EXACT, BEAM TRACING AND COMPLEX GEOMETRICAL OPTICS SOLUTIONS FOR THE PROPAGATION OF GAUSSIAN ELECTROMAGNETIC BEAMS

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Introduction. The diffractive broadening of collimated or focused electromagnetic wave beams can result in significant changes of the absorption profiles with respect to standard ray-tracing calculations, where diffraction is neglected [1, 2]. In this paper, the beam tracing (BT) method [3, 4] and the complex geometrical optics (CGO) approach [5, 6] are applied to study the propagation of a Gaussian beam in two-dimensional isotropic inhomogeneous media. Both methods rely on the use of a complex phase for the wave field, the imaginary part of which allows for the evolution of the beam profile. BT exploits a paraxial expansion of the wave phase around a reference ray to obtain a set of equations for the width of the beam and the curvature of the phase front. In CGO the eikonal equation, $(\frac{c}{\omega}\nabla S(\mathbf{r}))^2 = \epsilon(\mathbf{r},\omega)$, is analytically extended to the complex space, and solved by means of its characteristics which are rays in the complex space; the phase and the amplitude are thus calculated in the complex space and the numerical procedure of "ray aiming" is used to obtain results in real space. This method accounts for diffraction by introducing a correlation between near rays. For propagation in both a homogeneous and a lens-like medium, as well as in a plasma slab, analytic BT solutions are obtained and compared numerically with both analytic exact solutions of the corresponding Helmholtz equation, to asses the accuracy of the method, and CGO calculations, to study the validity of the numerical algorithm. With respect to the exact solution , it is found that BT yields the energy density very accurately, the behavior of the wave electric field on the wavelength scale being also discussed. The agreement of the CGO numerical procedure with the analytic BT solutions turns out to be excellent.

The BT solutions of the Helmholtz equation for the propagation in an isotropic medium. With reference to the Helmholtz equation for the (monochromatic) wave electric field $\mathbf{E}(\mathbf{r},\omega)$ in a slowly space-varying isotropic medium with dielectric function $\epsilon(\mathbf{r},\omega)$, i.e., $[\nabla^2 + (\omega/c)^2\epsilon(\mathbf{r},\omega)]\,\mathbf{E}(\mathbf{r},\omega) = 0$, the solution relative to a wave beam, obtained by applying the BT method [3], can be written as

$$\mathbf{E}(\mathbf{r},\omega) = \hat{\mathbf{e}}(\tau)\left[\frac{\epsilon(0)}{\epsilon(\tau)}\right]^{\frac{1}{4}}\left[\frac{w_1(0)w_2(0)}{w_1(\tau)w_2(\tau)}\right]^{\frac{1}{2}} e^{-\frac{1}{2}\left[(\xi^1)^2+(\xi^2)^2\right]} e^{iS(\tau,\xi^1,\xi^2)}\times$$

$$\times \sum_{m,n} \frac{c_{mn}}{\sqrt{\pi 2^{m+n} m! n!}} H_m(\xi^1) H_n(\xi^2) e^{-i(m+n+1)(\frac{c}{\omega})^2 \int_0^\tau d\tau \left[\frac{2}{w_1(\tau)^2} + \frac{2}{w_2(\tau)^2}\right]}, \quad (1)$$

where the Gaussian harmonics are defined in terms of the parabolic cylinder functions, $H_m(\xi)$ being the Hermite polynomials. From (1) it appears that the line $\xi^1 = \xi^2 = 0$ is the symmetry axis of each harmonic, and it is referred to as reference ray ($\Re$); the coordinate system (τ, ξ^1, ξ^2) is such that τ is the coordinate along the reference ray with $d\tau =| d\mathbf{r}/d\tau |^{-1} ds$, ds being the elementary arc length of $\Re$, and $\xi^i \equiv \sqrt{2} v^i / w_i(\tau)$, $i = 1, 2$, are the transverse (to $\Re$) coordinates normalized to the beam widths w_i. Because of the fast varying exponential factor in (1), the eikonal function S is obtained by making use of a paraxial expansion about the reference ray, so that

$$S(\tau, \xi^1, \xi^2) = \frac{\omega}{c} \int_0^\tau ds(\tau) \sqrt{\epsilon(\tau)} + \frac{1}{2} \left[\frac{\partial^2 S}{\partial \xi^i \partial \xi^j}\right]_\Re \xi^i \xi^j, \quad (2)$$

the eigenvalues of the 2×2 Hessian matrix of S being related to the two radii of curvature of the phase front. For the specific case for which the dependence with respect to ξ^1 and ξ^2 is separable and considering an axisimmetric wave beam, one gets the 2nd term in right-hand-side of (2)

$$\frac{1}{2} \left[\frac{\partial^2 S}{\partial \xi^i \partial \xi^j}\right]_\Re \xi^i \xi^j = \frac{\omega}{2c} \frac{\sqrt{\epsilon(\tau)}}{R(\tau)} \left((v^1)^2 + (v^2)^2\right), \qquad \frac{1}{R(\tau)} = \frac{\omega}{2c\sqrt{\epsilon(\tau)}} \frac{d}{d\tau} \ln w(\tau) \quad (3)$$

the second equality relating the the radius of curvature $R(\tau)$ of the axisymmetric beam to the corresponding width $w(\tau)$. The coefficients c_{mn} in (1) are to be determined from the value of the electric field amplitude on the initial ($\tau = 0$) phase front. The radii of curvature and the beam widths are obtained as solutions of a tensorial Riccati-type equation which governs the Hessian matrix of the complex eikonal $\bar{S}(\tau) = S(\tau) + i((\xi^1)^2 + (\xi^2)^2)/2$, the coefficient of such an equation depending on the derivatives of the dispersion function [1,7].

The BT solutions for 2D-propagation. We consider the specific case of a 2D-Gaussian beam in the (x, z)-plane propagating in a weakly inhomogeneous medium slab, symmetric in the transverse (with respect to the beam launching direction) x-direction, i.e., the dielectric function is $\epsilon(z, x) = \epsilon(z, -x)$. Under these conditions and for an initially ($z = z_0$) plane Gaussian beam the electric field of which is $\mathbf{E}(\mathbf{r}, \omega) = \hat{\mathbf{y}} u_0 e^{-x^2/w_0^2}$, the reference ray is the z-axis, and solution (1) reduces to,

$$\mathbf{E}(\mathbf{r}, \omega) = \hat{\mathbf{y}} u_0 \left[\frac{\epsilon_0(0)}{\epsilon_0(z)}\right]^{\frac{1}{4}} \left[\frac{w_0}{w(z)}\right]^{\frac{1}{2}} e^{-\frac{x^2}{w(z)^2}} e^{i\left[\frac{\omega}{c}\int_{z_0}^z dz \sqrt{\epsilon_0(z)} + \frac{\omega\sqrt{\epsilon_0(z)} x^2}{2cR(z)} - \frac{c}{\omega}\int_{z_0}^z \frac{dz}{\sqrt{\epsilon_0(z)} w(z)^2}\right]}, \quad (4)$$

$\epsilon_0(z) \equiv \epsilon(z, x = 0)$ is the dielectric function on the reference ray. A lens-like medium is characterized by the dielectric function $\epsilon(x) = \epsilon_0 \left[1 - (x/L)^2\right]$, the coordinate of the reference ray being $z = 2(c/\omega)\sqrt{\epsilon_0}\tau$, and $z_0 = 0$. The corresponding beam width is [7]

$$\frac{w(x)}{w_0} = \frac{1}{\sqrt{2}}\left[\left(1+\left(\frac{w_c}{w_0}\right)^4\right)+\left(1-\left(\frac{w_c}{w_0}\right)^4\right)\cos(2x/L)\right]^{1/2}, w_c \equiv \left(\frac{2cL}{\sqrt{\epsilon_0}\omega}\right)^{\frac{1}{2}}, \quad (5)$$

the curvature $1/R(z)$ being obtained by applying the 2nd of (3). As for the (real part of the) electric field amplitude, $u(z,x)$, from (4) one gets

$$\frac{u(z,x)}{u_0} = \left[\frac{w_0}{w(z)}\right]^{\frac{1}{2}} e^{-\frac{x^2}{w(z)^2}} \cos\left[\frac{\omega\sqrt{\epsilon_0}}{c}\left(z+\frac{x^2}{2R(z)}\right)-\frac{1}{2}\tan^{-1}\left(\left(\frac{w_c}{w_0}\right)^2\tan\frac{z}{L}\right)\right]. \quad (6)$$

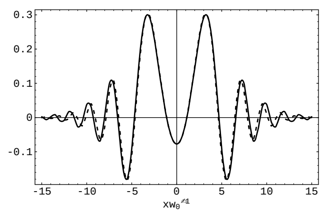
The case of a <u>homogeneous medium</u> is simply given by (5) and (6) to lowest order in $(z/L) \ll 1$. Let us consider now the case of a <u>plasma slab</u> of width a, with dielectric function $\epsilon(z,\omega) = 1 - \omega_p^2(z)/\omega^2$ and parabolic density. For an initially plane Gaussian beam, launched at $z_0 = -a$ along the density gradient direction, the width and the radius of curvature, expressed in terms of the ray coordinate τ, are equal to those calculated for the homogeneous medium, because of the plasma homogeneity in the transverse directions. For the chosen launching conditions, the beam propagation is unaffected by refraction, whereas the dispersive effects are embodied in

$$\tau(z) = \frac{\omega^2 a}{2c\omega_p}\left[\sinh^{-1}\left(\frac{\omega_p z}{\omega N a}\right)+\sinh^{-1}\left(\frac{\omega_p}{\omega N}\right)\right], \quad \omega_p = \omega_p(z=0), \quad N^2 = 1-\frac{\omega_p^2}{\omega^2}. \quad (7)$$

With $\epsilon_0(z) = N^2 + (\omega_p/\omega)^2(z/a)^2$, the integration occurring in (4) can be carried out explicitly.

Numerical results and discussion. A detailed numerical analysis of the BT solutions for the three foregoing specific cases is performed along with a comparison with the corresponding exact solutions. For a <u>homogeneous medium</u> , the x-profile of the BT solution is shown in Fig.(1a), dashed curve, for a fixed position on the reference ray. The oscillatory structure of the profile, weighted by the Gaussian factor e^{-x^2/w^2}, is due to the x^2-term in the cos-factor of (6) which describes the curvature of the phase front; diffraction effects avoid zeros of both $R(z)$ and $w(z)$, i.e., caustics. The corresponding x-profile of the exact solution of the 2D-Helmholtz equation [1] is also shown in Fig.(1a), full curve. For $z = 0$ the exact solution reduces just to the Fourier transform of a Gaussian, thus giving the initial field amplitude; for $z \neq 0$ an oscillating exponential sets in: this solution appears to be a superposition of modes, each one characterized by different wavelengths giving rise to the diffractive pattern shown in Fig.(1a); it appears that the BT solution is quite accurate, as quantitatively shown in Fig.(1b). With reference to a <u>lens-like medium</u> , the BT solution (5) and (6) is shown in Fig.(2a) as a function of z, for $x = 0$, dashed curve: the field oscillations on the scale of the wavelength appear to be shifted with respect to the exact solution (full curve).

Again the exact solution is a superposition of modes [7], no well-defined wavelength being thus identifiable. For a not too long propagation length, the arctan-term in (6) partially describes this shift, which, however, does not affect the energy density, as shown in Fig.(2b). For the propagation of a Gaussian beam in a lens-like medium the BT solution is also compared with the corresponding one obtained by means of the CGO approach and the agreement between the two solutions is found to be better than 0.02%. The BT solution for the beam propagation in a plasma slab exhibits an additional effect due to the density variation along $\Re$, which increases both the wavelength, the corresponding maximum being $\lambda_{max} = 2\pi c/(\omega N)$, and the electric field amplitude. This effect depends on the beam frequency ω (temporal dispersion). One should note that the BT method fails to describe the propagation of evanescent waves, i.e., for $\omega \approx \omega_p$, for which the CGO approach should be employed.



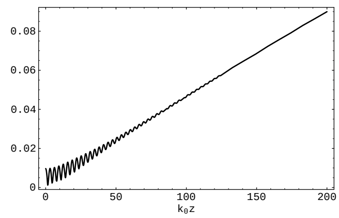
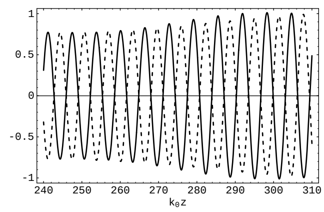


Fig.(1a): the BT (dashed curve) and exact (full curve) x-profile of a very narrow ($k_0 w_0 = 8$) beam propagating in a homogeneous medium; $k_0 z = 200$, $k_0 = \omega\sqrt{\epsilon_0}/c$.
Fig.(1b): the accuracy of the BT solution: $\Delta(k_0 z) = \int dx \mid u_{EX}(z,x) - u_{BT}(z,x) \mid^2$, vs $k_0 z$.



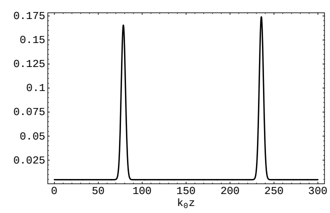


Fig.(2a): the z-dependence of the BT (dashed curve) and exact (full curve) solution on the beam axis ($x = 0$) for a narrow beam ($w_c/w_0 = 1.3$) propagating in a lens-like medium; $k_0 L = 50$.
Fig.(2b): the quantity $\mid 1 - \mathcal{E}_{BT}(k_0 z)/\mathcal{E}_{EX}(k_0 z) \mid$ vs. $k_0 z$, $\mathcal{E}_i(k_0 z) = \langle \int dx \mid u_i(x,z) \mid^2 \rangle$, with $i = BT, EX$ respectively, and $\langle\ \rangle$ is the average over the wavelength. Peaks occur at the foci predicted by geometrical optics.

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