

Stochasticity effects in quantum radiation reaction

N. Neitz and A. Di Piazza*

Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, D-69117 Heidelberg, Germany

When an ultrarelativistic electron beam collides with a sufficiently intense laser pulse, radiation-reaction effects can strongly alter the beam dynamics. In the realm of classical electrodynamics, radiation reaction has a beneficial effect on the electron beam as it tends to reduce its energy spread. Here, we show that when quantum effects become important, radiation reaction induces the opposite effect, i.e., the electron beam spreads out after interacting with the laser pulse. We identify the physical origin of this opposite tendency in the intrinsic stochasticity of photon emission, which becomes substantial in the full quantum regime. Our numerical simulations indicated that the predicted effects of the stochasticity can be measured already with presently available lasers and electron accelerators.

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A deep understanding of the dynamics of electric charges driven by electromagnetic fields is one of the most fundamental problems in physics, as it has implications in different fields, including accelerator, radiation and high-energy physics. Apart from its impact on practical issues, as the construction of new experimental devices (e.g., quantum x-free electron lasers [1]), the investigation of the dynamics of electric charges (electrons, for definiteness) is also of pure theoretical interest, as it involves in general a coupled dynamics of the electrons and of their own electromagnetic field.

In the realm of classical electrodynamics, radiation-reaction (RR) effects stem from the back reaction on the electron dynamics of the electromagnetic field generated by the electron itself while being accelerated by a background electromagnetic field [2, 3]. The Landau-Lifshitz (LL) equation has been recently identified as the classical equation of motion of an electron, with mass m and charge $e < 0$, which includes RR effects self-consistently [2–7], although alternative models have been suggested [8, 9]. The analytical solution of the LL equation in a plane-wave field [10] shows that if an electron impinges with initial four-momentum p_0^μ onto a plane-wave field (electric-field amplitude E_0 , central angular frequency ω_0 and propagating along the direction \mathbf{n}), RR effects substantially affect the electron dynamics, if the parameter $R_c = \alpha\chi_0\xi_0$ is of the order of unity (see also [11]). Here, $\alpha = e^2$ is the fine-structure constant, $\chi_0 = ((np_0)/m)E_0/E_{cr}$ is the so-called quantum nonlinearity parameter, with $n^\mu = (1, \mathbf{n})$ and $E_{cr} = m^2/|e| = 1.3 \times 10^{16}$ V/cm, and $\xi_0 = |e|E_0/m\omega_0$ is the classical nonlinearity or relativistic parameter (units with $\hbar = c = 1$ are used throughout). It is worth noting that, although χ_0 is much smaller than unity in the realm of classical electrodynamics [2], the parameter R_c can be of the order of unity [4, 10, 11]. The parameter R_c represents the average energy radiated by the electron in one laser period in units of the initial electron energy, and for an ultrarelativistic electron initially counterpropagating with respect to the laser field with en-

ergy ε , it is $R_c = 3.2\varepsilon[\text{GeV}]I_0[10^{23}\text{ W/cm}^2]/\omega_0[\text{eV}]$, with $I_0 = E_0^2/4\pi$ being the laser pulse peak intensity. The numerical value of the parameter R_c shows the generally demanding requirements to observe large RR effects and it explains why the LL equation still lacks an experimental confirmation (see [11–14] for recent experimental proposals). The expression of the parameter R_c is also in agreement with the well-known classical result that more energetic particles radiate more at given other conditions [15]. In turn, this explains physically the beneficial effect of RR when it is included, e.g., in the investigation of the production of electron [16] and ion [17–20] bunches in laser-plasma interaction. In fact, it is found that RR acts as a cooling mechanism and its effects render the energy spectra of the produced particle bunches more monochromatic than if RR is not included.

In this Letter we show that when quantum effects become important RR induces exactly the opposite behavior and makes the energy distribution of an electron beam initially counterpropagating with respect to a strong laser field broader as it was before the interaction. We explain this striking difference between classical and quantum RR relating it to the stochastic nature of the emission of radiation, which becomes substantial in the quantum regime, and indicating that quantum effects amount to add a stochastic term in the LL classical equation. By means of numerical simulations we show that the broadening of the electron distribution in the quantum regime, is measurable in principle with presently available technology even in an all-optical setup. Our results are relevant for future laser-based electron accelerators, indicating that one cannot rely on the beneficial effects of RR on the energy spread of the electron beam at sufficiently high electron energies that quantum effects become important. We note that the stochastic nature of photon emission has instead been shown to lower the laser intensity threshold at which electromagnetic cascades are generated [21].

Taking into account exactly RR in the full strong-field QED regime is a formidable task, as it amounts to deter-

mine completely the S -matrix, describing the interaction of the electron-positron field with the radiation field in the presence of the strong background electromagnetic field [2, 22]. Thus, we limit here to the so-called “nonlinear moderately-quantum” regime [22], where: 1) $\xi_0 \gg 1$, such that nonlinear effects in the laser field amplitude are large; 2) $\chi_0 \lesssim 1$, such that nonlinear QED effects are already important, but electron-positron pair production is still negligible. In this regime, RR effects on the electron dynamics in a strong plane-wave field mainly stem from the sequential emission of many photons by the electron, and they can be investigated by means of a kinetic approach [23–25] (see [22], for an alternative, microscopic approach). In this approach, the electrons and the photons are described by distribution functions in phase space, which obey to “kinetic” equations. Since electron-positron pair production is neglected: 1) the distribution function of positrons can be assumed to vanish identically; 2) the kinetic equation for the electron distribution function is not coupled to that of the photons [23–25]. Another realistic approximation, which allows us to avoid technical complications in favor of a clearer physical understanding, is to consider an electron bunch initially counterpropagating with respect to the laser field and with a typical energy $\varepsilon^* \gg m\xi_0$. This is the case, for example, in the realistic situation of a electron bunch with typical energy $\varepsilon^* = 1$ GeV colliding head-on with an optical ($\omega_0 = 1.55$ eV) laser field of intensity 10^{22} W/cm² [26] for which $m\xi_0 = 25$ MeV. The condition $\varepsilon^* \gg m\xi_0$ ensures that the transverse momentum of the electrons (with respect to the initial propagation direction) remains much smaller than the longitudinal one in passing through the plane wave [27], and this reduces the present problem to a one-dimensional one.

By assuming that the plane wave propagates along the positive y direction and that it is linearly-polarized along the z direction, we can write its electric field as $\mathbf{E}(\varphi) = E_0 f(\varphi) \hat{\mathbf{z}}$, where $\varphi = \omega_0(t - y)$ is the laser phase and $f(\varphi)$ is the pulse-shape function such that $|f(\varphi)|_{\max} = 1$. If $p^\mu = (\varepsilon, \mathbf{p})$ is the four-momentum of an electron, it is convenient to introduce the quantity $p_- = \varepsilon - p_y$, which is a constant of motion in the plane-wave field under consideration [27]. However, if the electron emits a photon with four-momentum $k^\mu = (\omega, \mathbf{k})$, then its four-momentum changes to $p'^\mu = (\varepsilon', \mathbf{p}')$ and $p'_- = p_- - k_-$, with $p'_- = \varepsilon' - p'_y$ and $k_- = \omega - k_y$. The single-photon emission probability per unit phase φ and per unit $u = k_-/(p_- - k_-)$ in the ultrarelativistic regime $\xi_0 \gg 1$ reads [28]

$$\frac{dP_{p_-}}{d\varphi du} = \frac{\alpha}{\sqrt{3}\pi} \frac{m^2}{\omega_0 p_-} \frac{1}{(1+u)^2} \left[\left(1 + u + \frac{1}{1+u} \right) \times K_{\frac{2}{3}} \left(\frac{2u}{3\chi(\varphi, p_-)} \right) - \int_{\frac{2u}{3\chi(\varphi, p_-)}}^{\infty} dx K_{\frac{1}{3}}(x) \right], \quad (1)$$

where $K_\nu(\cdot)$ is the modified Bessel function of ν th order and where $\chi(\varphi, p_-) = (p_-/m)|E(\varphi)|/E_{\text{cr}}$, with $E(\varphi) = E_0 f(\varphi)$. Since the probability in Eq. (1) depends only on the phase-space variables φ and p_- , it is possible to describe the electron beam via an electron distribution $n_e(\varphi, p_-)$, which satisfies the kinetic equation (see Ref. [24])

$$\frac{\partial n_e}{\partial \varphi} = \int_{p_-}^{\infty} dp'_- \frac{dP_{p'_-}}{d\varphi dp'_-} n'_e - \int_0^{p_-} dk_- \frac{dP_{p_-}}{d\varphi dk_-} n_e \quad (2)$$

with $n_e = n_e(\varphi, p_-)$, $n'_e = n_e(\varphi, p'_-)$ and

$$\frac{dP_{p'_-}}{d\varphi dp'_-} = \frac{p'_-}{p_-^2} \frac{dP_{p'_-}}{d\varphi du} \Big|_{u=\frac{p'_--p_-}{p_-}}, \quad (3)$$

$$\frac{dP_{p_-}}{d\varphi dk_-} = \frac{p_-}{(p_- - k_-)^2} \frac{dP_{p_-}}{d\varphi du} \Big|_{u=\frac{k_-}{p_- - k_-}}. \quad (4)$$

The kinetic equation (2) is an integro-differential equation, i.e., it is non-local in the momentum p_- . This occurrence is intimately connected to the quantum nature of the emission of radiation. In fact, the emission of radiation is described quantum mechanically as the emission of photons, which carry energy and momentum, such that, if an electron emits a photon with momentum k_- , its initial state with a given momentum $p_{0,-}$ will be coupled to that with momentum $p_{0,-} - k_-$, with k_- ranging from 0 to $p_{0,-}$.

In order to investigate the classical limit of Eq. (2) for $\chi(\varphi, p_-) \ll 1$, it is convenient to perform the change of variable $v = (p'_- - p_-)/p_- \chi(\varphi, p_-)$ ($v = k_-/(p_- - k_-) \chi(\varphi, p_-)$) in the first (second) integral in Eq. (2). By expanding the resulting equation in $\chi(\varphi, p_-)$ and by keeping terms up to the order $\chi^3(\varphi, p_-)$, one obtains the Fokker-Planck-like equation [29] (see also [25, 30])

$$\frac{\partial n_e}{\partial \varphi} = -\frac{\partial}{\partial p_-} [A(\varphi, p_-) n_e] + \frac{1}{2} \frac{\partial^2}{\partial p_-^2} [B(\varphi, p_-) n_e] \quad (5)$$

with a “drift” coefficient $A(\varphi, p_-)$ and a “diffusion” coefficient $B(\varphi, p_-)$ given by

$$A(\varphi, p_-) = -\frac{2\alpha m^2}{3\omega_0} \chi^2(\varphi, p_-) \left[1 - \frac{55\sqrt{3}}{16} \chi(\varphi, p_-) \right], \quad (6)$$

$$B(\varphi, p_-) = \frac{\alpha m^2}{3\omega_0} \frac{55}{8\sqrt{3}} p_- \chi^3(\varphi, p_-), \quad (7)$$

respectively. It is worth observing that this equation is no longer an integro-differential equation but rather a partial differential equation. In other words, when quantum photon-recoil effects become smaller and smaller, the distribution function of electrons with a momentum p_- depends only on its values close to p_- and its dynamics is local. On this respect, we also note that higher-order corrections in $\chi(\varphi, p_-)$ would result in the appearance

of terms proportional to higher and higher derivatives of $n_e(\varphi, p_-)$ with respect to p_- .

If we first consider only the terms proportional to $\chi^2(\varphi, p_-)$ in Eq. (5), the latter equation has the form of a Liouville equation:

$$\frac{\partial n_e}{\partial \varphi} = -\frac{\partial}{\partial p_-} \left(n_e \frac{dp_-}{d\varphi} \right), \quad (8)$$

where

$$\frac{dp_-}{d\varphi} = -\frac{I_{cl}(\varphi, p_-)}{\omega_0}, \quad (9)$$

with $I_{cl}(\varphi, p_-) = (2/3)\alpha m^2 \chi^2(\varphi, p_-)$ being the classical intensity of radiation [27]. Equation (9) is exactly the classical single-particle equation for the momentum p_- resulting from the LL equation [10] (see also [31]). In other words, the terms in Eq. (5) proportional to $\chi^2(\varphi, p_-)$ describe the classical dynamics of the electron distribution including RR. The fact that Eq. (8) has the form of a Liouville equation implies, as it must be, that the classical dynamics of the electron distribution is deterministic [29]. Also, since the single-particle equation (9) admits the analytical solution [10], $p_-(\varphi; p_{0,-}) = p_{0,-}/h(\varphi, p_{0,-})$ with

$$h(\varphi, p_{0,-}) = 1 + \frac{2}{3}\alpha \frac{p_{0,-}}{\omega_0} \frac{E_0^2}{E_{cr}^2} \int_0^\varphi d\varphi' f^2(\varphi') \quad (10)$$

for an electron with initial momentum $p^\mu(0) = p_0^\mu = (\epsilon_0, \mathbf{p}_0)$ ($p_{0,-} = \epsilon_0 - p_{0,y}$), one can write explicitly the exact analytical solution of Eq. (8) by means of the method of characteristics. If the distribution $n_e(0, p_-)$ at the initial phase $\varphi = 0$ is given, for example, by the Gaussian distribution $n_e(0, p_-) = N \exp[-(p_- - p_-^*)^2/2\sigma_{p_-}^2]$, where N is a normalization factor, p_-^* is the average value of p_- and σ_{p_-} is the standard deviation [32], then the solution of Eq. (8) reads

$$n_e(\varphi, p_-) = \frac{N}{g^2(\varphi, p_-)} \exp \left\{ -\frac{1}{2\sigma_{p_-}^2} \left[\frac{p_-}{g(\varphi, p_-)} - p_-^* \right]^2 \right\}, \quad (11)$$

with $g(\varphi, p_-) = h(\varphi, -p_-)$. Since $p_{0,-}$ in Eq. (10) is positive for finite values of $p_{0,y}$ and $p_{0,-} \rightarrow 0$ only at $p_y \rightarrow +\infty$, the function $g(\varphi, p_-)$ must be non-negative for all values of φ , and the equation $g(\varphi, p_{-,max}) = 0$ fixes the maximum value $p_{-,max} = p_{-,max}(\varphi)$ allowed for the variable p_- at each φ . Before passing to investigate the quantum corrections in Eq. (5), we observe that the classical solution in Eq. (10) is such that $0 < \partial p_-(\varphi; p_{0,-})/\partial p_{0,-} < 1$ for $\varphi > 0$ and this ensures that, due to RR effects, the difference $\Delta p_-(\varphi)$ between the momenta of two electrons decreases for increasing values of φ . This implies that RR effects tend to decrease the energy width of the electron distribution in agreement with previous results [19, 20]. Also, if

$\sigma_{p_-} \ll p_-^*$ in Eq. (11), it can be seen that the distribution $n_e(\varphi, p_-)$ is approximately a Gaussian with effective width $\sigma_{p_-}(\varphi) \approx \sigma_{p_-}/h^2(\varphi, p_-^*)$ decreasing at increasing φ 's.

The quantum corrections in Eq. (5) to the classical kinetic equation (8) stem from two different contributions. The first one affects the drift coefficient $A(\varphi, p_-)$ (see Eq. (6)) and it corresponds to the leading quantum correction to the total intensity of radiation found in [24, 28]. This correction, does not change the structure of the classical kinetic equation (8) but only the “effective” momentum change per unit phase. Since this leading quantum correction is negative, we expect that it tends to decrease the reduction of the width with respect to the classical prediction. However, by replacing the classical intensity of radiation $I_{cl}(\varphi, p_-)$ with the corresponding quantum one $I_q(\varphi, p_-)$ (see, e.g., Eq. (83) on pg. 522 in [28]), the resulting Liouville equation would still predict a reduction of the width of the electron distribution function. Although this statement can be proven mathematically, it can be intuitively understood as a physical consequence of the fact that more energetic electrons *on average* emit more radiation. On the other hand, however, the second leading quantum correction corresponds to the diffusion coefficient $B(\varphi, p_-)$ in Eq. (7) and it alters the structure of the classical kinetic equation. The appearance of a diffusion-like term in the kinetic equation of the electron distribution is intimately connected to the stochastic nature of the quantum emission of photons. According to the theory of stochastic differential equations, in fact, the Fokker-Planck-like equation (5) is related to the single-particle stochastic equation $dp_- = -A(\varphi, p_-)d\varphi + \sqrt{B(\varphi, p_-)}dW$, where dW represents an infinitesimal stochastic function [29]. The diffusion term in Eq. (5) is responsible of the broadening of the distribution function. In the case, for example, of a Gaussian distribution function assumed to be well peaked at φ around the classical value $p_-(\varphi; p_-^*)$ (see Eq. (10)), it can easily be shown, that if σ_{p_-} is its initial width, then

$$\sigma_{p_-}(\varphi) \approx \frac{1}{h^2(\varphi, p_-^*)} \left[\sigma_{p_-}^2 + \int_0^\varphi d\varphi' B(\varphi', p_-^*) \right]^{1/2}. \quad (12)$$

This result clearly shows the opposite influence of the classical drift term and of the quantum diffusion term on the width of the electron distribution. It is worth noting that the correction to the width arising from the quantum corrections in the drift coefficient $A(\varphi, p)$ is found to be smaller than the correction proportional to the diffusion term by a factor $\sigma_p^2/p_-^{*,2} \ll 1$, and it has been neglected in the approximated expression (12). We warn the reader about the fact that a formal solution of the Fokker-Planck equation, for example, in the case of an initial δ -like momentum distribution and vanishing drift term, would predict, due to the spreading in the mo-

mentum distribution, the spurious presence of particles with momentum larger than the initial one. This indicates that, for a completely consistent treatment, the full equation (2) has to be employed, which will be carried out below numerically.

The above effect on the broadening of the electron momentum distribution can also be interpreted in terms of the entropy $S(\varphi) = -\int_0^\infty dp_- n_e(\varphi, p_-) \ln[n_e(\varphi, p_-)/n_0]$ associated to the electron distribution, where the Boltzmann constant has been set equal to unity and where the physically ineffective constant n_0 can be chosen, for example, such that $S(0) = 0$. By employing this definition and Eqs. (5), (6) and (7), it results

$$\frac{dS}{d\varphi} = -\frac{4\alpha m^2}{3\omega_0} \int_0^\infty \frac{dp_-}{p_-} \chi^2(\varphi, p_-) n_e \left\{ 1 - \frac{55\sqrt{3}}{32} \chi(\varphi, p_-) \left[1 + \frac{1}{6} \frac{p_-^2}{n_e^2} \left(\frac{\partial n_e}{\partial p_-} \right)^2 \right] \right\}. \quad (13)$$

This result further corroborates the idea that, while the classical “deterministic” evolution of the electrons implies a reduction of the entropy of the electrons, quantum corrections tend to increase it.

In order to show that the effects discussed above can be in principle measured with presently available laser and electron accelerator technology, we consider below two numerical examples. In both cases we assume a laser pulse with $f(\varphi) = \sin^2(\varphi/2N_L) \sin(\varphi)$ for $\varphi \in [0, \varphi_f] = [0, 2N_L\pi]$ and zero elsewhere, where N_L is the number of laser cycles and with $\omega_0 = 1.55$ eV, and an initial Gaussian electron distribution with a total number of 1000 electrons.

In the first numerical example, we choose the laser and electron parameters such that quantum effects are negligible, whereas RR effects are relatively large. We set $I_0 = 4.3 \times 10^{20}$ W/cm², $p_-^* = 84$ MeV (note that $\varepsilon^* \approx p_-^*/2 = 42$ MeV) such that $\chi^* = (p_-^*/m)(E_0/E_{cr}) \approx 5 \times 10^{-3}$, $\sigma_{p_-} = 8.4$ MeV, and $N_L = 1600$, corresponding to a pulse duration of about 4 ps. The results for the initial and final distribution are shown in Fig. 1. As expected, the final distribution $n_e(\varphi_f, p_-)$, calculated by solving numerically Eq. (2) (solid, red line) and the classical analytical solution $n_e^{LL}(\varphi_f, p_-)$ (see Eq. (11)) are very similar and both show a reduction of the width from the initial value 8.4 MeV to the final one 4.7 MeV. In the second numerical example, instead, we want to probe the quantum regime and we set $I_0 = 2 \times 10^{22}$ W/cm² [26], $p_-^* = 2$ GeV ($\varepsilon^* \approx 1$ GeV) and $\sigma_{p_-} = 0.2$ GeV corresponding to $\chi^* = 0.8$, and $N_L = 10$ corresponding to about 30 fs. Electron beams with such energies are nowadays available not only in conventional accelerators but also by employing plasma-based electron accelerators [33] (see also [34]), allowing in principle for an all-optical setup. The results of our numerical simulations are shown in Fig. 2. The figure shows that in the

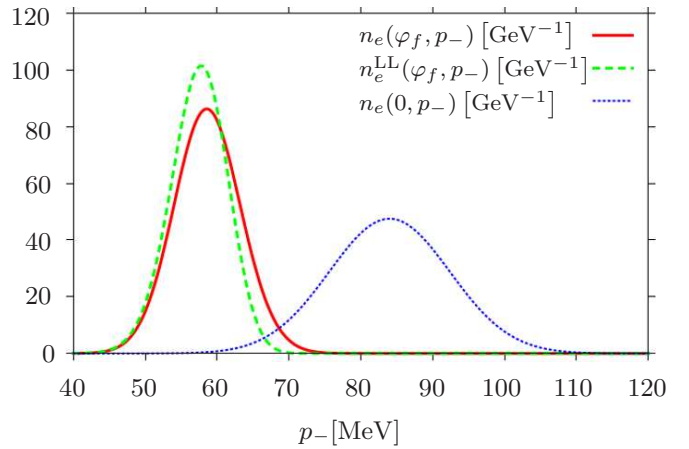


FIG. 1. Comparison of the initial electron distribution (dotted, blue line) and the final electron distribution according to Eq. (2) (solid, red line) and to Eq. (11) (dashed, green line). The laser and the electron distribution parameters are given in the text.

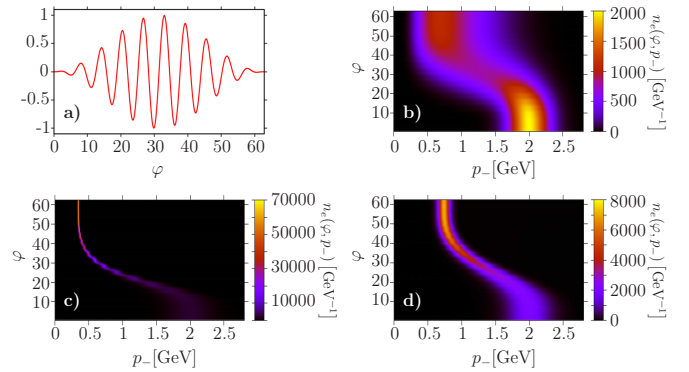


FIG. 2. Time evolution of the electron distribution for a 10-cycle \sin^2 -like laser pulse (part a)) according to Eq. (2) (part b)), to Eq. (11) (part c)) and to Eq. (8) and Eq. (9) with the replacement $I_{cl}(\varphi, p_-) \rightarrow I_q(\varphi, p_-)$ (part d)). The laser and the electron distribution parameters are given in the text.

quantum regime the full quantum calculations based on Eq. (2) predict a broadening of the electron distribution (Fig. 2b), according to our analysis above. Whereas, the classical calculations based on the exact solution in Eq. (11) (see Fig. 2c) predict a strong narrowing of the distribution. It is interesting to note that, according to the discussion above Eq. (12), if we consider the classical equation (8) and we substitute the classical intensity of radiation $I_{cl}(\varphi, p_-)$ with the quantum intensity $I_q(\varphi, p_-)$ (see, e.g., [24, 28]), the corresponding results (see Fig. 2d) still predict a narrowing of the distribution function. This clearly supports the idea that the broadening of the electron distribution is an effect of the importance of the stochasticity of the emission of radiation, which becomes substantial in the quantum regime.

In conclusion, we have demonstrated that the importance of the stochastic nature of the emission of radia-

tion in the quantum regime, has a profound impact on the evolution of an electron beam passing through an intense laser field. The stochasticity, in fact, induces a broadening on the electron momentum distribution, whereas classical theory of RR even predicts a narrowing of the distribution itself. A numerical example has shown the feasibility of measuring such effects by employing already demonstrated laser intensities and electron-beam energies.

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* dipiazza@mpi-hd.mpg.de

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