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Self-Consistent Neoclassical Transport Coefficients for Elongated Tokamaks

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Introduction:

For symmetric magnetic configurations (e.g. tokamaks), the flux-friction relation [1] is based on the equivalence of the $\underline{B}\cdot\underline{\nabla}B$ and the $\underline{B}\times\underline{\nabla}B$ terms in the drift-kinetic equation, the first being responsible for the friction of passing particles with trapped ones and the second for the radial drift, i.e. radial transport. In the conventional neoclassical theory, mono-energetic transport coefficients are defined by moments of the 1st order distribution function (involving flux-surface averaging and integration over the pitch, $p = v_\parallel/v$). The flux-friction relation leads to a coupling of the 3 mono-energetic transport coefficients, i.e. the particle transport coefficient, D_{11}, the bootstrap current coefficient, D_{31} (equivalent to the Ware pinch, D_{13} due to Onsager symmetry), and the parallel electric conductivity, D_{33}. With a database of elongated tokamak configurations, these mono-energetic transport coefficients are described self-consistently by analytic expressions.

Flux-Friction Relations:

The mono-energetic 1st order drift kinetic equation (DKE) is written in conservative form with spherical velocity-space coordinates, $p;\ v$

$$V(f) - C^p(f) = \underline{\nabla}\cdot\Big(\frac{\underline{B}}{B}\,pv + \frac{\underline{E}\times\underline{B}}{B_0^2}\Big)f + \frac{\partial}{\partial p}\,\dot{p}f - \nu\frac{\partial}{\partial p}\,(1-p^2)\,\frac{\partial f}{\partial p} \qquad (1)$$

with the incompressible form of the $\underline{E}\times\underline{B}$ drift, with $C^p(f)$ the Lorentz form of the pitch-angle collision term, and with

$$\dot{p} = -(1-p^2)\,v\,\frac{\underline{B}\cdot\underline{\nabla}B}{2B^2}.$$

The flux surface average, $\langle A\rangle$, and the averaged moment, $[\,A\,]$ are defined by

$$\langle A\rangle = \int A\frac{d\theta}{B}\cdot\Big(\int\frac{d\theta}{B}\Big)^{-1} \qquad \text{and} \qquad [\,A\,] = \int_{-1}^{1}\langle A\rangle\,dp.$$

Straightforward integration with

$$\Big\langle\,\underline{\nabla}\cdot\frac{\underline{B}}{B}\,f\,\Big\rangle = \Big\langle\,f\,\frac{\underline{B}\cdot\underline{\nabla}B}{B^2}\,\Big\rangle \qquad \text{and} \qquad \big\langle\,\underline{\nabla}\cdot(\underline{E}\times\underline{B})\,f\,\big\rangle = \big\langle\,(1-b)\,(\underline{E}\times\underline{B})\cdot\underline{\nabla}\,f\,\big\rangle$$

leads to

$$[\,pV(f) - pC^p(f)\,] = \Big[\,\frac{1+p^2}{2}\,v\,\frac{\underline{B}\cdot\underline{\nabla}B}{B^2}\,f\,\Big] + 2\nu\,[\,pf\,] \qquad (2)$$

where $b = B/B_0$, and the $\underline{E}\times\underline{B}$ drift effect can be neglected (2nd order in inverse aspect ratio, $\varepsilon = r/R$). For tokamaks, $B_\theta = \iota\varepsilon B_0$ with ι being the rotational transform yields

$$\underline{B}\cdot\underline{\nabla}B = -\iota\varepsilon\,(\underline{B}\times\underline{\nabla}B)\cdot\underline{e}_r$$

(with the unit vector $\underline{e}_r$ normal to the flux surface) coupling the radial flux, $\propto (\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r$, and the friction, $\propto \underline{B} \cdot \underline{\nabla} B$. This feature is not present in configurations with lacking symmetry.

The radial component of the ∇B-drift velocity is given by

$$\dot{r} = v_{\nabla B\,|r} = \frac{m}{qB}\,\frac{1+p^2}{2B^2} v^2\,(\underline{B} \times \underline{\nabla} B) \cdot \underline{e}_r,$$

and inserting in eq.(2) yields the flux-friction relation

$$v\,[\,pV(f) - pC^p(f)\,] = -\iota\varepsilon\,\omega_{c0}\,[\,b\,\dot{r}\,f\,] + 2\nu v\,[\,pf\,]. \tag{3}$$

By normalizing the DKE (1) ($\hat{X} = X\iota v/R$)

$$\hat{V} - \hat{C}_p(f) = \frac{\partial}{\partial\theta}\,\frac{p}{b}\,f + \frac{\partial b^{-1}}{\partial\theta}\,\frac{\partial}{\partial p}\,\frac{1-p^2}{2} f - \nu^*\,\frac{\partial}{\partial p}(1-p^2)\frac{\partial f}{\partial p}$$

with collisionality, $\nu^* = \nu R/\iota v$, the 1st order DKE for the radial transport and the bootstrap current is defined by

$$\hat{V}(f_1^*) - \hat{C}^p(f_1^*) = -\rho^*\,\frac{1+p^2}{4\iota\varepsilon}\,\frac{\partial b^{-2}}{\partial\theta}\,\frac{d\ln f_M}{d\ln r} \tag{4}$$

and for the ohmic current and the Ware pinch by

$$\hat{V}(g_1^*) - \hat{C}^p(g_1^*) = -\frac{u^*}{\iota}\,p \tag{5}$$

with $f_1^* = f_1/f_M$ ($g_1^* = g_1/f_M$), with “gyroradius” $\rho^* = v/r\omega_{c0}$, and “loop voltage” $u^* = (qR/TB)\,\underline{E} \cdot \underline{B}$ in normalized form. Both the radial transport (symmetric part in p of f_1^*) and the ohmic current (asymmetric part in p of g_1^*) are directly driven by the inhomogeneity. With eq.(3), the averaged moment equations are obtained

$$-\Gamma_{11}\,(1-\alpha_{11}) + 2\nu^*\,\Gamma_{31} = 0 \quad \text{and} \quad -\Gamma_{13}\,(1-\alpha_{13}) + 2\nu^*\,\Gamma_{33} = -\frac{2}{3}\,\frac{u^*}{\iota} \tag{6}$$

with the particle flux $\Gamma_{11} = [\dot{r}^* f_1^*]$, bootstrap current $\Gamma_{31} = [pf_1^*]$, Ware pinch $\Gamma_{13} = [\dot{r}^* g_1^*]$, and ohmic current $\Gamma_{33} = [pg_1^*]$; $\dot{r}^* = \dot{r}\,/\,\rho^* v = \frac{1}{2}\,(1+p^2)\,\partial b^{-1}/\partial\theta$, and the (small) finite aspect ratio corrections $\alpha_{11} = [(b-1)\,\dot{r}^* f_1^*]/[\dot{r}^* f_1^*]$ and $\alpha_{13} = [(b-1)\,\dot{r}^* g_1^*]/[\dot{r}^* g_1^*]$.

The Pfirsch-Schlüter (PS) contributions must be eliminated in both Γ_{11} and Γ_{33}. With the ansatz $f_1^* = \phi_0 + p\phi_1$ in the PS-regime, eq. (3) leads to

$$\phi_1 = \frac{\rho^*}{\iota\varepsilon}\,r(f_M)'\,\big(1 - \frac{1}{b^2}\big)\,b \quad \text{and} \quad {\Gamma_{11}}^{PS}(1-\alpha_{11}) = 2\nu^*[p^2\phi_1] = \frac{4}{3}\,\frac{\rho^*\nu^*}{\iota\varepsilon}\,\frac{d\ln f_M}{d\ln r}\,\Big\langle\big(1 - \frac{1}{b^2}\big)\,b\Big\rangle$$

For the simplified “standard model” (see Sec. 8.9 in ref.[2]) of an elongated tokamak $b = (1 + \kappa\varepsilon\,\cos\,\theta)^{-1}$ with the reduction of the toroidal curvature, κ (elongation $\simeq \kappa^{-2}$), $\langle(1-b^{-2})\,b\rangle = \kappa^2\varepsilon^2/2$ is obtained. The direct integration of the DKE (5) for large ν^* (neglecting the $\hat{V}$ contribution) leads to

$${\Gamma_{33}}^{PS} = -\frac{1}{3\iota\nu^*}\,\langle u^*\rangle.$$

Eliminating both PS contributions in eq.(6), dividing by the "thermodynamic forces" ($d \ln f_M / d \ln r$ and u^*, respectively), and using Onsager symmetry, $\hat{D}_{13} = -\hat{D}_{31}$, yields the system for the diffusion coefficients

$$-\left(\hat{D}_{11} - \hat{D}_{11}^{PS}\right)(1-\alpha_{11}) + 2\nu^* \hat{D}_{31} = 0 \tag{7}$$

$$\hat{D}_{31}(1-\alpha_{13}) + 2\nu^* \left(\hat{D}_{33} - \hat{D}_{33}^{PS}\right) = 0 \tag{8}$$

Database of elongated tokamak configurations:

The DKES code [3, 4] is used to calculate the 3 mono-energetic transport coefficients for a database of elongated tokamak configurations in an extended "standard model" $\underline{b} = \iota\varepsilon \underline{e}_\theta + (1+\kappa\varepsilon\cos\theta)^{-1}\underline{e}_\phi$. The database is defined in the ranges $0.26 \le \iota \le 1.04$, $0.0125 \le \varepsilon \le 0.4$, and $0.25 \le \kappa \le 1.0$. For up to 35 ν/v values ($10^{-8} \le \nu/v \le 10^3$) in 24 configurations, the diffusion coefficients are calculated with up to 250 Fourier modes and up to 1000 Legendre polynomials in the expansion of the distribution function at low collisionalities. At even lower ν/v, the accuracy strongly decreases. For estimating the PS contributions, 44 configurations are used (at $\nu/v = 10^3$ due to fast convergence to the PS limit). The test functions for the non-linear fitting are constructed mainly by "trial and error". This procedure is supported by the complete co-variance analysis of the least-squares fitting. The dependence on B and on R is known and therefore omitted for simplicity, i.e. $B = 1$ T and $R = 1$ m is used for the database ($D_{11} \propto 1/B^2$, $D_{31} \propto 1/B$, and D_{33} independent of B).

Fit results:

All representations of the mono-energetic transport coefficients, D_{ij}, are given in DKES notation and can be normalized by the plateau value for $\kappa = 1$, $D_{11}^n = \pi/(8\iota)$, the collisionless asymptote for $\kappa = 1$ and $\varepsilon \to 0$, $D_{31}^n = 0.9733/(\iota\sqrt{\varepsilon})$ [5], and the collisional limit, $D_{33}^n = 2v/3\nu$ (in DKES notation).
For the PS contributions, the best-fits are obtained by

$$D_{11}{}^{PS} = \frac{4\kappa^2}{3\iota^2}\,\frac{\nu}{v}\left(1 + 3.42\varepsilon^{3.6}(1 - 2.58\iota^{1.6}) - 0.6\varepsilon^2(1-\kappa^2)\right)$$

$$D_{33}{}^{PS} = \frac{2v}{3\nu}\left(1 - 1.18(\kappa\varepsilon)^{1.84} + 0.68\varepsilon^3\iota^{2.5}\right).$$

The averaged deviation of the fit to the DKES data is 0.7% (0.1%) for $D_{11}{}^{PS}$ ($D_{33}{}^{PS}$).
The convergence of the D_{31} coefficient to the collisionless asymptote is independently fitted in order to avoid the influence of (inaccurate) test functions in other collisionality regimes. For $\kappa = 1$ and in the limit $\varepsilon \to 0$, the result from analytic theory [5] is confirmed (i.e. D_{31}^n); the extension for elongation and finite ε in the banana regime is given by

$$D_{31}^b = 0.9733\sqrt{\frac{\kappa}{\varepsilon\iota^2}}\left(1 - 0.67(\kappa\varepsilon)^2\right)\cdot\left(1 + \frac{1.03}{\kappa\varepsilon^{2/3}\iota^{1/3}}\sqrt{\frac{\nu}{v}}\right)^{-1} \tag{9}$$

Finally, all 3 diffusion coefficients are fitted simultanously. With the contributions for the plateau and the PS regime, the bootstrap current coefficient is represented by

$$D_{31} = \left({D_{31}^{b}}^{-1.75} + {D_{31}^{pl}}^{-1.75} + {D_{31}^{PS}}^{-1.75} \right)^{-1/1.75}$$
$$\text{with} \quad D_{31}^{pl} = 0.39 \frac{\kappa^2 \varepsilon}{t} \frac{v t}{\nu} \quad \text{and} \quad D_{31}^{PS} = 0.068 \frac{\kappa^2 \varepsilon}{t} \left(\frac{v t}{\nu} \right)^2 ,$$

and the radial diffusion coefficient and the parallel electric conductivity by

$$D_{11} = {D_{11}}^{PS} + \frac{1}{\alpha} D_{31} \frac{\nu}{v} \; \text{ and } \; D_{33} = {D_{33}}^{PS} - \alpha \, D_{31} \frac{v}{\nu} \; \text{ with } \; \alpha = t\varepsilon \big(1 - 0.97 (\kappa \varepsilon)^{1.75} \big)$$

corresponding to eqs.(7,8). The average deviation of this representation for all 3 mono-energetic transport coefficients to the DKES data is 1.9%, and the difference is higher for the bootstrap current coefficient especially for large ε.

Discussion:

With the very accurate expansion of the 1st order distribution function, the D_{31}^{b} coefficient of eq. (9) calculated with DKES can be compared with other approaches. First of all, the analytical estimate of Ref. [5] is confirmed for $\kappa = 1$ and very small ε (the energy convolution leads to an additional factor 1.5 resulting in the given value of 1.46). In lowest order in ε and $\nu^* \to 0$, this representation is independent of the magnetic field model.

The situation is different, however, with respect to the finite ε-correction and the convergence to the collisionless asymptotic value. For the very low collisionalities, the ratio $D_{31}/D_{31}^{n} = (1+a_0)/(1+a_1\sqrt{\nu^*})$ corresponding to eq. (9) is fitted to DKES calculations for 3 different magnetic field models with $\varepsilon = 0.125$, $\kappa = 1$ and $t = 0.2614$: the model with only 1 b-Fourier mode, $b = 1 - \varepsilon \cos\theta$, yields $a_0 = -0.0347$ and $a_1 = 4.045$; the "standard" model, $b = (1 + \varepsilon\cos\theta)^{-1}$, yields $a_0 = -0.0091$ and $a_1 = 3.76$; and, finally, the model used in Ref. [6], $b = (1 + 2\varepsilon\cos\theta + 4\varepsilon^2\cos 2\theta)^{-1/2}$, yields $a_0 = -0.129$ and $a_1 = 4.98$. For the last model, $a_0 = 2.74\varepsilon \simeq +0.3425$ and $a_1 = 1$ obtained by a δf-Monte Carlo technique [6] is in contradiction to the results with DKES calculations. However, no access to the asymtotic lmfp-regime was possible in the δf-Monte Carlo simulations (which were restricted to $\nu^* \geq 10^3$).

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