguish these two possibilities, we think it notable that mathematical notation is structured in such a way that there is a systematic relationship between low-level visual features affecting and mathematical syntax. This alignment raises the possibility that perceived three-dimensional structure may be used as a cue to mathematical ordering.

As long as episodes of formal reasoning are indeed typically organized by attention-based interactions with external environments (Landy et al., 2008; Patsenko & Altmann), the alignment of perceptual factors such as visual grouping, salience, and depth may be significant factors in making symbolic mathematical notation such a powerful and successful system for supporting reasoning.

Three limitations of the current work are worth noting: one is that it does not indicate the strength of the judgment. Although judgments were significant and consistent in Experiments 1, 2A, and 3, participants made forced choice binary judgments. Thus, while we can conclude that people generally perceive multiplications as closer than additions, the current experiments give no indication of the magnitude of the perceived difference.

Another limitation is that there is no indication in the current studies of whether this perceived difference in depth has any effect on mathematical judgments. Future work should explore whether explicit manipulations of apparent depth disrupt mathematical reasoning processes. Finally, there is a possibly important confound in Experiment 1. In these stimuli, the laces on the baseballs overlap, and are obscured by, the addition sign slightly more than the multiplication sign. This might provide a stronger depth cue, causing subjects to see the baseball under the addition sign as farther away. This confound could not explain the difference between Experiment 1 and 2a, nor could it explain the effect in Experiment 3. In Experiment 3, the baseballs appeared under the letters. These were counterbalanced across condition, and so could not have led to differences in judgment.

Recognizing these limitations, nevertheless the existence of consistent depth cues in mathematical notations bolsters interpretations that treat mathematical reasoning as (sometimes) a form of spatial reasoning over symbolic objects. Accounts that treat symbolic reasoning as abstract rule learning cannot make systematic predictions about depth, such as those seen here. Understanding how and when such factors matter for reasoning promises to further illuminate our understanding of general formal reasoning processes.

Acknowledgments

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