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Relative Perturbation Theory for Quadratic Eigenvalue Problems

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Abstract

In this paper, we derive new relative perturbation bounds for eigenvectors and eigenvalues for regular quadratic eigenvalue problems of the form $\lambda^2 Mx + \lambda Cx + Kx = 0$, where M and K are nonsingular Hermitian matrices and C is a general Hermitian matrix. We base our findings on new results for an equivalent regular Hermitian matrix pair $A - \lambda B$. The new bounds can be applied to many interesting quadratic eigenvalue problems appearing in applications, such as mechanical models with indefinite damping. The quality of our bounds is demonstrated by several numerical experiments.

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1 Introduction

The quadratic eigenvalue problem (QEP) is to find scalars λ and nonzero vectors x satisfying

$$(\lambda^2 M + \lambda C + K)x = 0, \tag{1.1}$$

where M, C and K are $n \times n$ complex matrices. A major algebraic difference between the QEP and the standard (and also generalized) eigenvalue problem is that the QEP has $2n$ eigenvalues with up to $2n$ eigenvectors, and if there are more than n vectors they do not form a linearly independent set. The solution of the QEP is required in many applications arising in the dynamic analysis of structural mechanical and acoustic systems, in electronic circuit simulation, in fluid mechanics, in modeling microelectronic mechanical systems, and so on. The number

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of application of the QEP is constantly growing. In [11] and [19] an extensive theoretical background on the QEP and the other polynomial eigenvalue problems can be found, and in [28] one can found more about applications, mathematical properties, and a variety of numerical solution techniques for the QEP.

The aim of this paper is to present relative perturbation bounds for eigenvalues and eigenspaces of the QEP from (1.1), where M, C, and K are allowed to be indefinite Hermitian and C could be singular.

In general, the perturbation theory of the matrix or operator eigenvalue problems can be divided in two major parts. The first part belongs to the so-called standard or absolute perturbation theory which can be found in many well-known textbooks. Without minimizing the importance of the omitted titles, the important textbooks which contain the results on standard perturbation theories for matrices as well as operators are [17, 10, 3, 5, 25, 27].

On the other hand, in the late 80s and early 90s, the so-called relative perturbation theory becomes a very active research area. Again without minimizing the importance of the omitted titles, some important results of the relative perturbation theory can be found in [8, 2, 9]. The development of such a theory went back to as early as Kahan's technical report [16] in 1966.

Regarding the perturbation theory for the QEP in a general setting, the standard or absolute perturbation bounds are given, for example, in [24, 30, 28], but to the authors' knowledge there is no relative perturbation bounds for eigenvalues and especially for eigenspaces for the QEP, of which all three coefficient matrices could be arbitrary Hermitian.

Some results on the relative perturbations for the QEP can be found in [29]. There one can find the bounds for eigenvectors and eigenvalues for QEP (1.1), where the matrices M, C and K are positive definite Hermitian matrices and the condition

$$(x^H Cx)^2 - 4 \cdot x^H Mx x^H Kx > 0, \quad \forall x \in \mathbb{C}^n, x \neq 0,$$

is satisfied, which means that the corresponding QEP is hyperbolic. The more about hyperbolic QEP can be found in [28, 24].

In this paper we will derive similar bounds for the more general case, that is, QEP does not have to be hyperbolic and moreover we allow that the matrices M and K are nonsingular Hermitian and C is any Hermitian matrix. Such QEP arises in many applications. For example, in [26] negative definite matrix C causes increasing of the robotic system energy, and brake-squeal is based on the loss of stability of the brake-system which is caused by negative definite damping matrix C, see for example [1, 15]. Also, in [31, 18, 6], authors have considered stabilization of the unstable mechanical systems which is caused by indefinite damping matrix C. Therefore, we will illustrate our results on the numerical experiments motivated by these problems.

Our approach to the perturbations of the QEP will be based on a proper linearization and a construction of the appropriate relative perturbation bounds for obtained regular matrix pair. First, we will consider Hermitian regular matrix pairs $A - \lambda B$ with A and B Hermitian and derive corresponding relative perturbation bounds for eigenvalues and eigenspaces. Then, we will use these bounds to derive desirable eigenvalue and eigenvector bounds for QEP (1.1).

Until now, there is a vast amount of material in perturbation theory (covering absolute and relative perturbation results) for definite matrix pairs. Here we will list some of those results relating to the eigenspaces perturbations, where the distance between two eigensubspaces is measured by bounding the trigonometric function of the angle operator Θ associated with the eigensubspaces $\mathcal{X} = \mathrm{span}(X)$ and $\mathcal{Y} = \mathrm{span}(Y)$ of original and perturbed matrix pairs. The angle operator Θ is defined by

$$\Theta(\mathcal{X}, \mathcal{Y}) = \arcsin(P_{\mathcal{X}} - P_{\mathcal{Y}}),$$

where $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ are orthogonal projections on subspaces $\mathcal{X}$ and $\mathcal{Y}$, of the same dimension. The eigenvalues of the matrix Θ represent canonical angles. For example, in [27, 7] and in [14, 13, 29] one can find results from standard perturbation theory, and the relative perturbation bounds for eigenspaces, respectively. For the case that B is positive definite Hermitian in [14, 13] authors define angle operator Θ in the matrix-dependent scalar product $\langle x, y\rangle_B = y^T Bx$, $\forall x, y \in \mathbb{R}^n$ and measure the distance between the subspaces spanned by perturbed and original eigenvectors. For the case that the matrix B is Hermitian indefinite nonsingular and matrix pair (A, B) is definite, it is not possible to use this approach. For that case, as it has been shown in [29], it is possible to measure the distance between two subspaces, using the fact that $\|\sin\Theta(\mathcal{X}, \mathcal{Y})\| \to 0$ if and only if $\|Y^H BX\| \to 0$. For general regular matrix pairs, until now, there is no similar bound for eigenspaces and part of this paper is devoted to it.

This paper is outlined as follows. In Section 2 we derive our relative perturbation $\sin\Theta$ type theorems and a relative bound on eigenvalues for regular matrix pairs. We apply these bounds on QEP and derive relative perturbation bounds for eigenvectors and eigenvalues in Section 3. Numerical examples to illustrate our bounds are given in Section 4. Finally, some conclusions are summarized in Section 5.

Notations. Through this paper we use $\|\cdot\|_2$, $\|\cdot\|_F$ and $\|\cdot\|_{\mathrm{ui}}$ to denote the spectral matrix norm, the Frobenius norm and any unitary invariant matrix norm, respectively, where there is no danger of confusion. If A is a positive (semi-)definite Hermitian matrix, we will write $A \succ 0$ ($A \succeq 0$), and similarly $A \prec 0$ ($A \preceq 0$) for a negative (semi-)definite Hermitian A. We use the standard MATLAB notation $A(:, i)$ for the i-th column of the matrix A. Also, by F_m and G_m respectively we will denote the $m \times m$ matrices of the forms

$$F_m = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, G_m = \begin{bmatrix} & & 1 & 0 \\ & \iddots & \iddots & \\ 1 & \iddots & & \\ 0 & & & \end{bmatrix}.$$

Then the Jordan block of size m for eigenvalue λ is

$$J_m(\lambda) := \lambda I_m + G_m F_m = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}.$$

A matrix pair $A - \lambda B$ could also be written by (A, B). If A_1, B_1 have the same size and so do A_2, B_2, then the matrix pair $\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} - \lambda \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$ is also written by $(A_1 - \lambda B_1) \oplus (A_2 - \lambda B_2)$

or $(A_1, B_1) \oplus (A_2, B_2)$. For any matrices W, V of apt sizes, $W^H(A, B)V$ is used to denote $(W^H AV, W^H BV)$.

2 Relative perturbation bound for a Hermitian matrix pair

In this section, we will see that the bound of the angle between the eigenspaces of a regular Hermitian matrix pair is related to the bound of the solutions to the *structured Sylvester equations* $SX - XS' = T$, where S, S', T are special structured. Our approach goes more or less along the way of Davis and Kahan [7] and much more similarly to that of Li [22, 23].

In the very beginning, we state the spectral structure of a Hermitian matrix pair, which is the basis in the discussion of eigenspaces.

Lemma 2.1 ([12, Theorem 5.10.1], [20, Theorem 6.1]). *Every Hermitian matrix pair (A, B) is congruent to a Hermitian matrix pair of the form*

$$(0,0) \oplus \bigoplus_{j=1}^{p} (G_{2\varepsilon_j+1}, \begin{bmatrix} & & F_{\varepsilon_j} \\ & 0 & \\ F_{\varepsilon_j} & & \end{bmatrix}) \oplus \bigoplus_{j=1}^{r} \delta_j (F_{k_j}, G_{k_j}) \oplus \bigoplus_{j=1}^{q} \eta_j (\alpha_j F_{\ell_j} + G_{\ell_j}, F_{\ell_j}) \oplus \bigoplus_{j=1}^{s} (\begin{bmatrix} & \beta_j F_{m_j} + G_{m_j} \\ \overline{\beta_j} F_{m_j} + G_{m_j} & \end{bmatrix}, F_{2m_j}). \quad (2.1)$$

Here $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ and $k_1 \leq \cdots \leq k_p$ are positive integers, α_j are real numbers, β_j are complex nonreal numbers, δ_j, η_j are signs ($+1$ or -1). The form is uniquely determined by (A, B) up to a combination of permutations of the following types of blocks:

T0. $(0, 0)$;

T1. $(G_{2\varepsilon_j+1}, \begin{bmatrix} & & F_{\varepsilon_j} \\ & 0 & \\ F_{\varepsilon_j} & & \end{bmatrix}), \quad j = 1, \dots p$;

T2. $\delta_j (F_{k_j}, G_{k_j}), \quad j = 1, \dots r$;

T3. $\eta_j (\alpha_j F_{\ell_j} + G_{\ell_j}, F_{\ell_j}), \quad j = 1, \dots q$;

T4. $(\begin{bmatrix} & \beta_j F_{m_j} + G_{m_j} \\ \overline{\beta_j} F_{m_j} + G_{m_j} & \end{bmatrix}, F_{2m_j}), \quad j = 1, \dots s$, *with possible replacement of β_j by $\overline{\beta_j}$.*

Specifically, if (A, B) is a regular pair, then its canonical form only contains blocks of type **T2**, **T3**, **T4**.

To increase the similarity of the asymptotic behavior between these blocks and the corresponding eigenvalue, in the following we make an equivalent transformation:

1. For blocks of type **T3** with $\alpha_j \neq 0$, write $T = \operatorname{diag}(|\alpha_j|^{\frac{\ell_j-1}{2}}, |\alpha_j|^{\frac{\ell_j-1}{2}-1}, \dots, |\alpha_j|^{-\frac{\ell_j-1}{2}})$, and then $T^H F_{\ell_j} T = F_{\ell_j}$ and $T^H G_{\ell_j} T = |\alpha_j| G_{\ell_j}$, so the new block pair is of type $\eta_j(\alpha_j F_{\ell_j} + |\alpha_j| G_{\ell_j}, F_{\ell_j})$.

2. For blocks of type **T4**, write $T = \operatorname{diag}(|\beta_j|^{\frac{m_j-1}{2}}, |\beta_j|^{\frac{m_j-1}{2}-1}, \dots, |\beta_j|^{-\frac{m_j-1}{2}})$, and then $T^H F_{m_j} T = F_{m_j}$ and $T^H G_{m_j} T = |\beta_j| G_{m_j}$; noticing

$$\begin{bmatrix} T & \\ & T \end{bmatrix}^H \begin{bmatrix} & S \\ S^H & \end{bmatrix} \begin{bmatrix} T & \\ & T \end{bmatrix} = \begin{bmatrix} & T^H S T \\ T^H S^H T & \end{bmatrix},$$

   the new block pair is of type

$$(\begin{bmatrix} & \beta_j F_{m_j} + |\beta_j| G_{m_j} \\ \overline{\beta_j} F_{m_j} + |\overline{\beta_j}| G_{m_j} & \end{bmatrix}, F_{2m_j}).$$

Then we have this variant canonical form:

Lemma 2.2. *Every regular Hermitian matrix pair (A, B) is congruent to a Hermitian matrix pair of the form*

$$\bigoplus_{j=1}^{r} \delta_j (F_{k_j}, G_{k_j}) \oplus \bigoplus_{j=1}^{q'} \eta'_j (G_{\ell'_j}, F_{\ell'_j}) \oplus \bigoplus_{j=1}^{q} \eta_j (\alpha_j F_{\ell_j} + |\alpha_j| G_{\ell_j}, F_{\ell_j}) \oplus \bigoplus_{j=1}^{s} (\begin{bmatrix} & \beta_j F_{m_j} + |\beta_j| G_{m_j} \\ \overline{\beta_j} F_{m_j} + |\overline{\beta_j}| G_{m_j} & \end{bmatrix}, F_{2m_j}). \quad (2.2)$$

Here k_j are positive integers, α_j are real numbers, β_j are complex nonreal numbers, $\delta_j, \eta'_j, \eta_j$ are signs ($+1$ or -1). The form is uniquely determined by (A, B) up to a combination of permutations of the following types of blocks:

R1. $\delta_j (F_{k_j}, G_{k_j}), \quad j = 1, \dots r$;

R2. $\eta'_j (G_{\ell'_j}, F_{\ell'_j}), \quad j = 1, \dots q'$;

R3. $\eta_j (\alpha_j F_{\ell_j} + |\alpha_j| G_{\ell_j}, F_{\ell_j}), \quad j = 1, \dots q$;

R4. $(\begin{bmatrix} & \beta_j F_{m_j} + |\beta_j| G_{m_j} \\ \overline{\beta_j} F_{m_j} + |\overline{\beta_j}| G_{m_j} & \end{bmatrix}, F_{2m_j}), \quad j = 1, \dots s$, *with possible replacement of β_j by $\overline{\beta_j}$.*

Next, we begin to consider the structured Sylvester equations. Besides the references given above, structured Sylvester equations and their connection to the $\sin\Theta$ type theorems can be found in, e.g., [21]. According to our needs, we derive this result:

Lemma 2.3. *Given same-size matrices M, N. Suppose Λ, Ω and Λ', Ω' are block pairs of the type **R2**, **R3**, **R4** in (2.2). Write $\lambda(\Lambda,\Omega) = \{\underbrace{\lambda,\dots,\lambda}_{n}\}$ or $\{\underbrace{\lambda,\overline{\lambda},\dots,\lambda,\overline{\lambda}}_{2n}\}$ and $\lambda(\Lambda',\Omega') = \{\underbrace{\lambda',\dots,\lambda'}_{n'}\}$ or $\{\underbrace{\lambda',\overline{\lambda'},\dots,\lambda',\overline{\lambda'}}_{2n'}\}$. For any $\alpha > 0, m \in \mathbb{N}$, define*

$$\varphi_-(\alpha,m) := \alpha^{1-m}\frac{1-\alpha^m}{1-\alpha}, \quad \varphi_+(\alpha,m) := \alpha^{-m}\frac{1+\alpha-2\alpha^m}{1-\alpha}.$$

If $\lambda(\Lambda,\Omega)\cap\lambda(\Lambda',\Omega') = \emptyset$, then each of the equations

$$(\Omega'\Lambda')^H Y - Y\Omega\Lambda = -(\Omega'\Lambda')^H M + N\Omega\Lambda \tag{2.3a}$$
$$Y - (F_{n'}G_{n'})^H Y\Omega\Lambda = -M + (F_{n'}G_{n'})^H N\Omega\Lambda \tag{2.3b}$$

has a unique solution Y whose size is the same as M, N. Moreover, writing $\lambda' = \infty$ for (2.3b), then

$$\|Y\|_F \le \alpha_1(\lambda,\lambda',n,n')\|M\|_F + \alpha_2(\lambda,\lambda',n,n')\|N\|_F,$$

where

$$\begin{aligned}
\alpha_1(\lambda,\lambda',n,n') &= \tbinom{n'+n-1}{n}\varphi_-(\gamma,n)\varphi_+(\gamma',n'), & \alpha_2(\lambda,\lambda',n,n') &= \tbinom{n'+n-1}{n'}\varphi_-(\gamma',n')\varphi_+(\gamma,n);\\
\alpha_1(0,\lambda',n,n') &= \tbinom{n'+n-1}{n}|\lambda'|^{-1}\varphi_-(|\lambda'|,n-1)+1, & \alpha_2(0,\lambda',n,n') &= \tbinom{n'+n-1}{n}|\lambda'|^{-1}\varphi_-(|\lambda'|,n-1);\\
\alpha_1(\lambda,0,n,n') &= \tbinom{n'+n-1}{n'}|\lambda|^{-1}\varphi_-(|\lambda|,n'-1), & \alpha_2(\lambda,0,n,n') &= \tbinom{n'+n-1}{n'}|\lambda|^{-1}\varphi_-(|\lambda|,n'-1)+1;\\
\alpha_1(\lambda,\infty,n,n') &= 2|\lambda|\varphi_-(\frac{1}{2|\lambda|},n'-1)+1, & \alpha_2(\lambda,\infty,n,n') &= 2|\lambda|\varphi_-(\frac{1}{2|\lambda|},n'-1);\\
\alpha_1(0,\infty,n,n') &= \min\{n,n'\}, & \alpha_2(0,\infty,n,n') &= \min\{n,n'\}-1,
\end{aligned} \tag{2.4}$$

and $\gamma := \min\{\left|\frac{\lambda'-\lambda}{\lambda}\right|, \left|\frac{\overline{\lambda'}-\lambda}{\lambda}\right|\}, \gamma' := \min\{\left|\frac{\lambda'-\lambda}{\lambda'}\right|, \left|\frac{\overline{\lambda'}-\lambda}{\lambda'}\right|\}$. Note that in (2.4) $\lambda, \lambda' \notin \{0,\infty\}$ implicitly.

The proof of Lemma 2.3 is long and contains complicated computations, so we defer it to Appendix A.

In the next, we will present the relative perturbation theory for the Hermitian pairs, using the above results.

Consider two Hermitian matrix pairs (A,B) and $(\widetilde{A},\widetilde{B})$, and two nonsingular matrices $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and $\widetilde{X} = \begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 \end{bmatrix}$ for which these assumptions hold:

(A1) these two pairs are both regular.

(A2) $X, \widetilde{X}$ satisfy

$$X^H(A,B)X = (\Lambda,\Omega) = (\Lambda_1,\Omega_1)\oplus(\Lambda_2,\Omega_2),$$
$$\widetilde{X}^H(\widetilde{A},\widetilde{B})\widetilde{X} = (\widetilde{\Lambda},\widetilde{\Omega}) = (\widetilde{\Lambda}_1,\widetilde{\Omega}_1)\oplus(\widetilde{\Lambda}_2,\widetilde{\Omega}_2),$$

where $(\Lambda_i, \Omega_i), i = 1, 2$ and $(\widetilde{\Lambda}_i, \widetilde{\Omega}_i), i = 1, 2$ are of the form (2.2) and $\Lambda_i, \Omega_i, \widetilde{\Lambda}_i, \widetilde{\Omega}_i$ have the same size for $i = 1, 2$ respectively. In detail, we could write

$$(\Lambda_i, \Omega_i) = \bigoplus_{j_i=1}^{m_i} (\Lambda_{i,j_i}, \Omega_{i,j_i}), \quad i = 1, 2, (\widetilde{\Lambda}_i, \widetilde{\Omega}_i) = \bigoplus_{j_i=1}^{m_i} (\widetilde{\Lambda}_{i,j_i}, \widetilde{\Omega}_{i,j_i}), \quad i = 1, 2, \tag{2.5}$$

where $(\Lambda_{i,j_i}, \Omega_{i,j_i})$ are blocks in (2.2) of size n_{i,j_i} and the corresponding eigenvalue is λ_{i,j_i}, and similarly we have $(\widetilde{\Lambda}_{i,j_i}, \widetilde{\Omega}_{i,j_i}), \widetilde{n}_{i,j_i}, \widetilde{\lambda}_{i,j_i}$.

(A3) it holds that

$$\lambda(\Lambda_1, \Omega_1) \cap \lambda(\Lambda_2, \Omega_2) = \emptyset, \quad \lambda(\widetilde{\Lambda}_1, \widetilde{\Omega}_1) \cap \lambda(\widetilde{\Lambda}_2, \widetilde{\Omega}_2) = \emptyset,$$
$$\lambda(\Lambda_1, \Omega_1) \cap \lambda(\widetilde{\Lambda}_2, \widetilde{\Omega}_2) = \emptyset, \quad \lambda(\Lambda_2, \Omega_2) \cap \lambda(\widetilde{\Lambda}_1, \widetilde{\Omega}_1) = \emptyset.$$

(A4) it holds that

$$\infty \notin \lambda(\Lambda_1, \Omega_1), \quad 0 \notin \lambda(\Lambda_2, \Omega_2),$$
$$\infty \notin \lambda(\widetilde{\Lambda}_1, \widetilde{\Omega}_1), \quad 0 \notin \lambda(\widetilde{\Lambda}_2, \widetilde{\Omega}_2),$$

which implies $\Omega_1, \widetilde{\Omega}_1, \Lambda_2, \widetilde{\Lambda}_2$ are nonsingular.

Then we can state the central theorem in this paper.

Theorem 2.1. *Let (A, B) and $(\widetilde{A}, \widetilde{B}) = (A + \delta A, B + \delta B)$ be regular Hermitian matrix pairs and X and $\widetilde{X}$ be nonsingular matrices. If the assumptions (A1)–(A4) hold for perturbed and unperturbed matrix pairs, then we have*

$$\|\sin \Theta(\operatorname{span}(X_1), \operatorname{span}(\widetilde{X}_1))\|_F \le \alpha_1^m \kappa_2(X)\kappa_2(\widetilde{X})\|A^+\delta A\|_F + \alpha_2^m \|X\|_2^2 \|\widetilde{X}^{-1}\|_2^2 \|\widetilde{B}^+\delta B\|_F, \tag{2.6}$$

where

$$\alpha_1^m := \max_{j_1,j_2} \alpha_1(\widetilde{\lambda}_{1,j_1}, \lambda_{2,j_2}, \widetilde{n}_{1,j_1}, n_{2,j_2}), \quad \alpha_2^m := \max_{j_1,j_2} \alpha_2(\widetilde{\lambda}_{1,j_1}, \lambda_{2,j_2}, \widetilde{n}_{1,j_1}, n_{2,j_2}), \tag{2.7}$$

and $\alpha_1(\cdot,\cdot,\cdot,\cdot), \alpha_2(\cdot,\cdot,\cdot,\cdot)$ are defined in (2.4). Moreover, if B is nonsingular,

$$\|\sin \Theta(\operatorname{span}(X_1), \operatorname{span}(\widetilde{X}_1))\|_F \le \kappa_2(X)\kappa_2(\widetilde{X}) \left(\alpha_1^m \|A^+\delta A\|_F + \alpha_2^m \|B^{-1}\delta B\|_F\right). \tag{2.8}$$

Proof. First, similar to the discussion in [29, Lemma 2],

$$\|\sin \Theta(\operatorname{span}(X_1), \operatorname{span}(\widetilde{X}_1))\|_F \le \|(BX_2)^+\|_2 \|\widetilde{X}_1^+\|_2 \|\|X_2^H B\widetilde{X}_1\|_F.$$

Since $BX_2 = X^{-H}\begin{bmatrix} 0 \\ \Omega_2 \end{bmatrix}$, $\|BX_2\|_2 \le \|X^{-1}\|_2$ and $\|(BX_2)^+\|_2 = \|\begin{bmatrix} 0 & \Omega^H \end{bmatrix}^+ X\|_2 \le \|X\|_2$. Similarly, $\|\widetilde{B}\widetilde{X}_1\|_2 \le \|\widetilde{X}^{-1}\|_2$.

Then, to bound the $\|\sin\Theta(\text{span}(X_1),\text{span}(\widetilde{X}_1))\|_F$ we need to estimate $\|X_2^H B\widetilde{X}_1\|_F$. Using (2.5), define L, R by:

$$L = L_1 \oplus L_2 = \bigoplus_{i=1}^{2}\bigoplus_{j_i=1}^{m_i} L_{i,j_i}, \quad R = R_1 \oplus R_2 = \bigoplus_{i=1}^{2}\bigoplus_{j_i=1}^{m_i} R_{i,j_i},$$

where $L_{i,j_i} = \Lambda_{i,j_i}\Omega_{i,j_i}, R_{i,j_i} = I$ if the corresponding block pair is of type **R1** or $L_{i,j_i} = I, R_{i,j_i} = \Omega_{i,j_i}\Lambda_{i,j_i}$ if not. Then, it is easy to check that L and R are lower triangular matrices and also, $L_1 = I$. Note that

$$X^H AXL = \Lambda L = \Omega R = X^H BXR$$

and then

$$AXL = BXR, \quad \text{and similarly} \quad \widetilde{A}\widetilde{X}\widetilde{L} = \widetilde{B}\widetilde{X}\widetilde{R}. \tag{2.9}$$

Also, $\widetilde{L}$ and $\widetilde{R}$ are lower triangular matrices and $\widetilde{L}_1 = I$. Then

$$\begin{aligned} R_2^H X_2^H B\widetilde{X}_1\widetilde{L}_1 - L_2^H X_2^H B\widetilde{X}_1\widetilde{R}_1 &= R_2^H X_2^H B\widetilde{X}_1\widetilde{L}_1 - L_2^H X_2^H \widetilde{B}\widetilde{X}_1\widetilde{R}_1 + L_2^H X_2^H \delta B\widetilde{X}_1\widetilde{R}_1 \\ &= L_2^H X_2^H A\widetilde{X}_1\widetilde{L}_1 - L_2^H X_2^H \widetilde{A}\widetilde{X}_1\widetilde{L}_1 + L_2^H X_2^H \delta B\widetilde{X}_1\widetilde{R}_1 \\ &= -L_2^H X_2^H \delta A\widetilde{X}_1\widetilde{L}_1 + L_2^H X_2^H \delta B\widetilde{X}_1\widetilde{R}_1. \end{aligned}$$

Note that $AX_2 = X^{-H}\begin{bmatrix} 0 \\ \Lambda_2 \end{bmatrix}$. Since Λ_2 is nonsingular, AX_2 is column full rank, which infers $\text{span}(X_2) \cap \mathcal{N}(A) = \emptyset$, where $\mathcal{N}(A)$ is the nullspace of A. Note that $I - A^+A$ is the orthogonal projector onto $\mathcal{N}(A)$. Thus, $A^+AX_2 = X_2$. Then

$$\begin{aligned} R_2^H X_2^H B\widetilde{X}_1\widetilde{L}_1 - L_2^H X_2^H B\widetilde{X}_1\widetilde{R}_1 &= -L_2^H X_2^H AA^+\delta A\widetilde{X}_1\widetilde{L}_1 + L_2^H X_2^H \delta B\widetilde{X}_1\widetilde{R}_1 \\ &= -R_2^H X_2^H BA^+\delta A\widetilde{X}_1\widetilde{L}_1 + L_2^H X_2^H \delta B\widetilde{X}_1\widetilde{R}_1. \end{aligned} \tag{2.10}$$

To estimate norm of the solution of structured Sylvester equation (2.10) we will use this consideration: for the equation $S_1^H YT_1 - S_2^H YT_2 = S_1^H MT_1 - S_2^H NT_2$, the solution Y can be written as $Y = Y^{(1)} + Y^{(2)}$, where $Y^{(1)}$ and $Y^{(2)}$ are the solutions of equations $S_1^H YT_1 + S_2^H YT_2 = S_1^H MT_1$ and $S_1^H YT_1 + S_2^H YT_2 = -S_2^H NT_2$ respectively. Estimate these two $Y^{(i)}$ and then use $\|Y\| \le \|Y^{(1)}\| + \|Y^{(2)}\|$ (for any norm) to obtain the result. For $Y^{(1)}$, or equivalently $\delta B = 0$: Thus, for $j_1 = 1, \dots, m_1$ and $j_2 = 1, \dots, m_2$,

$$R_{2,j_2}^H Y_{j_2,j_1}^{(1)} \widetilde{L}_{1,j_1} - L_{2,j_2}^H Y_{j_2,j_1}^{(1)} \widetilde{R}_{1,j_1} = -R_{2,j_2}^H X_{2,j_2}^H BA^+\delta A\widetilde{X}_{1,j_1}\widetilde{L}_{1,j_1}.$$

Noticing $\widetilde{L}_1 = I$, this equation on $Y_{j_2,j_1}^{(1)}$ is of form in (2.3). By Lemma 2.3,

$$\|Y_{j_2,j_1}^{(1)}\|_F \le \alpha_1(\widetilde{\lambda}_{1,j_1}, \lambda_{2,j_2}, \widetilde{n}_{1,j_1}, n_{2,j_2})\|X_{2,j_2}^H BA^+\delta A\widetilde{X}_{1,j_1}\|_F.$$

Then

$$\|Y^{(1)}\|_F = \sqrt{\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2} \|Y_{j_2,j_1}^{(1)}\|_F^2}$$

$$
\begin{aligned}
&\leq \max \alpha_1 \sqrt{\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2} \|X_{2,j_2}^H BA^+\delta A\widetilde{X}_{1,j_1}\|_F^2} \\
&\leq \max \alpha_1 \|X_2^H BA^+\delta A\widetilde{X}_1\|_F \\
&\leq \max \alpha_1 \|BX_2\|_2 \|\widetilde{X}_1\|_2 \|A^+\delta A\|_F \\
&\leq \max \alpha_1 \|X^{-1}\|_2 \|\widetilde{X}\|_2 \|A^+\delta A\|_F.
\end{aligned}
$$

For $Y^{(2)}$,

$$\|Y^{(2)}\|_F \leq \max \alpha_2 \|X_2^H \delta B\widetilde{X}_1\|_F.$$

For the case that B is nonsingular,

$$\|Y^{(2)}\|_F \leq \max \alpha_2 \|X^{-1}\|_2 \|\widetilde{X}\|_2 \|B^{-1}\delta B\|_F.$$

Thus (2.8) holds.

For the general case, since Ω_1 is nonsingular, $\widetilde{B}^+\widetilde{B}\widetilde{X}_1 = \widetilde{X}_1$. Then

$$\|Y^{(2)}\|_F \leq \max \alpha_2 \|X\|_2 \|\widetilde{X}^{-1}\|_2 \|B^{-1}\delta B\|_F.$$

Thus (2.6) holds. □

Also, a more general but weaker result can be given for any unitarily invariant norm.

Theorem 2.2. *Given a Hermitian matrix pair (A, B) and its corresponding perturbed pair $(\widetilde{A}, \widetilde{B})$ with $B, \widetilde{B}$ nonsingular (which guarantees assumption (A1)). Under the assumptions (A2)–(A4), if there exist $\alpha \geq 0$ and $\delta > 0$ such that*

$$
\begin{aligned}
&\|\Omega_2\Lambda_2\|_2 \leq \alpha, \quad \|(\widetilde{\Omega}_1\widetilde{\Lambda}_1)^{-1}\|_2^{-1} \geq \alpha+\delta, \qquad \textit{or} \\
&\|(\Omega_2\Lambda_2)^{-1}\|_2^{-1} \geq \alpha+\delta, \quad \|\widetilde{\Omega}_1\widetilde{\Lambda}_1\|_2 \leq \alpha,
\end{aligned}
$$

then, for any unitarily invariant norm $\|\cdot\|_{\mathrm{ui}}$ and p, q where $p^{-1}+q^{-1}=1$,

$$\|\sin\Theta(\operatorname{span}(X_1), \operatorname{span}(\widetilde{X}_1))\|_{\mathrm{ui}} \leq \mu\kappa_2(X)\kappa_2(\widetilde{X}) \sqrt[q]{\|A^+\delta A\|_{\mathrm{ui}}^q + \|B^{-1}\delta B\|_{\mathrm{ui}}^q},$$

where $\mu = \frac{\delta}{\sqrt[p]{\alpha^p+(\alpha+\delta)^p}}$; also

$$\|\sin\Theta(\operatorname{span}(X_1), \operatorname{span}(\widetilde{X}_1))\|_{\mathrm{ui}} \leq \kappa_2(X)\kappa_2(\widetilde{X}) \left(\frac{\|A^+\delta A\|_{\mathrm{ui}}}{\frac{\delta}{\alpha}} + \frac{\|B^{-1}\delta B\|_{\mathrm{ui}}}{\frac{\delta}{\alpha+\delta}} \right).$$

Proof. The proof is similar to Theorem 2.1, except a modified version of [22, Lemma 2.3] is used to estimate (2.10) other than Lemma 2.3. The "modified" version is to get rid of the assumption "Ω, Γ are Hermitian". That assumption appears there for using [22, Lemma 2.2] to make clear that (2.10) has a unique solution. But now that is guaranteed by Lemma 2.3. □

Remark 2.1. Theorem 2.2 is not that useful for a general case. If there is no semi-simple eigenvalue, we have this bounds

$$\|\Omega_2\Lambda_2\|_2 \le 2\max_{i_2}|\lambda_{2,i_2}|, \quad \|(\widetilde{\Omega}_1\widetilde{\Lambda}_1)^{-1}\|_2 \le \max_{\widetilde{i}_1} n_{\widetilde{i}_1}|\widetilde{\lambda}^{-1}_{1,\widetilde{i}_1}|,$$

which means the gap must be bigger than $(2-\frac{1}{2})\alpha + 2\delta$, which is not the expected one, namely $k\delta$ for some scalar k. This theorem is good only for the case that all eigenvalues are semi-simple.

Besides, we can have eigenvalue perturbation bounds:

Theorem 2.3. *Given a Hermitian matrix pair (A, B) and its corresponding perturbed pair $(\widetilde{A}, \widetilde{B})$ with $B, \widetilde{B}$ nonsingular (which guarantees assumption (A1)). Under the assumptions (A2)–(A4), consider λ and $\widetilde{\lambda}$ are eigenvalues of the two matrix pairs respectively and X_λ and $\widetilde{X}_{\widetilde{\lambda}}$ are the corresponding invariant subspaces containing Jordan chains with dimensions $\ell, \widetilde{\ell}$ respectively. Suppose that $\lambda, \widetilde{\lambda}$ both lie in the upper half plane ($\mathrm{Im}(\lambda) \ge 0, \mathrm{Im}(\widetilde{\lambda}) \ge 0$) but nonzero, and write $\gamma = \left|\frac{\widetilde{\lambda}-\lambda}{\lambda}\right|, \widetilde{\gamma} = \left|\frac{\widetilde{\lambda}-\lambda}{\widetilde{\lambda}}\right|$. Then,*

1. *provided $\ell = \widetilde{\ell} = 1$, or equivalently, both eigenvalues are semi-simple respectively, and then rewriting the eigenvectors as $X_\lambda = x, \widetilde{X}_{\widetilde{\lambda}} = \widetilde{x}$,*

$$\gamma \le \frac{\|X^{-1}\|_2\|\widetilde{X}\|_2(\|A^{+}\delta A\|_{\mathrm{ui}} + \|B^{-1}\delta B\|_{\mathrm{ui}})}{|x^H\widetilde{B}\widetilde{x}|} \le \frac{\|X^{-1}\|_2\|\widetilde{X}\|_2(\|A^{+}\delta A\|_{\mathrm{ui}} + \|B^{-1}\delta B\|_{\mathrm{ui}})}{|x^H B\widetilde{x}| - \|X^{-1}\|_2\|\widetilde{X}\|_2\|B^{-1}\delta B\|_{\mathrm{ui}}};$$

2. *provided $\gamma \le \frac{5-\sqrt{17}}{4}, \widetilde{\gamma} \le \frac{5-\sqrt{17}}{4}$,*

$$\frac{\gamma^{\ell}\widetilde{\gamma}^{\widetilde{\ell}}}{\sqrt{\gamma^2+\widetilde{\gamma}^2}} \le 2\frac{\|X^{-1}\|_2\|\widetilde{X}\|_2}{\|X_\lambda^H B\widetilde{X}_{\widetilde{\lambda}}\|_F}\sqrt{\binom{\widetilde{\ell}+\ell-1}{\ell}^2\|A^{+}\delta A\|_F^2 + \binom{\widetilde{\ell}+\ell-1}{\widetilde{\ell}}^2\|B^{-1}\delta B\|_F^2}.$$

Proof. Adopt in the same definition of $L, R, \widetilde{L}, \widetilde{R}$ in the proof of Theorem 2.1. Since $B, \widetilde{B}$ are nonsingular, $L = I, \widetilde{L} = I$. By (2.9),

$$\begin{aligned} R^H X^H B\widetilde{X} - X^H B\widetilde{X}\widetilde{R} &= -X^H\delta A\widetilde{X} + X^H\delta B\widetilde{X}\widetilde{R} \\ &= -R^H X^H BA^{+}\delta A\widetilde{X} + X^H\delta B\widetilde{X}\widetilde{R} \end{aligned}$$

Then, for $j_1 = 1, \dots, m_1$ and $j_2 = 1, \dots, m_2$,

$$R_{j_2}^H X_{j_2}^H B\widetilde{X}_{j_1} - X_{j_2}^H B\widetilde{X}_{j_1}\widetilde{R}_{j_1} = -R_{j_2}^H X_{j_2}^H BA^{+}\delta A\widetilde{X}_{j_1} + X_{j_2}^H\delta B\widetilde{X}_{j_1}\widetilde{R}_{j_1}.$$

By $BX = X^{-H}\bigoplus_{i=1}^{m_2} F_{n_{2,i}}$,

$$R_{j_2}^H X_{j_2}^H B\widetilde{X}_{j_1} - X_{j_2}^H B\widetilde{X}_{j_1}\widetilde{R}_{j_1} = -R_{j_2}^H F_{n_{j_2}} X^{-1}A^{+}\delta A\widetilde{X}_{j_1} + F_{n_{j_2}} X^{-1}B^{-1}\delta B\widetilde{X}_{j_1}\widetilde{R}_{j_1}.$$

If $\ell = \widetilde{\ell} = 1$, then left-multiplying by $e_{n_{j_2}}^H$ and right-multiplying by $e_{n_{j_1}}$ (or $e_{\frac{n_{j_1}}{2}}$ to make $\lambda_{j_2}, \widetilde{\lambda}_{j_2}$ both in the upper half plane), or equivalently consider the element in the southeast corner, also getting rid of the subscripts to coincide with the lemma, the equation gives

$$\left(\lambda - \widetilde{\lambda}\right) x^H B\widetilde{x} = -\lambda e^H X^{-1} A^{+} \delta A \widetilde{x} + \widetilde{\lambda} e^H X^{-1} B^{-1} \delta B \widetilde{x}.$$

Thus,

$$\frac{\widetilde{\lambda} - \lambda}{\lambda} = \frac{e^H X^{-1} A^{+} \delta A \widetilde{x} - e^H X^{-1} B^{-1} \delta B \widetilde{x}}{x^H B\widetilde{x} + x^H \delta B \widetilde{x}} = \frac{e^H X^{-1} (A^{+} \delta A - B^{-1} \delta B) \widetilde{x}}{x^H \widetilde{B} \widetilde{x}},$$

which infers item 1.

Otherwise, by Lemma 2.3, also getting rid of the subscripts to coincide with the lemma,

$$\|X_\lambda^H B \widetilde{X}_{\widetilde{\lambda}}\|_F \le \alpha_1 \|F X^{-1} A^{+} \delta A \widetilde{X}_{\widetilde{\lambda}}\|_F + \alpha_2 \|F X^{-1} B^{-1} \delta B \widetilde{X}_{\widetilde{\lambda}}\|_F,$$

where, noticing that $\gamma \le \frac{5-\sqrt{17}}{4}, \widetilde{\gamma} \le \frac{5-\sqrt{17}}{4}$ give $\frac{1}{1-\gamma}\frac{1+\widetilde{\gamma}}{1-\widetilde{\gamma}} \le 2$,

$$\alpha_1 = \binom{\widetilde{\ell}+\ell-1}{\ell} \gamma^{1-\ell} \frac{1-\gamma^{\ell}}{1-\gamma} \widetilde{\gamma}^{-\widetilde{\ell}} \frac{1+\widetilde{\gamma}-2\widetilde{\gamma}^{\widetilde{\ell}}}{1-\widetilde{\gamma}} \le 2\binom{\widetilde{\ell}+\ell-1}{\ell} \gamma^{1-\ell} \widetilde{\gamma}^{-\widetilde{\ell}},$$

and similarly $\alpha_2 \le 2\binom{\widetilde{\ell}+\ell-1}{\widetilde{\ell}} \widetilde{\gamma}^{1-\widetilde{\ell}} \gamma^{-\ell}$, and then,

$$\|X_\lambda^H B \widetilde{X}_{\widetilde{\lambda}}\|_F \le 2\widetilde{\gamma}^{-\widetilde{\ell}} \gamma^{-\ell} \|X^{-1}\|_2 \|\widetilde{X}\|_2 [\gamma \binom{\widetilde{\ell}+\ell-1}{\ell} \|A^{+} \delta A\|_F + \widetilde{\gamma} \binom{\widetilde{\ell}+\ell-1}{\widetilde{\ell}} \|B^{-1} \delta B\|_F],$$

which infers item 2. □

3 Relative perturbation bound for a regular Hermitian QEP

In this section we will derive perturbation bounds for eigenvectors and eigenvalues for the QEP, using results from the previous section.

Given $M, C, K \in \mathbb{C}^{n\times n}$, the quadratic matrix polynomial of order n is defined by

$$Q(\lambda) = \lambda^2 M + \lambda C + K. \tag{3.1}$$

We denote the spectra of $Q(\lambda)$ by

$$\Lambda(Q) := \{\lambda \in \mathbb{C} : \det Q(\lambda) = 0\},$$

which is the multiset of eigenvalues of $Q(\lambda)$. $Q(\lambda)$ is called regular if $\det Q(\lambda)$ is not identically zero for $\lambda \in \mathbb{C}$ and singular otherwise. In this section we assume that $Q(\lambda)$ is regular. The QEP for $Q(\cdot)$ is to find $\lambda \in \mathbb{C}$ and nonzero $v \in \mathbb{C}$ such that

$$Q(\lambda)x = 0. \tag{3.2}$$

When this equation is satisfied, λ and x are called an eigenvalue and an eigenvector, respectively. All eigenvalues of $Q(\cdot)$ are the roots of $\det Q(\lambda) = 0$, which has $2n$ complex roots, counting multiplicities and including eigenvalues at infinity. If the matrix M in (3.1) is nonsingular. Also, we assume that matrices M, C and K are Hermitian, say, the QEP is Hermitian. In that case eigenvalues are real or they are coming in complex conjugate pairs $(\lambda, \overline{\lambda})$. Instead of the QEP (3.2), similarly as in [29], we will consider equivalent generalized eigenvalue problem

$$\mathscr{L}_Q(\lambda)y = 0, \quad y = \begin{bmatrix} x \\ \lambda x \end{bmatrix} \in \mathbb{C}^{2n}, \tag{3.3}$$

where $\mathscr{L}_Q(\lambda) := A - \lambda B$ is a matrix pencil. If M is nonsingular, we can obtain symmetric linearization

$$\mathscr{L}_Q(\lambda) = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} - \lambda \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \tag{3.4}$$

or if K is nonsingular

$$\mathscr{L}_Q(\lambda) = \begin{bmatrix} 0 & -K \\ -K & C \end{bmatrix} - \lambda \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}. \tag{3.5}$$

In linearizations (3.4) and (3.5) matrices A and B are Hermitian. For the more details about linearization of the QEP see [28].

More to the point, we consider QEP

$$\lambda^2 Mx + \lambda Cx + Kx = 0, \tag{3.6}$$

where M and K are nonsingular Hermitian, and C is Hermitian. The corresponding perturbed QEP is

$$\widetilde{\lambda}^2 \widetilde{M}\widetilde{x} + \widetilde{\lambda}\widetilde{C}\widetilde{x} + \widetilde{K}\widetilde{x} = 0, \tag{3.7}$$

where $\widetilde{M} = M + \delta M$ and $\widetilde{K} = K + \delta K$ are nonsingular Hermitian and $\widetilde{C} = C + \delta C$ is Hermitian. That means that we will consider Hermitian matrix pair (A, B), where

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \text{ and } B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}. \tag{3.8}$$

The corresponding perturbed pair $(\widetilde{A}, \widetilde{B})$ is such that

$$\widetilde{A} = \begin{bmatrix} -(K + \delta K) & 0 \\ 0 & M + \delta M \end{bmatrix} \text{ and } \widetilde{B} = \begin{bmatrix} C + \delta C & M + \delta M \\ M + \delta M & 0 \end{bmatrix}. \tag{3.9}$$

We assume that matrix pair (A, B) and also perturbed pair $(\widetilde{A}, \widetilde{B})$ satisfy assumptions (A1)–(A4).

Note that there exist nonsingular matrices X and $\widetilde{X}$ which simultaneously diagonalize matrix pairs (3.8) and (3.9), respectively. The columns of the matrices X and $\widetilde{X}$ are of the forms

$$X(:, i) = \begin{bmatrix} x_i \\ \lambda_i x_i \end{bmatrix} \quad \text{and} \quad \widetilde{X}(:, i) = \begin{bmatrix} \widetilde{x}_i \\ \widetilde{\lambda}_i \widetilde{x}_i \end{bmatrix}, \quad i = 1, \dots, 2n, \tag{3.10}$$

respectively, where λ_i, $\widetilde{\lambda}_i$ are eigenvalues and x_i, $\widetilde{x}_i \in \mathbb{C}^{n\times n}$ are corresponding eigenvectors of the QEPs (3.6) and (3.7).

In the next theorem we will derive a bound for $|\sin\vartheta(x_i,\widetilde{x}_i)|$, where ϑ is angle between the eigenvectors x_i and $\widetilde{x}_i$.

Theorem 3.1. *Given M, C, $K \in \mathbb{C}^{n\times n}$ as in (3.6). Let $\widetilde{M} = M + \delta M$, $C = C + \delta C$ and $\widetilde{K} = K + \delta K$ be the corresponding perturbed matrices and X and $\widetilde{X}$ be the nonsingular matrices that simultaneously diagonalize matrix pairs (A, B) and $(\widetilde{A},\widetilde{B})$, respectively. Under the assumptions (A1)–(A4) on matrix pairs (3.8) and (3.9), we have this bound*

$$|\sin\vartheta(x_i,\widetilde{x}_i)| \leq \kappa_2(X)\kappa_2(\widetilde{X})\left(\alpha_1^m\delta a_F + \alpha_2^m\delta b_F\right), \tag{3.11}$$

where α_1^m, α_2^m are the same as that in (2.7), and

$$\delta a_F = \sqrt{\|K^{-1}\delta K\|_F^2 + \|M^{-1}\delta M\|_F^2} \tag{3.12}$$

$$\delta b_F = \sqrt{2\cdot\|M^{-1}\delta M\|_F^2 + \|M^{-1}\delta C - M^{-1}CM^{-1}\delta M\|_F^2}. \tag{3.13}$$

Proof. Let us assume that x_i and $\widetilde{x}_i$ are normalized i.e. $\|x_i\|_2 = \|\widetilde{x}_i\|_2 = 1$. Then we have that $\cos\vartheta(x_i,\widetilde{x}_i) = |x_i^H\widetilde{x}_i|$ and also,

$$\cos\vartheta(X(:,i),\widetilde{X}(:,i)) = \frac{|X(:,i)^H\widetilde{X}(:,i)|}{\|X(:,i)\|_2\|\widetilde{X}(:,i)\|_2} = \frac{|1+\overline{\lambda_i}\widetilde{\lambda}_i|}{\sqrt{1+|\lambda_i|^2}\sqrt{1+|\widetilde{\lambda}_i|^2}}\cdot\cos\vartheta(x_i,\widetilde{x}_i), \qquad \vartheta\in[0,2\pi], \tag{3.14}$$

where $X(:,i)$ and $\widetilde{X}(:,i)$ are defined in (3.10). Since $|(1+\overline{\lambda_i}\widetilde{\lambda}_i)| \leq \sqrt{1+|\lambda_i|^2}\sqrt{1+|\widetilde{\lambda}_i|^2}$, it is easy to see that

$$\cos\vartheta(X(:,i),\widetilde{X}(:,i)) \leq \cos\vartheta(x_i,\widetilde{x}_i), \qquad \vartheta\in[0,2\pi]. \tag{3.15}$$

That means that

$$|\sin\vartheta(x_i,\widetilde{x}_i)| \leq |\sin\vartheta(X(:,i),\widetilde{X}(:,i))| \text{ for } \vartheta\in[0,2\pi]. \tag{3.16}$$

and the bound (3.1) follows from Theorem 2.1 simply by taking the Frobenius norm of matrices

$$A^{-1}\delta A = \begin{bmatrix} -K^{-1}\delta K & 0 \\ 0 & M^{-1}\delta M \end{bmatrix},$$

$$B^{-1}\delta B = \begin{bmatrix} -M^{-1}\delta M & 0 \\ M^{-1}\delta C - M^{-1}CM^{-1}\delta M & M^{-1}\delta M \end{bmatrix}.$$

□

The next theorem contains upper bound for the relative errors in the eigenvalues.

Theorem 3.2. *Given M, C, $K \in \mathbb{C}^{n\times n}$ as in (3.6). Let $\widetilde{M} = M + \delta M$, $\widetilde{C} = C + \delta C$ and $\widetilde{K} = K + \delta K$ be corresponding perturbed matrices and (A, B) and $(\widetilde{A}, \widetilde{B})$ be linearized Hermitian matrix pairs, respectively. Under the assumptions (A2)–(A4), consider λ and $\widetilde{\lambda}$ are the eigenvalues of the two matrix pairs, respectively, and X_λ and $\widetilde{X}_{\widetilde{\lambda}}$ are corresponding invariant subspaces containing Jordan chains with dimensions ℓ, $\widetilde{\ell}$, respectively. Suppose that $\lambda, \widetilde{\lambda}$ both lie in the upper half plane ($\mathrm{Im}\,(\lambda) \geq 0, \mathrm{Im}\,(\widetilde{\lambda}) \geq 0$) but nonzero, and write $\gamma = \left|\frac{\widetilde{\lambda}-\lambda}{\lambda}\right|$, $\widetilde{\gamma} = \left|\frac{\widetilde{\lambda}-\lambda}{\widetilde{\lambda}}\right|$, then:*

1. *provided $\ell = \widetilde{\ell} = 1$, or equivalently, both eigenvalues are semi-simple and then rewriting the eigenvectors as $X_\lambda = \begin{bmatrix} x \\ \lambda x \end{bmatrix}$, $\widetilde{X}_{\widetilde{\lambda}} = \begin{bmatrix} \widetilde{x} \\ \widetilde{\lambda}\widetilde{x} \end{bmatrix}$,*

$$\gamma \leq \frac{\|X^{-1}\|_2 \|\widetilde{X}\|_2 (\delta a + \delta b + \delta c)}{\delta d - \|X^{-1}\|_2 \|\widetilde{X}\|_2 (\delta b + \delta c)}, \tag{3.17}$$

   where

$$\begin{aligned}
\delta a &= \max\left\{\|K^{-1}\delta K\|_{\mathrm{ui}}, \|M^{-1}\delta M\|_{\mathrm{ui}}\right\}, \\
\delta b &= \|M^{-1}\delta M\|_{\mathrm{ui}}, \\
\delta c &= \|M^{-1}\delta C\|_{\mathrm{ui}} + \|M^{-1} C M^{-1}\delta M\|_{\mathrm{ui}}, \\
\delta d &= \left|(\overline{\lambda}\widetilde{\lambda} - 1) x^H M^{-1}\delta M \widetilde{x} + \overline{\lambda} x^H M^{-1}\delta C \widetilde{x} - \overline{\lambda} x^H M^{-1} C M^{-1}\delta M \widetilde{x}\right|.
\end{aligned}$$

2. *provided $\gamma \leq \frac{5-\sqrt{17}}{4}, \widetilde{\gamma} \leq \frac{5-\sqrt{17}}{4}$,*

$$\frac{\gamma^{\ell}\widetilde{\gamma}^{\widetilde{\ell}}}{\sqrt{\gamma^2 + \widetilde{\gamma}^2}} \leq 2\frac{\|X^{-1}\|_2 \|\widetilde{X}\|_2}{\|X_\lambda^H B \widetilde{X}_{\widetilde{\lambda}}\|_F} \sqrt{\binom{\widetilde{\ell}+\ell-1}{\ell}^2 \delta a_F^2 + \binom{\widetilde{\ell}+\ell-1}{\widetilde{\ell}}^2 \delta b_F^2}, \tag{3.18}$$

   where δa_F and δb_F are defined in (3.12).

Proof. Bounds (3.17) and (3.18) follows simply from Theorem 2.3 by using the facts that in item 1:

$$\|A^{-1}\delta A\|_{\mathrm{ui}} \leq \delta a, \qquad \|B^{-1}\delta B\|_{\mathrm{ui}} \leq \delta b + \delta c,$$

$$\left|\begin{bmatrix} x^H & \overline{\lambda} x^H \end{bmatrix} B^{-1} \begin{bmatrix} \widetilde{x} \\ \widetilde{\lambda}\widetilde{x} \end{bmatrix}\right| = \left|(\overline{\lambda}\widetilde{\lambda} - 1) x^H M^{-1}\delta M \widetilde{x} + \overline{\lambda} x^H M^{-1}\delta C \widetilde{x} - \overline{\lambda} x^H M^{-1} C M^{-1}\delta M \widetilde{x}\right|.$$

and in item 2:

$$\|A^{-1}\delta A\|_F = \delta a_F \quad \text{and} \quad \|B^{-1}\delta B\|_F = \delta b_F.$$

□

4 Numerical examples

In this section the perturbation bounds for regular Hermitian QEPs given in Section 3 will be illustrated by several numerical examples. The bound for hyperbolic QEPs given in [29] can be considered as a special case here, but as a main advantage, the new bound can be applied on the systems which are no longer overdamped. Although the new bound is restricted just on regular Hermitian QEPs, its solution is required in different mechanical systems, which are described by nonsingular Hermitian mass M and stiffness K matrices, and any Hermitian damping matrix C. In the following numerical examples, we will compare the new bound with the bound given in [29, Theorem 7], which is detailedly

$$|\sin\vartheta(X(:,i),\widetilde{X}(:,i))| \leq \kappa(X)\kappa(\widetilde{X})\left(\frac{\|A_0^{-1}\delta A_0\|_F}{\min\limits_{j\neq i}\dfrac{|\lambda_i-\widetilde{\lambda}_j|}{|\lambda_i|}}+\frac{\|J\delta J\|_F}{\min\limits_{j\neq i}\dfrac{|\lambda_i-\widetilde{\lambda}_j|}{|\widetilde{\lambda}_j|}}\right), \quad i=1,\ldots,2n, \tag{4.1}$$

where, instead of the matrix pair $A-\lambda B$ given in (3.4), authors considered the equivalent matrix pair $A_0-\dfrac{1}{\lambda}J$, where

$$A_0=\begin{bmatrix} L_K^{-1}CL_K^{-H} & L_K^{-1}L_M \\ L_M^H L_K^{-H} & 0\end{bmatrix}, \quad J=\begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix},$$

and then the corresponding perturbations are

$$\delta A=\begin{bmatrix} L_K^{-1}\delta CL_K^{-H} & L_K^{-1}\delta ML_M^{-H} \\ L_M^{-1}\delta ML_K^{-H} & 0\end{bmatrix}, \quad \delta J=\begin{bmatrix} L_K^{-1}\delta KL_K^{-H} & 0\\ 0 & L_M^{-1}\delta ML_M^{-H}\end{bmatrix}.$$

Here L_M and L_K are the matrices in Cholesky factorizations $M=L_ML_M^H$ and $K=L_KL_K^H$, respectively.

The numerical examples below will illustrate that new bound (3.11) is applicable in the cases that bound (4.1) is not.

Example 4.1. This is the problem `Wiresaw1` in the collection NLEVP [4]. It is a gyroscopic QEP arising in the vibration analysis of the wiresaw, for more details see [31]. It takes the form $G(\lambda)=(\lambda^2M+\lambda C+K)x=0$ where the coefficient matrices are defined by

$$M=\frac{1}{2}I_n, \quad K=\frac{\pi^2(1-\nu^2)}{2}\operatorname{diag}(j^2)_{j=1,\ldots,n},$$
$$C=-C^T=[c_{jk}]_{j,k=1,\ldots,n}, \quad \text{with} \quad c_{jk}=\begin{cases}\dfrac{4jk}{j^2-k^2}\nu, & \text{if } j+k \text{ is odd},\\ 0, & \text{otherwise},\end{cases}$$

where n is the size of the problem and ν is a real nonnegative parameter corresponding to the speed of wire. Clearly $M\succ 0$, K is definite Hermitian when $\nu\neq 1$, and C is skew-Hermitian. Then the QEP

$$Q(\lambda):=-G(-\mathrm{i}\lambda)=\lambda^2M+\lambda(\mathrm{i}C)-K \tag{4.2}$$

is regular and Hermitian, which means our bound (3.11) can be applied.

Note that for $0 < \nu < 1$, $K \succ 0$ and the QEP (4.2) is hyperbolic (but not overdamped). Then the corresponding linearization $A - \lambda B$ is positive definite and bound (4.1) for hyperbolic QEPs can be also applied. On the contrary, for $\nu > 1$, the QEP (4.2) is not hyperbolic, and then bound (4.1) can not be used.

First we choose $n = 5, \nu = 0.9 < 1$, and a group of random perturbations δM, δC and δK which satisfy

$$|(\delta M)_{ij}| \leq \eta |M_{ij}|, \quad |(\delta C)_{ij}| \leq \eta |C_{ij}|, \quad |(\delta K)_{ij}| \leq \eta |K_{ij}|, \tag{4.3}$$

where $\eta = 10^{-8}$ and matrices $M + \delta M$, $\mathrm{i}(C + \delta C)$, $K + \delta K$ are also Hermitian. Then we will compare our bound (3.11) and bound (4.1) for eigenvectors $x_1, \widetilde{x}_1$ which correspond to the eigenvalue $\lambda_1 = 21.9063$ of (4.2). New bound (3.11) reads

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \leq 4.9283 \cdot 10^{-5},$$

and it is not too much worse than bound (4.1) which reads

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \leq 1.1908 \cdot 10^{-6}.$$

Then we choose n and $\delta M, \delta C, \delta K$ in the same way but $\nu = 1.0019 > 1$. For the eigenvectors $x_1, \widetilde{x}_1$ that correspond to the eigenvalue $\lambda_1 = -22.7864$, bound (3.11) reads

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \leq 8.5756 \cdot 10^{-6},$$

while bound (4.1) cannot be applied. In comparison, the exact value is

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \approx 1.7615 \cdot 10^{-9}.$$

Example 4.2. This example is related to the model arising in the analysis of the behavior of the brake system, given in [1]. Brake squeal is the major problem in the automotive industry and it is based on the loss of stability of the brake system.

In this example we will consider negative-friction damping excitation mechanisms. Instability of the system is caused by a damping matrix $C \prec 0$.

In [1], authors consider mechanical models of dimension 1 or 2. Here this problem is generalized to the QEP of size n

$$(\lambda^2 M + \lambda C + K)x = 0, \tag{4.4}$$

where M, C and K are mass, damping and stiffness matrices, respectively, and defined as:

$$M = \mathrm{diag}(j)_{j=1,\dots,n}, \quad C = -\gamma I_n, \quad K = \begin{bmatrix} 10 & -5 & & \\ -5 & \ddots & \ddots & \\ & \ddots & \ddots & -5 \\ & & -5 & 10 \end{bmatrix}_{n\times n},$$

Clearly $M \succ 0, K \succ 0, C \prec 0$.

Here we choose $n = 4, \gamma = 0.1$ and random perturbations δM, δC and δK as in (4.3). Assume that perturbed matrices $\widetilde{M} = M + \delta M \succ 0, \widetilde{K} = K + \delta K \succ 0, \widetilde{C} = C + \delta C \prec 0$.

Contrary to the bounds in [29], which are obtained for hyperbolic QEPs whose eigenvalues are real, the new bounds can be applied on the complex eigenvalues and corresponding eigenvectors. This example is designed to illustrate the sensitivity of complex eigenvalues and corresponding eigenvectors for the QEP with negative damping.

Part 1: For the eigenvectors $x_1, \widetilde{x}_1$ which correspond to the eigenvalue $\lambda_1 = 0.0251 + 1.1701\mathrm{i}$, bound (3.11) reads

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \leq 6.9547 \cdot 10^{-6},$$

in comparison with the exact value

$$|\sin \vartheta(x_1, \widetilde{x}_1)| \approx 1.2738 \cdot 10^{-8}.$$

Part 2: In this part we will illustrate the performances of our eigenvalue bound (3.17). In comparison to the bounds for eigenvalues given in [30] and in [29], which holds only for overdamped QEPs, new bound is applicable for any regular Hermitian QEPs. Table 4.1 shows our bound and the exact relative error for all eigenvalues $\lambda_i \in \Lambda(Q)$, which appears in complex conjugate pairs.

λ_i	exact value	estimate (3.17)
0.0438±3.4550i	4.8239e-09	6.9836e-07
0.0224±2.3223i	6.0146e-10	3.3994e-07
0.0197±1.6640i	7.8488e-10	2.5006e-07
0.0183±0.8543i	3.0628e-09	3.3075e-07

Table 4.1: Relative perturbation bound (3.17) and exact relative error for eigenvalues.

Example 4.3. This example is constructed to show that our bound for eigenvalues and eigenvectors is also sensitive on perturbations of Jordan blocks if they appear in the canonical form of the matrix pair (A, B). In this small experimental example, M, C and K are chosen as:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad C = 2 \cdot K.$$

The Jordan form of the linearized pair (A, B) is

$$J = \operatorname{diag}\left\{-3.4142, -0.5858, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\right\}.$$

Under small perturbations δM, δC and δK as in (4.3), where $\mu = 10^{-7}$, we lose the size-2 Jordan block of the matrix pair (A, B) and all the eigenvalues of the perturbed matrix pair $(\widetilde{A}, \widetilde{B})$ are semi-simple. Table 4.2 shows that our bound in comparison to the exact value of the relative error in eigenvalues is good enough to detect the case that the structure of Jordan blocks is changed. The bounds given in [30] and in [29] can not be applied here.

λ_i	exact value	estimate (3.17)
-3.4142	2.2188e-07	2.7405e-05
-0.5858	3.8069e-08	8.0671e-07
λ_i	exact value	estimate (3.18)
-1	5.6518e-08	5.2899e-06

Table 4.2: Relative perturbation bounds (3.17) and (3.18) and exact relative error for eigenvalues.

Eigenvector perturbation bound (3.11) can not be derived directly for eigenvectors in QEP. Here we will measure distance between the subspaces. The subspace that we will consider is spanned by the columns of the matrices $X_1 = X(:, 1:2)$ of which one is the eigenvector and the other is the generalized eigenvector for the eigenvalue $\lambda_{1,2} = -1$. The perturbed subspace is spanned by the columns of the matrix $\widetilde{X}_1 = \widetilde{X}(:, 1:2)$ which are eigenvectors for two distinct eigenvalues. Our bound (3.11) gives

$$\|\sin(\mathrm{span}(X_1), \mathrm{span}(\widetilde{X}_1))\|_F \leq 7.9620 \cdot 10^{-2},$$

since $\kappa(X) \approx 3.4558$, $\kappa(\widetilde{X}) \approx 1.2357 \cdot 10^4$, $\alpha_1^m \approx 2.4142$ and $\alpha_2^m \approx 2.4143$. The exact bound is

$$\|\sin(\mathrm{span}(X_1), \mathrm{span}(\widetilde{X}_1))\|_F \approx 8.0927 \cdot 10^{-5},$$

which confirms the fact that Jordan blocks are very sensitive on small perturbations and small perturbations can significantly change the corresponding invariant subspace.

5 Conclusion

The main contributions of this paper are new relative perturbation bounds for the eigenvalues and their corresponding invariant subspaces for regular Hermitian quadratic eigenvalue problems based on the new corresponding bounds for the regular Hermitian pairs. The obtained bounds can be applied on many interesting problems, for example, on the quadratic eigenvalue problems which appear in many mechanical models, especially on the models with indefinite damping or mass matrices. The main advantage of the new bounds, over some earlier bounds, is that they are more general and can be applied not only on the hyperbolic quadratic eigenvalue problems, but also on the other regular quadratic eigenvalue problems. The quality of our bounds have been illustrated in several numerical examples.

A Proof of Lemma 2.3

Consider (2.3a). At first, we have the inverse of a kind of structured matrix, which will be used several times. For the block matrix $P = D_1 \otimes L + FD_2G \otimes I = [P_{i,j}]_{i,j=1,\dots,m}$ with $D_1 = \mathrm{diag}(d_{1,1}, \dots, d_{1,m}), D_2 = \mathrm{diag}(d_{2,1}, \dots, d_{2,m})$ and $L \in \mathbb{C}^{n\times n}$ lower triangular, it can also be written as

$$P_{i,j} = \begin{cases} 0, & i < j \text{ or } i > j+1, \\ d_{1,j} L, & i = j, \\ d_{2,m-j} I, & i = j+1. \end{cases}$$

Then its inverse $P^{-1} = [Q_{i,j}]_{i,j=1,\dots,m}$ satisfies

$$\begin{aligned} Q_{i,j} &= \begin{cases} \prod_{k=1}^{i-j} (-P^{-1}_{i-k+1,i-k+1} P_{i-k+1,i-k}) Q_{j,j}, & i \geq j, \\ 0, & i < j, \end{cases} \\ &= \begin{cases} (-1)^{i-j} \prod_{k=1}^{i-j+1} d^{-1}_{1,i-k+1} \prod_{k=1}^{i-j} d_{2,m-i+k} L^{j-i-1}, & i \geq j, \\ 0, & i < j. \end{cases} \end{aligned}$$

Now we turn back to the structured Sylvester equation (2.3a). Note that $(I \otimes \Omega' \Lambda' - \Omega\Lambda \otimes I)$ is a lower triangular matrix. The diagonal entry must be $\lambda - \lambda'$, so this matrix is nonsingular.

$$\begin{aligned} \operatorname{vec}(Y) &= (I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I)^{-H} [-(I \otimes \Omega'\Lambda')^H \operatorname{vec}(M) + (\Omega\Lambda \otimes I)^H \operatorname{vec}(N)] \\ &=: -W_1^H \operatorname{vec}(M) + W_2^H \operatorname{vec}(N), \end{aligned}$$

which shows (2.3a) has a unique solution Y, and $W_1 - W_2 = I$. There are eight cases in total:

1. (Λ, Ω) and (Λ', Ω') are both of type **R3**:

$$\begin{aligned} I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I &= I \otimes (\lambda' I + |\lambda'| FG) - (\lambda I + |\lambda| FG) \otimes I \\ &= I \otimes ([\lambda' - \lambda] I + |\lambda'| FG) - |\lambda| FG \otimes I, \end{aligned}$$

and then

$$(I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I)^{-1} = \begin{cases} |\lambda|^{i-j} ([\lambda' - \lambda] I + |\lambda'| FG)^{j-i-1}, & i \geq j, \\ 0, & i < j, \end{cases}$$

where

$$([\lambda' - \lambda] I + |\lambda'| FG)^{-1} = \begin{cases} (-|\lambda'|)^{i'-j'} (\lambda' - \lambda)^{j'-i'-1}, & i' \geq j', \\ 0, & i' < j'. \end{cases}$$

Then

$$([\lambda' - \lambda] I + |\lambda'| FG)^{j-i-1} = \begin{cases} \binom{i'-j'+i-j+1}{i-j+1} (-|\lambda'|)^{i'-j'} (\lambda' - \lambda)^{j'-i'+j-i-1}, & i' \geq j', \\ 0, & i' < j'. \end{cases} \tag{A.1}$$

Thus

$$W_1 = \begin{cases} |\lambda|^{i-j} (\lambda' I + |\lambda'| FG)([\lambda' - \lambda] I + |\lambda'| FG)^{j-i-1}, & i \geq j, \\ 0, & i < j, \end{cases}$$

where

$$
\begin{aligned}
&(\lambda' I + |\lambda'|FG)([\lambda'-\lambda]I + |\lambda'|FG)^{j-i-1} \\
&\quad = \begin{cases} (-|\lambda'|)^{i'-j'}(\lambda'-\lambda)^{j'-i'+j-i}[\binom{i'-j'+i-j+1}{i-j+1}\lambda'(\lambda'-\lambda)^{-1} - \binom{i'-j'+i-j}{i-j+1}], & i' \geq j', \\ 0, & i' < j'. \end{cases}
\end{aligned}
$$

Easy to see $\|W_1\|_1 = \|W_1\|_\infty = \|W_1 e_1\|_1$. Note that $\|W_1\|_2 \leq \sqrt{\|W_1\|_1 \|W_1\|_\infty} = \|W_1\|_1$. Thus,

$$
\begin{aligned}
\|W_1\|_2 &\leq \|W_1 e_1\|_1 \\
&= \sum_{i=1}^{n}\sum_{i'=1}^{n'} \left| |\lambda|^{i-1}(-|\lambda'|)^{i'-1}(\lambda'-\lambda)^{2-i'-i}[\tbinom{i'+i-1}{i}\lambda'(\lambda'-\lambda)^{-1} - \tbinom{i'+i-2}{i}] \right| \\
&\leq \sum_{i=1}^{n}\sum_{i'=1}^{n'} \gamma^{1-i}\gamma'^{1-i'}[\tbinom{i'+i-1}{i}\gamma'^{-1} + \tbinom{i'+i-2}{i}] \\
&= 2\sum_{i=1}^{n}\sum_{i'=1}^{n'-1} \gamma^{1-i}\gamma'^{-i'}\tbinom{i'+i-1}{i} + \sum_{i=1}^{n}\gamma^{1-i}\gamma'^{-n'}\tbinom{n'+i-1}{i} \\
&\leq \left(2\sum_{i=1}^{n}\sum_{i'=1}^{n'-1} \gamma^{1-i}\gamma'^{-i'} + \sum_{i=1}^{n}\gamma^{1-i}\gamma'^{-n'}\right)\tbinom{n'+n-1}{n} \\
&= \left(2\gamma'^{-1}\frac{1-\gamma'^{1-n'}}{1-\gamma'^{-1}}\frac{1-\gamma^{-n}}{1-\gamma^{-1}} + \gamma'^{-n'}\frac{1-\gamma^{-n}}{1-\gamma^{-1}}\right)\tbinom{n'+n-1}{n} \\
&= \gamma'^{-1}\frac{2-\gamma'^{1-n'}-\gamma'^{-n'}}{1-\gamma'^{-1}}\frac{1-\gamma^{-n}}{1-\gamma^{-1}}\tbinom{n'+n-1}{n} \\
&= \gamma'^{-n'}\gamma^{1-n}\frac{1+\gamma'-2\gamma'^{n'}}{1-\gamma'}\frac{1-\gamma^{n}}{1-\gamma}\tbinom{n'+n-1}{n} \\
&= \tbinom{n'+n-1}{n}\varphi_-(\gamma,n)\varphi_+(\gamma',n').
\end{aligned}
$$

Similarly,

$$
\|W_2\|_2 \leq \tbinom{n'+n-1}{n'}\varphi_-(\gamma',n')\varphi_+(\gamma,n).
$$

2. $(\mathit{\Lambda},\mathit{\Omega})$ and $(\mathit{\Lambda}',\mathit{\Omega}')$ are both of type **R4**:

$$
\begin{aligned}
I\otimes \mathit{\Omega}'\mathit{\Lambda}' - \mathit{\Omega}\mathit{\Lambda}\otimes I &= \begin{bmatrix} I & \\ & I \end{bmatrix} \otimes \begin{bmatrix} \lambda' I + |\lambda'|FG & \\ & \overline{\lambda'}I + |\overline{\lambda'}|FG \end{bmatrix} - \begin{bmatrix} \lambda I + |\lambda|FG & \\ & \overline{\lambda}I + |\overline{\lambda}|FG \end{bmatrix} \otimes \begin{bmatrix} I & \\ & I \end{bmatrix} \\
&= \operatorname{diag}(R_1, R_2, \overline{R_1}, \overline{R_2}),
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 &= [I\otimes(\lambda' I + |\lambda'|FG) - (\lambda I + |\lambda|FG)\otimes I], \\
R_2 &= [I\otimes(\overline{\lambda'} I + |\lambda'|FG) - (\lambda I + |\lambda|FG)\otimes I].
\end{aligned}
$$

Then

$$W_1 = \mathrm{diag}(R_1^{-1}(\lambda' I + |\lambda'|FG), R_2^{-1}(\overline{\lambda'} I + |\lambda'|FG), \overline{R_1}^{-1}(\lambda' I + |\lambda'|FG), \overline{R_2}^{-1}(\overline{\lambda'} I + |\lambda'|FG)).$$

Write $\gamma_1 := \left|\frac{\lambda'-\lambda}{\lambda}\right|, \gamma_1' := \left|\frac{\lambda'-\lambda}{\lambda'}\right|, \gamma_2 := \left|\frac{\overline{\lambda'}-\lambda}{\lambda}\right|, \gamma_2' := \left|\frac{\overline{\lambda'}-\lambda}{\lambda'}\right|$, then by the same calculation in case 1,

$$\|R_1^{-1}(\lambda' I + |\lambda'|FG)\|_2 \le \tbinom{n'+n-1}{n}\varphi_-(\gamma_1, n)\varphi_+(\gamma_1', n');$$
$$\|R_2^{-1}(\overline{\lambda'} I + |\lambda'|FG)\|_2 \le \tbinom{n'+n-1}{n}\varphi_-(\gamma_2, n)\varphi_+(\gamma_2', n').$$

Similar to the calculation in case 1,

$$\overline{R_1}^{-1}(\lambda' I + |\lambda'|FG) = \begin{cases} |\lambda|^{i-j}(\lambda' I + |\lambda'|FG)([\overline{\lambda'} - \overline{\lambda}]I + |\lambda'|FG)^{j-i-1}, & i \ge j, \\ 0, & i < j, \end{cases}$$

where

$$\begin{aligned}(\lambda' I &+ |\lambda'|FG)([\overline{\lambda'} - \overline{\lambda}]I + |\lambda'|FG)^{j-i-1} \\ &= \begin{cases} (-|\lambda'|)^{i'-j'}(\overline{\lambda'} - \overline{\lambda})^{j'-i'+j-i}[\tbinom{i'-j'+i-j+1}{i-j+1}\lambda'(\overline{\lambda'} - \overline{\lambda})^{-1} - \tbinom{i'-j'+i-j}{i-j+1}], & i' \ge j', \\ 0, & i' < j', \end{cases}\end{aligned}$$

and then

$$\|\overline{R_1}^{-1}(\lambda' I + |\lambda'|FG)\|_2 \le \tbinom{n'+n-1}{n}\varphi_-(\gamma_1, n)\varphi_+(\gamma_1', n');$$

similarly,

$$\|\overline{R_2}^{-1}(\overline{\lambda'} I + |\lambda'|FG)\|_2 \le \tbinom{n'+n-1}{n}\varphi_-(\gamma_2, n)\varphi_+(\gamma_2', n').$$

To sum up,

$$\|W_1\|_2 \le \tbinom{n'+n-1}{n}\varphi_-(\gamma, n)\varphi_+(\gamma', n')$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$. Similarly,

$$\|W_2\|_2 \le \tbinom{n'+n-1}{n'}\varphi_-(\gamma', n')\varphi_+(\gamma, n).$$

3. (Λ, Ω) is of type **R2**, while (Λ', Ω') is of type **R3**:

$$I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I = I \otimes (\lambda' I + |\lambda'|FG) - FG \otimes I,$$

and then

$$(I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I)^{-1} = \begin{cases} (\lambda' I + |\lambda'|FG)^{j-i-1}, & i \ge j, \\ 0, & i < j, \end{cases}$$

where

$$(\lambda' I + |\lambda'|FG)^{-1} = \begin{cases} (-|\lambda'|)^{i'-j'}\lambda'^{j'-i'-1}, & i' \ge j', \\ 0, & i' < j'. \end{cases}$$

Thus

$$W_1 = \begin{cases} (\lambda' I + |\lambda'|FG)^{j-i}, & i \ge j, \\ 0, & i < j, \end{cases}$$

where

$$(\lambda' I + |\lambda'| FG)^{j-i} = \begin{cases} (-|\lambda'|)^{i'-j'} \lambda'^{j'-i'+j-i} \binom{i'-j'+i-j}{i-j}, & i' \geq j', \\ 0, & i' < j', \end{cases} \quad \text{for } i > j.$$

Easy to see $\|W_1\|_1 = \|W_1\|_\infty = \|W_1 e_1\|_1$. Note that $\|W_1\|_2 \leq \sqrt{\|W_1\|_1 \|W_1\|_\infty} = \|W_1\|_1$. Thus,

$$\begin{aligned} \|W_1\|_2 &\leq \|W_1 e_1\|_1 \\ &= 1 + \sum_{i=2}^{n} \sum_{i'=1}^{n'} \left| (-|\lambda'|)^{i'-1} \lambda'^{2-i'-i} \binom{i'+i-2}{i-1} \right| \\ &= 1 + \sum_{i=2}^{n} \sum_{i'=1}^{n'} |\lambda'|^{1-i} \binom{i'+i-2}{i-1}. \end{aligned}$$

Noticing[1] $\sum_{i'=1}^{n'} \binom{i'+i-2}{i-1} = \binom{i-1}{i-1} + \sum_{i'=2}^{n'} \binom{i'+i-2}{i-1} = \binom{i}{i} + \sum_{i'=2}^{n'} \binom{i'+i-2}{i-1} = \cdots = \binom{n'+i-1}{i}$,

$$\begin{aligned} \|W_1\|_2 &\leq 1 + \sum_{i=2}^{n} |\lambda'|^{1-i} \binom{n'+i-1}{i} \\ &\leq 1 + \frac{|\lambda'|^{-1} - |\lambda'|^{-n}}{1 - |\lambda'|^{-1}} \binom{n'+n-1}{n} \\ &= 1 + \binom{n'+n-1}{n} |\lambda'|^{-1} \varphi_-(|\lambda'|, n-1). \end{aligned}$$

Then

$$\begin{aligned} \|W_2\|_2 &\leq \|W_1 e_1 - e_1\|_1 \\ &= \sum_{i=2}^{n} \sum_{i'=1}^{n'} \left| (-|\lambda'|)^{i'-1} \lambda'^{2-i'-i} \binom{i'+i-2}{i-1} \right| \\ &= \binom{n'+n-1}{n} |\lambda'|^{-1} \varphi_-(|\lambda'|, n-1). \end{aligned}$$

4. $(\mathit{\Lambda}, \mathit{\Omega})$ is of type **R2**, while $(\mathit{\Lambda}', \mathit{\Omega}')$ is of type **R4**:

$$\begin{aligned} I \otimes \mathit{\Omega}' \mathit{\Lambda}' - \mathit{\Omega} \mathit{\Lambda} \otimes I &= I \otimes \begin{bmatrix} \lambda' I + |\lambda'| FG & \\ & \overline{\lambda'} I + |\overline{\lambda'}| FG \end{bmatrix} - FG \otimes \begin{bmatrix} I & \\ & I \end{bmatrix} \\ &= \operatorname{diag}(R_1, \overline{R_1}), \end{aligned}$$

where

$$R_1 = [I \otimes (\lambda' I + |\lambda'| FG) - FG \otimes I].$$

Then

$$W_1 = \operatorname{diag}(R_1^{-1} (\lambda' I + |\lambda'| FG), \overline{R_1}^{-1} (\overline{\lambda'} I + |\lambda'| FG)).$$

[1] $\binom{p}{q-1} + \binom{p}{q} = \binom{p+1}{q}$ for any $q \in \mathbb{N}_+$.

By the same calculation in case 3,

$$\|R_1^{-1}(\lambda' I + |\lambda'|FG)\|_2 \le \binom{n'+n-1}{n}\varphi_-(|\lambda'|, n);$$

Note that

$$\|\overline{R_1}^{-1}(\overline{\lambda'} I + |\lambda'|FG)\|_2 = \|R_1^{-1}(\lambda' I + |\lambda'|FG)\|_2.$$

Thus

$$\|W_1\|_2 \le \binom{n'+n-1}{n}\varphi_-(|\lambda'|, n).$$

Similarly,

$$\|W_2\|_2 \le \binom{n'+n-1}{n}\varphi_-(|\lambda'|, n') - 1.$$

5. (Λ, Ω) is of type **R3**, while (Λ', Ω') is of type **R4**:

$$\begin{aligned} I \otimes \Omega'\Lambda' - \Omega\Lambda \otimes I &= I \otimes \begin{bmatrix} \lambda' I + |\lambda'|FG & \\ & \overline{\lambda'} I + |\overline{\lambda'}|FG \end{bmatrix} - (\lambda I + |\lambda|FG) \otimes \begin{bmatrix} I & \\ & I \end{bmatrix} \\ &= \operatorname{diag}(R_1, \overline{R_1}), \end{aligned}$$

where

$$R_1 = [I \otimes (\lambda' I + |\lambda'|FG) - (\lambda I + |\lambda|FG) \otimes I].$$

Then

$$W_1 = \operatorname{diag}(R_1^{-1}(\lambda' I + |\lambda'|FG), \overline{R_1}^{-1}(\overline{\lambda'} I + |\lambda'|FG)).$$

By the same calculation in case 1,

$$\|R_1^{-1}(\lambda' I + |\lambda'|FG)\|_2 \le \binom{n'+n-1}{n}\varphi_-(\gamma, n)\varphi_+(\gamma', n');$$

Note that

$$\|\overline{R_1}^{-1}(\overline{\lambda'} I + |\lambda'|FG)\|_2 = \|R_1^{-1}(\lambda' I + |\lambda'|FG)\|_2.$$

Thus

$$\|W_1\|_2 \le \binom{n'+n-1}{n}\varphi_-(\gamma, n)\varphi_+(\gamma', n').$$

Similarly,

$$\|W_2\|_2 \le \binom{n'+n-1}{n'}\varphi_-(\gamma', n')\varphi_+(\gamma, n).$$

6. (Λ, Ω) is of type **R3**, while (Λ', Ω') is of type **R2**: Note that there exist two permutation matrices P, Q to make $A \otimes B = P(B \otimes A)Q$. Thus, $\|A \otimes B\|_2 = \|B \otimes A\|_2$. So it is the same as case 3.

7. (Λ, Ω) is of type **R4**, while (Λ', Ω') is of type **R2**: the same as case 4.

8. (Λ, Ω) is of type **R4**, while (Λ', Ω') is of type **R3**: the same as case 5.

Consider (2.3b). Note that $(I \otimes I - \Omega\Lambda \otimes FG)$ is a lower triangular matrix. The diagonal entry must be 1, so this matrix is nonsingular. Then

$$\begin{aligned}\operatorname{vec}(Y) &= (I \otimes I - \Omega\Lambda \otimes FG)^{-H}[-(I \otimes I)^H \operatorname{vec}(M) + (\Omega\Lambda \otimes FG)^H \operatorname{vec}(N)] \\ &=: -W_1^H \operatorname{vec}(M) + W_2^H \operatorname{vec}(N),\end{aligned}$$

which shows (2.3b) has a unique solution Y, and also $W_1 - W_2 = I$. Note that there exist two permutation matrices P, Q to make $A \otimes B = P(B \otimes A)Q$. Thus, $\|A \otimes B\|_2 = \|B \otimes A\|_2$. Then

$$I \otimes I - FG \otimes \Omega\Lambda = \begin{bmatrix} I & & & \\ \Omega\Lambda & I & & \\ & \ddots & \ddots & \\ & & \Omega\Lambda & I \end{bmatrix},$$

and

$$W_0 := (I \otimes I - FG \otimes \Omega\Lambda)^{-1} = \begin{cases} (-\Omega\Lambda)^{i'-j'}, & i' \geq j', \\ 0, & i' < j'. \end{cases}$$

Then

$$\|W_1\|_2 = \|(I \otimes I - \Omega\Lambda \otimes FG)^{-1}\|_2 = \|(I \otimes I - FG \otimes \Omega\Lambda)^{-1}\|_2 = \|W_0\|_2.$$

There are three cases, for which it is easy to see that $\|W_0\|_1 = \|W_0\|_\infty = \|W_0 e_1\|_1$ and then $\|W_0\|_2 \leq \sqrt{\|W_0\|_1 \|W_0\|_\infty} = \|W_0\|_1 = \|W_0 e_1\|_1$:

1. if (Λ, Ω) is of **R2**:

$$(-\Omega\Lambda)^{i'-j'} = \begin{cases} (-1)^{i'-j'}, & i - j = i' - j', \\ 0, & i - j \neq i' - j'. \end{cases}$$

   Thus $\|W_1\|_2 \leq \|W_0 e_1\|_1 = \min\{n, n'\}$; and $\|W_2\|_2 \leq \|W_0 e_1 - e_1\|_1 = \min\{n, n'\} - 1$.

2. if (Λ, Ω) is of **R3**:

$$(-\Omega\Lambda)^{i'-j'} = \begin{cases} (-1)^{i'-j'} \lambda^{i'-j'-i+j} |\lambda|^{i-j} \binom{i'-j'}{i-j}, & i \geq j, \\ 0, & i < j, \end{cases} \quad \text{for } i' \geq j', \tag{A.2}$$

   Thus

$$\begin{aligned}\|W_1\|_2 &\leq \|W_0 e_1\|_1 \\ &= 1 + \sum_{i=1}^{n} \sum_{i'=2}^{n'} \left| (-1)^{i'-1} \lambda^{i'-i} |\lambda|^{i-1} \binom{i'-1}{i-1} \right| \\ &= 1 + \sum_{i=1}^{n} \sum_{i'=2}^{n'} |\lambda|^{i'-1} \binom{i'-1}{i-1} \\ &\leq 1 + \sum_{i'=2}^{n'} |\lambda|^{i'-1} 2^{i'-1}\end{aligned}$$

$$= 1 + \frac{2|\lambda| - (2|\lambda|)^{n'}}{1 - 2|\lambda|}$$
$$= 1 + 2|\lambda|\varphi(\frac{1}{2|\lambda|}, n' - 1);$$

and

$$\|W_2\|_2 \leq \|W_0 e_1 - e_1\|_1 = 2|\lambda|\varphi(\frac{1}{2|\lambda|}, n' - 1).$$

3. if (Λ, Ω) is of **R4**: $(-\Omega\Lambda)^{i'-j'} = \mathrm{diag}(R, \overline{R})$ and R is of the form (A.2). Thus,

$$\|W_1\|_2 \leq \max\{\|W_0 e_1\|_1, \|\overline{W_0} e_1\|_1\} = 2|\lambda|\varphi(\frac{1}{2|\lambda|}, n' - 1) + 1;$$

and

$$\|W_2\|_2 \leq \max\{\|W_0 e_1 - e_1\|_1, \|\overline{W_0} e_1 - e_1\|_1\} = 2|\lambda|\varphi(\frac{1}{2|\lambda|}, n' - 1).$$

After calculations, no matter what case it is, by

$$\|Y\|_F \leq \|W_1\|_2 \|N\|_F + \|W_2\|_2 \|M\|_F,$$

we have the result.

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