Eisenstein series and automorphic representations

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arXiv:1511.04265v1 [math.NT] 13 Nov 2015

We provide an introduction to the theory of Eisenstein series and automorphic forms on real simple Lie groups G, emphasising the role of representation theory. It is useful to take a slightly wider view and define all objects over the (rational) adeles $\mathbb{A}$, thereby also paving the way for connections to number theory, representation theory and the Langlands program. Most of the results we present are already scattered throughout the mathematics literature but our exposition collects them together and is driven by examples. Many interesting aspects of these functions are hidden in their Fourier coefficients with respect to unipotent subgroups and a large part of our focus is to explain and derive general theorems on these Fourier expansions. Specifically, we give complete proofs of Langlands' constant term formula for Eisenstein series on adelic groups $G(\mathbb{A})$ as well as the Casselman–Shalika formula for the p-adic spherical Whittaker vector associated to unramified automorphic representations of $G(\mathbb{Q}_p)$. Somewhat surprisingly, all these results have natural interpretations as encoding physical effects in string theory. We therefore introduce also some basic concepts of string theory, aimed toward mathematicians, emphasising the role of automorphic forms. In addition, we explain how the classical theory of Hecke operators fits into the modern theory of automorphic representations of adelic groups, thereby providing a connection with some key elements in the Langlands program, such as the Langlands dual group ${}^LG$ and automorphic L-functions. Our treatise concludes with a detailed list of interesting open questions and pointers to additional topics where automorphic forms occur in string theory.

Note to the reader

These notes have grown out of our endeavour to understand the theory of automorphic representations and the structure of Fourier expansions of automorphic forms with a particular emphasis on adelic methods and Eisenstein series. Our intention is also to open a path of communication between mathematicians and physicists, in particular string theorists, interested in these topics. Most of the results in these notes exist already in the literature and we benefitted greatly from [57, 80, 130–132, 218, 294, 308, 309]; our exposition differs, however, at places from the standard one. A few new results and examples are included as well, in particular we provide many techniques for working out aspects of the Fourier expansion of Eisenstein series. We intend to expand the material of these notes in the future and we would be very grateful to learn of any omissions and mistakes that we have made unintentionally.

Sections that are more advanced or explore topics beyond the main focus of the notes are marked with an asterisk.

Acknowledgements

Our understanding of the material presented here was greatly facilitated by numerous discussions with colleagues both from the mathematics community and from the string theory community. We are especially indebted to Guillaume Bossard, David Ginzburg, Stephen D. Miller, Hermann Nicolai, Bengt E.W. Nilsson and Boris Pioline for many clarifying and stimulating discussions over the past years. In addition, we gratefully acknowledge the exchanges with Olof Ahlén, Sergei Alexandrov, Marcus Berg, Benjamin Brubaker, Lisa Carbone, Martin Cederwall, Thibault Damour, Dennis Eriksson, Alex J. Feingold, Matthias Gaberdiel, Terry Gannon, Ori Ganor, Dmitri Gourevitch, Michael B. Green, Stefan Hohenegger, Ralf Köhl, Kyu-Hwan Lee, Gregory W. Moore, Jakob Palmkvist, Christoffer Petersson, Siddhartha Sahi, Per Salberger, Gordan Savin, Oliver Schlotterer, Philippe Spindel, Stefan Theisen, Pierre Vanhove, Roberto Volpato, Peter West and Martin Westerholt-Raum.

Contents

Chapter 1

Motivation and background

An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.

— Robert P. Langlands*

This text grew out of our endeavour to learn the adelic techniques used in the analysis of Eisenstein series in many mathematical works. Part of our motivation came from research problems in string theory were we faced the challenge of calculating certain Fourier coefficients of automorphic forms on exceptional Lie groups. The present text can be viewed as the culmination of the resulting journey through the world of automorphic forms. Even though none of the results that we present here are new, we felt that there might be an interest in such a survey since many of the original sources and textbooks do not make an easy first reading, especially for theoretical physicists like ourselves. Therefore we strove to be as pedagogical *and* precise as possible but will sometimes sacrifice rigour or generality for conveying ideas and explicit examples. The reader is referred to the many sources quoted if he wishes more details on a particular point.

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

1.1 Automorphic forms and Eisenstein series

Automorphic forms are functions $f(g)$ on a Lie group G that

(1) are invariant under the action of a discrete subgroup $\Gamma \subset G$: $f(\gamma \cdot g) = f(g)$ for all $\gamma \in \Gamma$,

(2) satisfy eigenvalue differential equations under the action of the ring of G-invariant differential operators and

(3) have well-behaved growth conditions.

A more explicit and refined form of these conditions will be given in chapter 5 when we properly define automorphic forms; here we content ourselves with a qualitative description based on examples. We will mainly be interested in automorphic forms $f(g)$ that are invariant under the action of the maximal compact subgroup K of G when acting from the right: $f(gk) = f(g)$ for all $k \in K$; such forms are called *K-spherical*. The automorphic forms are then functions on the coset G/K.

The prime example of an automorphic form is obtained when considering $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. The maximal compact subgroup is $K = SO(2,\mathbb{R})$ and the coset space G/K is a constant negative curvature space isomorphic to the Poincaré upper half plane $\mathbb{H} = \{z = x + iy \mid x \in \mathbb{R} \text{ and } y > 0\}$. A function satisfying the three criteria above is then given by the non-holomorphic function

$$f_s(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}}. \tag{1.1}$$

The sum converges absolutely for $\mathrm{Re}(s) > 1$. The action of an element $\gamma \in SL(2,\mathbb{Z})$ on $z \in \mathbb{H}$ is given by the standard fractional linear form (see appendix A)

$$\gamma \cdot z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.2}$$

Property (1) is then verified by noting that the integral lattice $(c,d) \in \mathbb{Z}^2$ is preserved by the action of $SL(2,\mathbb{Z})$ and acting with $\gamma \in SL(2,\mathbb{Z})$ in (1.1) merely reorders the terms in the absolutely convergent sum. Property (2) in this case reduces to a single equation since there is only a single primitive G-invariant differential operator for the real rank one group $SL(2,\mathbb{R})$. This operator is given by

$$\Delta = y^2\left(\partial_x^2 + \partial_y^2\right) \tag{1.3}$$

and corresponds to the Laplace–Beltrami operator on the upper half plane $\mathbb{H}$. In group theoretical terms it is the quadratic Casimir operator. Acting with it on the function (1.1) one finds

$$\Delta f_s(z) = s(s-1) f_s(z) \tag{1.4}$$

and hence $f_s(z)$ is an eigenfunction of Δ (and therefore of the full ring of differential operators generated by Δ). Condition (3) relating to the growth of the function here corresponds to the behavior of $f_s(z)$ near the boundary of the upper half plane, more particularly near the so-called *cusp at infinity* when $y \to \infty$.[1] The growth condition requires $f_s(y)$ to grow at most as a power law as $y \to \infty$. To verify this point it is easiest to consider the *Fourier expansion* of $f_s(y)$. This requires a bit more machinery and also paves the way to the general theory. We will introduce it heuristically in section 1.3 and in detail in chapter 6.

The form of the function $f_s(z)$ is very specific to the action of $SL(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}$. To pave the way for the more general theory of automorphic forms for higher rank Lie groups we shall now rewrite (1.1) in a more suggestive way. In fact, $f_s(z)$ is (almost) an example of an *Eisenstein series* on $G = SL(2,\mathbb{R})$. To see this, we first extract the greatest common divisor of the coordinates of the lattice point $(c,d) \in \mathbb{Z}^2$:

$$f_s(z) = \left(\sum_{k>0} k^{-2s}\right) \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}} = \zeta(2s) \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1}} \frac{y^s}{|cz+d|^{2s}} \tag{1.5}$$

where we have evaluated the sum over the common divisor k using the *Riemann zeta function* [318]

$$\zeta(s) = \sum_{n>0} n^{-s}. \tag{1.6}$$

Referring back to (1.2), we can rewrite the summand using an element of the group $SL(2,\mathbb{Z})$:

$$\frac{y^s}{|cz+d|^{2s}} = [\mathrm{Im}\,(\gamma\cdot z)]^s \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1.7}$$

For this to be possible, two things have to occur: (i) For any co-prime pair (c,d) such a matrix $\gamma \in SL(2,\mathbb{Z})$ must exist, and (ii) if several matrices exist we must form equivalence classes such that the sum over co-prime pairs (c,d) corresponds exactly to the sum over equivalence classes. For (i), we note that the condition that c and d be co-prime is necessary since it would otherwise be impossible to satisfy the determinant condition $ad - bc = 1$ over $\mathbb{Z}$. At the same time, co-primality is sufficient to guarantee existence of integers a_0 and b_0 that complete c and d to a matrix $\gamma \in SL(2,\mathbb{Z})$. In fact, there is a one-parameter family of solutions for γ that can be written as

$$\begin{pmatrix} a_0 + mc & b_0 + md \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c & d \end{pmatrix} \tag{1.8}$$

for any integer $m \in \mathbb{Z}$. (That these are all solutions to the determinant condition over $\mathbb{Z}$ is an elementary lemma of number theory, sometimes called *Bézout's lemma* [180].) The

[1]For $\Gamma = SL(2,\mathbb{Z})$ this is the only cusp up to equivalence. With this one means that the fundamental domain of the action of Γ on $\mathbb{H}$ only touches the boundary of the upper half plane at a single point. See appendix A for pictures and [28, 201, 213] more details on discrete subgroups of $SL(2,\mathbb{R})$.

form (1.8) tells us also how to resolve point (*ii*): We identify matrices that are obtained from each other by left multiplication by a matrix belonging to the *Borel subgroup*

$$B(\mathbb{Z}) = \left\{ \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\} \subset SL(2,\mathbb{Z}). \tag{1.9}$$

The interpretation of this group is that it is the stabiliser of the y-axis.

Summarising the steps we have performed, we find that we can write the function (1.1) as

$$f_s(z) = 2\zeta(2s) \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} [\mathrm{Im}\,(\gamma \cdot z)]^s\,. \tag{1.10}$$

Since we had included the matrix $-\mathbb{1}$ in the definition of $B(\mathbb{Z})$, an extra factor of 2 arises in this formula.

Dropping the multiplicative ζ-factor, we obtain the function

$$E(\chi_s, z) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi_s(\gamma \cdot z), \tag{1.11}$$

where we have also introduced the notation $\chi_s(z) = [\mathrm{Im}(z)]^s = y^s$. The reason for this notation is that χ_s is actually induced from a *character* on the real Borel subgroup. We will explain this in more detail below in chapter 5. Note that this way of writing the automorphic form makes the invariance under $SL(2,\mathbb{Z})$ completely manifest because it is a sum over images.

The form (1.11) is what we will call an *Eisenstein series* on $SL(2,\mathbb{R})$ and it is this form that generalises straight-forwardly to Lie groups $G(\mathbb{R})$ other than $SL(2,\mathbb{R})$ (whereas the form with the sum over a lattice does not, as we discuss in more detail in section 12.2). In complete analogy with (1.11) we define the (minimal parabolic) Eisenstein series on $G(\mathbb{R})$ invariant under the discrete group $G(\mathbb{Z})$ by[2]

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi(\gamma g) \tag{1.12}$$

where χ is (induced from) a character on the Borel subgroup $B(\mathbb{R})$ and $g \in G(\mathbb{R})$. Eisenstein series are the protagonists of the story we will develop.

1.2 Why Eisenstein series and automorphic forms?

Before delving into the further analysis of Eisenstein series, let us briefly step back and provide some motivation for their study.

[2]We will always take $G(\mathbb{Z})$ as the Chevalley group that is defined as the stabiliser (in $G(\mathbb{R})$) of a preferred integral basis (Chevalley basis) of the Lie algebra of $G(\mathbb{R})$, see section 4.1.4 below more details.

1.2.1 A mathematician's possible answer

Automorphic forms are of great importance in many mathematical fields such as number theory, representation theory and algebraic geometry. The various ways in which automorphic forms enter these seemingly disparate fields are connected by a web of conjectures collectively referred to as the Langlands program [98, 116, 196, 197, 216, 217].

Much of the arithmetic information is contained in the Fourier coefficients of automorphic forms. The standard examples correspond to modular forms on $G(\mathbb{R}) = SL(2,\mathbb{R})$, where these coefficients yield eigenvalues of Hecke operators (covered in chapter 11) and the counting of points on elliptic curves.

For arbitrary Lie groups $G(\mathbb{R})$ one considers the Hilbert space $L^2(\Gamma\backslash G(\mathbb{R}))$ of square-integrable functions that are invariant under a left action by a discrete subgroup $\Gamma \subset G(\mathbb{R})$. This space carries a natural action of $g \in G(\mathbb{R})$, called the right-regular action, through

$$\big[\pi(g)f\big](x) = f(xg) \tag{1.13}$$

where $f \in L^2\big(\Gamma\backslash G(\mathbb{R})\big)$, $g, x \in G(\mathbb{R})$ and $\pi : G(\mathbb{R}) \to \mathrm{Aut}\big(L^2\big(\Gamma\backslash G(\mathbb{R})\big)\big)$ is the right-regular representation map. Since the functions are square-integrable the representation is unitary. This representation-theoretic viewpoint on automorphic forms was first proposed by Gelfand, Graev and Piatetski-Shapiro [120] late developed considerably by Jacquet and Langlands [176]. This perspective provides the key to generalising the classical theory of modular forms on the complex upper half plane to higher rank Lie groups.

It is an immediate, important and difficult question as to what the decomposition of the space $L^2(\Gamma\backslash G(\mathbb{R}))$ into irreducible representations of $G(\mathbb{R})$ looks like. The irreducible constituents in this decomposition are called *automorphic representations*. This spectral problem was tackled and solved by Langlands [218]. The Eisenstein series (and their analytic continuations) form an integral part in the resolution although they themselves are not square-integrable.[3] Although we will not describe the full resolution of this problem in these notes, automorphic representations will play a prominent role in our discussion.

1.2.2 A physicist's possible answer

Many problems in quantum mechanics are characterised by discrete symmetries. At the heart of many of them lies Dirac quantisation where charges (e.g. electric or magnetic) of physical states are restricted to lie in certain lattices rather than in continuous spaces. The discrete symmetries preserving the lattice are often called dualities and can give very interesting different angles on a physical problem. This happens in particular in string theory, where such dualities mix perturbative and non-perturbative effects.

For the discrete symmetry to be a true symmetry of a physical theory, all observable quantities must be given by functions that are invariant under the discrete symmetry, corresponding to property (1) discussed at the beginning of section 1.1. Similarly, the

[3] A passing physicist might note that this is very similar to using non-normalisable plane waves as a 'basis' for wave functions in quantum mechanics. Indeed the piece $\chi(\gamma g)$ in (1.12) is exactly like a plane wave; the γ-sum is there to make it invariant under the discrete group so that $E(\chi, g)$ are the simplest Γ-invariant plane waves.

dynamics or other symmetries of the theory impose differential equations on the observables, corresponding to property (2), and the growth condition (3) is typically associated with having well-defined perturbative regimes of the theory. The main example we have in mind here comes from string theory and the construction of scattering amplitudes of type II strings in various maximally supersymmetric backgrounds [138–141, 195, 252]. However, the logic is not necessarily restricted to this, see also [6, 25, 229, 302] for some other uses of automorphic forms in physics.

For these reasons one is naturally led to the study of automorphic forms in physical systems with discrete symmetries. Via this route one is also led to the same spectral problem posed by the mathematician since one needs to determine which automorphic representation a given physical observable belongs to. Again, the Eisenstein series and their properties are the building blocks of such spaces and it is important to understand them well. Furthermore, in a number of examples from string theory it was actually possible to show that the observable is given by an Eisenstein series itself [138, 144].[4] A detailed discussion of automorphic forms in string theory is given in chapter 2.

1.3 Analysing automorphic forms and adelisation

We now return to the study of Eisenstein series defined by (1.12) and their properties, starting again with the very explicit example (1.1) for $SL(2,\mathbb{R})$.

1.3.1 Fourier expansion of the $SL(2,\mathbb{R})$ series

The discrete Borel subgroup $B(\mathbb{Z})$ of (1.9) acts on the variable $z = x + iy$ as translations:

$$\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \cdot z = z \pm m \quad \text{for } m \in \mathbb{Z} \tag{1.14}$$

and therefore any automorphic function (that is by definition invariant under any discrete transformation) is periodic in the x direction with period equal to 1 corresponding to the smallest non-trivial $m = 1$. This means that we can Fourier expand it in modes $e^{2\pi i n x}$. Applying this to (1.11) leads to

$$E(\chi_s, z) = \underbrace{C(y)}_{\substack{\text{constant term} \\ \text{zero mode}}} + \underbrace{\sum_{n\neq 0} a_n(y) e^{2\pi i n x}}_{\text{non-zero mode}} . \tag{1.15}$$

As we indicated, it is natural to divide the Fourier expansion into two parts depending on whether one deals with the zero Fourier mode (a.k.a. constant term) or with a non-zero mode. Since the Fourier expansion was only in the x direction, the Fourier coefficients still depend on the second variable y.[5]

[4] That Eisenstein series are mostly not square-integrable is no problem in these cases since the object computed (part of a scattering amplitude) is not a wavefunction and not required to be normalisable.

[5] If one dealt with an automorphic form holomorphic in z (called modular forms in chapter 5 below) this would not be true since the holomorphicity condition links the x and y dependence. The Fourier

Determining the explicit form of the Fourier coefficients is one of the key problems in the study of Eisenstein series. In the example of $SL(2,\mathbb{R})$ this can for instance be done by making recourse to the formulation in terms of a lattice sum that was given in (1.1) and using the technique of Poisson resummation. The calculation is reviewed in appendix B and leads to the following explicit expression

$$E(\chi_s, z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{m\neq 0} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}, \tag{1.16}$$

where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{1.17}$$

is the *completion of the Riemann zeta function* (1.6), $K_s(z)$ is the modified Bessel function of the second kind (that decreases exponentially for $z \to \infty$ in accordance with the growth condition) and

$$\sigma_{1-2s}(n) = \sum_{d|n} d^{1-2s} \tag{1.18}$$

is called a divisor sum (or the instanton measure in physics; see chapter 2 below) and given by a sum over the positive divisors of $n \neq 0$.

As is evident from (1.16), the explicit form of the Fourier expansion can appear quite complicated and involves special functions as well as number theoretic objects. For the case of more general $G(\mathbb{R})$ the method of Poisson resummation is not necessarily available as there is not always a form of the Eisenstein series as a lattice sum. *It is therefore desirable to develop alternative techniques for obtaining (parts of) the Fourier expansion under more general assumptions.*[6] This is achieved by lifting the Eisenstein series into an adelic context which we now sketch and explain in more detail in section 5.2.

1.3.2 Adelisation of Eisenstein series

A standard elementary technique in number theory for analysing equations over $\mathbb{Z}$ is by analysing them instead as congruences for every prime (and its powers) separately (sometimes known as the Hasse principle or the local-global principle based on the Chinese remainder theorem) [1, 251] (see also Appendix D for some examples). One way of writing all the terms together is by using the *ring of adeles* $\mathbb{A}$. The adeles can formally be thought of as an infinite tuple

$$a = (a_\infty; a_2, a_3, a_5, a_7, \ldots) \in \mathbb{A} = \mathbb{R} \times \prod_{p<\infty}{}' \mathbb{Q}_p, \tag{1.19}$$

coefficients in an expansion in $q = e^{2\pi i(x+iy)} = e^{2\pi i z}$ would be pure numbers. This is the origin of the name *constant term* for the zero mode in (1.15).

[6] Additional care has to be taken for the Fourier expansion for general $G(\mathbb{R})$ also because the translation group $B(\mathbb{Z})$ is in general not abelian. One can still define (abelian) Fourier coefficients as we will see, however, they fail to capture the full Eisenstein series. There are also non-abelian parts to the Fourier expansion.

where $\mathbb{Q}_p$ denotes the *p-adic numbers* that are a completion of the rational number $\mathbb{Q}$ that is inequivalent to the standard one (leading to $\mathbb{R}$) and that is parametrised by a prime number p and defined properly in section 3.1. The product is over all prime numbers and the prime on the product symbol indicates that the entries a_p in the tuple are restricted in a certain way (see (3.59) below for the exact statement). The real numbers $\mathbb{R}$ can be written as $\mathbb{Q}_\infty$ in this context and interpreted as the completion of $\mathbb{Q}$ at the 'prime' $p = \infty$. *Very* crudely, an adele can be thought of as summarising the information of an object modulo all primes.

Strong approximation is a similar method that lifts a general automorphic form from being defined on the space $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ to the space $G(\mathbb{Q})\backslash G(\mathbb{A})/K(\mathbb{A})$ so that $G(\mathbb{Q})$ plays the role of the discrete subgroup that was played by $G(\mathbb{Z})$ before. However, $G(\mathbb{Q})$ is a nicer group than $G(\mathbb{Z})$ since $\mathbb{Q}$ is a field whereas $\mathbb{Z}$ is only a ring. This facilitates the analysis and allows the application of many theorems for algebraic groups.

A consequence of using strong approximation and adeles is that the result of the calculation factorises according to (1.19) and one can do the calculation for all primes and $p = \infty$ separately. Indeed, the explicit form (1.16) for the Fourier expansion of the $SL(2, \mathbb{R})$ Eisenstein series already secretly had this form. This can be seen for example in the constant term since

$$\frac{\xi(2s-1)}{\xi(2s)} = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)} \prod_{p<\infty} \frac{1-p^{-2s}}{1-p^{1-2s}} \tag{1.20}$$

where we have used the definition of the completed zeta function from (1.17) and the *Euler product formula* for the Riemann zeta function [318]

$$\zeta(s) = \sum_{n>0} n^{-s} = \prod_{p<\infty} \frac{1}{1-p^{-s}}. \tag{1.21}$$

In (1.20) we clearly recognise a factorised form that is very similar to (1.19). That this is not an accident will be demonstrated in section 7.2 for $SL(2, \mathbb{R})$. For the other Fourier modes in (1.16) we get a similar factorisation with the modified Bessel function belonging to the $p = \infty$ factor and

$$\sigma_{1-2s}(m) = \prod_{p<\infty} \gamma_p(m) \frac{1-p^{-(2s-1)}\,|m|_p^{2s-1}}{1-p^{-(2s-1)}} \tag{1.22}$$

where $|m|_p$ is the p-adic norm of m defined in section 3.1 and $\gamma_p(m)$ selects all factors with $|m|_p \leq 1$ as shown in section 3.4. The complete derivation for the non-constant terms can be found in section 7.3 for the $SL(2, \mathbb{R})$ Eisenstein series.

The adelic methods are so powerful that one can obtain a closed, simple and group-theoretic formula for the constant term of Eisenstein series on any (split real) Lie group $G(\mathbb{R})$. This formula, known as the *Langlands constant formula* will be the topic of chapter 8.

For the (abelian) Fourier coefficients, the adelic methods also help to obtain fairly general results, in particular for the part that involves the finite primes $p < \infty$. For the contribution coming from the $\mathbb{R}$ in (1.19) the results are not quite as general; already for $SL(2, \mathbb{R})$ this is what gives the modified Bessel function. We discuss the Fourier coefficients in chapter 9.

1.4 Reader's guide and main theorems

Chapters 2, 3 and 4 provide preliminary background material that is needed in subsequent chapters. Chapter 2 gives a brief overview of how automorphic forms enter in computing scattering amplitudes in string theory. This is not intended as a comprehensive introduction to string theory, but its aim is rather to act as a first glimpse, primarily directed towards mathematicians, of a vast and fascinating topic that is closely tied to automorphic forms and representation theory. Throughout the main text we also offer remarks and pointers that indicate physical interpretations of various mathematical notions and results. Chapter 3, on the other hand, introduces the basic machinery of p-adic and adelic analysis which will be crucial for everything we do later. The main thrust of the chapter is provided by the numerous examples of computing p-adic integrals that will be used extensively in proving Langlands constant term formula, and computing Fourier coefficients of Einstein series. In chapter 4 we introduce some basic features of Lie algebras and Lie groups that will be used in the remainder of the text. We first discuss Lie groups and Lie algebras over $\mathbb{R}$ and then move on to algebraic groups over $\mathbb{Q}_p$ as well as adelic groups.

We shall now discuss the structure of the remainder of the text in a little more detail, with emphasis on the central results in each chapter.

- In chapter 5 we introduce the general theory of automorphic forms and automorphic representations. We start out gently by discussing how to pass from modular forms on the upper half plane to automorphic forms on the adelic group $SL(2,\mathbb{A})$. We then move on to the general case of arbitrary Lie groups. We define *Eisenstein series* $E(\chi, g)$ for general split real Lie groups $G(\mathbb{R})$ that are invariant under the discrete Chevalley subgroup $G(\mathbb{Z})$. The definition (1.12) requires the choice of a character χ of a parabolic subgroup P of G; alternatively, we can think of χ as being defined by a choice of weight vector λ of the (split real) Lie algebra of $G(\mathbb{R})$. We explain how this can be understood from the point of view of the representation theory of $G(\mathbb{R})$, and we show how to lift the function from being defined on $G(\mathbb{R})$ to a function defined on $G(\mathbb{A})$ where $\mathbb{A}$ are the *adeles* of the rational number field $\mathbb{Q}$.

- A major part of these notes are devoted to analysing Fourier expansions of automorphic forms. This is a highly non-trivial subject with many interesting connections to representation theory as well as in physics. In chapter 6 we introduce the general theory of Fourier coefficients and Whittaker vectors, with emphasis on the representation-theoretic viewpoint. Toward the end of the chapter we also introduce some more advanced topics, such as the nilpotent orbits and wavefront sets, as well as the Piatetski-Shapiro–Shalika formula.

- Chapter 7 we illustrate all the general techniques in the context of Eisenstein series on $SL(2)$. Specifically, using adelic techniques, we provide a detailed proof of the following classic theorem:

Theorem 1.1. *The complete Fourier expansion of the Eisenstein series $E(\chi_s, g)$ for $g \in SL(2,\mathbb{R}) \subset SL(2,\mathbb{A})$ is given by:*

$$E(\chi_s, g) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \sum_{m\neq 0} \frac{2y^{1/2}}{\xi(2s)} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{1.23}$$

Furthermore, the Eisenstein series satisfies the functional relation

$$E(\chi_s, g) = \frac{\xi(2s-1)}{\xi(2s)} E(\chi_{1-s}, g), \tag{1.24}$$

where ξ is the (completed) Riemann zeta function. The rest of the notation is explained in chapter 7.

- The first two terms in the Fourier expansion above correspond to the zeroth Fourier coefficients. These are often collectively referred to as the *constant term* of the Eisenstein series, A very important and general result in this context is provided by the so called Langlands constant term formula, which yields a remarkably simple expression for the complete constant term of Eisenstein series on arbitrary semi-simple Lie groups. In chapter 8 we give a complete proof of the following theorem of Langlands:

  Theorem 1.2 (Langlands' constant term formula). *Let G be a real semi-simple Lie group and $G(\mathbb{A})$ its adelisation. Let λ be a weight of the Lie algebra $\mathfrak{g}$, W the associated Weyl group, and N a maximal unipotent radical of G. We then have*

$$\int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng) dn = \sum_{w\in W} a^{w\lambda+\rho} \prod_{\alpha>0\,:\, w\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}, \tag{1.25}$$

  where a belongs to the Cartan torus $A \subset G$, and the product runs over positive roots α of $\mathfrak{g}$.

- The infinite sum in the Fourier expansion (1.23) correspond to the non-zero coefficients and this is generally referred to as the *non-constant term*. In chapter 9 we discuss the general structure of of Fourier coefficients of Eisenstein series on reductive groups G. For this part of the expansion much less is known explicitly. However, there exists a beautiful formula due to Kato–Shintani–Casselman–Shalika, commonly known as the *Casselman–Shalika formula*, which gives an explicit expression for the so-called p-adic Whittaker vector. This corresponds to a local version of the Fourier coefficient of the Eisenstein series, which can be used to reassemble the full (global) coefficient. A large part of chapter 9 is therefore devoted to proving the following theorem:

  Theorem 1.3 (Casselman–Shalika formula). *Let $G(\mathbb{Q}_p)(p < \infty)$ be a p-adic semi-simple Lie group, $N(\mathbb{Q}_p)$ a maximal unipotent radical of $G(\mathbb{Q}_p)$ and ψ an* *unramified unitary character on $N(\mathbb{Q}_p)$. The Casselman–Shalika formula is given by:*

$$\int_{N(\mathbb{Q}_p)} \chi(w_0 n a)\overline{\psi(n)} dn = \frac{\epsilon(\lambda)}{\xi(\lambda)} \sum_{w \in W} (\det(w)) |a^{w\lambda+\rho}| \prod_{\substack{\alpha>0 \\ w\alpha<0}} p^{\langle\lambda|\alpha\rangle} \tag{1.26}$$

- For certain special types of Fourier coefficients, so called *degenerate Whittaker vectors*, one can take one step further and compute the full global coefficient (and not just the p-adic version). In chapter 9 we also prove the following theorem which gives such a formula:

  Theorem 1.4 (degenerate Whittaker vector). *Let $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$ be a degenerate character associated subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$. Then the degenerate Whittaker vector on $G(\mathbb{A})$ is given by*

  $$W_\psi^\circ(\chi, a) = \sum_{w_c w'_{long} \in \mathcal{W}/\mathcal{W}'} a^{(w_c w'_{long})^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) W'^\circ_{\psi^a}(w_c^{-1}\lambda, \mathbb{1}), \tag{1.27}$$

  where W'°_ψ denotes a Whittaker function on the $G'(\mathbb{A})$ subgroup of $G(\mathbb{A})$. The weight $w_c^{-1}\lambda$ is given as a weight of $G'(\mathbb{A})$ by orthogonal projection.

  A more complete formulation is provided in section 9.5. In section 9.6 we also provide an extensive example of how to calculate Whittaker vectors for Eisenstein series on $SL(3, \mathbb{A})$.

- In chapter 10 we illustrate how to perform calculations with Eisenstein series in practice. More specifically we explain how to evaluate Langlands constant term formula in concrete examples, which, in particular, involves a detailed analysis of the functional equation. We also show how to perform similar evaluations of the Whittaker vectors that appear in the non-constant Fourier coefficients. We provide some explicit examples for exceptional Lie groups.

- It is interesting to note that both the Langlands constant term formula (1.25) and the Casselman–Shalika formula (1.26) have cunning similarities to the Weyl character formula. That this is not an accident is a central insight of Langlands. To understand this requires the additional machinery of *Hecke theory*, which is the topic of chapter 11. Here we explain how to pass from the classical notion of Hecke operators acting on modular forms to the general notion of *spherical Hecke algebras* on adelic groups. This analysis leads us to a reformulation of the Casselman–Shalika formula that clearly illustrates the intimate connection with the Weyl character formula. In this context we are naturally lead to the notion of the *Langlands dual group* ${}^LG$ and to the notion of *automorphic L-functions*, which form a central ingredient in the Langlands program. The chapter concludes with a discussion of the *Langlands-Shahidi method*, which is a powerful way to construct L-functions from automorphic representations.

- In the concluding chapter 12, we present various interesting directions which we have not been able to cover in detail. We emphasise open questions and conjectures, many of which has sprung out of problems in string theory, and we have tried to formalise them and phrased them in purely mathematical terms. We also discuss briefly some of the key ingredients in the Langlands program and we make various comments and conjectures regarding its extension to Kac–Moody groups.

We end by giving some disclaimers: All groups $G(\mathbb{R})$ that will be considered here are associated with split real forms and we also restrict to simple groups. The only base field that we will use for adelisation are the rational numbers $\mathbb{Q}$. Often we will perform formal manipulations of infinite sums and integrals without paying attention to whether the expressions are (absolutely) convergent or not. The expressions typically depend on a set of parameters and for some range of parameters convergence can be established. In many cases, the results can be extended by analytic continuation.

Chapter 2

String theory scattering and automorphic forms

In this chapter we will introduce some basic concepts of string theory with emphasis on scattering amplitudes of closed strings. Our main purpose is to illustrate the deep connection between string theory and automorphic forms, which may come as a surprise to a mathematician. The topic is far too vast for us to do it justice in just a single chapter, but hopefully this will provide a sufficient glimpse to spark the motivation for further studies. The main point we wish to convey is that the Fourier coefficients of automorphic forms on higher rank Lie groups capture essential information about string theory amplitudes. We should stress that scattering amplitudes is but one aspect of the relation between string theory and automorphic forms. Other connections, such as to BPS-state counting, are discussed in section 12.1. For more information about string theory we recommend the books [24, 150, 272, 326] and, for a brief introduction, the lecture notes [82, 311].

2.1 String theory concepts

String theory is a theory of one-dimensional extended objects propagating in a (Lorentzian) *target space-time* M. During their propagation, strings sweep out a two-dimensional *world-sheet* Σ and string theory can therefore be thought of as the dynamics of the embedding maps $X : \Sigma \to M$, where both Σ and M are endowed with additional structure (like a metric) that enter in the definition of the dynamics. In *superstring theory* —that we exclusively consider here and refer to generally as string theory— this additional structure includes *world-sheet supersymmetry* and the space of allowed world-sheets Σ is then the space of all closed, orientable super Riemann surfaces. We here also specialise to so-called type IIB superstrings; otherwise one might have to include boundaries and non-orientable surfaces as well. Riemann surfaces are classified by their *genus* $h \in \mathbb{Z}_{\geq 0}$, a topological invariant.

A fundamental parameter of string theory is the characteristic *length* ℓ_s *of a string*. More commonly, one uses the parameter $\alpha' = \ell_s^2$ which can be thought of as the scale of area of the string world-sheet Σ. For very small string lengths $\ell_s \to 0$, the strings

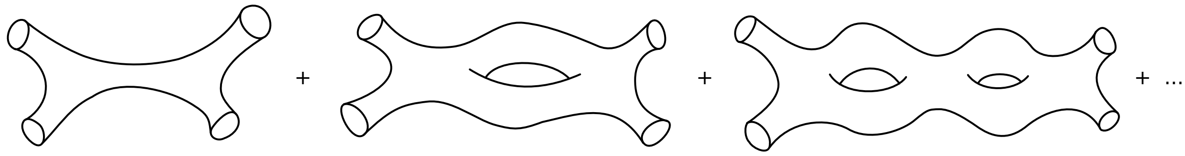


Figure 2.1: *String world-sheets as they appear in the scattering of four closed strings. Ignoring the asymptotic boundary states, the diagrams correspond to genera $h = 0, 1, 2$, respectively.*

look effectively like point particles. The (bosonic) spectrum of string excitations in flat ten-dimensional Minkowski space $M = \mathbb{R}^{1,9}$ is then given by states of mass

$$m^2 = \frac{2}{\alpha'}N, \tag{2.1}$$

where $N \in \mathbb{Z}_{\geq 0}$ is the excitation number of the string states. The lightest string states are massless and there is an infinite sequence of massive states with quantised masses and the separation between the masses is set by the *string scale* α'. At every mass level one has a finite number of degrees of freedom.

Strings can interact by various joining and splitting processes. Such interactions correspond to string world-sheets with asymptotic states coming together for a scattering process and then separating again. Examples with few splittings, corresponding to low genus world-sheets, are depicted in figure 2.1. In the limit $\alpha' \to 0$ the diagrams lose their waists and reduce to standard *Feynman diagrams* commonly used in the *quantum field theory* of *point particles*. At the same time, the massive states ($N > 0$) in (2.1) become infinitely heavy compared to the massless states ($N = 0$) in the limit $\alpha' \to 0$.

Among the quantities one wants to compute in string theory are *scattering amplitudes*. They provide information about the likelihood of a certain scattering process of strings to take place. Computing a scattering amplitude in *string perturbation theory* requires summing over possible world-sheets of all genera h. A given topology is then weighted by the so-called *string coupling* g_s with a weight $g_\mathrm{s}^{2(h-1)}$. The string coupling is a measure of the strength of string-string interaction, i.e., the joining and splitting of strings. Note that in string theory it is convention to use the term *loop* when referring to the genus of a world-sheet. The probability of a certain string scattering process is given by the modulus square of the scattering amplitude. The perturbative expansion is a power expansion in g_s.

Besides the data of the asymptotic states, the scattering amplitude depends on α', the string coupling and potentially other so-called *moduli fields*. These can be thought of as aspects of the target space M in the form of additional (scalar) fields living on them. Only their vacuum expectation values matter and we will denote them by

$$g \in \mathcal{M} \qquad \text{(moduli expectation values)}\,. \tag{2.2}$$

Here, $\mathcal{M}$ is the so-called *moduli space* of string theory. The string coupling constant g_s turns out to be related to one of the moduli fields called the *dilaton*, but in general there are many more moduli fields.

Table 2.1: *Table of Cremmer–Julia symmetry groups $G(\mathbb{R})$ with compact subgroup $K(\mathbb{R})$ and U-duality groups $G(\mathbb{Z})$ for compactifications of type IIB string theory on a d-dimensional torus T^d to $D = 10 - d$ dimensions.*

d	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$	D
0	$SL(2,\mathbb{R})$	$SO(2,\mathbb{R})$	$SL(2,\mathbb{Z})$	10
1	$GL(2,\mathbb{R})$	$SO(2,\mathbb{R})$	$SL(2,\mathbb{Z})$	9
2	$SL(2,\mathbb{R}) \times SL(3,\mathbb{R})$	$SO(2,\mathbb{R}) \times SO(2,\mathbb{R})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	8
3	$SL(5,\mathbb{R})$	$SO(5,\mathbb{R})$	$SL(5)$	7
4	$SO(5,5,\mathbb{R})$	$(SO(5,\mathbb{R}) \times SO(5,\mathbb{R}))/\mathbb{Z}_2$	$SO(5,5,\mathbb{Z})$	6
5	$E_6(\mathbb{R})$	$USp(8,\mathbb{R})/\mathbb{Z}_2$	$E_6(\mathbb{Z})$	5
6	$E_7(\mathbb{R})$	$SU(8,\mathbb{R})/\mathbb{Z}_2$	$E_7(\mathbb{Z})$	4
7	$E_8(\mathbb{R})$	$SO(16,\mathbb{R})/\mathbb{Z}_2$	$E_8(\mathbb{Z})$	3

The structure of moduli space is of central importance in understanding the possible forms of string scattering amplitudes. Much is known for flat target spaces of the type $M = \mathbb{R}^{1,9}$ (flat Minkowski space) or $M = \mathbb{R}^{1,9-d} \times T^d$ (toroidal compactification). In both cases one retains *maximal supersymmetry* strongly constraining the moduli space. The classical low energy moduli space is a symmetric space of the form

$$\mathcal{M}_{\text{class.}} = G(\mathbb{R})/K(\mathbb{R}) = E_{d+1}(\mathbb{R})/K(E_{d+1}(\mathbb{R})), \tag{2.3}$$

where E_{d+1} is the Cremmer–Julia sequence of duality groups [72, 73, 185] that are listed in table 2.1 and their Dynkin diagrams shown in figure 2.2, and $K(E_{d+1}(\mathbb{R}))$ are their maximal compact subgroups. Up to $d \leq 7$, these groups are finite-dimensional reductive groups and we restrict to this range first. We will come back to $d \geq 8$ in section 12.7. For other internal manifolds one can get a large variety of different Lie groups.

The classical low-energy effective theory in $D = 10 - d$ dimensions, which is described by supergravity, has a symmetry given by the non-compact real Lie group $G(\mathbb{R})$ [171, 253]. However, as mentioned in section 1.2.2, when passing to the quantum theory, the classical symmetries are generically broken because the (generalised) electro-magnetic charges of physical states become quantised according to the *Dirac-Schwinger-Zwanziger quantisation condition*, and take values in some integral lattice Γ [88, 250, 306, 307]. Although the classical symmetry group $G(\mathbb{R})$ is broken, there is a discrete subgroup of $G(\mathbb{R})$ that survives and remains a symmetry of the full quantum theory. This quantum symmetry is defined as the subgroup of $G(\mathbb{R})$ that preserves the lattice Γ [171]

$$\{g \in G(\mathbb{R}) \mid g\Gamma = \Gamma\}. \tag{2.4}$$

This quantum symmetry group is generally referred to as a *U-duality group* which unifies the previously known existing dualities called S- and T-duality and agrees with the Chevalley subgroup $G(\mathbb{Z})$ [299]. The U-duality groups $G(\mathbb{Z})$ for toroidal compactifications of type IIB string theory on a torus T^d are also listed in table 2.1.

Points in moduli space related by U-duality transformations give rise to equivalent string theories. This implies that the correct moduli space of quantum string theory is not

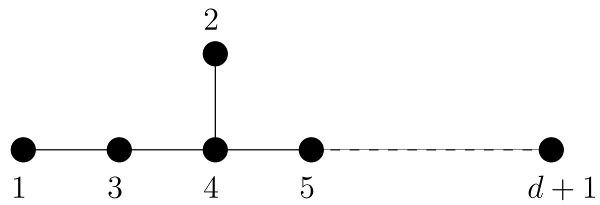


Figure 2.2: *The Dynkin diagram of the Cremmer–Julia symmetry group E_{d+1} with labelling of nodes in the 'Bourbaki convention'.*

the classical symmetric space (2.3) but

$$\mathcal{M} \equiv \mathcal{M}_{\text{quantum}} = G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})\,, \tag{2.5}$$

and all observables, like string scattering amplitudes, that are functions of the expectation values of the moduli are functions on this space.

Put differently; physical observables are $G(\mathbb{Z})$-invariant functions on $G(\mathbb{R})/K$. In addition, physical constraints, such as supersymmetry, typically force these observables to satisfy differential equations and have a prescribed asymptotic behaviour at infinity, thereby satisfying the conditions (1)–(3) in section 1.1, characterising automorphic forms.

2.2 Four-graviton scattering amplitudes

Let us now consider an explicit example. In ten-dimensional type IIB string theory the scattering of four massless string states called *gravitons* gives rise to a quantum correction to standard gravitational interactions in general relativity without the inclusion of strings.

The exact form of the full *four-graviton scattering* amplitude is not known but one can attempt to perform series expansions of the amplitude with respect to some of its arguments. There are two common expansions of the amplitude.

The first expansion is string perturbation theory, discussed above, in which one treats the string coupling constant g_s as small and computes the contributions to the amplitudes from Riemann surfaces of increasing genus. This involves an integral over the moduli space of all Riemann surfaces of a given genus and with a number of punctures corresponding to the number of asymptotic scattering states. These integrals have been studied up to two loops, see for instance [85], and become increasingly hard for increasing genus. One complication arises from the Schottky problem of parametrising the moduli space for large genus h. Another serious complication is that one should actually integrate over the moduli space of *super* Riemann surfaces since one is dealing with superstring theory and it is known that this integral cannot be reduced in a simple manner to an integral over ordinary Riemann surfaces for $h \geq 5$ [89]. Finally, the amplitude is not expected to be a convergent series in g_s, meaning that there are *non-perturbative effects* arising from instanton configurations [138, 268, 296]. These are roughly of the form e^{-1/g_s} and do not admit a Taylor series expansion around weak coupling $g_s = 0$ and therefore cannot be captured by string perturbation theory. The string coupling g_s is one of the coordinates on the moduli space $\mathcal{M}$ and the limit $g_s \to 0$ corresponds to approaching a cusp on $\mathcal{M}$.

The second expansion of the amplitude is the *low energy expansion* in which one considers the momenta of the scattering particles to be small, leading to an expansion in derivatives of the fields. Dimensionless expansion parameters are formed by multiplying momenta squared with α', which is why the expansion is also called the α'*-expansion*. It is this expansion that makes contact to automorphic forms and we will now study it in detail.

As asymptotic states, gravitons are characterised by their momenta $k_i \in \mathbb{R}^{1,9}$ ($i = 1,2,3,4$) and their polarisations $\epsilon_i \in S^2(\mathbb{R}^{1,9})$, which are symmetric second rank tensors subject to some constraints whose detail we do not require for the present discussion. Since gravitons are massless, the momenta satisfy $k_i^2 = 0$, where the norm-squared is computed using the Lorentzian metric on $\mathbb{R}^{1,9}$. Out of the four momenta k_i one forms the dimensionless Lorentz invariant *Mandelstam variables*

$$s = -\frac{\alpha'}{4}(k_1+k_2)^2, \quad t = -\frac{\alpha'}{4}(k_1+k_3)^2 \quad \text{and} \quad u = -\frac{\alpha'}{4}(k_1+k_4)^2. \tag{2.6}$$

Momentum conservation ($k_1+k_2+k_3+k_4 = 0$) and masslessness imply that $s+t+u = 0$. Any symmetric polynomial in s,t,u then is a polynomial in

$$\sigma_2 = s^2+t^2+u^2, \quad \text{and} \quad \sigma_3 = s^3+t^3+u^3. \tag{2.7}$$

The string scattering amplitude will therefore be a function of the momenta only through σ_2 and σ_3. Similar simplifications arise for the polarisation tensors ϵ_i that enter the final answer only in a particular combination that we will denote by $\mathcal{R}^4$. It can be expressed as the contraction of two copies of a standard rank 8 tensor t_8, whose precise form is for example given in [150, 156], and four copies of the linearised curvature tensor $\mathcal{R}_{\mu\nu\rho\sigma} \propto k_\mu \epsilon_{\nu\rho} k_\sigma$ (with permutations).

Our four graviton amplitude in $D = 10-d$ dimensions is therefore of the form

$$\mathcal{A}^{(D)}(s,t,u,\epsilon_i;g), \tag{2.8}$$

with $g \in \mathcal{M}$. We recall that the string scale α' was absorbed into the Mandelstam variables s,t,u.

The (analytic in α' part of the) four-graviton amplitude (in Einstein frame) in this expansion takes the form [140]

$$\mathcal{A}^{(D)}(s,t,u,\epsilon_i;g) = \left[\mathcal{E}^{(D)}_{(0,-1)}(g)\frac{1}{\sigma_3} + \sum_{p\geq 0}\sum_{q\geq 0}\mathcal{E}^{(D)}_{(p,q)}(g)\sigma_2^p\sigma_3^q\right]\mathcal{R}^4\,. \tag{2.9}$$

The interesting objects in this expression are the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}(g)$ that are functions on the moduli space $\mathcal{M}_{\text{class.}} = G(\mathbb{R})/K(\mathbb{R})$ in (2.3).

The first term in (2.9) plays a special role in that it is the only term that is not polynomial in σ_2 and σ_3. It is the lowest order term in the α'-expansion and it agrees with what one would calculate in a standard theory of gravity with Lagrangian given by the Ricci scalar only (referred to as the *Einstein–Hilbert term*). The coefficient function $\mathcal{E}^{(D)}_{(0,-1)}(g) = 3$ is constant. By contrast, the infinite series of terms in p and q come with higher powers of α' and they reflect the contribution of *massive* string states to the graviton scattering process [156].

The low energy effective theory is obtained by writing the field theory action whose classical interactions give rise to the same quantum corrected amplitudes obtained from string theory, order by order in α'. It gets corrections from the four-graviton amplitudes on the form [146]

$$\mathcal{E}^{(D)}_{(p,q)}(g) D^{2p+3q} R^4 \tag{2.10}$$

where D denotes a covariant derivative and R^4 (not to be confused with the Ricci curvature scalar) is a contraction of two t_8 tensors and four Riemann curvature tensors like the linearised version for the polarisation term $\mathcal{R}^4$ in the amplitude.

In other words, for the first few orders in α' we get the corrections

$$S = S_{\text{class.}} + \int d^D x \sqrt{-G} \Big((\alpha')^3 \mathcal{E}^{(D)}_{(0,0)}(g) R^4 + (\alpha')^5 \mathcal{E}^{(D)}_{(1,0)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}^{(D)}_{(0,1)}(g) D^6 R^4 + \dots \Big), \tag{2.11}$$

where $S_{\text{class.}}$ is the classical, zeroth order low energy effective action described by supergravity. We see why this expansion also is called the *derivative expansion*.

As the functions $\mathcal{E}^{(D)}_{(p,q)}(g)$ depend on the moduli $g \in \mathcal{M}$, they in particular depend on the string dilaton and thus on the string coupling g_{s} that controls string perturbation theory in terms of Riemann surfaces as discussed above. However, there is no reason that the dependence on g_{s} be analytic. Non-analytic terms in g_{s} are known as *non-perturbative effects* and they appear in $\mathcal{E}^{(D)}_{(p,q)}(g)$ through so-called *instanton contributions*. Their direct determination in terms of a string theory calculation is typically very hard. The action of the U-duality group $G(\mathbb{Z})$ also includes a transformation that mixes perturbative and non-perturbative effects and therefore using U-duality opens up the opportunity to access non-perturbative effects indirectly.

Because of U-duality, the coefficients $\mathcal{E}^{(D)}_{(p,q)}$ are, in fact, functions on the (quantum) moduli space $\mathcal{M}$: *They are functions on the symmetric space* $\mathcal{M}_{\text{class.}} = G(\mathbb{R})/K(\mathbb{R})$ *invariant under the left action of the discrete group* $G(\mathbb{Z})$.

This means that $\mathcal{E}^{(D)}_{(p,q)}$ satisfy (1) of the definition of automorphic forms in section 1.1 and, since they should also have perturbative expansions in the weak string coupling limit $g_{\text{s}} \to 0$ and other similar limits of the moduli space corresponding to cusps in G/K, they also satisfy the growth condition (3).

Let us now discuss the remaining condition (2) which requires that an automorphic form satisfies the eigenvalue equations of G-invariant differential operators.

String theory imposes differential conditions on the coefficient functions. This was analysed most thoroughly by Green and Sethi in the case of ten-dimensional ($D = 10$) type IIB string theory and $p = q = 0$ [141]. They found that $\mathcal{E}^{(10)}_{(0,0)}(g)$ has to satisfy a Laplace equation with an eigenvalue determined by supersymmetry considerations. This, together with the known value for the string tree level ($h = 0$ topology) scattering result, uniquely determined the coefficient function to be a non-holomorphic Eisenstein series on $SL(2,\mathbb{R})$ [261] as discussed by Green and Gutperle before in [138, 139].

In other dimensions D and for small values of p and q, there are strong arguments that the coefficient functions $\mathcal{E}^{(D)}_{(p,q)}(g)$ satisfy the differential equations [140]

$$R^4: \quad \left(\Delta_{G/K} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)}(g) = 6\pi\delta_{D,8}, \tag{2.12a}$$

$$D^4R^4: \quad \left(\Delta_{G/K} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)}(g) = 40\zeta(2)\delta_{D,7}, \tag{2.12b}$$

$$D^6R^4: \quad \left(\Delta_{G/K} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)}(g) = 40\zeta(3)\delta_{D,6} - (\mathcal{E}^{(D)}_{(0,0)}(g))^2, \tag{2.12c}$$

where $\Delta_{G/K}$ is the Laplace-Beltrami operator on G/K.

We see that the third equation is qualitatively very different from the first two since it has a non-constant function as a source on the right-hand side, and we will discuss this case in more detail in section 12.1.2.

The Kronecker delta contributions in all three equations in (2.12) are related to the existence of UV divergences in the underlying supergravity theory and the existence of supersymmetric counterterms. They arise in those dimensions where also the eigenvalue vanishes and signal logarithmic terms in the coefficient function $\mathcal{E}^{(D)}_{(p,q)}$. We refer the reader to [149] for further discussions of this point. There can be additional Kronecker delta contributions related to form factor divergences and these are discussed in [265].

Besides from these special cases, equations (2.12a) and (2.12b) correspond to eigenfunction conditions from (2) in section 1.1. For dimensions lower than ten, there are additional G-invariant differential operators other than $\Delta_{G/K}$ but the corresponding conditions are not fully known from string theory. A superspace analysis that generates the other differential equations was pioneered in [33, 34]. It is expected that the coefficient functions for R^4 and D^4R^4 satisfy all the required differential equations and hence are standard automorphic forms.

As shown in section 1.1 in the case of $G(\mathbb{R}) = SL(2,\mathbb{R})$, Eisenstein series are eigenfunctions to the Laplace-Beltrami operator, and comparing with computed scattering amplitudes in string theory one has been able to conjecture the exact forms of the coefficients $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ in terms of maximal parabolic Eisenstein series which will be defined in chapter 5. Parabolic subgroups, denoted by P, are introduced in section 4.1.3.

More precisely, in five, four and three dimensions, with symmetry groups E_6, E_7 and E_8 according to table 2.1, if one considers the maximal parabolic subgroups P that have semi-simple Levi parts $SO(5,5)$, $SO(6,6)$ and $SO(7,7)$, respectively, then the solutions

$$R^4: \qquad \mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3)E(\lambda_{s=3/2}, P, g), \tag{2.13a}$$

$$D^4R^4: \qquad \mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5)E(\lambda_{s=5/2}, P, g). \tag{2.13b}$$

to equations (2.12a) and (2.12b) are the conjectured coefficient functions appearing in the four-graviton amplitudes, or equivalently, as corrections to the effective action (2.11).

The weight λ_s specifies the character χ_s on P, which defines the Eisenstein series similar to (1.12). It is given by

$$\lambda_s = 2s\Lambda_P - \rho, \tag{2.14}$$

where Λ_P denotes the fundamental weight orthogonal to the Levi subgroup L of $P = LU$ and ρ the Weyl vector.

Remark 2.1. Looking at the coefficients of (12.3) we recognise the corresponding values from the tree level amplitudes which are computed in section 2.4. This means that the above functions are nothing but the (single) U-duality orbit of the tree level results. This is no longer true for the higher functions $\mathcal{E}^{(D)}_{(p,q)}$ as we will discuss in more detail in section 12.1.2.

These conjectures have been subjected to numerous consistency checks [144, 146, 263] and, particularly, capture the known results of scattering amplitudes in the weak coupling limit $g_s \to 0$ which we will discuss in the following section.

Remark 2.2. We would also like to point out that recent investigations of superstring scattering amplitudes at tree-level and one-loop for more than four particles have revealed very interesting different connections to number theory. Instead of single ζ-values like $\zeta(3)$ one will typically have so-called (elliptic) *multiple zeta values* governed by *Drinfeld associators* [41, 42, 93, 280]. We note that this structure is at fixed order in string perturbation theory whereas the U-duality invariant functions we are discussing here include all perturbative and non-perturbative effects.

2.3 Physical interpretation of Fourier expansion

We will know study the functions $\mathcal{E}^{(D)}_{(p,q)}$ which were found above as the quantum corrections to the low energy effective action in type IIB string theory on tori T^d. Since they are invariant under the discrete subgroup $G(\mathbb{Z})$, they are periodic functions and we can extract physical information from their Fourier expansions.

For concreteness, let us consider the R^4 and D^4R^4 coefficients $\mathcal{E}^{(10)}_{(0,0)}$ and $\mathcal{E}^{(10)}_{(1,0)}$ in ten dimensions, where $G(\mathbb{R}) = SL(2,\mathbb{R})$ — although the physical interpretations hold for general dimensions and coefficients.

As stated in section 1.1, the classical moduli space $G(\mathbb{R})/K(\mathbb{R}) = SL(2,\mathbb{R})/SO(2,\mathbb{R})$ is isomorphic to the Poincaré upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y = \operatorname{Im} z > 0\}$ parametrised by a complex scalar field called the *axio-dilaton* here denoted by $z = x + iy$ with x being the axion and where $y = g_s^{-1}$ is related to the dilaton. The U-duality group $G(\mathbb{Z}) = SL(2,\mathbb{Z})$ includes the translation invariance $z \to z + 1$.

To reproduce the right classical behaviour, the leading order term in the weak coupling limit $g_s \to 0$, i.e. $y \to \infty$, should be (in Einstein frame):

$$\begin{aligned} \mathcal{E}^{(10)}_{(0,0)} &\sim 2\zeta(3)y^{3/2} \\ \mathcal{E}^{(10)}_{(1,0)} &\sim \zeta(5)y^{5/2} \end{aligned} \qquad \text{as} \quad y \to \infty. \tag{2.15}$$

These weak coupling limits of the scattering amplitudes are computed in section 2.4.

The eigenvalue equations (2.12) for the coefficient functions are

$$\begin{aligned} \Delta \mathcal{E}^{(10)}_{(0,0)}(z) &= \frac{3}{4}\mathcal{E}^{(10)}_{(0,0)}(z) \\ \Delta \mathcal{E}^{(10)}_{(1,0)}(z) &= \frac{15}{4}\mathcal{E}^{(10)}_{(1,0)}(z) \end{aligned} \tag{2.16}$$

where $\Delta = y^2(\partial_x^2 + \partial_y^2)$ is the Laplace-Beltrami operator on $\mathbb{H}$ from (1.3). They are thus automorphic forms as defined in section 1.1.

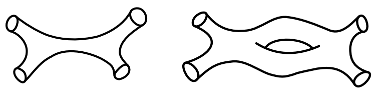
It was first realised by Green et al. in [138] and [143], that these conditions are solved by[7]

$$\begin{aligned} \mathcal{E}^{(10)}_{(0,0)}(z) &= f_{3/2}(z) = 2\zeta(3)E(s=3/2, z) \\ \mathcal{E}^{(10)}_{(1,0)}(z) &= \tfrac{1}{2}f_{5/2}(z) = \zeta(5)E(s=5/2, z) \end{aligned} \tag{2.17}$$

as seen from (1.16) and (1.4) with $f_s(z)$ defined in (1.1). This is exactly the $SL(2,\mathbb{R})$ variant of (2.13).

The Fourier expansion (1.16) then has a direct physical interpretation: the first two terms (constant terms) correspond to the perturbative quantum corrections (tree-level and one-loop), while the infinite series of non-constant coefficients encode non-perturbative effects. To see this we can expand the Bessel function $K_{s-1/2}(2\pi|n|y)$ as $y \to \infty$ which, for the R^4 coefficient, yields

$$\mathcal{E}^{(D)}_{(0,0)}(z) = \overbrace{\underbrace{2\zeta(3)y^{3/2}}_{\text{tree-level}} + \underbrace{4\zeta(2)y^{-1/2}}_{\text{one-loop}}}^{\text{perturbative terms}} \overbrace{+ 2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m)e^{-S_{\text{inst}}(z)}\left[1 + \mathcal{O}(y^{-1})\right]}^{\text{non-perturbative terms}}, \tag{2.18}$$



amplitudes in the presence of instantons

where we have defined the *instanton action*

$$S_{\text{inst}}(z) := 2\pi|m|y + 2\pi i m x. \tag{2.19}$$

It is clear from this expression that the infinite series is exponentially suppressed by e^{-y} in the limit $y \to \infty$. As $y = g_s^{-1}$ this corresponds to the weak coupling limit of the theory and this exponential suppression is characteristic for a non-perturbative, instanton effect.

In string theory, they arise from a background of so-called D-*instantons* [138, 142] where D stands for Dirichlet as in the Dirichlet boundary conditions that are imposed on strings attached to them. These are special cases of extended objects in string theory called D-*branes*, localised to a single point in space-time.

[7] A priori these solutions are only unique up to the addition of cusp forms but they were subsequently ruled out in [261] for the R^4 coupling.

From this perspective the divisor sum $\sigma_{-2}(n) = \sum_{d|n} d^{-2}$ is the *instanton measure* and, as described in [138], counts the number of ways the mode m, which is called the *instanton charge*, can be factorised into two integers: the winding number of a D-particle trajectory around a circle in the T-dual type IIA string theory, and the Ramond–Ramond charge of the same particle. When m is a negative integer it corresponds to an *anti-instanton*.

The higher order corrections in $g_s = y^{-1}$ in the non-perturbative terms are higher genus corrections to the scattering amplitude in the presence of instantons.

In summary, the discrete symmetry leading us to the study of automorphic forms gives a lot of information about string theory: it tells us that there are no perturbative corrections to the R^4 term for genera larger than one, and gives clues of how to compute general scattering amplitudes in the presence of instantons in string theory, which is otherwise very difficult to do.

The above example provided the first hint of an intriguing relation between quantum corrections in string theory and automorphic forms. There is by now a vast literature on this subject; see [9, 11–15, 140, 151, 152, 157, 159, 191, 211, 212, 252, 254, 261, 262, 266, 267, 269, 270] for a sample. In recent years also the representation theoretic aspects have proven to play an increasingly important role [95–97, 144, 146, 259, 263], thus providing ample motivation also from physics for the emphasis on automorphic representations in these notes.

Section 12.1, continues the discussions of this chapter with the topic of automorphic representations for the coefficients $\mathcal{E}^{(D)}_{(p,q)}$ and the D^6R^4 correction which differs from the lower order terms by also requiring a non-constant source term in the differential equation (2.12c).

2.4 Computing the four-graviton tree level amplitude*

A comprehensive discussion of string theory and its scattering amplitudes is beyond the scope of this work. We only give one indicative and, hopefully, illustrative example and refer to the string theory literature [150, 272] for more information.

The example is the four-graviton amplitude at string *tree level*. The closed string tree level topology is that of a sphere, and the four asymptotic graviton states correspond to four punctures in this sphere as pictured in figure 2.3. This configuration can be obtained by a homeomorphism of the left-most diagram in figure 2.1. By definition, the string scattering amplitude is given by an integral over the moduli space of all Riemann spheres and all possible insertion points for four punctures. The discussion below uses the *Ramond–Neveu–Schwarz (RNS) formalism*.

In terms of complex geometry, we can describe the sphere $S^2 = \mathbb{CP}^1$ by one complex variable $z \in \mathbb{C}$ everywhere except at the 'north-pole.' An asymptotic graviton state at a puncture z_i corresponds to a *vertex operator*. Due to a complication called *ghost picture* [99] that arises for the superstring, we actually require it in two related forms, one

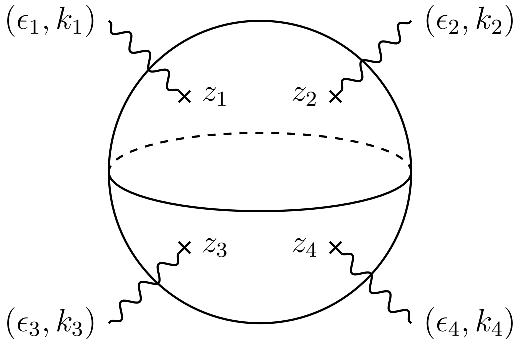


Figure 2.3: *Riemann sphere with four punctures as it appears in the four graviton scattering amplitude. An external massless graviton with polarisation ϵ_i and momentum k_i is located at each of the four punctures $z_i \in \mathbb{CP}^1$.*

called ghost picture 0 and ghost picture -1:

$$V_0(z_i; k_i, \epsilon_i) = \frac{2}{\alpha'} : \epsilon_{i,\mu\nu} \left(\partial X^\mu + \frac{\alpha'}{2} k_\rho \psi^\rho \psi^\mu \right) \left(\bar{\partial} X^\nu + \frac{\alpha'}{2} k_\sigma \bar{\psi}^\sigma \bar{\psi}^\nu \right) e^{ik_{i,\mu} X^\mu} :, \tag{2.20a}$$

$$V_{-1}(z_i; k_i, \epsilon_i) =: \epsilon_{i,\mu\nu} \psi^\mu \bar{\psi}^\nu e^{ik_{i,\mu} X^\mu} : . \tag{2.20b}$$

In these expressions, we are using the Einstein summation convention for repeated Lorentz indices $\mu = 0, \ldots, 9$ that label the ten directions of the Minkoswki target space in which the sphere is embedded via the embedding coordinates $X^\mu \equiv X^\mu(z, \bar{z})$. The polarisation tensor $\epsilon_i \equiv \epsilon_{i,\mu\nu}$ is a second-rank symmetric tensor of the Lorentz group $SO(1,9)$ and the external momentum of the graviton $k_i \equiv k_{i,\mu}$ is light-like in the Minkowski metric: $k_i^2 = k_{i,\mu} k_{i,\nu} \eta^{\mu\nu} = 0$ as the scattering gravitons are massless. The colons surrounding this expression indicate a specific normal ordering procedure necessary for the vertex operator to be well-defined on a Fock space vacuum. The field ψ^μ and its (Dirac) conjugate $\bar{\psi}^\mu$ correspond to the fermionic coordinates that accompany the bosonic X^μ in superstring theory. The derivatives ∂ and $\bar{\partial}$ are with respect to the world-sheet coordinate z.

The desired expression for the scattering amplitude is then roughly of the following form for $D = 10$ (in string frame)

$$\mathcal{A}^{(10)}_{\text{tree}}(s, t, u, \epsilon_i; g) = \int_{\mathcal{M}_{0,4}} d\mu \langle \prod_{i=1}^{4} V(z_i; k_i, \epsilon_i) \rangle_{S^2}, \tag{2.21}$$

where the angled brackets denote the *correlation function* of the vertex operators on the given sphere S^2 that we detail below. For reasons of ghost number saturation, two vertex operators have to be taken in ghost picture 0 (of the form V_0 in (2.20)) and two in ghost picture -1 (of the form V_{-1} in (2.20)). The integral $d\mu$ is over all Riemann spheres (genus $h = 0$) with four punctures, so that the positions z_i of the punctures is also integrated over. More concretely, the measure is given by the integral over all metrics that can be put on topological spheres up to diffeomorphisms and Weyl rescalings (local dilatations).

To make this well-defined in the path-integral sense one has to divide by this volume of this gauge group so that

$$d\mu = \frac{D\gamma_{\mathbb{CP}^1}}{\text{Vol}(\text{diff} \times \text{Weyl})}. \tag{2.22}$$

Here, $D\gamma_{\mathbb{CP}^1}$ indicates all possible metrics on the sphere $\mathbb{CP}^1$. According to the *Riemann–Roch theorem*, the sphere as a Riemann surface has no metric moduli and any metric $\gamma_{\mathbb{CP}^1}$ can be brought into the form of that of the round sphere

$$ds^2 = \frac{dzd\bar{z}}{(1+|z|^2)^2} \tag{2.23}$$

by diffeomorphisms and Weyl rescalings. There is no modulus in this expression and even this form is left invariant by the *conformal Killing group* of the sphere $PSL(2,\mathbb{C})$ that acts by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2,\mathbb{C}) : \qquad z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta} \tag{2.24}$$

on the coordinate z. So, even after fixing the diff $\times$ Weyl gauge-freedom to bring the metric to the above form (2.23), one has still the freedom to perform transformations from the conformal Killing group $PSL(2,\mathbb{C})$ that represent the residual gauge freedom. Without fixing it the integral in (2.21) is of the form

$$\int\limits_{\mathcal{M}_{0,4}} d\mu = \prod_{j=1}^{4} \int\limits_{\mathbb{CP}^1} d^2 z_j \Delta_{\text{FP}}, \tag{2.25}$$

where Δ_{FP} is a *Faddeev–Popov determinant* arising from the gauge-fixing and that is treated by introducing (super-)ghost systems. We will not be more specific on it here (see for example [272]) and only mention its effect on the calculation below. We see that, due to the absence of metric moduli for the sphere, the integral over the moduli space of four-punctured sphere reduces to an integral over the locations z_i of the four punctures. The complex three-dimensional conformal Killing group can be used to fix three of the puncture positions to one's favourite values; a standard choice being $0, 1, \infty$ and only as single integral over a single puncture position remains.

Let us return to the correlation function appearing in the expression (2.21) above. It is formally given by a path-integral over all (super-)embeddings X^μ of the (super-)Riemann sphere into the ten-dimensional target space. Schematically, one has

$$\langle \prod_{i=1}^{4} V(z_i; k_i, \epsilon_i) \rangle_{S^2} = g_{\text{s}}^{-\chi(S^2)} \int DXD\psi D\bar{\psi}\, e^{-S[X,\psi,\bar{\psi}]} \prod_{i=1}^{4} V(z_i; k_i, \epsilon_i), \tag{2.26}$$

where $S[X, \psi, \bar{\psi}]$ denotes the two-dimensional *σ-model* action that is basically the induced volume under the embedding. The *Euler number* $\chi(S^2) = 2$ of the sphere provides the standard topological weighting of different string diagrams that was mentioned above (cf. figure 2.1) and here evaluates to g_s^{-2}. The string coupling g_s is the only string theory modulus g of equation (2.2) of relevance in the present discussion. As our goal here is to give a heuristic derivation of the final formula (2.30) below, we are not displaying or discussing aspects related to so-called *pictures* of vertex operators associated with the ghosts arising from gauge-fixing. A proper treatment would modify the above equation [150, 272]; this modification will be taken into account in the final expression below.

The correlation function can be evaluated on the sphere explicitly in terms of the Green's function on the sphere. First, we note that for vertex operators in ghost picture 0 the fermionic integrals (over ψ and $\bar\psi$) pick out the contribution

$$\alpha' \epsilon_{i,\mu\nu} k_\rho k_\sigma e^{ik_{i,\tau}X^\tau} \propto \alpha' \mathcal{R}_{\rho\mu\nu\sigma} e^{ik_{i,\tau}X^\tau} \tag{2.27}$$

from $V(z_i; k_i, \epsilon_i)$ in (2.20a) and we recognise the linearised Riemann curvature tensor mentioned in section 2.2, and the full integral also provides the necessary contractions of the four Riemann tensors. For the vertex operator in picture -1 of (2.20b), the integral yield contributions that are roughly of the form

$$\epsilon_{i,\mu\nu} e^{ik_{i,\tau}X^\tau} \propto \frac{\mathcal{R}_{\rho\mu\nu\sigma}}{k_\rho k_\sigma} e^{ik_{i,\tau}X^\tau}. \tag{2.28}$$

The Green's function on the sphere now evaluates the product of two normal ordered exponentials to

$$: e^{ik_{i,\mu}X^\mu(z_i)} :: e^{ik_{j,\nu}X^\nu(z_j)} : \propto |z_i - z_j|^{\alpha' k_i \cdot k_j} : e^{i(k_1+k_2)_\mu X^\mu(z_i)} : + \dots \tag{2.29}$$

which is sometimes called the operator product expansion (or Wick contraction) and the dots denote subdominant terms in an expansion around $z_i \to z_j$. In this expression we have used the Lorentz inner product $k_i \cdot k_j = k_{i,\mu} k_j^\mu$ between the two external momenta k_i and k_j. Since the external momenta are massless, we can rewrite this product as $k_i \cdot k_j = \frac{1}{2}(k_i + k_j)^2$ and we already begin to recognise a connection to the Mandelstam variables s, t and u defined in (2.6). One has to perform the above operator product expansion of e^{ikX} for all four factors appearing in the correlation function and this will lead to a permutation invariant expression of the four Mandelstam variables.

We can now put all the pieces together: (*i*) The integral over the moduli space reduces to an integral over the four punctures, (*ii*) the conformal Killing groups allows to fix three of four punctures to fixed values that we choose to be 0, 1 and ∞, (*iii*) the integral over the fermionic variables in the correlation function produces the linearised curvature tensors times exponentials of the form e^{ikX}, (*iv*) these exponentials get converted into factors of the form $|z_i - z_j|^{\alpha' k_i \cdot k_j}$ in all possible ways and (*v*) include additional contributions from the ghost sector. The ultimate integral is of the form

$$\begin{aligned} \mathcal{A}^{(10)}_{\text{tree}}(s,t,u,\epsilon_i;g) &= \frac{4(\alpha')^2 g_s^{-2}}{\pi} \frac{\mathcal{R}^4}{(k_1\cdot k_3)^2} \delta(s+t+u) \int_{\mathbb{C}} d^2z |z|^{\alpha' k_1\cdot k_2 - 2} |1-z|^{\alpha' k_2 \cdot k_3 - 2} \\ &= (\alpha')^4 g_s^{-2} \delta(s+t+u) \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+u)\Gamma(1+t)} \mathcal{R}^4 \end{aligned} \tag{2.30}$$

We have introduced a few factors related to the normalisations of the various measures introduced above. The shift in the power is due to the conformal ghost sector that we have not discussed explicitly and the denominator $(k_1 \cdot k_3)^2 = \frac{4}{(\alpha')^2} t^2$ is due to (2.28). We note that the factor $\delta(s+t+u)$ expresses momentum conservation for massless states. As stated in section 2.2, the linearised curvature tensor $\mathcal{R}_{\mu\nu\rho\sigma} \propto k_\mu \epsilon_{\nu\rho} k_\sigma$ is determined by the graviton's momentum and polarisation, and $\mathcal{R}^4$ is a specific contraction of the linearised curvatures of the four gravitons. More precisely, it is given by $t_8 t_8 \mathcal{R}^4$ where the t_8 tensor contracts four powers of an antisymmetric matrix $M_{\mu\nu}$ according to $t^{\mu_1 \dots \mu_8} M_{\mu_1\mu_2} \cdots M_{\mu_7\mu_8} = 4\mathrm{Tr}(M^4) - (\mathrm{Tr}(M^2))^2$.

The amplitude (2.30) can be expanded for small values of s, t and u which we recall from (2.6) contain the string length $\alpha' = \ell_\mathrm{s}^2$. The result is (in string frame)

$$\mathcal{A}^{(10)}_{\mathrm{tree}}(s,t,u,\epsilon_i;g) = g_\mathrm{s}^{-2}(\alpha')^4 \left(\frac{3}{\sigma_3} + 2\zeta(3) + \zeta(5)\sigma_2 + \frac{2\zeta(3)^2}{3}\sigma_3 + \dots \right) \mathcal{R}^4 \tag{2.31}$$

where we have used $stu = \frac{1}{3}\sigma_3$ up to momentum conservation $s+t+u=0$. The above expression provides the tree level contributions to the functions $\mathcal{E}^{(10)}_{(0,-1)}$, $\mathcal{E}^{(10)}_{(0,0)}$, $\mathcal{E}^{(10)}_{(1,0)}$ and $\mathcal{E}^{(10)}_{(0,1)}$ appearing in (2.9) in ten dimensions. A prefactor of g_s^4 is to be attributed to the external states, so that the overall weight of the amplitude is g_s^{-2} as it should be for a tree level calculation. We note also that $g_\mathrm{s}^2(\alpha')^4 = \ell_{10}^8 = \kappa_{10}^2$ in terms of the ten-dimensional Planck scale ℓ_{10} and the ten-dimensional gravitational coupling κ_{10}. The fact that this amplitude is proportional to the square of the coupling is characteristic for a tree level scattering of four particles (in a theory that has cubic vertices).

We note also upon toroidal compactification to $D < 10$ dimensions integrals such as (2.30) receive additional contributions related to the structure of states on the torus T^d $(d = 10 - D)$ that is used in the compactification. This leads naturally to a relation of string amplitudes to *theta correspondences*.

Chapter 3

Preliminaries on p-adic and adelic technology

As seen in section 1.3.2 the Fourier expansion of the $SL(2,\mathbb{R})$ Eisenstein series factorises into an Euler product over all primes p. This number theoretic information is best captured by introducing the p-adic numbers which, for any prime p, are an extension of the rational numbers, and furthermore the ring of adeles, which encapsulates all the different p-adic extensions in a single product.

This chapter is intended as an introduction to these objects as well as providing example calculations that will be used throughout the remaining text. Additional reading can for example be found in [1, 80, 117, 251]. Readers familiar with the subject are welcome to proceed to the next chapter and come back to the explicit examples when needed later on in the text. Further reading can be found in [40] and [80].

For the whole of this chapter and most of the remaining text, let p be a prime number.

3.1 p-adic numbers

We start by providing the basic definitions and discussing some of the properties of p-adic numbers.

Definition 3.1 (Integers $\mathbb{Z}_p$). The *p-adic integers* $\mathbb{Z}_p$ are formal power series in p with coefficients between 0 and $p-1$

$$x \in \mathbb{Z}_p \quad \Longleftrightarrow \quad x = x_0 p^0 + x_1 p^1 + \dots \quad \text{with } x_i \in \mathbb{Z}/p\mathbb{Z} \cong \{0, 1, \dots, p-1\}\,. \tag{3.1}$$

The p-adic integers form a ring.

Arithmetic operations on the p-adic integers work in the usual manner. However, since all coefficients in the expansion are positive it may not be immediately obvious how the additive inverse (i.e. subtraction) works. As an example, consider the equation $x + 1 = 0$ that should have a solution over $\mathbb{Z}_p$. The inverse is given by the infinite power series in p

with all coefficients are equal to $p-1$:

$$x = \sum_{i=0}^{\infty}(p-1)p^i \tag{3.2}$$

This is a bit like evaluating the (non-converging) sum $x = \sum_{k=0}^{\infty} 10^k = 1 + 10 + 100 + \ldots$ in decimal notation to be an infinite string of 1s. Multiplying by 9 and then adding 1 creates a zero for every decimal place. Hence $9x + 1 = 0 \Leftrightarrow x = -1/9 = 1/(1-10)$ in agreement with a naive application of the geometric series definition.

Next we define the p-adic number field.

Definition 3.2 (Number field $\mathbb{Q}_p$). The associated number field is given by the *p-adic numbers* $\mathbb{Q}_p$ that are formal Laurent series in p with a finite number of terms of degree less than zero, i.e. finite polar part

$$x = x_k p^k + x_{k+1}p^{k+1} + \ldots \quad \text{with } x_k \neq 0\,, \tag{3.3}$$

where k is some integer not necessarily positive.

The p-adic numbers $\mathbb{Q}_p$ can be thought of as the completion of rational numbers $\mathbb{Q}$ with respect to the following norm.

Definition 3.3 (p-adic norm $|\cdot|_p$). The *p-adic norm* on $\mathbb{Q}_p$ is given by

$$|x|_p = p^{-k} \quad \Leftrightarrow \quad \text{with } x_k \neq 0\,. \tag{3.4}$$

The p-adic norm is multiplicative

$$|x \cdot y|_p = |x|_p |y|_p \tag{3.5}$$

and satisfies a stronger triangle inequality than generic norms, namely

$$|x + y|_p \leq \max(|x|_p, |y|_p)\,, \tag{3.6}$$

for $x, y \in \mathbb{Q}_p$. This second property is called *ultrametric property* and a space with a norm of this type is called *non-archimedean* in contrast with archimedean spaces satisfying the usual archimedean triangle inequality. The p-adic norm of 0 is $|0|_p = 0$.

The integer k in (3.4) is called the p-adic *valuation* of $\mathbb{Q}$ or $\mathbb{Q}_p$ and is often also denoted by $\nu_p(x)$. Two properties of the p-adic valuation, equivalent to the ones above for the p-adic norm, are

$$\nu_p(x \cdot y) = \nu_p(x) + \nu_p(y) \tag{3.7}$$

and

$$\nu_p(x + y) \geq \min(\nu_p(x), \nu_p(y))\,, \tag{3.8}$$

where in the last property equality is achieved if $\nu_p(x) \neq \nu_p(y)$.

The integers in the normed space $\mathbb{Q}_p$ can then be expressed as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \,|\, |x|_p \leq 1\} , \tag{3.9}$$

i.e. they have an exponent $k \geq 0$ of p. This shows that the p-adic integers are compactly embedded in $\mathbb{Q}_p$. The complementary set to $\mathbb{Z}_p$ in $\mathbb{Q}_p$ is given by

$$\mathbb{Q}_p \setminus \mathbb{Z}_p = \{x \in \mathbb{Q}_p \,|\, |x|_p > 1\} . \tag{3.10}$$

Let us provide two simple examples illustrating the p-adic expansion of a rational number.

Example 3.4: p-adic expansions

We consider the p-adic expansion of the rational number $x = \frac{1}{2} \in \mathbb{Q}$ for $p = 2$ and $p = 3$.

For $p = 2$ one has $|x|_2 = 2^1 = 2$ or $\nu_2(x) = -1$ and hence $\frac{1}{2}$ is not a 2-adic integer. As an element of $\mathbb{Q}_2$ one finds $\frac{1}{2} = 1 \cdot 2^{-1}$ as the expansion of the form (3.3).

For $p = 3$ one has $|x|_3 = 3^0 = 1$ or $\nu_3(x) = 0$ and hence $\frac{1}{2}$ is a 3-adic integer. Its expansion of the form (3.3) is $\frac{1}{2} = 2 \cdot 3^0 + \sum_{k>0} 3^k$.

Another useful property for the p-adic norm of the greatest common divisor of two integers which will be used in section 9.6 is introduced in the following example.

Example 3.5: Norm of a greatest common divisor

Let m and n be two integers, $d = \gcd(m, n)$, $m' = m/d$ and $n' = n/d$. Then $1 = \gcd(m', n')$ which, for a prime p, means that if $|m'|_p < 1$ (that is, $p \mid m'$) then $|n'|_p = 1$ (that is, $p \nmid n'$) and vice versa. Thus, $1 = |\gcd(m', n')|_p = \max(|m'|_p , |n'|_p)$. Hence,

$$|d|_p = |d|_p \max(|m'|_p , |n'|_p) = \max(|m'd|_p , |n'd|_p) = \max(|m|_p , |n|_p) . \tag{3.11}$$

We also define the multiplicatively invertible p-adic numbers of $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

Definition 3.6 (Multiplicatively invertible numbers $\mathbb{Z}_p^\times$ and $\mathbb{Q}_p^\times$). The set of *multiplicatively invertible elements* in $\mathbb{Z}_p$ will be denoted by

$$\mathbb{Z}_p^\times = \left\{x \in \mathbb{Z}_p \,\middle|\, x^{-1} \text{ exists in } \mathbb{Z}_p\right\} = \{x \in \mathbb{Z}_p \,|\, |x|_p = 1\} = \{x \in \mathbb{Q}_p \,|\, |x|_p = 1\} . \tag{3.12}$$

They correspond to those x in (3.1) for which $x_0 \neq 0$. The set of multiplicatively invertible elements $\mathbb{Q}_p^\times$ in $\mathbb{Q}_p$ is defined as

$$\mathbb{Q}_p^\times = \{x \in \mathbb{Q}_p \,|\, |x|_p \neq 0\} . \tag{3.13}$$

For p-adic numbers the case when p is the prime at infinity, i.e. $p = \infty$, is typically associated with standard calculus via

$$\mathbb{Q}_\infty = \mathbb{R} . \tag{3.14}$$

In accord with the terminology used for more general number fields the case of a finite prime, i.e. $p < \infty$, is sometimes referred to as the *non-archimedean place*, while $p = \infty$ is called the *archimedean place*.

The p-adic numbers were introduced in number theory by Hensel with the intention of transferring the powerful tools of complex analysis to power and Laurent series. A theorem by Ostrowski [200] states that any non-trivial norm on $\mathbb{Q}$ is either the standard Euclidean norm (leading to the real numbers upon completion) or one of the p-adic norms.

Remark 3.7 (Alternative construction of $\mathbb{Q}_p$). Another way of defining the p-adic numbers is through the following definition of the p-adic norm of an ordinary rational number $x \in \mathbb{Q}$:

$$|x|_p = p^{-k}\,, \tag{3.15}$$

where $k \in \mathbb{Z}$ is the largest integer such that $x = p^k y$ with $y \in \mathbb{Q}$ not containing any powers of p in its numerator or denominator (in cancelled form); this is often stated as p^k divides x. It is from this construction that one obtains $\mathbb{Q}_p$ as the completion of $\mathbb{Q}$ and one obtains an embedding of $\mathbb{Q}$ into $\mathbb{Q}_p$. The definition implies that for a prime q and $k \in \mathbb{Z}$

$$|q^k|_p = \begin{cases} p^{-k} & \text{if } p = q \\ 1 & \text{otherwise}\,. \end{cases} \tag{3.16}$$

3.2 p-adic integration

Integration on $\mathbb{Q}_p$ can be defined with respect to the *additive measure* dx that is invariant under translation and has a simple scaling transformation

$$d(x+a) = dx\,, \qquad d(ax) = |a|_p dx\,. \tag{3.17}$$

The measure is by convention normalised as to give the p-adic integers unit volume:

$$\int_{\mathbb{Z}_p} dx = 1\,. \tag{3.18}$$

We will now provide a series of examples of basic p-adic integrals. When evaluating such integrals it is often useful to employ different decompositions of $\mathbb{Z}_p$. One such decomposition is to write $\mathbb{Z}_p$ as a disjoint union

$$\mathbb{Z}_p = \bigsqcup_{k=0}^{p-1} C_k\,, \tag{3.19}$$

where C_k denotes the set of those p-adic integers with 'constant' coefficient (the coefficient of p^0 in (3.1)) equal to k. Another decomposition of $\mathbb{Z}_p$ employed is to write it as

$$\mathbb{Z}_p = \bigsqcup_{k=0}^{\infty} p^k \mathbb{Z}_p^\times\,, \tag{3.20}$$

Example 3.8: Volume of invertible integers $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p^\times} dx = \frac{p-1}{p}. \tag{3.21}$$

The integral is a simple consequence of the definition (3.18) and can be understood intuitively by noting that only $p-1$ out of the p choices for the constant coefficient of $x \in \mathbb{Z}_p$ correspond to elements in $\mathbb{Z}_p^\times$. For a more formal derivation we use decomposition (3.19) of $\mathbb{Z}_p$ and integrate over each C_k separately. By translation invariance of the measure (3.17) all C_k have the same volume $1/p$. Integrating over all C_k except for the one with $k = 0$ one thus obtains the above formula.

The following two examples explore the integration of the p-adic norm over different domains.

Example 3.9: Integration of the norm over $\mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} |x|_p^s dx = \frac{p-1}{p} \frac{1}{1-p^{-s-1}}\,, \tag{3.22}$$

with $s \in \mathbb{C}$. This is derived in a few steps:

$$\begin{aligned} \int_{\mathbb{Z}_p} |x|_p^s dx &= \sum_{k=0}^{\infty} \int_{p^k\mathbb{Z}_p^\times} |x|_p^s dx = \sum_{k=0}^{\infty} p^{-ks} \int_{p^k\mathbb{Z}_p^\times} dx = \sum_{k=0}^{\infty} p^{-ks} \int_{\mathbb{Z}_p^\times} p^{-k} dy \\ &= \frac{p-1}{p} \sum_{k=0} p^{-k(s+1)} = \frac{p-1}{p} \frac{1}{1-p^{-s-1}}. \end{aligned} \tag{3.23}$$

In the first step we have used the decomposition (3.20) of the p-adic integers. Then we have used the fact that for $x \in p^k\mathbb{Z}_p^\times$ the norm is $|x|_p = p^{-k}$. After that we have changed variables to $x = p^k y$ with $y \in \mathbb{Z}_p^\times$, used the resulting volume of $\mathbb{Z}_p^\times$ computed in exmaple 3.8 and carried out the geometric sum. The integral converges only for $\mathrm{Re}(s) > -1$.

Using the identity from the previous example we can also evaluate the following integral which will be used in chapter 7 and section 8.6.2.

Example 3.10: Integration of the norm over $\mathbb{Q}_p \setminus \mathbb{Z}_p$

$$\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^s dx = \frac{p-1}{p} \frac{p^{s+1}}{1-p^{s+1}}\,, \tag{3.24}$$

with $s \in \mathbb{C}$ and the domain of integration as defined in (3.10). The integral is then evaluated in the following steps

$$\begin{aligned}\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^s dx &= \int_{|x|_p>1} |x|_p^s dx = \sum_{k=1}^{\infty} p^{ks} \int_{p^{-k}\mathbb{Z}_p^\times} dx = \sum_{k=1}^{\infty} p^{k(s+1)} \int_{\mathbb{Z}_p^\times} dx \\ &= \frac{p-1}{p} \sum_{k=1}^{\infty} p^{k(s+1)} = \frac{p-1}{p} \frac{p^{s+1}}{1-p^{s+1}}. \end{aligned} \tag{3.25}$$

The integral converges for $\mathrm{Re}(s) < -1$. Note that the same integral over all of $\mathbb{Q}_p$ does not exist.

Remark 3.11 (Multiplicative measure $dx^\times$). We denote the *multiplicative measure* on $\mathbb{Q}_p^\times$ by $d^\times x$ with its defining relation

$$d^\times x = \frac{p}{p-1} \frac{dx}{|x|_p} . \tag{3.26}$$

It satisfies $\int_{\mathbb{Z}_p^\times} d^\times x = 1$. It transforms as $d^\times(ax) = d^\times x$. Integrating the function $|x|_p^s$ against the multiplicative measure $d^\times x$ the result (3.22) simplifies to

$$\int_{\mathbb{Z}_p} |x|_p^s \, d^\times x = \sum_{k=0}^{\infty} p^{-ks} \int_{\mathbb{Z}_p^\times} d^\times x = \sum_{k=0}^{\infty} p^{-ks} = \frac{1}{1-p^{-s}} , \tag{3.27}$$

where in the first step we used the property (3.20). Note that the same result is obtained if we restrict the integration domain to $\mathbb{Z}_p\setminus\{0\}$, which will be useful in the proof of proposition 3.26.

3.3 Characters and the Fourier transform

In this section we introduce the concept of a character which is then used to define the p-adic Fourier transform. As before we provide explicit computations of various integrals serving as prototypical examples for later calculations.

Definition 3.12 (Fractional part of a p-adic number). The *fractional part* $[y]_p$ of a p-adic number $y \in \mathbb{Q}_p$ is given by its class in $\mathbb{Q}_p/\mathbb{Z}_p$, or more concretely by the terms in its series expansion with negative powers of p:

$$\left[x_k p^k + \ldots + x_{-1}p^{-1} + x_0 p^0 + x_1 p^1 + \ldots\right]_p = \begin{cases} x_k p^k + \ldots x_{-1}p^{-1} & \text{if } k < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.28}$$

Note that we will often suppress the subscript p when there is no risk of confusion.

We will now show that given a rational number x, subtracting all the fractional parts of x with respect to all $\mathbb{Z}_p$ from x leaves a normal integer. This will, for example, be used in sections 3.5 and 9.6.

Proposition 3.13 (Normal integer of a p-adic number). *Let $x \in \mathbb{Q}$. Then,*

$$x - \sum_{p<\infty} [x]_p \in \mathbb{Z} \,. \tag{3.29}$$

Proof. By design, $x - [x]_p \in \mathbb{Z}_p$ and for any prime $q \neq p$

$$\left|[x]_q\right|_p \leq \max(\left|x_k q^k\right|_p, \ldots, \left|x_{-1} q^{-1}\right|_p) \leq 1 \tag{3.30}$$

if $k < 0$ (since $x_i \in \mathbb{Z}$) and otherwise $[x]_q = 0$ which means that $[x]_q \in \mathbb{Z}_p$. Hence, for any prime p

$$x - \sum_{q<\infty} [x]_q = (x - [x]_p) - \sum_{q \neq p} [x]_q \in \mathbb{Z}_p \tag{3.31}$$

which proves the statement. □

With the definition of the fractional part of a p-adic number introduced we can now provide the definition of an additive character.

Definition 3.14 (Additive characters). *Additive characters* on $\mathbb{Q}_p$ are defined by

$$\psi_p \equiv \psi_{p,u} \,:\, \mathbb{Q}_p \to U(1)\,, \quad \psi_{p,u}(x) = e^{-2\pi i [ux]_p} \qquad x, u \in \mathbb{Q}_p \,. \tag{3.32}$$

The additive characters of (3.32) satisfy the relations $\psi_{p,u}(x)\psi_{p,u}(y) = \psi_{p,u}(x+y)$ and $\psi_{p,u}\psi_{p,v} = \psi_{p,u+v}$, as well as $\overline{\psi_{p,u}(x)} = \psi_{p,-u}(x) = \psi_{p,u}(-x)$. The *conductor* of the character is its kernel $|u|_p \mathbb{Z}_p$, but often we simply call $|u|_p$ the conductor.

Note that in the following, we shorten the notation to $\psi_p \equiv \psi_{p,u}$ since it will be more important to keep track of the dependence on the prime p; the 'mode number' u will be given explicitly where needed. Also in the interest of simplicity of notation we will shall often drop the prime p script on the symbol for the fractional part $[\cdot]_p$ when writing out characters explicitly.

Let us work through some integrals over the additive character.

Example 3.15: Integration of a character over $p^k\mathbb{Z}_p$

For $k \in \mathbb{Z}$ one has

$$\int_{p^k\mathbb{Z}_p} e^{-2\pi i [ux]} dx = p^{-k} \gamma_p(up^k)\,, \tag{3.33}$$

where the characteristic function $\gamma_p(u)$ of $\mathbb{Z}_p$ in $\mathbb{Q}_p$ is defined as

$$\gamma_p(u) := \int_{\mathbb{Z}_p} e^{-2\pi i [ux]} dx = \begin{cases} 1 & \text{if } u \in \mathbb{Z}_p\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.34}$$

The function $\gamma_p(u)$ is also called the p-adic *Gaussian* which will be discussed in more detail in section 3.4.

In order to derive this result we start with the case when $k = 0$:

$$\int_{\mathbb{Z}_p} \psi_u(x) dx = \int_{\mathbb{Z}_p} e^{-2\pi i [ux]} dx \tag{3.35}$$

and the integral only depends on the conductor $|u|_p$. We then distinguish two cases: (i) $u \in \mathbb{Z}_p$ and (ii) $u \notin \mathbb{Z}_p$:

(i) If $u \in \mathbb{Z}_p$ then $[ux]_p = 0$ for $x \in \mathbb{Z}_p$ and hence the integral equals $\int_{\mathbb{Z}_p} dx = 1$.

(ii) If $u \notin \mathbb{Z}_p$ then we are effectively integrating a periodic function over a full period and hence the integral gives zero. More concretely, consider the example when $u = p^{-1}$; then

$$\int\limits_{\mathbb{Z}_p} e^{-2\pi i [p^{-1}x]} dx = \sum_{k=0}^{p-1} e^{-2\pi i k/p} \int\limits_{C_k} dx = \frac{1}{p} \sum_{k=0}^{p-1} e^{-2\pi i k/p} = 0 \tag{3.36}$$

with C_k defined as in (3.19) and where we have used the fact that $\int_{C_k} dx = 1/p$, c.f. also example 3.8. If u is 'more rational' one has to refine the summation region more but will always encounter sums that average to zero. We have thus derived (3.33) for the case of $k = 0$.

The result for the integral in the case when $k \neq 0$ then follows by a simple change of variables:

$$\int\limits_{p^k\mathbb{Z}_p} e^{-2\pi i [ux]} dx = p^{-k} \int\limits_{\mathbb{Z}_p} e^{-2\pi i [up^k x]} dx = p^{-k} \gamma_p(up^k). \tag{3.37}$$

We will also require the integral over 'shells' of p-adic numbers.

Example 3.16: Integration of a character over $p^k\mathbb{Z}_p^\times$

For $k \in \mathbb{Z}$ we have

$$\int\limits_{p^k\mathbb{Z}_p^\times} e^{-2\pi i [ux]} dx = \begin{cases} \frac{p-1}{p} p^{-k} & \text{for } |u|_p \leq p^k \\ -p^{-(k+1)} & \text{for } |u|_p = p^{k+1} \\ 0 & \text{for } |u|_p > p^{k+1} \end{cases} . \tag{3.38}$$

Starting as before with the case $k = 0$, this can be related to the preceding example by noting that $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus (p\mathbb{Z}_p)$:

$$\begin{aligned}
\int\limits_{\mathbb{Z}_p^\times} e^{-2\pi i [ux]} dx &= \int\limits_{\mathbb{Z}_p} e^{-2\pi i [ux]} dx - \int\limits_{p\mathbb{Z}_p} e^{-2\pi i [ux]} dx \\
&= \gamma_p(u) - p^{-1} \int\limits_{\mathbb{Z}_p} e^{-2\pi i [upx]} dx \\
&= \gamma_p(u) - p^{-1}\gamma_p(pu) \\
&= \begin{cases} \frac{p-1}{p} & \text{for } |u|_p \leq 1, \text{ i.e., } u \in \mathbb{Z}_p \\ -p^{-1} & \text{for } |u|_p = p \\ 0 & \text{for } |u|_p > p \end{cases} .
\end{aligned} \tag{3.39}$$

The result of $p^k\mathbb{Z}_p^\times$ for $k \neq 0$ then follows by a change of variables

$$\begin{aligned}
\int\limits_{p^k\mathbb{Z}_p^\times} e^{-2\pi i [ux]} dx &= p^{-k} \int\limits_{\mathbb{Z}_p^\times} e^{-2\pi i [up^k x]} dx = p^{-k}\gamma_p(up^k) - p^{-(k+1)}\gamma_p(up^{k+1}) \\
&= \begin{cases} \frac{p-1}{p} p^{-k} & \text{for } |u|_p \leq p^k \\ -p^{-(k+1)} & \text{for } |u|_p = p^{k+1} \\ 0 & \text{for } |u|_p > p^{k+1} \end{cases} .
\end{aligned} \tag{3.40}$$

Note also that this implies that

$$\int\limits_{\mathbb{Q}_p \setminus \mathbb{Z}_p} e^{-2\pi i[ux]} dx = \sum_{k=-1}^{-\infty} \int\limits_{p^k \mathbb{Z}_p^\times} e^{-2\pi i[ux]} dx = -\gamma_p(u) \,. \tag{3.41}$$

An important comment here concerns the integral of a character over all of $\mathbb{Q}_p$: Since $\mathbb{Q}_p$ is formally the sum of $p^k\mathbb{Z}_p^\times$ over all $k \in \mathbb{Z}$, we see from the above result that for any $u \in \mathbb{Q}_p$ we obtain formally

$$\int\limits_{\mathbb{Q}_p} e^{-2\pi i[ux]} dx = 0 \qquad \text{(not well-defined!)} \tag{3.42}$$

which is the analogue of the incorrect equation $\int_{\mathbb{R}} e^{2\pi iux} dx = 0$ which could be derived by splitting up $\mathbb{R}$ into an infinite number of intervals of length $1/u$ on each of which the integral vanishes. As is well-known, the integral over the whole real line of $e^{2\pi iux}$ is not well-defined but rather yields a δ-distribution. We will now see that something similar is true for the p-adic character $e^{-2\pi i[ux]}$ integrated over $\mathbb{Q}_p$.

Before introducing the concept of a p-adic Fourier transform let us make a short comment about function spaces used. The functions which we will be integrating are elements of $\mathcal{S}(\mathbb{Q}_p)$ which is the *Schwartz-Bruhat* space. These functions generalise the *Schwartz functions* which are infinitely differentiable, with rapidly decreasing derivatives.

Definition 3.17 (Fourier transform). One defines the *Fourier transform* over $\mathbb{Q}_p$ by integrating a function f_p on $\mathbb{Q}_p$ against the additive character $\psi_p(x) \equiv \psi_{p,u}(x)$:

$$\tilde{f}_p(u) = \int\limits_{\mathbb{Q}_p} f_p(x)\psi_p(x) dx = \int\limits_{\mathbb{Q}_p} f_p(x) e^{-2\pi i[ux]} dx \,. \tag{3.43}$$

The inverse transform uses the conjugate character

$$f_p(x) = \int\limits_{\mathbb{Q}_p} \tilde{f}_p(u)\overline{\psi_p(x)} du = \int\limits_{\mathbb{Q}_p} \tilde{f}_p(u) e^{2\pi i[ux]} du \,. \tag{3.44}$$

One can now ask for which functions f_p the transform is well-defined and can actually be inverted. As a first step we calculate the composition of the transforms of the characteristic function of a ball $p^k\mathbb{Z}_p \subset \mathbb{Q}_p$, i.e., $f_p(x) = \gamma_p(p^{-k}x)$

$$\begin{aligned} \int\limits_{\mathbb{Q}_p} \overline{\psi_p(x)} \int\limits_{\mathbb{Q}_p} \psi_p(y)\gamma_p(p^{-k}y) dy du &= \int\limits_{\mathbb{Q}_p} \overline{\psi_p(x)} \int\limits_{p^k\mathbb{Z}_p} \psi_p(y) dy du \\ &= \int\limits_{\mathbb{Q}_p} e^{2\pi i[ux]} p^{-k}\gamma_p(up^k) du = \int\limits_{\mathbb{Z}_p} e^{2\pi i[up^{-k}x]} du = \gamma_p(p^{-k}x) \,. \end{aligned} \tag{3.45}$$

From this calculation we see that restricting to compactly supported (and bounded) functions makes the integrals well-defined. We can relax the assumption of compact support if the function decreases sufficiently fast for larger and larger balls. This is for instance the case when $f_p(x) = |x|_p^s$ with $\mathrm{Re}(s)$ sufficiently negative. However, since in this case $|x|_p^s$ blows up for $|x|_p \to 0$ one has to cut out that region or replace $f_p(x)$ by a different function there. In summary, the p-adic Fourier transform is only well-defined on functions that are locally constant (i.e. constant on each $p^k\mathbb{Z}_p^\times$) and have compact support or a sufficiently fast decrease when $|x|_p \to \infty$.

Let us go through a series of interesting examples.

Example 3.18: Fourier transform of $|x|_p^s$ over $\mathbb{Q}_p \setminus \mathbb{Z}_p$

Here, we cut out the compact region of the integers and consider the effect of the damping function $|x|_p^s$ with $\mathrm{Re}(s) < -1$. The result is

$$\int\limits_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^s \psi_p(x) dx = \gamma_p(u) \left((1-p^s) \frac{1-p^{s+1}|u|_p^{-s-1}}{1-p^{s+1}} - 1 \right). \tag{3.46}$$

To show this, we denote the integral by I and distinguish two cases: (i) u integral and (ii) u non-integral:

(i): If $u \in \mathbb{Z}_p$ and has conductor p^k with $k \geq 0$, then we evaluate the integral as

$$\begin{aligned} I &= \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^s e^{-2\pi i [p^k x]} dx = \sum_{\ell=1}^{\infty} p^{s\ell} \int_{p^{-\ell}\mathbb{Z}_p^\times} e^{-2\pi i [p^k x]} dx \\ &= \sum_{\ell=1}^{\infty} p^{(s+1)\ell} \int_{\mathbb{Z}_p^\times} e^{-2\pi i [p^{k-\ell} x]} dx \underset{\text{Ex. 3.16}}{=} \frac{p-1}{p} \sum_{\ell=1}^{k} p^{(s+1)\ell} - \frac{1}{p} p^{(k+1)(s+1)} \\ &= (1-p^s) \frac{1-p^{s+1}|u|_p^{-s-1}}{1-p^{s+1}} - 1 \end{aligned} \tag{3.47}$$

(ii): If $u \notin \mathbb{Z}_p$, so that the character has conductor p^k with $k < 0$ we find

$$I = \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |x|_p^s e^{-2\pi i [p^k x]} dx = \sum_{\ell=1}^{\infty} p^{(s+1)\ell} \int_{\mathbb{Z}_p^\times} e^{-2\pi i [p^{k-\ell} x]} dx = 0 \tag{3.48}$$

by example 3.16 since $k - \ell < -1$ for all $\ell \geq 1$.

Example 3.19: Fourier transform of $|x|_p^s$ over $\mathbb{Z}_p$

Here we cut out the region $\mathbb{Q}_p \setminus \mathbb{Z}_p$. In order to have a bounded function for $|x|_p \to 0$ we now require $\mathrm{Re}(s) > -1$. The Fourier transform now evaluates to

$$\int\limits_{\mathbb{Z}_p} |x|_p^s e^{-2\pi i [xu]} dx = \gamma_p(u) \frac{p-1}{p} \frac{1}{1-p^{-s-1}} + (1-\gamma_p(u)) |u|_p^{-s-1} \frac{1-p^s}{1-p^{-s-1}}. \tag{3.49}$$

This can be derived in a few steps

$$\int\limits_{\mathbb{Z}_p} |x|_p^s e^{-2\pi i[xu]}dx = \sum_{\ell=0}^{\infty} p^{-s\ell} \int_{p^\ell \mathbb{Z}_p^\times} e^{-2\pi i[ux]}dx = \sum_{\ell=0}^{\infty} p^{-(s+1)\ell} \int_{\mathbb{Z}_p^\times} e^{-2\pi i[up^\ell x]}dx$$
$$= \gamma_p(u)\frac{p-1}{p}\frac{1}{1-p^{-s-1}} + (1-\gamma_p(u))|u|_p^{-s-1}\frac{1-p^s}{1-p^{-s-1}}, \tag{3.50}$$

where we have treated the cases $u \in \mathbb{Z}_p$ and $u \notin \mathbb{Z}_p$ separately and used equation (3.22) and example 3.16.

3.4 p-adic Gaussian and Bessel function

In this section we will discuss two special functions: the *p-adic Gaussian* and the *p-adic Bessel function*, which will play a role later on in the text.

The p-adic analogue of the Gaussian $e^{-\pi x^2}$ is given by the function

$$\gamma_p(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}_p, \text{ i.e. } |x|_p \leq 1 \\ 0 & \text{if } x \notin \mathbb{Z}_p, \text{ i.e. } |x|_p > 1\,, \end{cases} \tag{3.51}$$

which we have already encountered in example 3.15 of the previous section. In order to see why it is the generalisation of the real Gaussian $e^{-\pi x^2}$ we recall that the real Gaussian is invariant under Fourier transformation. Using (3.33) this property is then also easily checked for the p-adic version:

$$\tilde{\gamma}_p(u) = \int\limits_{\mathbb{Q}_p} \psi_u(x)\gamma_p(x)dx = \int\limits_{\mathbb{Z}_p} e^{-2\pi i[ux]}dx = \gamma_p(u)\,. \tag{3.52}$$

Let us also note the following useful property of the finite product of the p-adic Gaussian which will be used in chapter 7 in the computation of the Fourier coefficients of Eisenstein series on $SL(2,\mathbb{A})$, c.f. equation (7.78):

$$\prod_{p<\infty} \gamma_p(m) = \begin{cases} 1 & \text{if } m \in \mathbb{Z} \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.53}$$

In order to introduce the p-adic version of the Bessel function, recall that the real (modified) Bessel function K_s can be written as the (inverse) Fourier transform of the function $(1+u^2)^{-2s} \equiv ||(1,u)||^{-2s} = (1+u^2)^{-s}$ via

$$\int_{\mathbb{R}} (1+u^2)^{-s} e^{-2\pi i m u}du = \frac{2\pi^s}{\Gamma(s)}|m|^{s-1/2}K_{s-1/2}\left(2\pi|m|\right), \tag{3.54}$$

where $\Gamma(s)$ is the standard Gamma function. The p-adic generalisation of the integrand is through $||(1,u)||_p^{-2s} = (\max(1,|u|_p))^{-2s}$. The normalisation to be chosen is [190, 191]

$$\tilde{f}(u) = \frac{1}{1-p^{-2s}}(\max(1,|u|_p))^{-2s}\,. \tag{3.55}$$

The Fourier transform of this function is

$$f(x) = \gamma_p(x)\frac{1 - p^{-2s+1}|x|_p^{2s-1}}{1 - p^{-2s+1}}\,. \tag{3.56}$$

which we call the p-adic Bessel function and will be used, for example, in section 7.3.

To demonstrate these properties of the p-adic Bessel function we perform the Fourier transform in the following example.

Example 3.20: Fourier transform of p-adic Bessel function

Consider the following calculation

$$\begin{aligned}&\int\limits_{\mathbb{Q}_p} e^{2\pi i[ux]}\frac{1}{1-p^{-2s}}(\max(1,|u|_p))^{-2s}du \\ &= \frac{1}{1-p^{-2s}}\int\limits_{\mathbb{Z}_p} e^{2\pi i[ux]}du + \frac{1}{1-p^{-2s}}\int\limits_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |u|_p^{-2s}e^{2\pi i[ux]}du\,.\end{aligned} \tag{3.57}$$

We have separated the integral according to the two possible cases of the max function. The first integral is given in example 3.15 and the second one in example 3.18. Combining the results we obtain

$$\int\limits_{\mathbb{Q}_p} e^{2\pi i[ux]}\frac{1}{1-p^{-2s}}(\max(1,|u|_p))^{-2s}du = \gamma_p(x)\frac{1-p^{-2s+1}|x|_p^{2s-1}}{1-p^{-2s+1}}\,. \tag{3.58}$$

We note that the integral converges for $\mathrm{Re}(s) > 1/2$.

3.5 Adeles

In the previous sections we have introduced the concept of the p-adic completions $\mathbb{Q}_p$ of $\mathbb{Q}$ and we have shown in a number of examples how integration can be carried out locally and also that the real Gaussian and Bessel function have p-adic counterparts. The next step will be to organise the completions $\mathbb{Q}_p$ of the rational numbers $\mathbb{Q}$ into a global field called the adeles of $\mathbb{Q}$, denoted by $\mathbb{A}$, which comprises the p-adic completions at all primes, including the prime at infinity, at the same time. The introduction of the adeles as a global number field is in line with the so-called local-to-global principle. For a brief summary highlighting the power of this principle see appendix D.

Definition 3.21 (Adeles $\mathbb{A}$). The *adeles* $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$ are defined as a *restricted direct product*

$$\mathbb{A} = \mathbb{R} \times {\prod_{p<\infty}}' \mathbb{Q}_p\,, \tag{3.59}$$

where the restriction on the product (signified by the prime) means that $\mathbb{A}$ consists of those elements

$$a = (a_p) = (a_\infty; a_2, a_3, a_5, a_7, \dots) \tag{3.60}$$

such that for almost all finite primes p one has $a_p \in \mathbb{Z}_p$.

The restriction on the direct product in the definition of the adeles makes them locally compact which is needed for the existence of a Haar measure. Also as a consequence of the definition, the adeles are endowed with a natural topology, and they are in fact a locally compact ring. We refer the reader to [117] for more details on these issues. It will sometimes be useful to talk about the *finite* adeles $\mathbb{A}_f$ which are defined as the restricted direct product over the finite primes:

$$\mathbb{A}_f = \prod_{p<\infty}{}' \mathbb{Q}_p, . \tag{3.61}$$

We also define the set of invertible elements of the adeles.

Definition 3.22 (Ideles $\mathbb{A}^\times$). The *ideles* $\mathbb{A}^\times$ are the set of invertible elements in $\mathbb{A}$. They are defined as:

$$\mathbb{A}^\times = \mathbb{R}^\times \times \prod_{p<\infty}{}' \mathbb{Q}_p^\times . \tag{3.62}$$

The norm for the adeles is induced directly from the local norms.

Definition 3.23 (Global norm $|\cdot|_\mathbb{A}$). The *global norm* $|\cdot|_\mathbb{A}$ on $\mathbb{A}$ is induced from the norm $|\cdot|_p$ on the local factors $\mathbb{Q}_p$ according to the formula

$$|a|_\mathbb{A} = \prod_{p\leq\infty} |a_p|_p. \tag{3.63}$$

This is in fact a finite product since almost all $a_p \in \mathbb{Z}_p$ and hence satisfy $|a_p|_p = 1$.

The *strong approximation principle*[8] states that the set

$$J = \mathbb{R}_+ \times \prod_{p<\infty} \mathbb{Z}_p^\times \tag{3.64}$$

is a fundamental domain for $\mathbb{Q}^\times \backslash \mathbb{A}^\times$. Hence we can write the ideles as the (disjoint) union

$$\mathbb{A}^\times = \bigcup_{k\in\mathbb{Q}^\times} k \cdot J. \tag{3.65}$$

The rational numbers *embed diagonally* into the adeles, i.e. $\mathbb{Q} \hookrightarrow \mathbb{A}$, by simply taking

$$\mathbb{Q} \ni x \longmapsto (x; x, x, x, \dots) \in \mathbb{A} . \tag{3.66}$$

One can see that this is indeed and element of the adeles since for $x \in \mathbb{Q}$ the norm $|x|_p$ is non-trivial only for the finite number of p's which divide x in the sense of (3.15). In other words, the prime factorisations of the coprime numerator and denominator of x contain only a finite number of primes. By factorising $x \in \mathbb{Q}$ into its prime factors we see that

$$|x|_\mathbb{A} = |x|_\infty \prod_{p<\infty} |x|_p = |x|_\infty \, |x|_\infty^{-1} = 1 . \tag{3.67}$$

Following [80], we will now show that with this embedding $\mathbb{Q}$ sits discretely inside $\mathbb{A}$, mimicking the way the integers $\mathbb{Z}$ are embedded as a lattice inside $\mathbb{R}$. As we will see, this fact lies at the heart of the analysis in subsequent sections.

[8] A higher rank version of strong approximation is proven in section 4.2.2.

Proposition 3.24 (Discrete embedding of $\mathbb{Q}$ in $\mathbb{A}$). *$\mathbb{Q}$ sits discretely inside $\mathbb{A}$.*

Proof. Let us first consider $0 \in \mathbb{Q}$ and construct

$$V = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \prod_{p<\infty} \mathbb{Z}_p \subset \mathbb{A} \,. \tag{3.68}$$

The subgroup $\mathbb{Z}_p$ is an open ball in $\mathbb{Q}_p$ since $|x|_p$ takes only a discrete set of values, that is, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\} = \{x \in \mathbb{Q}_p \mid |x|_p < \alpha\}$ for any $1 < \alpha < p$.

Thus, V is an open neighbourhood of 0 in $\mathbb{A}$ and for any $x \in V \cap \mathbb{Q}$ we have that $|x|_p \leq 1$ for all $p < \infty$ which means that $x \in \mathbb{Z}$, and $|x|_\infty < \frac{1}{2}$ which then gives that $x = 0$. Hence, we have found an open neighbourhood V to 0 in $\mathbb{A}$ such that $V \cap \mathbb{Q} = \{0\}$. For a general point $r \in \mathbb{Q}$ these arguments generalise by instead considering $r + V$, which makes $\mathbb{Q}$ discrete in $\mathbb{A}$. □

With the definition of the adeles as the collection of all local factors at hand, we will now see how to turn a set of local functions into a global one.

3.6 Adelisation

One can extend a collection of local functions f_p on $\mathbb{Q}_p$ to a global function $f_\mathbb{A}$ on $\mathbb{A}$:

$$f_\mathbb{A}(a) = f_\mathbb{A}(a_\infty; a_f) \,, \qquad a_f = (a_2, a_3, a_5, \dots) \in \mathbb{A}_f \,, \tag{3.69}$$

via an Euler product

$$f_\mathbb{A}(a) = \prod_{p\leq\infty} f_p(a_p) \,. \tag{3.70}$$

Starting from $f_\mathbb{A}$ we can recover a function on $\mathbb{R}$ by setting

$$f_\infty(a_\infty) = f_\mathbb{A}(a_\infty; 1, 1, 1, \dots) \,. \tag{3.71}$$

One says that $f_\mathbb{A}$ is the *adelisation* of $f_\mathbb{R}$. Similarly we can extend to $\mathbb{A}$ the notion of local additive characters ψ on $\mathbb{Q}_p$.

Let $u = (u_\infty, u_2, u_3, \dots) \in \mathbb{A}$ and $\psi_p : \mathbb{Q}_p \to U(1)$ be an additive character, such that for finite p this coincides with the character

$$\psi_p(x_p) = e^{-2\pi i [u_p x_p]_p}, \qquad u_p, x_p \in \mathbb{Q}_p \,, \tag{3.72}$$

defined in section 3.3, while for $p = \infty$ this is the standard character on $\mathbb{R}$:

$$\psi_\infty(x_\infty) = e^{2\pi i u_\infty x_\infty}, \qquad u_\infty, x_\infty \in \mathbb{R}. \tag{3.73}$$

We can then consider a global character

$$\psi_\mathbb{A} \,:\, \mathbb{A} \,\to\, U(1) \tag{3.74}$$

as the adelisation of $\psi_{\mathbb{R}}$, i.e., as the Euler product

$$\psi_{\mathbb{A}}(x) = \prod_{p\leq\infty} \psi_p(x_p) = e^{2\pi i u_\infty x_\infty} \prod_{p<\infty} e^{-2\pi i [u_p x_p]_p} \,, \tag{3.75}$$

which we will denote as $\psi_{\mathbb{A}}(x) = e^{2\pi i u x}$ for short.

The sign difference in the exponentials of the characters at the archimedean and non-archimedean places have been introduced for the following reason. For $u = m \in \mathbb{Q}$ diagonally in $\mathbb{A}$, the character $\psi_{\mathbb{A}}$ is periodic in $\mathbb{Q}$ since for $x \in \mathbb{A}$ and $r \in \mathbb{Q}$

$$\psi_{\mathbb{A}}(x+r) = \psi_{\mathbb{A}}(x)\psi_{\mathbb{A}}(r) \tag{3.76}$$

with

$$\psi_{\mathbb{A}}(r) = \prod_{p\leq\infty} \psi_p(r) = \exp\Big(2\pi i \big(mr - \sum_{p<\infty} [mr]_p\big)\Big) = 1 \tag{3.77}$$

using proposition 3.13. Thus, for rational u, $\psi_{\mathbb{A}}$ is a character on $\mathbb{Q}\backslash\mathbb{A}$. That these are all the characters on $\mathbb{Q}\backslash\mathbb{A}$ is shown in [80].

Integration over the adeles is similarly defined using Euler products [219]. For instance, the integral over an adelic function $f_{\mathbb{A}}(x)$ can be written as

$$\int_{\mathbb{A}} f_{\mathbb{A}}(x)\, dx = \left(\int_{\mathbb{R}} f_{\mathbb{R}}(x)\, dx\right) \left(\prod_{p<\infty} \int_{\mathbb{Q}_p} f_p(x)\, dx\right). \tag{3.78}$$

Definition 3.25 (Adelic Fourier transform). The adelic Fourier transform is defined using the global character $\psi_{\mathbb{A}}$ as follows:

$$\tilde{f}_{\mathbb{A}}(u) = \int_{\mathbb{A}} f_{\mathbb{A}}(x) \overline{\psi}_{\mathbb{A}}(x) dx. \tag{3.79}$$

We will perform several integrals of this type in subsequent sections.

In the following section we will illustrate the usefulness of the adelic framework in the context of the Riemann zeta function.

3.7 Adelic analysis of the Riemann zeta function

In this section we will illustrate the power of the adelic formalism by analysing the Riemann zeta function from this point of view. This was one of the main points of the celebrated thesis of Tate [305] which first introduced the notion of Fourier analysis over the adeles.

3.7.1 The completed Riemann zeta function

The first task will be to illustrate how the completed Riemann zeta function is a much more natural object from an adelic perspective, than the ordinary zeta function. Recall first that the completed Riemann zeta function takes the form:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{3.80}$$

We now have:

Proposition 3.26 (Tate's global Riemann integral [305]). *The completed Riemann zeta function $\xi(s)$ can be written in the following global form:*

$$\xi(s) = \int_{\mathbb{A}^\times} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s d^\times x, \tag{3.81}$$

where $s \in \mathbb{C}$ and $\gamma_{\mathbb{A}} = \prod_{p\leq\infty} \gamma_p$ with γ_p the p-adic Gaussian (3.51) and $\gamma_\infty = e^{-\pi x^2}$.

Proof. Splitting the integral into an Euler product yields

$$\int_{\mathbb{A}^\times} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s d^\times x = \left(\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|_\infty^s d^\times x\right) \prod_{p<\infty} \int_{\mathbb{Q}_p^\times} \gamma_p(x)|x|_p^s d^\times x. \tag{3.82}$$

The archimedean integral can be evaluated in terms of a Gamma function:

$$\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|_\infty^s d^\times x = \int_{\mathbb{R}} e^{-\pi x^2}|x|_\infty^{s-1} dx = \pi^{-s/2}\Gamma(s/2), \tag{3.83}$$

where we made use of (3.26).

Due to the γ_p-factor, the p-adic integrals localise on the p-adic integers

$$\int_{\mathbb{Q}_p^\times} \gamma_p(x)|x|_p^s d^\times x = \int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s d^\times x. \tag{3.84}$$

By remark 3.11, this yields

$$\int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s d^\times x = \int_{\mathbb{Z}_p} |x|_p^s d^\times x = \frac{1}{1-p^{-s}}. \tag{3.85}$$

Combining everything and performing the product over primes we obtain

$$\int_{\mathbb{A}^\times} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s d^\times x = \pi^{-s/2}\Gamma(s/2) \prod_{p<\infty} \frac{1}{1-p^{-s}} = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(s). \tag{3.86}$$

□

Remark 3.27. The above result illustrates that the adelic approach gives an elegant integral representation of the completed zeta function, where the normalization factor corresponds to the contribution from the archimedean place $p = \infty$. Such integrals were first considered in the thesis of Tate [305], and then developed further by Jacquet and Langlands [176].

Remark 3.28. It is common to define the archimedean zeta factor by $\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$ and write the global Euler product form of the completed Riemann zeta function as

$$\xi(s) = \prod_{p\leq\infty} \zeta_p(s). \tag{3.87}$$

Anticipating later notions, the completed Riemann zeta function can be thought of as an automorphic form on the group $GL(1, \mathbb{A})$.

3.7.2 The functional relation

We shall now take the analysis one step further and prove the following famous theorem using the adelic framework.

Theorem 3.29 (Functional relation for the completed Riemann zeta function). *The completed Riemann zeta function satisfies the functional relation*

$$\xi(s) = \xi(1-s). \tag{3.88}$$

Proof. To prove the theorem using the approach of Tate [305] we first need the following Lemma (our proof follows the structure of [304]):

Lemma 3.30 (Adelic Poisson resummation). *For any (sufficiently nice) function $f_{\mathbb{A}}$ we have the Poisson summation formula*

$$\sum_{\gamma\in\mathbb{Q}} f_{\mathbb{A}}(\gamma) = \sum_{\gamma\in\mathbb{Q}} \tilde{f}_{\mathbb{A}}(\gamma). \tag{3.89}$$

Proof. The proof is similar to the proof of the ordinary Poisson summation formula so we will be brief. Define

$$F_{\mathbb{A}}(x) = \sum_{\gamma\in\mathbb{Q}} f_{\mathbb{A}}(x\gamma). \tag{3.90}$$

This function is periodic by construction and so has a Fourier expansion. The Fourier coefficients F_{ψ_γ} of $F_{\mathbb{A}}$ with respect a unitary character ψ_γ precisely equals the Fourier transform $\tilde{f}_{\mathbb{A}}(\gamma)$ of the seed function $f_{\mathbb{A}}(\gamma)$ and so we can write

$$F_{\mathbb{A}}(x) = \sum_{\gamma\in\mathbb{Q}} \tilde{f}_{\mathbb{A}}(\gamma)\psi_\gamma(x). \tag{3.91}$$

Putting $x = 1$ in this formula equating it with $F_{\mathbb{A}}(1)$ from the definition (3.90) then establishes the result. □

To complete the proof of the theorem we need also the following lemma:

Lemma 3.31. *The global theta function*

$$\Theta_{\mathbb{A}}(x) = \sum_{k\in\mathbb{Q}} \gamma_{\mathbb{A}}(kx) \tag{3.92}$$

satisfies the functional relation

$$\Theta_{\mathbb{A}}(x) = \frac{1}{|x|_{\mathbb{A}}}\Theta_{\mathbb{A}}(1/x), \qquad \forall x \in \mathbb{A}^\times. \tag{3.93}$$

Proof. This follows from applying the Poisson summation formula and the fact that the global Gaussian $\gamma_{\mathbb{A}}(x)$ is invariant under Fourier transform. □

Now let J be fundamental domain for $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, as in (3.64). By Lemma 3.31 we then have

$$\int_J \Theta_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s d^\times x = \int_J \Theta_{\mathbb{A}}(1/x)|x|_{\mathbb{A}}^{s-1} d^\times x = \int_J \Theta_{\mathbb{A}}(x)|x|_{\mathbb{A}}^{1-s} d^\times x, \tag{3.94}$$

where in the last step we used the fact that the multiplicative measure is invariant under $x \to x^{-1}$. Finally, using the factorisation $\mathbb{A}^\times = \bigcup_{k\in\mathbb{Q}^\times} k \cdot J$ (see (3.64)), and the fact that $|x|_{\mathbb{A}} = 1$ for $x \in \mathbb{Q}$, we can rewrite (3.94) as

$$\int_{\mathbb{A}^\times} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^s d^\times x = \int_{\mathbb{A}^\times} \gamma_{\mathbb{A}}(x)|x|_{\mathbb{A}}^{1-s} d^\times x, \tag{3.95}$$

thus establishing the functional relation (3.88) for the completed Riemann zeta function. □

Chapter 4

Basic notions from Lie algebras and Lie groups

We will make use of some standard terminology from the theory of Lie groups and Lie algebras that we briefly summarise for definiteness. We first address complex and real Lie algebras and groups before we turn to the adelic setting with emphasis on the strong approximation theorem.

4.1 Real Lie algebras and real Lie groups

The material reviewed in this section can be found for example in [103, 168, 172, 186].

4.1.1 Split real simple Lie algebras and root systems

Let $\mathfrak{g}(\mathbb{C})$ be a finite-dimensional and simple complex Lie algebra from the Cartan–Killing classification. We will consider here only the split real form $\mathfrak{g} \equiv \mathfrak{g}(\mathbb{R})$ of the Lie algebra. We choose a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, that is, a maximally abelian subalgebra of semi-simple elements. This means that we can decompose $\mathfrak{g}$ into eigenspaces of $\mathfrak{h}$ in what is called the *root space decomposition*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \tag{4.1}$$

where the root space $\mathfrak{g}_\alpha$ for a generalised eigenvalue $\alpha : \mathfrak{h} \to \mathbb{R}$ is given by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x \quad \text{for all } h \in \mathfrak{h}\}\,. \tag{4.2}$$

The set of $\alpha \neq 0$ for which $\mathfrak{g}_\alpha \neq \{0\}$ is called the set of roots Δ. By our assumption on the Lie algebra $\mathfrak{g}$ we have that $\dim(\mathfrak{g}_\alpha) = 1$ for all $\alpha \in \Delta$. Since $\mathfrak{g}_\alpha$ is one-dimensional there is, for each root $\alpha \in \Delta$, a unique element $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ such that $\alpha(H_\alpha) = 2$.

In the set of roots $\Delta \subset \mathfrak{h}^*$ we choose a system of *simple roots*

$$\Pi = \{\alpha_1, \dots, \alpha_r\}, \tag{4.3}$$

where $r = \dim(\mathfrak{h})$ is the *rank* of the Lie algebra. Then any root $\alpha \in \Delta$ can be written as an integral linear combination of the simple roots

$$\alpha = \sum_{i=1}^{r} m_i \alpha_i, \tag{4.4}$$

where either all $m_i \geq 0$ (and α is called a *positive root*: $\alpha > 0$) or all $m_i \leq 0$ (and α is called a *negative root*: $\alpha < 0$). The set of positive/negative roots is denoted by $\Delta_\pm$ and they satisfy $\Delta_- = -\Delta_+$. There is a unique *highest root* $\theta \in \Delta$ for which the *height* $\mathrm{ht}(\alpha) = \sum_i m_i$ is maximal. Another important element is the *Weyl vector* (which is not necessarily an element of Δ)

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{4.5}$$

We define the spaces of *positive/negative step operators* by

$$\mathfrak{n} \equiv \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha, \tag{4.6}$$

as well as the *(upper) Borel subalgebra*

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}. \tag{4.7}$$

The spaces $\mathfrak{n}_\pm$ are nilpotent subalgebras of $\mathfrak{g}$; the Borel subalgebra $\mathfrak{b}$ is solvable. One can think of $\mathfrak{n}_\pm$ as strictly upper/lower triangular matrices and $\mathfrak{h}$ as diagonal matrices.

More formally, the notions of nilpotency and solvability for Lie algebras are defined as follows. A nilpotent Lie algebra is one whose *lower central series* $D_k(\mathfrak{g}) := [\mathfrak{g}, D_{k-1}(\mathfrak{g})]$ vanishes for some finite k. A solvable Lie algebra is one whose *derived series* $D^k(\mathfrak{g}) := \big[D^{k-1}(\mathfrak{g}), D^{k-1}(\mathfrak{g})\big]$ vanishes for some finite k. The Borel subalgebra $\mathfrak{b}$ includes semi-simple elements whence the lower central series does not vanish. The semi-simple elements disappear in $D^1(\mathfrak{b}) = [\mathfrak{b}, \mathfrak{b}]$ and thus the derived series vanishes, rendering $\mathfrak{b}$ solvable. The derived series will play an role when discussing Fourier expansions of automorphic forms in chapter 6.

On $\mathfrak{g}$ one can define an *invariant bilinear form* that we will write as $\langle x|y\rangle$ for $x, y \in \mathfrak{g}$. Invariance means compatibility with the Lie bracket:

$$\langle [x, y]\,|z\rangle = \langle x|\,[y, z]\rangle. \tag{4.8}$$

This form is proportional to the Killing metric. We have that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ implies that $H_\alpha = [X_\alpha, Y_\alpha]$ for some $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$.

Then, for any $h \in \mathfrak{h}$

$$\langle H_\alpha|h\rangle = \langle [X_\alpha, Y_\alpha]|h\rangle = \langle X_\alpha|[Y_\alpha, h]\rangle = \alpha(h)\langle X_\alpha|Y_\alpha\rangle\,. \tag{4.9}$$

With $h = H_\alpha$ this becomes

$$\langle H_\alpha|H_\alpha\rangle = \alpha(H_\alpha)\langle X_\alpha|Y_\alpha\rangle = 2\langle X_\alpha|Y_\alpha\rangle \tag{4.10}$$

Thus, by insertion into (4.9)

$$\alpha(h) = \frac{2\langle H_\alpha|h\rangle}{\langle H_\alpha|H_\alpha\rangle} \tag{4.11}$$

Sometimes we will also use the notation $\langle\alpha|h\rangle$ for $\alpha(h)$.

The Cartan element $T_\alpha = 2H_\alpha/\langle H_\alpha|H_\alpha\rangle$ can then be used to define an inner product on $\mathfrak{h}^*$ by

$$\langle\alpha|\beta\rangle = \langle T_\alpha|T_\beta\rangle = \alpha(T_\beta) = \beta(T_\alpha)\,. \tag{4.12}$$

Since $\mathfrak{g}$ is finite-dimensional and simple this bilinear form on $\mathfrak{h}^*$ is positive definite and can be used to define the lengths of root vectors α. We normalise it such that the highest root θ has length $\theta^2 := \langle\theta|\theta\rangle = 2$.

The bilinear form on $\mathfrak{h}^*$ (spanned over $\mathbb{R}$ by the simple roots) can be used to define a basis of $\mathfrak{h}^*$ dual to the simple roots. The corresponding basis elements are called the *fundamental weights* Λ_i and satisfy

$$\langle\Lambda_i|\alpha_j\rangle = \frac{1}{2}\langle\alpha_i|\alpha_i\rangle\delta_{ij} \quad \text{for } i,j = 1,\ldots,r. \tag{4.13}$$

In terms of the fundamental weights one can re-express the Weyl vector of equation (4.5) as $\rho = \sum_{i=1}^r \Lambda_i$. A general element of $\mathfrak{h}^*$ will be called a *weight* and denoted by λ.

Associated with the choice of simple roots α_i is also a realisation of the *Weyl group* of $\mathfrak{g}$. This is a finite Coxeter group that is generated by the *fundamental reflections* w_i $(i = 1,\ldots,r)$ that are defined through their action on weights λ by

$$w_i(\lambda) = \lambda - \frac{2\langle\lambda|\alpha_i\rangle}{\langle\alpha_i|\alpha_i\rangle}\alpha_i, \tag{4.14}$$

so that in particular $w_i(\rho) = \rho - \alpha_i$. A general word of the Weyl group is given by a succession of fundamental reflections $w = w_{i_1}\cdots w_{i_\ell}$ and we call $\ell = \ell(w)$ the *length* of the Weyl word w. This assumes that the expression is in reduced form, i.e., that the relations between the generating fundamental w_i have been used to make the word as short as possible. We denote the Weyl group by $\mathcal{W} \equiv \mathcal{W}(\mathfrak{g})$ and its distinguished longest element by w_{long}. The *longest Weyl word* has the property $w_{\text{long}}(\Delta_+) = \Delta_-$; all other Weyl words map some positive roots to other positive roots.

Since $\dim(\mathfrak{g}_\alpha) = 1$ for all roots $\alpha \in \Delta$ and Δ_- is opposite to Δ_+ we can define for any $\alpha > 0$ a triplet

$$(E_\alpha, H_\alpha, F_\alpha) \quad \in \mathfrak{g}_\alpha \times \mathfrak{h} \times \mathfrak{g}_{-\alpha} \tag{4.15}$$

such that the triplet forms a standard $\mathfrak{sl}(2,\mathbb{R})$ subalgebra of $\mathfrak{g}$. The relations of one such $\mathfrak{sl}(2,\mathbb{R})$ algebra are

$$[H_\alpha, E_\alpha] = 2E_\alpha, \quad [H_\alpha, F_\alpha] = -2F_\alpha, \quad [E_\alpha, F_\alpha] = H_\alpha. \tag{4.16}$$

We also use the notation $E_{-\alpha} = F_\alpha$, Furthermore, we introduce the following notation for the $\mathfrak{sl}(2,\mathbb{R})$ triples associated with the simple roots α_i for $i = 1,\ldots,r$:

$$e_i \equiv E_{\alpha_i}, \quad f_i \equiv F_{\alpha_i}, \quad h_i \equiv H_{\alpha_i}. \tag{4.17}$$

The h_i form a basis of the Cartan subalgebra $\mathfrak{h}$. The r triples (e_i, h_i, f_i) are sometimes referred to as the *simple Chevalley generators*.

The *Cartan matrix* A is an $r \times r$ matrix defined by the elements

$$A_{ij} = \frac{2\langle\alpha_i|\alpha_j\rangle}{\langle\alpha_i|\alpha_i\rangle} = \frac{2\alpha_j(h_i)}{\alpha_i(h_i)} = \alpha_j(h_i). \tag{4.18}$$

The Lie algebra $\mathfrak{g}$ has a *compact subalgebra* $\mathfrak{k}$ that is spanned by $E_\alpha - E_{-\alpha}$. It is of dimension equal to the number of positive roots. All its elements have negative norm in the invariant bilinear form discussed above.

4.1.2 Split real Lie groups and highest weight representations

Many of the notions just introduced carry over to the group level. Let $G(\mathbb{R})$ be a real Lie group with Lie algebra $\mathfrak{g}$ of the type just discussed. The link between the Lie algebra and Lie group is given by the standard exponential map (in the identity component of $G(\mathbb{R})$).

The Cartan subalgebra $\mathfrak{h}$ of commuting elements is the Lie algebra of an abelian subgroup $A(\mathbb{R}) \subset G(\mathbb{R})$ that we take to be the exponential of $\mathfrak{h}$. Topologically, $A(\mathbb{R}) \cong (GL(1,\mathbb{R})_+)^r$, where the + subscript indicates that we restrict to positive elements. An important remark here is that there is a larger abelian subgroup, sometimes called the (split) *Cartan torus* that is of the form $(GL(1,\mathbb{R}))^r$ and covers $A(\mathbb{R})$. We will sometimes abuse notation and refer to $A(\mathbb{R})$ as the Cartan torus or even refer to the Cartan torus as $A(\mathbb{R})$ as it should always be clear from the context which abelian subgroup is meant.

The space of nilpotent elements $\mathfrak{n} \equiv \mathfrak{n}_+$ is the Lie algebra of a *unipotent* subgroup $N(\mathbb{R}) \subset G(\mathbb{R})$. The compact subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of a (maximal) *compact* subgroup $K(\mathbb{R}) \subset G(\mathbb{R})$.

The *Iwasawa decomposition* states that one can write any element $g \in G(\mathbb{R})$ uniquely as the product of elements of the three subgroups just introduced, i.e.,

$$G(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R}) \tag{4.19}$$

with uniqueness of decomposition [168].

The split real Lie algebras $\mathfrak{g}(\mathbb{R})$ have irreducible finite-dimensional representations labelled by a *dominant highest weight* Λ. This is an element of $\mathfrak{h}^*$ that has integral non-negative coefficients when expanded in the basis of fundamental weights Λ_i that was introduced in (4.13). In other words, a dominant highest weight Λ satisfies

$$\langle\Lambda|\alpha_i\rangle \in \mathbb{N}_0 \quad \text{for all } i = 1,\ldots,r. \tag{4.20}$$

We denote the *highest weight representation* of a dominant highest weight Λ by V_Λ. The notion of highest weight implies that there is a vector $v_\Lambda \in V_\Lambda$ that satisfies

$$h \cdot v_\Lambda = \Lambda(h)v_\Lambda \qquad \text{for all } h \in \mathfrak{h}, \tag{4.21a}$$

$$E_\alpha \cdot v_\Lambda = 0 \qquad \text{for all positive roots } \alpha \in \Delta_+. \tag{4.21b}$$

The first condition reflects that the vector v_Λ is in the Λ-eigenspace of the action of $\mathfrak{g}$ (hence it is a weight vector) and the second condition shows that it is annihilated by all raising operators (hence at highest weight). Here, we have denoted the action of $\mathfrak{g}$ on the representation space V_Λ by $\cdot$ for brevity.

The structure of highest weight representations V_Λ can be conveniently summarized in terms of its character

$$\mathrm{ch}_\Lambda = \sum_{\mu\in\mathfrak{h}^*} \mathrm{mult}_{V_\Lambda}(\mu) e^\mu, \tag{4.22}$$

where $\mathrm{mult}_{V_\Lambda}(\mu)$ denotes the *weight multiplicity* of a weight $\mu \in \mathfrak{h}^*$ in the representation V_Λ, i.e., the dimension of the μ-eigenspace of the action of $\mathfrak{g}$ on V_Λ. The expression e^μ denotes an element of the group algebra of $\mathfrak{h}^*$ and satisfies $e^{\mu_1}e^{\mu_2} = e^{\mu_1+\mu_2}$ for two weights μ_1 and μ_2. Any representation has a character but the advantage of highest weight representations is that there is a nice compact formula that determines the character $\mathrm{ch}(V_\Lambda)$ in terms of Λ, the root structure of $\mathfrak{g}$ and its Weyl group. This formula is the *Weyl character formula* [103, 172]:

$$\mathrm{ch}_\Lambda = \frac{\sum_{w\in\mathcal{W}} \epsilon(w) e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})}. \tag{4.23}$$

The product in the denominator is over all positive roots $\alpha \in \Delta_+$ of the algebra $\mathfrak{g}$ and ρ is the Weyl vector defined in (4.5). The sign $\epsilon(w) = (-1)^{\ell(w)}$ gives the signature of w as an even or odd element in $\mathcal{W}$. As a special case for $\Lambda = 0$ one obtains the one-dimensional *trivial representation* with $\mathrm{ch}(V_0) = 1$. This implies the *denominator formula*

$$\sum_{w\in\mathcal{W}} \epsilon(w) e^{w(\rho)-\rho} = \prod_{\alpha>0}(1-e^{-\alpha}) \tag{4.24}$$

that ties the structure of the Weyl group to the structure of the root system. There is an alternative form of the character formula that will play a rôle later on. This based on observing that

$$w\left(e^\rho \prod_{\alpha>0}(1-e^{-\alpha})\right) = \epsilon(w) e^\rho \prod_{\alpha>0}(1-e^{-\alpha}) \tag{4.25}$$

is $\mathcal{W}$ skew-invariant, as follows for example from the denominator identity. This implies that one can write the character ch_Λ alternatively as

$$\mathrm{ch}_\Lambda = \sum_{w\in\mathcal{W}} w\left(\frac{e^{\lambda+\rho}}{e^\rho \prod_{\alpha>0}(1-e^{-\alpha})}\right) = \sum_{w\in\mathcal{W}} w\left(\frac{e^{\Lambda}}{\prod_{\alpha>0}(1-e^{-\alpha})}\right). \tag{4.26}$$

The character ch_Λ is not only a formal object but can actually be interpreted as a function $\mathrm{ch}_\Lambda : \mathfrak{h}(\mathbb{C}) \to \mathbb{C}$ on the Cartan subalgebra $\mathfrak{h}(\mathbb{C})$ by replacing $e^\Lambda(h) = e^{\Lambda(h)}$ etc. everywhere. The resulting expression converges everywhere on the complexified Cartan subalgebra. We can also evaluate the character on elements of the maximal torus by the exponential map. Let $a \in A$ then

$$\mathrm{ch}_\Lambda(a) = \frac{\sum_{w\in\mathcal{W}} \epsilon(w) a^{w(\lambda+\rho)} a^{-\rho}}{\prod_{\alpha>0}(1-a^{-\alpha})} = \sum_{w\in\mathcal{W}} w\left(\frac{a^\Lambda}{\prod_{\alpha>0}(1-a^{-\alpha})}\right). \tag{4.27}$$

Remark 4.1. The highest weight representations V_Λ for split real $G(\mathbb{R})$ are finite-dimensional, but not unitary. For complex $G(\mathbb{C})$ the representation V_Λ is irreducible and *unitarizable* for dominant highest weights.

Remark 4.2. For Kac–Moody algebras with symmetrizable Cartan matrix, convergence is restricted to the interior of the complexified *Tits cone* [186, §10.6]. Since we will not be dealing with this case, we refer the reader to literature.

4.1.3 Borel and parabolic subgroups

An important notion for the development of automorphic representations will be that of Borel and parabolic subgroups. The *(upper) Borel subgroup* is given by

$$B(\mathbb{R}) = A(\mathbb{R})N(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R}). \tag{4.28}$$

Here, the abelian group $A(\mathbb{R})$ denotes the full Cartan torus that covers the exponential of the Cartan subalgebra $\mathfrak{h}$.

A *(standard) parabolic subgroup* $P(\mathbb{R})$ of $G(\mathbb{R})$ is a proper subgroup that contains the standard Borel subgroup $B(\mathbb{R})$. If we think of $B(\mathbb{R})$ as consisting of upper triangular matrices (in $G(\mathbb{R})$) then a parabolic subgroup $P(\mathbb{R})$ contains all upper triangular matrices as well as some lower triangular ones. The discussion of this section is valid for both $\mathbb{R}$ and $\mathbb{C}$ and from here on we will suppress the notation of the underlying field.

Standard parabolic subgroups can be described by choosing a subset Σ of the simple roots Π of $\mathfrak{g}$ [71]. The subset $\Sigma \subset \Pi$ generates a root system $\langle\Sigma\rangle$ which defines a *parabolic subalgebra* as follows

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{p})} \mathfrak{g}_\alpha \qquad \text{where } \Delta(\mathfrak{p}) = \Delta_+ \cup \langle\Sigma\rangle\,. \tag{4.29}$$

For clarity of notation, we suppress typically the dependence on the subset Σ.

The parabolic subalgebra can be decomposed into semi-simple *Levi subalgebra* $\mathfrak{l}$ and a *nilpotent subalgebra* $\mathfrak{u}$

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \tag{4.30}$$

which is called a *Levi decomposition*.

Explicitly,

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha \qquad \mathfrak{u} = \bigoplus_{\alpha\in\Delta_+\setminus\langle\Sigma\rangle_+} \mathfrak{g}_\alpha\,, \tag{4.31}$$

where $\langle\Sigma\rangle_+ = \Delta_+ \cap \langle\Sigma\rangle$. Henceforth we will often denote the set difference $\Delta_+ \setminus \langle\Sigma\rangle_+$ as $\Delta(\mathfrak{u})$. We also note that $\mathfrak{l}$ has the same rank as $\mathfrak{g}$.

The reductive Levi subalgebra is often decomposed further into

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{a}_P \tag{4.32}$$

with $\mathfrak{m} = [\mathfrak{l}, \mathfrak{l}]$ being semi-simple and $\mathfrak{a}_P \subset \mathfrak{h}$ being abelian. The decomposition

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a}_P \oplus \mathfrak{u} \tag{4.33}$$

of the parabolic subalgebra is referred to as the *Langlands decomposition*.

Note that we have decorated $\mathfrak{a}_P$ with a subscript P to distinguish its corresponding group A_P from the A in the Iwasawa decomposition. Recall that we use $\mathfrak{h}$ (and not $\mathfrak{a}$) for the Cartan subalgebra of $\mathfrak{g}$.

Explicitly we have that

$$\begin{aligned} \mathfrak{a}_P &= \{h \in \mathfrak{h} \mid \alpha(h) = 0 \text{ for all } \alpha \in \Sigma\} \\ \mathfrak{m} &= [\mathfrak{l}, \mathfrak{l}] = \mathfrak{a}_P^{\perp} \oplus \bigoplus_{\alpha \in \langle \Sigma \rangle} \mathfrak{g}_\alpha , \end{aligned} \tag{4.34}$$

where the orthogonal complement $\mathfrak{a}_P^{\perp}$ is taken within $\mathfrak{h}$ with respect to the invariant bilinear form $\langle \cdot | \cdot \rangle$.

Example 4.3: Parabolic subgroups of $\mathfrak{sl}(3, \mathbb{R})$

As an example we consider the Lie algebra $\mathfrak{g}(\mathbb{R}) = \mathfrak{sl}(3, \mathbb{R})$ of type A_2. It has two simple roots $\Pi(\mathfrak{g}) = \{\alpha_1, \alpha_2\}$ and positive roots given by

$$\Delta_+(\mathfrak{g}) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} . \tag{4.35}$$

Choosing the subset $\Sigma = \{\alpha_1\}$ defines a parabolic subalgebra $\mathfrak{p}(\mathbb{R}) \subset \mathfrak{sl}(3, \mathbb{R})$ with root system

$$\Delta(\mathfrak{p}) = \underbrace{\{\alpha_1, -\alpha_1\}}_{\Delta(\mathfrak{l})} \cup \underbrace{\{\alpha_2, \alpha_1 + \alpha_2\}}_{\Delta(\mathfrak{u})} . \tag{4.36}$$

The Levi subalgebra $\mathfrak{l}(\mathbb{R})$ consists of the embedded $\mathfrak{sl}(2, \mathbb{R})$ associated with the simple root α_1, together with an additional abelian element:

$$\mathfrak{l}(\mathbb{R}) = \underbrace{\mathfrak{sl}(2, \mathbb{R})}_{\mathfrak{m}(\mathbb{R})} \oplus \underbrace{\mathbb{R}}_{\mathfrak{a}(\mathbb{R})} \tag{4.37}$$

The nilpotent part $\mathfrak{u}(\mathbb{R})$ is a two-dimensional abelian Lie algebra and transforms in the two-dimensional representation of $\mathfrak{l}(\mathbb{R})$.

As (traceless) (3×3)-matrices the elements of $\mathfrak{p}(\mathbb{R})$, $\mathfrak{l}(\mathbb{R})$ and $\mathfrak{u}(\mathbb{R})$ take the forms

$$\mathfrak{p}(\mathbb{R}) : \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \mathfrak{l}(\mathbb{R}) : \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \mathfrak{u}(\mathbb{R}) : \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.38}$$

At the level of Lie groups there are corresponding notions. Let P be a connected group having $\mathfrak{p}$ as its Lie algebra. Then there are (unique) decompositions

$$P = LU = MA_PU , \tag{4.39}$$

also called the *Levi decomposition* and *Langlands decomposition*. The subgroup L is called the *Levi subgroup* and U the *unipotent subgroup* or *unipotent radical* of the parabolic subgroup $P \subset G$.

A particularly important class of parabolic subgroups is furnished by the so-called *maximal parabolic subgroups*. These are in a sense the largest (proper) parabolic subgroups and are characterised by choosing as a defining set Σ all simple roots of G but one: $\Sigma = \Pi \setminus \{\alpha_{i_*}\}$, where we denoted the simple root that is left out by α_{i_*}. We will use the notation P_{i_*} to denote the maximal parabolic subgroup associated with such a choice. For maximal parabolic subgroups one has that

$$L = GL(1) \times M \tag{4.40}$$

where M is a semi-simple Lie group. The Dynkin diagram of its Lie algebra is obtained by removing the node i_* from the Dynkin diagram of $\mathfrak{g}$. The parabolic subgroup of example 4.3 is maximal and corresponds to the choice $i_* = 2$.

4.1.4 Chevalley group notation and discrete subgroups

Using the exponential map, we will often parametrise group elements in terms of some basic elements. Concretely, we define for roots $\alpha \in \Delta$ and $u \in \mathbb{R}$ (or another base field of the split Lie algebra)

$$x_\alpha(u) = \exp(uE_\alpha), \tag{4.41}$$

where E_α is the distinguished element of the root space $\mathfrak{g}_\alpha$ that appears in the Chevalley basis constructed in (4.15). The one-parameter group generated by $x_\alpha(u)$ for $u \in \mathbb{R}$ will be denoted by $N_\alpha(\mathbb{R})$.

Furthermore, let

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u) \quad \text{and} \quad h_\alpha(u) = w_\alpha(u)w_\alpha(1)^{-1}. \tag{4.42}$$

The notation $w_\alpha(u)$ is connected to the Weyl group defined above by noting that the $w_{\alpha_i}(1)$ (for simple α_i) generate a cover of the Weyl group [186]. For $u \approx 1$, the element $h_\alpha(u)$ yields H_α.

These elements so defined satisfy

$$x_\alpha(u)x_\alpha(v) = x_\alpha(u+v) \quad \text{and} \quad h_\alpha(u)h_\alpha(v) = h_\alpha(uv) \tag{4.43}$$

and for $\alpha \neq -\beta$

$$x_\alpha(u)x_\beta(v)x_\alpha(u)^{-1}x_\beta(v)^{-1} = \prod_{\substack{m,n>0 \\ m\alpha+n\beta\in\Delta}} x_{m\alpha+n\beta}(c^{\alpha\beta}_{mn}u^m v^n), \tag{4.44}$$

which is the exponentiation of the relation $[E_\alpha, E_\beta] \propto E_{\alpha+\beta}$. The constants $c^{\alpha\beta}_{mn}$ depend on the chosen order in the product and the structure constants of the basis $\{E_\alpha\}$. If $\alpha = -\beta$ we obtain instead

$$w_\alpha(u)x_\alpha(v)w_\alpha(-u) = x_{-\alpha}(-uv^2). \tag{4.45}$$

This is related to the commutator $[E_\alpha, E_{-\alpha}] = H_\alpha$.

We will always take the discrete subgroup $G(\mathbb{Z})$ that is generated by the $x_\alpha(u)$ and $h_\alpha(u)$ for integer u. This group is called the (adjoint) Chevalley group. Another way of obtaining the group is to consider the integer lattice of elements spanned by the generators E_α, H_α and F_α (for $\alpha > 0$). Since the structure constants of $\mathfrak{g}$ are integral in this basis (whence Chevalley basis), this lattice actually defines a Lie algebra $\mathfrak{g}(\mathbb{Z})$ over the ring $\mathbb{Z}$. The group $G(\mathbb{R})$ acts on $\mathfrak{g}(\mathbb{R})$ via the adjoint action. The alternative definition of $G(\mathbb{Z})$ is as the stabiliser of $\mathfrak{g}(\mathbb{Z})$ under this action. The group $G(\mathbb{Z})$ contains representatives of the Weyl group.

4.2 p-adic and adelic groups

In this section we introduce some basic properties of linear algebraic groups defined over a number field $\mathbb{F}$, which can be either local or global. In our treatment we shall always take $\mathbb{F}$ to be either $\mathbb{Q}$, $\mathbb{Q}_p$ or the ring of adeles $\mathbb{A}$. For more details and proofs, see for instance [27, 57, 117, 120].

Recall that a Lie group $G = G(\mathbb{C})$ defined over $\mathbb{C}$ is a differentiable manifold with a compatible group structue. More generally, one can consider *algebraic groups* $G(\mathbb{F})$ over any number field $\mathbb{F}$. Formally, the group $G(\mathbb{F})$ is an (affine) algebraic variety equipped with a group structure given by polynomial operations with values in $\mathbb{F}$. We will be interested in linear algebraic groups over $\mathbb{F}$, which are subgroups $G(\mathbb{F})$ of the group $GL(n, \mathbb{F})$ of invertible $n \times n$ matrices with entries in $\mathbb{F}$. As we shall see, the notion of algebraic group extends to local fields, like $\mathbb{F} = \mathbb{Q}_p$, or global fields, like $\mathbb{F} = \mathbb{Q}$, or even the adeles $\mathbb{F} = \mathbb{A}$.

4.2.1 p-adic groups

We shall now take a closer look at algebraic groups G over the local field of p-adic numbers $\mathbb{Q}_p$. At the infinite place, $\mathbb{Q}_\infty = \mathbb{R}$, this is just a real Lie group $G(\mathbb{R})$ corresponding to a real form of a complex Lie group $G(\mathbb{C})$. We will always take this to be the split real form discussed in the preceding section.

Let us focus on the non-archimedean completions $\mathbb{Q}_{p<\infty}$ of $\mathbb{Q}$, comparing with the more familiar situation of real Lie groups $G(\mathbb{R})$ where it is appropriate. If $G(\mathbb{Q})$ is a linear algebraic subgroup of $GL(n, \mathbb{C})$ defined by polynomial conditions with coefficients in $\mathbb{Q}$, then we can also speak of the local linear algebraic group $G(\mathbb{Q}_p)$ defined by the same polynomial conditions, but now taken over $\mathbb{Q}_p$. The typical example of a linear algebraic group is $SL(n, \mathbb{C})$ that is defined as the subgroup of $GL(n, \mathbb{C})$ such that the polynomial equation $\det(g) = 1$ is satisfied. As this equation has rational coefficient, one can define the local linear algebraic groups $GL(n, \mathbb{Q}_p)$ and $SL(n, \mathbb{Q}_p)$, which are simply the corresponding groups of $n \times n$ matrices with entries in $\mathbb{Q}_p$.

An important fact is that the notion of maximal compact subgroup carries over to the local setting. Recall that for a real Lie group $G(\mathbb{R})$ in its split real form the maximal compact subgroup $K(G)$ is defined as the fixed point set of G under the Chevalley involution. For example, in the case of $G(\mathbb{R}) = SL(n, \mathbb{R})$ we have $K(G) = SO(n)$. To understand the analogous notion of maximal compact subgroup of $G(\mathbb{Q}_p)$, recall that the p-adic integers $\mathbb{Z}_p$ form a compact ring inside $\mathbb{Q}_p$. It follows that the subgroup of integer points

$$G(\mathbb{Z}_p) = G \cap GL(n, \mathbb{Z}_p), \tag{4.46}$$

sits compactly inside $G(\mathbb{Q}_p)$. Hence, for finite primes the maximal compact subgroup of $G(\mathbb{Q}_p)$ is $K_p = G(\mathbb{Z}_p)$.

For real Lie groups $G(\mathbb{R})$ we always have a unique *Iwasawa decomposition*

$$G(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R}), \tag{4.47}$$

where $K(\mathbb{R})$ is the maximal compact subgroup, $A(\mathbb{R})$ is the Cartan torus and $N(\mathbb{R})$ is the nilpotent subgroup generated by the positive Chevalley generators of the Lie algebra of G. The notion of Iwasawa decomposition carries over to the local situation, where we have a decomposition of the form

$$G(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)G(\mathbb{Z}_p). \tag{4.48}$$

In contrast to the case of real groups, the local Iwasawa decomposition is *not* unique, however its restriction to A is, and this fact will play a crucial role later.

Example 4.4: Iwasawa decompositions in $SL(2, \mathbb{Q}_p)$ for $p \leq \infty$

We now consider in more detail the example of $G(\mathbb{Q}_p) = SL(2, \mathbb{Q}_p)$. The maximal compact subgroup is $K_p = SL(2, \mathbb{Z}_p)$ and the Iwasawa decomposition reads

$$SL(2, \mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)SL(2, \mathbb{Z}_p), \tag{4.49}$$

where

$$N(\mathbb{Q}_p) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \middle| \, x \in \mathbb{Q}_p \right\}, \qquad A(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{Q}_p^\times \right\}. \tag{4.50}$$

To illustrate this further, let us consider the explicit Iwasawa decomposition of a specific element

$$g = \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \in SL(2, \mathbb{Q}_p), \tag{4.51}$$

which will be of relevance for the analysis in subsequent sections. First notice that if $u \in \mathbb{Z}_p$ then g is already in $SL(2, \mathbb{Z}_p)$ and the decomposition is trivial. Consider therefore the case when $u \in \mathbb{Q}_p \backslash \mathbb{Z}_p$, for which one could write a $g = nak$ decomposition as follows

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & u^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & \\ & u \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & u^{-1} \end{pmatrix}. \tag{4.52}$$

Notice that for $u \in \mathbb{Q}_p \backslash \mathbb{Z}_p$, the element $u^{-1} \in \mathbb{Z}_p$ and therefore the matrix on the right is in $K_p = SL(2, \mathbb{Z}_p)$ such that this represents *a* valid Iwasawa decomposition.

An important remark is that the Iwasawa decomposition for groups over non-archimedean fields is *not unique*. In the present example, all possible Iwasawa decompositions are of the form

$$\begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{-1}u^{-1} - ke^{-2}u^{-2} \\ & 1 \end{pmatrix} \begin{pmatrix} (eu)^{-1} & \\ & eu \end{pmatrix} \begin{pmatrix} k & ke^{-1}u^{-1} - 1 \\ 1 & e^{-1}u^{-1} \end{pmatrix} \tag{4.53}$$

for arbitrary $k \in \mathbb{Z}_p$ and $e \in \mathbb{Z}_p^\times$. We note that since $|e|_p = 1$, the norms of the entries of the element $a \in A(\mathbb{Q}_p)$ is unambiguously defined even though the full Iwasawa decomposition is not unique. The relation (4.52) corresponds to $k = 0$ and $e = 1$. (One can render the p-adic Iwasawa decomposition unique by imposing further restrictions on the individual elements [131]. We will not use this here.)

It is illuminating to compare (4.52) with the decomposition of the analogous element in $SL(2,\mathbb{R})$. Thus, take

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \in SL(2,\mathbb{R}), \tag{4.54}$$

so that in this case $x \in \mathbb{R}$. The *unique* Iwasawa decomposition of this element is

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x}{1+x^2} \\ & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{1+x^2} & \\ & \sqrt{1+x^2} \end{pmatrix} k, \tag{4.55}$$

with

$$k = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 1 & -x \\ x & 1 \end{pmatrix} \in SO(2,\mathbb{R}). \tag{4.56}$$

Hence, the component along the Cartan torus in (4.52) is in fact simpler in the Iwasawa decomposition of $SL(2,\mathbb{Q}_p)$ compared with that of $SL(2,\mathbb{R})$.

4.2.2 Adelisation and strong approximation

We now discuss the central notion of strong approximation that allows the reformulation of many questions concerning $G(\mathbb{R})$ and its automorphic forms in terms of questions on the adelic group $G(\mathbb{A})$. The description in this section is general; the following section 4.2.3 gives more details for the case of $G = SL(2)$.

Given an algebraic group G defined over $\mathbb{Q}$ we can consider its adelisation $G(\mathbb{A})$ as the restricted direct product

$$G(\mathbb{A}) = G(\mathbb{R}) \times G_f, \tag{4.57}$$

where

$$G_f = \prod_{p<\infty}{}' G(\mathbb{Q}_p), \tag{4.58}$$

consisting of elements $g = (g_p) = (g_\infty; g_2, g_3, g_5, \dots)$ such that all but finitely many $g_p \in G(\mathbb{Z}_p)$. We further set

$$K_f = \prod_{p<\infty} G(\mathbb{Z}_p) \tag{4.59}$$

and we then have the notion of maximal compact subgroup $K_\mathbb{A}$ of $G(\mathbb{A})$ defined as

$$K_\mathbb{A} = K_\infty \times K_f, \tag{4.60}$$

where K_∞ is the maximal compact subgroup of $G(\mathbb{R})$. The adelic version of the Iwasawa decomposition thus reads

$$G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K_\mathbb{A}. \tag{4.61}$$

When G is split of rank r, the adelic Cartan torus is given by

$$A(\mathbb{A}) = GL(1,\mathbb{A}) \times \cdots \times GL(1,\mathbb{A}) \cong (\mathbb{A}^\times)^r. \tag{4.62}$$

Since $\mathbb{Q}$ is discrete in $\mathbb{A}$ according to proposition 3.24, it follows that $G(\mathbb{Q})$ is a discrete subgroup of $G(\mathbb{A})$. This implies that the arithmetic coset space $G(\mathbb{Q})\backslash G(\mathbb{A})$ corresponds to the adelisation of $G(\mathbb{Z})\backslash G(\mathbb{R})$. In fact, topologically $G(\mathbb{Q})\backslash G(\mathbb{A})$ is the total space of a fiber bundle over $G(\mathbb{Z})\backslash G(\mathbb{R})$ [120]:

$$\begin{array}{ccc} K_f & \hookrightarrow & G(\mathbb{Q})\backslash G(\mathbb{A}) \\ & & \downarrow \\ & & G(\mathbb{Z})\backslash G(\mathbb{R}) \end{array} \tag{4.63}$$

One way of stating *strong approximation* then asserts that

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/K_f. \tag{4.64}$$

This has the very useful consequence that any function $\phi_\mathbb{R}$ on $G(\mathbb{Z})\backslash G(\mathbb{R})$ can be lifted to a function $\phi_\mathbb{A}$ on the adelisation $G(\mathbb{Q})\backslash G(\mathbb{A})$ where $\phi_\mathbb{A}$ is characterized by being right-invariant under K_f. The consequences of this for automorphic forms will be discussed in chapter 5.

The strong approximation theorem (4.64) can be stated even more generally for open subgroups K_Γ of K_f according to [80] (see also [271, 273]):

Theorem 4.5 (Strong approximation theorem). *Let G be a topological group with $G(\mathbb{Q})$ dense in G_f, let K_Γ be an open subgroup of K_f and $\Gamma = K_\Gamma \cap G(\mathbb{Q})$. Then*

$$\begin{aligned} \phi : \Gamma\backslash G(\mathbb{R}) &\to G(\mathbb{Q})\backslash G(\mathbb{A})/K_\Gamma \\ \Gamma x_\infty &\mapsto G(\mathbb{Q})(x_\infty; \mathbb{1})K_\Gamma \end{aligned} \tag{4.65}$$

is a homeomorphism. Here, $G(\mathbb{Q})$ is diagonally embedded in $G(\mathbb{A})$; $G(\mathbb{R})$ is embedded as $(x_\infty; \mathbb{1})$ and K_Γ as $(\mathbb{1}; k_p)$.

An assumption in the theorem is that $G(\mathbb{Q})$ should be dense in G_f and this is equivalent to the statement that for all open subsets U of G_f we have that $U \cap G(\mathbb{Q}) \neq \emptyset$. An example of such a group G that will be useful for us is $SL(n)$ [57].

Proof.

- ϕ is well-defined (independent of coset representative)

  Let $x_\infty, y_\infty \in G(\mathbb{R})$ such that $\Gamma x_\infty = \Gamma y_\infty$, that is, there exists a $\gamma \in \Gamma$ such that $x_\infty = \gamma y_\infty$.

  We have that $\Gamma = K_\Gamma \cap G(\mathbb{Q})$. Denoting a double coset in $G(\mathbb{Q})\backslash G(\mathbb{A})/K_\Gamma$ by square brackets (with the real and finite places given separately) this leads to

$$\phi(\Gamma x_\infty) = [x_\infty; \mathbb{1}] = [\gamma y_\infty; \mathbb{1}] = [\gamma y_\infty; \gamma\gamma^{-1}] \overset{(a)}{=} [y_\infty; \gamma^{-1}] \overset{(b)}{=} [y_\infty; \mathbb{1}] = \phi(\Gamma y_\infty) \tag{4.66}$$

  where we have used (a) that $\gamma \in G(\mathbb{Q})$ and (b) that $\gamma^{-1} \in K_\Gamma$

- ϕ is injective

  Assume $\phi(\Gamma x_\infty) = \phi(\Gamma y_\infty)$. Then $G(\mathbb{Q})(x_\infty; \mathbb{1})K_\Gamma = G(\mathbb{Q})(y_\infty; \mathbb{1})K_\Gamma$, that is, there exists a $\gamma \in G(\mathbb{Q})$ and $k \in K_\Gamma$ such that $(x_\infty; \mathbb{1}) = \gamma(y_\infty; \mathbb{1})k = (\gamma y_\infty; \gamma k)$.

  This means that $x_\infty = \gamma y_\infty$ and $\gamma = k^{-1}$. Since $\gamma \in G(\mathbb{Q})$ and $k \in K_\Gamma$ we then have that $\gamma = k^{-1} \in K_\Gamma \cap G(\mathbb{Q}) = \Gamma$. Thus, $x_\infty = \gamma y_\infty$ implies that $\Gamma x_\infty = \Gamma y_\infty$.

- ϕ is surjective $\big(G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K_\Gamma\big)$

  Let $x = (x_\infty; x_f)$ be an arbitrary element in $G(\mathbb{A}) = G(\mathbb{R}) \times G_f$.

  We will now show that since $G(\mathbb{Q})$ is dense in G_f, there exists a $\gamma \in G(\mathbb{Q})$ such that $\gamma^{-1}x_f \in K_\Gamma$.

  Consider the continuous map $f : G_f \to G_f \; g \mapsto g x_f$. K_Γ is an open set around $\mathbb{1}$ in G_f. Since f is continuous $U = f^{-1}(K_\Gamma)$ is an open set in G_f around x_f^{-1}. Let $\gamma^{-1} \in U \cap G(\mathbb{Q})$ which is non-empty as $G(\mathbb{Q})$ is dense in G_f. Then $\gamma^{-1}x_f = f(\gamma^{-1}) \in f(U) = K_\Gamma$.

  Let $k = \gamma^{-1}x_f \in K_\Gamma$. Then,

$$x = (x_\infty; x_f) = (x_\infty; \gamma k) = \gamma(\gamma^{-1}x_\infty; k) = \gamma(\gamma^{-1}x_\infty; \mathbb{1})k \in G(\mathbb{Q})G(\mathbb{R})K_\Gamma \tag{4.67}$$

□

Remark 4.6. The generalisation to open subgroups K_Γ is important since it allows to treat different discrete subgroups Γ in a uniform way. Typically these subgroups are associated with arithmetically defined congruence subgroups.

4.2.3 Strong approximation for $SL(2, \mathbb{R})$

In this section, we illustrate the concepts of the preceding section in some examples involving $G = GL(2)$ and $G = SL(2)$.

Example 4.7: Discreteness of $GL(2,\mathbb{Q})$ in $GL(2,\mathbb{A})$

We will now show that $GL(2,\mathbb{Q})$ is discrete in $GL(2,\mathbb{A})$ by first considering the identity element. The line of reasoning is analogous to the case of $\mathbb{Q}$ being discretely embedded in $\mathbb{A}$ that was treated in proposition 3.24.

Let $U \subset GL(2,\mathbb{R})$ be an open neighbourhood of $\mathbb{1}$ such that $U \cap GL(2,\mathbb{Z}) = \{\mathbb{1}\}$. Then

$$V = U \times \prod_{p<\infty} GL(2,\mathbb{Z}_p) \subset GL(2,\mathbb{A}) \tag{4.68}$$

is an open neighbourhood of $\mathbb{1}$ in $GL(2,\mathbb{A})$.

Since, with the diagonal embedding,

$$GL(2,\mathbb{Q}) \cap \prod_{p<\infty} GL(2,\mathbb{Z}_p) = GL(2,\mathbb{Z}) \tag{4.69}$$

we then have that $GL(2,\mathbb{Q}) \cap V = \{\mathbb{1}\}$. For an arbitrary element $g \in GL(2,\mathbb{Q})$ these arguments generalise directly by instead considering gV. We have then that $GL(2,\mathbb{Q})$ is discrete in $GL(2,\mathbb{A})$.

Perhaps more importantly for our further calculations, it can similarly be shown that $SL(2,\mathbb{Q})$ is discrete in $SL(2,\mathbb{A})$.

The next example discusses how the strong approximation theorem 4.5 works for $SL(2)$ and different choices of subgroup Γ.

Example 4.8: Strong approximation for $SL(2)$

Let $G = SL(2)$. Firstly, let $K_\Gamma = K_f = \prod_{p<\infty} G(\mathbb{Z}_p)$. Then $\Gamma = K_\Gamma \cap G(\mathbb{Q}) = G(\mathbb{Z}) = SL(2,\mathbb{Z})$, the standard modular group. From the above theorem we then get that

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/K_f. \tag{4.70}$$

The second example addresses the principal congruence subgroup $\Gamma_0(N)$. Let locally

$$\Gamma_0(N)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_p) : c \equiv 0 \mod N\mathbb{Z}_p \right\} \tag{4.71}$$

and $K_\Gamma(N) = K_0(N) := \prod_{p<\infty} K_\Gamma(N)_p$ where

$$K_\Gamma(N)_p = \begin{cases} SL(2,\mathbb{Z}_p) & p \nmid N \\ \Gamma_0(N)_p & p \mid N \end{cases} \tag{4.72}$$

Since $K_\Gamma \subset K_f = \prod_{p<\infty} SL(2,\mathbb{Z}_p)$ we know that $\Gamma = K_\Gamma \cap SL(2,\mathbb{Q}) \subset SL(2,\mathbb{Z})$.

That $c \equiv 0 \mod N\mathbb{Z}_p$ for all divisors p of N means that (with $c \in \mathbb{Z}$)

$$\begin{aligned} c \in N\mathbb{Z}_p \quad \forall p \mid N &\iff \left|\frac{c}{N}\right|_p \leq 1 \quad \forall p \mid N \\ &\iff \frac{c}{N} \text{ has no } p \text{ in the denominator} \quad \forall p \mid N \\ &\iff c \equiv 0 \mod N \end{aligned} \tag{4.73}$$

Thus $\Gamma = \Gamma_0(N)$ and from the above theorem

$$\Gamma_0(N)\backslash SL(2,\mathbb{R}) \cong SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})/K_\Gamma(N). \tag{4.74}$$

We finally exhibit a isomorphism of cosets of the discrete subgroups in $G(\mathbb{A})$ with cosets of discrete subgroups in $G(\mathbb{R})$. This will be central for the adelic lift of Eisenstein series in section 5.1.6.

Example 4.9: Bijection of Borel cosets

In this example we will (based on the notes of [113]) show that

$$\begin{aligned} \phi : B(\mathbb{Z})\backslash SL(2,\mathbb{Z}) &\to B(\mathbb{Q})\backslash SL(2,\mathbb{Q}) \\ B(\mathbb{Z})\gamma &\mapsto B(\mathbb{Q})\gamma \end{aligned} \tag{4.75}$$

is an isomorphism, where

$$B(\mathbb{F}) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \cap SL(2,\mathbb{F}). \tag{4.76}$$

The mapping is well-defined since if $B(\mathbb{Z})\gamma' = B(\mathbb{Z})\gamma$ then $B(\mathbb{Q})\gamma' = B(\mathbb{Q})\gamma$ as $B(\mathbb{Z}) \subset B(\mathbb{Q})$.

It is injective because if $B(\mathbb{Q})\gamma' = B(\mathbb{Q})\gamma$ then there exists a b in $B(\mathbb{Q})$ such that $\gamma' = b\gamma$, but then $b = \gamma'\gamma^{-1} \in SL(2,\mathbb{Z})$ which means that $b \in B(\mathbb{Q}) \cap SL(2,\mathbb{Z}) = B(\mathbb{Z})$. Thus, $B(\mathbb{Z})\gamma' = B(\mathbb{Z})\gamma$.

For the surjectivity we need to show that every $B(\mathbb{Q})g$ with $g \in SL(2,\mathbb{Q})$ has a representative in $SL(2,\mathbb{Z})$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Q}) \qquad b = \begin{pmatrix} q & m \\ 0 & q^{-1} \end{pmatrix} \in SL(2,\mathbb{Q}) \qquad bg = \begin{pmatrix} qa+mc & qb+md \\ q^{-1}c & q^{-1}d \end{pmatrix} \tag{4.77}$$

where $c = c_1/c_2$ and $d = d_1/d_2$ with $c_i, d_i \in \mathbb{Z}$ in shortened form with positive denominators. Now set $q = \gcd(c_1 d_2, c_2 d_1)/(c_2 d_2)$ which makes $q^{-1}c$ and $q^{-1}d$ coprime integers, and thus there exist integers α and β such that $\alpha q^{-1} d - \beta q^{-1} c = 1$ by Bézout's lemma.

If $c = 0$ then $d \neq 0$, $a = 1/d$ and $q = \gcd(0, c_2 d_1)/(c_2 d_2) = |c_2 d_1|/(c_2 d_2) = |d|$ meaning that $qa = q^{-1}d = \pm 1$ and we may choose m such that $qb + md$ is integer. On the other hand, if $c \neq 0$ we may choose $m = (\alpha - qa)/c$ giving $qa + mc = \alpha$ and $qb + md = \beta$ which are both integers. This completes the proof.

Chapter 5

Automorphic forms and representation theory

In this section we explain how to think about automorphic forms as functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$, as opposed to the more familiar concept of $G(\mathbb{Z})$-invariant functions on a real Lie group $G(\mathbb{R})$. This leads naturally to the notion of automorphic representations and the close connection with studying the unitary action of $G(\mathbb{A})$ on the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We discuss in detail the theory of Eisenstein series from the point of view of representations induced from parabolic subgroups $P(\mathbb{A}) \subset G(\mathbb{A})$. As a warmup we begin in section 5.1 by discussing the passage from classical modular forms on the upper-half plane $\mathbb{H}$ to automorphic forms on the adelic group $SL(2, \mathbb{A})$. This will serve to illustrate some of the main points in a simple and explicit setting.

5.1 From classical modular forms to (adelic) automorphic forms

In this section we show how to pass from the classical notion of a modular form as a function on the complex upper-half plane $\mathbb{H}$, to an automorphic form as a function on a Lie group G. Here we focus on the example of $SL(2)$, leaving the generalisation to higher rank groups to subsequent sections. We begin by going from $\mathbb{H}$ to $SL(2, \mathbb{R})$ and then further to the adelic group $SL(2, \mathbb{A})$.

5.1.1 Holomorphic modular forms

Let $\mathbb{H}$ be the complex upper-half plane $\{z = x + iy \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. This carries an action of $SL(2, \mathbb{R})$ given by the Möbius transformation

$$z \to g \cdot z = \frac{az + b}{cz + d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \tag{5.1}$$

The appearance of the $SL(2,\mathbb{R})$-action is very natural since we have in fact an isomorphism $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$, where $SO(2,\mathbb{R}) \subset SL(2,\mathbb{R})$ is the stabiliser of the point $i \in \mathbb{H}$. This is reviewed in appendix A.

A *holomorphic modular form* of weight $w \geq 0$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which transforms according to

$$f\left(\frac{az+d}{cz+d}\right) = (cz+d)^w f(z), \tag{5.2}$$

under the discrete action of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{5.3}$$

If $f(z)$ has zero weight, $w = 0$, we call it a *modular function*. The prefactor $(cz+d)^w$ in (5.2) is often referred to as *factor of automorphy*. The defining eq. (5.2) implies that f is periodic $f(z+1) = f(z)$ (for any weight w) and thus has a Fourier expansion of the form

$$f(z) = \sum_{n\in\mathbb{Z}} a(n) q^n, \qquad q := e^{2\pi i z}. \tag{5.4}$$

Decomposing $q = e^{2\pi i z} = e^{2\pi i x} e^{-2\pi y}$ the Fourier coefficients can be computed from the standard Fourier transform

$$a(n) e^{-2\pi n y} = \int_0^1 e^{-2\pi i n x} f(x+iy) dx. \tag{5.5}$$

This formula (and its generalisations) will play a key role in subsequent chapters.

The moderate growth condition mentioned in section 1.1 can be formulated as the statement that

$$|f(x+iy)| \leq C \cdot y^N \tag{5.6}$$

for some constants C, N as $y \to \infty$ for any $x \in \mathbb{R}$. For holomorphic modular forms this is in fact equivalent to the statement that all negative Fourier coefficients $a(n), n < 0$, in (5.4) vanish. To see this we simply use the integral representation (5.5) for the Fourier coefficient and calculate its norm:

$$|a(n) e^{-2\pi n y}| = \left| \int_0^1 e^{-2\pi i n x} f(x+iy) dx \right| \tag{5.7}$$

Removing the oscillating exponential we obtain a sequence of inequalities

$$\left| \int_0^1 e^{-2\pi i n x} f(x+iy) dx \right| \leq \int_0^1 |f(x+iy)|\, dx \leq \int_0^1 C \cdot y^N dx = C \cdot y^N. \tag{5.8}$$

Thus we arrive at the inequality

$$|a(n) e^{-2\pi n y}| \leq C \cdot y^N. \tag{5.9}$$

and when $n < 0$ the exponential $e^{-2\pi n y}$ blows up as $y \to \infty$ so therefore we must have $a(n) = 0$ for $n < 0$ as claimed.

Example 5.1: Classical holomorphic Eisenstein series

Classic examples of holomorphic modular forms on $\mathbb{H}$ are provided by the *holomorphic Eisenstein series* defined by

$$E_{2w}(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{1}{(cz+d)^{2w}}. \tag{5.10}$$

One can check that this satisfies all the criteria stated above for integral $w \geq 2$. The (finite-dimensional) space $\mathcal{M}_{2w}(SL(2,\mathbb{Z})$ of weight $2w$ holomorphic modular forms is a ring, famously generated by the Eisenstein series $E_4(z)$ and $E_6(z)$ (see, e.g., [323] for a proof). The Fourier expansions of $E_4(z)$ and $E_6(z)$ are given by

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + \cdots \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 + \cdots \end{aligned} \tag{5.11}$$

where

$$\sigma_s(n) = \sum_{d|n} d^s \tag{5.12}$$

is the divisor function. For proofs see for example the classic book by Serre [287].

5.1.2 Modular forms for congruence subgroups*

It is often of interest in number theory to consider holomorphic modular forms for congruence subgroups $\Gamma \subset SL(2,\mathbb{Z})$ (a good reference is the book by Diamond and Shurman [86]). These satisfy an analogous relation to (5.2) but with extra restrictions on the transformation matrix and possibly with a character appearing on the right hand side. Consider for example the congruence subgroups

$$\begin{aligned} \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, N|c, a-1, d-1 \right\} \\ \Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, N|c \right\}, \end{aligned} \tag{5.13}$$

where $\Gamma_0(N)$ contains $\Gamma_1(N)$ as a normal subgroup of finite index $\phi(N)$ (the Euler totient function). The space of weight w modular forms for $\Gamma_1(N)$ (resp. $\Gamma_0(N)$) is then denoted by $\mathcal{M}_w(\Gamma_1(N))$ (resp. $\mathcal{M}_w(\Gamma_0(N))$). Since the quotient $\Gamma_0(N)/\Gamma_1(N)$ is isomorphic to the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$ of order $\phi(N)$, one can relate modular forms on these different congruence subgroups through the introduction of Dirichlet characters. A *Dirichlet character* χ is a group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \tag{5.14}$$

where the product between two Dirichlet characters χ_1 and χ_2 is defined by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for $g \in (\mathbb{Z}/N\mathbb{Z})^\times$. One can then decompose the space $\mathcal{M}_w(\Gamma_1(N))$ in terms of modular forms for the larger group $\Gamma_0(N)$, at the expense of introducing Dirichlet characters:

$$\mathcal{M}_w(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_w(\Gamma_0(N), \chi), \tag{5.15}$$

where functions in the χ-eigenspace $\mathcal{M}_w(\Gamma_0(N), \chi)$ obey a generalisation of (5.2):

$$f\left(\frac{az+d}{cz+d}\right) = \chi(d)(cz+d)^w f(z), \qquad f \in \mathcal{M}_w(\Gamma_0(N), \chi). \tag{5.16}$$

Functions in $\mathcal{M}_w(\Gamma_0(N), \chi)$ are said to be of *level* N.

Example 5.2: Classical level N Eisenstein series

An example of a modular form for $\Gamma_0(N)$ is provided by the level N, weight $2w$ Eisenstein series

$$E_{2w}(z;\chi) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)=1}} \frac{\chi(n)}{(mz+n)^{2w}}, \tag{5.17}$$

generalising the classical series (5.10).

5.1.3 From holomorphic modular forms to automorphic forms on $SL(2,\mathbb{R})$

We shall now see how to adapt the theory of holomorphic modular forms on $\mathbb{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$ to the more general framework of automorphic forms defined on $G = SL(2,\mathbb{R})$, invariant under the left action of $SL(2,\mathbb{Z})$.

Given a weight w holomorphic modular form $f : \mathbb{H} \to \mathbb{C}$ we define a new (complex) function on $SL(2,\mathbb{R})$ through the assignment

$$f \longmapsto \varphi_f(g) = (ci+d)^{-w} f(g\cdot i), \tag{5.18}$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. The prefactor here is chosen in such away as to cancel the factor of automorphy in (5.2) in order for the function φ_f to be *invariant* under $SL(2,\mathbb{Z})$:

$$\varphi_f(\gamma g) = \varphi_f(g), \qquad \gamma \in SL(2,\mathbb{Z}). \tag{5.19}$$

According to our definition in section 1.1 φ_f is thus an automorphic function on $SL(2,\mathbb{R})$. Note that the condition of moderate growth is satisfied automatically since the seed function f is holomorphic.

We can make the *lift* (5.18) *from* $\mathbb{H}$ *to* $SL(2,\mathbb{R})$ more explicit by making use of the Iwasawa decomposition of an element $g \in SL(2,\mathbb{R})$:

$$g = nak = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{5.20}$$

with $n \in N(\mathbb{R}), a \in A(\mathbb{R}), k \in SO(2,\mathbb{R})$. Acting with g on the point i one finds then

$$g\cdot i = x + iy \equiv z. \tag{5.21}$$

We recall that $K = SO(2,\mathbb{R})$ leaves the point i invariant. Plugging the Iwasawa decomposition of g into the right-hand side of (5.18) we can write φ_f as a function of the three variables (x, y, θ):

$$\varphi_f(g) = \varphi_f(x, y, \theta) = e^{iw\theta} y^{w/2} f(x + iy). \tag{5.22}$$

Moreover, under the right-action of

$$k = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in SO(2,\mathbb{R}) \tag{5.23}$$

it transforms by a phase:

$$\varphi_f(gk) = e^{iw\vartheta}\varphi_f(g). \tag{5.24}$$

This implies that the original transformation property (5.2) of f under $SL(2,\mathbb{Z})$ has been traded for the above phase transformation of $\varphi_f(g)$ under $K = SO(2,\mathbb{R})$. While f itself was invariant under $SO(2,\mathbb{R})$ one instead says that φ_f is *K-finite*, implying that the action of K on f generates a finite-dimensional vector space; in the present example this is represented by the one-dimensional space of characters $\sigma : k \mapsto e^{iw\vartheta}$ through

$$\varphi_f(gk) = \sigma(k)\varphi_f(g). \tag{5.25}$$

Next, we address the question how the automorphic form φ_f incorporates the holomorphy of f on $\mathbb{H}$:

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0, \tag{5.26}$$

where $z = x + iy$. The corresponding statement for φ_f is that it satisfies

$$F\varphi_f = -2ie^{-2i\theta}\left(y\frac{\partial}{\partial \bar{z}} - \frac{1}{4}\frac{\partial}{\partial \theta}\right)\varphi_f = 0. \tag{5.27}$$

We will now give a group-theoretic interpretation to this differential condition. The space of smooth functions φ on $G = SL(2,\mathbb{R})$ is acted upon by the *right-regular action* of $SL(2,\mathbb{R})$ denoted by ρ

$$\left(\rho(g)\varphi\right)(h) = \varphi(hg) \qquad \text{for } g \in SL(2,\mathbb{R}). \tag{5.28}$$

For smooth functions one can linearise the action and obtain a realisation of the $\mathfrak{sl}(2,\mathbb{R})$ generators in terms of (linear) differential operators. The derivation of these operators is reviewed in appendix A.3. Using the expressions from there one recognises that the particular operator annihilating φ_f in (5.27) corresponds to the generator F in the so-called compact realisation of $\mathfrak{sl}(2,\mathbb{R})$. The three basis generators in this realisation are given by

$$H = i\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \tag{5.29}$$

and satisfy

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H. \tag{5.30}$$

This basis is unitarily equivalent to the standard real basis of (2×2)-matrices, see appendix A for details.

Thus, the differential operator F in (5.27) may in fact be identified with the lowering operator in the basis (E, F, H) of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. This implies that φ_f may be viewed as a lowest weight state of a representation of $\mathfrak{sl}(2, \mathbb{R})$. Furthermore, we note that the (Hermitian) generator H in this basis corresponds to

$$e^{i\theta H} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2, \mathbb{R}), \tag{5.31}$$

and hence corresponds to the differential operator $H = -i\partial_\theta$. Therefore, H is diagonal on φ_f with eigenvalue w:

$$H\varphi_f = w\varphi_f. \tag{5.32}$$

From the commutation relations we further deduce that E raises the H-eigenvalue w by $+2$, while F lowers it by the same amount. This implies that the holomorphic Eisenstein series E_{2w} can be viewed as lowest weight vectors in the holomorphic discrete series of $SL(2, \mathbb{R})$, providing our first glimpse of the general connection between automorphic forms and representation theory, a topic that will be discussed in more generality in section 5.4 and onwards. See also section 5.5 for some more details on the specific case of $SL(2)$ treated above.

Before we proceed we shall mention one final important property of φ_f, namely that it is an eigenfunction of the Laplacian on $SL(2, \mathbb{R})$. This Laplacian is derived in appendix A.3 and reads

$$\Delta_{SL(2,\mathbb{R})} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta}. \tag{5.33}$$

Acting on φ_f one obtains that

$$\Delta_{SL(2,\mathbb{R})}\varphi_f = \frac{w}{2} \left(\frac{w}{2} - 1 \right) \varphi_f. \tag{5.34}$$

As we will see, all the properties of φ_f discussed above will have counterparts in the general theory of automorphic forms.

The automorphic lift of weight w, level N modular forms for $\Gamma_0(N)$ will be treated in section 5.2.1.

Example 5.3: Lift of a holomorphic Eisenstein series

The lift of the holomorphic Eisenstein series $E_{2w}(z)$ to an automorphic form on $SL(2, \mathbb{R})$ can be written explicitly as follows. For f a weight w holomorphic modular form, $g \in SL(2, \mathbb{R})$, define the *slash operator* $f|_w g : \mathbb{H} \to \mathbb{C}$ by

$$(f|_w g)(z) := (cz + d)^{-w} f \left(\frac{az + b}{cz + d} \right), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}). \tag{5.35}$$

Using the slash operator the defining relation (5.2) can be written simply as

$$(f|_w \gamma)(z) = f(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{5.36}$$

By a calculation similar to the one used in proving equation (1.7) we can now rewrite the Eisenstein series $E_{2w}(z)$ in (5.10) directly as a function on $SL(2,\mathbb{R})$. Parametrising an element $g \in SL(2,\mathbb{R})$ in Iwasawa form as $g = nak$ (see 5.20) we obtain

$$\varphi_f(nak) = (f|_w nak)(i) = e^{iw\theta} y^{w/2} \sum_{\gamma \in N(\mathbb{Z})\backslash SL(2,\mathbb{Z})} (1|_w\gamma)(nak \cdot i). \tag{5.37}$$

Remark 5.4. We would like to make a cautionary remark regarding the generalisation of the above discussion to arbitrary groups $G(\mathbb{R})$. Modular forms are holomorphic functions on $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$ with simple transformation properties under $SL(2,\mathbb{Z})$, and it seems natural to try and generalise this construction to higher rank real Lie groups $G(\mathbb{R})$. One might suspect a generalisation to holomorphic functions $f : G(\mathbb{R})/K \to \mathbb{C}$, where K is the maximal compact subgroup of $G(\mathbb{R})$, transforming with some weight under the action of a discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$. However, this only works whenever the coset $G(\mathbb{R})/K$ carries a complex structure. In the case above this complex structure is provided by the fact that the maximal compact subgroup $K = SO(2,\mathbb{R}) \cong U(1)$. In general, the maximal subgroup K of some $G(\mathbb{R})$ does not have an isolated $U(1)$ factor that can provide a complex structure on $G(\mathbb{R})/K$ and therefore we could not expect to have a general theory of holomorphic modular forms on G/K. A standard example with a complex structure is provided by $G = Sp(2n,\mathbb{R})$, $K = U(n)$, in which case $Sp(2n;\mathbb{R})/U(n)$ is a hermitian symmetric domain known as the Siegel upper half space. This leads to the theory of holomorphic Siegel modular forms (see, e.g., [54] for a review).

5.1.4 Maass forms and non-holomorphic Eisenstein series

As just discussed, it is in general too restrictive (and often impossible) to consider holomorphic modular forms. It is therefore called for to look for a theory of arbitrary (*non-holomorphic*) functions $f : G(\mathbb{R})/K \to \mathbb{C}$ which transform nicely under the action of some discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$. This leads to the notion of an *automorphic form* that we will now discuss for $SL(2,\mathbb{R})$.

In addition to the holomorphic modular forms, the classical theory also contains an interesting class of *non-holomorphic functions* $f : SL(2,\mathbb{Z})\backslash\mathbb{H} \to \mathbb{R}$. These non-holomorphic functions are eigenfunctions of the Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H} = SL(2,\mathbb{R})/SO(2,\mathbb{R})$ (that is simply obtained from (5.33) since $\partial_\theta = 0$ on $\mathbb{H}$):

$$\Delta_{\mathbb{H}} f = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = \lambda f \tag{5.38}$$

and by definition are *invariant* under $SL(2,\mathbb{Z})$:

$$f(\gamma \cdot z) = f(z); \tag{5.39}$$

there is no non-trivial weight w compared to (5.2). Similarly, to the holomorphic case we require that $f(z)$ is of moderate growth, i.e. that it grows at most polynomially for $y \to \infty$ (see (5.6)).

Functions on $SL(2,\mathbb{R})$ satisfying these conditions are called *Maass (wave) forms*, and they can also be fit into the general framework of automorphic forms, with even less effort than for the holomorphic modular forms. Given a Maass form f on $\mathbb{H}$ we lift this to a function φ_f on $SL(2,\mathbb{R})$ according to (5.22)

$$f \longmapsto \varphi_f(g) = \varphi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k\right) = f(x+iy), \tag{5.40}$$

where we used the Iwasawa decomposition $g = nak \in SL(2,\mathbb{R})$ given in eq. (5.159). The lift in this case is trivial since $w = 0$.

The associated function $\varphi_f(g)$ then satisfies

$$\varphi_f(\gamma g k) = \varphi_f(g), \qquad \gamma \in SL(2,\mathbb{Z}), \quad k \in SO(2,\mathbb{R}), \tag{5.41}$$

and so is indeed an automorphic form on $SL(2,\mathbb{R})$.

Important examples of Maass forms are provided by the *non-holomorphic Eisenstein series* with parameter $s \in \mathbb{C}$

$$E(s,z) = \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}}. \tag{5.42}$$

This converges absolutely for $\mathrm{Re}(s) > 1$, but according to Langlands general theory [218] it can be analytically continued to a meromorphic function of $s \in \mathbb{C}\backslash\{0,1\}$. This crucial fact relies on the *functional relation*

$$\xi(s)E(s,z) = \xi(1-s)E(1-s,z), \tag{5.43}$$

where $\xi(s)$ is the completed Riemann zeta function (1.17).

One can verify that the Eisenstein series $E(s,z)$ indeed defines an $SL(2,\mathbb{Z})$-invariant eigenfunction of the Laplacian $\Delta_{\mathbb{H}}$ with eigenvalue $\lambda = s(s-1)$. The non-holomorphic Eisenstein series $E(s,z)$ provides the simplest example of the class of Eisenstein series on a group $G(\mathbb{R})$ that will be our main concern in the following.

It is instructive to rewrite $E(s,z)$ as defined in (5.42). We parametrise an arbitrary group element $g \in SL(2,\mathbb{R})$ according to the same Iwasawa decomposition $g = nak$ as in (5.20). Then introduce a character $\chi_s : B = NA \to \mathbb{C}^\times$ defined by

$$\chi_s(na) = y^s, \qquad n \in N,\ a \in A, \tag{5.44}$$

and extend it to all of $SL(2,\mathbb{R})$ by requiring it to be trivial on $SO(2,\mathbb{R})$: $\chi_s(nak) = \chi_s(na)$. The Eisenstein series $E(s,\tau)$ can now be equivalently written as

$$E(s,g) = \sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi_s(\gamma g), \tag{5.45}$$

where the quotient by the discrete Borel subgroup $B(\mathbb{Z}) = \{\left(\begin{smallmatrix} \pm 1 & m \\ & \pm 1 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\}$ is needed since it leaves χ_s invariant. (This was also explained in the introduction.) It should be apparent that this reformulation of the Eisenstein series is well suited for generalisations to higher rank Lie groups $G(\mathbb{R})$. This will be discussed in section 5.3 below.

5.1.5 Maass forms of non-zero weight*

One can generalise the definition of Maass form given above to include non-holomorphic functions which transform with a weight. We define a *weight w Maass form* to be a non-holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+d}{cz+d}\right) = \left(\frac{cz+d}{|cz+d|}\right)^w f(z), \qquad w \in \mathbb{Z}. \tag{5.46}$$

A weight w Maass form is furthermore an eigenfunction of the weight w Laplacian Δ_w which is a modification of (5.38):

$$\Delta_{\mathbb{H}}^w := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iwy\frac{\partial}{\partial x}. \tag{5.47}$$

We can elucidate the meaning of this differential operator by lifting the weight w Maass form f to an automorphic form φ_f on $SL(2,\mathbb{R})$ through a straightforward generalisation of (5.40):

$$f \longmapsto \varphi_f(g) := \left(\frac{ci+d}{|ci+d|}\right)^{-w} f(g\cdot i), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{5.48}$$

Rewriting this in Iwasawa form (5.20) yields

$$\varphi_f(g) = \varphi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k\right) = e^{iw\theta} f(x+iy). \tag{5.49}$$

We then recognise the weight w Laplacian $\Delta_{\mathbb{H}}^w$ in (5.47) as nothing by the full Laplacian on $SL(2,\mathbb{R})$ (5.33) after evaluating the derivative on θ:

$$\Delta_{SL(2,\mathbb{R})}\varphi_f(g) = e^{iw\theta}\left[y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iwy\frac{\partial}{\partial x}\right] f(x+iy) = e^{iw\theta}\Delta_{\mathbb{H}}^w f(z). \tag{5.50}$$

Example 5.5: Non-holomorphic Eisenstein series of weight w

A classic example of a weight w Maass form is the following generalisation of the non-holomorphic Eisenstein series (5.42):

$$E_w(s,z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}}\left(\frac{cz+d}{c\bar{z}+d}\right)^w, \tag{5.51}$$

which transforms as

$$E_k\left(s, \frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{w/2} E_k(s,z) = \left(\frac{cz+d}{|cz+d|}\right)^w E_k(s,z). \tag{5.52}$$

We will come back to this Eisenstein series in section 5.5.

5.1.6 Adelisation of non-holomorphic Eisenstein series

As we discussed in section 4.2.2, strong approximation ensures that we can always lift a function on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ to an adelic function on $SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})$, where the role of the discrete subgroup is now played by $SL(2,\mathbb{Q})$. Recall that this lift also requires that the resulting function is right-invariant under $K_f = \prod_{p<\infty} SL(2,\mathbb{Z}_p)$. It is now a simple matter to generalise the Eisenstein series $E(s,g)$ to such an adelic function. First extend the definition of χ_s to a function $\chi_s : B(\mathbb{A}) \to \mathbb{C}$, which is invariant under the left action of $B(\mathbb{Q})$. We extend it to all of $SL(2,\mathbb{A})$ using the global Iwasawa decomposition $SL(2,\mathbb{A}) = B(\mathbb{A})K_{\mathbb{A}}$ and demanding it to be trivial on $K_{\mathbb{A}} = SO(2,\mathbb{R}) \times K_f$. Note that this automatically takes care of the required condition of K_f-invariance on the right. The adelic Eisenstein series then takes the form

$$E(s, g_{\mathbb{A}}) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \chi_s(\gamma g_{\mathbb{A}}), \tag{5.53}$$

which is a function on $SL(2,\mathbb{A})$ satisfying

$$E(s, \gamma g_{\mathbb{A}} k_{\mathbb{A}}) = E(s, g_{\mathbb{A}}), \qquad \gamma \in SL(2,\mathbb{Q}), \quad k_{\mathbb{A}} \in K_{\mathbb{A}}. \tag{5.54}$$

As shown in Example 4.9, the range of the sum in (5.53) is in fact in bijection with the range of summation in (5.45):

$$B(\mathbb{Q})\backslash SL(2,\mathbb{Q}) \cong B(\mathbb{Z})\backslash SL(2,\mathbb{Z}). \tag{5.55}$$

Therefore, if we restrict to elements $g_{\mathbb{A}} = (g_\infty; 1, 1, \dots) \in SL(2,\mathbb{A})$, with $g_\infty \in SL(2,\mathbb{R})$, then the adelic Eisenstein series reduces to the real Eisenstein series (5.45). More details of this procedure can be found in section 5.5.

With a little more effort one can also obtain the adelisation of the function $\varphi_f(g)$ in (5.18), for f a weight w holomorphic modular form on $\mathbb{H}$. This analysis is done in Example 5.2.1.

Even though our main interest often lies with automorphic forms on real Lie groups $G(\mathbb{R})$, the adelic reformulation turns out to be extremely convenient for many purposes, not the least of which being the calculational advantages that it brings when computing Fourier expansions of automorphic forms, a topic which we will be concerned with in the second half of these notes.

5.2 Adelic automorphic forms

We shall now give the definition of an automorphic form in the adelic framework. There are various degrees of generality here; for instance, one can define the theory of automorphic forms over the adeles $\mathbb{A}_{\mathbb{F}}$ of an arbitrary number field $\mathbb{F}$. However, we shall continue to assume that $\mathbb{F} = \mathbb{Q}$ in what follows. The framework of adelic automorphic forms was originally developed in the books by Gelfand-Graev-Piatetski-Shapiro [120], and Jacquet–Langlands [176]. Good introductions can be found in the books by Gelbart [117], Bump [57] and Goldfeld-Hundley [131, 132].

In what follows we let G be a split algebraic group defined over $\mathbb{Q}$ and $G(\mathbb{A})$ its adelisation. The typical example we have in mind is $G(\mathbb{A}) = SL(n, \mathbb{A})$. Let us now state our definition of an automorphic form:

Definition 5.6 (Automorphic form). An *automorphic form* is a smooth function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:

1. *left $G(\mathbb{Q})$-invariance:* $\varphi(\gamma g) = \varphi(g), \qquad \gamma \in G(\mathbb{Q})$,
2. *right K-finiteness:* $\dim_{\mathbb{C}}\langle \varphi(gk) | k \in K_{\mathbb{A}} \rangle < \infty$,
3. *$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$-finiteness:* $\dim\langle X\varphi(g) | X \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rangle < \infty$,
4. *φ is of moderate growth:* for any norm $|| \cdot ||$ on $G(\mathbb{A})$ there exists a positive integer n and a constant C such that $|\varphi(g)| \leq C||g||^n$.

Remark 5.7. We denote by $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ the space of automorphic forms satisfying definition 5.6. This is a subspace of the space $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of *smooth* functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. An adelic function $\varphi(g)$, with $g = (g_\infty; g_f) \in G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$, is said to be smooth if it is C^∞ with respect to the archimedean variables $g_\infty \in G(\mathbb{R})$ and locally constant with respect to the non-archimedean variables $g_f \in G(\mathbb{A}_f)$.

Let us now elaborate a little on the definition 5.6 of an adelic automorphic form:

- Condition (1) ensues as a straightforward generalisation of invariance of the function under a discrete subgroup of $G(\mathbb{A})$.
- The condition of right K-finiteness means that the vector space V spanned by the functions $k \mapsto \varphi(gk)$, $k \in K_{\mathbb{A}}$, is finite-dimensional. We have already seen an example of a non-trivial K representation in (5.25). When φ is K-invariant on the right, it would be more appropriate to refer to φ as an automorphic function rather than form but we will use the more general term.
- In condition (3), $\mathfrak{g}$ is the Lie algebra associated with the group G and $\mathcal{Z}(\mathfrak{g})$ is the center of its universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. The center $\mathcal{Z}(\mathfrak{g})$ is the space of bi-invariant differential operators on G, i.e. the quadratic Casimir and higher-order operators. The condition of $\mathcal{Z}(\mathfrak{g})$-finiteness then implies that φ is contained in a $\mathcal{Z}(\mathfrak{g})$-invariant finite dimensional subspace of $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Equivalently, if $X \in \mathcal{Z}(\mathfrak{g})$ then $\mathcal{Z}(\mathfrak{g})$-finiteness implies that there exists a polynomial $R(X)$ such that $R(X)\varphi = 0$.

Remark 5.8. It is sometimes useful to specify the transformation properties of an automorphic form with respect to the center $Z(\mathbb{A})$ of $G(\mathbb{A})$. To this end, let ω be a *central character*, i.e. a homomorphism $\omega : Z(\mathbb{A}) \to \mathbb{C}^\times$, which is trivial on $Z(\mathbb{Q})$. An automorphic form f is then said to have central character ω if it satisfies conditions (1)-(4) along with the additional condition

5. $f(zg) = \omega(z)f(g)$.

We shall now illustrate these defining properties of an automorphic form, by giving two examples. First we look at the non-holomorphic Eisenstein series (5.53) and we will verify its properties according to the above definition.

Example 5.9: Verification of automorphic properties of an Eisenstein series on $SL(2,\mathbb{A})$

Consider the case when $G(\mathbb{A}) = SL(2,\mathbb{A})$. We now verify the conditions (1)-(4) in definition 5.6 for the non-holomorphic Eisenstein series defined in (5.53):

$$E(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \chi_s(\gamma g), \qquad g \in SL(2,\mathbb{A}). \tag{5.56}$$

- By construction, $E(s,g)$ is left $SL(2,\mathbb{Q})$-invariant and so satisfies condition (1).
- Moreover, by definition the function χ_s is invariant under any $k_p \in SL(2,\mathbb{Z}_p)$, $\chi_s(gk_p) = \chi_s(g)$, and hence condition (2) is also satisfied.
- To understand condition (3) we recall that $E(s,g)$ is an eigenfunction of the $\mathfrak{g}$-invariant Laplacian $\Delta_{\mathbb{H}}$ on $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ with eigenvalue $\lambda = s(s-1)$. Hence, $E(s,g)$ is in the kernel of the operator $(\Delta - \lambda) \in \mathcal{Z}(\mathfrak{g})$, and since for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{A})$, $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\Delta_{\mathbb{H}}]$, we have that condition (3) is satisfied.
- The final part consists in verifying the moderate growth condition (4). To this end one must translate the classical moderate growth condition (5.6) to the adelic picture. A norm $||\,||$ on $SL(2,\mathbb{A})$ can be defined as follows (see, e.g., [131]):

$$||g|| := \prod_{p\leq\infty} \max\left\{|a|_p, |b|_p, |c|_p, |d|_p, |ad-bc|_p^{-1}\right\}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.57}$$

where it is understood that $|a|_p = |a_p|_p$ etc., with a_p the p:th component of the adele $a = (a_\infty, a_2, a_3, \dots) \in \mathbb{A}$. For a proof that the the moderate growth condition of $E(s,g)$ with respect to this norm follows from the classical moderate growth on $SL(2,\mathbb{R})$, see p. 122-123 of [131].

Before we move on to analysing Eisenstein series on arbitrary reductive groups we shall give an additional important definition:

Definition 5.10 (Cusp form). An automorphic form $f \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a *cusp form* if for all parabolic subgroups $P(\mathbb{A}) \subset G(\mathbb{A})$ it satisfies

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)du = 0, \tag{5.58}$$

where U is the unipotent radical in the Levi decomposition $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$, and du is the left-invariant Haar measure on U. The subspace of cusp forms will be denoted by $\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A})) \subset \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

This definition generalises the notion of cusp form found in the classical theory, namely holomorphic modular forms $f(\tau)$ whose Fourier expansion in $q = e^{2\pi i\tau}$ contains no term of order q^0. An example is provided by Ramanujan's discriminant $\Delta(z)$ of weight $w = 12$.

The integral in (5.58) can be thought of as the zeroth Fourier coefficient of $f(g)$ with respect to U; by analogy with the classical theory it is called the "constant term" of $f(g)$, although in general it is by no means constant. From this perspective a cusp form is simply an automorphic form with vanishing constant term. Constant terms are analyzed in detail for $SL(2,\mathbb{A})$ in chapter 7 and in full generality in chapter 8.

5.2.1 Adelic lift of a holomorphic modular form with Hecke character*

We shall now construct the adelic lift of a holomorphic modular form f. To illustrate the power of the adelic formalism we will consider the general case addressed in section 5.1.2, namely let $f \in \mathcal{M}_w(\Gamma_0(N),\chi)$, i.e a level N holomorphic modular form for $\Gamma_0(N)$ with Dirichlet character χ. We can now use strong approximation (see sec. 4.2.2) to lift f to a function on $SL(2,\mathbb{A})$. Recall from section 4.2.2 (see in particular Example 4.8) that strong approximation implies that any $g \in SL(2,\mathbb{A})$ can be (non-uniquely) written as

$$g = \gamma g_\infty k_f, \qquad \gamma \in SL(2,\mathbb{Q}),\ g_\infty \in SL(2,\mathbb{R}),\ k_f \in K_0(N), \tag{5.59}$$

where $K_0(N) \subset K_f = \prod_{p<\infty} SL(2,\mathbb{Z}_p)$ was defined in Example 4.8. In order to define a lift to $SL(2,\mathbb{A})$ we must first lift the Dirichlet character χ to the adelic setting. This can be done using the $GL(1,\mathbb{A}) = \mathbb{A}^\times$-version of strong approximation:

$$\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_+ \prod_{p<\infty} \mathbb{Z}_p^\times . \tag{5.60}$$

This implies that any Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^\times$ has a canonical lift to an adelic (Hecke) character

$$\omega_\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times. \tag{5.61}$$

Indeed, such a character can be decomposed as

$$\omega_\chi = \omega_{\chi,\infty} \prod_{p<\infty} \omega_{\chi,p}, \tag{5.62}$$

where the archimedean factor $\omega_{\chi,\infty}$ is taken to be trivial, and each local factor $\omega_{\chi,p}$ equals the Hecke character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for N a power of the prime p.

Next, we lift the local character $\omega_{\chi,p}$ to a character on $SL(2,\mathbb{Z}_p)$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \omega_{\chi,p}(d)$. The adelic lift of the holomorphic modular form f is then defined by:

$$\varphi_f(g) := (ci+d)^{-w} f(g_\infty \cdot i) \omega_\chi(k_f), \tag{5.63}$$

where $g_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. We can also write this in terms of the slash operator used in Example 5.3:

$$\varphi_f(g) = (f|_w g_\infty)(i)\, \omega_\chi(k_f). \tag{5.64}$$

Having extended the definition of φ_f to an adelic automorphic form we wish to verify the conditions (1)-(4) of the definition:

- Condition (1) is satisfied by construction: $\varphi_f(\gamma g) = \varphi_f(g)$, for any $g \in SL(2,\mathbb{A})$ and $\gamma \in SL(2,\mathbb{Q})$.

- Condition (2), concerning right K-finiteness can be seen as follows. Finiteness under the non-archimedean K_f is a consequence of the relation
$$\varphi_f(gk_f) = \varphi_f(g)\omega_\chi(k_f) \tag{5.65}$$
while at the archimedean place we have
$$\varphi_f(gk_\infty) = \varphi_f(g)e^{iw\theta}, \tag{5.66}$$
where $k_\infty = k_\infty(\theta) \in SO(2,\mathbb{R})$ as in (5.31).
- $\mathcal{Z}(\mathfrak{g})$-finiteness (Condition 3) again follows from the fact that φ_f is an eigenfunction of the Laplacian:
$$\Delta\varphi_f = \frac{w}{2}\left(\frac{w}{2}-1\right)\varphi_f. \tag{5.67}$$
- Finally the condition of moderate growth (Condition 4) is satisfied if the coefficients $a(n)$ in the q-expansion of $f(\tau)$ satisfy $a(n) = 0$ whenever $n < 0$ which holds since f is holomorphic.

Finally, we shall see that φ_f is in fact an example of an automorphic form *with central character*, as in the supplementary Condition (5) mentioned in remark 5.8. To this end we must first view φ_f as a function on $GL(2,\mathbb{A})$ as opposed to $SL(2,\mathbb{A})$. The defining relation (5.63) is still valid for $g \in GL(2,\mathbb{A})$ and conditions (1)-(4) go through without change. Our aim is now to check how $\varphi_f(g)$ transforms under the non-trivial centre $Z(GL(2,\mathbb{A})) = \mathbb{A}^\times$. An element $z \in Z(GL(2,\mathbb{A}))$ can be represented by the diagonal matrix
$$z = \begin{pmatrix} r & \\ & r \end{pmatrix}, \qquad r \in \mathbb{A}. \tag{5.68}$$
Strong approximation then yields the decompositions
$$\begin{aligned} g &= \gamma g_\infty k_f, \qquad \gamma \in GL(2,\mathbb{Q}),\ g_\infty \in GL(2,\mathbb{R})^+,\ k_f \in K_0(N), \\ r &= \alpha\, r_\infty\, r_f, \qquad \alpha \in \mathbb{Q}^\times,\ r_\infty \in \mathbb{R}_+,\ r_f \in \prod_{p<\infty} \mathbb{Z}_p^\times, \end{aligned} \tag{5.69}$$
and consequently
$$zg = \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \gamma \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix} g_\infty \begin{pmatrix} r_f & \\ & r_f \end{pmatrix} k_f \in GL(2,\mathbb{A}). \tag{5.70}$$
We can now proceed to calculate the action of Z on the automorphic form φ_f:
$$\varphi_f(zg) = \left(f|_w \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix} g_\infty\right)(i)\, \omega_\chi\left(\begin{pmatrix} r_f & \\ & r_f \end{pmatrix} k_f\right). \tag{5.71}$$
To evaluate this we first notice that
$$\left(f|_w \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix}\right) = f, \tag{5.72}$$

and hence

$$\left(f|_w\begin{pmatrix}r_\infty & \\ & r_\infty\end{pmatrix}g_\infty\right)(i) = (f|_w g_\infty)(i). \tag{5.73}$$

Using the multiplicative property of the Hecke character we further have

$$\omega_\chi\left(\begin{pmatrix}r_f & \\ & r_f\end{pmatrix}k_f\right) = \omega_\chi\left(\begin{pmatrix}r_f & \\ & r_f\end{pmatrix}\right)\omega_\chi(k_f). \tag{5.74}$$

By definition the Hecke character ω_χ is trivial on $\mathbb{Q}^\times$ and at the archimedean place. Thus, using strong approximation we can write

$$\omega_\chi(r_f) = \omega_\chi(\alpha r_\infty r_f) = \omega_\chi(z), \qquad z \in Z(GL(2,\mathbb{A}). \tag{5.75}$$

Combining everything we then find

$$\varphi_f(zg) = \omega_\chi(z)\varphi_f(g), \tag{5.76}$$

verifying that φ_f is an automorphic form with central character $\omega = \omega_\chi$ as in remark 5.8.

5.3 Eisenstein series and multiplicative characters

We now want to generalise the construction of adelic Eisenstein series given in section 5.1 to arbitrary reductive groups $G(\mathbb{A})$. To this end we must first recall the process of constructing representations of G via induction from a standard parabolic subgroup $P \supset B$. In this section we shall take $P = B$, the Borel subgroup which is the minimal parabolic subgroup. The case of arbitrary (standard) parabolic subgroups will be treated in the subsection 5.6.

5.3.1 Adelic multiplicative characters

Fix a Borel subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ with Levi decomposition $B(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})$. Recall that since $G(\mathbb{A})$ is split, $A(\mathbb{A}) \cong (\mathbb{A}^\times)^{\operatorname{rank}\mathfrak{g}}$. Introduce a multiplicative character

$$\chi\,:\,B(\mathbb{Q})\backslash B(\mathbb{A}) \to \mathbb{C}^\times, \tag{5.77}$$

defined by

$$\chi(na) = \chi(a), \qquad\qquad n \in N(\mathbb{A}),\, a \in A(\mathbb{A}). \tag{5.78}$$

Using the Iwasawa decomposition we can extend χ to all of $G(\mathbb{A})$ by demanding it to be trivial on $K_\mathbb{A}$:

$$\chi(g) = \chi(nak) = \chi(na) = \chi(an) = \chi(a), \qquad k \in K_\mathbb{A}. \tag{5.79}$$

Although we extend the character to all of $G(\mathbb{A})$ it is only multiplicative on $B(\mathbb{A})$:

$$\chi(bb') = \chi(b)\chi(b') = \chi(a)\chi(a'), \qquad b, b' \in B(\mathbb{A}). \tag{5.80}$$

On the other hand, to evaluate it on a product of two elements $g, g' \in G(\mathbb{A})$ we have

$$\chi(gg') = \chi(bkb'k') = \chi(bkb') = \chi(b\tilde{b}\tilde{k}) = \chi(b\tilde{b}) = \chi(b)\chi(\tilde{b}), \tag{5.81}$$

where $\tilde{b}\tilde{k}$ is the Iwasawa decomposition of kb'. From this we see also

$$\chi(bg) = \chi(b)\chi(g), \qquad b \in B(\mathbb{A}), g \in G(\mathbb{A}). \tag{5.82}$$

The global character splits into an Euler product over local factors:

$$\chi(g) = \prod_{p\leq\infty} \chi_p(g_p), \qquad\qquad g_p \in G(\mathbb{Q}_p). \tag{5.83}$$

There is a one-to-one correspondence between such characters and weights of the Lie algebra $\mathfrak{g}(\mathbb{R})$, or, more precisely, complex linear functionals $\lambda \in \mathfrak{h}^\star_{\mathbb{C}} = \mathfrak{h}(\mathbb{R})^\star \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathfrak{h}(\mathbb{R})$ is the Cartan subalgebra of $\mathfrak{g}(\mathbb{R})$. We define a logarithm map H as follows:

$$H : G(\mathbb{A}) \to \mathfrak{h}(\mathbb{R}), \tag{5.84}$$

defined by

$$H(g) = H(nak) = H(a) = \log|a|. \tag{5.85}$$

The absolute value is defined as follows. Parametrise the group element $a \in A(\mathbb{A})$ by

$$a = \exp\left(\sum_{\alpha\in\Pi} u_\alpha H_\alpha\right), \qquad H_\alpha \in \mathfrak{h}(\mathbb{R}), \quad u_\alpha \in \mathbb{A}, \tag{5.86}$$

where Π denotes the set of simple roots of $\mathfrak{g}(\mathbb{R})$.

Then we define

$$\log|a| := \log\exp\left(\sum_{\alpha\in\Pi} |u_\alpha| H_\alpha\right) = \sum_{\alpha\in\Pi} |u_\alpha| H_\alpha = \sum_{\alpha\in\Pi}\left(\prod_{p\leq\infty} |u_{\alpha,p}|_p\right) H_\alpha, \tag{5.87}$$

where each $u_{\alpha,p} \in \mathbb{Q}_p$.

The choice of character χ can now be parametrised by the choice of linear functional λ according to the formula:

$$\chi(g) = e^{\langle\lambda+\rho|H(g)\rangle} = |a^{\lambda+\rho}|. \tag{5.88}$$

Here, we introduced a convenient short-hand notation. The translation by the Weyl vector ρ constitutes a convenient choice of normalization.

Remark 5.11 (Modulus character). The map

$$b \mapsto e^{\langle 2\rho|H(b)\rangle} \equiv \delta_B(b), \qquad b \in B(\mathbb{A}), \tag{5.89}$$

is often called the *modular function* (or "modulus character") of B. It is defined by

$$\delta_B(b) = \Big|\det \operatorname{ad}(b)\big|_{\mathfrak{n}}\Big|. \tag{5.90}$$

In words, it is the modulus of the determinant of the adjoint representation of $b \in B(\mathbb{A})$, restricted to the Lie algebra $\mathfrak{n}$ of the unipotent radical N. By virtue of the properties (5.85) of the logarithm map we have

$$\delta_B(b) = \delta_B(na) = \delta_B(a). \tag{5.91}$$

The modulus character corresponds to the Jacobian that relates the left- and right-invariant Haar measures on B. This implies in particular that under conjugation by $a \in A(\mathbb{A})$, i.e.

$$n \mapsto ana^{-1}, \tag{5.92}$$

the Haar measure dn on $N(\mathbb{A})$ transforms by

$$dn \mapsto \delta_B(a)dn. \tag{5.93}$$

This fact will play a crucial role in our calculations in chapter 7 and onwards.

See Example 5.12 for an explicit description for $SL(2, \mathbb{A})$. Using the modulus character we can write χ in the alternative form

$$\chi(g) = e^{\langle \lambda | H(g) \rangle} \delta_B^{1/2}(g). \tag{5.94}$$

This form of the character is common in the mathematical literature.

Example 5.12: Haar measure and modulus character for the Borel subgroup of $SL(2, \mathbb{A})$

For $SL(2, \mathbb{A})$ we can take

$$b = na = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}, \tag{5.95}$$

in which case the right-invariant Haar measure is

$$dnda = \frac{dudv}{v}, \tag{5.96}$$

and the modulus character is given by

$$\delta_B(na) = \delta_B \left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} \right) = |v|^2. \tag{5.97}$$

5.3.2 Eisenstein series

With the definition of the character χ on the Borel subgroup, we are now in a position to state Langlands's definition of an Eisenstein series for an arbitrary reductive groups $G(\mathbb{A})$. It is defined as the sum over images of the coset $B(\mathbb{Q})\backslash G(\mathbb{Q})$:

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g)\,, \tag{5.98}$$

and using the explicit parametrisation (5.88), the definition reads

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g) \rangle}. \tag{5.99}$$

The series defined here is not the only possible type of Eisenstein series that one can define, although it is the one that we will be most interested in. However, in section 5.4.4 we will treat Eisenstein series in the context of automorphic representations. This will then provide us with a way of deriving different types of Eisenstein series, including the one just defined.

For the above series Godement [29, 129] proved that the sum converges absolutely whenever λ lies in the open subset

$$\{\lambda \in \mathfrak{h}^\star_{\mathbb{C}} \mid \mathrm{Re}(\lambda) \in \rho + (\mathfrak{h}^\star)^+\}, \tag{5.100}$$

where the positive chamber $(\mathfrak{h}^\star)^+$ is defined by

$$(\mathfrak{h}^\star)^+ = \{\Lambda \in \mathfrak{h}^\star \mid \langle \Lambda, H_\alpha \rangle > 0, \, \forall \, \alpha \in \Pi\}, \tag{5.101}$$

so that we require $\langle \lambda, H_\alpha \rangle > 1$ for all simple roots α. For discussing the spectral decomposition of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, see also section 5.4.5, one is mainly interested in the case when $\lambda \in i\mathfrak{h}^\star$. This choice for λ is motivated in section 5.4.3 by the fact that the inducing representation is unitary in this case. However, such values lie outside the domain of absolute convergence (5.100) of $E(\lambda, g)$, which seems worrisome for the spectral decomposition. This puzzle is resolved by the remarkable result of Langlands that the Eisenstein series $E(\lambda, g)$ can in fact be analytically continued outside of the domain (5.100) to a meromorphic function on all of $\mathfrak{h}^\star_{\mathbb{C}}$. To establish the analytic continuation a crucial property of $E(\lambda, g)$ is its *functional relation* which relates its value at λ to its value at the Weyl-transform of λ:

$$E(\lambda, g) = M(w, \lambda) E(w\lambda, g)\,, \qquad w \in \mathcal{W}(\mathfrak{g}), \tag{5.102}$$

where $M(w, \lambda)$ is a known function. The functional relation will be discussed in more detail in chapter 8. Another important property is that $E(\lambda, g)$ is an eigenfunction of the Laplace operator $\Delta_{G/K}$:

$$\Delta_{G/K} E(\lambda, g) = \frac{1}{2} \left(\langle \lambda | \lambda \rangle - \langle \rho | \rho \rangle \right) E(\lambda, g). \tag{5.103}$$

This formula is derived in appendix C. In fact, the Eisenstein series $E(\lambda, g)$ is a common eigenfunction of all $G(\mathbb{A})$-invariant differential operators which is a reflection of its $\mathcal{Z}(\mathfrak{g})$-finiteness. The following is a useful property of Eisenstein series:

Proposition 5.13. *In the special case when $\lambda = -\rho$ we have*

$$E(-\rho, g) = 1. \tag{5.104}$$

Proof. Note first that by (5.103), $E(-\rho, g)$ is an eigenfunction of the Laplacian $\Delta_{G/K}$ with eigenvalue zero; hence it must be a constant function. To fix the constant to unity, we note that by Langlands constant term formula (see chapter 8), the constant term of $E(-\rho, g)$ with respect to the maximal unipotent radical N is

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(-\rho, ng) dn = 1, \tag{5.105}$$

from which the claim follows. □

5.4 Automorphic representations

In this section we introduce the concept of an automorphic representation associated with an adelic group $G(\mathbb{A})$. Since this is a rather difficult concept to grasp at first sight, we shall begin with a heuristic discussion before we delve into the technical definition. Our main focus will then lie with the so called principal series which is the representation relevant for general Eisenstein series. We also provide some remarks on the problem of classifying all automorphic representations of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

5.4.1 Automorphic forms and representation theory: a first glance

We have already seen hints in section 5.1 that automorphic forms on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ are intimately related to the representation theory of $SL(2,\mathbb{R})$. Here we will further develop this point of view and also generalise it to the adelic framework.

The main idea is that the space of smooth functions $\varphi : SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}) \to \mathbb{C}$ carries several actions:

- First, we have the action π of $SL(2,\mathbb{R})$ by *right-translation*:

$$[\pi(h)\varphi](g) := \varphi(gh), \qquad g, h \in SL(2,\mathbb{R}). \tag{5.106}$$

- Second, we have the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ by differential operators:

$$(D_X \cdot \varphi)(g) := \frac{d}{dt}\varphi(g \cdot e^{tX})\Big|_{t=0}, \qquad X \in \mathcal{U}(\mathfrak{sl}(2,\mathbb{C})). \tag{5.107}$$

Whenever one has a group action on a space it is natural to look for a decomposition into irreducible representations of the group. Moreover, since the centre $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ commutes with $SL(2,\mathbb{R})$ it is also natural to distinguish the irreducible components in terms of their eigenvalues with respect to differential operators in $\mathcal{Z}$. For these reasons the theory of automorphic forms on $SL(2,\mathbb{R})$ is closely related to the decomposition of the space $\mathcal{A}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ into irreducible representations with respect to the right-regular action of $SL(2,\mathbb{R})$, compatible with the action by $\mathcal{U}$. To get an idea of what this entails, let us now look at an extremely simplified, though still enlightening, example.

Example 5.14: Fourier analysis on $\mathbb{Z}\backslash\mathbb{R}$

In this example we will look at the abelian situation where the space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ is replaced by $\mathbb{Z}\backslash\mathbb{R}$, the circle group. This is the setting of classical Fourier analysis. The space of smooth functions $C^\infty(\mathbb{Z}\backslash\mathbb{R})$ is then just Fourier series where the coefficients are constrained to decay rapidly with increasing Fourier number. Let us now formalise this a little and try to analyse it in the spirit of automorphic forms. Consider the unitary character $\chi_k : \mathbb{Z}\backslash\mathbb{R} \to U(1)$, defined by $\chi_k(x) = e^{2\pi i k x}$ for $x \in \mathbb{R}, k \in \mathbb{Z}$. Any function $f \in C^\infty(\mathbb{Z}\backslash\mathbb{R})$ can then be expanded in a Fourier series in terms of these characters

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \chi_k(x), \tag{5.108}$$

Recall that an automorphic form is also required to satisfy a moderate growth condition. In the present setting we can choose square-integrability of $f(x)$ as a suitable condition for moderate growth. Thus the space of automorphic forms on $\mathbb{Z}\backslash\mathbb{R}$ can be taken to be the Hilbert space $L^2(\mathbb{Z}\backslash\mathbb{R}) \subset C^\infty(\mathbb{Z}\backslash\mathbb{R})$ where the Fourier coefficients satisfy the square-integrability condition

$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty. \tag{5.109}$$

Each character χ_k generates a one-dimensional irreducible subspace $V_k = \mathbb{C}\chi_k \subset L^2(\mathbb{Z}\backslash\mathbb{R})$ and the *regular representation* π of $\mathbb{R}$ defined by

$$(\pi(y)f)(x) := f(x+y), \qquad x, y \in \mathbb{R}, \tag{5.110}$$

is diagonalized by the subspaces V_k:

$$\pi(y) \cdot v = \chi_k(y)v, \qquad v \in V_k,\ y \in \mathbb{R}. \tag{5.111}$$

The set of equivalence classes of unitary representations of a group G is called the *unitary dual*, usually denoted $\widehat{G}$. In our example, the unitary dual $\widehat{\mathbb{R}}$ is simply the space of Fourier coefficients subject to the condition (5.109):

$$\widehat{\mathbb{R}} = L^2(\mathbb{Z}) = \{(c_k) \mid \sum_k |c_k|^2 < \infty\}. \tag{5.112}$$

This gives the spectral decomposition of the Hilbert space $L^2(\mathbb{Z}\backslash\mathbb{R})$. The fact that the spectrum is discrete is a general feature of spectral theory on compact spaces like $S^1 \cong \mathbb{Z}\backslash\mathbb{R}$.

Before we proceed with the adelic perspective we shall consider one more simple example that illustrates another feature that has a counterpart on the general theory of automorphic forms.

Example 5.15: Fourier analysis on $\mathbb{R}$

Consider the same setting as in the previous example, namely $G = \mathbb{R}$, but we now take the discrete subgroup $G(\mathbb{Z})$ to be trivial. In other words, we are interested in the regular representation of $\mathbb{R}$ on the Hilbert space $L^2(\mathbb{R})$. The regular action π is defined in the same way as in (5.110), but now this action is diagonalised on a continuous family of characters $\chi_\zeta : \mathbb{R} \to U(1)$ defined by $\chi_\zeta(x) = e^{2\pi i\zeta x}$ with $\zeta, x \in \mathbb{R}$. On the irreducible subspaces $V_\zeta = \mathbb{C}\chi_\zeta$ we then have

$$\pi(x) \cdot v = \chi_\zeta(x)v, \qquad \zeta, x \in \mathbb{R},\ v \in V_\zeta. \tag{5.113}$$

The unitary dual in this case is a "continuous direct sum", or *direct integral* of irreducible representations, meaning that any function $f \in L^2(\mathbb{R})$ can be written as a continuous version of a Fourier series:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\zeta)\chi_\zeta(x)d\zeta, \tag{5.114}$$

where $\hat{f}(\zeta)$ is the standard Fourier transform of f with respect to the character χ_ζ.

From the above analysis we conclude that the spectral decomposition of $L^2(\mathbb{R})$ with respect to the regular action of $\mathbb{R}$ has only a continuous part, in stark contrast with the situation in example 5.14

above. The appearance of a continuous spectrum is a general feature of spectral analysis on non-compact spaces, just like the discrete spectrum always appear for compact spaces. We also note a curious feature, namely that although the characters χ_k are used to decompose the spectrum $L^2(\mathbb{R})$ they are in fact *not* square-integrable. This is not a problem since the Fourier transform always preserves square integrability. An elaborate version of this phenomenon will reappear later in this chapter.

In the previous examples we have illustrated how the spectral analysis on compact or non-compact spaces the spectrum has very different properties. When we generalise this to the non-abelian setting of $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ we actually combine these properties in the following sense. Consider for a moment the space of square-integrable automorphic forms $L^2(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ which is a subspace of all automorphic forms where the moderate growth condition is replaced by the square-integrability condition. The space $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ is certainly non-compact and we therefore expect that a spectral analysis would give rise to a continuous spectrum. In addition, and in contrast to the abelian case $\mathbb{Z}\backslash\mathbb{R}$, the quotient $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ has *finite volume* and therefore also give rise to a discrete spectrum. Indeed, it was proven by Selberg that

$$L^2(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})) = \mathbb{C} \oplus L^2_{cusp}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})) \oplus L^2_{cont}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})), \tag{5.115}$$

where:

- the first factor $\mathbb{C}$ represents the constant functions (these are considered to be part of the discrete spectrum and arise also as the residue of the non-holomorphic Eisenstein series $E(s,z)$ at $s=1$, see also section 10.1.1),
- the remainder of the discrete spectrum is $L^2_{cusp}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ which is spanned by Maass cusp forms φ with a discrete set of eigenvalues $\lambda_n, n = 1,2,3,\ldots,$ with respect to the Laplacian $\Delta_{\mathbb{H}}$,
- the continuous spectrum $L^2_{cont}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ is a direct integral of non-holomorphic Eisenstein series $E(s,z)$ (5.42).

Remark 5.16. Note that the non-holomorphic Eisenstein series $E(s,\tau)$ are not-square integrable and so are not themselves part of $L^2(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ they nevertheless play a key role in parametrising the unitary dual, in a very similar vein as the non-square integrable, continuous characters χ_ζ occurred in the spectral decomposition of $L^2(\mathbb{R})$ in example 5.15. The spectral decomposition (5.115) form a crucial ingredient in the Selberg trace formula (see [5] for a nice introduction).

Although Eisenstein series are not square integrable they are still important for the representation theoretic aspects of automorphic forms and therefore it is natural to enhance the space of automorphic forms to include non-square integrable objects. One then replaces the square-integrable condition with a more general moderate growth condition, leading to the full space of automorphic forms $\mathcal{A}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$. We thus have the inclusions

$$L^2_{cusp}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})) \subset L^2(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})) \subset \mathcal{A}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})). \tag{5.116}$$

The representation-theoretic aspects of automorphic forms is however not yet complete, as the above treatment is missing an important ingredient. The space $\mathcal{A}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ carries an additional action by *Hecke operators*, which has not yet been taken into account. We will treat Hecke operators in detail in chapter 11 so here we shall merely offer some qualitative remarks. A Hecke operator is an operation $T_p : \mathcal{A} \to \mathcal{A}$, parametrised by a prime number $p < \infty$. The set of all Hecke operators $\{T_p\}_{p<\infty}$ forms a commutative ring, called the *Hecke algebra*. An element $\varphi \in \mathcal{A}(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R}))$ which is an eigenvector for all Hecke operators

$$T_p\varphi = \lambda_p\varphi, \tag{5.117}$$

is called a *Hecke eigenform*. Here the eigenvalues λ_p carry the arithmetic information contained in φ. However, the right-regular action of $SL(2,\mathbb{R})$ on $\mathcal{A}$ cannot be used to accommodate the action of the Hecke algebra and so the analysis of automorphic forms in terms of the representation theory of $SL(2,\mathbb{R})$ is incomplete. One of the reasons for passing to the adelic picture is precisely to remedy this problem. The basic idea is this: if we consider the right-regular action of $SL(2,\mathbb{A})$ on the space $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$, then the Hecke eigenvalues $\{\lambda_p\}$ parametrise the irreducible representations of the right regular action of the local subgroups $SL(2,\mathbb{Q}_p)$ on $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$. Thus, from the adelic perspective the Hecke algebra plays the same role at the non-archimedean places of $SL(2,\mathbb{A})$, as the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ does at the archimedean place. The Hecke action is thus implicitly already taken into account in the general definition 5.6 of an adelic automorphic form.

Example 5.17: The action of Hecke operators on non-holomorphic Eisenstein series

For illustration we consider here a simple example of how the Hecke operators act on the non-holomorphic Eisenstein series $E(s,z)$. For any integer $n > 0$ we define the operator T_n as follows:

$$(T_nE)(s,z) := \frac{1}{n}\sum_{d|n}\sum_{b=0}^{d-1} E\left(s, \frac{nz+bd}{d^2}\right). \tag{5.118}$$

In chapter 11 we will show that

$$(T_nE)(s,z) = \lambda_n E(s,z), \tag{5.119}$$

with

$$\lambda_n = n^{s-1/2}\sigma_{1-2s}(n). \tag{5.120}$$

This is precisely the numerical Fourier coefficient in the Fourier expansion of $E(s,z)$; see eq. (1.16). The operators T_n further satisfy the following basic relation

$$T_mT_n = \sum_{d|(m,n)} \frac{1}{d} T_{mn/d^2}, \tag{5.121}$$

characterising the Hecke algebra. This algebra is generated by the subset of Hecke operators T_p for p a (finite) prime, hence the fundamental information is contained in the prime eigenvalues λ_p, as claimed in the main text. In chapter 11 we provide much more details on Hecke operators and explain their link with the representation theory of $SL(2,\mathbb{Q}_p)$.

We now want to make sense of combined action on $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ of $SL(2,\mathbb{A})$ by right-translation as well as the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$

by differential operators. To this end it is useful to distinguish between the action of the finite part $SL(2,\mathbb{A}_f)=\prod_{p<\infty} SL(2,\mathbb{Q}_p)$ and the archimedean part $SL(2,\mathbb{R})$. For any $\varphi\in\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ we then have

$$\begin{aligned}(\pi(h_f)\varphi)(g) &= \varphi(gh_f), \qquad g\in SL(2,\mathbb{A}),\ h_f\in SL(2,\mathbb{A}_f).\\ (\pi(h_\infty)\varphi)(g) &= \varphi(gh_\infty), \qquad g\in SL(2,\mathbb{A}),\ h_\infty\in SL(2,\mathbb{R}).\end{aligned} \tag{5.122}$$

Here it is understood that the elements h_f and h_∞ are embedded in the canonical way into the adelic group. For instance, when we write gh_∞ we really mean

$$g\cdot\left(h_\infty,\begin{pmatrix}1&\\&1\end{pmatrix},\begin{pmatrix}1&\\&1\end{pmatrix},\dots,\begin{pmatrix}1&\\&1\end{pmatrix}\right),\qquad g\in SL(2,\mathbb{A}),\ h_\infty\in SL(2,\mathbb{R}). \tag{5.123}$$

These two actions of course commute with each other. The action by $SL(2,\mathbb{A}_f)$ at the non-archimedean places also commutes with the $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$-action at the archimedean place, and so this gives a well-defined representation. On the other hand, the right-regular action of $K_\infty=SO(2,\mathbb{R})\subset SL(2,\mathbb{R})$ does *not* commute with $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$. Rather, for $X\in\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ and $k_\infty\in SO(2,\mathbb{R})$ one has

$$D_X\cdot\pi(k_\infty)=\pi(k_\infty)\cdot D_{k_\infty^{-1}Xk_\infty}, \tag{5.124}$$

where D_X is the differential operator (5.107). One can check this by a direct calculation:

$$\begin{aligned}D_X\cdot\left(\pi(k_\infty)\varphi\right)(g) &= D_X\cdot\varphi(gk_\infty)\\ &= D_X\cdot\varphi\left(g\cdot\left(k_\infty,\begin{pmatrix}1&\\&1\end{pmatrix},\begin{pmatrix}1&\\&1\end{pmatrix},\dots,\begin{pmatrix}1&\\&1\end{pmatrix}\right)\right)\end{aligned} \tag{5.125}$$

where $k_\infty\in SO(2,\mathbb{R})$. Now using the definition of D_X we find that the right hand side can be written as

$$\frac{d}{dt}\varphi\left(g\cdot\left(e^{Xt}k_\infty,\begin{pmatrix}1&\\&1\end{pmatrix},\begin{pmatrix}1&\\&1\end{pmatrix},\dots,\begin{pmatrix}1&\\&1\end{pmatrix}\right)\right)\bigg|_{t=0}. \tag{5.126}$$

Inserting the identity $k_\infty k_\infty^{-1}$ and using the following property of the matrix exponential

$$k_\infty^{-1}e^{Xt}k_\infty=e^{k_\infty^{-1}Xk_\infty t}, \tag{5.127}$$

we can rewrite equation (5.126) as

$$\frac{d}{dt}\varphi\left(gk_\infty\cdot\left(k_\infty^{-1}e^{Xt}k_\infty,\begin{pmatrix}1&\\&1\end{pmatrix},\begin{pmatrix}1&\\&1\end{pmatrix},\dots,\begin{pmatrix}1&\\&1\end{pmatrix}\right)\right)\bigg|_{t=0}=\pi(k_\infty)\cdot\left(D_{k_\infty^{-1}Xk_\infty}\cdot\varphi(g)\right), \tag{5.128}$$

which is the right hand side of (5.124). This turns out to be the characteristic property of a so called $(\mathfrak{g},K_\infty)$-*module*, a notion which will be properly defined in the next section.

Remark 5.18. To ensure that the space $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ is preserved under all three actions defined above, one must of course verify that they are compatible with definition 5.10. In other words one should check that the three functions

$$D_X\cdot\varphi(g),\qquad(\pi(h_\infty)\varphi)(g),\qquad(\pi(h_f)\varphi)(g), \tag{5.129}$$

all satisfy Conditions (1)-(4) in definition 5.10. See, e.g., section 5.1 of [131] for a detailed check of this.

When speaking about an *automorphic representation* of $SL(2,\mathbb{A})$ one really refers to a structure that carries a standard group representation with respect to the finite part $SL(2,\mathbb{A}_f)$, and a $(\mathfrak{g}, K_\infty)$-module structure at the archimedean place. In the following section we will give the precise definition for an arbitrary reductive group $G(\mathbb{A})$ and discuss some central features of automorphic representations.

5.4.2 Formal definition

In this section we shall give the precise definition of an automorphic representation of an adelic group $G(\mathbb{A})$ and present some of the key features that will be important in subsequent chapters. Just as in the $SL(2,\mathbb{A})$-discussion of the previous section, we are interested in the combined actions of $G(\mathbb{A}_f)$ and K_∞ by right-translation and the action of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ by differential operators. The general analysis goes through in a similar vein as above and the conclusion is that the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ does not carry a group representation with respect the whole group $G(\mathbb{A})$, but only with respect to the finite part $G(\mathbb{A}_f)$. At the real place one has instead that $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ carries the structure of a $(\mathfrak{g}, K_\infty)$-module, whose definition we will now recall.

Definition 5.19 ($(\mathfrak{g}, K)$-module). A $(\mathfrak{g}, K_\infty)$-*module* is a complex vector space V which carries an action of both the Lie algebra $\mathfrak{g}$ and K_∞, such that all vectors $v \in V$ are K-finite, i.e. $\dim \langle k_\infty \cdot v \,|\, k_\infty \in K_\infty \rangle < \infty$. The actions of $\mathfrak{g}$ and K_∞ are furthermore required to be compatible in the following sense

$$X \cdot k_\infty \cdot v = k_\infty \cdot \mathrm{Ad}_{k_\infty^{-1}}(X) \cdot v, \qquad k_\infty \in K_\infty,\ X \in \mathfrak{g}. \tag{5.130}$$

Remark 5.20. In our context the complex vector space V is $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, the action by $X \in \mathfrak{g}$ is by the differential operator D_X and the action by $k_\infty \in K_\infty$ is by right-translation. In this setting $k_\infty \cdot \mathrm{Ad}_{k_\infty^{-1}}(X)$ means $\pi(k_\infty) \cdot D_{k_\infty^{-1} X k_\infty}$ and hence equation (5.124) is precisely the compatibility condition (5.130) for a $(\mathfrak{g}, K_\infty)$-module.

Remark 5.21. Let us offer some remarks on the usefulness of $(\mathfrak{g}, K_\infty)$-modules. The notion of $(\mathfrak{g}, K_\infty)$-module was introduced by Harish-Chandra in his efforts on "algebraisation" of representations. Function spaces on groups are themselves typically not specific enough and there can be many function spaces that share the same algebraic features. For example, one can consider continuous functions on a group manifold and they are a perfectly nice representation of G. However, unless the functions are differentiable, this representations does not give rise to a representation of the Lie algebra $\mathfrak{g}$ that would be represented by an algebra of differential operators. There are many different types of differentiable functions on G and the notion of $(\mathfrak{g}, K_\infty)$-module mainly serves to eliminate the ambiguities related to choosing a type.

With the above concepts introduced, we are now ready to state the definition of an automorphic representation.

Definition 5.22 (Automorphic representation). A representation π of $G(\mathbb{A})$ is called an *automorphic representation* if it occurs as an irreducible constituent in the decomposition of $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ with respect to the simultaneous action by

$$(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f), \tag{5.131}$$

where K_∞ and $G(\mathbb{A}_f)$ acts by right-translation and $\mathfrak{g}$ by differential operators at the archimedean place.

We shall for short denote by V the complex vector space on which an automorphic representation π acts. Then V is simultaneously a $(\mathfrak{g}_\infty, K_\infty)$-module and a $G(\mathbb{A}_f)$-module and the automorphic representation is also often denoted by the pair (π, V).

Let $K_f \subset G(\mathbb{A}_f)$ be a compact open subgroup (not necessarily maximal) and σ an irreducible representation of $K_\infty \times K_f$. Denote by $V[\sigma]$ the space of vectors in V that transforms according to σ under the action of $K_\infty \times K_f$. We then have the following important definition:

Definition 5.23 (Admissible representation). A $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$-module V is called *admissible* if the subspace $V[\sigma] \subset V$ is finite-dimensional for all σ.

It is then a central result of Flath [94], known as *tensor decomposition theorem*, that for admissible automorphic representations (π, V) there exists an Euler product decomposition into local factors

$$(\pi, V) = \bigotimes_{p\leq\infty}(\pi_p, V_p), \tag{5.132}$$

where the archimedean component (π_∞, V_∞) is a $(\mathfrak{g}_\infty, K_\infty)$-module according to the discussion above, while the non-archimedean components (π_p, V_p) furnish representations of $G(\mathbb{Q}_p)$.

Let us finally also introduce the notion of a spherical representation and vector.

Definition 5.24 (Unramified representation). An automorphic representation π_p is called *unramified* (or *spherical*) if V_p contains a non-zero vector f_p which is invariant under $K_p = G(\mathbb{Z}_p)$. We then call such an f_p a *spherical vector*. Globally one has the important notion that if (π, V) is a spherical automorphic representation if π_p is spherical for almost all p.

5.4.3 Principal series representation

Fix a Borel subgroup B and a quasi-character $\chi : B \to \mathbb{C}^\times$ defined as in (5.88):

$$\chi = e^{\langle\lambda+\rho|H\rangle}. \tag{5.133}$$

Consider now the following space of smooth functions on $G(\mathbb{A})$:

$$I(\chi) = \left\{ f : G(\mathbb{A}) \to \mathbb{C} \,\middle|\, f(bg) = \chi(b)f(g),\, b \in B(\mathbb{A}) \right\}. \tag{5.134}$$

This is the function space of an induced representation of $G(\mathbb{A})$ called the *principal series representation*; it is also often denoted by $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$. The principal series $I(\chi)$ provides an important example of an automorphic representation thanks to the theory of Eisenstein series which will be discussed in section 5.4.4 below. In general $I(\chi)$ is a reducible representation. However, one can show that when $\chi = \otimes_p \chi_p$ is an unramified character, $I(\chi)$ is an irreducible, admissible representation and so affords a decomposition into local factors:

$$I(\lambda) = \bigotimes_{p\leq\infty} I_p(\lambda) = \bigotimes_{p\leq\infty} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p. \tag{5.135}$$

Remark 5.25. The space $I(\lambda)$ can be viewed as the total space of a fiber bundle $I(\lambda) \to \mathfrak{h}_{\mathbb{C}}^{\star}$, with the fiber over each point $\lambda \in \mathfrak{h}_{\mathbb{C}}^{\star}$ consisting of the space of functions on $G(\mathbb{A})$ which transform by the character $e^{\langle\lambda+\rho|H\rangle}$ under the left action of $B(\mathbb{A})$.

Definition 5.26 (Standard section). An element $\mathsf{f}_\lambda \in I(\lambda)$ is called a *standard section* if it is $K_{\mathbb{A}}$-finite and its restriction to $K_{\mathbb{A}}$ is independent of λ.

By virtue of (5.135), any standard section $\mathsf{f}_\lambda \in I(\lambda)$ splits into a product of local factors

$$\mathsf{f}_\lambda = \bigotimes_{p\leq\infty} \mathsf{f}_{\lambda,p}. \tag{5.136}$$

Although the principal series representations $I_p(\lambda)$ are infinite-dimensional, one can still attach to them a notion of "size", which is called the *functional, or Gelfand–Kirillov, dimension* and denoted by GKdim. This is defined as the smallest number of variables on which we can realize the functions in $I_p(\lambda)$. For example, the functional dimension of the Hilbert space $L^2(\mathbb{R}^n)$ is n. Similarly, by the Iwasawa decomposition $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)G(\mathbb{Z}_p)$ the sections $\mathsf{f}_{\lambda,p} \in I_p(\lambda)$ are determined by their restriction to $B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p) = G(\mathbb{Z}_p)$ and hence the functional dimension is

$$\mathrm{GKdim}(I_p(\lambda)) = \dim_{\mathbb{Q}_p} B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p). \tag{5.137}$$

5.4.4 Eisenstein series and induced representations

Let us now discuss the definition of Eisenstein series from the perspective of induced representations. One can think of an Eisenstein series as providing a $G(\mathbb{A})$-equivariant map from the induced representation $I(\lambda)$ of (5.134) into the space of automorphic forms:

$$E\,:\, I(\lambda) \,\to\, \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})). \tag{5.138}$$

For any standard section $\mathsf{f}_\lambda \in I(\lambda)$ the construction of the corresponding Eisenstein series is given by

$$E(\mathsf{f}_\lambda, g) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \mathsf{f}_\lambda(\gamma g). \tag{5.139}$$

As f_λ varies in the fiber of $I(\lambda) \to \mathfrak{h}_{\mathbb{C}}^{\star}$ we thus obtain a family of Eisenstein series $E(\mathsf{f}_\lambda, g)$ that satisfy all the conditions of definition 5.58 for an automorphic form; in particular, $K_{\mathbb{A}}$-finiteness follows from the fact that f_λ is a standard section. By virtue of the decomposition

$\mathsf{f}_\lambda = \otimes_{p\leq\infty}\mathsf{f}_{\lambda,p}$, the Eisenstein series $E(\mathsf{f}_\lambda, g)$ can be defined by choosing all the local factors $\mathsf{f}_{\lambda,p}$ separately. This gives a lot of freedom in defining the Eisenstein series and is one of the main reasons why the adelic formalism is so powerful (see section 5.5 for a detailed demonstration in the case of $SL(2,\mathbb{A})$).

In order to recover the particular Eisenstein series of definition (5.99), one chooses the standard section f_λ to be equal to the inducing character, $\mathsf{f}_\lambda = e^{\langle\lambda+\rho|H\rangle} = \chi$.

In section 5.5 we will illustrate for $G(\mathbb{A}) = SL(2,\mathbb{A})$ how the more general construction in (5.139) interpolates between holomorphic and non-holomorphic Eisenstein series on the upper-half plane $\mathbb{H}$.

5.4.5 Classifying automorphic representations

It is one of the central unsolved problems in the theory of automorphic forms to classify all automorphic representations. In fact, all admissible automorphic representations have been classified (see, e.g., [57]). This includes in particular the spherical, or unramified, representations.

The task of decomposing $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ into irreducible representations is closely connected to the problem of decomposing the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ under the unitary action of G. A priori this might seem a little surprising since an automorphic form need not be square-integrable; indeed the Eisenstein series $E(s,g)$ considered in section 5.1 provide an example of a non-square integrable automorphic form. The decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ splits into two orthogonal spaces

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{\text{discrete}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_{\text{continuous}}(G(\mathbb{Q})\backslash G(\mathbb{A})), \tag{5.140}$$

corresponding respectively to the discrete and continuous parts of the spectrum. It turns out that the discrete spectrum is spanned by cusp forms and residues of Eisenstein series [218, 243].

It is a fundamental result in the spectral theory of automorphic forms that the space $\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the subspace of $L^2_{\text{discrete}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ corresponding to smooth, cuspidal, K-finite and $\mathcal{Z}(\mathfrak{g})$-finite vectors occurring in the decomposition of the unitary representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. This is the reason that cusp forms constitute an essential part in the theory of automorphic forms.

While the discrete spectrum can be understood in this way as a direct sum of invariant subspaces spanned by cusp forms, the space $L^2_{\text{continuous}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ rather decomposes into a *direct integral* over principal series representation of $G(\mathbb{R})$. Such integrals turn out to be parametrised by Eisenstein series, even though these by themselves are not square integrable (see the following section and also [117] for more on the continuous spectrum and the relation with Eisenstein series). This situation is a generalisation of the problem of decomposing the Hilbert space $L^2(\mathbb{R})$ via Fourier analysis in terms of the non-square integrable characters (Fourier modes) $e^{2\pi ixy}$, $x, y \in \mathbb{R}$, as discussed in example 5.15. The construction of Eisenstein series on $G(\mathbb{Q})\backslash G(\mathbb{A})$, generalising the function $E(g,s)$ of section 5.1, therefore constitutes an equally important part of the theory of automorphic forms as that of analyzing the space of cusp forms. Moreover, the complement of $\mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$

inside the discrete spectrum $L^2_{\text{discrete}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is conjecturally spanned by *residues* of Eisenstein series $E(\lambda, g)$ for special values of the weight λ. Thus, one expects that the discrete spectrum decomposes according to:

$$L^2_{\text{discrete}}(G(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A})). \tag{5.141}$$

Arthur has outlined a set of conjectures that characterise precisely which weights λ for which the representation becomes square-integrable [3] (for proofs of Arthur's conjectures in some cases, see [175, 194, 233, 242, 244]). See also section 10.2.3 for further discussions of square-integrability of Eisenstein series.

Example 5.27: A representation-theoretic viewpoint on Eisenstein series on $SL(2,\mathbb{A})$

We now analyze the general Eisenstein series $E(\lambda, g)$ more explicitly for $G(\mathbb{A}) = SL(2,\mathbb{A})$. In this case the space of (complex) weights $\mathfrak{h}^\star_{\mathbb{C}}$ is one-dimensional and spanned by the fundamental weight Λ dual to the unique simple root α of the Lie algebra $\mathfrak{sl}(2,\mathbb{A})$. The Weyl vector ρ is also identical to $\Lambda = \alpha/2$. Therefore, we can parametrise the weight appearing in (5.99) with a single parameter $s \in \mathbb{C}$ according to

$$\lambda = 2s\Lambda - \rho = (2s-1)\Lambda \quad \Longrightarrow \quad \lambda + \rho = 2s\Lambda\,. \tag{5.142}$$

The character $\chi : B(\mathbb{Q})\backslash B(\mathbb{A}) \to \mathbb{C}^\times$ in (5.88) can now be written as

$$\chi_s(g) \equiv e^{\langle \lambda+\rho | H(g)\rangle} = e^{\langle 2s\Lambda | H(a)\rangle}\,. \tag{5.143}$$

We can write all these objects explicitly in the fundamental representation of $\mathfrak{sl}(2,\mathbb{A})$:

$$g = nak = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} k \tag{5.144}$$

with $k \in K_{\mathbb{A}}$. Evaluated on the group element (5.144) we then find

$$\chi_s(g) = e^{2s\langle \Lambda | H(a)\rangle} = |v|^{2s} \tag{5.145}$$

since $H(a) = \log|v| \cdot H_\alpha$ where H_α is the Cartan generator of $\mathfrak{sl}(2,\mathbb{A})$. The general Eisenstein series $E(g;\lambda)$ in (5.99) now becomes

$$E(s,g) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \chi_s(\gamma g), \tag{5.146}$$

which is indeed equivalent to (5.53). This Eisenstein series is attached to the induced representation

$$I(s) = \text{Ind}_{B(\mathbb{A})}^{SL(2,\mathbb{A})} \chi_s. \tag{5.147}$$

This representation is unitary when $s = \frac{1}{2} + it \in \frac{1}{2} + i\mathbb{R}_+$. In this simple example one can also give a more explicit description of the spectral problem of decomposing the space $L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ with respect to the right-regular action of $SL(2,\mathbb{A})$. The decomposition (5.140) becomes in this case

$$L^2(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})) = L^2_0(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})) \oplus \mathbb{C} \oplus \int_0^\infty I\left(\tfrac{1}{2}+it\right) dt, \tag{5.148}$$

where the discrete spectrum L^2_{discrete} is represented by the space of cusp forms L^2_{cusp} together with the space $\mathbb{C}$ of constant functions that correspond to the residual spectrum from the pole of $E(s,g)$ at $s=1$. The continuous spectrum $L^2_{\text{continuous}}$ corresponds to the integral over the principal series $I(s)$, restricted to the unitary domain $s \in \frac{1}{2} + i\mathbb{R}_+$. For a proof of this statement, see the book by Gelbart [117].

5.5 Embedding of the discrete series in the principal series

Our aim in this section is to illustrate in great detail the construction of the general Eisenstein series $E(\mathsf{f}_\lambda, g)$ in (5.139) for the special case of $SL(2,\mathbb{A})$. We will in particular demonstrate that when restricted to a function on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ this yields a generalisation of the classical non-holomorphic Eisenstein series, which in fact interpolates between the non-holomorphic function $E(s,z)$, $z \in \mathbb{H}$, and the weight w holomorphic Eisenstein series $E_w(z)$. We explain how to understand this representation-theoretically in terms of the embedding of the holomorphic discrete series of $SL(2,\mathbb{R})$ into the principal series.

5.5.1 Eisenstein series for arbitrary standard sections

Let $I(\lambda) = \mathrm{Ind}_{B(\mathbb{A})}^{SL(2,\mathbb{A})} \chi_s$ be the induced representation (5.134) for $SL(2,\mathbb{A})$. As in Example 5.27 we take the inducing character $\chi_s : B(\mathbb{Q})\backslash B(\mathbb{A}) \to \mathbb{C}^\times$ (extended to all of $SL(2,\mathbb{A})$) to be defined by

$$\chi_s(bk) = \chi_s(b) = \chi_s \begin{pmatrix} v & \star \\ & v^{-1} \end{pmatrix} = |v|^{2s}, \qquad s \in \mathbb{C}, \tag{5.149}$$

where $b \in B(\mathbb{A})$ and $k \in K_\mathbb{A} = SO(2,\mathbb{R}) \times \prod_{p<\infty} SL(2,\mathbb{Z}_p)$.

Let $\mathsf{f}_\lambda = \otimes_p \mathsf{f}_{\lambda,p} \in I(\lambda)$ with each local factor

$$\mathsf{f}_{\lambda,p} \in I_p(\lambda) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{SL(2,\mathbb{Q}_p)} \chi_p, \tag{5.150}$$

determined by its restriction to $SL(2,\mathbb{Z}_p) = B(\mathbb{Q}_p)\backslash SL(2,\mathbb{Q}_p)$. For the purposes of this example we shall now fix these local sections as follows.

- For the non-archimedean places $p < \infty$ we choose the section $\mathsf{f}_{\lambda,p}$ to be the unique (normalized) spherical vector $\mathsf{f}^\circ_{\lambda,p}$ in $I_p(\lambda)$ defined by (see also section 6.3.3)

$$\mathsf{f}^\circ_{\lambda,p}(g_p) = \mathsf{f}^\circ_{\lambda,p}(b_p k_p) = \chi_{s,p}(b_p), \qquad \mathsf{f}^\circ_{\lambda,p}(k_p) = \mathsf{f}^\circ_{\lambda,p}(1) = 1, \tag{5.151}$$

  where $b_p \in B(\mathbb{Q}_p)$ and $k_p \in SL(2,\mathbb{Z}_p)$.

- For the archimedean place $p = \infty$ we define $\mathsf{f}_{\lambda,\infty} \in I_\infty(\lambda)$ according to

$$\mathsf{f}_{\lambda,\infty}(b_\infty) = \chi_{s,\infty}(b_\infty), \qquad \mathsf{f}_{\lambda,\infty}(g_\infty k_\infty) = e^{iw\theta}\mathsf{f}_{\lambda,\infty}(g_\infty), \qquad \mathsf{f}_{\lambda,\infty}(1) = 1, \tag{5.152}$$

where $w \in \mathbb{Z}$, $g_\infty \in SL(2,\mathbb{R})$, $b_\infty \in B(\mathbb{R})$ and

$$k_\infty = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2,\mathbb{R}). \tag{5.153}$$

Notice that with this definition $\mathsf{f}_{\lambda,\infty}$ is a $K_\infty = SO(2,\mathbb{R})$-finite, but *non-spherical* section of $I_\infty(\lambda)$.

With these definitions of the local factors, the product

$$\mathsf{f}_\lambda = \mathsf{f}_{\lambda,\infty} \otimes \bigotimes_{p<\infty} \mathsf{f}^\circ_{\lambda,p} \tag{5.154}$$

becomes a standard section (because λ and w are independent parameters) of the global representation $I(\lambda)$.

With this choice of section we now construct the Eisenstein series

$$E(\mathsf{f}_\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \left(\mathsf{f}_{\lambda,\infty}(\gamma g_\infty) \times \prod_{p<\infty} \mathsf{f}^\circ_{\lambda,p}(\gamma g_p) \right), \tag{5.155}$$

with $g = (g_\infty; g_2, g_3, \dots) \in SL(2,\mathbb{A}) = SL(2,\mathbb{R}) \times \prod'_{p<\infty} SL(2,\mathbb{Q}_p)$. We now want to analyze the restriction of this adelic Eisenstein series to a function on $SL(2,\mathbb{R})$. To this end we fix the adelic group element to be the identity at all finite places:

$$g = (g_\infty; 1, 1, \cdots) \in SL(2,\mathbb{A}). \tag{5.156}$$

In example 4.9 we showed the bijection of cosets $B(\mathbb{Q})\backslash SL(2,\mathbb{Q}) \cong B(\mathbb{Z})\backslash SL(2,\mathbb{Z})$ with each $B(\mathbb{Q})g$ coset having a representative in $SL(2,\mathbb{Z})$ for $g \in SL(2,\mathbb{Q})$. We will now use this write the Eisenstein series as a sum over $\gamma \in SL(2,\mathbb{Z})$. At the finite places $SL(2,\mathbb{Z})$ embeds into $SL(2,\mathbb{Z}_p)$, and hence, by (5.151), we have

$$\mathsf{f}^\circ_{\lambda,p}(\gamma) = 1 \qquad (\text{for } \gamma \in SL(2,\mathbb{Z}) \text{ and } p < \infty). \tag{5.157}$$

The Eisenstein series $E(\mathsf{f}_\lambda, g)$ therefore restricts to

$$\begin{aligned} E(\mathsf{f}_{\lambda,\infty}, g_\infty) &= \sum_{B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \mathsf{f}_{\lambda,\infty}(\gamma g_\infty) \cdot \prod_{p<\infty} \mathsf{f}^\circ_{\lambda,p}(\gamma) \\ &= \sum_{B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \mathsf{f}_{\lambda,\infty}(\gamma g_\infty) \cdot \prod_{p<\infty} \mathsf{f}^\circ_{\lambda,p}(\gamma) \\ &= \sum_{B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \mathsf{f}_{\lambda,\infty}(\gamma g_\infty). \end{aligned} \tag{5.158}$$

To relate this to a function on the upper-half plane $\mathbb{H}$, we use the Iwasawa decomposition

$$g_\infty = b_\infty k_\infty = n_\infty a_\infty k_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{5.159}$$

which yields

$$E(\mathsf{f}_{\lambda,\infty}, g_\infty) = e^{iw\theta} \sum_{B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \mathsf{f}_{\lambda,\infty}(\gamma b_\infty). \tag{5.160}$$

It remains to evaluate $\mathsf{f}_{\lambda,\infty}(\gamma b_\infty)$. To this end we perform an additional Iwasawa decomposition of γb_∞ with the result:

$$\gamma b_\infty = b'_\infty k'_\infty, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{5.161}$$

with

$$b'_\infty = \begin{pmatrix} \frac{y^{1/2}}{|cz+d|} & \star \\ 0 & \frac{|cz+d|}{y^{1/2}} \end{pmatrix}, \qquad k'_\infty = \begin{pmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{pmatrix} = \frac{1}{|cz+d|} \begin{pmatrix} cx+d & -cy \\ cy & cx+d \end{pmatrix}, \tag{5.162}$$

and $z = x + iy = b_\infty \cdot i$. Further using that $e^{i\theta} = \cos\theta + i \sin\theta$ we find

$$e^{i\theta'} = \frac{|cz+d|}{cz+d}, \tag{5.163}$$

and hence, by (5.149) and (5.152), the section in (5.160) evaluates to

$$\mathsf{f}_{\lambda,\infty}(\gamma b_\infty) = \mathsf{f}_{\lambda,\infty}(b'_\infty k'_\infty) = \left(\frac{|cz+d|}{cz+d}\right)^w \chi_{s,\infty}(b'_\infty) = \left(\frac{|cz+d|}{cz+d}\right)^w \frac{y^s}{|cz+d|^{2s}}. \tag{5.164}$$

We thereby arrive at the following explicit expression for the Eisenstein series

$$E(\mathsf{f}_{(s,w),\infty}, g_\infty) = e^{iw\theta} \sum_{(c,d)=1} \frac{y^s}{(cz+d)^w |cz+d|^{2s-w}}. \tag{5.165}$$

This is a non-holomorphic function on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$ with weight $e^{iw\theta}$ under the right action of $k_\infty \in SO(2,\mathbb{R})$.

5.5.2 Representation theoretic interpretation

Let us now analyse the Eisenstein series (5.165) a little closer. First observe that $E(\mathsf{f}_{(s,w),\infty}, g_\infty)$ interpolates between the classical holomorphic and non-holomorphic Eisenstein series on the upper-half plane. Indeed, restricting the value of s to $s = w/2$ we obtain

$$E(\mathsf{f}_{(w/2,w),\infty}, g_\infty) = e^{iw\theta} y^{w/2} \sum_{(c,d)=1} \frac{1}{(cz+d)^w} = e^{iw\theta} y^{w/2} E_w(z), \tag{5.166}$$

which we recognize as $\varphi_f(g_\infty)$ in the terminology of section 5.1 (see eq. (5.22)) with $f = E_w(z)$ being the classical weight w holomorphic Eisenstein series on $\mathbb{H}$. Similarly, restricting to $w = 0$ in (5.165) we obtain the classical non-holomorphic Eisenstein series

$$E(\mathsf{f}_{(s,0),\infty}, g_\infty) = \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}} = E(s,z). \tag{5.167}$$

Note that this is compatible with the fact that fixing $w = 0$ is equivalent to choosing the local section $\mathsf{f}_{\lambda,p}$ to be spherical also at the archimedean place, $\mathsf{f}_{\lambda,\infty} = \mathsf{f}^{\circ}_{\lambda,\infty}$. The Eisenstein series $E(\mathsf{f}_\lambda, g)$ in (5.99) then reduces to (5.53) which is indeed the adelisation of $E(s, z)$.

While the non-holomorphic Eisenstein series $E(s, z)$ is naturally associated with the principal series $\mathrm{Ind}_{B(\mathbb{R})}^{SL(2,\mathbb{R})}\chi_s$, the holomorphic Eisenstein series $E_w(z)$ is rather associated with the so-called *holomorphic discrete series* $\mathcal{D}(w)$ of $SL(2,\mathbb{R})$. Let us recall some properties of the discrete series. For $w \in \mathbb{Z}_+$, let $\mathcal{H}(w)$ be the Hilbert space of holomorphic square-integrable functions on the upper-half plane $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$. Then the holomorphic discrete series $\mathcal{D}(w)$ representation of $SL(2,\mathbb{R})$ consists of functions $f \in \mathcal{H}(w)$ transforming as follows:

$$f(z) \longmapsto (cz+d)^w f\left(\frac{az+b}{cz+d}\right), \tag{5.168}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. This is precisely the transformation property of holomorphic modular forms in $\mathcal{M}_w(SL(2,\mathbb{Z}))$. To understand more precisely the role of Eisenstein series in this context, let us note that the following differential operator

$$L_w := -(z-\bar{z})\frac{\partial}{\partial \bar{z}} - \frac{w}{2} \tag{5.169}$$

annihilates E_w:

$$L_w \cdot E_w(z) = 0. \tag{5.170}$$

Moreover, we also observe that another differential operator

$$R_w := (z-\bar{z})\frac{\partial}{\partial z} + \frac{w}{2} \tag{5.171}$$

raises the weight of the Eisenstein series by 2,

$$R_w \cdot E_w(z) = E_{w+2}(z). \tag{5.172}$$

The operators (R_w, L_w) are called *Maass operators* and they can be understood representation-theoretically as follows. Let φ_f be the lift of $f = E_w$ to $SL(2,\mathbb{R})$ as in section 5.1.3. Recall that φ_f satisfies the differential equation

$$D_F \cdot \varphi_f = -2ie^{-2i\theta}\left(y\frac{\partial}{\partial \bar{z}} - \frac{1}{4}\frac{\partial}{\partial \theta}\right)\varphi_f = 0, \tag{5.173}$$

where D_F is the differential operator-realisation of the negative Chevalley generator F of $SL(2,\mathbb{R})$ in the compact basis (5.29) (see also appendix A for more details). Using (5.166) we can rewrite this as

$$D_F \cdot \varphi_f = e^{iw\theta} y^{w/2} L_w \cdot E_w(z) = 0, \tag{5.174}$$

revealing that the Maass operator L_w is nothing but the Chevalley generator F after evaluating the derivative on θ. Similarly, the Maass operator R_w is the positive Chevalley operator E. We conclude from this that the holomorphic Eisenstein series $E_w(z)$ is the lowest weight vector in the holomorphic discrete series of $SL(2,\mathbb{R})$, with weight w as measured by the Cartan generator H (c.f. (5.29)):

$$D_H \cdot \varphi_f = w\varphi_f. \tag{5.175}$$

One can also check that

$$D_H\cdot(D_E\cdot\varphi_f) = [D_H, D_E]\cdot\varphi_f + D_E\cdot(D_H\cdot\varphi_f) = 2D_E\cdot\varphi_f + wD_E\cdot\varphi_f = (w+2)D_E\cdot\varphi_f, \tag{5.176}$$

This means that φ_f is the lowest weight state in a representation of $\mathfrak{sl}(2,\mathbb{R})$, whose states are obtained by acting successively with the raising operator D_E:

$$\{\varphi_f, D_E\cdot\varphi_f, D_E^2\cdot\varphi_f, \dots\}. \tag{5.177}$$

Here each vector $D_E^n\cdot\varphi_f$, $n\geq 0$, is an automorphic form on $SL(2,\mathbb{A})$, and hence belongs to the space $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$. The span of the states (5.177) is a subspace V of $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ that is clearly invariant under the $\mathfrak{sl}(2,\mathbb{R})$-action. It is furthermore preserved by $K = SO(2,\mathbb{R})$, since each vector $D_E^n\cdot\varphi_f \in \mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ is K-finite by definition 5.6. Thus, the vector space V spanned by (5.177) is a $(\mathfrak{g}, K)$-module. This is the $(\mathfrak{g}, K)$-module underlying the holomorphic discrete series $\mathcal{D}(w)$.

In general the principal series $\mathrm{Ind}_{B(\mathbb{R})}^{SL(2,\mathbb{R})}\chi_s$ is not a lowest (or highest) weight representation; indeed the general Eisenstein series (5.165) is not annihilated by either of D_E or D_F. However, as we restrict to the integer points $s = w/2$ of the complex weight space where χ_s lives, we land on an irreducible submodule of $\mathrm{Ind}_{B(\mathbb{R})}^{SL(2,\mathbb{R})}\chi_s$ which can be identified with the holomorphic discrete series $\mathcal{D}(w)$. In other words, we have discovered the well-known fact that the holomorphic discrete series can be embedded into the principal series for special values of the inducing character:

$$\mathcal{D}(w) \subset \mathrm{Ind}_{B(\mathbb{R})}^{SL(2,\mathbb{R})}\chi_s\big|_{s=w/2}. \tag{5.178}$$

We should in fact be a little more careful. In (5.166) it is understood that the weight is restricted to be a positive integer $w > 0$. We should therefore distinguish between positive and negative weights in the spherical vector (5.152). The case $w > 0$ leads to (5.166) as we just discussed. The negative weight case $w < 0$ leads to the same conclusion, except that the restriction (5.166) now corresponds to the *anti-holomorphic* Eisenstein series $\overline{E}_w(\bar{z})$. This Eisenstein series is then naturally associated with the anti-holomorphic discrete series $\overline{\mathcal{D}}(w)$ of $SL(2,\mathbb{R})$ which is defined analogously to (5.168) for antiholomorphic functions $\overline{f}(\bar{z})$. The anti-holomorphic Eisenstein series $\overline{E}_w(\bar{z})$ lifts to a function $\varphi_{\overline{f}}$ which is annihilated by D_E, rather than D_F and can therefore be interpreted as a *highest weight vector* of $\overline{\mathcal{D}}(w)$ with weight $-w$. The negative Chevalley generator D_F then lowers the weight by 2.

The above discussion shows that both the holomorphic and anti-holomorphic discrete series can be embedded into the principal series. The complement is a finite-dimensional representation of $SL(2,\mathbb{R})$, known as Sym^{w-1}; this is the w-dimensional symmetric power representation of $SL(2,\mathbb{R})$ acting on homogeneous degree $w-1$ polynomials in two real

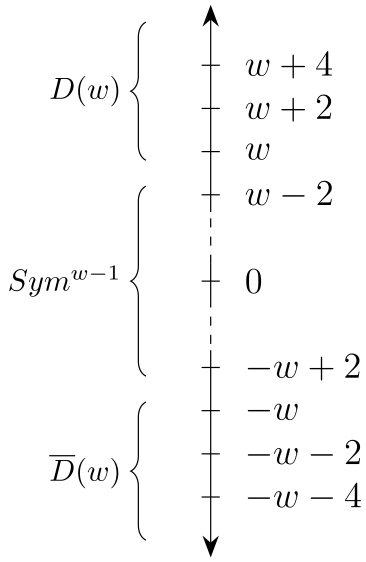


Figure 5.1: *Weight diagram for $SL(2,\mathbb{R})$.*

variables. Indeed, it is easy to see from the weight diagram in figure 5.1 that the number of weights that are excluded from the holomorphic and anti-holomorphic discrete series are precisely equal to w. This implies that Sym^{w-1} as the following quotient of the principal series by the discrete series:

$$\mathrm{Ind}_{B(\mathbb{R})}^{SL(2,\mathbb{R})}\chi_{w/2}\Big/\Big(\mathcal{D}(w)\oplus\overline{\mathcal{D}}(w)\Big) = Sym^{w-1}. \tag{5.179}$$

5.6 Eisenstein series for non-minimal parabolics*

With a little effort one can generalise the construction of Eisenstein series to any parabolic subgroup. In what follows we restrict to standard parabolics, that is, those that contain the Borel subgroup $B(\mathbb{A}) = A(\mathbb{A})N(\mathbb{A})$ as discussed in section 4.1.3. Fix such a standard parabolic subgroup $P(\mathbb{A}) \subset G(\mathbb{A})$ with Langlands decomposition as in (4.39)

$$P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A}) = M(\mathbb{A})A_P(\mathbb{A})U(\mathbb{A}) \tag{5.180}$$

The full group $G(\mathbb{A})$ factorises (non-uniquely) as

$$G(\mathbb{A}) = M(\mathbb{A})A_P(\mathbb{A})U(\mathbb{A})K_{\mathbb{A}}. \tag{5.181}$$

For an arbitrary element of $G(\mathbb{A})$ we write

$$g = luk = ma_Puk, \qquad l\in L(\mathbb{A}),\ m\in M(\mathbb{A}),\ a_P\in A_P(\mathbb{A}),\ u\in U(\mathbb{A}),\ k\in K_{\mathbb{A}}. \tag{5.182}$$

See Example 5.28 for some details of these decompositions in the cases $G(\mathbb{A}) = GL(n,\mathbb{A})$ and $G(\mathbb{A}) = SL(n,\mathbb{A})$.

5.6.1 Multiplicative characters

We now want to define multiplicative characters on $P(\mathbb{A})$ analogously to what we did for the Borel subgroup in section 5.3.1. These will be homomorphisms

$$\chi_P : P(\mathbb{Q})\backslash P(\mathbb{A}) \to \mathbb{C}^\times . \tag{5.183}$$

determined by their restriction to the Levi subgroup

$$\chi_P(lu) = \chi_P(l) \qquad l \in L(\mathbb{A}),\, u \in U(\mathbb{A})\,. \tag{5.184}$$

As in the minimal parabolic case the characters can be described by roots, but now the image in root space will be $\mathfrak{a}_P^*$, the lie algebra of A_P, instead of $\mathfrak{h}^*$ and we will use a generalisation of the logarithm map H from (5.85).

Let $H_P : P(\mathbb{A}) \to \mathfrak{a}_P(\mathbb{R})$ be defined by

$$H_P(p) = H_P(lu) = H_P(ma_Pu) = H_P(a_P) = \log|a_P| \qquad a_P \in A_P \subseteq A \tag{5.185}$$

where the absolute value is defined as in (5.87).

A character can then be defined using a weight $\lambda_P \in \mathfrak{a}_P^*$ as

$$\chi_P(l) = e^{\langle \lambda_P + \rho_P | H_P(l) \rangle} \tag{5.186}$$

where ρ_P is now the restriction of the full Weyl vector to the positive roots $\Delta(\mathfrak{u}) = \Delta_+ \backslash \langle \Sigma \rangle\rangle_+$ of $\mathfrak{g}$ from section 4.1.3

$$\rho_P = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{u})} \alpha. \tag{5.187}$$

In Example 5.28 we give some details of the above construction for the case of $GL(n, \mathbb{A})$.

Example 5.28: Parabolic subgroups and characters for $GL(n, \mathbb{A})$

For $G(\mathbb{A}) = GL(n, \mathbb{A})$ there is a bijection between standard parabolic subgroups $P(\mathbb{A})$ and ordered partitions $(n_1, \ldots, n_q)$ of n. It is then sometimes useful to start from this point of view when parametrising the subgroup P instead of the one based on subsets $\Sigma \subset \Pi$ from section 4.1.3.

For a given such partition we then have that $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A}) = M(\mathbb{A})A_P(\mathbb{A})U(\mathbb{A})$ can be expressed explicitly as

$$L(\mathbb{A}) = \left\{ \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_q \end{pmatrix} \middle| \, l_i \in GL(n_i, \mathbb{A}) \right\} \qquad U(\mathbb{A}) = \left\{ \begin{pmatrix} \mathbb{1}_{n_1} & \star & \star \\ & \ddots & \star \\ 0 & & \mathbb{1}_{n_q} \end{pmatrix} \right\} \tag{5.188}$$

$$M(\mathbb{A}) = \left\{ \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_q \end{pmatrix} \middle| \, m_i \in SL(n_i, \mathbb{A}) \right\} \qquad A_P(\mathbb{A}) = \left\{ \begin{pmatrix} a_1 \mathbb{1}_{n_1} & & 0 \\ & \ddots & \\ 0 & & a_q \mathbb{1}_{n_q} \end{pmatrix} \middle| \, a_i \in GL(1, \mathbb{A}) \right\}, \tag{5.189}$$

where $\mathbb{1}_n$ denotes the $n \times n$ identity matrix.

Similarly, instead of working with the Chevalley basis for $\mathfrak{a}_P$ it is useful to choose a basis that reflects the block form in the parametrisation above.

We choose a basis $\tilde{H}_i$ for $\mathfrak{a}_P$ such that H_P from (5.185) becomes

$$H_P\Big(\begin{pmatrix} a_1\mathbb{1}_{n_1} & & 0 \\ & \ddots & \\ 0 & & a_q\mathbb{1}_{n_q}\end{pmatrix}\Big) = \sum_{i=1}^{q} n_i \log|a_i|\, \tilde{H}_i\,. \tag{5.190}$$

For

$$l = \begin{pmatrix} l_1 & & 0 \\ & \ddots & \\ 0 & & l_q \end{pmatrix} \tag{5.191}$$

we then obtain that

$$H_P(l) = \sum_{i=1}^{q} \log|\det l_i|\, \tilde{H}_i\,. \tag{5.192}$$

Now introduce a basis $\tilde{\Lambda}_i$ for $\mathfrak{a}_P^*$ dual to $\tilde{H}_i$, that is, $\langle \tilde{\Lambda}_i | \tilde{H}_j\rangle = \delta_{ij}$, and let λ_P and ρ_P in $\mathfrak{a}_P^*$ be parametrised by $\lambda_P = \sum_{i=1}^{q} s_i \tilde{\Lambda}_i$ and $\rho_P = \sum_{i=1}^{q} \rho_i \tilde{\Lambda}_i$ with $s_i, \rho_i \in \mathbb{C}$. Note that since $\tilde{H}_i$ is not the Chevalley basis, the $\tilde{\Lambda}_i$ are not the standard fundamental weights.

Any character on $P(\mathbb{Q})\backslash P(\mathbb{A})$ can then be constructed by

$$\chi_P(lu) = \chi_P(l) = e^{\langle \lambda_P + \rho_P | H_P(l)\rangle} = \prod_{i=1}^{q} |\det l_i|^{s_i+\rho_i}\,. \tag{5.193}$$

For $G = SL(n, \mathbb{A})$ we have the restriction $\prod_{i=1}^{q} a_i = 1$ which reduces the number of independent elements in the sum (5.190) spanning $\mathfrak{a}_P$. In the same way, the parameters in $\lambda_P = \sum_{i=1}^{q} s_i \tilde{\Lambda}_i$ are also restricted. A general character on $P(\mathbb{Q})\backslash P(\mathbb{A})$ for $SL(n, \mathbb{A})$ can thus be seen as special cases of (5.193). Explicitly, we require that $\langle \lambda_P | \mathbb{1}_n\rangle = 0$ and, since $\mathbb{1}_n = \sum_i n_i \tilde{H}_i$, we get the restriction that $\sum_i n_i s_i = 0$.

5.6.2 Parabolically induced representations

Associated with the parabolic subgroup $P(\mathbb{A})$ we now consider the following space of functions

$$I_P(\lambda) = \{ f : G(\mathbb{A}) \to \mathbb{C} \,|\, f(gp) = e^{\langle \lambda + \rho_P | H_P(p)\rangle} f(g),\ g \in G(\mathbb{A}),\ p \in P(\mathbb{A})\} \tag{5.194}$$

where $\lambda \in \mathfrak{a}_P^*$. Note that we have suppressed the subscript P on λ for brevity.

This is the function space of the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi_P = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} e^{\langle \lambda + \rho_P | H_P\rangle}$. We will also refer to this as the *principal series*, although strictly speaking that name should be reserved for the case $P(\mathbb{A}) = B(\mathbb{A})$, the Borel subgroup. In that case we have $I_B(\lambda) = I(\lambda)$ from (5.134). The generic functional dimension of the representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \chi_P$ is, similarly to (5.137), given by

$$\mathrm{GKdim}(I_P(\lambda)) = \dim(G) - \dim(P). \tag{5.195}$$

It is now straightforward to construct an Eisenstein series associated with the induced representation $I_P(\lambda)$. It takes the form

$$E(\lambda, P, g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho_P | H_P(\gamma g)\rangle}. \tag{5.196}$$

In case P is the Borel subgroup we write $E(\lambda, B, g) = E(\lambda, g)$ and we recover the Eisenstein series in (5.139).

Just as for the Borel subgroup, we can of course also start from any standard section $\mathsf{f}_\lambda \in I_P(\lambda)$ and obtain another Eisenstein series

$$E(\mathsf{f}_\lambda, P, g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \mathsf{f}_\lambda(\gamma g). \tag{5.197}$$

One can generalise the Eisenstein series $E(\lambda, P, g)$ even further by modifying the induced representation $I_P(\lambda)$ as follows. For $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$ let σ be a representation of $L(\mathbb{A})$ and define

$$\sigma_\lambda(l) = \sigma(l) e^{\langle \lambda + \rho_P | H_P(l)\rangle}, \qquad l \in L(\mathbb{A}),\ \lambda \in \mathfrak{a}_P^\star(\mathbb{C}). \tag{5.198}$$

We then consider the associated induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\sigma_\lambda$. This corresponds to automorphic forms on $L(\mathbb{Q})\backslash L(\mathbb{A})$, extended to $P(\mathbb{A})$ by triviality on $U(\mathbb{A})$. More specifically, it is the space of functions

$$\phi\ :\ \big(L(\mathbb{Q})U(\mathbb{A})\big)\backslash G(\mathbb{A}) \to \mathbb{C}, \tag{5.199}$$

such that for each fixed $g \in G(\mathbb{A})$ the function

$$\phi_g\ :\ l \to \phi(lg) \tag{5.200}$$

is a vector in the finite-dimensional space V of automorphic forms on $L(\mathbb{Q})\backslash L(\mathbb{A})$ transforming according to the representation σ.

For each $\phi \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\sigma_\lambda,\ \lambda \in \mathfrak{a}_P^\star(\mathbb{C})$ we now have the Eisenstein series

$$E(\lambda, P, \phi, g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) e^{\langle \lambda + \rho_P | H_P(\gamma g)\rangle}. \tag{5.201}$$

In the mathematical literature, one often takes $\phi \in \mathcal{A}_0(L(\mathbb{Q})\backslash L(\mathbb{A}))$, the space of cusp forms on $L(\mathbb{A})$. In this case, Langlands has proven the analytic continuation and functional relation for $E(\lambda, P, \phi, g)$ [218].

Proposition 5.29. *The Eisenstein series $E(\lambda, g)$, induced from the Borel subgroup B, is a special case of the Eisenstein series $E(\lambda, P_{i_*}, \phi, g)$, where P_{i_*} is a maximal parabolic subgroup associated with the simple root α_{i_*} (see section 4.1.3).*

Proof. To see this, we follow the argument in [144]. First note that $B_{L_{i_*}} = L_{i_*} \cap B$ is a Borel subgroup of the Levi $L_{i_*} \subset P_{i_*}$. This implies that any $\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})$ can be uniquely decomposed as $\gamma = \gamma_1\gamma_2$, with

$$\gamma_1 \in B_{L_{i_*}}(\mathbb{Q})\backslash L_{i_*}(\mathbb{Q}), \qquad \gamma_2 \in P_{i_*}(\mathbb{Q})\backslash G(\mathbb{Q}). \tag{5.202}$$

We can thus rewrite the Eisenstein series $E(\lambda, g)$as follows:

$$E(\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma g)\rangle} = \sum_{\gamma_1 \in B_{L_{i_*}}(\mathbb{Q})\backslash L_{i_*}(\mathbb{Q})} \sum_{\gamma_2 \in P_{i_*}(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma_1\gamma_2 g)\rangle}. \tag{5.203}$$

Any complex weight $\lambda \in \mathfrak{a}^\star(\mathbb{C})$ can be decomposed as $\lambda = \lambda_{i_*} + \lambda_{i_*}^{\perp}$, where λ_{i_*} is a complex linear combination of all simple roots different from α_{i_*} and $\lambda_{i_*}^{\perp}$ is the orthogonal complement to these simple roots. Furthermore, for any $l \in L_{i_*}$ generated by only positive roots in the Lie algebra $\mathfrak{l}_{i_*}$ we have

$$\langle \lambda_{i_*} + \rho_{i_*} | H(l) \rangle = 1. \tag{5.204}$$

Hence, in the domain of absolute convergence, we may decompose the summation as

$$\begin{aligned} E(\lambda, g) &= \sum_{\gamma_1 \in B_{L_{i_*}}(\mathbb{Q}) \backslash L_{i_*}(\mathbb{Q})} \sum_{\gamma_2 \in P_{i_*}(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho | H(\gamma_1 \gamma_2 g) \rangle} \\ &= \sum_{\gamma_2 \in P_{i_*}(\mathbb{Q}) \backslash G(\mathbb{Q})} \left[\sum_{\gamma_1 \in B_{L_{i_*}}(\mathbb{Q}) \backslash L_{i_*}(\mathbb{Q})} e^{\langle \lambda_{i_*} + \rho_{i_*} | H(\gamma_2 g) \rangle} \right] e^{\left\langle \lambda_{i_*}^{\perp} + \rho_{i_*}^{\perp} | H(\gamma_2 g) \right\rangle} \\ &= \sum_{\gamma_2 \in P_{i_*}(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\left\langle \lambda_{i_*}^{\perp} + \rho_{i_*}^{\perp} | H(\gamma_2 g) \right\rangle} \phi(\gamma_2 g), \end{aligned} \tag{5.205}$$

where the function

$$\phi(g) = \sum_{\gamma_1 \in B_{L_{i_*}}(\mathbb{Q}) \backslash L_{i_*}(\mathbb{Q})} e^{\langle \lambda_{i_*} + \rho_{i_*} | H(g) \rangle} \tag{5.206}$$

is an Eisenstein series on the Levi L_{i_*}, induced from the Borel subgroup $B_{L_{i_*}}$. □

Remark 5.30. Proposition 5.29 can be straightforwardly generalised to give a relation between Eisenstein series $E(\lambda, g)$, induced from the Borel subgroup B, and Eisenstein series $E(\lambda, P, \phi, g)$ induced from an arbitrary parabolic subgroup P (not necessarily maximal).

To illustrate the general analysis of this section, we shall conclude with two explicit examples dealing with the case of maximal parabolic subgroups $P(\mathbb{A})$. This is the opposite extreme compared to the Borel subgroup, which we recall is a minimal parabolic.

Example 5.31: Eisenstein series on $SL(n, \mathbb{A})$ induced from a maximal parabolic P

Again we consider $SL(n, \mathbb{A})$ and in this example we take $P(\mathbb{A})$ to be a *maximal parabolic*. Maximal parabolic subgroups are simply classified by partitions $n \mapsto (n_1, n_2)$. The Levi decomposition is therefore $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$ with Levi subgroup given by

$$L(\mathbb{A}) = \left\{ \begin{pmatrix} l_1 & \\ & l_2 \end{pmatrix} | \, l_i \in GL(n_i, \mathbb{A}) \right\}. \tag{5.207}$$

The character χ_P evaluates to

$$\chi_P(luk) = \chi_P(l) = |\det l_1|^{s_1 + \rho_1} |\det l_2|^{s_2 + \rho_2}, \tag{5.208}$$

for $l \in L(\mathbb{A}), u \in U(\mathbb{A}), k \in K_{\mathbb{A}}$. Since the restriction to $SL(n, \mathbb{A})$ require $n_1 s_1 + n_2 s_2 = 0$ we effectively only have one independent parameter $s \in \mathbb{C}$.

Let now $\phi \in \mathrm{Ind}_{P(\mathbb{A})}^{SL(n,\mathbb{A})} \sigma_s$ such that

$$\phi(luk) = \phi(l). \tag{5.209}$$

The associated Eisenstein series is

$$E(\lambda, P, s, g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g)\chi_P(\gamma g). \tag{5.210}$$

The final example discusses a construction using a 5-grading of $\mathfrak{g}$ that is possible for all simply-laced ADE groups but $SL(2, \mathbb{A})$.

Example 5.32: Eisenstein series on $E_6.E_7, E_8$ induced from Heisenberg parabolic subgroups

Let now $G(\mathbb{A})$ be the adelisation of either E_6, E_7 or E_8 with Lie algebra $\mathfrak{g}$. We shall analyze the above construction for a very special type of maximal parabolic subgroup of G, known as the *Heisenberg parabolic*, henceforth denoted by P_{Heis}. This parabolic subgroup is associated with the highest root θ of $\mathfrak{g}$. Similar arguments can be made for the ADE-series of simple Lie algebras [191, 192] but then not necessarily resulting in P_{Heis} being maximal.

Associated with θ the Lie algebra exhibits a canonical 5-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{5.211}$$

where the subscript indicates the eigenvalue under the Cartan generator H_θ associated with θ, and $\mathfrak{g}_{\pm 2}$ are one-dimensional subspaces spanned by the corresponding Chevalley generators $E_{\pm\theta}$. The triple $(H_\theta, E_\theta, E_{-\theta})$ generates an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra:

$$[H_\theta, E_\theta] = 2E_\theta, \qquad [H_\theta, F_\theta] = -2F_\theta, \qquad [E_\theta, F_\theta] = H_\theta. \tag{5.212}$$

The zeroth subspace $\mathfrak{g}_0$ is of the form $\mathfrak{m}_{\text{Heis}} \oplus \mathbb{C}H_\theta$, where $\mathfrak{m}_{\text{Heis}} \subset \mathfrak{g}$ is a reductive Lie algebra corresponding to the commutant of the $\mathfrak{sl}(2, \mathbb{R})$-algebra (5.212) inside $\mathfrak{g}$. The nilpotent subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Heisenberg algebra of dimension $\dim_{\mathbb{R}} \mathfrak{g}_1 + 1 \equiv 2d + 1$, with commutator

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2. \tag{5.213}$$

We set

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{5.214}$$

which is the Lie algebra of a maximal parabolic subgroup $P_{\text{Heis}} \subset G$, the *Heisenberg parabolic*. Its Levi and Langlands decompositions are

$$P_{\text{Heis}} = L_{\text{Heis}} U_{\text{Heis}} = M_{\text{Heis}} A_{\text{Heis}} U_{\text{Heis}}, \tag{5.215}$$

where the Levi subgroup $L_{\text{Heis}} = M_{\text{Heis}} A_{\text{Heis}}$ is the exponentiation of $\mathfrak{g}_0$ further decomposing into $\mathfrak{a}_{\text{Heis}} = \mathbb{C}H_\theta$ and $\mathfrak{m}_{\text{Heis}}$ above, and the unipotent radical U_{Heis} is the Heisenberg group whose Lie algebra is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

For P_{Heis} we define a logarithm map $H_P : P_{\text{Heis}}(\mathbb{A}) \to \mathfrak{a}_{\text{Heis}}(\mathbb{R}) = \mathbb{R}H_\theta$ according to (5.185). Where for $a_P = \exp(vH_\theta)$ with $v \in \mathbb{A}$

$$H_P(p) = H_P(m a_P u) = H_P(a_P) = |v| H_\theta\,. \tag{5.216}$$

Let Λ_θ be the weight dual to H_θ, i.e. defined by

$$\langle \Lambda_\theta \mid H_\theta \rangle = 1, \tag{5.217}$$

and parametrise an arbitrary linear functional $\lambda \in \mathfrak{a}^*_{\text{Heis}}(\mathbb{C})$ by $\lambda = 2s\Lambda_\theta - \rho_{P_{\text{Heis}}}$ with $s \in \mathbb{C}$. The Weyl vector is $\rho_{P_{\text{Heis}}} = \Lambda_\theta$ so we have

$$\lambda = (2s-1)\Lambda_\theta, \qquad \lambda + \rho_{P_{\text{Heis}}} = 2s\Lambda_\theta. \tag{5.218}$$

Putting this together we obtain a character

$$\chi_{P_{\text{Heis}}} \equiv \chi_s : P_{\text{Heis}}(\mathbb{Q})\backslash P_{\text{Heis}}(\mathbb{A}) \to \mathbb{C}^\times \tag{5.219}$$

defined by

$$\chi_s = e^{\langle 2s\Lambda_\theta | H_P \rangle}. \tag{5.220}$$

We extend it to all of $G(\mathbb{A})$ by demanding that it is trivial on $K_\mathbb{A}$ by virtue of the decomposition (5.181). Explictly, we have

$$\chi_s(g) = \chi_s(ma_P uk) = \chi_s(a_P) = e^{\langle 2s\Lambda_\theta | H_P(a_P)\rangle} = |v|^{2s}. \tag{5.221}$$

The associated induced representation $\text{Ind}_{P_{\text{Heis}}(\mathbb{A})}^{G(\mathbb{A})}\chi_s$ is called the *degenerate principal series*. At the infinite place it has functional dimension

$$\text{GKdim}\,\text{Ind}_{P_{\text{Heis}}(\mathbb{R})}^{G(\mathbb{R})}\chi_s = \dim P_{\text{Heis}}(\mathbb{R})\backslash G(\mathbb{R}) = \dim \mathfrak{g}_1 \oplus \mathfrak{g}_2 = 2d+1, \tag{5.222}$$

and depends on a single complex parameter s. In contrast, the generic principal series induced from the Borel subgroup B depends on $r = \text{rank}\,\mathfrak{g}$ parameters $(s_1,\dots,s_r) \in \mathbb{C}^r$. Formally one can view $\text{Ind}_{P_{\text{Heis}}(\mathbb{A})}^{G(\mathbb{A})}\chi_s$ as the limit of $\text{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})} e^{\langle \sum_{i=1}^r s_i\Lambda_i | H\rangle}$ when projecting onto the complement of a complex co-dimension one locus in $\mathbb{C}^r$.

For any standard section $\mathsf{f}_s \in \text{Ind}_{P_{\text{Heis}}(\mathbb{A})}^{G(\mathbb{A})}\chi_s$ the Eisenstein series attached to the degenerate principal series is

$$E(s, P_{\text{Heis}}, g) = \sum_{\gamma \in P_{\text{Heis}}(\mathbb{Q})\backslash G(\mathbb{Q})} \mathsf{f}_s(\gamma g). \tag{5.223}$$

This Eisenstein series has interesting properties because its residues at the poles in the complex s-plane give rise to automorphic forms attached to special types of (unipotent) representations of G which have very small functional dimensions (typically of dimension less than $2d+1$). The smallest such representation is known as the *minimal representation* of G and it has functional dimension $d+1$. Automorphic forms attached to minimal representations were analyzed from this point of view in [126], and has also played an important role in physical applications [146, 158, 159, 191, 262, 263, 267]. See also sections 12.1 and 12.4.

Chapter 6

Whittaker vectors and Fourier coefficients

In this chapter, we analyse the general structure of the Fourier expansions of automorphic forms, with particular emphasis on Eisenstein series and the associated theory of Whittaker vectors. We will discuss both local and global aspects. As advanced topics we introduce the useful notion of wave-front set [230, 239–241] and discuss the method of Piatetski-Shapiro and Shalika [260, 295]. General references are [57, 130, 174] and we also found the discussions in [121, 146, 177, 199, 235] very useful.

6.1 Preliminary example: $SL(2,\mathbb{R})$ Whittaker vectors

In section 1.3, we discussed the Fourier expansion of the non-holomorphic Eisenstein series $E(s,z)$ where $z = x+iy$ is on the upper half plane $\mathbb{H}$. Invariance under $SL(2,\mathbb{Z})$ implies the periodicity of the series in the real x-direction:

$$E(s, x+1+iy) = E(s, x+iy), \tag{6.1}$$

and hence we have a Fourier expansion of the form

$$E(s, x+iy) = \sum_{m\in\mathbb{Z}} a_m(y) e^{2\pi i m x}, \tag{6.2}$$

where the y-dependent Fourier coefficients $a_m(y)$ can be extracted from the explicit expansion stated in (1.16) and will be derived in detail in chapter 7 using adelic methods. Let us now reinterpret $E(s,z)$ as a function on $SL(2,\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})K(\mathbb{R})$ according to the prescription in section 5.1.4. To this end we define

$$\varphi_E(g) = \varphi_E(nak) = \varphi_E\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\right) = E(s, x+iy), \tag{6.3}$$

where $n \in N(\mathbb{R})$, $a \in A(\mathbb{R})$, $k \in K(\mathbb{R}) = SO(2,\mathbb{R})$. From this point of view, the periodicity (6.1) of $E(s,z)$ in the variable x is equivalent to the invariance of $\varphi_E(g)$ under discrete left-translations: $\varphi_E(ng) = \varphi_E(g)$, $n \in N(\mathbb{Z})$. This follows from the simple calculation for $n = \left(\begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi_E\left(\begin{pmatrix}1 & 1\\ & 1\end{pmatrix} g\right) &= \varphi_E\left(\begin{pmatrix}1 & x+1\\ & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right)\\ &= E(s, x+1+iy),\end{aligned} \tag{6.4}$$

which equals $\varphi_E(g) = E(s, x+iy)$ by left $N(\mathbb{Z})$-invariance.

More generally, we can consider an automorphic form φ on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$, satisfying

$$\varphi(\gamma g k) = \sigma(k)\varphi(g), \qquad \gamma \in SL(2,\mathbb{Z}),\, k \in K(\mathbb{R}) = SO(2,\mathbb{R}), \tag{6.5}$$

where σ can be a non-trivial finite-dimensional representation of $K(\mathbb{R})$. When σ is non-trivial, the function φ depends on all three coordinates (x, y, θ). When σ is trivial and hence φ independent of k, the function is spherical.

The automorphy of φ includes invariance under $N(\mathbb{Z})$ and therefore $\varphi(g) = \varphi(x, y, \theta)$ will have a Fourier expansion of the same form as the one for $E(s, z)$, although the precise coefficients will of course be different depending on the choice of φ. To pave the way for higher rank groups, we now wish to recast this expansion in a form that can be easily generalised.

To this end, let $\psi : N(\mathbb{Z})\backslash N(\mathbb{R}) \to U(1)$ be a *unitary multiplicative character* on $N(\mathbb{R})$ which is trivial on $N(\mathbb{Z})$. The space of such characters is $\mathrm{Hom}\left(N(\mathbb{Z})\backslash N(\mathbb{R}), U(1)\right) \cong \mathbb{Z}$ and we can parametrise the choice of character by a single integer m via

$$\psi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) = e^{2\pi i m x}, \qquad m \in \mathbb{Z},\, x \in \mathbb{R}. \tag{6.6}$$

This is therefore nothing but a set of Fourier modes. If ψ is *non-trivial*, i.e. $m \neq 0$, we say that ψ is *generic*. For higher rank groups, if a character is non-trivial, it does not necessarily mean that it is also generic. In definition 6.10 we will extend our concept of this notion to the case of higher rank groups by introducing a more refined notion of generic vs. non-generic (or degenerate) characters.

Then, due to the periodicity of the automorphic form, $\varphi(ng) = \varphi(g)$, $n \in N(\mathbb{Z})$, we can write $\varphi(g)$ as a Fourier expansion along $N(\mathbb{R})$:

$$\varphi(g) = \sum_{\psi \in \mathrm{Hom}(N(\mathbb{Z})\backslash N(\mathbb{R}), U(1))} W_\psi(g)\,, \tag{6.7}$$

where the sum runs over all possible characters ψ and hence over $m \in \mathbb{Z}$. We have also defined the *Whittaker vector*

$$W_\psi(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(ng)\overline{\psi(n)}dn, \tag{6.8}$$

with dn the *Haar measure* on $N(\mathbb{Z})\backslash N(\mathbb{R})$. The Haar measure is normalized such that $\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} dn = 1$. The expansion (6.7) is a reformulation of (6.2) as we will now illustrate.

By the Iwasawa decomposition $g = nak$ it follows that $W_\psi(g)$ is determined by its restriction to $A(\mathbb{R})$:

$$\begin{aligned}W_\psi(nak) &= \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(n'nak)\overline{\psi(n')}dn' \\ &= \sigma(k)\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(\tilde{n}a)\overline{\psi(\tilde{n}n^{-1})}d\tilde{n} \\ &= \psi(n)\sigma(k)W_\psi(a),\end{aligned} \tag{6.9}$$

where we used the multiplicativity of ψ as well as the invariance of the Haar measure under translations by $N(\mathbb{R})$. In particular, this allows us to rewrite the expansion in a way that is more akin to the classical form (6.7):

$$\varphi(g) = \sum_\psi W_\psi(ak)\psi(n). \tag{6.10}$$

Note that contrary to standard *harmonic analysis* the function $W_\psi(g)$ is not a numerical coefficient, but also contains explicitly the Fourier variable(s) that one is expanding in. This is made explicit by the factor $\psi(n)$ appearing in (6.9) and (6.10).

Remark 6.1. Strictly speaking, it is more accurate to refer to $W_\psi(g)$ as a *Whittaker function*, leaving the phrase Whittaker vector for the representation-theoretic counterpart, which is an element of a vector space, called the *Whittaker model* (see Definition 6.19). However, we will take the liberty to abuse terminology and often refer to $W_\psi(g)$ as a Whittaker vector.

By an explicit Iwasawa parametrisation of g in terms of the variables (x, y, θ) as in (5.159) and the character ψ in terms of an integer (6.6), the integral (6.8) takes the more familiar form

$$W_\psi(ak) = W_m(y, \theta) = \int_0^1 \varphi(x, y, \theta)e^{-2\pi imx}dx. \tag{6.11}$$

(For trivial $\sigma(k)$ the integral is independent of θ and equal to $a_m(y)$ of (6.2).)

The general $SL(2,\mathbb{R})$ expansion (6.10) contains two types of terms, corresponding to $m = 0$ ($\psi = 1$) and $m \neq 0$ ($\psi \neq 1$) that are useful to distinguish:

Definition 6.2 (Constant terms and Whittaker vectors for $SL(2,\mathbb{R})$). The sum (6.10) can be split into

$$\varphi(g) = W_1(ak) + \sum_{\psi\neq 1} W_\psi(g), \tag{6.12}$$

where the first term is independent of n and called the *constant term*. It is defined by

$$W_1(ak) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \varphi(nak)dn. \tag{6.13}$$

and we will sometimes also denote it by $C(ak) \equiv W_1(ak)$. The functions $W_\psi(g)$ for non-trivial characters ψ ($m \neq 0$) are the proper Whittaker vectors.

Remark 6.3. The functions $W_\psi(g)$ were termed Whittaker functions by Jacquet [174] because they reduce to the classical Whittaker function $W_{k,m}(y)$ for the group $GL(2,\mathbb{R})$. For $SL(2,\mathbb{R})$ they are given basically by modified Bessel functions that arise for the special case $k = 0$; see also appendix C.2. For higher rank groups $G(\mathbb{R})$, Whittaker functions define more complicated special functions as we will study in sections 6.2 and 9.6.

Example 6.4: Fourier and q expansion of holomorphic Eisenstein series

Consider now the example when $\varphi = \varphi_f$ with $f(z) = E_{2w}(z)$ being a weight $2w$ holomorphic Eisenstein series

$$E_{2w}(z) = \frac{1}{2}\sum_{(c,d)=1}\frac{1}{(cz+d)^{2w}}, \qquad \tau \in \mathbb{H},\ w > 1,\ w \in \mathbb{Z}. \tag{6.14}$$

This function is spherical (θ-independent) and has a well-known Fourier expansion

$$E_{2w}(z) = 1 + \sum_{m=1}^{\infty} a_m q^m, \qquad q = e^{2\pi i z}, \tag{6.15}$$

where the coefficients are given by

$$a_m = \frac{2}{\zeta(1-2w)}\sigma_{2w-1}(m), \tag{6.16}$$

with $\sigma_{2w-1}(m)$ the sum over positive divisors as in (1.18)

$$\sigma_s(m) = \sum_{d|m} d^s. \tag{6.17}$$

The coefficients can be alternatively be expressed in terms of *Bernoulli numbers*, see for example [2]. In this case the constant term and Whittaker vectors are given by

$$\begin{aligned} C(a) &\equiv W_1(a) = 1,\\ W_\psi(z) &\equiv W_\psi(na) = W_m(\tau) = \frac{2}{\zeta(1-2w)}\sigma_{2w-1}(m)q^m, \qquad m > 0. \end{aligned} \tag{6.18}$$

Notice that the holomorphicity of $E_{2w}(z)$ requires that $W_m(z)$ vanishes unless $m > 0$. As mentioned in section 5.5, this is due to the holomorphic Eisenstein series' E_{2w} being associated with the discrete series representation of $SL(2,\mathbb{R})$.

For completeness, we also recall the constant terms and Whittaker vectors for the non-holomorphic Eisenstein series $E(s,z)$ on $SL(2,\mathbb{R})$ from the introduction.

Example 6.5: Fourier expansion of non-holomorphic Eisenstein series

In the case when $\varphi = \varphi_E$, with $E(s,z)$ the non-holomorphic Eisenstein series on $\mathbb{H}$, the constant term $W_1(a)$ and Whittaker function $W_\psi(na)$ will be derived in chapter 7 with the result

$$\begin{aligned} W_1(a) &= W_1(y) = y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s}\\ W_\psi(na) &= W_m(x,y) = \frac{2y^{1/2}}{\xi(2s)}|m|^{s-1/2}\sigma_{1-2s}(m)K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}, \qquad \text{with } m \neq 0. \end{aligned} \tag{6.19}$$

In contrast to the holomorphic case, the 'constant term' here is not really constant; it is a function on the Cartan torus that is parametrised by the imaginary part y of $z = x + iy$. As we will see below, this is in fact a general feature, namely the constant term of a spherical automorphic function is only constant with respect to the coordinates along the unipotent radical $N(\mathbb{R})$ of the Borel subgroup $B(\mathbb{R}) \subset G(\mathbb{R})$.

6.2 Fourier expansions and unitary characters

We now turn to the general analysis of Fourier coefficients of automorphic forms on semi-simple Lie groups G, and we also switch to the adelic framework. For this we first require the notion of a unitary character ψ on a unipotent subgroup $U \subset G$ that generalises the Fourier mode $e^{2\pi imx}$ in (6.6). This is discussed in detail in section 6.2.1. We will then discuss the notion of Fourier expansion for different types of unipotent groups U in the sequel.

6.2.1 Unitary characters

Definition 6.6. Let $U(\mathbb{A})$ be a unipotent subgroup of the adelic group $G(\mathbb{A})$. A *unitary character* on $U(\mathbb{A})$ is a group homomorphism

$$\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1) \tag{6.20}$$

and we also require it to be trivial on the discrete subgroup $U(\mathbb{Q}) = U(\mathbb{A}) \cap G(\mathbb{Q})$ since we will study in the context of automorphic forms on $G(\mathbb{A})$ that are invariant under the discrete subgroup $G(\mathbb{Q})$. The space of all unitary characters on $U(\mathbb{A})$ that are trivial on $U(\mathbb{Q})$ are called the integral points of the *character variety*.

Remark 6.7. Unipotent groups are required if one wants to have non-trivial unitary characters. On the simple group $G(\mathbb{A})$ there are no non-trivial unitary characters.

Definition 6.6 generalises (6.6). As ψ is a group homomorphism to the abelian group $U(1)$, it is trivial on the *commutator subgroup*

$$[U, U] = \left\{ u_1 u_2 u_1^{-1} u_2^{-1} \;\middle|\; u_1, u_2 \in U \right\}. \tag{6.21}$$

In other words,

$$\psi([U, U]) = 1, \tag{6.22}$$

such that ψ is sensitive only to the *abelianisation* $[U, U]\backslash U$. We note that $[U, U]$ equals the second member of the derived series of U defined in section 4.1.1. We will discuss the relevance of the derived series for Fourier expansions in more detail below in section 6.2.3.

It is convenient to have a more explicit parametrisation of possible unitary characters ψ. To this end we restrict to the case where U is the unipotent of a standard parabolic subgroup $P = LU$ as defined in section 4.1.3. As always we are working with a fixed choice of split Cartan torus $A \subset G$. Such unipotent groups U can be generated from the product of *one-parameter subgroups*

$$U_\alpha = \left\{ x_\alpha(u_\alpha) = \exp(u_\alpha E_\alpha) \mid u_\alpha \in \mathbb{A} \right\}, \tag{6.23}$$

with α ranging over the subset $\Delta(\mathfrak{u})$ of positive roots of $\mathfrak{g}$ corresponding to the Lie algebra $\mathfrak{u}$ of U:

$$U = \prod_{\alpha \in \Delta(\mathfrak{u})} U_\alpha. \tag{6.24}$$

The restriction of ψ to any of the one-parameter subgroups U_α then yields a unitary character

$$\psi_\alpha : U_\alpha(\mathbb{Q})\backslash U_\alpha(\mathbb{A}) \to U(1). \tag{6.25}$$

As any one-parameter subgroup U_α is abelian and satisfies the isomorphism

$$U_\alpha(\mathbb{Q})\backslash U_\alpha(\mathbb{A}) \cong \mathbb{Q}\backslash\mathbb{A}, \tag{6.26}$$

the unitary character ψ_α can therefore be parametrised by a rational number $m_\alpha \in \mathbb{Q}$ as discussed in section 3.5, see also [80, Thm 5.4.3], and can be thought of as the global function

$$\psi_\alpha\left(x_\alpha(u_\alpha)\right) = e^{2\pi i m_\alpha u_\alpha} \tag{6.27}$$

and we will sometimes refer to the m_α as *mode numbers* or *instanton charges* as this is their interpretation in a string theory context, see chapter 2.

The triviality (6.22) of ψ can then be restated as

$$\psi\left(\prod_{\alpha \in \Delta([\mathfrak{u},\mathfrak{u}])} U_\alpha\right) = 1 \tag{6.28}$$

and the non-trivial unitary characters are therefore sensitive only to the one-parameters subgroups U_α such that α is a 'root' of $\mathfrak{u}$ but not of $[\mathfrak{u},\mathfrak{u}]$. This means that the parametrisation of different unitary characters ψ on U only requires the knowledge of the mode numbers m_α for the positive roots α that belong to $\Delta(\mathfrak{u})$ but not to $\Delta([\mathfrak{u},\mathfrak{u}])$. We define

$$\Delta^{(1)}(\mathfrak{u}) := \Delta(\mathfrak{u}) \setminus \Delta([\mathfrak{u},\mathfrak{u}]) \tag{6.29}$$

to be these roots.

Remark 6.8. The notation $\Delta^{(1)}(\mathfrak{u})$ indicates that these are the 'roots' of the abelianisation $[U,U]\backslash U$ of the degree one piece $U = U^{(1)}$ in the *derived series* of U. See section 6.2.3 for a more detailed discussion of the relevance of the derived series of U for Fourier expansions and section 4.1.1 for the notion of derived series.

The above considerations lead to

Proposition 6.9 (Parametrisation of unitary characters). *Let $U(\mathbb{A})$ be a unipotent subgroup of $G(\mathbb{A})$. Unitary characters $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$ can be parametrised uniquely by a set of mode numbers $\{m_\alpha \in \mathbb{Q} \mid \alpha \in \Delta^{(1)}(\mathfrak{u})\}$. The unitary character is then given by*

$$\psi\left(\prod_{\alpha\in\Delta^{(1)}(\mathfrak{u})} x_\alpha(u_\alpha)\right) = \exp\left(2\pi i \sum_{\alpha\in\Delta^{(1)}(\mathfrak{u})} m_\alpha u_\alpha\right). \tag{6.30}$$

It factorises into local places as in (3.75).

Proof. The triviality of ψ on the commutator subgroup $[U, U]$ shows that it suffices to define ψ on the abelianisation that is constructed from the one-parameter subgroups U_α with $\alpha \in \Delta^{(1)}(\mathfrak{u})$ for which the characters were determined in (6.27) above. The group homomorphism property of ψ then yields the proposition. □

The following notions will be important in the sequel.

Definition 6.10 (Generic and degenerate characters). Let $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to U(1)$ be a *global* character as in (6.30).

(i) ψ is called *generic* if $m_\alpha \neq 0$ for all $\alpha \in \Delta^{(1)}(\mathfrak{u})$, i.e. if the character is non-trivial on each one-parameter subgroups $U_\alpha(\mathbb{A})$ for $\alpha \in \Delta^{(1)}(\mathfrak{u})$.

(ii) If $m_\alpha = 0$ for all $\alpha \in \Delta^{(1)}(\mathfrak{u})$, the character ψ is called *trivial*.

(iii) Furthermore, if $m_\alpha \neq 0$ for at least one, but not all, $\alpha \in \Delta^{(1)}(\mathfrak{u})$, the character ψ is called *non-generic* or *degenerate*.

We illustrate these notions by the following example.

Example 6.11: Unitary characters on the maximal unipotent of $SL(n, \mathbb{A})$

Consider the case $G = SL(n, \mathbb{A})$ and $U(\mathbb{A}) = N(\mathbb{A})$ to be the (maximal) unipotent subgroup of the Borel subgroup $B(\mathbb{A})$, implying $\mathfrak{n} = \mathfrak{u}$. The set $\Delta(\mathfrak{n})$ is given by all positive roots Δ_+ of $\mathfrak{sl}(n)$ and the set $\Delta^{(1)}(\mathfrak{n})$ equals the $(n-1)$ simple roots $\Pi \subset \Delta_+$. In the fundamental representation we can write elements of $n \in N$ as $(n \times n)$-matrices of the form

$$n = \begin{pmatrix} 1 & u_1 & * & * & \cdots \\ & 1 & u_2 & * & \cdots \\ & & \cdots & & \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{pmatrix}. \tag{6.31}$$

The starred entries are of no relevance for the discussion of unitary characters as they are associated with the commutator subgroup $[N, N]$. A character ψ on N is determined by $n-1$ rational numbers m_i $(i = 1, \ldots, n-1)$ such that

$$\psi(n) = \exp(2\pi i \sum_{i=1}^{n-1} m_i u_i). \tag{6.32}$$

The character ψ is generic when all $m_i \neq 0$. It is degenerate when some m_i vanish and then it does not depend on the corresponding one-parameter subgroups.

We recall from section 3.5 that a global unitary character ψ_α on $\mathbb{Q}\backslash\mathbb{A}$ as in (6.27) factorises as

$$\psi_\alpha = \prod_{p\leq\infty} \psi_{\alpha,p}\,, \tag{6.33}$$

where for $p < \infty$

$$\psi_{\alpha,p} : U(\mathbb{Z}_p)\backslash U(\mathbb{Q}_p) \to U(1), \qquad\qquad \psi_{\alpha,p}(x_\alpha(u)) = e^{-2\pi i[m_\alpha u]} \tag{6.34}$$

in terms of the fractional part (3.28) of a p-adic number, and for $p = \infty$

$$\psi_\infty : U(\mathbb{Z})\backslash U(\mathbb{R}) \to U(1), \qquad\qquad \psi_{\alpha,p}(x_\alpha(u)) = e^{2\pi i m_\alpha u}. \tag{6.35}$$

This factorisation extends to characters ψ on unipotent groups U:

$$\psi = \prod_{p\leq\infty} \psi_p. \tag{6.36}$$

Definition 6.10 extends to all local characters ψ_p. Moreover, we have the following notion:

Definition 6.12 (Unramified unitary character). A generic *local* character ψ_p for $p < \infty$ is called *unramified* if for all $\alpha \in \Delta^{(0)}(\mathfrak{u})$ one has

$$\psi_{\alpha,p}\left(e^{uE_\alpha}\right) = e^{-2\pi i[u]}, \qquad u \in \mathbb{Q}_p. \tag{6.37}$$

Equivalently, this means that all instanton charges $|m_\alpha|_p = 1$ in (6.30). We call a global character unramified if $m_\alpha = 1$ for all α.

6.2.2 Fourier coefficients vs. Whittaker vectors

Now that we have the Fourier modes in terms of characters ψ on unipotent subgroups U, it is possible to define Fourier coefficients of automorphic forms.

Definition 6.13 (Fourier coefficient). Let φ be an automorphic form on $G(\mathbb{A})$, i.e., an element of the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, and $U(\mathbb{A})$ a unipotent subgroup of $G(\mathbb{A})$. The *Fourier coefficient* of φ with respect to the unitary character ψ on U is given by:

$$F_\psi(\varphi, g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du. \tag{6.38}$$

(du denotes the invariant Haar measure on U.) The Fourier coefficient can be viewed either as a function on $G(\mathbb{A})$ for fixed φ or as a functional on $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. When it is clear from the context which fixed φ is meant, we may write simply $F_\psi(g)$ for conciseness.

A short calculation similar to (6.9) shows that Fourier coefficients satisfy

$$F_\psi(\varphi, ug) = \psi(u) F_\psi(\varphi, g) \qquad \text{for all } u \in U. \tag{6.39}$$

We make the additional definitions for the case $U(\mathbb{A}) = N(\mathbb{A})$.

Definition 6.14 (Whittaker vector). Let φ be an automorphic form on $G(\mathbb{A})$, $N(\mathbb{A})$ be the maximal unipotent subgroup of a fixed Borel $B(\mathbb{A})$ and ψ be a unitary character on $N(\mathbb{A})$.

(*i*) The integral

$$W_\psi(\varphi, g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ug)\overline{\psi(u)} du. \tag{6.40}$$

is called the *Whittaker vector* of φ with respect to ψ.

(*ii*) If φ is $K_\mathbb{A}$ invariant, the Whittaker vector $W_\psi(\varphi, g)$ is right-invariant under $K_\mathbb{A}$ and the Whittaker vector is then called *spherical*. We denote it by $W_\psi^\circ(\varphi, g)$. The spherical Whittaker vector is completely determined by its values on the Cartan torus $A(\mathbb{A})$: Writing $g = nak$ in Iwasawa decomposed form one has

$$W_\psi^\circ(\varphi, nak) = \psi(n) W_\psi^\circ(\varphi, a). \tag{6.41}$$

This is the case for Eisenstein series.

Remark 6.15. Even though definition 6.14 is a special case of 6.13, it is useful to distinguish this case notationally. Throughout this work, we will denote Whittaker vectors (i.e., Fourier coefficients along the maximal unipotent N) by W_ψ and reserve the notation F_ψ for the case when the unipotent U is different from N. Whittaker vectors, i.e., Fourier coefficients associated with N, will be the main focus of this work and studied in detail in chapter 9 for Eisenstein series.

We note that if U is the unipotent of some standard parabolic subgroup $P = LU$ and φ $K_\mathbb{A}$-invariant, the general Fourier coefficient $F_\psi(\varphi, g)$ is determined by its values on the Levi subgroup L and one could define a spherical Fourier coefficient F_ψ° but we will not make use of this notion.

Definition 6.16 (Constant term). (*i*) The Fourier coefficient of an automorphic form φ with respect to the trivial character $\psi = \mathbb{1}$ on U is called the *constant term along U*:

$$C_U(\varphi, g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug) du. \tag{6.42}$$

It is independent of $u \in U$: $C_U(\varphi, ug) = C_U(\varphi, g)$.

(*ii*) For the case $U = N$, we will call it simply the *constant term* and denote it by

$$C(\varphi, g) \equiv C_N(\varphi, g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)dn. \tag{6.43}$$

If φ is spherical, the constant term is a function only of the Cartan torus $A(\mathbb{A})$: Using Iwasawa decomposition $C(\varphi, nak) = C(\varphi, a)$.

6.2.3 Abelian vs. non-abelian Fourier expansions

In the $SL(2)$ example of section 6.1, the Whittaker vectors W_ψ were used in (6.7) to give a *complete* Fourier expansion of an automorphic form φ by summing over all possible unitary characters ψ.

It is a natural question how this carries over to higher rank groups $G(\mathbb{A})$. In view of proposition 6.9, we can already anticipate that the Fourier expansion with unitary characters ψ on a unipotent group U will in general be incomplete since the characters ψ only depend on the abelianisation $[U, U]\backslash U$, see (6.22). For $SL(2)$ the (maximal) unipotent group N is abelian and we did not have to consider this subtlety. The general statement is

Proposition 6.17 (Partial Fourier sum). *Let $U(\mathbb{A})$ be a unipotent subgroup of $G(\mathbb{A})$ and φ be an automorphic form on $G(\mathbb{A})$. Then the sum of Fourier coefficients over all unitary characters ψ on U yields*

$$\sum_{\psi} F_\psi(\varphi, g) = \int\limits_{[U,U](\mathbb{Q})\backslash[U,U](\mathbb{A})} \varphi(ug)du. \tag{6.44}$$

In other words, the sum of the Fourier coefficients reconstitutes only the average of the automorphic form over the commutator subgroup $[U, U]$. If U is abelian, the Fourier expansion is complete.

Proof. See [235]. □

In order to obtain a complete Fourier expansion when the unipotent U is non-abelian, one has to consider the *derived series* of U (cf. also section 4.1.1):

$$U^{(i+1)} = [U^{(i)}, U^{(i)}], \qquad U^{(1)} = U. \tag{6.45}$$

Since U is unipotent, the derived series trivializes after finitely many steps: $U^{(i_0)} = \{\mathbb{1}\}$ for some $i_0 \geq 1$ and we assume i_0 to be the smallest integer for which $U^{(i_0)} = \{\mathbb{1}\}$. If U is abelian, one has $i_0 = 2$. The successive quotients $U^{(i+1)}\backslash U^{(i)}$ are the abelianisations of the unipotent groups $U^{(i)}$ for any integer $i \geq 1$. A unitary character $\psi^{(i)}$ on $U^{(i)}$ is trivial on $U^{(i+1)}$. One can define Fourier coefficients for any of the $U^{(i)}$ by the same formula as in definition 6.13:

$$F_{\psi^{(i)}}(\varphi, g) = \int\limits_{U^{(i)}(\mathbb{Q})\backslash U^{(i)}(\mathbb{A})} \varphi(ug)\overline{\psi^{(i)}(u)}du. \tag{6.46}$$

As an immediate analogue of proposition 6.17 one has that

$$\sum_{\psi^{(i)}} F_{\psi^{(i)}}(\varphi, g) = \int_{U^{(i+1)}(\mathbb{Q})\backslash U^{(i+1)}(\mathbb{A})} \varphi(ug)du. \tag{6.47}$$

We observe that the right-hand side is nothing but the constant term of φ along $U^{(i+1)}$, corresponding to $\psi^{(i+1)} = 1$. It is therefore natural that the complete *non-abelian Fourier expansion* of φ along U is given by

$$\varphi(g) = C_U(\varphi, g) + \sum_{\psi^{(1)}\neq 1} F_{\psi^{(1)}}(\varphi, g) + \sum_{\psi^{(2)}\neq 1} F_{\psi^{(2)}}(\varphi, g) + \ldots + \sum_{\psi^{(i_0)}\neq 1} F_{\psi^{(i_0)}}(\varphi, g). \tag{6.48}$$

The trivial character $\psi^{(i)} = 1$ is always excluded because the sum of the preceding terms reconstitutes the constant term along $U^{(i)}$ by (6.47). Note that unitary characters $\psi^{(1)}$ are characters on $U^{(1)} = U$ and therefore equal the unitary characters we have been discussing in definition 6.13. We will sometimes refer to the Fourier coefficients in (6.48) associated with $U^{(i)}$ and $i \geq 2$ as *non-abelian Fourier coefficients* and the ones associated with $U^{(1)} = U$ as the *abelian Fourier coefficient*.

The same structure of the expansion and terminology arises for the case when the unipotent U is given by the maximal unipotent N. Then we have

$$\varphi(g) = \underbrace{C(g)}_{\text{constant term}} + \underbrace{\sum_{\psi^{(1)}\neq 1} W_{\psi^{(1)}}(g)}_{\text{abelian term}} + \underbrace{\sum_{\psi^{(2)}\neq 1} W_{\psi^{(2)}}(g)}_{\text{non-abelian term}} + \cdots, \tag{6.49}$$

where we have suppressed the fixed automorphic function φ on the right-hand side.

Remark 6.18. Our main interest in this work lies with the abelian Whittaker vectors $W_{\psi^{(1)}}$ and we will discuss them in more detail in the following sections and in particular in chapter 9. Non-abelian Fourier expansions have been carried out in detail for $SL(3, \mathbb{R})$ in [57,249,267,314] and this will be reviewed in section 9.6. Non-abelian Fourier expansions for the non-split real group $SU(2, 1)$ can be found in [12,173] and some further comments on the non-abelian coefficients will be offered in chapter 12.

6.3 Induced representations and Whittaker models

We now specialise to the case then the automorphic form $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is an Eisenstein series

$$E(\mathsf{f}_\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \mathsf{f}_\lambda(\gamma g), \qquad g \in G(\mathbb{A}), \tag{6.50}$$

constructed from a standard section f_λ of the (in general, non-unitary) principal series $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$, cf. section 5.4.3. Here $\chi = e^{\langle \lambda+\rho|H\rangle}$ is the inducing character on the Borel subgroup $B(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})$, as defined in section 5.3.1. For the constant term of $E(\mathsf{f}_\lambda, g)$ one can derive an explicit formula; this is done in great detail for $SL(2, \mathbb{A})$ in chapter 7. The formula for arbitrary split groups $G(\mathbb{A})$, due to Langlands, will be derived in chapter 8. Here we are interested in the representation theoretic properties of the non-constant (abelian) Whittaker coefficients of $E(\mathsf{f}_\lambda, g)$.

6.3.1 Global considerations

For a character ψ on $N(\mathbb{Q})\backslash N(\mathbb{A})$ the abelian coefficients of $E(\mathsf{f}_\lambda, g)$ are given by the Whittaker function W_ψ of the type (6.40). Plugging $E(\mathsf{f}_\lambda, g)$ from (6.50) into (6.40) and exchanging the order of summation and integration we obtain the formula

$$W_\psi(\mathsf{f}_\lambda, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathsf{f}_\lambda(\gamma n g)\overline{\psi(n)}dn. \tag{6.51}$$

Representation theoretically, $W_\psi(\mathsf{f}_\lambda, g)$ belongs to the *induced representation*

$$\mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi = \left\{ W_\psi : G(\mathbb{A}) \to \mathbb{C} \,\middle|\, W_\psi(ng) = \psi(n)W_\psi(g),\ n \in N(\mathbb{A}) \right\}. \tag{6.52}$$

Equation (6.51) thus gives an embedding

$$I(\lambda) = \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi \hookrightarrow \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi. \tag{6.53}$$

Definition 6.19 (Whittaker model). The space

$$Wh_\psi(\lambda) = \{W_\psi(\mathsf{f}_\lambda) | \mathsf{f}_\lambda \in I(\lambda)\} \subset \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi \tag{6.54}$$

is called a *Whittaker model* of $I(\lambda)$, and its elements *Whittaker vectors*. The associated map

$$\mathsf{f}_\lambda \mapsto W_\psi(\mathsf{f}_\lambda), \tag{6.55}$$

is an *intertwiner* between the principal series $I(\lambda)$ and its Whittaker model $Wh_\psi(\lambda)$.

Remark 6.20. An important result about Whittaker models is their uniqueness: for each fixed section $\mathsf{f}_\lambda \in I(\lambda)$ and fixed generic character ψ there exists a *unique* Whittaker vector $W_\psi(\mathsf{f}_\lambda)$ (see, e.g., [57, 68]). This property is known *multiplicity one* and was shown originally for $GL(n)$ locally for archimedean and non-archimedean fields in [176, 295]. We note that it does not hold for $SL(n)$ if $n > 2$ [23].

In chapter 9 we will show that, *for generic* ψ, the Whittaker vector can be written as a single integral rather than a sum. The argument relies on the Bruhat decomposition of $G(\mathbb{Q})$, which allows one to trade the sum over $\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})$ for a sum over the Weyl group $\mathcal{W}(\mathfrak{g})$. The end result is that the Whittaker function may be written as

$$W_\psi(\mathsf{f}_\lambda, g) = \int\limits_{N(\mathbb{A})} \mathsf{f}_\lambda(w_{\text{long}} n g)\overline{\psi(n)}dn. \tag{6.56}$$

This expression is sometimes known as a *Jacquet–Whittaker integral* [174]. The sum over γ has reduced to a single contribution represented by w_{long}, the longest element in the Weyl group $\mathcal{W}(\mathfrak{g})$ (for the details see chapter 9).

6.3.2 Local considerations

Recall from section 5.4.2 that by Flath's tensor product theorem the principal series decomposes into a product over all places [94]

$$\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi = \bigotimes_{p\leq\infty} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p, \tag{6.57}$$

and we have a similar decomposition for $\mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi$:

$$\mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}\psi = \bigotimes_{p\leq\infty} \mathrm{Ind}_{N(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_p. \tag{6.58}$$

To each standard section $\mathsf{f}_{\lambda,p} \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$ we then have a local (p-adic) Whittaker vector

$$W_{\psi_p}(\mathsf{f}_{\lambda,p}, g) = \int_{N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p)} \mathsf{f}_{\lambda,p}(w_{\mathrm{long}}ng)\overline{\psi_p(n)}dn, \quad g \in G(\mathbb{Q}_p) \tag{6.59}$$

and the global Whittaker model $Wh_\psi(\chi)$ splits accordingly

$$Wh_\psi(\chi) = \bigotimes_{p\leq\infty} Wh_{\psi_p}(\chi_p). \tag{6.60}$$

In chapter 9 we will derive an explicit formula (*Casselman–Shalika formula*) for the p-adic Whittaker function $W_{\psi_p}(\mathsf{f}_{\lambda,p}, g)$, $p < \infty$, in the special case when $W_{\psi_p}(\mathsf{f}_{\lambda,p}, g)$ is *spherical* and ψ unramified, notions that were defined in definitions 6.14 and 6.12, respectively.

For generic characters ψ, the global Whittaker function can then be recovered as an Euler product over all places

$$W_\psi(\mathsf{f}_\lambda, g) = \prod_{p\leq\infty} W_{\psi_p}(\mathsf{f}_{\lambda,p}, g_p), \qquad g \in G(\mathbb{A}),\ g_p \in G(\mathbb{Q}_p). \tag{6.61}$$

It is sometimes useful to separate the finite places $p < \infty$ from the infinite place $p = \infty$ and make the following definition:

Definition 6.21 (finite Whittaker vector). Consider the Whittaker vector W_ψ^{fin} obtained by taking the product over all the *finite places*:

$$W_\psi^{\mathrm{fin}}(\mathsf{f}_\lambda^{\mathrm{fin}}, g_f) = \prod_{p<\infty} W_{\psi_p}(\mathsf{f}_{\lambda,p}, g_p), \qquad g_f = (1; g_2, g_3, \dots) \in G(\mathbb{A}_f). \tag{6.62}$$

We call this the *finite Whittaker vector*.

Remark 6.22. The finite Whittaker vector plays an important role in string theory where it contributes to the *instanton measure*, as we illustrate in example 6.25 below and as was discussed in chapter 2.

6.3.3 Spherical Whittaker vectors

Here we will introduce a special class of Whittaker vectors which are spherical in an appropriate sense. Assume that $\mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ is *unramified*, i.e. for almost all places p the local component $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$ is *spherical*. This implies that there exists a *unique* (up to normalization) section $\mathsf{f}^\circ_{\lambda,p} \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$ that satisfies

$$\mathsf{f}^\circ_{\lambda,p}(bk) = \chi_p(b), \qquad \mathsf{f}^\circ_{\lambda,p}(k) = \mathsf{f}^\circ_{\lambda,p}(1) = 1, \tag{6.63}$$

where $b \in B(\mathbb{Q}_p)$ and $k \in G(\mathbb{Z}_p)$.

Definition 6.23 (spherical vector). We call $\mathsf{f}^\circ_{\lambda,p} \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p$, defined by (6.63), a *spherical vector*.

Definition 6.24 (spherical Whittaker vector). To each spherical vector $\mathsf{f}^\circ_{\lambda,p}$ and generic character ψ_p we can associate a *spherical Whittaker vector* $W^\circ_{\psi_p} \in Wh_{\psi_p}(\chi_p)$, defined by

$$W^\circ_{\psi_p}(\lambda, g) = \int_{N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p)} \mathsf{f}^\circ_{\lambda,p}(w_{\text{long}} n g)\overline{\psi_p(n)}dn, \qquad g \in G(\mathbb{Q}_p). \tag{6.64}$$

As before, The spherical Whittaker vector satisfies the relation

$$W^\circ_{\psi_p}(\lambda, nak) = \psi_p(n) W^\circ_{\psi_p}(\lambda, a), \tag{6.65}$$

where $n \in N(\mathbb{Q}_p)$, $a \in A(\mathbb{Q}_p)$, $k \in G(\mathbb{Z}_p)$. This again implies that $W^\circ_{\psi_p}(\lambda, g)$ is completely determined by its restriction to the Cartan torus $A(\mathbb{Q}_p)$, where it equals

$$W^\circ_{\psi_p}(\lambda, a) = \int_{N(\mathbb{Q}_p)} \mathsf{f}^\circ_{\lambda,p}(w_{\text{long}} n a)\overline{\psi_p(n)}dn. \tag{6.66}$$

Example 6.25: Spherical Whittaker vector for $SL(2,\mathbb{A})$

We now illustrate the discussion for the Eisenstein series $E(s,g)$ on $SL(2,\mathbb{A})$. The results below are all derived in section 7.3. Recall from example 5.27 that the Eisenstein series is obtained by choosing the standard section f_λ to be the spherical vector $\mathsf{f}^\circ_\lambda = \mathsf{f}^\circ_s$, such that

$$E(\mathsf{f}^\circ_s, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \mathsf{f}^\circ_s(\gamma g) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \chi_s(\gamma n a), \tag{6.67}$$

where $\chi_s = e^{\langle 2s\Lambda|H\rangle}$, $\Lambda = \alpha/2$ with α the simple root of $\mathfrak{sl}(2,\mathbb{R})$. The local spherical Whittaker vector (6.66) is

$$W^\circ_{\psi_p}(s, a) = \int_{N(\mathbb{A})} \chi_s(w_{\text{long}} n a)\overline{\psi(n)}dn, \qquad a \in A(\mathbb{Q}_p). \tag{6.68}$$

As will be shown in detail in section 7.3, the integral equals

$$W^\circ_{\psi_\infty}(s, y) = \frac{2\pi^s}{\Gamma(s)} y^{1/2}|m|^{s-1/2}K_{s-1/2}(2\pi|m|y), \tag{6.69}$$

at the archimedean place $p = \infty$ (see (7.73)). Here, $m \in \mathbb{Z}^\times$, and we parametrised the Cartan torus $A(\mathbb{R})$ according to

$$\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}, \qquad y \in \mathbb{R}_{>0}. \tag{6.70}$$

At the non-archimedean places, the integral becomes (cf. (7.77))

$$W^\circ_{\psi_p}(s, v) = |v|_p^{-2s+2} \gamma_p(mv^2)(1 - p^{-2s}) \frac{1 - p^{-2s+1}|mv^2|_p^{2s-1}}{1 - p^{-2s+1}}, \tag{6.71}$$

with $p < \infty$ and $m \in \mathbb{Q}^\times$, and we parametrised the torus $A(\mathbb{Q}_p)$ by

$$\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}, \qquad v \in \mathbb{Q}_p^\times \tag{6.72}$$

for all $p < \infty$.

The associated finite Whittaker vector (6.62), evaluated at the identity $v = 1$, is only non-vanishing for $m \in \mathbb{Z}^\times$ because of the γ_p factors as is seen in section 3.4. For $m \in \mathbb{Z}^\times$

$$W^{\circ,\text{fin}}_\psi(s, 1) = \prod_{p<\infty} W^\circ_{\psi_p}(s, 1) = \left(\prod_{p<\infty} (1 - p^{-2s}) \right) \left(\prod_{p<\infty} \frac{1 - p^{-(2s-1)}\,|m|_p^{2s-1}}{1 - p^{-(2s-1)}} \right). \tag{6.73}$$

The first factor is simply the Euler product (1.21) of the (inverse of the) Riemann zeta function $\zeta(2s)^{-1}$. We will now show that the second factor is actually the divisor sum $\sigma_t(m)$ defined in (1.18) denoting $t = 1 - 2s$ for brevity.

Assume first that $m = p^a$ for some prime p and positive integer a. Then

$$\sigma_t(m) = \sum_{d|m} d^t = 1 + p^t + p^{2t} + \ldots + p^{at} = \frac{1 - p^{(a+1)t}}{1 - p^t}. \tag{6.74}$$

For $m = p^a q^b$ we get

$$\begin{aligned} \sigma_t(m) &= 1 + p^t + q^t + p^{2t} + q^{2t} + p^t q^t + \ldots + p^{at} q^{bt} \\ &= (1 + p^t + \ldots + p^t)(1 + q^t + \ldots + q^{bt}) = \sigma_t(p^a)\sigma_t(q^b). \end{aligned} \tag{6.75}$$

Similarly, for the general case with m having the prime factorisation $m = p_1^{a_1} \cdots p_r^{a_r}$,

$$\sigma_t(m) = \sigma_t(p_1^{a_1}) \cdots \sigma_t(p_r^{a_r}) = \prod_{i=1}^r \frac{1 - p_i^t p_i^{a_i t}}{1 - p_i^t} = \prod_{p<\infty} \frac{1 - p^t\,|m|_p^{-t}}{1 - p^t} \tag{6.76}$$

since $|m|_p = p_j^{-a_j}$ for $p = p_j$ (some j) and otherwise $|m|_p = 1$. In other words, the finite spherical Whittaker vector for $SL(2, \mathbb{A})$ (and the divisor sum σ_t) are *multiplicative*.

Thus, for non-zero integer m

$$W^{\circ,\text{fin}}_\psi(s, 1) = \frac{1}{\zeta(2s)} \sigma_{1-2s}(m). \tag{6.77}$$

Comparing with the discussion in section 1.3 we conclude that the finite Whittaker vector $W_\psi^{\circ,\text{fin}}$, defined in (6.62), is closely related to the *instanton measure* in string theory. More precisely, when evaluating the finite Whittaker vector at the identify in $SL(2,\mathbb{A}_f)$ we obtain the divisor sum which is characteristic for so-called D(−1)-instanton effects in string theory (see [138]). This in fact also holds for more general groups $G(\mathbb{A})$ and gives a strong physics motivation for the detailed analysis of the Casselman–Shalika formula presented in section 9.3.

6.4 Fourier coefficients and nilpotent orbits*

When considering the Fourier expansion along a unipotent radical $U(\mathbb{A})$ that is part of a standard parabolic subgroup $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$, one can group the Fourier integrals (6.38) into orbits of the Levi factor $L(\mathbb{Q})$, see for example [146, 235]. There is a close connection to the theory of nilpotent orbits of the adjoint action of $G(\mathbb{C})$ on its Lie algebra $\mathfrak{g}(\mathbb{C})$ and the notion of wavefront sets through the work of Moeglin–Waldspurger [239, 241], Matumoto [230], Ginzburg–Rallis–Soudry [122–124], Jiang–Liu–Savin [177] and many others.

Remark 6.26. The discussion of the present section only applies to Fourier expansions along unipotent radicals U of non-minimal parabolic subgroups; for expansions along $N(\mathbb{A})$ contained in the (minimal parabolic) Borel subgroup $B(\mathbb{A})$ the orbits under the abelian Levi factor become single points.

6.4.1 Character variety orbits

Let ψ denote a unitary character on $U(\mathbb{A})$ that is trivial on $U(\mathbb{Q})$ and consider the Fourier integral $F_\psi(g) \equiv F_\psi(\varphi, g)$ of an automorphic form φ as defined in definition 6.13. We consider φ fixed for the following discussion and will suppress it in the notation $F_\psi(g)$. Under the action of an element $\gamma \in L(\mathbb{Q})$, that is an element γ of the intersection of the discrete subgroup with the Levi factor, the Fourier coefficient changes as follows

$$\begin{aligned} F_\psi(\gamma g) &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(u\gamma g)\overline{\psi(u)}du = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(\gamma^{-1}u\gamma g)\overline{\psi(u)}du \\ &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(\gamma u\gamma^{-1})}du = F_{\psi^\gamma}(g) \end{aligned} \tag{6.78}$$

where we have used the fact that φ is invariant under discrete transformations as well as the fact that the change of coordinates $u \to \gamma^{-1}u\gamma$ is uni-modular since γ is in the discrete subgroup. In the last step, we have defined the transformed character

$$\psi^\gamma(u) := \psi(\gamma u\gamma^{-1}) \tag{6.79}$$

and identified its Fourier coefficient. The transformed character ψ^γ is well-defined since the Levi component $L(\mathbb{Q})$ acts on $U(\mathbb{Q})$ by conjugation. In view of the terminology introduced in definition 6.6, the orbits thus produced are called *character variety orbits*. We also introduce the following notion:

Definition 6.27. Let ψ be a unitary character on the unipotent subgroup $U(\mathbb{A})$ of a standard parabolic subgroups $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$. The set

$$C_\psi = \{\gamma \in L(\mathbb{Q}) \mid \psi^\gamma = \psi\} \tag{6.80}$$

is called the *stabiliser* of the character ψ. We will sometimes use the same terminology when referring to the action of $L(\mathbb{R})$ or $L(\mathbb{C})$ on the corresponding character variety.

The calculation (6.78) shows that the Fourier coefficient F_ψ is invariant (automorphic) under the stabiliser subgroup C_ψ.

The adjoint action of $L(\mathbb{Q})$ on $U(\mathbb{Q})$ can be described more explicitly by realizing the original character ψ in terms of a weight vector similar to proposition 6.9. The Lie algebra $\mathfrak{u}$ consists of nilpotent elements $X \in \mathfrak{u}$ and we can write an element $u \in U$ as $u = e^X$. A unitary character ψ on U is then given by an element ω of the dual space $\mathfrak{u}^*$ via

$$\psi(e^X) = \exp\left(2\pi i \omega(X)\right) \tag{6.81}$$

and the triviality (6.22) of ψ on the commutator subgroup $[U, U]$ enforces that

$$\omega\left([\mathfrak{u}, \mathfrak{u}]\right) = 0, \tag{6.82}$$

so that ω is not an arbitrary element of $\mathfrak{u}^*$ but one associated with the Lie algebra of the abelianisation $[U, U]\backslash U$. Clearly, the abelianisation $[U, U]\backslash U$ is preserved by the adjoint action of $L(\mathbb{Q})$ on $U(\mathbb{A})$ and $L(\mathbb{Q})$ therefore acts dually on the space of allowed ω. By virtue of (6.78), the Fourier coefficients for all characters in one orbit are related and it suffices to calculate the Fourier coefficient of one representative of an orbit. In practice, it is more convenient to take the dual of ω and study the adjoint nilpotent orbits of the action of $L(\mathbb{Q})$ on $\mathfrak{u}(\mathbb{Q})$, where one can also restrict to the abelian quotient $[\mathfrak{u}, \mathfrak{u}]\backslash \mathfrak{u}$.

Remark 6.28. Let Σ be the subset of the simple roots Π that defines a standard parabolic subgroup $P = LU$, cf. section 4.1.3. The nilpotent Lie algebra $\mathfrak{u}$ of U has a (finite) graded decomposition

$$\mathfrak{u} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{u}_j, \quad \text{with} \quad \mathfrak{u}_j = \left\langle E_\alpha \;\middle|\; \alpha = \sum_{\beta \in \Pi} n_\beta \beta \in \Delta_+ \quad \text{and} \quad \sum_{\alpha \in \Pi \backslash \Sigma} n_\alpha = j \right\rangle. \tag{6.83}$$

Each space $\mathfrak{u}_j$ is preserved by the adjoint action of L and the space of characters ψ on U is dual to $\mathfrak{u}_1$. The character variety orbits can therefore be viewed dually in $\mathfrak{u}_1$. The space $\mathfrak{u}_1$ is isomorphic (as a vector space) to $[\mathfrak{u}, \mathfrak{u}]\backslash \mathfrak{u}$.

Example 6.29: Mirabolic subgroups of $GL(n, \mathbb{R})$

Consider $G(\mathbb{R}) = SL(n, \mathbb{R})$ that can be represented by $(n \times n)$-matrices. A maximal parabolic subgroup can be chosen with Levi factor $L(\mathbb{R}) = GL(n-1, \mathbb{R})$ through the following matrices

$$L(\mathbb{R}) = \left\{ \begin{pmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & & & & \vdots \\ 0 & * & * & \cdots & * & * \end{pmatrix} \right\} = \left\{ \begin{pmatrix} r & 0 \\ 0 & m \end{pmatrix} \,\middle|\, r \in \mathbb{R},\, m \in GL(n-1, \mathbb{R}) \text{ such that } \det(m) = r^{-1} \right\} \tag{6.84}$$

and associated $(n-1)$-dimensional unipotent radical

$$U(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 & u^T \\ 0 & \mathbb{1}_{n-1} \end{pmatrix} \,\middle|\, u \in \mathbb{R}^{n-1} \right\}. \tag{6.85}$$

The unipotent radical is abelian in the present case and acted upon by $L = GL(n-1, \mathbb{R})$. Characters ψ can be thought of as being given by $(n-1)$-column vectors ω that contract into $X \in \mathrm{Lie}(U) = \mathfrak{u}$ and define the character via (6.81). These parabolic subgroups are sometimes referred to as *mirabolic subgroups*.

For the local transformation of the Fourier coefficients (6.78) at the archimedean place one needs to restrict to orbits under $L(\mathbb{Z})$ that force $r = 1$ and $m \in SL(n-1, \mathbb{Z})$ in (6.84) (or $r = -1$ and $\det(m) = -1$ but this does not influence the discussion below). The action of the Levi subgroup $L(\mathbb{Z})$ on $U(\mathbb{R})$ is by

$$\begin{pmatrix} 1 & \\ & m \end{pmatrix} \begin{pmatrix} 1 & u^T \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & m^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u^T m^{-1} \\ & 1 \end{pmatrix}. \tag{6.86}$$

The group $L(\mathbb{Z})$ then acts on the character variety $\mathfrak{u}^*$ modelled by a vector $\omega \in \mathbb{R}^n$ by

$$\omega \mapsto m^{-1} \omega, \tag{6.87}$$

that is, simply by left multiplication of the column vector. The character variety $\{\omega \in \mathbb{R}^{n-1}\}$ decomposes into infinitely many orbits with representatives

$$\begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for} \quad \sigma \in \mathbb{R}_{\geq 0} \tag{6.88}$$

under this action. If the character is trivial on integral points (as is the case for unitary characters trivial on $U(\mathbb{Z})$) the representatives are labelled by $\sigma \in \mathbb{Z}_{\geq 0}$:

$$\mathbb{Z}^{n-1} = \bigcup_{\sigma \in \mathbb{Z}_{\geq 0}} \left\{ SL(n-1, \mathbb{Z}) \cdot \begin{pmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{6.89}$$

An arbitrary vector $\omega \in \mathbb{Z}^{n-1}$ belongs to the orbit with $\sigma = \gcd(\omega)$.

Classifying the orbits of the action of L on U over $\mathbb{Z}$ or $\mathbb{Q}$ is in general a difficult task, see [22, 178, 207, 279] for some results. A slightly coarser description can be obtained by complexifying the Levi subgroup to $L(\mathbb{C})$ and studying the complex orbits. All such complex orbits have been determined in the literature [79, 222, 235], using the methods of Dynkin [91], Kostant [204, 205], Bala–Carter [7, 8], Vinberg [313] and Kac [114].

6.4.2 Wavefront sets and vanishing theorems for Fourier coefficients

There are many different choices of parabolic subgroup $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$ and associated Fourier expansions along their unipotents U. All the different character variety orbits of the action of $L(\mathbb{Q})$ on unitary characters on $U(\mathbb{A})$ are associated with nilpotent elements $\omega \in \mathfrak{u}^* \subset \mathfrak{g}^*$. A given character variety orbit therefore lies in a *coadjoint nilpotent orbit* of the action of $G(\mathbb{R})$ on elements of $\mathfrak{g}^*$ that are dual to nilpotent elements. Properties of automorphic representations of $G(\mathbb{A})$ are only associated with structures arising from G, implying that the character variety orbits (under the action of L) are less fundamental than the *nilpotent orbit* they embed into. We will not fully develop the theory of nilpotent orbits here but refer the reader to the books [62, 71, 300] for a detailed exposition. Below we will mention only some aspects that are of relevance to our discussion.

The approach using nilpotent orbits is useful because it sometimes allows to determine that certain Fourier coefficients must vanish identically without actually calculating them. At the heart of this is the notion of the (complexified) *wavefront set* of an irreducible automorphic representation π (cf. definition 5.22).

Definition 6.30 (Wavefront set). Let π_p be an automorphic representation of $G(\mathbb{Q}_p)$ at a local place p. The *wavefront set* of π_p is given by

$$\mathrm{WF}(\pi_p) = \bigcup_{i\in I} \overline{\mathcal{O}}_i, \tag{6.90}$$

where the $\mathcal{O}_i$ are a finite collection of complex nilpotent orbits and the closure is with respect to the Zariski topology naturally defined on the set of nilpotent elements of $\mathfrak{g}(\mathbb{C})$. The $\mathcal{O}_i$ appearing in the sum are characterized by admitting a non-trivial Fourier coefficient [124, 239].

The wavefront set therefore is the closure of a (set of) nilpotent orbits [30, 183]. Originally, it is defined as the *annihilator ideal* associated with the representation π_p; in the case of the so-called minimal representation it is also referred to as the *Joseph ideal*.

Remark 6.31. We will also use the notion of a *global wavefront set* of an adelic representation $\pi = \otimes_{p\leq\infty}\pi_p$ of $G(\mathbb{A})$. It is a priori not guaranteed that the local wavefront set $\mathrm{WF}(\pi_p)$ does not vary as p varies and therefore one has to treat this notion with care. For Eisenstein series induced by characters of the form (5.88) this does not happen. Global wavefront sets have been discussed for example in [177] where it was also shown that the maximal orbits in wavefront sets have to be so-called *special* orbits. This property was known for local wavefront sets due to [239].

A nilpotent orbit for a Lie algebra $\mathfrak{g}$ is the orbit of a nilpotent element $X \in \mathfrak{g}$ under the action of the adjoint group G with Lie algebra $\mathfrak{g}$, see for example [71, 300] for an introduction. The theorems of [30, 183] show that one can associate (the closure of) a unique nilpotent orbit in $\mathfrak{g}$ to any irreducible automorphic representation π of G, meaning that the wavefront set of irreducible automorphic representation is given by the closure of unique maximal orbit (w.r.t. the partial closure ordering). One can also consider the action of the adjoint group G on the dual Lie algebra $\mathfrak{g}^*$ and study coadjoint nilpotent orbits. Using the non-degenerate Killing form, we can identify adjoint and coadjoint nilpotent orbits. By the correspondence (6.81) one can view characters ψ on some unipotent U as elements of $\mathfrak{g}^*$ and the character variety orbits lie therefore in coadjoint nilpotent orbits.

The link to the $L(\mathbb{C})$-orbits of Fourier coefficients F_{ψ_U} of an automorphic function φ is provided by the theorems of Mœglin–Waldspurger and Matumoto [230, 235, 241] that assert that a Fourier coefficient can only be non-zero if its associated character variety orbit in $\mathfrak{u}^*$ (under the action of $L(\mathbb{C})$) intersects a coadjoint nilpotent orbit in $\mathfrak{g}^* \supset \mathfrak{u}^*$ (under the action of $G(\mathbb{C})$) that belongs to the wavefront set associated with the automorphic representation to which φ belongs.

Example 6.32: Minimal representation of E_6

Suppose φ belongs to the minimal representation of the exceptional Lie group $E_6(\mathbb{R})$ of dimension 78. Then its associated wavefront set is the closure of the minimal orbit (of dimension 22). The minimal representation of E_6 can be realized as a special point in the degenerate principal series representation that is associated with a maximal parabolic subgroup $P = LU$ with Levi factor $L = SO(5,5) \times GL(1)$. The unipotent U in this case is a Heisenberg group of dimensions 21. The (dualized) character variety $\mathfrak{u}_1 = [\mathfrak{u}, \mathfrak{u}] \backslash \mathfrak{u}$ has dimension 20 and is acted upon by $L(\mathbb{R})$. After complexification one finds that $\mathfrak{u}_1$ breaks up into five different character variety orbits under $GL(6, \mathbb{C})$ [235]. Of these only the trivial and the smallest non-trivial one intersect the closure of the minimal coadjoint nilpotent orbit. One concludes that the Fourier coefficients in the remaining three character variety orbits must vanish in the minimal representation.

We also note that the Gelfand–Kirillov dimension of the degenerate principal series in this case is $21 = 20 + 1$, corresponding to the dimension of the Heisenberg group. At the special point where the minimal representation is realized the 20-dimensional space can be polarized into 10 ‘coordinates’ and 10 ‘momenta’ and the Heisenberg algebra is realized on functions of the 10 coordinate variables on which the momenta act as derivative operators. This action of the Heisenberg group extends to all of $E_6(\mathbb{R})$ and can also be given an oscillator realization [158].

This example is based on [126, 146, 235, 263] where more information can be found. The minimal representation discussed here is an example of a small representation that we will discuss in more detail in sections 10.3.2 and 12.1.1.

The connection between nilpotent orbits and Fourier coefficients is made more concrete in the work of Ginzburg [123]. We follow [160] in the following discussion. To the nilpotent orbit $\mathcal{O} \equiv \mathcal{O}_X$ of a nilpotent element $X \in \mathfrak{g}$ one can associate a *Jacobson–Morozov triple* $H, X, Y \in \mathfrak{g}$ that satisfies the standard $\mathfrak{sl}(2)$ Lie algebra relations. The orbit is uniquely characterised by the (unique) Weyl chamber image of H under the action of the Weyl group. This leads to a labelling of nilpotent orbits by weighted Dynkin diagrams, where the weights are non-negative integers. (These integers are less than or equal to two but this does not matter for our discussion.) Any integrally weighted Dynkin diagram gives rise to a graded decomposition

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \tag{6.91}$$

where $\mathfrak{g}_i$ is the space of elements in $\mathfrak{g}$ with eigenvalue i under the adjoint action of the H that lies in the Weyl chamber. All $\mathfrak{g}_i$ are of finite dimension and there are only finitely many non-trivial $\mathfrak{g}_i$ since $\mathfrak{g}$ is finite-dimensional. We define

$$\mathfrak{l}_{\mathcal{O}} = \mathfrak{g}_0, \quad \mathfrak{u}_{\mathcal{O}} = \bigoplus_{i\geq 1} \mathfrak{g}_i \quad \text{and} \quad \mathfrak{v}_{\mathcal{O}} = \bigoplus_{i\geq 2} \mathfrak{g}_i. \tag{6.92}$$

Let $L_{\mathcal{O}}$, $U_{\mathcal{O}}$ and $V_{\mathcal{O}}$ be corresponding subgroups of G. A nilpotent orbit $\mathcal{O}$ has a unique stabiliser $C_{\mathcal{O}} \subset L_{\mathcal{O}}$ that is a reductive group.

Definition 6.33 (orbit Fourier coefficient). Let $\mathcal{O}$ be a non-trivial nilpotent orbit and let $\psi_V : V_{\mathcal{O}}(\mathbb{Q})\backslash V_{\mathcal{O}}(\mathbb{A}) \to U(1)$ be a unitary character on $V_{\mathcal{O}}$. We require ψ_V to have the same stabiliser type under the action of $L_{\mathcal{O}}$ as the stabiliser $C_{\mathcal{O}}$ of the orbit $\mathcal{O}$. Then the *orbit Fourier coefficient* of an automorphic form φ belonging to some automorphic representation π is defined as

$$F_{\mathcal{O}}(\varphi, \psi_V, g) = \int\limits_{V_{\mathcal{O}}(\mathbb{Q})\backslash V_{\mathcal{O}}(\mathbb{A})} \varphi(vg)\overline{\psi_V(v)}dv. \tag{6.93}$$

For the trivial orbit $\mathcal{O} = \{0\}$ we define the orbit Fourier coefficient to be the constant term along the maximal unipotent $N(\mathbb{A})$ as in definition 6.16.

The orbit Fourier coefficients vanish when the orbit does not belong to the wavefront set and allow a rewriting of the Fourier expansion of an automorphic function in terms of a sum over nilpotent orbits. This is similar to the expansion of the *Howe–Harish-Chandra* expansion of the *character distribution* of an automorphic representation [162, 170]:

$$\mu(\pi) = \sum_{\mathcal{O}\in \mathrm{WF}(\pi)} c_{\mathcal{O}}\mu_{\mathcal{O}} \tag{6.94}$$

For local automorphic representations π_p, the numbers $c_{\mathcal{O}}$ are computed by the Moeglin–Waldspurger theorem [241, 277].

Remark 6.34. It is often possible to relate the orbit Fourier coefficients to (degenerate) Whittaker vectors and this was done for example in [107, 123, 160]. Turning the argument around, one might suspect that the wavefront set can be effectively computed by studying the degenerate Whittaker vectors with charges defining the parabolic subgroups defining a nilpotent orbit in the Bala–Carter classification. This is borne out for minimal representations [107, 235] and also well supported for some other small representations relevant for string theory [32, 34, 160], see also the discussion in sections 12.1.1 and 12.1.3.

Degenerate Whittaker vectors and their relation to small representations in the local and global case have also been discussed in detail recently by Gourevitch–Sahi [134, 136, 137].

6.5 Method of Piatetski-Shapiro and Shalika*

The grouping of Fourier coefficients into orbits under a Levi subgroup L discussed in the previous section is a powerful tool for analyzing automorphic forms. This is at the heart of the method of Piatetski-Shapiro and Shalika that expresses an automorphic form on $GL(n,\mathbb{R})$ completely in terms of its Whittaker vectors (with respect to N) [260, 295]. We briefly explain how this connection between Fourier coefficients along U and Whittaker vectors along N comes about in the case of $GL(n,\mathbb{R})$ following [105, 235]. Generalisations to some other groups have been discussed by Miller and Sahi [235].

Let $P = LU$ be a parabolic subgroup of G. According to proposition 6.17 we have for a spherical automorphic form $\varphi(g) = \varphi(gk)$ in an automorphic representation π that

$$\sum_{\psi} F_{\psi}^{\circ}(g) = \int\limits_{U^{(2)}(\mathbb{Q})\backslash U^{(2)}(\mathbb{A})} \varphi(ug)du, \tag{6.95}$$

where $U^{(2)} = [U,U]$ is the derived group of U and

$$F_{\psi}(g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}du \tag{6.96}$$

is the Fourier coefficient of φ along U for the character ψ.

Now we want to group the sum over the characters ψ into complex orbits thanks to (6.78). A given character ψ can have a stabiliser $C_{\psi}(\mathbb{A}) \subset L(\mathbb{A})$ under the action of $L(\mathbb{A})$ and $C_{\psi}(\mathbb{Q}) = C(\mathbb{Q}) \cap L_{\psi_U}(\mathbb{A})$ is a discrete subgroup of it. Writing the set of complex character orbits as $\mathrm{WF}(\pi)$ one can write (6.95) as [235]

$$\sum_{\psi} F_{\psi}^{\circ}(g) = \sum_{\mathcal{O}\in\mathrm{WF}(\pi)} \sum_{\psi\in\mathcal{O}} \sum_{\gamma\in C_{\psi}(\mathbb{Q})\backslash L(\mathbb{Q})} F_{\psi}(\gamma g), \tag{6.97}$$

where the sum over $\psi \in \mathcal{O}$ denotes *single representatives* of the different integral orbits contained in the complex orbit $\mathcal{O} \in \mathrm{WF}(\pi)$. The method of Piatetski-Shapiro and Shalika then uses the fact that $F_{\psi}(\gamma g)$ is a function on the reductive stabiliser $C_{\psi}(\mathbb{A})$ and automorphic under $C_{\psi}(\mathbb{Q})$ and so can be expanded in the same manner, yielding an iterative procedure for determining the Fourier expansion.

In the case of $GL(n,\mathbb{R})$ this can be done very successfully in terms of iterations of parabolic subgroups of the type discussed in example 6.29. As was explained there, the unipotent subgroup is abelian and therefore the Fourier expansion (6.95) recovers the whole automorphic function φ. Moreover, there is a unique non-trivial complex orbit of $GL(n-1,\mathbb{C})$ acting on the $(n-1)$-dimensional $U(\mathbb{C})$. The trivial orbit corresponds to trivial $\psi = 1$ and corresponds to the constant term in the expansion along U, cf. definition 6.16. In order to avoid having to include this term at every iteration step, we now assume until the end of this section that φ is a cusp form. Then the sum over complex orbits in $\mathrm{WF}(\pi)$ has only a single element.

The integral orbits contained in the single complex orbit can also be identified easily in this case. They are represented by (non-zero) integers m_{α_1} and the representatives can be chosen to be such that

$$\psi(u) = e^{2\pi i m_{\alpha_1} u_{\alpha_1}} \tag{6.98}$$

where α_1 is the first simple root. (Allowing m_{α_1} to be integral instead of integral and positive actually overcounts the integral orbits by a factor of two but this has no impact on the final result.) In terms of matrices as in example 6.29 this can be written as

$$u = \exp\begin{pmatrix} 0 & u_{\alpha_1} & u_{\alpha_1+\alpha_2} & \cdots & u_{\alpha_1+\ldots+\alpha_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \omega = (m_{\alpha_1}, 0, 0, \ldots, 0)^T \in \mathfrak{u}^* \tag{6.99}$$

The stabiliser of such a character in $L(\mathbb{Q}) = GL(n-1,\mathbb{Q})$ is given by $GL(n-2,\mathbb{Q}) \subset GL(n-1,\mathbb{Q})$. At the present stage we have therefore from (6.97)

$$\varphi(g) = \sum_{m_{\alpha_1}\in\mathbb{Z}} \sum_{\gamma\in GL(n-2,\mathbb{Q})\backslash GL(n-1,\mathbb{Q})} F_\psi(\gamma g). \tag{6.100}$$

The Fourier coefficient $F_\psi(\gamma g)$ appearing in (6.100) is therefore an automorphic form on $GL(n-1,\mathbb{Q})$ automorphic under $GL(n-2,\mathbb{Q})$. The iteration now consists in repeating the same process for this smaller subgroup. What this will produce is a sum over $m_{\alpha_2} \in \mathbb{Z}$ and an automorphic form on $GL(n-2,\mathbb{A})$ and so on. At the end of the iteration we obtain

$$\varphi(g) = \sum_{m_{\alpha_1},\ldots m_{\alpha_{n-1}}\in\mathbb{Z}} \sum_{\gamma\in N(n-1,\mathbb{Q})\backslash GL(n-1,\mathbb{Q})} W_\psi(\gamma g) \tag{6.101}$$

where

$$W_\psi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\overline{\psi(n)}dn \tag{6.102}$$

is a standard Whittaker vector on N for the character with instanton charges m_α for the simple roots $\alpha \in \Pi$ and as defined in (6.40). (The integration domain is enlarged from U to N by combining some of the intermediate sums over cosets.) Reassembling the sum over all these characters we therefore can also also write

$$\varphi(g) = \sum_{\gamma\in N(n-1,\mathbb{Z})\backslash GL(n-1,\mathbb{Z})} \int_{N^{(2)}(\mathbb{Z})\backslash N^{(2)}(\mathbb{R})} \varphi(n\gamma g)dn \tag{6.103}$$

where $N^{(2)}$ is the derived group of N, cf. (6.45), and we have projected back down to $\mathbb{R}$. The power of the formula (6.101) is that it allows us to reconstruct the whole (cuspidal) automorphic form from its standard Whittaker vectors by taking suitable translates of them. This is important since, according to (6.49), an automorphic function also contains terms beyond the standard Whittaker vectors in its expansion. The result of Piatetski-Shapiro and Shalika tells us how to compute these non-abelian terms as translates of abelian terms. We will see a similar structure later when we study the case of $SL(3)$ in detail in section 9.6.

Chapter 7

Fourier coefficients of Eisenstein series on $SL(2,\mathbb{A})$

In this chapter we apply the formalism developed in chapter 6 to the classical theory of non-holomorphic Eisenstein series $E(s,z)$ on the double coset $SL(2,\mathbb{Z})\backslash\mathbb{H}$, with $z\in\mathbb{H}=SL(2,\mathbb{R})/SO(2,\mathbb{R})$ which was already presented as a canonical example in the introduction. Following the analysis in the previous section, we will consider the adelic treatment of this Eisenstein series. The purpose of this chapter is to give an example for the Fourier expansion of an Eisenstein series, where the method can be made explicit and is still fully tractable. We try to carefully introduce every step in the calculation, however the explanation of some of the underlying theory is postponed to the next chapter, where Langlands' constant term formula is derived in full detail. Where appropriate in this chapter, we refer the reader to the next chapter for more detailed explanations.

7.1 Statement of theorem

Before we state the theorem, let us introduce some of the necessary terminology. Recall from chapter 4 that the adelic group $SL(2,\mathbb{A})$ has the maximal compact subgroup $K_{\mathbb{A}}$, defined by

$$K_{\mathbb{A}}=K(\mathbb{R})\times\prod_{p<\infty}K(\mathbb{Q}_p)=SO(2,\mathbb{R})\times\prod_{p<\infty}SL(2,\mathbb{Z}_p)\,, \tag{7.1}$$

see for example [219]. We then have, by strong approximation (4.64), that

$$SL(2,\mathbb{A})=SL(2,\mathbb{Q})SL(2,\mathbb{R})K_{\mathbb{A}}. \tag{7.2}$$

This decomposition ensures that any automorphic form φ on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$ corresponds to an automorphic form on $SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})/K_{\mathbb{A}}$.

For the adelic group $SL(2,\mathbb{A})$ we have the Iwasawa decomposition

$$SL(2,\mathbb{A})=N(\mathbb{A})A(\mathbb{A})K_{\mathbb{A}}\,. \tag{7.3}$$

Given a generic group element $g = nak$ of $SL(2,\mathbb{A})$, we define the character χ on $SL(2,\mathbb{A})$ in analogy with the general definition (5.88) such that

$$\chi(nak) = |a^{\lambda+\rho}|\,, \tag{7.4}$$

where λ is some weight vector of $\mathfrak{sl}(2,\mathbb{A})$ and ρ is the Weyl vector. In the case of $SL(2,\mathbb{A})$ the space of (complex) weights is one-dimensional and spanned by the fundamental weight Λ_1 dual to the unique simple root α_1 of $\mathfrak{sl}(2,\mathbb{A})$. The Weyl vector ρ is also identical to Λ_1. Therefore, we can parametrise the weight appearing in (7.4) with a single parameter $s \in \mathbb{C}$ as

$$\lambda = 2s\Lambda_1 - \rho = (2s-1)\Lambda_1 \quad \Rightarrow \quad \lambda + \rho = 2s\Lambda_1\,. \tag{7.5}$$

Furthermore, we make use of the function $H(g)$ of (5.84), which denotes the Lie algebra element associated with the abelian part a in the Iwasawa decomposition of $g = nak$, such that $a = \exp(H(g))$. With this function the character can now be written as

$$\chi_s(g) \equiv |a^{\lambda+\rho}| = e^{(\lambda+\rho)(H(g))} = e^{(\lambda+\rho)(H(a))} = e^{2s\Lambda_1(H(a))}\,, \tag{7.6}$$

where we have introduced the notation χ_s for the character parametrised by $s \in \mathbb{C}$. In the case of $SL(2,\mathbb{A})$ we can write all these objects explicitly (in the fundamental representation) as (2×2)-matrices as follows:

$$g = nak = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} k \tag{7.7}$$

with $k \in K_{\mathbb{A}}$ of (7.1). Here, u and v are adelic numbers. In keeping with the identification of the upper half plane (cf. also appendix A)

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} = SL(2,\mathbb{R})/SO(2,\mathbb{R}), \tag{7.8}$$

this requires that at the archimedean place $p = \infty$ we have to use the following parametrisation

$$g_\infty = n_\infty a_\infty k_\infty = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\infty\,, \tag{7.9}$$

with $y > 0$ and $k_\infty \in SO(2)$. Evaluated on the group element (7.7), the character (7.6) yields

$$\chi_s(g) = e^{2s\Lambda_1(H(a))} = |v|^{2s} \tag{7.10}$$

since $H(a) = \log|v| \cdot H_1$ where H_1 is the Cartan generator of $SL(2,\mathbb{A})$ and the norm is the adelic one. For the archimedean place this implies with (7.9) that $\chi_s(g_\infty) = y^s$, where we have embedded g_∞ into $G(\mathbb{A})$ as $g = (g_\infty, 1, 1, \ldots)$.

The adelic Eisenstein series $E(\chi, g)$ is then defined by summing the character over a coset according to $(g \in SL(2,\mathbb{A}))$

$$E(\chi_s, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} e^{(\lambda+\rho)(H(\gamma g))}\,, \tag{7.11}$$

where the Borel subgroup $B(\mathbb{Q}) = N(\mathbb{Q})A(\mathbb{Q})$. Recall that this is the definition of the $SL(2,\mathbb{A})$ Eisenstein series attached to the induced representation which was given in equation (5.146) of example 5.27. The sum converges absolutely for $\text{Re}(s) > 1$.

We are now ready to state the theorem:

Theorem 7.1 (Fourier expansion $SL(2,\mathbb{A})$ Eisenstein series). *The expansion of $E(\chi_s, g)$ with respect to the unipotent radical N of $SL(2,\mathbb{A})$ is given for $g = g_\infty \in SL(2,\mathbb{R}) \subset SL(2,\mathbb{A})$ by:*

$$E(\chi_s, g) = \sum_\psi W_\psi(s,g) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$
$$+ \sum_{m\neq 0} \frac{2}{\xi(2s)} y^{1/2} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}, \quad (7.12)$$

where there terms on the right-hand side of the first line constitute the constant term and the second line provides the non-constant terms. Here, we have used the parametrisation (7.9) for $g_\infty \in SL(2,\mathbb{R})$.

Furthermore, the Eisenstein series satisfies the functional relation

$$E(\chi_s, g) = \frac{\xi(2s-1)}{\xi(2s)} E(\chi_{1-s}, g). \quad (7.13)$$

Proof. The proof of this theorem constitutes the rest of the present chapter. □

To prove the theorem we now wish to analyse the Fourier expansion of $E(\chi_s, g)$ along the unipotent radical N. This was already outlined in section 6.1. According to the general discussion of the previous chapter, we have the following expansion

$$E(\chi_s, g) = \sum_{\psi \in \text{Hom}(N(\mathbb{Q})\backslash N(\mathbb{A}), U(1))} W^\circ_\psi(s,g). \quad (7.14)$$

We recall from section 6.2 that the superscript indicates that the Fourier coefficient (Whittaker vector) W°_ψ is spherical, i.e., K-independent: $W^\circ_\psi(nak) = W^\circ_\psi(na)$. We shall distinguish the 'constant' Fourier coefficient $W^\circ_1(s,g)$ corresponding to the special case of a trivial character $\psi \equiv 1$

$$W^\circ_1(s,g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi_s, ng) dn\,, \quad (7.15)$$

and the remaining 'non-constant' coefficients given by

$$W^\circ_\psi(s,g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi_s, ng)\overline{\psi(n)} dn. \quad (7.16)$$

The expressions $W_1(s,g)$ and $W_\psi(s,g)$ are sometimes simply referred to as the 'constant term' and the 'Fourier coefficients', respectively.

Plugging-in the definition of the Eisenstein series, and interchanging the sum and integration, we can rewrite the coefficients in the following form

$$W_1^\circ(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(\gamma n g)dn, \tag{7.17a}$$

$$W_\psi^\circ(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(\gamma n g)\overline{\psi(n)}dn, \tag{7.17b}$$

where we recall that the sums converge absolutely for $\mathrm{Re}(s) > 1$.

We now proceed with the analysis of the constant and non-constant terms, starting with the constant term.

7.2 Constant term

As seen in more detail in section 8.2, the constant term (7.17a) can be re-written as

$$\begin{aligned} W_1^\circ(s,g) &= \sum_{\gamma\in B(\mathbb{Q})\backslash SL(2,\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(\gamma n g)dn \\ &= \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})} \; \sum_{\delta\in\gamma^{-1}B(\mathbb{Q})\gamma\cap B(\mathbb{Q})\backslash B(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(\gamma\delta n a)dn. \end{aligned} \tag{7.18}$$

Because of the quotient by $B(\mathbb{Q})$ on the left in the original γ sum, we must make sure to not overcount the coset representatives δ and this is achieved by the restriction on the δ sum. To simplify the integral further we shall need the following result:

Proposition 7.2 (Bruhat decomposition).

$$SL(2,\mathbb{Q}) = \bigcup_{w\in\mathcal{W}} B(\mathbb{Q})wB(\mathbb{Q}). \tag{7.19}$$

Proof. To establish (7.19) we begin by noting that for the first coset representative w, we have the double coset $BwB = B$ and for the second coset representative we get

$$\begin{aligned} B\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}B &= \left\{ \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \tilde{a} & \tilde{b} \\ & \tilde{a}^{-1} \end{pmatrix} \,:\, b,\tilde{b}\in\mathbb{Q},\, a,\tilde{a}\in\mathbb{Q}^\times \right\} \\ &= \left\{ \begin{pmatrix} -b\tilde{a} & a\tilde{a}^{-1} - b\tilde{b} \\ -a^{-1}\tilde{a} & -a^{-1}\tilde{b} \end{pmatrix} \,:\, b,\tilde{b}\in\mathbb{Q},\, a,\tilde{a}\in\mathbb{Q}^\times \right\} \\ &= \left\{ \begin{pmatrix} a & \frac{ad-1}{c} \\ c & d \end{pmatrix} \,:\, a,d\in\mathbb{Q},\, c\in\mathbb{Q}^\times \right\} \end{aligned} \tag{7.20}$$

and hence the Bruhat decomposition (7.19) corresponds to the division of $SL(2,\mathbb{Q})$ into those matrices with lower left entry equal to zero and those where it is non-zero. □

Using this we can now unfold the δ sum in (7.18) to the integration domain by enlarging it, which yields

$$W_1^\circ(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash SL(2,\mathbb{A})/B(\mathbb{Q})} \int\limits_{\gamma^{-1}B(\mathbb{Q})\gamma\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(\gamma n g)dn. \tag{7.21}$$

The measure on this larger space is induced from the embedding $N(\mathbb{Q}) \to N(\mathbb{A})$.

We can simplify the summation even further by using the embedding of the Weyl group $\mathcal{W}$ into $K(\mathbb{Q})$; the sum over cosets has only two contributions arising from the trivial and non-trivial coset representatives that can be chosen as

$$w = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{7.22}$$

These correspond precisely to the fundamental Weyl reflections of the Weyl group $\mathcal{W}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{Q})$. Denoting the coset representatives by Weyl words w, we can therefore write the constant term as

$$W_1^\circ(s,g) = \sum_{w\in\mathcal{W}} C_w = \sum_{w\in\mathcal{W}} \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(wng)dn\,, \tag{7.23}$$

where we have defined individual contributions C_w to the constant term that are labelled by elements of the Weyl group.

7.2.1 Trivial Weyl word

In the case when the Weyl word is the trivial Weyl reflection, i.e. $w = \mathbb{1}$, the integral reduces to

$$\begin{aligned} C_{\mathbb{1}} = \int\limits_{B(\mathbb{Q})\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(ng)dn &= \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(ng)dn \\ &= |v|^{2s} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} dn = |v|^{2s}, \end{aligned} \tag{7.24}$$

where we have used the fact that the Haar measure on $N(\mathbb{Q})\backslash N(\mathbb{A})$ is normalized to 1, and applied the definition (7.10) of the character χ_s for the Iwasawa decomposed group element g.

7.2.2 Non-trivial Weyl word

When w is the non-trivial Weyl reflection in (7.22), it is clear that we have a trivial intersection

$$w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q}) = \{\mathbb{1}\} \tag{7.25}$$

and hence the integral for the non-trivial Weyl word simplifies to

$$C_w = \int\limits_{w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q}) \backslash N(\mathbb{A})} \chi_s(wng)dn = \int\limits_{N(\mathbb{A})} \chi_s(wng)dn. \tag{7.26}$$

To evaluate the integral we first note that we can restrict the argument to $\chi_s(wng) = \chi_s(wna)$ since we integrate over $N(\mathbb{A})$ and χ_s is trivial on $K_\mathbb{A}$. Therefore, we have to evaluate

$$\int\limits_{N(\mathbb{A})} \chi_s(wna)dn. \tag{7.27}$$

Now we choose a parametrisation of $N(\mathbb{A})$ by

$$N(\mathbb{A}) = \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \mid u \in \mathbb{A} \right\}, \tag{7.28}$$

and of a as in (7.7) to write the integral explicitly as

$$\int\limits_{N(\mathbb{A})} \chi_s(wna)dn = \int\limits_{A} \chi_s\Bigg(\underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{w}\underbrace{\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}}_{n}\underbrace{\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}}_{a}\Bigg)du \tag{7.29}$$

We now want to separate out how the integral depends on a. This is done by writing

$$wna = waa^{-1}na = (waw^{-1})w(a^{-1}na). \tag{7.30}$$

The a-dependence comes from both parentheses in this relation. The factor in the first parenthesis is

$$waw^{-1} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} -v^{-1} & \\ & -v \end{pmatrix} \tag{7.31}$$

and lies in $A(\mathbb{A})$. It can therefore be extracted from the character χ_s using $\chi_s(waw^{-1}) = |v|^{-2s}$ by the definition (7.10) of χ_s and using the multiplicative properties of χ_s.

The factor in the second parenthesis in (7.30) is a conjugation of $N(\mathbb{A})$ by a diagonal element a and can be undone by a change of integration variable. Explicitly, we have

$$a^{-1}na = \begin{pmatrix} v^{-1} & \\ & v \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix} = \begin{pmatrix} 1 & v^{-2}u \\ & 1 \end{pmatrix} \tag{7.32}$$

Making the change of variables $u \to v^{-2}u$ that maps the (Haar) measure $du \to |v|^2 du$, we can combine the contributions from the two parentheses in (7.30) to obtain the a-dependence

$$\int\limits_{\mathbb{A}} \chi_s(wna)dn = \underbrace{|v|^{-2s}}_{\chi_s(waw^{-1})} \underbrace{|v|^2}_{\text{change of } du} \int\limits_{\mathbb{A}} \chi_s\Bigg(\underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{w}\underbrace{\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}}_{n}\Bigg)du. \tag{7.33}$$

In order to evaluate the remaining integral, we rewrite the character according to

$$\chi_s(wn) = \chi_s(wnw^{-1}w) = \chi(wnw^{-1}), \tag{7.34}$$

where we have used that we have embedded the Weyl group in $K_{\mathbb{A}}$ and the fact that χ_s is right invariant under $K_{\mathbb{A}}$. Inserting the explicit parametrisations for w and n we find

$$wnw^{-1} = \begin{pmatrix} 1 & \\ -u & 1 \end{pmatrix}. \tag{7.35}$$

We see that the Weyl transformation w maps the upper triangular element into a lower triangular element as expected since the non-trivial w maps the (unique) positive root of $SL(2,\mathbb{A})$ to the unique negative root. To evaluate the character χ_s we will need to perform an Iwasawa decomposition of its argument. By Langlands' theory, see [219], the integral (7.33 enjoys complete factorisation into a product

$$\int\limits_{\mathbb{A}} \chi_s\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\right)du = \prod_{p\leq\infty}\int_{\mathbb{Q}_p}\chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right)du \tag{7.36}$$

such that one can analyse the integrals for each prime p separately.

Archimedean place $p = \infty$. We first prove the following result for the archimedean integral corresponding to the real prime at infinity $\mathbb{Q}_\infty = \mathbb{R}$.

Lemma 7.3 (Gindikin–Karpelevich formula for $SL(2,\mathbb{R})$). *At the archimedean place the integral (7.36) evaluates to:*

$$\int_{\mathbb{R}}\chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right)du = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}. \tag{7.37}$$

Proof. At the archimedean place, we denote the parameters of $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ by x and $y^{1/2}$ rather than u and v as shown in (7.9). The integral then becomes

$$\int_{-\infty}^{\infty}\chi_{s,\infty}\left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\right)dx. \tag{7.38}$$

In order to evaluate this we must bring the argument of the character into Iwasawa form, i.e. we must find $n \in N$ and $a \in A$ such that

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = nak, \tag{7.39}$$

for some $k \in K_\infty = SO(2,\mathbb{R})$. This was done in example 4.4 with the result (cf. (4.55)) that

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-x}{1+x^2} \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{1+x^2}^{-1} & \\ & \sqrt{1+x^2} \end{pmatrix}k \tag{7.40}$$

and the character therefore evaluates to

$$\chi_{s,\infty}\left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\right) = \chi_{s,\infty}\left(\begin{pmatrix} \sqrt{1+x^2}^{-1} & \\ & \sqrt{1+x^2} \end{pmatrix}\right) = (\sqrt{1+x^2})^{-2s}. \tag{7.41}$$

We then find for the integral

$$\int_{-\infty}^{\infty} \chi_{s,\infty}\left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\right) dx = \int_{-\infty}^{\infty} (\sqrt{1+x^2})^{-2s} dx = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\,. \tag{7.42}$$

□

Non-archimedean places $p < \infty$. We shall now prove the corresponding result for finite primes:

Lemma 7.4 (Gindikin–Karpelevich formula for $SL(2,\mathbb{Q}_p)$ [219]).

$$\int_{\mathbb{Q}_p} \chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right) du = 1 + \frac{p-1}{p}\frac{p^{-2s+1}}{1-p^{-2s+1}} = \frac{1-p^{-2s}}{1-p^{-2s+1}}. \tag{7.43}$$

Proof. We now consider the terms in (7.36) for which p is a finite prime:

$$\int_{\mathbb{Q}_p} \chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right) du. \tag{7.44}$$

The Iwasawa decomposition of this element was also discussed in example 4.4. When $u \in \mathbb{Z}_p$ we have

$$\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} \in SL(2,\mathbb{Z}_p)\,. \tag{7.45}$$

The compact part of $SL(2,\mathbb{Q}_p)$ is $K_p = SL(2,\mathbb{Z}_p)$ and hence, since χ_s is trivial on K, we have

$$\chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right) = 1\,. \tag{7.46}$$

We may thus split the integral into

$$\int_{\mathbb{Z}_p} du + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right) du\,, \tag{7.47}$$

where the first term is unity by choice of normalisation (3.18) for the measure du. When $u \in \mathbb{Q}_p$ but not in $\mathbb{Z}_p$, i.e. $|u|_p > 1$, we write the matrix in an Iwasawa decomposition (which is not unique, cf. example 4.4)

$$\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & \\ & u \end{pmatrix} k. \tag{7.48}$$

The remaining integral becomes

$$\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} \chi_{s,p}\left(\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & \\ & u \end{pmatrix} k\right) du = \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |u|_p^{-2s} du. \tag{7.49}$$

We recognize this as an integral of the type (3.25) that we already evaluated, so the result is

$$\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |u|_p^{-2s} du = \frac{p-1}{p}\frac{p^{-2s+1}}{1-p^{-2s+1}}. \tag{7.50}$$

Combining this with the first term in (7.47) proves the claim. □

Remark 7.5. Integrals of this type will be evaluated more generally inequation (8.36), leading to a more general *Gindikin–Karpelevich formula*.

7.2.3 The global form of the full constant term

We are now ready to assemble all the pieces and write down the complete result for the constant term $W_1(s,g)$. The only remaining step is to compute the product over all finite primes in (7.36). Recalling the Euler product representation of the Riemann zeta function (1.21), we find

$$\prod_{p<\infty}\int_{\mathbb{Q}_p} \chi_{s,p}\left(\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}\right) du = \prod_{p<\infty}\frac{1-p^{-2s}}{1-p^{-2s+1}} = \frac{\zeta(2s-1)}{\zeta(2s)}. \tag{7.51}$$

Combining this with the result from the archimedean integral (7.42), including the overall pre-factor from (7.33), as well as the contribution from the trivial Weyl word in (7.24) we finally find

$$W_1^\circ(s,g) = |v|^{2s} + \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}|v|^{-2s+2}. \tag{7.52}$$

Here, we have left $v\in\mathbb{A}$. Restricting to $g\in SL(2,\mathbb{R})$ we can write this in terms of $v=y^{1/2}$ instead, with the result that the first term scales like y^s while the second term scales as y^{1-s}. The relation between the exponents is that induced from the non-trivial Weyl reflection:

$$s\to 1-s\,. \tag{7.53}$$

Referring back to our particular parametrisation (7.4) and (7.5) of the character χ_s, we recall that for w being trivial, we have $\lambda+\rho = 2s\Lambda_1$, while for w being the non-trivial Weyl word we obtain $w\lambda+\rho = 2(1-s)\Lambda_1$. Hence the parameter s is seen to be related by the above transformation. Recall from section 3.7.1 that the completed zeta function is given by

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{7.54}$$

and satisfies the functional relation

$$\xi(s) = \xi(1-s). \tag{7.55}$$

Using this we can write the constant term in the following compact way

$$W_1^\circ(s,g) = |v|^{2s} + \frac{\xi(2s-1)}{\xi(2s)}|v|^{-2s+2}. \tag{7.56}$$

For $v = y^{1/2}$ this agrees with the constant term in (1.16) and the statement of theorem 7.1. We note that constant terms therefore satisfy the functional relation

$$W_1^\circ(s,g) = \frac{\xi(2-2s)}{\xi(1-2s)}C(1-s,g) = \frac{\xi(2s-1)}{\xi(2s)}C(1-s,g), \tag{7.57}$$

where the functional relation for the completed Riemann zeta function (3.88) has been used.

7.3 The non-constant Fourier coefficients

The Whittaker vector W_ψ° of (7.17b) we want to compute is given by

$$W_\psi^\circ(s,g) = \sum_{w\in\mathcal{W}} F_{w,\psi} = \sum_{w\in\mathcal{W}} \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(wng)\overline{\psi(n)}dn\,. \tag{7.58}$$

Using the Iwasawa decomposition of $g = n'ak$ and performing a change of variables this expression can be re-written as

$$W_\psi^\circ(s,g) = \overline{\psi(n'^{-1})}\sum_{w\in\mathcal{W}} \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(wna)\overline{\psi(n)}dn\,. \tag{7.59}$$

As for the constant term case, the sum over the Weyl group has two contributions, one each, for when the Weyl word is trivial, $\mathbb{1}$, and non-trivial, w,

$$W_\psi^\circ = F_{\mathbb{1},\psi} + F_{w,\psi}\,. \tag{7.60}$$

The two contributions will be treated separately below.

Given our standard parametrisation of n by the variable u, we define the character ψ (against which we integrate) as a direct product

$$\psi = \prod_{p\leq\infty}\psi_p \tag{7.61}$$

with

$$\psi_p(u) = \begin{cases} e^{2\pi i m u} & \text{for } p=\infty\,,\\ e^{-2\pi i[mu]} & \text{for } p<\infty\,. \end{cases} \tag{7.62}$$

The function $[\,\cdot\,]$ returns the fractional part of a p-adic number as defined in (3.28). An important point here is that we are interested in characters ψ of the continuous group $N(\mathbb{Q})$ embedded diagonally in $N(\mathbb{A})$. Therefore the coefficient m is a rational number and identical in all ψ_p.

In equation (7.62), we have not indicated the conductor m as a subscript on the character ψ_p in contrast to section 3.3 in order to keep the notation light. It is always understood that the conductor is m. Note that for the pre-factor in (7.59) we have $\overline{\psi(n'^{-1})} = \psi(n')$.

7.3.1 Trivial Weyl word

In the trivial case, i.e. when w is equal to the identity matrix, the integral in (7.59) takes the form

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(na)\overline{\psi(n)}dn\,. \tag{7.63}$$

As before, we use the definition $\chi_s(na) = |v|^{2s}$.

The complete expression for the 'trivial' term of the Fourier coefficient is then given by

$$F_{\mathbb{1},\psi}(s,g) = \psi(n')|v|^{2s} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \overline{\psi(n)}dn\,. \tag{7.64}$$

We now proceed to write this expression as a product over all primes, including the place at infinity, as

$$F_{\mathbb{1},\psi}(s,g) = \prod_{p\leq\infty} \psi_p(n')|v|_p^{2s} \int\limits_{N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p)} \overline{\psi_p(u)}du\,. \tag{7.65}$$

This has to be evaluated separately for each prime $p \leq \infty$. Starting with the archimedean $p = \infty$, the domain of integration is $\mathbb{Z}\backslash\mathbb{R} \cong [0,1]$. This leads to the integral

$$F_{\mathbb{1},\psi,\infty}(s,g) = \psi_\infty(n')|v|_\infty^{2s} \int_0^1 e^{-2\pi i m u}du = 0 \tag{7.66}$$

since this is the integral of a periodic function (with mean value zero) over a full period. Therefore, the full Fourier coefficient vanishes for the trivial Weyl word:

$$F_{\mathbb{1},\psi}(s,g) = 0. \tag{7.67}$$

This is the reflection of a general phenomenon that will be discussed in section 9.1 below.

7.3.2 Non-trivial Weyl word

In the case when the Weyl word is non-trivial, with representation

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7.68}$$

the corresponding term in the Fourier coefficient reads

$$F_{w,\psi}(s,g) = \psi(n') \int\limits_{N(\mathbb{A})} \chi_s(wna)\overline{\psi(n)}dn\,. \tag{7.69}$$

We now perform the same transformation (7.30) to remove the a-dependence from χ_s. Under a change of variables $n \to ana^{-1}$, the integration measure transforms as $dn \to |v|^2 dn$, and we obtain

$$\psi(n')|v|^2 \int\limits_{N(\mathbb{A})} \chi_s(wan)\overline{\psi(ana^{-1})}dn\,. \tag{7.70}$$

Inserting $w^{-1}w$ in the argument before and after n, we find for the character $\chi_s(wan) = \chi_s(waw^{-1})\chi_s(wnw^{-1}w) = |v|^{-2s}\chi_s(wnw^{-1})$, where we have again used the fact that χ is right invariant under a Weyl group transformation. The full expression then takes the form

$$\psi(n')|v|^{-2s+2} \int\limits_{N(\mathbb{A})} \chi_s(wnw^{-1})\overline{\psi(ana^{-1})}dn\,. \tag{7.71}$$

Now we write the expression in the standard way as a product over all places

$$F_{w,\psi}(s,g) = \prod_{p\leq\infty} \psi_p(n')|v|_p^{-2s+2} \int\limits_{N(\mathbb{A})} \chi_{s,p}(wnw^{-1})\overline{\psi_p(ana^{-1})}dn \tag{7.72}$$

and evaluate the archimedean and non-archimedean places separately.

The archimedean place $p=\infty$: The Iwasawa decomposition of wnw^{-1} is as in (7.41) and again leads to $\chi_{s,\infty}(wnw^{-1}) = \sqrt{1+x^2}^{-2s}$. Furthermore, the character evaluates to $\psi_\infty(ana^{-1}) = \exp(2\pi imyx)$, such that overall we obtain

$$\begin{aligned} F_{w,\psi,\infty} &= \psi_\infty(n')|y|_\infty^{-s+1} \int\limits_{-\infty}^{\infty} (1+x^2)^{-s} e^{-2\pi imyx}dx \\ &= \frac{2\pi^s}{\Gamma(s)} y^{1/2}|m|^{s-1/2}K_{s-1/2}(2\pi|m|y)\psi_\infty(n'), \end{aligned} \tag{7.73}$$

where we have used the integral representation of the modified Bessel function given in equation (3.54) and $y>0$.

The non-archimedean places $p < \infty$: We have to analyse the integral

$$F_{w,\psi,p} = \psi_p(n')|v|_p^{-2s+2} \int\limits_{\mathbb{Q}_p} \chi_{s,p}(wnw^{-1})\overline{\psi_p(ana^{-1})}dn\,. \tag{7.74}$$

We will set $a = n' = 1$ along the finite primes since we are interested in the Eisenstein series as a function on $SL(2,\mathbb{R})$ only. From the Iwasawa decomposition of wnw^{-1} following (7.44) we know that $\chi_{s,p}(wnw^{-1})$ is given by

$$\chi_{s,p}(wnw^{-1}) = \max(1, |u|_p)^{-2s}, \tag{7.75}$$

where u parametrises $N(\mathbb{Q}_p)$ as in (7.28) and we have to integrate this against the appropriate character

$$\int\limits_{\mathbb{Q}_p} \max(1, |u|_p)^{-2s} e^{2\pi i[mu]}du\,. \tag{7.76}$$

Using example 3.20, this integral evaluates to

$$\int\limits_{\mathbb{Q}_p} \max(1, |u|_p)^{-2s} e^{2\pi i[mu]}du = \gamma_p(m)(1-p^{-2s})\frac{1-p^{-2s+1}|m|_p^{2s-1}}{1-p^{-2s+1}}\,. \tag{7.77}$$

Taking the product over all finite places yields

$$\prod_{p<\infty} F_{w,\psi,p} = \left(\prod_{p<\infty}(1-p^{-2s})\right)\left(\prod_{p<\infty}\gamma_p(m)\frac{1-p^{-(2s-1)}|m|_p^{2s-1}}{1-p^{-(2s-1)}}\right). \tag{7.78}$$

The first factor is equal to $\zeta(2s)^{-1}$ by virtue of (1.21). We can restrict to $m \in \mathbb{Z}$ due to the occurrence of the p-adic Gaussian $\gamma_p(m)$ for all $p < \infty$ as seen in section 3.4. Writing m then in terms of its unique prime factorisation $m = \prod q_i^{k_i}$ with q_i primes, we can rewrite the second factor (cf. example 6.25). Consider first the case when $m = q^k$ for a single prime q. Then (3.16) implies that the second factor can be written as a sum over (positive) divisors of q^k

$$\begin{aligned}\left(\prod_{p<\infty}\frac{1-p^{-(2s-1)}|q^k|_p^{2s-1}}{1-p^{-(2s-1)}}\right) &= \frac{1-q^{-(2s-1)}q^{-k(2s-1)}}{1-q^{-(2s-1)}} = \frac{1-q^{-(k+1)(2s-1)}}{1-q^{-(2s-1)}}\\ &= \sum_{d|q^k} d^{-(2s-1)}.\end{aligned} \tag{7.79}$$

By multiplicativity of the expressions, we therefore obtain for a general integral m

$$\left(\prod_{p<\infty}\gamma_p(m)\frac{1-p^{-(2s-1)}|m|_p^{2s-1}}{1-p^{-(2s-1)}}\right) = \sum_{d|m} d^{-2s+1} =: \sigma_{1-2s}(m), \tag{7.80}$$

where we have used the general divisor sum $\sigma_{1-2s}(m)$ over positive divisor of an integer (cf. (1.18)).

Putting everything together we therefore obtain a non-vanishing coefficient only for integral m with value

$$W_\psi^\circ(s,g) = F_{w,\psi}(s,g) = \frac{2}{\xi(2s)} y^{1/2}|m|^{s-1/2}\sigma_{1-2s}(m)K_{s-1/2}(2\pi|m|y)\psi_\infty(n'), \tag{7.81}$$

where we have used the definition $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ for the completed Riemann ζ-function (1.17).

Finally, we address the functional relation (7.13) for the non-zero Fourier coefficients. The modified Bessel function has the property $K_{s-1/2}(w) = K_{1/2-s}(w)$ for all $w > 0$. For the divisor sum one finds similarly

$$\sigma_{1-2s}(m) = \sum_{d|m} d^{1-2s} = |m|^{1-2s} \sum_{d|m} \left(\frac{|m|}{d}\right)^{2s-1} = |m|^{1-2s}\sigma_{2s-1}(m). \tag{7.82}$$

Putting this together, we obtain

$$W_\psi^\circ(s,g) = \frac{\xi(2s-1)}{\xi(2s)} W_\psi^\circ(1-s,g), \tag{7.83}$$

where again the functional relation (3.88) of the Riemann zeta function was used. This concludes the proof of theorem 7.1.

Chapter 8

Langlands constant term formula

In this chapter we shall provide a detailed proof of the Langlands constant term formula for Eisenstein series on an arbitrary reductive group G. This generalises the results of the previous chapter for $G = SL(2)$. We will also discuss the general functional relation satisfied by Eisenstein series and we explain how to define and evaluate constant terms with respect to non-maximal unipotent radicals.

8.1 Statement of theorem

We start from the following definition of the minimal parabolic Eisenstein series

$$E(\chi, g) = \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g)\,. \tag{8.1}$$

This is a valid rewriting since the cosets $B(\mathbb{Q})\backslash G(\mathbb{Q})$ are in bijection with those of $B(\mathbb{Z})\backslash G(\mathbb{Z})$ (see Example 4.9 for a proof of this for $SL(2)$). When writing (8.1), we can allow $g \in G(\mathbb{A})$. The real function (1.12) is re-obtained by setting $g = (g_\infty, 1, 1, \ldots)$, i.e., setting the components along $G(\mathbb{Q}_p)$ equal to the identity for $p \neq \infty$.

As in section 5.3 we parametrise the character χ by

$$\chi(nak) = a^{\lambda+\rho} \tag{8.2}$$

in terms of a weight λ of the Lie algebra and ρ is the Weyl vector.

Our interest in the present section is to evaluating the so-called constant terms in the minimal parabolic subalgebra B (standard Borel); that is we shall prove:

Theorem 8.1 (Langlands' constant term formula). *The constant term of $E(\lambda, g)$ with respect to the unipotent radical $N \subset B$ is given by:*

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)dn = \sum_{w\in\mathcal{W}} a^{w\lambda+\rho} \prod_{\alpha>0\,|\,w\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}, \tag{8.3}$$

where dn is the Haar measure that is normalised such that $N(\mathbb{Q})\backslash N(\mathbb{A})$ has unit volume.

Proof. The proof of this theorem constitutes the greater part of the present chapter and is contained in sections 8.2 to 8.7. □

Clearly, the constant term (8.4) depends only on a: For $g = n'ak$ in Iwasawa form, right K-invariance and a change of integration variation reduce the dependence to a. In what follows we shall therefore define

$$C(\chi, a) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)dn = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, na)dn\,, \tag{8.4}$$

and we view the integral as a function on the Cartan torus.

8.2 Bruhat decomposition

The first step in evaluating (8.4) is to use the Bruhat decomposition [27]:

$$G(\mathbb{Q}) = \bigcup_{w\in\mathcal{W}} B(\mathbb{Q})wB(\mathbb{Q}) \tag{8.5}$$

that describes the group $G(\mathbb{Q})$ as a disjoint union of double cosets by the Borel subgroup $B(\mathbb{Q}) \subset G(\mathbb{Q})$. The group $\mathcal{W}$ is the Weyl group of $G(\mathbb{R})$. Clearly, we could restrict the group $B(\mathbb{Q})$ on the right to the subgroup generated by those positive step operators that are mapped to negative step operators by the action of w. The ones that stay positive are already contained in the Borel subgroup on the left. One can think of the Bruhat decomposition as the extension of the tessellation of the Cartan subalgebra into Weyl chambers to the full group.

Using the same trick as in section 7.2 we can rewrite the constant term as

$$\begin{aligned} C(\chi, a) &= \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma na)dn \\ &= \sum_{w\in\mathcal{W}} \int_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)dn\,. \end{aligned} \tag{8.6}$$

Continuing from (8.6) we look at the individual terms

$$C_w = \int_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)dn \tag{8.7}$$

and note that the integration domain can be simplified to

$$C_w = \int_{N_w(\mathbb{A})} \chi(wna)dn\,, \tag{8.8}$$

where $N_w(\mathbb{A})$ is generated from a product over the positive roots that are mapped to negative roots by the given Weyl word w

$$N_w(\mathbb{A}) = \prod_{\alpha>0\,|\,w\alpha<0} N_\alpha(\mathbb{A}) \tag{8.9}$$

and $N_\alpha(\mathbb{A})$ is the subgroup of $G(\mathbb{A})$ generated by the step operator E_α and its dimension is given by the length ℓ of the reduced Weyl word w. This simplification of (8.7) uses two facts:

(i) (Upper) Borel elements that get mapped to lower Borel elements by w have trivial intersection with $N(\mathbb{Q})$ and therefore the quotient becomes trivial and leaves an integral over all of $\mathbb{A}$ in that direction.

(ii) If an (upper) Borel element is mapped to another upper Borel element by the action of w, one is left with the integral over the corresponding quotient. However, since the part wn of the argument is then still a Borel element, the character is insensitive to it and one is left with the volume of corresponding Borel directions which is normalised to unity.

Therefore, in (8.8) we have carried out many trivial integrals and are only left with the non-trivial integrals where wn is really a lower Borel element.

8.3 Parametrising the integral

We will eventually evaluate integral (8.8) using a recursive method and we start by parametrising it conveniently. First, we need to know something about $N_w(\mathbb{A})$ defined in (8.9). We fix a reduced expression $w = w_{i_1} w_{i_2} \cdots w_{i_\ell}$ for the Weyl word w of length ℓ. The subscripts refer to the nodes of the Dynkin diagram of $G(\mathbb{Q})$ and w_i are the fundamental reflections that generate the Weyl group. Then one can explicitly enumerate all positive roots that are mapped to negative roots by the action of w as follows. Define

$$\gamma_k = w_{i_\ell} w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k} \tag{8.10}$$

where α_{i_k} is the i_kth simple root. That this gives a valid description of the positive roots generating U_w can be checked easily by induction. Therefore we have

$$\{\alpha > 0\,|\,w\alpha < 0\} = \{\gamma_i\,|\,i = 1, \ldots, \ell\}\,. \tag{8.11}$$

We also note that there is a simple expression for the sum of all these roots:

$$\gamma_1 + \ldots + \gamma_\ell = \rho - w^{-1}\rho \tag{8.12}$$

which can again be checked by induction. We note in particular $\gamma_\ell = \alpha_{i_\ell}$.

In the next step we use the Chevalley basis notation to write elements $u \in N_w(\mathbb{A})$ as

$$u = x_{\gamma_1}(u_1) \cdots x_{\gamma_\ell}(u_\ell) \tag{8.13}$$

with the Chevalley generator $x_\alpha(v)$ being defined by

$$x_\alpha(v) = e^{vE_\alpha}\,, \tag{8.14}$$

where E_α is the generator of the α root space normalised to unity (for both short and long roots) and $v \in \mathbb{A}$ is the parameter of the group element. With this parametrisation, we can rewrite our individual term C_w of (8.8) as

$$C_w = \int_{\mathbb{A}^\ell} \chi\left(w x_{\gamma_1}(u_1)\cdots x_{\gamma_\ell}(u_\ell)a\right) du_1\cdots du_\ell\,. \tag{8.15}$$

8.4 Obtaining the a dependence of the integral

It is now possible to extract the dependence on a from the integral (8.15). This is done by conjugating the abelian element a through to the left in the argument of χ. The result is

$$C_w = \int_{\mathbb{A}^\ell} \chi\left(w a x_{\gamma_1}(a^{-\gamma_1}u_1)\cdots x_{\gamma_\ell}(a^{-\gamma_\ell}u_\ell)\right) du_1\cdots du_\ell\,, \tag{8.16}$$

where we have used the fact that Cartan elements act diagonally on the E_{γ_i} root spaces. In the next step we can move w past a in the argument of χ and employ the multiplicative property (5.80) of the character to obtain

$$\begin{aligned} C_w &= \chi(waw^{-1}) \int_{\mathbb{A}^\ell} \chi\left(w x_{\gamma_1}(a^{-\gamma_1}u_1)\cdots x_{\gamma_\ell}(a^{-\gamma_\ell}u_\ell)\right) du_1\cdots du_\ell \\ &= \chi(waw^{-1})a^{\gamma_1+\ldots\gamma_\ell} \int_{\mathbb{A}^\ell} \chi\left(w x_{\gamma_1}(u_1)\cdots x_{\gamma_\ell}(u_\ell)\right) du_1\cdots du_\ell\,, \end{aligned} \tag{8.17}$$

where we have also rescaled the u-variables and moved the Jacobi factor outside of the integral. In the form (8.17), one can read off the full a-dependence of the constant term. Using (8.2) and (8.12) we can rewrite the a-dependence of the constant term, remembering that χ is $\mathcal{W}$-invariant from the right, as

$$\begin{aligned} C_w &= a^{w^{-1}(\lambda+\rho)}a^{\rho-w^{-1}\rho}I_w \\ &= a^{w^{-1}\lambda+\rho}I_w\,, \end{aligned} \tag{8.18}$$

where the remaining, a-independent integral is

$$I_w = \int_{\mathbb{A}^\ell} \chi\left(x_{w\gamma_1}(u_1)\cdots x_{w\gamma_\ell}(u_\ell)\right) du_1\cdots du_\ell\,, \tag{8.19}$$

and we have applied w to all the Chevalley elements and used again K-invariance of χ on the right.

8.5 Solving the remaining integral by induction

We will now solve (8.19) by induction. First, we note that $w\gamma_i$ is a negative root for all i by virtue of the definition of γ_i. Therefore, the factors $x_{w\gamma_i}(u_i)$ appearing in (8.19) are elements of the lower triangular Borel subgroup of $G(\mathbb{A})$. To evaluate the character χ in (8.19) according to (8.2), we need to perform an Iwasawa decomposition and isolate the $A(\mathbb{A})$ part of the argument of the character χ. We start by Iwasawa decomposing the last Chevalley factor in the argument of the character according to

$$x_{w\gamma_\ell}(u_\ell) = n(u_\ell)a(u_\ell)k(u_\ell)\,. \tag{8.20}$$

The (negative) step operator $E_{w\gamma_\ell}$ that enters in $x_{w\gamma_\ell}(u_\ell)$ is part of an $SL(2,\mathbb{A})$ subgroup of $G(\mathbb{A})$ and the Iwasawa decomposition (8.20) takes place in that subgroup. We choose to label the $SL(2,\mathbb{A})$ subgroup by its *positive* root $-w\gamma_\ell$, so that the corresponding Cartan generator is proportional to $H_{-w\gamma_\ell}$. The problem of Iwasawa decomposing the $SL(2,\mathbb{A})$ associated with $-w\gamma_\ell$ is different for $p=\infty$ and $p<\infty$ and will be treated separately in 8.6.1 and 8.6.2 below.

Inserting the Iwasawa decomposed (8.20) into (8.19), we can again drop the compact element on the right. Then we can conjugate $n(u_\ell)$ through to the left. For this we have to pass through the negative step operators $x_{w\gamma_i}(u_i)$ for $i=1,\dots,\ell-1$. This produces an element that can be arranged as a product of nilpotent elements on the left times negative step operators $x_{w\gamma_i}(u_i')$ for $i=1,\dots,\ell-1$ with different u_i'. The nilpotent elements disappear in the character and the transformation of the space of parameters u_1 to $u_{\ell-1}$ is uni-modular.We tacitly perform the corresponding change of variables $u_i \to u_i'$. We therefore obtain that $n(u_\ell)$ can be completely absorbed and we are left with:

$$I_w = \int_{\mathbb{A}^\ell} \chi\left(x_{w\gamma_1}(u_1)\cdots x_{w\gamma_{\ell-1}}(u_{\ell-1})a(u_\ell)\right) du_1\cdots du_\ell. \tag{8.21}$$

In the next step, we conjugate $a(u_\ell)$ to the left. This rescales again the u variables with the result

$$\begin{aligned} I_w &= \int_{\mathbb{A}^\ell} \chi\left(a(u_\ell)x_{w\gamma_1}(a(u_\ell)^{-w\gamma_1}u_1)\cdots x_{w\gamma_{\ell-1}}(a(u_\ell)^{-w\gamma_{\ell-1}}u_{\ell-1})\right) du_1\cdots du_\ell \\ &= \int_{\mathbb{A}} \chi(a(u_\ell))a(u_\ell)^{w(\gamma_1+\ldots\gamma_{\ell-1})}du_\ell\cdot I_{w'}\,, \end{aligned} \tag{8.22}$$

where we have undone the scaling of the variables at the cost of introducing a Jacobi factor and introduced w' through

$$w = w'w_{i_\ell}, \tag{8.23}$$

i.e., it is obtained from the Weyl word w by removing the right-most fundamental reflection. This is the recursion formula we are after. All that remains now is to evaluate one integral over $\mathbb{A}$.

8.6 The Gindikin–Karpelevich formula

Using the expression (8.2) for the character χ, the desired integral is

$$I_\ell = \int_{\mathbb{A}} a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots\gamma_{\ell-1})} du_\ell = \prod_{p\leq\infty} \int_{\mathbb{Q}_p} a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots\gamma_{\ell-1})} du_\ell, \tag{8.24}$$

and can be evaluated for each finite and infinite prime $\leq \infty$ as follows. For this one needs the explicit Iwasawa decomposition expressions for $a(u_\ell)$ that will be derived below for $p = \infty$ and $p < \infty$. We also introduce the notation

$$a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots\gamma_{\ell-1})} = |\phi_\ell|^{z_\ell+1} \tag{8.25}$$

with (8.2) and

$$z_\ell = (\lambda + w'\rho)(H_{-w\gamma_\ell}) - 1 = -\langle\lambda|w\gamma_\ell\rangle, \tag{8.26}$$

where we have used $a(u_\ell) = e^{\log(\phi_\ell)H_{-w\gamma_\ell}}$ to introduce the Cartan generator $H_{-w\gamma_\ell}$ of the $SL(2,\mathbb{A})$ associated with the $-w\gamma_\ell$ positive root space. We have also used

$$w(\gamma_1 + \ldots + \gamma_{\ell-1}) = w'\rho - \rho \tag{8.27}$$

and

$$(w'\rho)(H_{-w\gamma_\ell}) = -\langle w'\rho|w\gamma_\ell\rangle = -\langle\rho|w_\ell\gamma_\ell\rangle = \langle\rho|\alpha_{i_\ell}\rangle = 1\,, \tag{8.28}$$

since $\gamma_\ell = \alpha_{i_\ell}$ and we have normalised the symmetric bilinear form such that ρ has unit inner product with all simple roots. The precise value of ϕ_ℓ depends on whether one is at $Q_\infty = \mathbb{R} \subset \mathbb{A}$ or at $Q_p \subset \mathbb{A}$ for $p < \infty$.

8.6.1 Integral over $\mathbb{R}$: $p = \infty$

At the archimedean place we have result:

Lemma 8.2 (archimedean Gindikin–Karpelevich formula).

$$\int_{\mathbb{Q}_p} a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots\gamma_{\ell-1})} du_\ell = \sqrt{\pi}\frac{\Gamma(z_\ell/2)}{\Gamma\left((z_\ell+1)/2\right)}. \tag{8.29}$$

Proof. If $u = u_\ell \in \mathbb{R}$, the Iwasawa decomposition (8.20) is (see Example 4.4)

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{u}{1+u^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{1+u^2} & 0 \\ 0 & \sqrt{1+u^2} \end{pmatrix} k \tag{8.30}$$

with

$$k = \frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 1 & -u \\ u & 1 \end{pmatrix} \in SO(2,\mathbb{R}). \tag{8.31}$$

The diagonal matrix is $a(u_\ell)$. Substituting it into the integral (8.24) for $u = u_\ell \in \mathbb{R}$, one obtains with (8.25)

$$\int_{\mathbb{R}} (1+u^2)^{-(z_\ell+1)/2} du = \sqrt{\pi}\frac{\Gamma(z_\ell/2)}{\Gamma\left((z_\ell+1)/2\right)}. \tag{8.32}$$

□

8.6.2 Integral over $\mathbb{Q}_p$ for finite p

We now prove the corresponding result for finite primes:

Lemma 8.3 (non-archimedean Gindikin–Karpelevich formula). *The non-archimedean integral evaluates to*

$$\int_{\mathbb{Q}_p} a(u_\ell)^{\lambda+\rho+w(\gamma_1+\ldots\gamma_{\ell-1})} du_\ell = \frac{1-p^{-z_\ell-1}}{1-p^{-z_\ell}}, \tag{8.33}$$

where $z_\ell > 0$.

Proof. If $u = u_\ell \in \mathbb{Z}_p$, the matrix

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \tag{8.34}$$

is in $SL(2,\mathbb{Z}_p)$ which is the compact part of $SL(2,\mathbb{Q}_p)$. Therefore $a(u_\ell) = 1$ in this case and the integral is trivial.

If $u \in \mathbb{Q}_p \backslash \mathbb{Z}_p$, the (non-unique) Iwasawa decomposition yields (see Example 4.4)

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} k\,. \tag{8.35}$$

Even though the Iwasawa decomposition is not unique, $a(u_\ell)$ is defined uniquely. We therefore obtain

$$\begin{aligned} \int_{\mathbb{Q}_p} |\phi_\ell|_p^{z_\ell+1} du &= \int_{\mathbb{Z}_p} dx + \int_{\mathbb{Q}_p\backslash\mathbb{Z}_p} |u|_p^{-z_\ell-1} du \\ &= 1 + \frac{p-1}{p}\frac{p^{-z_\ell}}{1-p^{-z_\ell}} = \frac{1-p^{-z_\ell-1}}{1-p^{-z_\ell}}\,, \end{aligned} \tag{8.36}$$

where we have used the integral (3.25) and the normalisation of the measure (3.18). □

8.6.3 The global formula

Putting finite and infinite contributions together, we therefore obtain (with z_ℓ as in (8.26))

$$I_\ell = \sqrt{\pi}\frac{\Gamma(z_\ell/2)}{\Gamma\left((z_\ell+1)/2\right)} \prod_{p<\infty} \frac{1-p^{-z_\ell-1}}{1-p^{-z_\ell}} = \frac{\xi(z_\ell)}{\xi(z_\ell+1)}\,, \tag{8.37}$$

where we have used the Euler product formula (1.21) for the Riemann zeta function and its completion

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{8.38}$$

that satisfies the functional relation (cf. (3.88))

$$\xi(s) = \xi(1-s)\,. \tag{8.39}$$

8.7 Assembling the constant term

We can now write the final formula for the constant term (8.4) by assembling (8.18) and the result (8.37) inserted into the recursion relation (8.22). The answer is

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)dn = \sum_{w\in\mathcal{W}} a^{w^{-1}\lambda+\rho} \prod_{\alpha>0\,|\,w\alpha<0} \frac{\xi(-\langle\lambda|w\alpha\rangle)}{\xi(1-\langle\lambda|w\alpha\rangle)}\,. \tag{8.40}$$

By relabelling the sum by $w \to w^{-1}$ we obtain the standard Langlands formula for the constant term in the minimal parabolic

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)dn = \sum_{w\in\mathcal{W}} a^{w\lambda+\rho} \prod_{\alpha>0\,|\,w\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}\,. \tag{8.41}$$

We note that the inner product here is normalised such that $\langle\rho|\alpha_i\rangle = 1$ for all simple roots α_i. Often one denotes the intertwining coefficient by

$$M(w,\lambda) = \prod_{\alpha>0\,|\,w\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}\,. \tag{8.42}$$

This concludes the proof of theorem 8.1.

Remark 8.4. The derivation above of the constant term formula made heavy use of the adeles $\mathbb{A}$. This was most noticeable when evaluating the integral (8.24) that yielded the completed Riemann ζ-functions in their Euler product form. Still, one may wonder whether this level of abstraction was really necessary. For $SL(2,\mathbb{R})$ one can obtain the constant terms alternatively by Poisson resummation techniques, see appendix B for a summary, without ever making reference to p-adic numbers. What this requires, however, is an explicit understanding of the sum over the cosets $B(\mathbb{Z})\backslash G(\mathbb{Z})$ in the definition of the Eisenstein series and their relation to sums over integer lattices. In the general case, the description of these cosets is not easy to characterise and the lattice sum descriptions typically involve representation theoretic constraints on the sum and this is discussed in more detail in section 12.2. None of these details are required for obtaining the constant term formula when using the p-adic description and this is where the power of the method lies.

8.8 Functional relations for Eisenstein series

The definition of the Eisenstein series (8.1) is initially restricted to the domain of (absolute) convergence of the defining sum. As we mentioned before this requires that the weight λ entering in the definition of the character χ through the relation (8.2) has sufficiently large real parts. More precisely, it is required to lie in the so-called Godemont range

$$\mathrm{Re}\langle\lambda|\alpha_i\rangle > 1 \qquad \text{for all simple roots } \alpha_i. \tag{8.43}$$

The Eisenstein series $E(\chi, g)$ can then be defined by analytic continuation in the complexified weight λ to almost all values of λ. Note that a complex weight λ corresponds to a character χ taking values not in $U(1)$ but in $\mathbb{C}^\times$. We are mainly interested in real weights. As shown in [214, 218], this continuation is possible everywhere except for certain hyperplanes that are related to the integral weight lattice.

An important property of the Eisenstein series is that they obey functional relations. More precisely one has:

Theorem 8.5 (Functional relation for Eisenstein series [218]). *For each $w \in \mathcal{W}$ the Eisenstein series $E(\lambda, g)$ satisfies the functional relation:*

$$E(\lambda, g) = M(w, \lambda) E(w\lambda, g)\,. \tag{8.44}$$

In other words, the Eisenstein series along the Weyl orbit of a character are all proportional to each other.

Proof. To prove this, note first that the coefficients $M(w, \lambda)$ given by (8.42) satisfy the following property:

Lemma 8.6.

$$M(w_1 w_2, \lambda) = M(w_1, w_2\lambda) M(w_2, \lambda)\,, \qquad , \forall\, w_1, w_2 \in \mathcal{W}. \tag{8.45}$$

Proof. Assume first that w_1 and w_2 are fundamental Weyl reflections, say w_i and w_j. This yields

$$\begin{aligned} M(w_i, w_j\lambda) M(w_j, \lambda) &= \prod_{\alpha>0\,|\,w_i\alpha<0} \frac{\xi(\langle w_j\lambda|\alpha\rangle)}{\xi(1+\langle w_j\lambda|\alpha\rangle)} \prod_{\alpha>0\,|\,w_j\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)} \\ &= \prod_{w_j\alpha'>0\,|\,w_iw_j\alpha'<0} \frac{\xi(\langle\lambda|\alpha'\rangle)}{\xi(1+\langle\lambda|\alpha'\rangle)} \prod_{\alpha>0\,|\,w_j\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)}, \end{aligned} \tag{8.46}$$

where in the second step we made the substitution $\alpha' = w_j\alpha$ and used the invariance of the bilinear form: $\langle w_j\lambda|\alpha\rangle = \langle\lambda|w_j\alpha\rangle$.

We want to show that the two factors combine into the left hand side of (8.45). To this end we rewrite the two disjoint sets of roots

$$\begin{aligned} A &= \{\alpha > 0 \wedge w_j\alpha < 0\} \\ B &= \{w_j\alpha > 0 \wedge w_iw_j\alpha < 0\} \end{aligned} \tag{8.47}$$

using Lemma 3.7 from [186] which states that if any $\alpha \in \Delta_+$ satisfies $w_i\alpha < 0$ for some fundamental reflection w_i then $\alpha = \alpha_i$; the root corresponding to w_i.

Thus, $\alpha > 0 \wedge w_j\alpha < 0 \implies \alpha = \alpha_j \implies w_iw_j\alpha = -w_i\alpha_j < 0$ and

$$A = \{\alpha > 0 \wedge w_iw_j\alpha < 0 \wedge w_j\alpha < 0\}\,. \tag{8.48}$$

Similarly, $\alpha' = w_j\alpha > 0 \wedge w_i\alpha' = w_iw_j\alpha < 0 \implies \alpha' = \alpha_i \implies \alpha = w_j\alpha' = w_j\alpha_i > 0$ and

$$B = \{\alpha > 0 \wedge w_iw_j\alpha < 0 \wedge w_j\alpha > 0\}\,. \tag{8.49}$$

This gives the disjoint union

$$A \cup B = \{\alpha > 0 \wedge w_i w_j \alpha < 0\} = \{\alpha > 0 \wedge w\alpha < 0\} \tag{8.50}$$

and it follows that

$$\begin{aligned} &\prod_{w_j\alpha'>0\,|\,w_iw_j\alpha'<0} \frac{\xi(\langle\lambda|\alpha'\rangle)}{\xi(1+\langle\lambda|\alpha'\rangle)} \prod_{\alpha>0\,|\,w_j\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)} \\ = &\prod_{\alpha>0\,|\,w_iw_j\alpha<0} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)} = M(w_i, w_j\lambda). \end{aligned} \tag{8.51}$$

The general formula (8.45) for arbitrary products of fundamental reflections follows by iterating this procedure. □

The functional relation (8.44) for the constant term of the Eisenstein series, now follows from this result applied to the constant term formula (8.41). The fact that this also extends to the non-constant terms was shown by Langlands [218]. □

Remark 8.7. The functional relation (8.44) shows the limitations of the analytic continuation: For weights λ and Weyl words $w \in \mathcal{W}$ for which $M(w, \lambda)$ is not a non-zero finite number, the relation appears ill-defined. This can only happen for λ on certain hyperplanes as indicated above. Another apparent limitation of the functional relation is that for choosing a Weyl word w that stabilises the weight λ one would require $M(w, \lambda)$. This is not guaranteed to be true. For generic λ, (8.44) provides a valid relation. The remaining cases are those when $E(\lambda, g)$ actually develops poles (as a function of λ) and one has to consider appropriate normalising factors in the functional relation to make it well-defined.

8.9 Expansion in maximal parabolics*

In the previous sections we have explained how to expand Eisenstein series along a minimal parabolic subgroup, with unipotent radical N. It is also possible to make an expansion along different parabolic subgroups with smaller unipotent radical. The analogue of the constant term (8.4) then retains a dependence on the coordinates on $N(\mathbb{A})$ since only a subset of them is integrated out. In In this section we state and prove a theorem giving the formula for the constant terms of $E(\chi, g)$ in an expansion along a maximal parabolic subgroup.

The maximal parabolic subgroup, which we denote by $P_{j_\circ}$, is defined with respect to a particular choice of simple root $\alpha_{j_\circ}$ (i.e node $j_\circ$ in the Dynkin diagram) of G. This choice defines a subset of the simple roots, $\Pi_{j_\circ} = \Pi \backslash \{\alpha_{j_\circ}\}$, where Π is the set of all simple roots. From this we furthermore define $\Gamma_{j_\circ}$ to be the set of all positive roots of G, which are given by linear combinations of the simple roots contained in $\Pi_{j_\circ}$, i.e., those roots of G that do not contain $\alpha_{j_\circ}$. Using the standard Langlands decomposition, we can write a parabolic subgroup as the product of the Levi subgroup L and the unipotent radical U.

In the present case this reads $P_{j_\circ} = L_{j_\circ} U_{j_\circ}$. At the level of the corresponding Lie algebras one obtains $\mathfrak{p}_{j_\circ} = \mathfrak{l}_{j_\circ} \oplus \mathfrak{u}_{j_\circ}$ (not a direct sum of Lie algebras) with

$$\mathfrak{p}_{j_\circ} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta_+ \cup (-\Gamma_{j_\circ})} \mathfrak{g}_\alpha \tag{8.52}$$

and

$$\mathfrak{l}_{j_\circ} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Gamma_{j_\circ} \cup -\Gamma_{j_\circ}} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{u}_{j_\circ} = \bigoplus_{\alpha \in \Delta_+ \setminus \Gamma_{j_\circ}} \mathfrak{g}_\alpha \,, \tag{8.53}$$

respectively. Note that the case of the minimal parabolic above can be recovered by setting $\Gamma_{j_\circ} = \emptyset$.

Proceeding in analogy with the definitions of the minimal parabolic expansion we define the constant term part of the maximal parabolic expression as

$$C_{j_\circ} = \int\limits_{U_{j_\circ}(\mathbb{Q}) \backslash U_{j_\circ}(\mathbb{A})} E(\lambda, ug) du \,. \tag{8.54}$$

The subscript $j_\circ$ indicates the restriction to the maximal parabolic subgroup. A similar constant term formula as in the case of the minimal parabolic expansion can also be derived for this case. Upon deleting the $j_\circ$th node from the Dynkin diagram of G, we will denote the group associated with the Dynkin diagram which is left, by G'. We also note that the Levi factor $L_{j_\circ}$ of $P_{j_\circ}$ can then be written as $L_{j_\circ} = GL(1) \times G'$, where the one-parameter group $GL(1)$ is parametrised by a single variable in $\mathbb{R}^\times$.

Let us then state the formula for the constant term in this maximal parabolic:

Theorem 8.8 (Constant term formula for maximal parabolics [243]). *The constant term of $E(\lambda, g)$ with respect to the unipotent radical $U_{j_\circ} \subset P_{j_\circ}$ is given by:*

$$\int\limits_{U_{j_\circ}(\mathbb{Q}) \backslash U_{j_\circ}(\mathbb{A})} E(\lambda, ug) du = \sum_{w \in \mathcal{W}_{j_\circ} \backslash \mathcal{W}} e^{\langle (w\lambda + \rho)_{\parallel j_\circ} | H(g) \rangle} M(w, \lambda) E^{G'}(\chi_w, g) \,. \tag{8.55}$$

Below we provide a concise explanation of the notation used and give a proof of the formula.

In equation (8.55) the Weyl group of $P_{j_\circ}$ is denoted by $\mathcal{W}_{j_\circ}$ and the sum on the right-hand-side is then restricted to a sum over a coset of the Weyl group, in contrast to the minimal parabolic case. Furthermore the projection operators $(\cdot)_{\parallel j_\circ}$ and $(\cdot)_{\perp j_\circ}$ are defined as follows when acting on a weight $\lambda \in \mathfrak{a}^*$:

$$\lambda_{\parallel j_\circ} := \frac{\langle \Lambda_{j_\circ} | \lambda \rangle}{\langle \Lambda_{j_\circ} | \Lambda_{j_\circ} \rangle} \Lambda_{j_\circ}, \tag{8.56a}$$

$$\lambda_{\perp j_\circ} := \lambda - (\lambda)_{\parallel j_\circ}. \tag{8.56b}$$

These correspond, respectively, to the component of λ parallel and orthogonal to the fundamental weight $\Lambda_{j_\circ}$. The orthogonal component is given by a linear combination of simple roots of G'. The character χ_w follows the definition (8.2), however with λ now replaced by the weight $(w\lambda)_{\perp j_\circ}$. The G' invariant Eisenstein Series on the right-hand-side of the equation is independent of the $GL(1)$ factor in the decomposition of the Levi subgroup L_{j_*}, as this dependence is projected out using the $(\cdot)_{\perp j_\circ}$ operator and appears solely through the exponential prefactor. Let us also note that for simplicity of notation we have put g in the argument of the Eisenstein series on the right, even though g lies effectively in G'.

We also note that the formula (8.55) is well-defined and independent of the choice of coset representative due to the functional relation (8.44).

Proof. Sources for the analysis presented here are [126, 243]. Consider two parabolic subgroups $P_1(\mathbb{A})$ and $P_2(\mathbb{A})$ of $G(\mathbb{A})$. The first one we take to be the one defining an Eisenstein series through

$$E(\chi, g) = \sum_{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} \chi(\gamma g), \tag{8.57}$$

where $\chi : P_1(\mathbb{Q})\backslash P_1(\mathbb{A}) \to \mathbb{C}^\times$ is a character on the parabolic $P_1(\mathbb{A})$. The parabolic subgroup $P_2(\mathbb{A}) = L_2(\mathbb{A})U_2(\mathbb{A})$ is used to define the constant term along $U_2(\mathbb{A})$ via

$$C_{U_2}(\chi, g) = \int\limits_{U_2(\mathbb{Q})\backslash U_2(\mathbb{A})} E(\chi, ug)du. \tag{8.58}$$

By the definition of the integral, the result is determined by its dependence on $g = l \in L_2(\mathbb{A})$ and we will restrict to the Levi factor now. Most of the steps in the evaluation of (8.58) will be very similar to those in section 8.2.

Using the Bruhat decomposition, we can rewrite the integral (8.58) as

$$C_{U_2}(\chi, l) = \sum_{w \in \mathcal{W}_1\backslash \mathcal{W}/\mathcal{W}_2} C_{w,U_2}(\chi, l)\,, \tag{8.59}$$

where the individual of the double coset of the Weyl group $\mathcal{W}$ is

$$C_{w,U_2}(\chi, l) = \sum_{\delta \in w^{-1}P_1(\mathbb{Q})w\cap P_2(\mathbb{Q})\backslash P_2(\mathbb{Q})} \int\limits_{U_2(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(w\delta ul)du. \tag{8.60}$$

Now the sum over $\delta \in P_2(\mathbb{Q})$ can be split into the Levi and unipotent part according to $\delta = \gamma_l\gamma_u$ and then one can unfold the sum over γ_u onto the integration domain as in (8.6). The result is

$$C_{w,U_2}(\chi, l) = \sum_{\gamma_l \in w^{-1}P_1(\mathbb{Q})w\cap L_2(\mathbb{Q})\backslash L_2(\mathbb{Q})} \int\limits_{w^{-1}P_1(\mathbb{Q})w\cap U_2(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(w\gamma_l ul)du. \tag{8.61}$$

We now specialize to $P_2(\mathbb{A})$ being maximal parabolic. Then the Levi factor takes the form (cf. 4.40)

$$L_2(\mathbb{A}) = GL(1, \mathbb{A}) \times M_2(\mathbb{A}) \tag{8.62}$$

with M_2 reductive and we parametrise the $L_2(\mathbb{A})$ element as $l = rm$. We next separate out the dependence on the $GL(1,\mathbb{A})$ element r by moving it to the left within χ. This leads to

$$C_{w,U_2}(\chi, rm) = r^{w^{-1}\lambda+\rho} \sum_{\gamma_m \in w^{-1}P_1(\mathbb{Q})w \cap M_2(\mathbb{Q})\backslash M_2(\mathbb{Q})} \int_{w^{-1}P_1(\mathbb{Q})w\cap U_2(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wu\gamma_m m)du, \tag{8.63}$$

by combining the contribution from $\chi(wrw^{-1})$ and the change of the measure du. Note also that we changed the summation over the $L_2(\mathbb{Q})$ cosets to one over $M_2(\mathbb{Q})$ cosets since the two agree. We have also interchanged γ_m and u as the corresponding change of variables is uni-modular ($\gamma_m \in M_2(\mathbb{Q})$ is discrete).

Let us analyze the properties of the integral

$$\mathcal{I} = \int_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wu\gamma_m m)du \tag{8.64}$$

that is a function from $M_2(\mathbb{A}) \to \mathbb{C}$. We also defined for simplicity

$$M_2^w(\mathbb{Q}) := w^{-1}P_1(\mathbb{Q})w \cap M_2(\mathbb{Q}) \quad \text{and} \quad U_2^w(\mathbb{Q}) := w^{-1}P_1(\mathbb{Q})w \cap U_2(\mathbb{Q}) \tag{8.65}$$

and we note that $M_2^w(\mathbb{Q})$ is a parabolic subgroup of $M_2(\mathbb{Q})$. The integral (8.64) is invariant under $\epsilon \in M_2^w(\mathbb{Q})$ according to

$$\int_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wu\epsilon\gamma_m m)du = \int_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(w\epsilon u\gamma_m m)du = \mathcal{I} \tag{8.66}$$

since $\epsilon = w^{-1}p_1 w$ for some $p_1 \in P_1(\mathbb{Q})$ and the character $\chi : P_1(\mathbb{Q})\backslash P_1(\mathbb{A}) \to \mathbb{C}^\times$ is invariant under $P_1(\mathbb{Q})$. But this means that

$$\chi_2^w(\gamma_m m) = \int_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wu\gamma_m m)du \tag{8.67}$$

is a character $\chi_2^w : M_2^w(\mathbb{Q})\backslash M_2(\mathbb{A}) \to \mathbb{C}^\times$ if it is non-zero. (The integral $\mathcal{I}$ serves as an intertwiner from characters on $P_1(\mathbb{A})$ to characters on $M_2^w(\mathbb{A})$.)

Inserting this back into the individual constant term (8.63) we obtain

$$C_{w,U_2}(\chi, rm) = r^{w^{-1}\lambda+\rho} \sum_{\gamma_m \in M_w^2(\mathbb{Q})\backslash M_2(\mathbb{Q})} \chi_2^w(\gamma_m m). \tag{8.68}$$

The sum over γ_m now produces an Eisenstein series on $M_2(\mathbb{A})$ so that in all

$$C_{U_2}(\chi, rm) = \sum_{w \in \mathcal{W}_1\backslash\mathcal{W}/\mathcal{W}_2} r^{w^{-1}\lambda+\rho} E^{M_2}(\chi_2^w, m), \tag{8.69}$$

where we indicated that the Eisenstein series is on $M_2(\mathbb{A})$. This expression can be simplified a bit more by identifying the character χ_2^w in terms of a weight of $\mathfrak{m}_2 = \mathrm{Lie}(M_2)$. To this end we evaluate (8.67) at a semi-simple element of $M_2(\mathbb{A})$, i.e., $m = a$. This leads to

$$\begin{aligned}\chi_2^w(a) &= \int\limits_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wua)du = \chi(waw^{-1})\delta_{\bar{U}_2^w}(a) \int\limits_{U_2^w(\mathbb{Q})\backslash U_2(\mathbb{A})} \chi(wu)du \\ &= M(w^{-1},\lambda)a^{(w^{-1}\lambda+\rho)_{M_2}},\end{aligned} \tag{8.70}$$

where the last symbol denotes the orthogonal projection onto the space of M_2 weights. The exponent comes about as follows. The character $\chi(waw^{-1})$ evaluates to $a^{w^{-1}(\lambda+\rho)}$ and the modulus character on $\bar{U}_2^w$ is determined by the sum over all roots of U_2 that are not mapped to roots of P_1, i.e., the total exponent of a

$$w^{-1}(\lambda+\rho) + \sum_{\alpha\in\Delta(\mathfrak{u}_2)|w\alpha\notin\Delta(\mathfrak{p}_1)} \alpha = (w^{-1}\lambda+\rho)_{M_2}. \tag{8.71}$$

We have made furthermore made use of our knowledge of the Gindikin–Karpelevich type integral, cf. the evaluation of (8.19). In summary we arrive at

$$C_{U_2}(\chi, rm) = \sum_{w\in\mathcal{W}_1\backslash\mathcal{W}/\mathcal{W}_2} r^{w^{-1}\lambda+\rho} M(w^{-1},\lambda)E^{M_2}((w^{-1}\lambda)_{M_2}, m), \tag{8.72}$$

in agreement with (8.55) if one replaces w by w^{-1} which maps the double coset to $\mathcal{W}_2\backslash\mathcal{W}/\mathcal{W}_1$. □

Chapter 9

Whittaker vectors of Eisenstein series

In this chapter, we derive theorem 9.1 that states the formula of Casselman–Shalika [64] (see also [294]) for the *local* abelian Fourier coefficients in the minimal parabolic (Borel) subgroup $B(\mathbb{A}) \subset G(\mathbb{A})$ for the Eisenstein series $E(\chi, g)$. This formula is used to evaluate Fourier integrals with a generic character ψ. By the discussion in chapter 6 the global form of these Fourier coefficients are captured by the spherical Whittaker vector

$$W_\psi^\circ(\chi, g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)\overline{\psi(n)}dn \tag{9.1}$$

for a (quasi-)character $\chi : B(\mathbb{A}) \to \mathbb{C}^\times$ and a general unitary character $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$. For $SL(2, \mathbb{A})$ we have already evaluated this integral in section 7.3. As there, a useful strategy is to factorise the integral and perform it at all places separately. It will turn out that only for the finite primes $p < \infty$ and so-called generic and *unramified characters* ψ (to be defined below) a nice and compact formula exists. In sections 9.4 and 9.5, we will explain how to also evaluate (9.1) for arbitrary generic or even degenerate characters ψ.

9.1 Reduction of the integral and the longest Weyl word

To begin with, we bring the integral (9.1) into a form that is more amenable to evaluation. As discussed in section 6.3.3, the spherical Whittaker vector satisfies

$$W_\psi^\circ(\chi, ngk) = \psi(n)W_\psi^\circ(\chi, g) \tag{9.2}$$

and is therefore determined by its values on $A(\mathbb{A})$ due to the Iwasawa decomposition (4.19) and (4.48). Hence, we will only consider it for $g = a \in A(\mathbb{A})$ in the sequel. For the discussion in this subsection we assume ψ to be generic (see definition 6.10), i.e., it does not vanish on any simple root generator.

We start evaluating (9.1) by applying the Bruhat decomposition as for the constant term to obtain

$$\begin{aligned} W_\psi^\circ(\chi, a) &= \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma n a)\overline{\psi(n)} dn \\ &= \sum_{w \in \mathcal{W}} \int_{w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)\overline{\psi(n)} dn \,. \end{aligned} \tag{9.3}$$

From the last line let us define $W_\psi^\circ(\chi, a) = \sum_{w \in \mathcal{W}} F_{w,\psi}$, where

$$F_{w,\psi} = \int_{w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)\overline{\psi(n)} dn \tag{9.4}$$

is the contribution from the Weyl word w.

Let us start with an analysis of the integration range of the Fourier integral (9.4), given by the coset $w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A})$, and the corresponding contribution to $W_\psi^\circ(\chi, a)$. It is clear that the intersection in the denominator of this coset consists of those elements of the (upper) Borel subgroup that are mapped to upper Borel elements under the action of the Weyl element w. For the whole denominator we can therefore write

$$w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q}) = \prod_{\substack{\alpha > 0 \\ w\alpha > 0}} N_\alpha(\mathbb{Q}) \,. \tag{9.5}$$

With this, the integration range conveniently splits up in the following way

$$w^{-1}B(\mathbb{Q})w \cap N(\mathbb{Q})\backslash N(\mathbb{A}) \simeq \left(\prod_{\substack{\beta > 0 \\ w\beta > 0}} N_\beta(\mathbb{Q})\backslash N_\beta(\mathbb{A}) \right) \cdot \left(\prod_{\substack{\gamma > 0 \\ w\gamma < 0}} N_\gamma(\mathbb{A}) \right) \,. \tag{9.6}$$

Let us introduce some notation. We denote the union in the first parenthesis as

$$N^w_{\{\beta\}} := \left(\prod_{\substack{\beta > 0 \\ w\beta > 0}} N_\beta(\mathbb{Q})\backslash N_\beta(\mathbb{A}) \right) \tag{9.7}$$

and the union in the second parenthesis as

$$N^w_{\{\gamma\}} := \left(\prod_{\substack{\gamma > 0 \\ w\gamma < 0}} N_\gamma(\mathbb{A}) \right) \,. \tag{9.8}$$

Here, the sets of roots $\{\beta\}$ and $\{\gamma\}$ contain precisely those roots, which satisfy the conditions imposed on the products in (9.7) and (9.8), respectively. It is important to note that there is a qualitative difference in the two sets: $N^w_{\{\gamma\}}$ is non-compact while $N^w_{\{\beta\}}$

is *compact*. With this splitting of the integration range the contribution F_w then takes the following form

$$\begin{aligned} F_{w,\psi} &= \int\limits_{N^w_{\{\beta\}} N^w_{\{\gamma\}}} \chi(wna)\overline{\psi(n)}dn \\ &= \int\limits_{N^w_{\{\beta\}}} \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\beta n_\gamma a)\overline{\psi(n_\beta n_\gamma)}dn_\beta dn_\gamma \,. \end{aligned} \tag{9.9}$$

Inserting $w^{-1}w$ between n_β and n_γ and splitting the character up into two factors, we obtain

$$\int\limits_{N^w_{\{\beta\}}} \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\beta w^{-1} wn_\gamma a)\overline{\psi(n_\beta)}\,\overline{\psi(n_\gamma)}dn_\beta dn_\gamma \,. \tag{9.10}$$

Let us note that the character χ is left invariant under any subgroup that is given by the exponential of positive root generators. In particular this applies to elements n of the (upper) Borel subgroup. Hence by definition of $N^w_{\{\beta\}}$, the character χ is insensitive to the factor $wn_\beta w^{-1}$ in the argument and we can split-off the integral over n_β, leaving us with

$$F_{w,\psi} = \int\limits_{N^w_{\{\beta\}}} \overline{\psi(n_\beta)}dn_\beta \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\gamma a)\overline{\psi(n_\gamma)}dn_\gamma \,. \tag{9.11}$$

Given the form of (9.11), we see that in the integral over n_β, effectively a periodic function is integrated over a full period in the compact space $N^w_{\{\beta\}}$. Provided that the character ψ is non-trivial along at least one simple root contained in $\{\beta\}$, this means that the whole integral will vanish. Since this is always true for generic ψ and we arrive at the conclusion

$$F_{w,\psi} = 0 \quad \text{unless } w = w_{\text{long}}. \tag{9.12}$$

This is true since all Weyl transformations except for the longest Weyl word w_{long} leave at least one simple root positive. Therefore we arrive at the following expression for the Whittaker function for a generic ψ:

$$W^\circ_\psi(\chi, a) = \int\limits_{N(\mathbb{A})} \chi(w_{\text{long}} na)\overline{\psi(n)}dn. \tag{9.13}$$

We will also refer to this expression as the *Jacquet integral*, see [174].

9.2 Unramified local Whittaker vectors

The integral (9.13) should now be evaluated for all places separately, that is

$$\begin{aligned}\int\limits_{N(\mathbb{A})} \chi(w_{\text{long}}na)\overline{\psi(n)}dn &= \int_{N(\mathbb{R})} \chi_\infty(w_{\text{long}}na)\overline{\psi_\infty(n)}dn \times \prod_{p<\infty} \int_{N(\mathbb{Q}_p)} \chi_p(w_{\text{long}}na)\overline{\psi_p(n)}dn \\ &= W^\circ_{\psi_\infty} \times \prod_{p<\infty} W^\circ_{\psi_p}. \end{aligned} \tag{9.14}$$

However, while we will derive a nice closed formula for the local places $p < \infty$, a general expression for the real place is not known to the best of our knowledge. In the case $SL(2, \mathbb{A})$, the resulting expression was given by the Bessel function (7.73) and for $SL(3, \mathbb{A})$ it is known that the triple integral over the three unipotent generators gives convoluted integrals of Bessel functions [60, 314], see also section 9.6. For general groups with 'more non-abelian' unipotent subgroups a proliferation of this nested structure of special functions is to be expected. On the other hand, the Whittaker vectors for finite $p < \infty$ contain the essential number theoretic information that is reflected as instanton measures in string theory applications [138, 248]. Therefore we will from now on consider only the group $G(\mathbb{Q}_p)$ for $p < \infty$ and calculate the local Whittaker vectors. To ease notation we shall suppress all subscripts involving primes, hence in the remainder of this section we write ψ for ψ_p and χ for χ_p.

9.2.1 Unramified characters ψ

The formula of Casselman and Shalika for the local Whittaker vectors is most conveniently stated when one restricts the character ψ to be *unramified* (see definition 6.12 and [64, p. 219]). Recall from definition 6.12 that this means that the character $\psi : N(\mathbb{Q}_p) \to U(1)$ has what a physicist might call 'unit instanton charges', i.e., when the element n is expanded in terms of positive step operators in a Chevalley basis as

$$n = \left(\prod_{\alpha\in\Delta_+\setminus\Pi} x_\alpha(u_\alpha) \right) \left(\prod_{\alpha\in\Pi} x_\alpha(u_\alpha) \right) \in N(\mathbb{Q}_p), \tag{9.15}$$

where we have ordered the individual factors in a convenient way. Note that for evaluating $\psi(n)$ the order does not matter since ψ is a homomorphism between abelian groups. An unramified character ψ is then one that satisfies

$$\psi(n) = \exp\left(2\pi i \left[\sum_{\alpha\in\Pi} m_\alpha u_\alpha \right] \right) \tag{9.16}$$

with $m_\alpha = 1$ for all simple roots $\alpha \in \Pi$. An unramified character is automatically generic.

The local Whittaker vector for an unramified vector will be denoted simply by

$$W^\circ(\chi, a) = \int\limits_{N(\mathbb{Q}_p)} \chi(w_{\text{long}}na)\overline{\psi(n)}dn, \tag{9.17}$$

where the reference to ψ has been suppressed for notational convenience and we do not display the fact that we are using a fixed $p < \infty$. Standard manipulations similar to (8.17) on (9.17) lead to

$$\begin{aligned} W^\circ(\chi, a) &= \chi(w_{\text{long}} a w_{\text{long}}^{-1}) \int\limits_{N(\mathbb{Q}_p)} \chi(w_{\text{long}} a^{-1} n a) \overline{\psi(n)} dn \\ &= \chi(w_{\text{long}} a w_{\text{long}}^{-1}) \delta(a) \int\limits_{N(\mathbb{Q}_p)} \chi(w_{\text{long}} n) \overline{\psi(a n a^{-1})} dn \\ &= |a^{w_{\text{long}}\lambda + \rho}| \int\limits_{N(\mathbb{Q}_p)} \chi(w_{\text{long}} n) \overline{\psi^a(n)} dn, \end{aligned} \tag{9.18}$$

where we defined $\psi^a(n) := \psi(a n a^{-1})$.

9.2.2 Vanishing properties

One advantage of restricting to unramified characters is that it is very simple to determine the support of $W^\circ(\chi, a)$ (cf. also [64, Lemma 5.1]). Consider an element $n \in N(\mathbb{Z}_p) \subset K_p$; then, by right K_p-invariance and the transformation properties (9.2),

$$W^\circ(\chi, a) = W^\circ(\chi, an) = W^\circ(\chi, a n a^{-1} a) = \psi(a n a^{-1}) W^\circ(\chi, a). \tag{9.19}$$

Therefore, $W^\circ(\chi, a)$ can only be non-vanishing if $\psi(a n a^{-1}) = 1$ which requires that $a^\alpha \in \mathbb{Z}_p$ for all positive roots α since ψ is unramified.

9.3 The Casselman–Shalika formula

We are now ready to state the main theorem of this section:

Theorem 9.1 (Casselman–Shalika formula). *The local unramified Whittaker vector $W^\circ(\chi, a)$, defined by the integral (9.14) for each $p < \infty$, is given by*

$$\int_{N(\mathbb{Q}_p)} \chi(w_{long} n a) \overline{\psi(n)} dn = \frac{\epsilon(\lambda)}{\zeta(\lambda)} \sum_{w \in \mathcal{W}} (\det(w)) |a^{w\lambda + \rho}| \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} p^{\langle \lambda | \alpha \rangle} \tag{9.20}$$

Proof. The proof of this theorem will constitute the remainder of this section 9.3. ☐

When translated to our notation, the Casselman–Shalika formula found for $W^\circ(\chi, a)$ in [64, Thm. 5.4] takes the form

$$W^\circ(\chi, a) = \frac{\epsilon(\lambda)}{\zeta(\lambda)} \sum_{w \in \mathcal{W}} (\det(w)) \Bigg(\prod_{\substack{\alpha > 0 \\ w\alpha < 0}} p^{\langle \lambda | \alpha \rangle} \Bigg) |a^{w\lambda + \rho}| = \frac{1}{\zeta(\lambda)} \sum_{w \in \mathcal{W}} \epsilon(w\lambda) |a^{w\lambda + \rho}| \tag{9.21}$$

with

$$\zeta(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{-(\langle\lambda|\alpha\rangle+1)}}, \quad \epsilon(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}}. \tag{9.22}$$

The latter identity of (9.21) follows from

$$\epsilon(w_i\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{\langle\lambda|w_i\alpha\rangle}} = \frac{1}{1 - p^{-\langle\lambda|\alpha_i\rangle}} \prod_{\substack{\alpha>0\\ \alpha\neq\alpha_i}} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}} = -p^{\langle\lambda|\alpha_i\rangle}\epsilon(\lambda) \tag{9.23}$$

where w_i is a fundamental reflection switching the sign of α_i and permuting the remaining positive roots. Recall that $\det w = (-1)^{\ell(w)}$ where $\ell(w)$ is the length of w as introduced in section 4.1.1. Formula (9.21) is valid only for unramified ψ and we have used χ and λ interchangeably.

Our strategy for proving theorem 9.1 will be a mixture of the works of Jacquet [174] and Casselman–Shalika [64]. The argument consists of the following steps:

1. Derivation of a functional equation for the Whittaker function under Weyl transformations on χ
2. Use this to show that a suitable multiple of the Whittaker function is Weyl invariant and write it as a sum over Weyl images
3. Determine one term in this sum and derive all other terms from it. This will yield formula (9.21)

Finally we will also show in section 9.4 how the formula (9.21) can be used to derive the Whittaker vectors for all generic characters ψ.

However, as a preparatory 'step 0', we first recall and slightly extend some results from chapter 7 where the Fourier coefficients for Eisenstein series on $SL(2,\mathbb{A})$ were discussed. Namely, after equation (7.74) we derived the Whittaker vector at finite places, evaluated at the identity $a = 1 \in A(\mathbb{Q}_p)$, for general ψ. Repeating the same steps but (i) keeping a arbitrary and (ii) choosing an unramified character ($m = 1$) leads to

$$\begin{aligned} G(\mathbb{Q}_p) = SL(2,\mathbb{Q}_p): \quad F_{w_{\text{long}},\psi,p} &= \int_{N(\mathbb{Q}_p)} \chi(w_{\text{long}}na)\overline{\psi(n)}dn \\ &= \chi(w_{\text{long}}aw_{\text{long}}^{-1})\delta(a) \int_{N(\mathbb{Q}_p)} \chi(w_{\text{long}}n)\overline{\psi(ana^{-1})}dn \\ &= \gamma_p(v^2)(1-p^{-2s})\frac{|v|^{-2s+2} - p^{-2s+1}|v|^{2s}}{1-p^{-2s+1}} \end{aligned} \tag{9.24}$$

with $\chi(a) = |a|^{2s}$ and $a = \mathrm{diag}(v, v^{-1}) = v^{H_\alpha}$ in terms of the unique positive root α of $\mathfrak{sl}(2,\mathbb{R})$. This formula, after dividing by $(1-p^{-2s})$ exhibits invariance under the Weyl reflection $s \leftrightarrow 1-s$. We will see how this feature generalises to arbitrary G and why it is basically a consequence of this $SL(2,\mathbb{Q}_p)$ calculation. Equation (9.24) also manifestly exhibits the vanishing property of section 9.2.2 since the p-adic Gaussian vanishes unless $|v^2| = |a^\alpha| \leq 1$.

Before embarking on the proof proper, we also record the following

Proposition 9.2 (Holomorphy of local Whittaker vectors [64]). *The local Whittaker vector $W^\circ(\chi, a)$ depends holomorphically on the quasi-character χ.*

Proof. Inspection of formula (9.21) immediately reveals holomorphy when χ is in the Godement domain (5.100) of absolute convergence. This extends to all χ by virtue of the functional relation derived below. □

The holomorphy of the Whittaker vector in the case of $SL(2,\mathbb{Q}_p)$ (as a function of s) can also be seen from the explicit expression (9.24) above. For $s \to \frac{1}{2}$, the expression stays finite. We will comment in much more detail on the behaviour of Eisenstein series in chapter 10.

9.3.1 Functional relation for the local Whittaker vector

We follow Jacquet's thesis [174]. First one defines a function associated to the Whittaker vector by

$$F(\lambda, g) = W^\circ_\psi(\lambda, w_{\text{long}}^{-1} g) \tag{9.25}$$

for $g \in G(\mathbb{Q}_p)$. This leads to the integral expression

$$F(\lambda, g) = \int\limits_{N_-(\mathbb{Q}_p)} \chi(n_- g)\overline{\psi_-(n_-)} dn_- \tag{9.26}$$

for the associated function. Here, objects with a minus subscript refer to the unipotent opposite to the standard unipotent $N(\mathbb{Q}_p)$. In other words, $N_-(\mathbb{Q}_p)$ designates the subgroup of $G(\mathbb{Q}_p)$ generated by the exponentials of the *negative* roots, whereas the usual $N(\mathbb{Q}_p)$ is associated with the positive roots. The reason that the opposite group arises here is because w_{long} maps all positive roots to negative ones (possibly combined with an outer automorphism). We will derive a functional relation for F under Weyl transformations which by (9.25) will imply one for the Whittaker vector.

The method for deriving the functional relation will be by reducing to the functional relation for $SL(2,\mathbb{Q}_p)$ that is manifest in (9.24) and then using the fact that $G(\mathbb{Q}_p)$ is made up of $SL(2,\mathbb{Q}_p)$ subgroups.

Let α_i be a simple positive root of $G(\mathbb{Q}_p)$. Then define for $g \in \mathbb{Q}_p$

$$F_i(\lambda, g) = \int\limits_{N_{i,-}(\mathbb{Q}_p)} \chi(n_{i,-} g)\overline{\psi_{i,-}(n_{i,-})} dn_{i,-}, \tag{9.27}$$

where the integral is now only over the one-dimensional subgroup generated by $x_{-\alpha_i}(u)$ and similarly the character $\psi_{i,-}$ is one of the (lower) unipotent of the $SL(2,\mathbb{Q}_p)$ associated with α_i and can be obtained from ψ_- by restriction to the subgroup $N_{i,-}$. The function F_i is useful because for any α_i we can write

$$N_-(\mathbb{Q}_p) = N_{i,-}(\mathbb{Q}_p)\hat{N}_-(\mathbb{Q}_p), \tag{9.28}$$

where $\hat{N}_-(\mathbb{Q}_p)$ are the lower unipotent elements that are not of the form $x_{-\alpha_i}(u)$ for some $u \in \mathbb{Q}_p$. Associated with the factorisation above is a unique decomposition $n_- = n_{i,-}\hat{n}_-$ and then the integral (9.26) leads to

$$F(\lambda, g) = \int\limits_{\hat{N}_-(\mathbb{Q}_p)} F_i(\hat{n}_- g)\overline{\psi_-(\hat{n}_-)} d\hat{n}_- \tag{9.29}$$

by carrying out the integral over $dn_{i,-}$.

The $SL(2,\mathbb{Q}_p)$ projected function (9.27) has the following invariances

$$F_i(\lambda, \hat{n}gk) = F_i(\lambda, g) \quad \text{for } \hat{n} \in \hat{N}(\mathbb{Q}_p) \text{ and } k \in K_p = G(\mathbb{Z}_p), \tag{9.30}$$

where $\hat{N}(\mathbb{Q}_p)$ is the unipotent subgroup opposite to $\hat{N}_-(\mathbb{Q}_p)$. It is generated by all positive roots but α_i. The set of these roots is invariant under the Weyl reflection w_i. Let P_i be the next-to-minimal parabolic subgroup defined by the (non-unique) decomposition

$$G(\mathbb{Q}_p) = P_i(\mathbb{Q}_p)K_p = \hat{N}(\mathbb{Q}_p)L_i(\mathbb{Q}_p)K_p = \hat{N}(\mathbb{Q}_p)\hat{A}(\mathbb{Q}_p)SL(2,\mathbb{Q}_p)_{\alpha_i}K_p \tag{9.31}$$

with $\hat{A}(\mathbb{Q}_p)$ the part of the split torus $A(\mathbb{Q}_p)$ that is not contained in the torus of the embedded $SL(2,\mathbb{Q}_p)_{\alpha_i}$. Using this decomposition and the invariances of F_i, one finds that the function $F_i(g)$ is determined by its values on elements of the form $g = \hat{a}g_i$ with $\hat{a} \in \hat{A}(\mathbb{Q}_p)$ and $g_i \in SL(2,\mathbb{Q}_p)_{\alpha_i}$. On such values one has that

$$F_i(\lambda, \hat{a}g_i) = |\hat{a}^{\lambda+\rho-\alpha_i}| \int\limits_{N_{i,-}(\mathbb{Q}_p)} \chi(n_{i,-}g_i)\overline{\psi^{\hat{a}}_{i,-}(n_{i,-})} dn_{i,-} \tag{9.32}$$

with $\psi^{\hat{a}}_-(n_{i,-}) = \psi_-(\hat{a}n_{i,-}\hat{a}^{-1})$. The integral is basically the integral we have done in (9.24) with the only change that χ is now defined on all of $G(\mathbb{Q}_p)$ in which $SL(2,\mathbb{Q}_p)_{\alpha_i}$ is embedded. The result is determined by diagonal a_i and reads for $\hat{a} = 1$

$$\int\limits_{N_{i,-}(\mathbb{Q}_p)} \chi(n_{i,-}a_i)\overline{\psi_{i,-}(n_{i,-})} dn_{i,-} = \gamma_p(a_i^{-\alpha_i})(1 - p^{-(\langle\lambda|\alpha_i\rangle+1)})\frac{1 - p^{-\langle\lambda|\alpha_i\rangle}|a_i^{-\alpha_i}|^{\langle\lambda|\alpha_i\rangle}}{1 - p^{-\langle\lambda|\alpha_i\rangle}}|a_i^{\lambda+\rho-\alpha_i}|. \tag{9.33}$$

Under the Weyl reflection w_i one has $w_i\lambda = \lambda - \langle\lambda|\alpha_i\rangle\alpha_i$ and the function F_i therefore satisfies

$$F_i(w_i\lambda, g) = F_i(\lambda, g)\frac{1 - p^{-(1+\langle w_i\lambda|\alpha_i\rangle)}}{1 - p^{-(1+\langle\lambda|\alpha_i\rangle)}}, \tag{9.34}$$

where one also must keep track of the non-trivial $\hat{a}$ given by the prefactor in (9.32). The relation (9.29) then gives immediately the same transformation under w_i for $F(\lambda, g)$ and therefore for the unramified Whittaker vector:

$$W^\circ(w_i\lambda, a) = \frac{\zeta_p(w_i, \lambda)}{\zeta_p(w_i, -\lambda)} W^\circ(\lambda, a) \tag{9.35}$$

where we defined the local ζ factor

$$\zeta_p(w, \lambda) = \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{1}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}}. \tag{9.36}$$

For a general Weyl transformation $w \in \mathcal{W}$ we find therefore

$$W^\circ(w\lambda, a) = \frac{\zeta_p(w, \lambda)}{\zeta_p(w, -\lambda)} W^\circ(\lambda, a) \tag{9.37}$$

which we check in example 9.3.

This is not surprisingly the same factor that appeared in (the functional relation for) the constant term, see (8.36).

Example 9.3

Let us check (9.37) with $w = w_i w_j$ starting from (9.35) where w_i and w_j are two (different) fundamental reflections. Using (9.35) twice we have that

$$W^\circ(w\lambda, a) = W^\circ(w_i w_j\lambda, a) = \frac{\zeta_p(w_i, w_j\lambda)}{\zeta_p(w_i, -w_j\lambda)} W^\circ(w_j\lambda, a) = \frac{\zeta_p(w_i, w_j\lambda)}{\zeta_p(w_i, -w_j\lambda)} \frac{\zeta_p(w_j, \lambda)}{\zeta_p(w_j, -\lambda)} W^\circ(\lambda, a)\,. \tag{9.38}$$

Consider now the factor

$$\begin{aligned} \frac{\zeta_p(w_i, w_j\lambda)}{\zeta_p(w_i, -w_j\lambda)} \frac{\zeta_p(w_j, \lambda)}{\zeta_p(w_j, -\lambda)} &= \prod_{\substack{\alpha'>0 \\ w_i\alpha'<0}} \frac{1 - p^{-(1-\langle w_j\lambda|\alpha'\rangle)}}{1 - p^{-(1+\langle w_j\lambda|\alpha'\rangle)}} \prod_{\substack{\alpha>0 \\ w_j\alpha<0}} \frac{1 - p^{-(1-\langle\lambda|\alpha\rangle)}}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}} \\ &= \prod_{\substack{w_j\alpha>0 \\ w_i w_j\alpha<0}} \frac{1 - p^{-(1-\langle\lambda|\alpha\rangle)}}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}} \prod_{\substack{\alpha>0 \\ w_j\alpha<0}} \frac{1 - p^{-(1-\langle\lambda|\alpha\rangle)}}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}} \end{aligned} \tag{9.39}$$

where we have made the substitution $\alpha' = w_j\alpha$ and used the fact that $\langle w_j\lambda|w_j\alpha\rangle = \langle\lambda|\alpha\rangle$. Applying the same argument as in the proof of Lemma 8.6 we can then combine the products into

$$W^\circ(w\lambda, a) = \prod_{\substack{\alpha>0 \\ w\alpha<0}} \frac{1 - p^{-(1-\langle\lambda|\alpha\rangle)}}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}} W^\circ(\lambda, a) = \frac{\zeta_p(w, \lambda)}{\zeta_p(w, -\lambda)} W^\circ(\lambda, a)\,, \tag{9.40}$$

as claimed.

9.3.2 Weyl invariant combination

As for the constant term (and the full Eisenstein series), one can obtain a Weyl invariant form by multiplying through by the denominator of ζ factors associated with the longest

Weyl word. Denoting

$$\zeta(\lambda) \equiv \zeta_p(w_{\text{long}}, \lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{-(1+\langle\lambda|\alpha\rangle)}} \tag{9.41}$$

one has that the function

$$\zeta(\lambda) W^\circ(\lambda, a) \tag{9.42}$$

is Weyl invariant. This is checked simply by combining (9.37) with the transformation of $\zeta_p(\lambda)$ that can be derived straightforwardly.

Because of the Weyl invariance of (9.42), we write it as a sum over Weyl images as

$$\zeta(\lambda) W^\circ(\lambda, a) = \sum_{w\in\mathcal{W}} c(w\lambda) |a^{w\lambda+\rho}| \tag{9.43}$$

since the invariant function has to be a polynomial in $a^{\lambda+\rho}$ (and its images). The fact that the local Whittaker vector is a single Weyl orbit follows from the considerations in [64].

9.3.3 Determining a special coefficient

Next we determine $c(w\lambda)$ for $w = w_{\text{long}}$ which is the coefficient of $|a^{w_{\text{long}}\lambda+\rho}|$ in (9.43). Referring back to (9.18) we see that the coefficient of $|a^{w_{\text{long}}\lambda+\rho}|$ in $W^\circ(\lambda, a)$ is obtained as the a-independent part of the integral

$$\int_{N(\mathbb{Q}_p)} \chi(w_{\text{long}} n) \overline{\psi^a(n)} dn. \tag{9.44}$$

The integral is a polyonmial in a and we can obtain its a-independent part formally by sending a to zero. (This is only formal because, of course, $0 \notin A(\mathbb{Q}_p)$.) Therefore, the a-independent part of this integral can be obtained by removing the character ψ^a from the integral and then one is left with the same integral as in the constant term (8.19) for $w = w_{\text{long}}$. The result then is the same as the local factor for $\mathbb{Q}_p$ in the constant term formula (8.41), viz.

$$c(w_{\text{long}}\lambda) = \zeta(\lambda) \prod_{\alpha>0} \frac{1 - p^{-(\langle\lambda|\alpha\rangle+1)}}{1 - p^{-\langle\lambda|\alpha\rangle}} = \prod_{\alpha>0} \frac{1}{1 - p^{-\langle\lambda|\alpha\rangle}} = \prod_{\alpha>0} \frac{1}{1 - p^{\langle w_{\text{long}}\lambda|\alpha\rangle}} = \epsilon(w_{\text{long}}\lambda). \tag{9.45}$$

This means that the general coefficient is given by

$$c(\lambda) = \epsilon(\lambda) = \prod_{\alpha>0} \frac{1}{1 - p^{\langle\lambda|\alpha\rangle}} \tag{9.46}$$

and the general formula for the Whittaker vector for an unramified character is

$$W^\circ(\lambda, a) = \frac{1}{\zeta(\lambda)} \sum_{w\in\mathcal{W}} \epsilon(w\lambda) |a^{w\lambda+\rho}|, \tag{9.47}$$

thus demonstrating (9.21). This concludes the proof of theorem 9.1.

9.4 Whittaker vectors for generic ψ

Theorem 9.1 is only valid for unramified character ψ, but we will now show that it can also be used for generic characters indirectly. Recall from definition 6.12 that for an unramified character $m_\alpha = 1$ for all $\alpha \in \Pi$ and that a generic character has $m_\alpha \neq 0$ for all α.

Let us take a closer look at the so called 'twisted' character $\psi^a(n) = \psi(ana^{-1})$ introduced above where ψ without superscript a is the unramified character. We note that periodicity of ψ^a is of course different from the one of ψ, but this will not influence our reasoning.

From (9.15) and (9.16) we have that

$$\psi(n) = \exp\left(-2\pi i \left[\sum_{\alpha\in\Pi} u_\alpha\right]\right), \qquad n = \left(\prod_{\alpha\in\Delta_+\setminus\Pi} x_\alpha(u_\alpha)\right)\left(\prod_{\alpha\in\Pi} x_\alpha(u_\alpha)\right) \tag{9.48}$$

where $x_\alpha(u_\alpha) = \exp(u_\alpha E_\alpha)$.

With insertions of aa^{-1}, the expression for ana^{-1} splits into factors of $ax_\alpha(u_\alpha)a^{-1}$. Using the Baker-Campbell-Hausdorff formula, these factors can be found as

$$ax_\alpha(u_\alpha)a^{-1} = \exp(e^t\, u_\alpha E_\alpha), \tag{9.49}$$

where t is defined by

$$[\log a, \log x_\alpha(u_\alpha)] = t \log x_\alpha(u_\alpha)\,. \tag{9.50}$$

Let a be parametrised as

$$a = \exp\left(\sum_{\beta\in\Pi} \log(v_\beta) H_\beta\right), \tag{9.51}$$

which gives

$$[\log a, \log x_\alpha(u_\alpha)] = \sum_{\beta\in\Pi} \log(v_\beta) u_\alpha [H_\beta, E_\alpha] = \underbrace{\sum_{\beta\in\Pi} \alpha(H_\beta)\log(v_\beta)}_{=t}\, u_\alpha E_\alpha\,. \tag{9.52}$$

Thus,

$$ana^{-1} = \left(\prod_{\alpha\in\Delta_+\setminus\Pi} x_\alpha(u'_\alpha)\right)\left(\prod_{\alpha\in\Pi} x_\alpha(u'_\alpha)\right), \qquad u'_\alpha = e^t u_\alpha = \left(\prod_{\beta\in\Pi} (v_\beta)^{\alpha(H_\beta)}\right) u_\alpha \tag{9.53}$$

and, finally, by listing the simple roots as $\alpha_i \in \Pi$ for $i = 1, \dots, r$ and denoting the associated elements u_{α_i} and v_{α_j} as u_i and v_j respectively

$$\psi^a(n) = \exp\left(-2\pi i \left[\sum_{i=1}^r u'_i\right]\right) = \exp\left(-2\pi i \left[\sum_{i=1}^r \left(\prod_{j=1}^r (v_j)^{A_{ji}}\right) u_i\right]\right) \tag{9.54}$$

where we have introduced the Cartan matrix A_{ij} defined in (4.18).

We now note that this is really a generic character with

$$m_i = m_{\alpha_i} = \prod_{j=1}^{r} (v_j)^{A_{ji}} \tag{9.55}$$

and that any generic character can be expressed in this way with the inverse relation

$$v_j = \prod_{i=1}^{r} (m_i)^{A^{-1}_{ij}} \tag{9.56}$$

where A^{-1}_{ij} is the inverse Cartan matrix.

Now that we can express a generic character in terms of the unramified character, we would like to find the Whittaker vector for ψ^a using (9.47) indirectly. More specifically, we ultimately want to find $W^\circ_{\psi^a}(\chi, a')$ with $a' = 1$ along the finite primes where $g = na'k$ and $a' \in A(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$. For each p this gives a contributing factor to the instanton measure as discussed in example 6.25.

This will bring us one step closer to finding the Fourier coefficients of the Eisenstein series with general instanton charges m_α in (9.13) and not only the restricted case of an unramified character.

Using similar steps as taken in (9.18), but in reverse order, we obtain

$$W^\circ_{\psi^a}(\chi, \mathbb{1}) = \big(\chi(w_{\text{long}} a w^{-1}_{\text{long}})\delta(a)\big)^{-1} W^\circ(\chi, a) = |a^{-(w_{\text{long}}\lambda + \rho)}| W^\circ(\chi, a)\,. \tag{9.57}$$

Therefore, the local instanton measure for a generic character ψ^a with instanton charges m_α can be expressed through the local instanton measure evaluated for an unramified character at non-trivial $a = \prod_{\alpha\in\Pi} v_\alpha^{H_\alpha} \in A(\mathbb{Q}_p)$.

Example 9.4

We illustrate formula (9.57) by recovering the result (7.77) for $SL(2, \mathbb{A})$. In this case, there is only one simple root α and $A_{\alpha\alpha} = 2$. The unramified Whittaker vector is as given in (9.24). If we want to get the Whittaker vector for a character ψ^a with instanton charge m, then (9.56) tells us that we have $v = m^{1/2}$ and from (9.57) we find that

$$\begin{aligned} W^\circ_{\psi^a}(\chi, \mathbb{1}) &= |v|^{2s-2}\gamma_p(v^2)(1 - p^{-2s})\frac{|v|^{-2s+2} - p^{-2s+1}|v|^{2s}}{1 - p^{-2s+1}} \\ &= \gamma_p(m)(1 - p^{-2s})\frac{1 - p^{-2s+1}|m|^{2s-1}}{1 - p^{-2s+1}} \end{aligned} \tag{9.58}$$

in agreement with (7.77).

9.5 Degenerate Whittaker vectors

While the Casselman–Shalika formula (9.47) provides an elegant expression for unramified *local* characters and, via (9.57), also for Fourier coefficients of generic characters, it is desirable to understand also Fourier coefficients for *non*-generic characters ψ. These are also sometimes referred to as *degenerate Whittaker vectors* in the literature [135, 241, 324] and have the property that they only depend on a subset of the simple roots of $G(\mathbb{R})$ rather than all simple roots.

In this section, we will prove the following theorem that holds for *global* characters ψ [97, 165]:

Theorem 9.5. *Let $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$ be a degenerate character with $\mathrm{supp}(\psi) = \Pi' \neq \Pi$ with associated subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$. Let $w_c w'_{long}$ be the representatives of the coset $\mathcal{W}/\mathcal{W}'$ defined below in (9.65). Then the degenerate Whittaker vector on $G(\mathbb{A})$ is given by*

$$W^\circ_\psi(\chi, a) = \sum_{w_c w'_{long} \in \mathcal{W}/\mathcal{W}'} a^{(w_c w'_{long})^{-1}\lambda + \rho} M(w_c^{-1}, \lambda) W'^\circ_{\psi^a}(w_c^{-1}\lambda, \mathbb{1}), \tag{9.59}$$

where W'°_ψ denotes a Whittaker function on the $G'(\mathbb{A})$ subgroup of $G(\mathbb{A})$. The weight $w_c^{-1}\lambda$ is given as a weight of $G'(\mathbb{A})$ by orthogonal projection.

Remark 9.6. In this theorem and in the remainder of the chapter we suppress the adelic absolute value on $|a^\mu|$ in order to ease the notation.

Before embarking on the proof, we explain the notation used here. For a global character

$$\psi\left(\prod_{\alpha\in\Pi} x_\alpha(u_\alpha)\right) = \exp\left(2\pi i \sum_{\alpha\in\Pi} m_\alpha u_\alpha\right), \tag{9.60}$$

we call

$$\mathrm{supp}(\psi) = \{\alpha \in \Pi \,|\, m_\alpha \neq 0\} \subset \Pi \tag{9.61}$$

determined by the non-vanishing m_α the *support of the character* ψ. With this notion, the definition 6.10 becomes

$$\begin{aligned} \mathrm{supp}(\psi) = \Pi \quad &\Longleftrightarrow \quad \psi \text{ generic,} \\ \mathrm{supp}(\psi) \neq \Pi \quad &\Longleftrightarrow \quad \psi \text{ non-generic or degenerate.} \end{aligned}$$

We note that a degenerate character $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$ canonically defines a simple proper subgroup $G' \subset G$. This subgroup G' is the one with simple root system $\Pi' = \mathrm{supp}(\psi)$; its Dynkin diagram is the subdiagram of the Dynkin diagram of G obtained by restricting to the nodes corresponding to $\mathrm{supp}(\psi)$. The subgroup G' has a Weyl group $\mathcal{W}'$ with longest Weyl word w'_{long}.

Proof. Using the Bruhat decomposition, the spherical Whittaker vector $W^\circ_\psi(\chi, a)$ can be written as a sum over the Weyl group $\mathcal{W}$ of G as in (9.3)

$$W^\circ_\psi(\chi, a) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, na)\overline{\psi(n)} dn = \sum_{w\in\mathcal{W}} F_{w,\psi}(\chi, a) \tag{9.62}$$

with

$$F_{w,\psi}(\chi,a) = \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(wna)\overline{\psi(n)}dn = \int\limits_{N^w_{\{\beta\}}} \overline{\psi(n_\beta)}dn_\beta \int\limits_{N^w_{\{\gamma\}}} \chi(wn_\gamma a)\overline{\psi(n_\gamma)}dn_\gamma\,. \tag{9.63}$$

The various $F_{w,\psi}$ can be analysed as in section 9.1 and we have used the w-dependent split of positive roots of N into two sets of $\{\beta\}$ and $\{\gamma\}$ as in (9.7) and (9.8). Importantly, for degenerate ψ the integral over the compact domain $N^w_{\{\beta\}}$ can be non-vanishing for Weyl words w different from w_{long}: if ψ is trivial on all the n_β in (9.63), the corresponding integral yields unity rather than zero. This means that the Weyl word w must map all elements in $\text{supp}(\psi)$ to negative roots in order to avoid the vanishing of $F_{w,\psi}$ and the sum over $\mathcal{W}$ in (9.62) can be restricted to the subset

$$\mathcal{C}_\psi = \{w \in \mathcal{W} \,|\, w\alpha < 0 \quad \text{for all } \alpha \in \text{supp}(\psi)\}\,. \tag{9.64}$$

(If ψ is generic, one recovers $\mathcal{C}_\psi = \{w_{\text{long}}\}$ in agreement with the discussion of section 9.1.)

We will now parametrise the set $\mathcal{C}_\psi$ explicitly. Denote by $\mathcal{W}'$ the Weyl subgroup generated by the fundamental reflections associated with $\Pi' = \text{supp}(\psi)$ only. It is the Weyl group of G' and has its own longest Weyl word that we denote by w'_{long}. The longest Weyl word w'_{long} has the desired property that it maps all elements in $\text{supp}(\psi)$ to negative roots and it is the only Weyl word in $\mathcal{W}'$ with this property. In fact, any element in $\mathcal{C}_\psi$ can be represented in a form that involves the longest word w'_{long} of $\mathcal{W}'$:

$$w \in \mathcal{C}_\psi \quad \Longleftrightarrow \quad w = w_c w'_{\text{long}}. \tag{9.65}$$

Here, $w_c \in \mathcal{W}$ must satisfy

$$w_c\alpha > 0 \quad \text{for all } \alpha \in \text{supp}(\psi) \tag{9.66}$$

in order for $w = w_c w'_{\text{long}}$ to belong to $\mathcal{C}_\psi$.

The words $w_c \in \mathcal{W}$ can be constructed as carefully chosen representatives of the coset $\mathcal{W}/\mathcal{W}'$. Consider the weight

$$\Lambda_\psi = \sum_{i\,:\,\alpha_i \notin \text{supp}(\psi)} \Lambda_i, \tag{9.67}$$

i.e., the sum of fundamental weights of G that are not associated with the support of the degenerate character ψ. The weight Λ_ψ is stabilised by $\mathcal{W}'$ and its $\mathcal{W}$-orbit is in bijection with the coset $\mathcal{W}/\mathcal{W}'$. A standard result for Weyl groups is that if $w(\alpha_i) < 0$ for some simple root, then $\ell(ww_i) < \ell(w)$ [186, Lemma 3.11]. Therefore, if $w(\alpha_i) < 0$ and $\alpha_i \in \text{supp}(\psi)$ we have

$$w(\Lambda_\psi) = (ww_i)(\Lambda_\psi) \tag{9.68}$$

since Λ_ψ is stabilised by the fundamental reflections from $\text{supp}(\psi)$ (these generate $\mathcal{W}'$). This means that if $w(\alpha_i) < 0$ there is a shorter Weyl word ww_i leading to the same point in the $\mathcal{W}$-orbit of Λ_ψ as the word w does. By induction, the shortest word leading to a given point in the Weyl orbit of Λ_ψ must be those $w_c \in \mathcal{W}$ that satisfy $w_c\alpha > 0$ for all $\alpha \in \mathrm{supp}(\psi)$. Hence, the words w_c appearing in (9.66) are the shortest words leading to the points of the $\mathcal{W}$-orbit of Λ_ψ. Such shortest words are not necessarily unique; for a given $\mathcal{W}$-orbit point any shortest word w_c will do. An explicit construction of the w_c can be achieved by the same orbit method as in section 10.3, see also [95, 97].

With the parametrisation $w = w_c w'_{\text{long}}$ of the elements of $\mathcal{C}_\psi$ we thus arrive at the following expression for the degenerate Whittaker integral (9.62):

$$W^\circ_\psi(\chi, a) = \sum_{w_c w'_{\text{long}} \in \mathcal{W}/\mathcal{W}'} F_{w_c w'_{\text{long}}, \psi}(\chi, a), \tag{9.69}$$

where it is understood that $w_c w'_{\text{long}}$ is the specific coset representative described above.

The quantities $F_{w_c w'_{\text{long}}, \psi}(\chi, a)$ can be evaluated by reducing them to Whittaker vectors of the subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$ associated with $\mathrm{supp}(\psi)$ as follows. First, we separate out the a-dependence as usual by conjugating it to the left and using the multiplicativity of χ

$$\begin{aligned} F_{w_c w'_{\text{long}}, \psi}(\chi, a) &= \int\limits_{(w_c w'_{\text{long}})^{-1} B(\mathbb{Q}) w_c w'_{\text{long}} \cap N(\mathbb{Q}) \backslash N(\mathbb{A})} \chi(w_c w'_{\text{long}} n a) \overline{\psi(n)} dn \\ &= a^{(w_c w'_{\text{long}})^{-1}\lambda + \rho} \int\limits_{(w_c w'_{\text{long}})^{-1} B(\mathbb{Q}) w_c w'_{\text{long}} \cap N(\mathbb{Q}) \backslash N(\mathbb{A})} \chi(w_c w'_{\text{long}} n) \overline{\psi^a(n)} dn \end{aligned} \tag{9.70}$$

with $\psi^a(n) = \psi(ana^{-1})$ as before. We can also rewrite the integration into the two sets $N^w_{\{\beta\}}$ and $N^w_{\{\gamma\}}$ (for $w = w_c w'_{\text{long}}$) as in (9.63) and we know that by construction the integral over $N^w_{\{\beta\}}$ gives unity.

The remaining integral over $N^w_{\{\gamma\}}$ is then over all positive roots γ that are mapped to negative roots by the action of $w = w_c w'_{\text{long}}$ and we drop the γ subscript for ease of notation. The particular form of w implies that we can parametrise the unipotent element as $n = n_c n'$ where $n' \in N'(\mathbb{A})$ is the (full) unipotent radical of the standard minimal Borel subgroup $B'(\mathbb{A})$ of $G'(\mathbb{A})$ that is determined by ψ; and n_c are the remaining elements whose total space we call $N_c(\mathbb{A})$. We note also that $w'_{\text{long}} n_c (w'_{\text{long}})^{-1}$ is generated exactly by the positive roots that are mapped to negative roots by w_c alone. The degenerate character ψ only depends on n', i.e., $\psi^a(n_c n') = \psi^a(n')$.

Putting these observations together one obtains

$$F_{w_c w'_{\text{long}}, \psi}(\chi, a) = a^{(w_c w'_{\text{long}})^{-1}\lambda + \rho} \int\limits_{N_c(\mathbb{A})} \int\limits_{N'(\mathbb{A})} \chi(w_c w'_{\text{long}} n_c n') \overline{\psi^a(n')} dn_c dn'. \tag{9.71}$$

As the next step one can rewrite the argument of the character χ as

$$\chi(w_c w'_{\text{long}} n_c n') = \chi\left(w_c w'_{\text{long}} n_c (w_c w'_{\text{long}})^{-1} w_c w'_{\text{long}} n'\right) = \chi\left(w_c w'_{\text{long}} n_c (w_c w'_{\text{long}})^{-1} w_c \tilde{n} \tilde{a}\right), \tag{9.72}$$

where we have performed an Iwasawa decomposition (in $G'(\mathbb{A})$) of $w'_{\text{long}}n' = \tilde{n}\tilde{a}\tilde{k}$ and used left-invariance of χ under $K'(\mathbb{A}) \subset K(\mathbb{A})$ in the last step. In the next step we want to perform another Iwasawa decomposition (now in $G(\mathbb{A})$) of

$$w_c\tilde{n}\tilde{a} = \hat{n}\hat{a}\hat{k}. \tag{9.73}$$

The important observation now is that $\tilde{n} \in N'(\mathbb{A})$ and w_c satisfies (9.66) which implies that $w_c N'(\mathbb{A}) w_c^{-1} \subset N(\mathbb{A})$. Therefore, the Iwasawa decomposition (9.73) has

$$\hat{n} = w_c\tilde{n}w_c^{-1}, \quad \hat{a} = w_c\tilde{a}w_c^{-1}, \quad \hat{k} = w_c. \tag{9.74}$$

Inserting this back into the integral (9.71) one can bring the element $\hat{n} \in N(\mathbb{A})$ to the left. This will induce a uni-modular change of the integration variables dn_c as in section 8.5. Conjugating the element $\hat{a} \in A(\mathbb{A})$ to the left will induce a non-trivial change of measure ($w = w_c w'_{\text{long}}$):

$$\begin{aligned}\int\limits_{N_c(\mathbb{A})} \chi(wn_cw^{-1}\hat{n}\hat{a})dn_c &= \int\limits_{N_c(\mathbb{A})} \chi(\hat{n}\hat{a}wn_cw^{-1})\hat{a}^{w_c\rho-\rho}dn_c = \int\limits_{N_c(\mathbb{A})} \chi(wn_cw^{-1})\tilde{a}^{w_c^{-1}\lambda-\rho}dn_c \\ &= \tilde{a}^{w_c^{-1}\lambda-\rho}\int\limits_{N_c(\mathbb{A})} \chi(wn_cw^{-1})dn_c = \chi'(\tilde{a})\int\limits_{N_c(\mathbb{A})} \chi(wn_cw^{-1})dn_c.\end{aligned} \tag{9.75}$$

We have evaluated the character χ on $\hat{n}\hat{a}$ in the second step according to $\chi(\hat{n}\hat{a}) = \hat{a}^{\lambda+\rho} = \tilde{a}^{w_c^{-1}\lambda+w_c^{-1}\rho}$ due to (9.74). In the last step, we have used that $\tilde{a}$ does not depend on n_c and can therefore be taken out of the integral and defined the character

$$\chi'(\tilde{a}) = \tilde{a}^{w_c^{-1}\lambda+\rho} = \chi'(\tilde{a}) = \chi'(w'_{\text{long}}n'), \tag{9.76}$$

on the group $G'(\mathbb{A})$. In the last step we have used the definition of $\tilde{a}$.

Putting everything together in (9.71) one obtains the factorised expression

$$F_{w_cw'_{\text{long}},\psi}(\chi, a) = a^{(w_cw'_{\text{long}})^{-1}\lambda+\rho}\int\limits_{N_c(\mathbb{A})} \chi(w_cw'_{\text{long}}n_c)dn_c \cdot \int\limits_{N'(\mathbb{A})} \chi'(w'_{\text{long}}n')\overline{\psi^a(n')}dn'. \tag{9.77}$$

The two separate integrals are both of types we have encountered before: The $N_c(\mathbb{A})$ integral is precisely the Gindikin–Karpelevich expression (8.19) for the Weyl word $w_c \in \mathcal{W}$ and so gives a factor $M(w_c^{-1}, \lambda)$ defined in (8.42), and the second integral is the generic Whittaker vector (9.13) for the subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$ with *generic* Fourier character ψ^a, in the representation given by the weight $w_c^{-1}\lambda$, projected orthogonally to $G'(\mathbb{A})$ and evaluated at the identity $\mathbb{1} \in A'(\mathbb{A})$. This completes the proof of theorem 9.5. □

As a consequence of the theorem, Whittaker vectors of non-generic characters ψ can be evaluated as sums over Whittaker vectors of subgroups on which the character is generic. We stress again that the choice of coset representative of $\mathcal{W}/\mathcal{W}'$ is important here. If the full Whittaker vector on the subgroup is known, the above formula provides the explicit expression for any character ψ. Thanks to the Casselman–Shalika formula, this means that the *local* Whittaker vector ($p < \infty$) can be calculated for *any* character, generic or not. The archimedean part is typically more intricate.

Remark 9.7. Theorem 9.5 of course also remains true in the case of generic ψ since then $\mathcal{W}' = \{\mathbb{1}\}$ is trivial and the sum on the right-hand side is just the decomposition of the generic Whittaker vector into Bruhat cells and nothing is gained. The power of the theorem arises in cases where one deals with an Eisenstein series that does not have any generic Whittaker vectors and one can then use (9.59) to determine the degenerate ones. This will be explored in more detail in section 10.4 below.

9.6 Whittaker vectors on $SL(3,\mathbb{A})$

We illustrate the general considerations above through the explicit example of $SL(3,\mathbb{A})$. The Eisenstein series on $SL(3,\mathbb{R})$, $GL(3,\mathbb{R})$ and this group have been studied in great detail in the literature [60, 314] by various techniques.

The split real group $SL(3,\mathbb{R})$ has rank two and we denote the two simple roots by α_1 and α_2. The corresponding Cartan generators will be called $H_1 \equiv H_{\alpha_1}$ and $H_2 \equiv H_{\alpha_2}$. A general element $a \in A(\mathbb{A})$ will be written as

$$a = v_1^{H_1} v_2^{H_2}. \tag{9.78}$$

The Eisenstein series is determined by the weight

$$\lambda = (2s_1 - 1)\Lambda_1 + (2s_2 - 1)\Lambda_2 \tag{9.79}$$

in terms of the fundamental weights dual to the simple roots.

The Weyl group consists of six elements:

$$\mathcal{W} = \{\mathbb{1}, w_1, w_2, w_1w_2, w_2w_1, w_1w_2w_1\}. \tag{9.80}$$

We will first compute the constant term using Langlands constant term formula of chapter 8. Then, using the results of sections 9.1–9.3, we find the local part of a Whittaker vector with an unramified character, which, with the help of section 9.4, can then be used to compute the local part of any Whittaker vector with a generic character. The remaining, degenerate, Whittaker vectors are then found following the arguments of section 9.5. Lastly, we will comment on the non-abelian Whittaker vectors.

9.6.1 Constant terms

We first evaluate the Langlands constant term formula (8.41). This yields a sum of six terms:

$$\begin{aligned}\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng)dn &= v_1^{2s_1} v_2^{2s_2} + \frac{\xi(2s_1-1)}{\xi(2s_1)} v_1^{2-2s_1} v_2^{2s_1+2s_2-1} + \frac{\xi(2s_2-1)}{\xi(2s_2)} v_1^{2s_1+2s_2-1} v_2^{2-2s_2} \\ &+ \frac{\xi(2s_1-1)\xi(2s_1+2s_2-2)}{\xi(2s_1)\xi(2s_1+2s_2-1)} v_1^{2s_2} v_2^{3-2s_1-2s_2} + \frac{\xi(2s_2-1)\xi(2s_1+2s_2-2)}{\xi(2s_2)\xi(2s_1+2s_2-1)} v_1^{3-2s_1-2s_2} v_2^{2s_1} \\ &+ \frac{\xi(2s_1-1)\xi(2s_2-1)\xi(2s_1+2s_2-2)}{\xi(2s_1)\xi(2s_2)\xi(2s_1+2s_2-1)} v_1^{2-2s_2} v_2^{2-2s_1}.\end{aligned} \tag{9.81}$$

Here, v_1 and v_2 are real positive parameters.

9.6.2 Generic Whittaker vectors

We first determine the *local* Whittaker vector for an unramified character ψ by using the Casselman–Shalika formula in the form (9.21). The quantities $1/\zeta(\lambda)$ and $\epsilon(\lambda)$ of (9.22) evaluate to

$$\frac{1}{\zeta(\lambda)} = (1-p^{-2s_1})(1-p^{-2s_2})(1-p^{1-2s_1-2s_2}), \tag{9.82a}$$

$$\epsilon(\lambda) = \frac{1}{(1-p^{2s_1-1})(1-p^{2s_2-1})(1-p^{2s_1+2s_2-2})} \tag{9.82b}$$

and the full unramified local coefficient is then

$$\begin{aligned} W^\circ(\lambda, a) = \frac{\epsilon(\lambda)}{\zeta(\lambda)}\Big(&|v_1|^{2s_1}|v_2|^{2s_2} - p^{2s_1-1}|v_1|^{2-2s_1}|v_2|^{2s_1+2s_2-1} - p^{2s_2-1}|v_1|^{2s_1+2s_2-1}|v_2|^{2-2s_2} \\ &+ p^{4s_1+2s_2-3}|v_1|^{2s_2}|v_2|^{3-2s_1-2s_2} + p^{2s_1+4s_2-3}|v_1|^{3-2s_1-2s_2}|v_2|^{2s_1} \\ &- p^{4s_1+4s_2-4}|v_1|^{2-2s_2}|v_2|^{2-2s_1}\Big). \end{aligned} \tag{9.83}$$

Here, v_1 and v_2 are in $\mathbb{Q}_p$.

From (9.83) we can deduce the Whittaker vector for a generic character with non-zero instanton charges m_1 and m_2, i.e., one that satisfies

$$\psi^a\left(x_{\alpha_1}(u_1)x_{\alpha_2}(u_2)\right) = \exp\left(2\pi i[m_1u_1 + m_2u_2]\right) \tag{9.84}$$

by exploiting (9.57). For this we require $v_1 = m_1^{2/3}m_2^{1/3}$ and $v_2 = m_1^{1/3}m_2^{2/3}$ in the expression above as well as the prefactor $|a^{-(w_{\text{long}}\lambda+\rho)}| = |v_1|^{2s_2-2}|v_2|^{2s_1-2}$. The result is

$$\begin{aligned} W^\circ_{\psi^a}(\chi, \mathbb{1}) = \frac{\epsilon(\lambda)}{\zeta(\lambda)}\Big(&|m_1|^{2s_1+2s_2-2}|m_2|^{2s_1+2s_2-2} - p^{2s_1-1}|m_1|^{2s_2-1}|m_2|^{2s_1+2s_2-2} \\ &- p^{2s_2-1}|m_1|^{2s_1+2s_2-2}|m_2|^{2s_1-1} + p^{4s_1+2s_2-3}|m_1|^{2s_2-1} + p^{2s_1+4s_2-3}|m_2|^{2s_1-1} \\ &- p^{4s_1+4s_2-4}\Big). \end{aligned} \tag{9.85}$$

As is well-known [60], this can also be expressed in terms of a Schur polynomial in (m_1, m_2) which here encodes the character of a highest weight representation of $\mathfrak{sl}(3,\mathbb{C})$. Taking the product over all $p < \infty$ produces double divisor sums.

In this case, we can also work out the archimedean Whittaker vector. The Whittaker vector at $p = \infty$ can be explicitly written as a convoluted integral of two modified Bessel functions as we will now show.

Starting from (9.14) and using the same standard manipulations as in (9.18) we have that

$$W^\circ_{\psi_\infty}(\chi_\infty, a) = \int_{N(\mathbb{R})} \chi_\infty(w_{\text{long}}na)\overline{\psi_\infty(n)}dn = |a^{w_{\text{long}}\lambda+\rho}| \int_{N(\mathbb{R})} \chi_\infty(w_{\text{long}}n)\overline{\psi_\infty(ana^{-1})}dn \tag{9.86}$$

with the generic character ψ_∞ given by two integers m_1 and m_2 through

$$\begin{aligned}\psi_\infty\big(x_{\alpha_1}(u_1)x_{\alpha_2}(u_2)\big) &= \exp\big(2\pi i(m_1u_1+m_2u_2)\big)\\ \chi_\infty(v_1^{H_1}v_2^{H_2}) &= |v_1|^{2s_1}\,|v_2|^{2s_2}\\ \left|a^{w_{\text{long}}\lambda+\rho}\right| &= |v_1|^{2-2s_2}\,|v_2|^{2-2s_1}\\ n = \left(\begin{smallmatrix}1&u_1&z\\0&1&u_2\\0&0&1\end{smallmatrix}\right) \qquad & w_{\text{long}} = \left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right).\end{aligned} \tag{9.87}$$

Evaluating the integrand we obtain

$$\begin{aligned}\int_{N(\mathbb{R})}\chi_\infty(w_{\text{long}}n)\overline{\psi(ana^{-1})}dn = \\ \int_{\mathbb{R}^3}\big(1+(1+u_1^2)u_2^2-2u_1u_2z+z^2\big)^{-s_1}\big(1+u_1^2+z^2\big)^{-s_2}\times \\ \times\exp\left(-2\pi i\frac{v_1^3m_1u_1+v_2^3m_2u_2}{v_1v_2}\right)du_1du_2dz\,.\end{aligned} \tag{9.88}$$

Using the variable substitution $u_2\to(u_2+u_1z)/(1+u_1^2)$ and integrating over u_2 we get

$$\begin{aligned}\frac{2\pi^{s_1}}{\Gamma(s_1)}\left|\frac{m_2v_2^2}{v_1}\right|^{s_1-\frac12}\int_{\mathbb{R}^2}\frac{(1+u_1^2+z^2)^{\frac14-\frac12s_1-s_2}}{\sqrt{1+u_1^2}}K_{s_1-\frac12}\left(2\pi\left|\frac{m_2v_2^2}{v_1}\right|\frac{\sqrt{1+u_1^2+z^2}}{1+u_1^2}\right)\times \\ \times\exp\left(-2\pi i\left(\frac{m_1v_1^2}{v_2}u_1+\frac{m_2v_2^2}{v_1}\frac{u_1z}{1+u_1^2}\right)\right)du_1dz\,.\end{aligned} \tag{9.89}$$

With standard manipulations (see for example [314, Lemma 7]), this integral can be expressed as a convoluted integral of two Bessel functions giving $W^\circ_{\psi_\infty}$ as

$$\begin{aligned}W^\circ_{\psi_\infty}(\chi_\infty,a) = \frac{4\pi^{2s_3+\frac12}\,|v_1v_2|}{\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)}\,|m_1m_2|^{s_3-\frac12}\left|\frac{v_1}{v_2}\right|^{s_1-s_2}\times \\ \times\int_0^\infty K_{s_3-\frac12}\left(2\pi\left|\frac{m_1v_1^2}{v_2}\right|\sqrt{1+1/x}\right)K_{s_3-\frac12}\left(2\pi\left|\frac{m_2v_2^2}{v_1}\right|\sqrt{1+x}\right)x^{\frac{s_2-s_1}{2}}\frac{dx}{x}\end{aligned} \tag{9.90}$$

where we have introduced $s_3=s_1+s_2-\frac12$ for compactness.

9.6.3 Degenerate Whittaker vectors

We now evaluate the Whittaker vectors for non-generic characters, i.e., those where either m_1 or m_2 vanishes. Note that it is not trivially possible to obtain this result from the generic one above by setting some parameters to zero. We will employ theorem 9.5 and perform this in the example $m_2=0$. Then the support of the character is only on the first simple root, so that $w'_{\text{long}}=w_1$ and the subgroup $G'(\mathbb{A})$ is the one associated with the first simple root only. The possible Weyl words that contribute to (9.59) are

$$w = w_cw'_{\text{long}}\in\left\{\mathbb{1}w'_{\text{long}},w_2w'_{\text{long}},w_1w_2w'_{\text{long}}\right\} = \{w_1,w_2w_1,w_1w_2w_1\}\,. \tag{9.91}$$

As a first step, we calulcate the projected weights $w_c^{-1}\lambda$ and $M(w_c^{-1},\lambda)$ factors that appear in (9.59) for the three choices:

$$w_c = \mathbb{1}: \qquad \lambda' = (w_c^{-1}\lambda)_{G'} = (2s_1-1)\Lambda_1', \quad M(w_c^{-1},\lambda) = 1, \tag{9.92a}$$

$$w_c = w_2: \qquad \lambda' = (2s_1+2s_2-2)\Lambda_1', \qquad M(w_c^{-1},\lambda) = \frac{\xi(2s_2-1)}{\xi(2s_2)}, \tag{9.92b}$$

$$w_c = w_1w_2: \quad \lambda' = (2s_2-1)\Lambda_1', \qquad M(w_c^{-1},\lambda) = \frac{\xi(2s_1-1)\xi(2s_1+2s_2-2)}{\xi(2s_1)\xi(2s_1+2s_2-1)}. \tag{9.92c}$$

where $\Lambda_1' = \alpha_1/2$ is the fundamental weight for $G'(\mathbb{A})$.

This will need to be combined with

$$\psi^a(x_{\alpha_1}(u_1)) = \psi(ax_{\alpha_1}(u_1)a^{-1}) = \exp\left(2\pi i a^{\alpha_1}u_1m_1\right) = \exp\left(2\pi i v_1^2v_2^{-1}m_1u_1\right) \tag{9.93}$$

and the $SL(2,\mathbb{A})$ Whittaker vector for $\lambda' = (2s'-1)\Lambda_1'$ given by (cf. (7.81))

$$W'^{\circ}_{\psi^a}(\lambda',\mathbb{1}) = \frac{2(2\pi)^{1/2-s'}}{\xi(2s')}\sigma_{2s'-1}(m_1)\mathcal{K}_{1/2-s'}(2\pi|m_1|v_1^2v_2^{-1}), \tag{9.94}$$

where we introduced the short-hand $\mathcal{K}_t(x) = x^{-t}K_{-t}(x)$ in order to facilitate comparison with [267]. Recall also the compact notation $s_3 = s_1+s_2-\frac{1}{2}$. The resulting expression for the $(m_1,0)$ degenerate Whittaker vector is then

$$\begin{aligned} W^\circ_\psi(\chi,a) = {} & \frac{2(2\pi)^{1/2-s_1}}{\xi(2s_1)}v_1^{2-2s_1}v_2^{2s_1+2s_2-1}\sigma_{2s_1-1}(m_1)\mathcal{K}_{1/2-s_1}(2\pi|m_1|v_1^2v_2^{-1}) \\ & + \frac{2(2\pi)^{1/2-s_3}}{\xi(2s_3)}\frac{\xi(2s_2-1)}{\xi(2s_2)}v_1^{3-2s_1-2s_2}v_2^{2s_1}\sigma_{2s_3-1}(m_1)\mathcal{K}_{1/2-s_3}(2\pi|m_1|v_1^2v_2^{-1}) \\ & + \frac{2(2\pi)^{1/2-s_2}}{\xi(2s_2)}\frac{\xi(2s_1-1)\xi(2s_3-1)}{\xi(2s_1)\xi(2s_3)}v_1^{2-2s_2}v_2^{2-2s_1}\sigma_{2s_2-1}(m_1)\mathcal{K}_{1/2-s_2}(2\pi|m_1|v_1^2v_2^{-1}). \end{aligned} \tag{9.95}$$

This matches also the expressions in [267] if one adapts the conventions and corrects a typo there. More precisely, one uses $v_1 = \nu^{-1/6}\tau_2^{1/2}$, $v_2 = \nu^{-1/3}$ and exchanges s_1 and s_2 to find the $\Psi_{0,q}$ coefficient in [267, Eq. (3.45)], if one fixes the third summand there. A similar calculation can be carried out for the degenerate Whittaker vector associated with instanton charges $(0,m_2)$; it simply amounts to interchanging the subscripts 1 and 2 everywhere thanks to the Dynkin diagram automorphism of $\mathfrak{sl}(3,\mathbb{R})$.

9.6.4 Non-abelian Fourier coefficients

So far in this section we have only studied Fourier coefficients (or Whittaker vectors) on N, but since the characters on N are trivial on the centre $Z = N^{(2)} = [N,N]$ they do not capture the complete Fourier expansion of $E(\chi,g)$ as discussed in section 6.2.3. To have a complete expansion we also need Fourier coefficients on Z, that is, Whittaker vectors on Z, with non-trivial characters $\psi_Z : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to U(1)$ parametrised by $k \in \mathbb{Q}^\times$

$$\psi_Z(n_{(2)}) = e^{2\pi ikz} \qquad n_{(2)} = \left(\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in Z(\mathbb{A}) . \tag{9.96}$$

To avoid ambiguities, we will denote the character on Z by ψ_Z and the characters on N by ψ_N.

Recalling (6.2.3), the Whittaker vectors on Z are defined by

$$W^\circ_{\psi_Z}(\chi, g) = \int\limits_{Z(\mathbb{Q})\backslash Z(\mathbb{A})} E(\chi, n_{(2)}g)\overline{\psi_Z(n_{(2)})}\, dn_{(2)} . \tag{9.97}$$

For the remaining parts of this section, we will drop the superscript for the spherical property and write the charge explicitly as $W^{(k)}_{\psi_Z}$ for clarity.

We will now show that these Whittaker vectors on Z are determined by the Whittaker vectors on N (that we calculated above) but before we can make an exact statement we need to make a few definitions.

Let $k, m_2 \in \mathbb{Q}$ with $k = a_1/b_1$ and $m_2 = a_2/b_2$ in shortened form where $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}$. Define

$$d = d(k, m_2) := \frac{\gcd(a_1b_2, a_2b_1)}{b_1b_2} \tag{9.98}$$

which is then strictly positive since $k \neq 0$ and let $k' := k/d = a_1b_2/\gcd(a_1b_2, a_2b_1) \in \mathbb{Z}$ and $m_2' := m_2/d = a_2b_1/\gcd(a_1b_2, a_2b_1) \in \mathbb{Z}$. Then, there exists integers α and β such that

$$\alpha m_2' - \beta k' = \gcd(k', m_2') = 1 . \tag{9.99}$$

The ambiguity in the definition of α and β is discussed in the proof of the following proposition.

Proposition 9.8. *Let $k \in \mathbb{Q}^\times$ with α, β, k' and m_2' defined as above. Then*

$$W^{(k)}_{\psi_Z}(\chi, g) = \sum_{m_1, m_2 \in \mathbb{Q}} W^{(m_1, d)}_{\psi_N}(\chi, lg) \qquad l = \begin{pmatrix} \alpha & \beta & 0 \\ k' & m_2' & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}) \tag{9.100}$$

where $g = (g_\infty, g_2, g_3, \ldots)$ is an arbitrary element of $G(\mathbb{A})$.

We will consider the restriction $g = (g_\infty, \mathbb{1}, \mathbb{1}, \ldots)$ giving integer charges in proposition 9.9. By $W^{(m_1,d)}_{\psi_N}$ we mean the Whittaker vectors on N given by ψ_N with instanton charges m_1 and d for the simple roots, which were calculated in (9.85), (9.90) and (9.95).

Proof. To show (9.100), let first l be defined as in that equation. We can expand $W^{(k)}_{\psi_Z}(\chi, g)$ further as

$$W^{(k)}_{\psi_Z}(\chi, g) = \int\limits_{\mathbb{Q}\backslash\mathbb{A}} E(\chi, \left(\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) g) e^{-2\pi ikz}\, dz = \sum_{m_2 \in \mathbb{Q}} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \left(\begin{smallmatrix} 1 & 0 & z \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{smallmatrix}\right) g) e^{-2\pi i(kz + m_2x_2)}\, dz dx_2 . \tag{9.101}$$

Using the automorphic invariance of $E(\chi, g)$ we can make the following conjugation with l

$$\begin{aligned} W_{\psi_Z}^{(k)}(\chi, g) &= \sum_{m_2} \int_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, l\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} l^{-1} l g) e^{-2\pi i(kz + m_2 x_2)}\, dz dx_2 \\ &= \sum_{m_2} \int_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \begin{pmatrix} 1 & 0 & -d(x_2 - \alpha(kz+m_2x_2)/d)/k \\ 0 & 1 & (kz+m_2x_2)/d \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i(kz + m_2 x_2)}\, dz dx_2 \\ &= \sum_{m_2} \int_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i d x_3}\, dx_2 dx_3\,, \end{aligned} \tag{9.102}$$

where we have made the substitution $(kz + m_2x_2)/d \to x_3$ and then $-d(x_2 - \alpha x_3)/k \to x_2$ leaving the integration domain the same. According to (3.17) and (3.67) the measure is also unchanged. We note that the ambiguity in α simply results in an extra shift in the periodic variable x_2.

We expand one more time

$$\begin{aligned} W_{\psi_Z}^{(k)}(\chi, g) &= \sum_{m_1, m_2 \in \mathbb{Q}} \int_{(\mathbb{Q}\backslash\mathbb{A})^3} E(\chi, \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i(m_1 x_1 + d x_3)}\, d^3x \\ &= \sum_{m_1, m_2} \int_{(\mathbb{Q}\backslash\mathbb{A})^3} E(\chi, \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i(m_1 x_1 + d x_3)}\, d^3x \\ &= \sum_{m_1, m_2} W_{\psi_N}^{(m_1, d)}(\chi, lg) \end{aligned} \tag{9.103}$$

where we, in the second step, have made the substitution $x_2 + x_1x_3 \to x_2$. □

Note that when inserting $g = (g_\infty, \mathbb{1}, \mathbb{1}, \ldots) \in G(\mathbb{A})$ into (9.100), reducing the adelic Eisenstein series on the left hand side to the real Eisenstein series, the arguments on the right hand side become non-trivial at the finite places. To be able to use the expressions for W_{ψ_N} above which require trivial arguments at the finite places, we need to factor out these effects.

Proposition 9.9. *Let $\tau = u_1 + iv_1^2/v_2 \in \mathbb{H}$ and*

$$\gamma = \begin{pmatrix} \alpha & \beta \\ k' & m_2' \end{pmatrix} \in SL(2, \mathbb{Z}) \qquad \gamma(\tau) = \frac{\alpha\tau + \beta}{k'\tau + m_2'} \qquad a'_{\operatorname{Im}\gamma(\tau)} = \begin{pmatrix} v_1' & 0 & 0 \\ 0 & v_2'/v_1' & 0 \\ 0 & 0 & 1/v_2' \end{pmatrix} \tag{9.104}$$

with $v_1' = \sqrt{v_2' \operatorname{Im}\gamma(\tau)}$ and $v_2' = v_2$, and $g = (g_\infty, \mathbb{1}, \ldots) \in G(\mathbb{A})$. Then $W_{\psi_Z}^{(k)}$ is non-vanishing only for $k \in \mathbb{Z}$ for which

$$W_{\psi_Z}^{(k)}\big(\chi, (g_\infty, \mathbb{1}, \ldots)\big) = \sum_{m_1, m_2 \in \mathbb{Z}} W_{\psi_N}^{(m_1, d)}\big(\chi, (a'_{\operatorname{Im}\gamma(\tau)}, \mathbb{1}, \ldots)\big) e^{-2\pi i(m_1 \operatorname{Re}\gamma(\tau) + m_2 u_2 + kz)} \tag{9.105}$$

Note that the sums over rationals have collapsed to sums over integers and that

$$l = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}. \tag{9.106}$$

This proves the results of [314] and [274] reviewed in [258, 267] with only a few manipulations using the compact framework of adelic automorphic forms.

Proof. In (9.100) the argument for $W_{\psi_N}^{(m_1,d)}$ is $lg = (lg_\infty; l, l, \ldots)$ and since $W_{\psi_N}(\chi, n'a'k') = \psi_N(n')W_{\psi_N}(\chi, a')$, we factorise lg at the archimedean and non-archimedean places into their respective Iwasawa decompositions. We have that $l \in SL(3, \mathbb{Z})$ which makes the p-adic Iwasawa decomposition trivial with $l \in K_p$. This was the reason for choosing l on this particular form.

We then use the following relation, similar to (9.19), to obtain conditions for m_1 and m_2. For $\hat{n} = (\mathbb{1}; \hat{n}_2, \hat{n}_3, \ldots)$ with $\hat{n}_p \in N(\mathbb{Z}_p) \subset K_p$ we have that $\hat{n} \in K_\mathbb{A}$ and

$$W_{\psi_N}(\chi, a) = W_{\psi_N}(\chi, a\hat{n}) = W_{\psi_N}(\chi, a\hat{n}a^{-1}a) = \psi_N(a\hat{n}a^{-1})W_{\psi_N}(\chi, a) \tag{9.107}$$

which requires that $\psi_N(a\hat{n}a^{-1}) = 1$ for $W_{\psi_N}(\chi, a)$ to be non-vanishing.

Specifically, for $W_{\psi_N}^{(m_1,d)}(\chi, a)$ with $a = (a_\infty; \mathbb{1}, \mathbb{1}, \ldots)$ we require that

$$1 = \psi_N(a\hat{n}a^{-1}) = \psi_{N,\infty}(\mathbb{1}) \prod_{p<\infty} \psi_{N,p}(\hat{n}_p) = \exp\left(-2\pi i \sum_{p<\infty} [m_1 u_1 + d u_2]_p\right) \tag{9.108}$$

for all $u_1, u_2 \in \mathbb{Z}_p$ where

$$\hat{n}_p = \begin{pmatrix} 1 & u_1 & z \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9.109}$$

This implies that $\sum_{p<\infty} [m_1]_p \in \mathbb{Z}$ and $\sum_{p<\infty} [d]_p \in \mathbb{Z}$, which, according to proposition 3.13, gives that $m_1, d \in \mathbb{Z}$. That d is integer means that, for all primes p

$$1 \geq |d|_p = \frac{\max(|a_1 b_2|_p, |a_2 b_1|_p)}{|b_1 b_2|_p} = \max(|k|_p, |m_2|_p) \tag{9.110}$$

according to (3.11), and hence, that k and m_2 are also integers.

For the archimedean place we have the Iwasawa decomposition

$$lg_\infty = l n_\infty a_\infty k_\infty = l \begin{pmatrix} 1 & u_1 & z \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2/v_1 & 0 \\ 0 & 0 & 1/v_2 \end{pmatrix} k_\infty = n'_\infty a'_\infty k'_\infty \qquad \text{with}$$

$$u'_1 = -\frac{d^2(m_2 + k u_1) v_2^2}{k^3 v_1^4 + k(m_2 + k u_1)^2 + v_2^2} + \frac{d\alpha}{k} \qquad u'_2 = \frac{m_2 u_2 + k z}{d} \tag{9.111}$$

$$v'_1 = \frac{v_1 v_2 d}{\sqrt{k^2 v_1^4 + (m_2 + k u_1)^2 v_2^2}} \qquad v'_2 = v_2$$

We define $\tau = u_1 + i v_1^2/v_2 \in \mathbb{H}$, which, under the l-translation on g_∞ above, transforms as $\tau \to \tau'$ with

$$\tau' = u'_1 + i\frac{(v'_1)^2}{v'_2} = \gamma(\tau) \qquad \gamma = \begin{pmatrix} \alpha & \beta \\ k' & m'_2 \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{9.112}$$

Putting it all together we obtain for $k \in \mathbb{Z}^\times$ and $g = (g_\infty; \mathbb{1}, \mathbb{1}, \dots)$

$$\begin{aligned} W^{(k)}_{\psi_Z}(\chi, (g_\infty; \mathbb{1}, \dots)) &= \\ &= \sum_{m_1, m_2 \in \mathbb{Z}} W^{(m_1,d)}_{\psi_N}(\chi, a') \psi^{(m_1,d)}_N(n') \\ &= \sum_{m_1, m_2 \in \mathbb{Z}} W^{(m_1,d)}_{\psi_N}(\chi, (a'_\infty; \mathbb{1}, \dots)) \Big(\prod_{p<\infty} \psi^{(m_1,d)}_{N,p}(n'_p) \Big) \psi^{(m_1,d)}_{N,\infty}(n'_\infty) \\ &= \sum_{m_1, m_2 \in \mathbb{Z}} W^{(m_1,d)}_{\psi_N}(\chi, (a'_{\operatorname{Im}\gamma(\tau)}; \mathbb{1}, \dots)) e^{-2\pi i (m_1 \operatorname{Re}\gamma(\tau) + m_2 u_2 + kz)} \end{aligned} \tag{9.113}$$

where $a'_{\operatorname{Im}\gamma(\tau)}$ is defined in (9.104). □

The remaining Whittaker vectors on N with trivial arguments at the non-archimedean places were computed in sections 9.6.2 and 9.6.3.

Remark 9.10. The physical intepretation of the $SL(2,\mathbb{Z})$ action described by γ is described by S-duality of type IIB string theory compactified on a Calabi-Yau threefold [259, 267, 276]. In this setting the parameters z and u_2 are scalar fields sourced by D5- and NS5-branes with charges m_2 and k (more generally denoted by p and q). The branes form bound states that are often reffered to as (p,q) 5-branes. The two scalar fields transform as an $SL(2)$-doublet under S-duality mirrored by their transformation under $g \to lg$ and the charges p and q, which appear as m_2 and k in (9.105), break the classical $SL(2,\mathbb{R})$ symmetry of the supergravity theory to the discrete $SL(2,\mathbb{Z})$ symmetry of the quantum corrected effective action described by Eisenstein series. In short, this tells us that if we can compute the effects from a $(p,0)$ 5-brane, the results for any (p,q) 5-brane follow from S-duality which is mirrored in the sum over matrices γ in (9.105).

9.7 The Casselman–Shalika formula and Langlands duality*

In this section we provide an alternative view on the Casselman–Shalika formula (9.21). For the present analysis it is useful to separate out the modulus character contribution $a^\rho = \delta^{1/2}(a)$ in formula (9.21) and write

$$a^{w\lambda+\rho} = a^\rho a^{w\lambda} = \delta^{1/2}(a) a^{w\lambda}. \tag{9.114}$$

Let ψ be an unramified character on N. The Casselman–Shalika formula (9.21) for the p-adic spherical Whittaker vector on $\mathbb{Q}_p$ evaluated at $a \in A(\mathbb{Q}_p)$ is

$$W^\circ(\lambda, a) = \frac{1}{\zeta(\lambda)} \delta^{1/2}(a) \sum_{w \in \mathcal{W}} w \left(\frac{|a^\lambda|}{\prod_{\alpha>0}(1 - p^{\langle\lambda|\alpha\rangle})} \right) \tag{9.115}$$

with

$$\frac{1}{\zeta(\lambda)} = \prod_{\alpha>0} \left(1 - p^{-1} p^{-\langle\lambda|\alpha\rangle}\right). \tag{9.116}$$

The sum over the Weyl group in (9.115) resembles closely the Weyl character formula (4.26) for highest weight modules. In order to make this resemblance exact, we compare with the rewritten character formula in (4.27) that we reproduce here for convenience:

$$\mathrm{ch}_\Lambda(b) = \sum_{w\in\mathcal{W}} w\left(\frac{b^\Lambda}{\prod_{\beta>0}(1-b^{-\beta})}\right), \tag{9.117}$$

where β runs over the roots of the group whose representation is being constructed and b is an element of its Cartan torus.

An important first observation now is that because of the way λ appears in the numerator and in the denominator of (9.115) the comparison can only work if the character we are trying to match onto is one of the *Langlands dual group* ${}^LG$, or L-group for short, which is a complex algebraic group canonically associated to G [216]. See also sections 11.7 and 12.5 for more details. The L-group is obtained by interchanging roots and co-roots [196], see also [127] for a realisation in physics. The root systems of G and ${}^LG$ are in bijection and the two groups have isomorphic Weyl groups.

Denoting the roots of the Langlands dual group by $\alpha^\vee$ instead of β, we are therefore looking for an element b of the dual torus ${}^LA$ such that $|b^{-\alpha^\vee}| = p^{\langle\alpha|\lambda\rangle}$. This condition fixes uniquely an element $b = a_\lambda \in {}^LA$, where we emphasise that the particular element depends on λ. To ensure that the numerator matches the character of an irreducible highest weight module V_Λ of ${}^LG$ we also need to evaluate (9.115) at a very specific point $a \equiv a_\Lambda$ of $A(\mathbb{Q}_p)$. This element a_Λ is fixed by the requirement that the following (duality) relation hold

$$a_\lambda^\Lambda = a_\Lambda^\lambda, \tag{9.118}$$

where the left-hand side derives from evaluating the character ch_Λ at the place a_λ and the right-hand side is what one obtains by evaluating the spherical Whittaker vector at the special point $a_\Lambda \in A$.

We observe that Λ parametrises points in the space of co-roots $\mathfrak{h}$ of G. By contrast, λ is an element of the space of roots (or weights) $\mathfrak{h}^*$ of G from the start, so that one has to consider a_λ as an element of the *dual torus* ${}^LA$ of the *Langlands dual group* ${}^LG$. Putting everything together we can write the spherical Whittaker vector evaluated at a_Λ in terms of the character of the highest weight representation V_Λ of ${}^LG$ as

$$W^\circ(\lambda, a_\Lambda) = \begin{cases} \frac{1}{\zeta(\lambda)}\delta^{1/2}(a_\Lambda)\mathrm{ch}_\Lambda(a_\lambda) & \text{if } \Lambda \text{ a dominant integral weight of } {}^LG, \\ 0 & \text{otherwise.} \end{cases} \tag{9.119}$$

The vanishing for non-dominant weights Λ of ${}^LG$ is a consequence of the vanishing properties of Whittaker vectors discussed in section 9.2.2.

To summarise the main result of this section: Local spherical Whittaker vectors for a principal series representations parametrised by a weight λ of G and evaluated at special points a_Λ associated with dominant weights Λ of the Langlands dual group ${}^LG$ can be evaluated in terms of the character ch_Λ of the irreducible highest weight V_Λ of ${}^LG$ evaluated at a point a_λ determined by the parameter of the principal series.

The parameter $a_\lambda \in {}^LA$ is called the *Satake–Langlands parameter* of the principal series representation of $G(\mathbb{Q}_p)$ determined by the weight λ and we will come back to it in a slightly different guise in section 11.7. We also note that the element $a_\Lambda \in A(\mathbb{Q}_p)$ actually corresponds to an equivalence class $A(\mathbb{Q}_p)/A(\mathbb{Z}_p)$ due to sphericality (right $K(\mathbb{Q}_p)$ invariance) of the Whittaker function.

For the case of $GL(n,\mathbb{Q}_p)$ one has ${}^LG = GL(n,\mathbb{C})$. If one considers the case when Λ is the highest weight of the fundamental n-dimensional representation $\mathbb{C}^n$, then the character ch_Λ is given by the degree n Schur polynomial S_n

$$W^\circ(\lambda, a_\Lambda) = \frac{1}{\zeta(\lambda)}\delta^{1/2}(a_\Lambda)S_n(\alpha_1,\ldots,\alpha_n), \qquad G = GL(n,\mathbb{Q}_p). \tag{9.120}$$

Here, λ is thought of as the diagonal matrix $\lambda = \mathrm{diag}(\alpha_1,\ldots,\alpha_n)$. This formula for $GL(n)$ was first proven by Shintani in 1976 [298], and it was subsequently generalised by Casselman–Shalika in 1980 to (11.130) which holds for any G. Remarkably the general formula was in fact conjectured by Langlands already in 1967 in a letter to Godement [220], a fact that was apparently unknown to Casselman and Shalika at the time of their proof [63].

Example 9.11: $SL(2,\mathbb{Q}_p)$ spherical Whittaker vector and $SL(2,\mathbb{C})$ characters

For the case $SL(2,\mathbb{Q}_p)$ the spherical Whittaker vector for unramified ψ was given explicitly in (9.24) for $\lambda = (2s-1)\rho$ and general $a = v^{H_\alpha}$ as

$$W^\circ(\lambda, a) = \gamma_p(v^2)(1-p^{-2s})\frac{|v|^{-2s+2} - p^{-2s+1}|v|^{2s}}{1-p^{-2s+1}}. \tag{9.121}$$

In order to verify the expression (9.119) we need to evaluate them at the special values a_Λ where $\Lambda = NH_\alpha/2$ is a dominant integral weight of ${}^LSL(2,\mathbb{Q}_p) = SL(2,\mathbb{C})$ for $N \in \mathbb{Z}_{\geq 0}$. This means $v^2 = p^N$ and the Whittaker vector evaluates to

$$W^\circ(\lambda, a_\Lambda) = (1-p^{-2s})\frac{p^{\frac{N}{2}(2s-2)} - p^{-2s+1-Ns}}{1-p^{-2s+1}}. \tag{9.122}$$

The Whittaker vector vanishes if N is not in $\mathbb{Z}_{\geq 0}$ because of the factor $\gamma_p(v^2)$.

Let us now determine the right-hand side of (9.119). For $\Lambda = \frac{N}{2}H_\alpha$, the character of the $(N+1)$-dimensional highest weight representation of ${}^LSL(2,\mathbb{Q}_p) \cong PSL(2,\mathbb{C})$ is

$$\mathrm{ch}_\Lambda = e^{NH_\alpha/2} + e^{(N-2)H_\alpha/2} + \ldots + e^{-NH_\alpha/2} = \frac{e^{-NH_\alpha/2} - e^{(N+2)H_\alpha/2}}{1-e^{H_\alpha}}. \tag{9.123}$$

This has to be evaluated at $a_\lambda = p^\lambda = p^{(2s-1)\Lambda_\alpha}$ which leads to

$$\mathrm{ch}_\Lambda(a_\lambda) = \frac{p^{N(2s-1)/2} - p^{-(N+2)(2s-1)/2}}{1-p^{-2s+1}}, \tag{9.124}$$

where we recall that the p-adic characters are evaluated with the p-adic norm such that for instance $e^{H_\alpha}(a_\lambda) = |p^{2s-1}| = p^{-2s+1}$. For $v^2 = p^N$, the modulus character evaluates to $\delta^{1/2}(a_\Lambda) = |p^{N/2}| = p^{-N/2}$

and one also has $\frac{1}{\zeta(\lambda)} = 1 - p^{-2s}$ from (9.116). Putting everything together in (9.119) leads to

$$W^\circ(\lambda, a_\Lambda) = (1-p^{2s})p^{-N/2}\frac{p^{N(2s-1)/2} - p^{-(N+2)(2s-1)/2}}{1-p^{-2s+1}} = (1-p^{-2s})\frac{p^{N(2s-2)/2} - p^{-2s+1-sN}}{1-p^{-2s+1}} \quad (9.125)$$

which equals (9.122).

Remark 9.12. Using formula (9.57) we can also reinterpret (9.119) in terms of a Whittaker vector for the twisted character

$$\psi_\Lambda(n) := \psi(a_\Lambda n a_\Lambda^{-1}) \quad (9.126)$$

as

$$W^\circ_{\psi_\Lambda}(\lambda, \mathbb{1}) = a_\Lambda^{-w_{\text{long}}\lambda-\rho} W^\circ(\lambda, a_\Lambda) = \frac{1}{\zeta(\lambda)} a_\lambda^{-w_{\text{long}}\Lambda} \text{ch}_\Lambda(a_\lambda), \quad (9.127)$$

where we used (9.118).

Chapter 10

Working with Eisenstein series

After having developed the formal theory of Eisenstein series and their Fourier expansion in the previous chapters we would like to discuss Eisenstein series from a more practical point of view in this chapter. In concrete examples this typically means obtaining as much information as possible for a particular Eisenstein series $E(\chi, g)$, that is a particular given χ. Many of the general theorems either simplify for such a χ or have to be evaluated with much care as $E(\chi, g)$ might be divergent for the chosen χ. This chapter deals with developing methods for addressing these issues. In particular, we exhibit methods for efficiently evaluating the constant term formula (8.41) and formula (9.59) for the Whittaker vectors of a given Eisenstein series. We will also discuss the pole structure of Eisenstein series (as a function of χ) in examples, their residues as well as different normalisations. In this chapter, the emphasis is on illustrating different methods through many examples; for proofs of general statements we will typically refer to the appropriate literature.

Many of the properties of Eisenstein series are controlled by the completed Riemann zeta function whose properties we briefly recall.

Proposition 10.1 (Properties of completed Riemann zeta function). *As a function of $s \in \mathbb{C}$, the completed Riemann zeta function $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has simple poles at $s = 0$ and $s = 1$ with residues -1 and $+1$, respectively. It is non-zero everywhere else. It satisfies the functional relation $\xi(s) = \xi(1-s)$.*

Proof. The first statements follow directly from the definition and the properties of gamma and zeta functions. The functional relation was shown originally by Riemann using analytic continuation [275]. □

10.1 The $SL(2,\mathbb{R})$ Eisenstein series as a function of s

We begin with the $SL(2,\mathbb{R})$ Eisenstein series $E(s, z)$ that was analysed in great detail in chapter 7 with its complete Fourier expansion given in theorem 7.1. We repeat the result here for convenience:

$$E(s,z) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2}{\xi(2s)} y^{1/2} \sum_{m\neq 0} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}, \tag{10.1}$$

where $z = x + iy$ is an element of the upper half plane $\mathbb{H} = SL(2,\mathbb{R})/SO(2)$. The original definition of $E(s,z)$ only converged for $\mathrm{Re}(s) > 1$ but by virtue of the functional relation (cf. theorem 7.1)

$$E(s,z) = \frac{\xi(2s-1)}{\xi(2s)} E(1-s,z) \tag{10.2}$$

or through analytic continuation of the Fourier expansion (10.1) one can define $E(s,z)$ for almost all complex s. We restrict our discussion to real s for simplicity.

10.1.1 Limiting values in original normalisation

From the explicit form (10.1) one sees that special things might happen for the values $s = 0$, $s = \frac{1}{2}$ and $s = 1$. All of them are outside the original domain of convergence. Let us note that the region $0 \leq \mathrm{Re}(s) \leq 1$ is often called the critical strip.

- $s = 0$: This is the limit where the inducing character $\chi_s(z) = y^s$ becomes trivial. Taking the limit in the expression (10.1) for the Fourier expansion one also sees that all terms go to zero except for the first. This is due to the factors $\frac{1}{\xi(2s)}$ that vanish linearly for $s \to 0$ while everything else stays bounded. The proper limiting behavoiur is therefore

$$E(s,z) = 1 + O(s). \tag{10.3}$$

  The constant value 1 could have been expected from the triviality of the inducing character but the definition in terms of a *Poincaré sum* is ill-defined. Only after analytic continuation of the sum one obtains the constant $E(0,z) = 1$.

  Representation theoretically, the function $E(s,z)$ in the limit $s \to 0$ belongs to the trivial representation of $SL(2,\mathbb{R})$.

- $s = \frac{1}{2}$: Inspection of the Fourier expansion (10.1) shows that the non-zero Fourier modes disappear in this limit due to the $\frac{1}{\xi(2s)}$ prefactor. For the constant terms one has to take the limit of the quotient of ξ-functions which is found to be -1 and the two contributions to the constant term cancel, leading to

$$E(s,z) = 0 + O\left(s - \tfrac{1}{2}\right). \tag{10.4}$$

  The first order term is a member of the principle series. It is on the critical line and is therefore almost unitary.

- $s = 1$: This is the most interesting case. The Fourier expansion (10.1) shows that the second constant term diverges in the limit $s \to 1$ while all other terms remain finite. The residue at the simple pole can be calculated easily from the completed Riemann zeta functions:

$$E(s,z) = \frac{3}{\pi(s-1)} + O\left((s-1)^0\right). \tag{10.5}$$

The residue is therefore a constant function and is therefore also of the same type as the limit $s \to 0$ discussed above. This is not surprising since the functional relation (10.2) relates the values $s = 0$ and $s = 1$ and one sees that the prefactor introduces the additional pole. Representation theoretically, the residue of the series $E(s,z)$ at the simple pole $s = 1$ belongs to the trivial representation of $SL(2,\mathbb{R})$.

The term at order $(s-1)^0$ can also be evaluated from the Fourier expansion using the fact that the modified Bessel function $K_{1/2}$ has an exact asymptotic expansion in terms of a simple exponential. One finds

$$\begin{aligned} E(s,z) = \frac{3}{\pi(s-1)} - \frac{6}{\pi}\bigg(&-\frac{\pi}{6}y + \log(4\pi\sqrt{y}) - 12\log A \\ &- \sum_{m>0}\sigma_{-1}(m)e^{2\pi i m(x+iy)} - \sum_{m>0}\sigma_{-1}(m)e^{2\pi i m(x-iy)}\bigg) + O(s-1). \end{aligned} \tag{10.6}$$

Here, A is the *Glaisher–Kinkelin constant* that satisfies $\log A = \frac{1}{12} - \zeta'(1)$. The expression can be rewritten by using the *Dedekind η function*

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{10.7}$$

where $q = e^{2\pi i z} = e^{2\pi i(x+iy)}$ on the right-hand side. From the product formula for $\eta(z)$ one concludes

$$\begin{aligned} \log\eta(z) &= \frac{1}{24}\log q + \sum_{n>0}\log(1-q^n) = \frac{\pi i}{12}(x+iy) - \sum_{n>0}\sum_{k>0}k^{-1}q^{kn} \\ &= \frac{\pi i}{12}(x+iy) - \sum_{m>0}\sum_{d|m}d^{-1}q^m = \frac{\pi i}{12}(x+iy) - \sum_{m>0}\sigma_{-1}(m)q^m. \end{aligned} \tag{10.8}$$

The s-independent term in (10.6) can therefore be written as

$$\begin{aligned} &-\frac{6}{\pi}\left(-\frac{\pi}{6}y + \log(4\pi\sqrt{y}) - 12\log A - \sum_{m>0}\sigma_{-1}(m)q^m - \sum_{m>0}\sigma_{-1}(m)\bar{q}^m\right) \\ &= -\frac{6}{\pi}\Big(-12\log A + \log(4\pi) + \log\left(\sqrt{y}|\eta(z)|^2\right)\Big), \end{aligned} \tag{10.9}$$

leading to

$$E(s,z) = \frac{3}{\pi(s-1)} + \frac{6}{\pi}\Big(12\log A - \log(4\pi) - \log\left(\sqrt{y}|\eta(z)|^2\right)\Big) + O(s-1). \tag{10.10}$$

This formula is known as the (first) *Kronecker limit formula*. Even though neither $\eta(z)$ nor $|\eta(z)|^2$ are $SL(2,\mathbb{Z})$ invariant, the particular combination appearing in this expression is invariant.

Remark 10.2. The particular combination of constants in the Kronecker limit formula (10.10) depends on the way the Eisenstein series is normalised. The formula is more commonly stated for the $SL(2,\mathbb{Z})$ invariant lattice sum (cf. (1.1)) for which one finds

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}} = 2\zeta(2s)E(s,z) \tag{10.11}$$

$$= \frac{\pi}{s-1} + 2\pi\Big(\gamma_{\rm E} - \log(2) - \log\left(\sqrt{y}|\eta(z)|^2\right)\Big) + O(s-1) \tag{10.12}$$

if one uses the following relation between the Glaisher–Kinkelin constant A and the *Euler–Mascheroni constant* $\gamma_{\rm E}$: $12\log A - \log(4\pi) = \gamma_{\rm E} - \log 2 - \frac{\zeta'(2)}{\zeta(2)}$.

10.1.2 Weyl symmetric normalisation

The functional relation (10.2) suggests to define a *completed Eisenstein series* in analogy with the completed Riemann zeta function by the definition

$$E^\star(s,z) = \xi(2s)E(s,z). \tag{10.13}$$

This then has the simple property that

$$E^\star(s,z) = E^\star(1-s,z) \tag{10.14}$$

and we call this the *Weyl symmetric normalisation* as it yields a function invariant under Weyl transformations acting on the character. Indeed, the non-trivial Weyl reflection w of $SL(2,\mathbb{R})$ acts on the weight $\lambda_s = (2s-1)\rho$ by

$$w\lambda_s = -(2s-1)\rho = (2(1-s)-1)\rho = \lambda_{1-s} \tag{10.15}$$

and so exchanges s and $1-s$. This was of course already used and apparent in the constant terms in (10.1).

Since the normalising factor has poles and zeroes of its own, the discussion of the behaviour of $E^\star(s,z)$ as a function of s is slightly changed from the one above. More precisely, the completed function $E^\star(s,z)$ has a simple poles at $s=0$ and $s=1$, whereas it has a non-trivial limit for $s=\frac{1}{2}$. Representation theoretically, $E^\star(\frac{1}{2},z)$ belongs to the principal series.

10.2 Properties of Eisenstein series

The behaviour of the $SL(2,\mathbb{R})$ Eisenstein series at the special values of s above was completely controlled by the constant terms. This is a general feature due to the holomorphy of the Fourier coefficients, see proposition 9.2. As we have full control of the constant terms thanks to the Langlands constant term formula (theorem 8.1), we can in principle completely determine the behaviour of an Eisenstein series $E(\lambda, g)$ on a group $G(\mathbb{R})$ as a function of λ. As the number of constant terms is generically equal to the order of the Weyl group $\mathcal{W}$ of G this can be quite tedious due to the large number of terms that have to be considered. In section 10.3, we will present a method that makes the problem more tractable for the case of non-generic λ when the Eisenstein series $E(\lambda, g)$ is not attached to the full principal series but to a degenerate principal series. The prime example of this is when it becomes a maximal parabolic Eisenstein series as defined in section 5.6. Before focussing on these cases in section 10.3, we offer a few general and cautionary remarks.

10.2.1 Validity of functional relation

As Langlands showed in his seminal work [218], the functional equation (8.44) repeated here for convenience

$$E(\lambda, g) = M(w, \lambda) E(w\lambda, g) \tag{10.16}$$

is valid for almost all $\lambda \in \mathfrak{h}^*(\mathbb{C})$. The exceptions are affine hyperplanes in the complex vector space $\mathfrak{h}^*(\mathbb{C})$. These affine hyperplanes are associated with poles and zeroes of the intertwining factor $M(w, \lambda)$. Since all poles and zeroes are of finite order, one can make sense of the functional relation even on these planes by treating also the Eisenstein series as meromorphic functions with finite order poles and singularities.

Example 10.3: Functional relation for $SL(2,\mathbb{R})$ Eisenstein series at a simple zero and pole

For $SL(2,\mathbb{R})$ and general $\lambda = (2s-1)\rho$, the intertwining factor for the non-trivial Weyl element $w = w_{\text{long}}$ is

$$M(w_{\text{long}}, \lambda) = \frac{\xi(2s-1)}{\xi(2s)} \tag{10.17}$$

and has a simple zero at $s = 0$ and a simple pole at $s = 1$. The functional relation (10.16) remains valid even at these places if one considers

$$\begin{aligned} E(s,z) &= 1 + s\hat{E}_0(z) + O\left(s^2\right), \\ E(s,z) &= \frac{3}{\pi(s-1)} + \hat{E}_1(z) + O\left(s-1\right) \end{aligned} \tag{10.18}$$

around $s = 0$ and $s = 1$, respectively. The expansion of the intertwining factor around these values is

$$\frac{\xi(2s-1)}{\xi(2s)} = -\frac{\pi s}{3} + O(s^2) = \frac{3}{\pi(s-1)} + O\left((s-1)^0\right), \tag{10.19}$$

such that

$$E(s,z) = 1 + s\hat{E}_0(z) + O(s^2) = \left(-\frac{\pi s}{3} + O(s^2)\right)\left(-\frac{3}{\pi s} + \hat{E}_1 + O(s)\right)$$
$$= 1 - \frac{\pi s}{3}\hat{E}_1 + O(s^2) \tag{10.20}$$

and so the functional relation relates $\hat{E}_0(z)$ and $\hat{E}_1(z)$ (as well as all higher order terms).

Of interest are also fixed planes of the action of the Weyl group action. In these cases, the functional relation (10.16) constrains the Eisenstein series on the fixed plane.

Example 10.4: Functional relation for $SL(2,\mathbb{R})$ Eisenstein series with $\lambda = 0$

For $SL(2,\mathbb{R})$ and $E(s,z)$ the fixed plane is $s = \frac{1}{2}$, corresponding to $\lambda = \lambda_{1/2} = 0$. The intertwining factor at this place takes the value $M(w,0) = -1$ such that the functional relation implies

$$E(\tfrac{1}{2},z) = -E(\tfrac{1}{2},z) \quad \Longrightarrow \quad E(\tfrac{1}{2},z) = 0, \tag{10.21}$$

consistent with the analysis in section 10.1.1.

We consider also a few examples of functional relations for higher rank groups.

Example 10.5: Functional relation for $SL(3,\mathbb{R})$ Eisenstein series

The most general Eisenstein series on $G = SL(3,\mathbb{R})$ is given by a weight

$$\lambda_{s_1,s_2} = 2s_1\Lambda_1 + 2s_2\Lambda_2 - \rho \tag{10.22}$$

that is parametrised by two complex parameters s_1 and s_2. The Λ_i are as always the fundamental weights. We denote the corresponding character by $\chi_{s_1,s_2}(a) = a^{\lambda_{s_1,s_2}+\rho}$ and the Eisenstein series by

$$E(s_1,s_2,g) = \sum_{B(\mathbb{Z})\backslash SL(3,\mathbb{Z})} \chi_{s_1,s_2}(\gamma g). \tag{10.23}$$

The sum is absolutely convergent for $\mathrm{Re}(s_1) > 1$ and $\mathrm{Re}(s_2) > 1$ [60]. The Weyl group of $SL(3,\mathbb{R})$ is isomorphic to the symmetric group on three letters and hence consists of six elements and the constant terms were already given in (9.81). Denoting the fundamental reflections by w_1 and w_2 one finds that

$$w_1\lambda_{s_1,s_2} = (1-2s_1)\Lambda_1 + 2(s_1+s_2-1)\Lambda_2 = \lambda_{1-s_1,s_1+s_2-\frac{1}{2}}, \tag{10.24a}$$
$$w_2\lambda_{s_1,s_2} = \lambda_{s_1+s_2-\frac{1}{2},1-s_2}. \tag{10.24b}$$

The other Weyl images can be obtained similarly. One functional relation is therefore

$$E(s_1,s_2,g) = M(w_1,\lambda_{s_1,s_2})E(1-s_1,s_1+s_2-\tfrac{1}{2},g) = \frac{\xi(\langle\alpha_1|\lambda_{s_1,s_2}\rangle)}{\xi(\langle\alpha_1|\lambda_{s_1,s_2}\rangle+1)}E(1-s_1,s_1+s_2-\tfrac{1}{2},g)$$
$$= \frac{\xi(2s_1-1)}{\xi(2s_1)}E(1-s_1,s_1+s_2-\tfrac{1}{2},g). \tag{10.25}$$

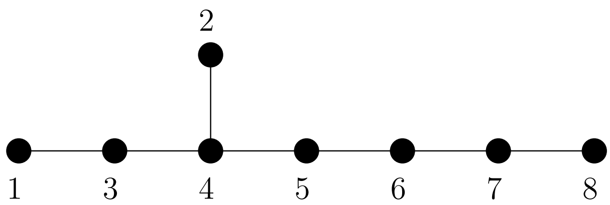


Figure 10.1: *The Dynkin diagram of E_8 with labelling of nodes in the 'Bourbaki convention'.*

That this is a valid relation can be checked on the constant terms from (9.81). We can consider the limit $s_1 \to \frac{1}{2}$ to conclude

$$E(\tfrac{1}{2}, s_2, g) = -E(\tfrac{1}{2}, s_2, g) \quad \Longrightarrow \quad E(\tfrac{1}{2}, s_2, g) = 0. \tag{10.26}$$

This is exactly as in the $SL(2,\mathbb{R})$ case in example 10.4 above. Again $s_1 = \frac{1}{2}$ corresponds to a fixed plane of a fundamental reflection and in this case one always obtains a vanishing Eisenstein series.

More involved examples are obtained for exceptional groups. These will play an important role in section 12.1 in the context of string theory.

Example 10.6: Functional relation for $E_8(\mathbb{R})$ maximal parabolic Eisenstein series

Consider a (maximal parabolic) Eisenstein series on E_8 with Dynkin diagram given in figure 10.1. For the weight

$$\lambda_s = 2s\Lambda_8 - \rho \tag{10.27}$$

the associated character $\chi_s(a) = a^{\lambda_s+\rho} = a^{2s\Lambda_8}$ is invariant under the maximal parabolic subgroup with semi-simple part E_7. (Here, Λ_8 denotes as always the fundamental weight associated with node 8.) We therefore have a family of maximal parabolic Eisenstein series

$$E(s,g) \equiv E(\lambda_s, P, g) = \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_s(\gamma g). \tag{10.28}$$

There are many functionally related Eisenstein series. The Weyl group of E_8 has order $|\mathcal{W}(E_8)| = 696\,729\,600$, but for the particular choice of parameter λ_s in (10.27) not all Weyl images of λ_s give different Eisenstein series. Instead it suffices to consider elements of the coset $\mathcal{W}(E_8)/\mathcal{W}(E_7)$ and representatives that do not end on an element of $\mathcal{W}(E_7)$. (This will be the main theme of section 10.3.) We consider as an example the Weyl word

$$w = w_1 w_3 w_4 w_5 w_6 w_7 w_8. \tag{10.29}$$

Then

$$w(2s\Lambda_8 - \rho) = (8-2s)\Lambda_1 + (2s-5)\Lambda_2 - \rho. \tag{10.30}$$

The intertwining factor is

$$M(w, 2s\Lambda_8 - \rho) = \frac{\xi(2s-7)}{\xi(2s)} \tag{10.31}$$

and therefore

$$E(2s\Lambda_8 - \rho, g) = \frac{\xi(2s-7)}{\xi(2s)} E((8-2s)\Lambda_1 + (2s-5)\Lambda_2 - \rho, g). \tag{10.32}$$

At $s = \frac{5}{2}$ this specialises to

$$E(5\Lambda_8 - \rho, g) = \frac{\xi(3)}{\xi(5)} E(3\Lambda_1 - \rho, g) \tag{10.33}$$

and therefore relates a specific maximal parabolic Eisenstein series 'on node 8' to another specific maximal parabolic Eisenstein series, this time 'on node 1'. The former appears in the discussion of minimal theta series for E_8 [126] while the latter version appears commonly in string theory, see chapter 2 and section 12.1, and both are related to the minimal unitary representation as we will discuss more in sections 10.3.2 and 12.1.1 below.

10.2.2 Weyl symmetric normalisation

For a general (minimal) Eisenstein series $E(\lambda, g)$ on a split real simple group G one can define a completed version according to

$$E^\star(\lambda, g) = \underbrace{\left[\prod_{\alpha>0} \xi(\langle\lambda|\alpha\rangle + 1)\right]}_{N_\lambda} E(\lambda, g). \tag{10.34}$$

The *normalising factor* N_λ is the denominator in $M(w_{\text{long}}, \lambda)$. This function is completely invariant under the action of the Weyl group $\mathcal{W}$: For any $w \in \mathcal{W}$ one has

$$E^\star(w\lambda, g) = E^\star(\lambda, g). \tag{10.35}$$

To see this it is sufficient to consider the action of a fundamental reflection $w_i \in \mathcal{W}$:

$$E^\star(w_i\lambda, g) = N_{w_i\lambda} E(w_i\lambda, g) = N_{w_i\lambda} M(w_i, \lambda)^{-1} E(\lambda, g) = N_\lambda E(\lambda, g) = E^\star(\lambda, g). \tag{10.36}$$

The prefactor works as follows

$$\begin{aligned} N_{w_i\lambda} M(w_i, \lambda)^{-1} &= \frac{\xi(\langle\lambda|\alpha_i\rangle + 1)}{\xi(\langle\lambda|\alpha_i\rangle)} \prod_{\alpha>0} \xi(\langle\lambda|w_i\alpha\rangle + 1) \\ &= \frac{\xi(\langle\lambda|\alpha_i\rangle + 1)}{\xi(\langle\lambda|\alpha_i\rangle)} \xi(-\langle\lambda|\alpha_i\rangle + 1) \prod_{\substack{\alpha>0 \\ \alpha\neq\alpha_i}} \xi(\langle\lambda|\alpha\rangle + 1) \\ &= \prod_{\alpha>0} \xi(\langle\lambda|\alpha\rangle + 1) = N_\lambda, \end{aligned} \tag{10.37}$$

where we have used the fact that w_i permutes the set of positive roots $\Delta_+ \setminus \{\alpha_i\}$ as well as the functional relation of the completed Riemann zeta function.

The normalising factor for minimal Eisenstein series has as many factors as the order of $\mathcal{W}$. When the character defined by a weight λ has a stabiliser that is larger than the

Borel subgroup B, it is sufficient to use a normalising factor with fewer factors to obtain a suitably symmetric combination. Consider the case of a non-minimal parabolic Eisenstein series $E(\lambda, g)$ where the stabiliser is given by a parabolic subgroup $P(\mathbb{Z})$, see section 5.6. Then the semi-simple part M of the Levi subgroup L of $P = LU$ has a Weyl group $\mathcal{W}(M)$. The normalising factor $N_{P,\lambda}$ in this case can be chosen to be the denominator (after cancelling all factors) of $M(w, \lambda)$ where w is defined by $w_{\text{long}}(G) = ww_{\text{long}}(M)$ through the longest words in $\mathcal{W}(G)$ and $\mathcal{W}(M)$. An alternative definition of w is as the longest Weyl word in the $\mathcal{W}(G)$ orbit of $\lambda + \rho$. The *normalised Eisenstein series*

$$E^\star(\lambda, P, g) = N_{P,\lambda} E(\lambda, P, g). \tag{10.38}$$

This completed function then is either invariant under a reflection group isomorphic to the Weyl group generated by the simple reflections in $\mathcal{W}(G)$ that do not belong to $\mathcal{W}(M)$, or it maps to a similar one that is obtained by intertwining additionally by an (outer) Dynkin diagram automorphism. Furthermore, the Weyl normalised series has a different pole structure compared to the Eisenstein series $E(\lambda, P, g)$ with standard normalisation as the normalising factor has zeroes and poles.

In the case of maximal parabolic Eisenstein series with weight

$$\lambda = 2s\Lambda_{i_*} - \rho \tag{10.39}$$

this means a reflection symmetry $s \leftrightarrow \frac{\langle \rho | \Lambda_{i_*} \rangle}{\langle \Lambda_{i_*} | \Lambda_{i_*} \rangle} - s$. We illustrate this by two examples.

Example 10.7: Weyl normalisation of $SL(3, \mathbb{R})$ maximal parabolic Eisenstein series

The first example contains a non-trivial diagram automorphism. Consider the group $G = SL(3, \mathbb{R})$ and the weight

$$\lambda_s = 2s\Lambda_1 - \rho. \tag{10.40}$$

The associated character $\chi_s(a) = a^{\lambda_s + \rho} = a^{2s\Lambda_1}$ is invariant under a maximal parabolic subgroup P with Levi factor $L = GL(1, \mathbb{R}) \times M$ with $M = SL(2, \mathbb{R})$. Denote the associated maximal parabolic Eisenstein series by

$$E(s, g) \equiv E(\lambda_s, P, g) = \sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi_s(\gamma g). \tag{10.41}$$

The Weyl word w that enters in the definition of the normalising factor N_{P,λ_s} is given by the relation

$$\underbrace{w_2 w_1 w_2}_{w_{\text{long}}(G)} = \underbrace{w_2 w_1}_{w} \underbrace{w_2}_{w_{\text{long}}(M)} \tag{10.42}$$

such that

$$M(w, \lambda_s) = \frac{\xi(2s-2)\xi(2s-1)}{\xi(2s-1)\xi(2s)} = \frac{\xi(2s-2)}{\xi(2s)} \quad \Longrightarrow N_{P,\lambda_s} = \xi(2s). \tag{10.43}$$

The normalised series then has a reflection symmetry $s \leftrightarrow \frac{3}{2} - s$ but it maps to the maximal parabolic Eisenstein series associated with the parabolic subgroup P' obtained by the diagram automorphism. In other words, the characters that are being related are

$$2s\Lambda_1 - 1 \quad \leftrightarrow 2\left(\tfrac{3}{2} - s\right)\Lambda_2 - \rho. \tag{10.44}$$

The second example does not have any non-trivial automorphisms.

Example 10.8: Weyl normalisation of $E_8(\mathbb{R})$ maximal parabolic Eisenstein series

We consider the E_8 Eisenstein series from example 10.6 with weight

$$\lambda_s = 2s\Lambda_8 - \rho. \tag{10.45}$$

The parabolic subgroup leaving the associated character invariant has semi-simple part E_7. The normalising factor in this case is associated with a Weyl word w of length $\ell(w) = 57$ that we do not spell out. The normalising factor turns out to be

$$N_{P,\lambda_s} = \xi(2s)\xi(2s-5)\xi(2s-9)\xi(4s-28) \tag{10.46}$$

and the thus normalised Eisenstein series $E^\star(s,g) = N_{P,\lambda}E(s,g)$ is invariant under the reflection w_8 that leads to the reflection law

$$E^\star(s,g) = E^\star\left(\tfrac{29}{2} - s, g\right). \tag{10.47}$$

This example is also discussed in [126] and we will say more about below in example 10.19. Here, we note that the normalising factor (10.46) has introduced a pole at $s = \frac{5}{2}$ (and also for other values). This means that the special Eisenstein series from example 10.6 now appears as a *residue of an Eisenstein series*.

10.2.3 Square-integrability of Eisenstein series

Langlands provided a criterion for Eisenstein series to be *square integrable* [218, §5]. We state the criterion for Eisenstein series $E(\lambda, g)$ such that they are finite at λ, meaning that they do not have a zero or pole as a meromorphic function of λ for the λ chosen.

Remark 10.9. If $E(\lambda, g)$ has a pole or zero at a given λ one has to consider a one-parameter family in the neighbourhood of λ and study a suitably normalised version such that the zeroth order term becomes finite [233]. In the case of $SL(2,\mathbb{R})$ and $E(s,z)$ this means multiplying by $(s-1)$ if one wants to study the square integrability of $E(s,z)$ at $s = 1$. Similarly, one would have to multiply by s^{-1} for the $s = 0$ case.

Under the assumption of a finite $E(\lambda, g)$, the constant term formula of theorem 8.1 implies that

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\lambda, ng)dn = \sum_{w\in\mathcal{W}} M(w,\lambda)a^{w\lambda+\rho} \tag{10.48}$$

is well-defined and non-vanishing function of a.

Proposition 10.10 (Square integrability of Eisenstein series [218]). *A finite Eisenstein series $E(\lambda, g)$ in the sense just described is square integrable if and only if*

$$\mathrm{Re}\langle w\lambda|\Lambda_i\rangle < 0 \qquad \textit{for all } i = 1,\ldots,\mathrm{rank}(G) \tag{10.49}$$

for all $w \in \mathcal{W}$ such that $M(w,\lambda) \neq 0$.

The intuition behind this proposition is that the condition ensures that all terms fall off fast enough as one approaches any cusp of $G(\mathbb{Z})\backslash G(\mathbb{R})$. A proof can be found in [218] and we content ourselves here with some examples.

Example 10.11: Non square integrability of $SL(2,\mathbb{R})$ Eisenstein series

Consider square integrability of Eisenstein series on $SL(2,\mathbb{R})$. For $\lambda_s = (2s-1)\rho$ one has to check the condition (10.49) for the Weyl words $w = \mathbb{1}$ and $w = w_{\text{long}}$. Plugging in the explicit expressions leads to

$$\operatorname{Re}\langle\lambda_s|\Lambda_1\rangle = \operatorname{Re}\frac{1}{2}(2s-1) < 0 \quad \text{and} \quad \operatorname{Re}\langle w_{\text{long}}\lambda_s|\Lambda_1\rangle = -\operatorname{Re}\frac{1}{2}(2s-1) < 0. \tag{10.50}$$

Clearly, these two conditions cannot be satisfied simultaneously and therefore we recover the well-known result that non-holomorphic Eisenstein series on $SL(2,\mathbb{R})$ are never square integrable. There is, however, a limiting case $\operatorname{Re} s = \frac{1}{2}$, where the conditions are almost satisfied. This corresponds to the Eisenstein series on the critical line $s = \frac{1}{2} + it$ (for $t \in \mathbb{R}$) that are δ-function normalisable. See for instance [117,308] for a discussion of these properties of Eisenstein series on $SL(2,\mathbb{R})$.

More interesting is the case when there are non-trivial square integrable functions within a degenerate principal series.

Example 10.12: Square integrability of E_8 maximal parabolic Eisenstein series for special λ

Consider the maximal parabolic E_8 Eisenstein series $E(s,g)$ with weight

$$\lambda = 2s\Lambda_8 - \rho \tag{10.51}$$

that was introduced in example 10.6. Computing the constant term one finds 240 non-vanishing $M(w,\lambda)$. Checking the criterion (10.49) one finds that it is satisfied for the values

$$s = \frac{5}{2}, \quad s = \frac{9}{2}, \quad s = 7 \tag{10.52}$$

and hence these are normalisable Eisenstein series for E_8 that belong to the discrete spectrum of the Laplacian on $G(\mathbb{R})/K(\mathbb{R})$ for $G = E_8$.

The value $s = \frac{5}{2}$ was also discussed in example 10.6 and it was shown there that $E(s,g)$ for this value is functionally related to another known normalisable maximal parabolic Eisenstein series [146] that appears in string theory for the R^4 correction, see also section 12.1.

The value $s = \frac{9}{2}$ can be analysed using the functional relation

$$E(2s\Lambda_8 - \rho) = \frac{\xi(2s-10)\xi(2s-13)}{\xi(2s)\xi(2s-5)} E(2(7-s)\Lambda_1 + 2(s-9/2)\Lambda_2 - \rho) \tag{10.53}$$

that shows that for $s = \frac{9}{2}$, the adjoint E_8 series is connected to the maximal parabolic series on node 1 with $s = \frac{5}{2}$. This is the case that appears in string theory for the D^4R^4 correction and it is known that the function is associated with the next-to-minimal series [146].

The value $s = 7$ is interesting because it does not represent any simplification in the wavefront set (compared to generic s) and so is just part of the residual discrete spectrum with orbit type A_2. These cases were also analysed in [233].

10.3 Evaluating constant term formulas

Langlands' constant term formula (cf. theorem 8.1)

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} E(\lambda, ng)dn = \sum_{w\in\mathcal{W}} M(w,\lambda)a^{w\lambda+\rho} \tag{10.54}$$

is nice and compact but evaluating it will a priori produce as many terms as there are different elements in the Weyl group. Since the order of the Weyl group becomes large very quickly as the rank of $G(\mathbb{R})$ grows this can render the resulting expressions rather unwieldy. However, by dint of choice of the parameter λ of the (degenerate) principal series the sum over Weyl elements may simplify as then some of the coefficients $M(w,\lambda)$ appearing in (10.54) vanish. For convenience we also recall that the definition of the intertwiner

$$M(w,\lambda) = \prod_{\substack{\alpha>0\\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(\langle\lambda|\alpha\rangle+1)} \tag{10.55}$$

and its multiplicative property

$$M(w_1w_2,\lambda) = M(w_1,w_2\lambda)M(w_2,\lambda) \quad \text{for any } w_1,w_2\in\mathcal{W}. \tag{10.56}$$

A convenient method for evaluating the Langlands constant term formula can then be developed by exploiting the multiplicative relation (10.56). We first adumbrate this method that we will refer to as the *orbit method*. Then we discuss a number of examples and finally mention further simplifications that arise for constant terms in non-maximal unipotent subgroups $U\subset N$. The corresponding constant term formula was given in section 8.9.

10.3.1 The orbit method

The factor $M(w,\lambda)$ is, by its definition in (10.55), given by the product of factors of the form

$$c(k) = \frac{\xi(k)}{\xi(k+1)}, \tag{10.57}$$

where $k=\langle\lambda|\alpha\rangle$ and α runs over all positive roots that satisfy $w\alpha<0$. The function $c(k)$ is sometimes referred to as the *Harish-Chandra c-function*. It has a simple pole at $k=1$ and a simple zero at $k=-1$; otherwise it takes finite non-zero values for real k and satisfies $c(k)c(-k)=1$ as well as $c(0)=-1$. For vanishing $M(w,\lambda)$, we are therefore particularly interested in roots α which satisfy $w\alpha<0$ and $\langle\lambda|\alpha\rangle=-1$.

To characterize these α further, let us define the *stabiliser of the weight* λ

$$\text{stab}(\lambda) = \{\alpha\in\Pi \,|\, \langle\lambda+\rho|\alpha\rangle = 0\}, \tag{10.58}$$

so that it is the subset of the simple roots Π for which $\lambda+\rho$ has vanishing Dynkin labels.

Remark 10.13. If $\mathrm{stab}(\lambda) \neq \{\}$, the corresponding Eisenstein series $E(\lambda, g)$ belongs to a *degenerate* principal series. Let $P \subset G$ be the parabolic subgroup corresponding to $\mathrm{stab}(\lambda) \subset \Pi$ as defined in section 4.1.3. Then

$$E(\lambda, g) = E(\lambda, P, g) = \sum_{\gamma \in P(\mathbb{Z})\backslash G(\mathbb{Z})} e^{\langle \lambda + \rho_P | H_P(\gamma g) \rangle} \tag{10.59}$$

as explained in section 5.6.2.

Example 10.14: Stabiliser of a maximal parabolic λ

As an example and referring back to section 5.6 we note that maximal parabolic Eisenstein series have very large stabilisers, corresponding to $\mathrm{stab}(\lambda) = \Pi \setminus \{\alpha_{i_*}\}$ for the value i_* that determines the maximal parabolic subgroup under which $\chi(a) = a^{\lambda+\rho}$ is left-invariant.

If w_i is the fundamental Weyl reflection in the simple root α_i defined in (4.14), it clearly maps $w_i(\alpha_i) = -\alpha_i$ and this is the only positive root that is mapped to a negative root by the fundamental reflection w_i [186]. If furthermore $\alpha_i \in \mathrm{stab}(\lambda)$, then

$$\alpha_i \in \mathrm{stab}(\lambda) \;\Leftrightarrow\; \langle \lambda | \alpha_i \rangle = -1 \quad \Rightarrow \quad M(w_i, \lambda) = c(-1) = 0. \tag{10.60}$$

By the multiplicative property (10.56) one can then deduce that all Weyl words w that end (on the right) on a fundamental reflection w_i with α_i in $\mathrm{stab}(\lambda)$ obey $M(w, \lambda) = 0$, see also [146]. Another way of putting this is that only those w can have non-vanishing $M(w, \lambda)$ that lie in

$$\mathcal{C}(\lambda) = \{ w \in \mathcal{W} \,|\, w\alpha > 0 \quad \text{for all } \alpha \in \mathrm{stab}(\lambda) \}\,. \tag{10.61}$$

Depending on $\mathrm{stab}(\lambda)$ this set can be much smaller than $\mathcal{W}$. In fact, its order is given by $|\mathcal{C}(\lambda)| = |\mathcal{W}|/|\mathcal{W}(\mathrm{stab}(\lambda))|$, where $\mathcal{W}(\mathrm{stab}(\lambda))$ is the subgroup of $\mathcal{W}$ that is generated by taking only words in the fundamental reflections associated with $\mathrm{stab}(\lambda) \subset \Pi$. Moreover, one then has the simplified constant term formula

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(\chi, ng) dn = \sum_{w \in \mathcal{C}(\lambda)} a^{w\lambda+\rho} M(w, \lambda). \tag{10.62}$$

The elements in $\mathcal{C}(\lambda)$ can be constructed using the Weyl orbit of a dominant weight Λ that is defined as follows:

Definition 10.15. Let $\lambda \in \mathfrak{h}^*$ be a weight with stabiliser $\mathrm{stab}(\lambda)$ as defined in (10.58) and $r = \dim \mathfrak{h}^*$ denote the rank of the underlying group. Let $I \subset \{1, \ldots, r\}$ be such that a simple root α_i of $\mathfrak{g}$ belongs to $\mathrm{stab}(\lambda)$ if and only if $i \in I$. Let $\bar{I}$ be the complement of I in $\{1, \ldots, r\}$. Then the dominant weight Λ associated to λ is defined as a sum over fundamental weights as

$$\Lambda = \sum_{i \in \bar{I}} \Lambda_i. \tag{10.63}$$

In other words, one considers the $\lambda + \rho$ and replaces all non-zero Dynkin labels by 1 to obtain Λ.

Clearly, $\mathcal{W}(\text{stab}(\lambda))$ stabilises Λ thus defined and the number of distinct points in the orbit $\mathcal{W} \cdot \Lambda$ equals $|\mathcal{W}|/|\mathcal{W}(\text{stab}(\lambda))|$. Therefore the points in the Weyl orbit are in bijection with the set $\mathcal{C}(\lambda)$.

In order to establish the bijection, we use the fact that for each element μ in the Weyl orbit of Λ there is a shortest element $w \in \mathcal{W}$ that satisfies $w\Lambda = \mu$. These elements w are exactly the Weyl words that make up the set $\mathcal{C}(\lambda)$. They can also be seen as specific representatives of the coset $\mathcal{W}/\mathcal{W}(\text{stab}(\lambda))$ whose size was already argued above to determine the number of summands in (10.62). The shortest element leading to an element μ is not necessarily unique but all choices of the same shortest length yield the same factor $M(w, \lambda)$.

A standard algorithm for constructing the Weyl orbit $\mathcal{W} \cdot \Lambda$ of a dominant weight Λ is as follows:

1. Define the initial set of orbit points as $\mathcal{O} = \{\Lambda\}$. This is the 'highest' element (with respect to the *height function* $\text{ht}(\mu) = \langle \rho | \mu \rangle$ on $\mathfrak{h}^*$) in the orbit and others will be constructed by using lowering Weyl reflections.

2. For a given $\mu \in \mathcal{O}$ compute the Dynkin labels $p_i = \langle \mu | \alpha_i \rangle$ with respect to all simple roots α_i.

3. If $p_i > 0$ for some $i = 1, \ldots, \text{rank}(G)$, then construct $\mu' = w_i \mu$ where w_i is the fundamental Weyl reflection in the simple root α_i. If μ' is not already in the orbit $\mathcal{O}$, add it.

4. For any weight μ in $\mathcal{O}$ for which steps 2 and 3 have not been carried out go to step 2.

Remark 10.16. As for the initial dominant weight Λ the Dynkin labels p_i are zero for all simple roots $\alpha_i \in \text{stab}(\lambda)$, any Weyl word thus constructed will end on a letter (fundamental Weyl reflection) w_i that does not belong to $\mathcal{W}(\text{stab}(\lambda))$.

Remark 10.17. In practice, it is very advisable to think of the Weyl orbit of Λ in terms of a graph where nodes correspond to weights μ that lie in the orbit $\mathcal{O}$ and links are labelled by the fundamental reflections that relate two such weights. This graph can be constructed algorithmically starting from Λ which corresponds to the identity element $\mathbb{1} \in \mathcal{W}$ by the above algorithm and one also keeps track of the corresponding Weyl words in this way. Elements μ that are farther from the dominant weight in this graph correspond to longer Weyl words.

With the Weyl orbit $\mathcal{W} \cdot \Lambda$ one has constructed all Weyl words that belong to $\mathcal{C}(\lambda)$ and can therefore evaluate the constant term formula (10.62). We consider an example to illustrate the method.

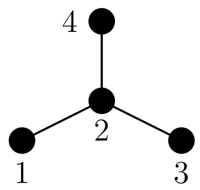


Figure 10.2: *The Dynkin diagram of $SO(4,4)$ with labelling of simple roots.*

Example 10.18: Orbit method for $SO(4,4;\mathbb{R})$ maximal parabolic Eisenstein series

We consider the group $SO(4,4;\mathbb{R})$ with Dynkin diagram of type D_4 shown in figure 10.2. The Weyl group $\mathcal{W}$ is of order 192 in this case. Taking

$$\lambda = 2s\Lambda_1 - \rho \tag{10.64}$$

yields a maximal parabolic Eisenstein series and

$$\text{stab}(\lambda) = \{\alpha_2, \alpha_2, \alpha_3\}\,, \tag{10.65}$$

such that $\mathcal{W}(\text{stab}(\lambda))$ is of type $\mathcal{W}(A_3)$ and order 24. The dominant weight Λ of (10.63) equals the sum of all fundamental weights Λ_i such that $\alpha_i \notin \text{stab}(\lambda)$ and thus $\Lambda = \Lambda_1$.

The Weyl orbit of $\Lambda = \Lambda_1$ consists of only eight points. Figure 10.3 shows the graph of this Weyl orbit.

The Weyl orbit can be calculated by starting from the highest weight Λ_1 and applying fundamental Weyl reflections in those simple roots whose Dynkin labels are positive. This is the implementation of the algorithm above. For the example shown in figure 10.3 this allows only w_1 acting on Λ_1.

Considering now the weight $\lambda = 2s\Lambda_1 - \rho$ that defines the $SO(4,4)$ Eisenstein series $E(\lambda, g)$, we see that the eight Weyl elements potentially contributing to the constant term formula (10.62) are

$$\mathcal{C}(\lambda) = \{\mathbb{1}, w_1, w_2w_1, w_3w_2w_1, w_4w_2w_1, w_4w_3w_2w_1, w_2w_4w_3w_2w_1, w_1w_2w_4w_3w_2w_1\}\,. \tag{10.66}$$

The corresponding factors $M(w,\lambda)$ are:

w	$M(w,\lambda)$
$\mathbb{1}$	1
w_1	$c(2s-1)$
w_2w_1	$c(2s-1)c(2s-2)$
$w_3w_2w_1$	$c(2s-1)c(2s-2)c(2s-3)$
$w_4w_2w_1$	$c(2s-1)c(2s-2)c(2s-3)$
$w_4w_3w_2w_1$	$c(2s-1)c(2s-2)^2c(2s-3)^2$
$w_2w_4w_3w_2w_1$	$c(2s-1)c(2s-2)^2c(2s-3)^2c(2s-4)$
$w_1w_2w_4w_3w_2w_1$	$c(2s-1)c(2s-2)^2c(2s-3)^2c(2s-4)c(2s-5)$

(10.67)

The table clearly reflects the multiplicative property (10.56) of the factors $M(w,\lambda)$: Moving one step down the Weyl orbit adds a single factor $c(k)$ to $M(w,\lambda)$.

Depending on the value of s some of the factors $M(w,\lambda)$ can vanish leading to a further reduction in the number of constant terms in (10.62). As already argued based on the multiplicative property (10.56) we should start at the top of the Weyl orbit. Let us look at a few examples, keeping in mind that we are looking for values of s where there are more factors $c(-1)$ than $c(+1)$ in the product.

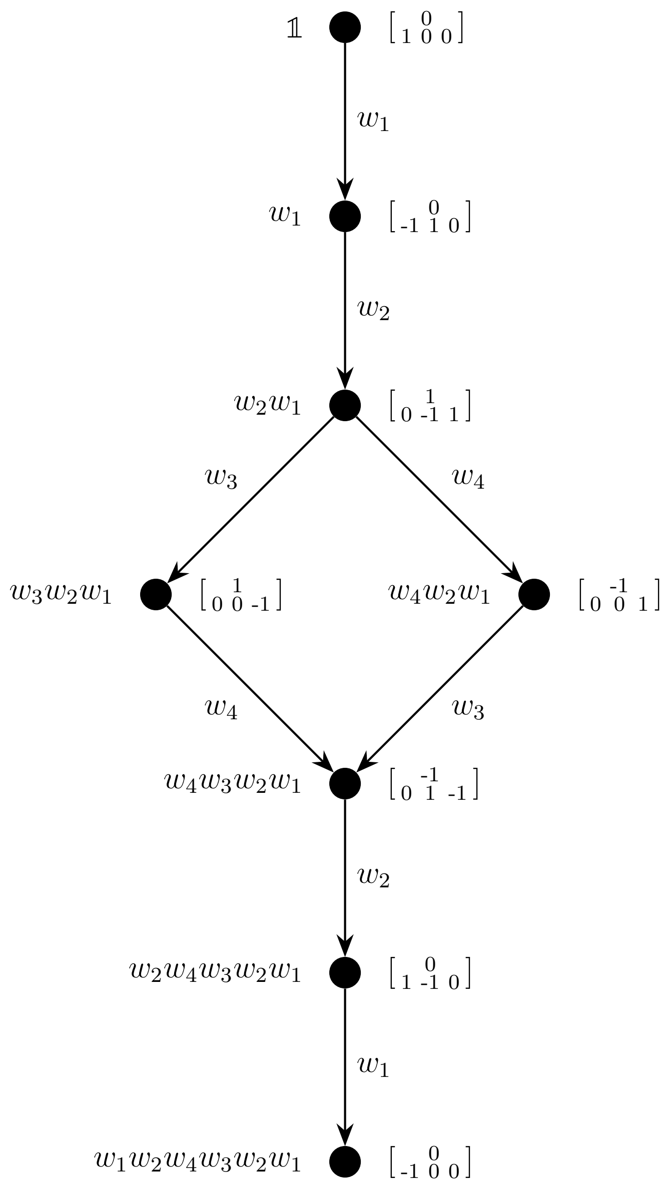


Figure 10.3: *The Weyl orbit of the fundamental weight* $\Lambda_1 = \left[\begin{smallmatrix} & 0 & \\ 1 & 0 & 0\end{smallmatrix}\right]$ *under the* D_4 *Weyl group. For each image point in the orbit, we have listed the Dynkin labels and a choice of shortest Weyl word that leads to the given point. The shortest Weyl words are those that make up the set* $\mathcal{C}(\lambda)$ *for* $\lambda = 2s\Lambda_1 - \rho$ *that contribute to the constant terms in (10.62).*

The simplest case is of course $s = 0$. Then only $w = \mathbb{1}$ has a non-vanishing $M(w, \lambda) = 1$ and this is the whole constant term. This is not surprising since $s = 0$ corresponds to $\lambda = -\rho$, yielding the trivial constant automorphic function $E(-\rho, g) \equiv 1$.

The next simplest case is $s = \frac{1}{2}$. For this choice one the two Weyl words $w = \mathbb{1}$ and $w = w_1$ contribute to the constant term (10.62). Working out their contributions one finds that they cancel (using $c(0) = -1$) and the constant term vanishes. (The same things happens for the $SL(2, \mathbb{R})$ series; see section 10.1.1.)

For the value $s = 1$ the factor $c(2s - 3)$ leads to a vanishing contribution but the factor $c(2s - 1)$ has a pole so one needs to the limit carefully. In all there are five non-vanishing contributions to the constant term since the last three orbit points (out of the total eight) contain the factor $c(2s - 3)^2$. Summing up the non-vanishing contributions leads to

$$v_1^2 + \frac{6v_2}{\pi}\left(\gamma_E - \log(4\pi) - \log\left(v_1 v_2^{-2} v_3 v_4\right)\right) + v_3^2 + v_4^2 \tag{10.68}$$

where we parametrised $a = v_1^{h_1} v_2^{h_2} v_3^{h_3} v_4^{h_4}$. The logarithms arise when taking the limit $s \to 1$ and reflect the confluence of the eigenvalues of two polynomial eigenfunctions of the Laplace operator.

Further simplifications occur for $s = \frac{3}{2}$ and $s = 2$ that we leave to the reader to evaluate.

If one had started with a fixed value of s for which simplifications occur, it would have been sufficient to construct the Weyl orbit up to the points where the $M(w, \lambda) = 0$.

10.3.2 Special λ-values and $E(\lambda, g)$

As we have seen in the $SO(4, 4)$ example 10.18 just now and in section 10.1.1, there can be special points $\lambda \in \mathfrak{h}^*$ where the constant terms (and the whole Eisenstein series) simplify. Parametrising the weight λ in terms of (complex) parameters s_i, these special points correspond to specific values for the s_i. These simplifications were already observed for the exceptional group G_2 by Langlands in his original work [218], see also [194].

In order to detect such simplifications, it is not efficient to calculate the whole set $\mathcal{C}(\lambda)$ and then the coefficients $M(w, \lambda)$ as in (10.67). Due to the partially ordered structure of the Weyl orbit and the multiplicative property (10.56) it suffices to also calculate the factor $M(w, \lambda)$ at the same as one constructs w using the Weyl orbit method of section 10.3.1. One need not construct further any path of the graph (of increasing word length) where one of the intermediate words satisfies $M(w, \lambda) = 0$. This simplifies the calculation of the constant term formula considerably [95].

Example 10.19: Orbit method for E_8 maximal parabolic Eisenstein series in the minimal representation

Consider again the maximal parabolic E_8 Eisenstein series of example 10.6 with weight $\lambda_s = 2s\Lambda_8 - \rho$. For calculating the constant term, we require the Weyl orbit of the dominant weight $\Lambda_8 = [0, 0, 0, 0, 0, 0, 0, 1]$ in Dynkin label notation. In total, the Weyl orbit of Λ_8 has 240 elements. The beginning of the Weyl orbit, computed with the orbit method is depicted in figure 10.4.

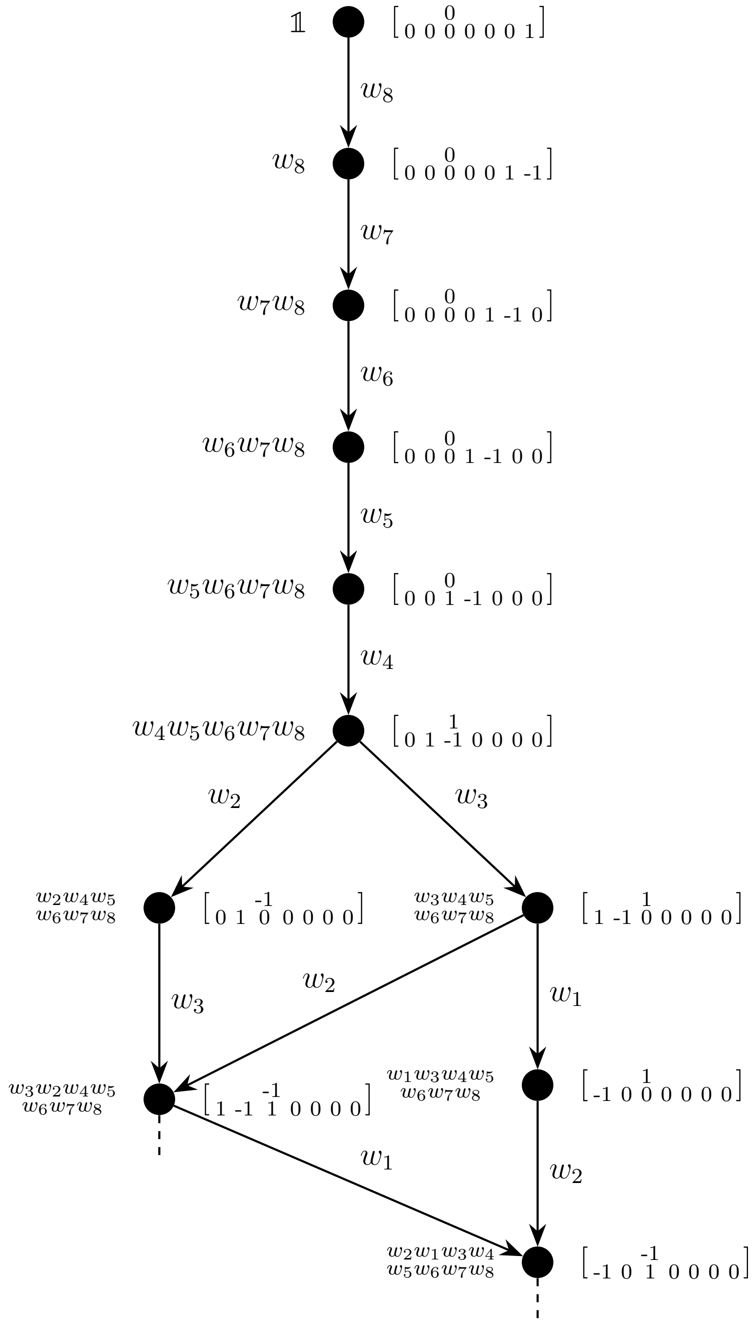


Figure 10.4: *The Weyl orbit of the fundamental weight* $\Lambda_1 = \left[\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{smallmatrix}\right]$ *under the* E_8 *Weyl group. For each image point in the orbit, we have listed the Dynkin labels and a choice of shortest Weyl word that leads to the given point. The shortest Weyl words are those that make up the set* $\mathcal{C}(\lambda)$ *that contribute to the constant terms in example 10.19.*

The corresponding factors $M(w,\lambda)$ for $\lambda = 2s\Lambda - \rho$ are given by the following table:

$$\begin{array}{r|l}
w & M(w,\lambda) \\
\hline
\mathbb{1} & 1 \\
w_8 & c(2s-1) \\
w_7w_8 & c(2s-1)c(2s-2) \\
w_6w_7w_8 & c(2s-1)c(2s-2)c(2s-3) \\
w_5w_6w_7w_8 & c(2s-1)c(2s-2)c(2s-3)c(2s-4) \\
w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5) \\
w_2w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5)c(2s-6) \\
w_3w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5)c(2s-6) \\
w_3w_2w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5)c(2s-6)^2 \\
w_1w_3w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5)c(2s-6)c(2s-7) \\
w_2w_1w_3w_4w_5w_6w_7w_8 & c(2s-1)c(2s-2)^2c(2s-3)c(2s-4)c(2s-5)c(2s-6)^2c(2s-7) \\
\vdots & \vdots
\end{array} \tag{10.69}$$

Simplifications arise as always for $s = 0$ and $s = \frac{1}{2}$. Another interesting case is $s = \frac{5}{2}$. One sees that the nineth and eleventh entry in the table contain a factor $c(2s-6)^2$ that makes the corresponding intertwiner vanish. Since the two Weyl words are the two bottom words in the orbit constructed thus far in figure 10.4 one knows that all remaining Weyl words coming from the orbit method applied to Λ_8 will have vanishing $M(w,\lambda)$ and therefore the constant term consists of nine terms. Taking the limit $s \to \frac{5}{2}$ carefully again gives logarithmic terms as in (10.68) that we do not display here. The value $s = \frac{5}{2}$ gives the simplest possible constant and the Eisenstein series is attached to the minimal representation as was already mentioned before.

Remark 10.20. Simplifications in the constant term have corresponding simplifications in the Whittaker vectors as we will see in section 10.4. They are typically associated with subrepresentations in the (degenerate) principal series called *small representations*. At these places the functional dimension of the automorphic representation reduces. This is discussed in more detail in section 12.1.1.

Example 10.21: Minimal representation of $SL(3,\mathbb{R})$

The Eisenstein series on $SL(3,\mathbb{A})$ introduced in section 9.6 with weight

$$\lambda = 2s_1\Lambda_1 + 2s_2\Lambda_2 - \rho \tag{10.70}$$

simplifies for special values of the parameters s_i. Putting

$$s_1 = 0 \quad \text{or} \quad s_2 = 0 \quad \text{or} \quad s_3 = s_1 + s_2 - \frac{1}{2} = 0 \tag{10.71}$$

or

$$s_1 = 1 \quad \text{or} \quad s_2 = 1 \quad \text{or} \quad s_3 = s_1 + s_2 - \frac{1}{2} = 1 \tag{10.72}$$

makes the generic Eisenstein series into one on a maximal parabolic subgroup as defined in section 4.1.3; the case $s_1 = 0$ corresponds to inducing from the maximal parabolic subgroup $P_1(\mathbb{A})$. In this case the Fourier expansion simplifies considerably as can be seen be inspecting the expression of section 9.6. This is already manifest from (9.82a) that appears in the expression of any generic Whittaker coefficient. Precisely for the choices above $1/\zeta(\lambda)$ vanishes identically, implying that all generic Whittaker coefficients vanish.

For the degenerate Whittaker coefficients one also obtains shorter expressions: Out of the three Weyl elements displayed in (9.91) two have a vanishing and one is left with a single modified Bessel function with associated divisor sum. The non-abelian Whittaker vector also simplifies as is shown in [267].

10.3.3 Constant terms in maximal parabolic subgroups

Given a unipotent $U \subset N$ and an Eisenstein series $E(\lambda, g)$ on a group G one can define the constant term along U, c.f. equation 6.16 and section 8.9 by

$$C_U = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\lambda, ug)du. \tag{10.73}$$

When $U = U_{j_\circ}$ is the unipotent of a maximal parabolic subgroup $P_{j_\circ} = L_{j_\circ}U_{j_\circ}$ associated with node $j_\circ$-th simple root, a general formula for this constant term was given in theorem 8.8:

$$\int_{U_{j_\circ}(\mathbb{Z})\backslash U_{j_\circ}(\mathbb{R})} E(\lambda, ug)du = \sum_{w\in\mathcal{W}_{j_\circ}\backslash\mathcal{W}} e^{\langle(w\lambda+\rho)_{\parallel j_\circ}|H(g)\rangle} M(w,\lambda)E^{M_{j_\circ}}((w\lambda)_{\perp j_\circ}, m). \tag{10.74}$$

The Eisenstein series on the right-hand side is one on the semi-simple part $M_{j_\circ}$ of the Levi subgroup $L_{j_\circ} = GL(1,\mathbb{R}) \times M_{j_\circ}$ and the exponential prefactor is a function only on the $GL(1,\mathbb{R})$ factor. We note that it is the same numerical coefficient $M(w,\lambda)$ as in (10.54) that controls this constant term. As we have explained in section 10.3.1, one can restrict the Weyl words to the set $\mathcal{C}(\lambda)$ that is in bijection with the Weyl orbit of $\lambda+\rho$ (or an equivalent dominant weight Λ defined in (10.63)). This bijection implies that it suffices to consider Weyl words w in the left coset $\mathcal{W}/\mathcal{W}(\text{stab}(\lambda)) \cong \mathcal{C}(\lambda)$. The addition quotient in formula (10.74) then allows the restriction to the double coset [95, 126]

$$w \in \mathcal{W}_{j_\circ}\backslash\mathcal{W}/\mathcal{W}(\text{stab}(\lambda)). \tag{10.75}$$

This double coset typically has very few representatives that allow for a swift evaluation of formula (10.74).

Remark 10.22. The double coset (10.75) also depends on λ and there can therefore be similar simplifications as those discussed in section 10.3.2.

Example 10.23: Constant term of $SO(4,4;\mathbb{R})$ Eisenstein series with respect to a maximal parabolic subgroup

For the $SO(4,4;\mathbb{R})$ Eisenstein series considered in example 10.18 with weight $\lambda = 2s\Lambda_1 - \rho$ we compute the constant term along the unipotent U_3 of the maximal parabolic subgroup P_3. The Weyl group of the semi-simple Levi part is again of type $\mathcal{W}(A_3)$ and generated by the fundamental reflections w_1, w_2 and w_4. Inspecting the list (10.66) of elements of $\mathcal{C}(\lambda)$ shows that the double coset (10.75) in this case has only two representatives, namely

$$\mathbb{1} \quad \text{and} \quad w_3w_2w_1. \tag{10.76}$$

If we denote the coordinate on $GL(1,\mathbb{R})$ by

$$r = e^{\langle \Lambda_3 | H(g) \rangle}, \tag{10.77}$$

we obtain for trivial representative the decomposition

$$(2s\Lambda_1)_{\parallel 3} = \frac{\langle 2s\Lambda_1|\Lambda_3\rangle}{\langle \Lambda_3|\Lambda_3\rangle}\Lambda_3 = s\Lambda_3, \quad (2s\Lambda_1 - \rho)_{\perp 3} = (2s-1)\Lambda_1^{M_3} - \Lambda_2^{M_3} - \Lambda_3^{M_3} \tag{10.78}$$

(where $\Lambda_i^{M_3}$ are the three fundamental weights of $M_3(\mathbb{R}) = SL(4,\mathbb{R})$) and a similar decomposition for the other representative the following constant term

$$\int\limits_{U_3(\mathbb{Z})\backslash U_3(\mathbb{R})} E(\lambda, ug)du = r^s E([2s-1,-1,-1],m) + r^{3-s}\frac{\xi(2s-3)}{\xi(2s)}E([-1,-1,2(s-1)-1],m), \tag{10.79}$$

where we have indicated the weight on the semi-simple subgroup M_3 by its Dynkin labels and have evaluated the intertwiner $M(w_3w_2w_1,\lambda)$ using (10.67).

10.4 Evaluating spherical Whittaker vectors

We now turn the question of efficiently evaluating degenerate Whittaker vectors that are given by theorem 9.5 whose result we briefly recall. The final formula there was

$$W_\psi^\circ(\lambda, a) = \sum_{w_c w'_{\text{long}} \in \mathcal{W}/\mathcal{W}'} a^{(w_c w'_{\text{long}})^{-1}\lambda + \rho} M(w_c^{-1},\lambda) W'^\circ_{\psi^a}(w_c^{-1}\lambda, \mathbb{1}). \tag{10.80}$$

We briefly recall the notation used in this formula. ψ denotes a degenerate character on the maximal unipotent $N \subset B \subset G$. It has support $\text{supp}(\psi) \subset \Pi$ given by (9.61) and this subset of simple roots defines a semi-simple subgroup $G' \subset G$ with Weyl group $\mathcal{W}' = \mathcal{W}(\text{supp}(\psi))$. The longest element $\mathcal{W}'$ is called w'_{long} and $w_c \in \mathcal{W}$ satisfies $w_c\alpha > 0$ for all $\alpha \in \text{supp}(\psi)$. The representative w_c of the coset $\mathcal{W}/\mathcal{W}'$ can be constructed using the orbit method of section 10.3.1. For any $a \in A(\mathbb{A})$, the twisted character $\psi^a(n) = \psi(ana^{-1})$ restricted to the unipotent $N' \subset G'$ is generic and $W'^\circ_{\psi^a}(w_c^{-1}\lambda, \mathbb{1})$ is the (generic) spherical Whittaker vector on the G' of the prinicipal series representation given by the restriction of the weight $w_c^{-1}\lambda$ to G' and evaluated at the identity. An example of this formula was worked out for $SL(3,\mathbb{A})$ in section 9.6.

One sees from formula (10.80) that it is again an intertwining coefficient $M(w_c^{-1},\lambda)$ that controls possible simplifications in the degenerate Whittaker vectors. We know from the discussion of the orbit method in section 10.3.1 that only those w_c^{-1} give a non-trivial $M(w_c^{-1},\lambda)$ that are the minimal representatives of the coset $\mathcal{W}/\mathcal{W}(\text{stab}(\lambda))$ (constructed by the orbit method). Due to the inverse w_c^{-1} we are therefore again faced with a double coset

$$\mathcal{W}(\text{supp}(\psi))\backslash\mathcal{W}/\mathcal{W}(\text{stab}(\lambda)). \tag{10.81}$$

This has very few representatives one has to consider when evaluating (10.80) for a non-generic λ and ψ.

For which Eisenstein series $E(\lambda, g)$ does formula (10.80) actually offer the prospect of helping find complete information about the Fourier expansion? As mentioned already in remark 9.7 this will happen when λ is such that $E(\lambda, g)$ is not in the generic principal series but in a degenerate one or even at one of the special λ values discussed in section 10.3.2.

10.4.1 Degenerate principal series and degenerate Whittaker vectors

If λ is such that $\mathrm{stab}(\lambda) \neq \{\}$, the Eisenstein series $E(\lambda, g)$ is associated to the degenerate principal series, see section 5.6. Prime example are maximal parabolic Eisenstein series when $\mathrm{stab}(\lambda) = \Pi \setminus \{\alpha_{j_*}\}$ where α_{j_*} is a single simple root that defines a maximal parabolic subgroup $P_{j_*} \subset G$. More generally, we define a parabolic subgroup

$$P_\lambda = L_\lambda U_\lambda, \tag{10.82}$$

such that the semi-simple part $M_\lambda \subset L_\lambda$ has the simple root system given by $\mathrm{stab}(\lambda)$. Then the Eisenstein series $E(\lambda, g)$ belongs to the degenerate principal series with Gelfand–Kirillov dimension

$$\mathrm{GKdim}(I(\lambda)) = \dim\left(P_\lambda \backslash G\right) \tag{10.83}$$

or even a subrepresentation of this in case λ sits at a special value.

For automorphic functions in a degenerate principal series one has that typically not all Whittaker vectors are non-zero and often the generic ones are absent. In order to determine which Whittaker vectors are non-zero we recall the notion of a wavefront set introduced in section 6.4. The wavefront set is the set of nilpotent orbits of $G(\mathbb{R})$ such that there are non-trivial Fourier coefficients (or Whittaker vectors) associated with it. Characters ψ are associated with nilpotent elements of $\mathfrak{g} = \mathfrak{g}(\mathbb{R})$ and one has to consider their $G(\mathbb{R})$ orbits in an automorphic representation.

Nilpotent orbits of $\mathfrak{g}$ under $G(\mathbb{R})$ come with a certain (even) dimension and they must be able to 'fit into' the automorphic representation of $E(\lambda, g)$ for a non-trivial (non-vanishing) Fourier coefficient to exist. There is a symplectic structure on a nilpotent orbit [71] and only a Lagrangian subspace corresponds to the character ψ of a Fourier mode or Whittaker vector. Let $X \in \mathfrak{g}$ be a nilpotent element and $\mathcal{O}_X$ its corresponding nilpotent orbit under $G(\mathbb{R})$. The constraint just explained means that there can be non-trivial Fourier coefficients only if

$$\frac{1}{2}\dim_{\mathbb{R}} \mathcal{O}_X \leq \dim\left(P_\lambda \backslash G\right). \tag{10.84}$$

Remark 10.24. If λ is generic such that $E(\lambda, g)$ is in the full principal series, then P_λ equals the standard Borel subgroup $B \subset G$ and the Gelfand–Kirillov dimension equals $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{h})$. At the same time, the largest nilpotent orbit (called the principal orbit) has dimension $\dim \mathfrak{g} - \dim \mathfrak{h}$, confirming the fact that such a generic Eisenstein series will have generic Fourier coefficients in general.

The condition (10.84) puts strong constraints on the orbits one has to consider for a degenerate principal series representation. Moreover, if a nilpotent orbit $\mathcal{O}_X$ has a representative X that lies completely in $[N,N]\backslash N$ such that there is a character $\psi_X : N \to U(1)$ associated with it, one can test whether $\mathcal{O}_X$ belongs to the wavefront set by computing the (degenerate) Whittaker vector for ψ_X with formula (10.80).

Example 10.25: Minimal orbit and A_1-type Whittaker vectors

Any simple group $G(\mathbb{R})$ has a unique minimal non-trivial nilpotent orbit $\mathcal{O}_{\text{min}}$ that is given by the orbit of a generator E_θ from the root space of the highest root θ. If $G(\mathbb{R})$ is simply-laced, one can alternatively choose as a nilpotent representative any simple step operator $X = E_{\alpha_i}$ where α_i is a simple root. The corresponding character ψ_X is maximally degenerate and has $\text{supp}(\psi_X) = \{\alpha_i\}$ and one can compute the associated degenerate Whittaker vectors using (10.80) in terms of Whittaker vectors on $SL(2,\mathbb{R})$ (which are completely known). The minimal orbit is called type A_1 in Bala–Carter terminology [7] and this relates to the subgroup G' that appears in the formula for degenerate Whittaker vectors.

As another example we consider the consequences of the condition (10.84) on a degenerate principal series of the group E_8.

Example 10.26: Wavefront sets of the adjoint E_8 series

Consider the maximal parabolic Eisenstein series $E(\lambda, g)$ of $E_8(\mathbb{R})$ given by the weight

$$\lambda = 2s\Lambda_8 - \rho \tag{10.85}$$

as in example 10.6. The degenerate principal series that $E(\lambda, g)$ belongs to is of Gelfand–Kirillov dimension (for generic s)

$$\text{GKdim}\, I(\lambda) = \dim(E_8) - \dim(P_\lambda) = 248 - (133 + 1 + 56 + 1) = 57. \tag{10.86}$$

According to (10.84), the largest nilpotent orbit that can contribute to the wavefront set of $E(\lambda, g)$ is therefore of dimension 114. Here is a list of nilpotent orbits of (split) $E_8(\mathbb{R})$ of small dimension [71, 256]

$\dim \mathcal{O}$	Bala–Carter label	weighted diagram over $\mathbb{C}$
0	0	$\left[\begin{smallmatrix} & & 0 & & & & \\ 0&0&0&0&0&0&0 \end{smallmatrix}\right]$
58	A_1	$\left[\begin{smallmatrix} & & 0 & & & & \\ 0&0&0&0&0&0&1 \end{smallmatrix}\right]$
92	$2A_1$	$\left[\begin{smallmatrix} & & 0 & & & & \\ 1&0&0&0&0&0&0 \end{smallmatrix}\right]$
112	$3A_1$	$\left[\begin{smallmatrix} & & 0 & & & & \\ 0&0&0&0&0&1&0 \end{smallmatrix}\right]$
114	A_2 and $(4A_1)''$	$\left[\begin{smallmatrix} & & 0 & & & & \\ 0&0&0&0&0&0&2 \end{smallmatrix}\right]$
$\vdots$	$\vdots$	$\vdots$

The last entry corresponds to a single complex orbit of type A_2 that splits into two real orbits [256]. All these orbits have representatives X in $[N,N]\backslash N$ and the associated Whittaker vectors can be calculated using (10.80). More complicated Whittaker vectors are absent in this degenerate principal series. For special s values not all the orbits in the above table appear in the wavefront set.

10.4.2 Whittaker vectors of maximal parabolic Eisenstein series

For maximal parabolic Eisenstein series one can also make statements about the vanishing of some Whittaker vector. Consider an Eisenstein series on $G(\mathbb{A})$ induced from a character

$\chi : P_{i_*}(\mathbb{A}) \to \mathbb{C}^\times$, i.e., one that is in the degenerate principal series and is parametrised by a single complex parameter $s \in \mathbb{C}$ through the weight $\lambda = 2s\Lambda_{i_*} - \rho$, see proposition 5.29. The Whittaker integral for an arbitrary character $\psi : N(\mathbb{Q})\backslash N(\mathbb{A}) \to U(1)$ leads to (cf. (9.11))

$$W_\psi(\chi, a) = \sum_{w \in \mathcal{W}(P_{i_*})\backslash\mathcal{W}} \int_{N^w_{\{\beta\}}} \overline{\psi(n_\beta)} dn_\beta \cdot \int_{N^w_{\{\gamma\}}} \chi(wn_\gamma a)\overline{\psi(n_\gamma)} dn_\gamma, \tag{10.87}$$

where the important point is now that the set of contributing Weyl words is restricted to the quotient $\mathcal{W}(P_{i_*})\backslash\mathcal{W}$ from the outset. Again, the integral over $N^w_{\{\beta\}}$ can make the whole expression vanish and imposes constraints on w and ψ. Now the set of positive roots β that appear in that integral is

$$\left\{\beta > 0 \,|\, w\beta \in \Delta_{P_{i_*}}\right\}, \tag{10.88}$$

where $\Delta_{P_{i_*}}$ denotes the subset of all roots in Δ that belong to the maximal parabolic $P_{i_*}(\mathbb{A})$; it involves all positive roots and some negative roots. The set of β now always involves a simple root for any w, therefore for a generic ψ the integral over n_β will vanish and we conclude that the generic Whittaker vector for a maximal parabolic Eisenstein series vanishes.

Another way to see this is by noting that the factor $1/\zeta(\lambda)$ that appears in (9.22) contains generally factors $(1 - p^{-(\langle\lambda|\alpha_i\rangle+1)}) = (1 - p^{-2s_i})$ for all simple roots α_i. For any degenerate principal series Eisenstein series one of these factors vanishes identically, and there is no pole a the same s_i values in the factor $\epsilon(\lambda)$ (see (9.22)). This is guaranteed by the holomorphy of the local Whittaker vector [64].

We will come back to this in the discussion in section 10.4.4.

10.4.3 Examples of degenerate Whittaker vectors

We now present some explicit expressions for degenerate Whittaker vectors calculated with the help of (10.80). The examples are taken mainly from [97]. The following notation will be used

$$B_m(s, v) = \frac{1}{\xi(2s)}\tilde{B}_m(s, v) = \frac{2}{\xi(2s)}|v|^{s-1/2}|m|^{s-1/2}\sigma_{1-2s}(m)K_{s-1/2}(2\pi|m|v) \tag{10.89}$$

for a Whittaker vector on an $SL(2,\mathbb{R})$ subgroup. For an $SL(3,\mathbb{R})$ subgroup we write similarly

$$B_{m_1,m_2}(s_1, s_2, v_1, v_2) = \frac{1}{\xi(2s_1)\xi(2s_2)\xi(2s_1 + 2s_2 - 1)}\tilde{B}_{m_1,m_2}(s_1, s_2, v_1, v_2). \tag{10.90}$$

The explicit expression for $\tilde{B}_{m_1,m_2}(s_1, s_2, v_1, v_2)$ in terms of an integral over two Bessel functions can be found in [60, 267]. See also section 9.6. For our purposes we only need to know that it is finite and non-zero for all values of s_1 and s_2. The same is true for $\tilde{B}_m(s, v)$.

Example 10.27: Degenerate Whittaker vectors of type A_1 for Eisenstein series of $E_{n\geq 6}(\mathbb{R})$

We consider maximal parabolic Eisenstein series of the finite-dimensional exceptional groups $E_n(\mathbb{R})$ with weight vector $\lambda = 2 \cdot \frac{3}{2}\Lambda_1 - \rho$ for $n = 6, 7, 8$. As before we use the standard Bourbaki labelling of the nodes of the Dynkin diagram. In these examples all non-vanishing (abelian) Whittaker vectors turn out to be given by a finite sum of n Whittaker vectors on the $SL(2,\mathbb{R})$ subgroup associated with each node of the Dynkin diagram. The full expression for the Fourier coefficients will be given by

$$\sum_{\psi\neq 0} W^\circ_\psi(\lambda, na) = \sum_{\alpha\in\Pi}\sum_{\psi_\alpha} c_\alpha(a) W'^\circ_{\psi^a_\alpha}(\lambda'_\alpha, \mathbb{1})\psi_\alpha(n)\,, \tag{10.91}$$

where W°_ψ on the left-hand side is given by (10.80) and a detailed derivation of the right-hand side can be found in [97]. For the maximally degenerate character ψ_α associated with the simple roots α, m_α is the only non-zero charge and $c_\alpha(a)$ is a function of the variables parametrising the Cartan torus. Furthermore, $W'^\circ_{\psi^a_\alpha}$ is a generic Whittaker vector on $SL(2,\mathbb{R})$ subgroup associated with the simple roots α and λ'_α is the projection of the weight λ onto this subgroup. We can then provide lists of the degenerate Whittaker vectors for each case:

- E_6:

ψ_α	$c_\alpha(a)W'^\circ_{\psi^a_\alpha}(\chi'_\alpha, \mathbb{1})$
$(m,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m)$	$\frac{\xi(2) v_6^3 B_{-1/2,m}\left(v_6^2 v_5^{-1}\right)}{\xi(3)}$

- E_7 :

ψ_α	$c_\alpha(a)W'^\circ_{\psi^a_\alpha}(\chi'_\alpha, \mathbb{1})$
$(m,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{\frac{3}{2},m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0)$	$\frac{\xi(2) v_6^3 v_7^{-2} B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,m)$	$v_7^4 B_{-1,m}\left(v_7^2 v_6^{-1}\right)$

- E_8 :

ψ_α	$c_\alpha(a)W'^{\circ}_{\psi^a_\alpha}(\chi'_\alpha, \mathbb{1})$
$(m,0,0,0,0,0,0,0)$	$v_3^2 v_1^{-1} B_{3/2,m}\left(v_1^2 v_3^{-1}\right)$
$(0,m,0,0,0,0,0,0)$	$\frac{v_2^2 \tilde{B}_{0,m}(v_2^2 v_4^{-1})}{\xi(3)}$
$(0,0,m,0,0,0,0,0)$	$\frac{\xi(2) v_4 B_{1,m}\left(v_3^2 v_1^{-1} v_4^{-1}\right)}{\xi(3)}$
$(0,0,0,m,0,0,0,0)$	$\frac{v_4 \tilde{B}_{1/2,m}(v_4^2 v_2^{-1} v_3^{-1} v_5^{-1})}{\xi(3)}$
$(0,0,0,0,m,0,0,0)$	$\frac{v_5^2 \tilde{B}_{0,m}(v_5^2 v_4^{-1} v_6^{-1})}{\xi(3) v_6}$
$(0,0,0,0,0,m,0,0)$	$\frac{\xi(2) v_6^3 v_7^{-2} B_{-1/2,m}\left(v_6^2 v_5^{-1} v_7^{-1}\right)}{\xi(3)}$
$(0,0,0,0,0,0,m,0)$	$\frac{v_7^4 B_{-1,m}\left(v_7^2 v_6^{-1} v_8^{-1}\right)}{r_8^3}$
$(0,0,0,0,0,0,0,m)$	$\frac{\xi(4) v_8^5 B_{-3/2,m}\left(v_8^2 v_7^{-1}\right)}{\xi(3)}$

The following provides and example of a degenerate Whittaker vector of type A_2.

Example 10.28: Degenerate Whittaker vectors of type A_2 for Eisenstein series of $E_8(\mathbb{R})$

For the E_8 series of example 10.26 we compute the Whittaker vector associated with the degenerate character ψ on N with 'charges'

$$\psi \quad \leftrightarrow \quad \left[{}_{0\,0\,0\,0\,0\,m\,n}^{\;\;\;\;0}\right]. \tag{10.92}$$

This choice of character ψ is associated with the 114-dimensional nilpotent orbit of type A_2 from the table in example 10.26. For simplicity we put the torus element $a = \mathbb{1}$. Then formula (10.80) gives

$$W_\psi(\lambda, \mathbb{1}) = \frac{\xi(2s-11)\xi(2s-14)\xi(2s-18)\xi(4s-29)}{\xi(2s)\xi(2s-5)\xi(2s-9)\xi(4s-28)} B_{6-s,\frac{19}{2}-s,m,n}\Big(1,1\Big). \tag{10.93}$$

(The contributing Weyl word has length 30 and we do not spell it out here.) As was argued in example 10.19, the value $s = \frac{5}{2}$ corresponds to a simpler Eisenstein series where the constant term simplifies. From the above formula we can see this also in the Whittaker vector of type A_2. The prefactor tends to zero for $s \to \frac{5}{2}$ while the $SL(3, \mathbb{R})$ Whittaker function stays finite and hence the degenerate Whittaker vector (10.93) disappears.

In the case $s = \frac{5}{2}$ one check similarly that all Whittaker vectors but the ones of type A_1 vanish, consistent with the fact that the corresponding Eisenstein series belongs to an automorphic realisation of the minimal representation.

10.4.4 Relation between Fourier coefficients and Whittaker vectors

In this section we investigate, based on the methods of Miller–Sahi [235] and Ginzburg [123], how to compute Fourier coefficients F_{ψ_U} on unipotent subgroups U (from definition 6.13) in terms of Whittaker vectors W_{ψ_N}, the latter of which are known using the methods of chapter 9. Details can be found in [160] and similar ideas were discussed in section 6.5. As we are discussing characters and Fourier coefficients on different subgroups we will adopt the subscript notation used in section 9.6.4 to avoid ambiguities.

Since both Whittaker vectors on N and the constant term simplifies for autormorphic forms in small representations as seen in the examples above, it is natural to also expect simplifications for more general Fourier coefficients. Using the wavefront set and the arguments in the previous sections one can tell for which representations a Fourier coefficient is non-vanishing, but by rewriting F_{ψ_U} in terms of W_{ψ_N} it is also possible to see how a non-vanishing F_{ψ_U} simplifies for smaller representations.

We have already seen an example of such a computation in proposition 9.8 where we, for $G = SL(3)$, showed how the Whittaker vector on $Z = [N, N]$ can be expressed as a sum of G-translated Whittaker vectors on N. When restricting to Eisenstein series in the minimal representation this expression simplifies as follows.

Example 10.29: Non-abelian $SL(3)$ Whittaker vector in minimal representation

For the example $G = SL(3)$ with $\lambda = (2s_1 - 1)\Lambda_1 + (2s_2 - 1)\Lambda_2$ the generic principal series has Gelfand-Kirillov dimension $\mathrm{GKdim}(I(\lambda)) = \dim G - \dim B = 3$, but for $(s_1, s_2) = (s, 0)$ or $(s_1, s_2) = (0, s)$ it reduces to $\mathrm{GKdim}(I(\lambda_{\text{min}})) = 2$ for which the Eisenstein series belongs to the minimal automorphic representation.

The orbits of $SL(3, \mathbb{C})$ are [71]

$\dim \mathcal{O}$	Bala-Carter label	weighted diagram
0	0	0 ●—● 0
4	A_1	2 ●—● 2
6	A_2	1 ●—● 1

This means that for π_{min} we only have Whittaker vectors on N of type A_1, that is, maximally degenerate Whittaker vectors charged under a single simple root.

In proposition 9.8 it was shown that

$$W^{(k)}_{\psi_Z}(\chi, g) = \sum_{m_1, m_2 \in \mathbb{Q}} W^{(m_1,d)}_{\psi_N}(\chi, lg) \tag{10.94}$$

where $d = d(k, m_2)$ as defined in (9.98) and l depends on k and m_2 as described in the proposition. Recall from section 9.6 that $W^{(m_1,d)}_{\psi_N}$ for $m_1 \neq 0$ contains a convoluted integral of two Bessel functions, while $W^{(0,d)}_{\psi_N}$ is simpler being proportional to a single Bessel function.

When restricting χ to χ_{min} (parametrised by s), we get that the sum over charges in (10.94) collapses to $m_1 = 0$ since $d \neq 0$, simplifying the expression for $W^{(k)}_{\psi_Z}$ which then only contains single Bessel functions. Note though that the sum over m_2 (that is, over l-translates) still remains, that is,

$$W^{(k)}_{\psi_Z}(\chi_{\text{min}}, g) = \sum_{m_2 \in \mathbb{Q}} W^{(0,d)}_{\psi_N}(\chi_{\text{min}}, lg)\,. \tag{10.95}$$

When inserting the argument $g = (g_\infty, \mathbb{1}, \mathbb{1}, \ldots)$ the sum over rational charges becomes a sum over integers, similar to what happens in proposition 9.9, since $l \in SL(3, \mathbb{Z})$ by design.

Let us now consider another example for $G = SL(3)$, but with Fourier coefficients on a maximal parabolic subgroup. Here, the expression simplifies even further in the minimal representation resulting in a single translated maximally degenerate Whittaker vector on N.

Example 10.30: Maximal parabolic Fourier coefficient in minimal representation

Continuing with $G = SL(3)$, we will now see that a Fourier coefficient F_{ψ_U} on the maximal parabolic subgroup corresponding to the first simple root

$$P = P_1 = LU = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad L = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad U = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10.96}$$

can be expressed as a single L-translated, maximally degenerate Whittaker vector in the minimal representation. Let

$$u = \begin{pmatrix} 1 & u_1 & u_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U \qquad \psi_U(u) = e^{2\pi i(m_1 u_1 + m_2 u_2)} \qquad m_1, m_2 \in \mathbb{Q} \qquad m_1 m_2 \neq 0\,. \tag{10.97}$$

Then, $d = d(m_1, m_2)$ as defined in (9.98) is strictly positive and $m_i' := m_i/d \in \mathbb{Z}$ with $\gcd(m_1', m_2') = 1$ which tells us that there exist integers α and β such that

$$l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -m_2' & m_1' \end{pmatrix} \in L(\mathbb{Z})\,. \tag{10.98}$$

Now we conjugate the Fourier coefficient with l as follows (cf. (6.78))

$$\begin{aligned} F_{\psi_U}^{(m_1,m_2)}(\chi, g) &:= \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \begin{pmatrix} 1 & u_1 & u_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g) e^{-2\pi i(m_1 u_1 + m_2 u_2)}\, du^2 \\ &= \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \begin{pmatrix} 1 & (m_1 u_1 + m_2 u_2)/d & -bu_1 + au_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i(m_1 u_2 + m_2 u_2)}\, du^2 \\ &= \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E(\chi, \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i d x_1}\, dx^2 = F_{\psi_U}^{(d,0)}(\chi, lg) \end{aligned} \tag{10.99}$$

where we have made the substitutions $(m_1 u_1 + m_2 u_2)/d \to x_1$ and $-bu_1 + au_2 \to x_2$. Note that there are other matrices $l \in L(\mathbb{Q})$ (explicitly given by m_1 and m_2) that would accomplish similar results, but if $l \in L(\mathbb{Z})$ the p-adic Iwasawa decomposition simplifies when inserting $g = (g_\infty, \mathbb{1}, \ldots, \mathbb{1})$ as in section 9.6.4.

Expanding further we obtain

$$F_{\psi_U}^{(m_1,m_2)}(\chi, g) = \sum_{m_3 \in \mathbb{Q}} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} E(\chi, \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} lg) e^{-2\pi i(d x_1 + m_3 x_3)}\, d^3x = \sum_{m_3 \in \mathbb{Q}} W_{\psi_N}^{(d,m_3)}(\chi, lg) \tag{10.100}$$

with $d > 0$ and where $W_{\psi_N}^{(d,m_3)}$ is a Whittaker vector on N with charges d and m_3 for the two simple roots.

Now restricting to $\chi_{\min}$, only the maximally degenerate Whittaker vectors are non-vanishing. This collapses the sum above to $m_3 = 0$ giving

$$F_{\psi_U}^{(m_1,m_2)}(\chi_{\min}, g) = W_{\psi_N}^{(d,0)}(\chi_{\min}, lg)\,. \tag{10.101}$$

We note that for $m_2 = 0$ and positive m_1 we have that $d = m_1$ and $l = \mathbb{1}$ giving

$$F_{\psi_U}^{(m_1,0)}(\chi_{\min}, g) = W_{\psi_N}^{(m_1,0)}(\chi_{\min}, g)\,. \tag{10.102}$$

The same statement can be made for negative m_1 as well and can be derived by directly making a further expansion as in (10.100) without a conjugation with l. We conclude that, in the minimal representation, a maximal parabolic Fourier coefficient charged only on the simple root α_1 simplifies to the maximally degenerate Whittaker vector charged on the same root.

In [160] it was similarly shown for $SL(3)$ and $SL(4)$ that all non-trivial Fourier coefficients on any maximal parabolic subgroup automorphic forms in the minimal representation simplify to a single translated maximally degenerate Whittaker vector on N.

This was accomplished by relating Fourier coefficients to the orbit Fourier coefficients defined in definition 6.33 and which vanish when the orbit does not belong to the wavefront set of the considered automorphic representation. Then, the orbit coefficients were expanded as sums of translated Whittaker vectors which were found to be maximally degenerate for the minimal orbit coefficients and Whittaker vectors charged under two strongly orthogonal roots for the next-to-minimal orbit coefficients. In the minimal representation, the maximal parabolic Fourier coefficients picked up only one of these maximally degenerate Whittaker vectors in the minimal orbit coefficient. In the same paper the next-to-minimal representation for $SL(4)$ is discussed as well.

Also, it was shown in [235] that, for E_6 and E_7, Fourier coefficients on certain maximal parabolic subgroups of automorphic forms in $\pi_{\min}$ are determined by maximally degenerate Whittaker vectors. From their proof, one may also deduce that, concretely, such a Fourier coefficient is exactly a translate of a maximally degenerate Whittaker vector similar to the results of [160] for $SL(3)$ and $SL(4)$.

From this it was conjectured in [160] that Fourier coefficients on maximal parabolic subgroups for other simply-laced, simple Lie groups simplify in a similar way and that each may be given in terms of a single, translated, maximally degenerate Whittaker vector on N. In the remaining parts of this section we will explore some applications and verifications of this statement.

We have seen in sections 9.2 and 9.4 that generic Whittaker vectors factorise

$$W_{\psi_N}(\chi, g) = \prod_{p\leq\infty} W_{\psi_{N,p}}(\chi_p, g)\,, \tag{10.103}$$

with $W_{\psi_{N,p}} \in \operatorname{Ind}_{N(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_{N,p}$, but that degenerate Whittaker vectors, in general, do not and are expressed as sums of factorising terms as seen in (10.80). As such, we cannot expect that all Fourier coefficients on any parabolic subgroup should factorise, that is, we cannot a priori expect that

$$\operatorname{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}\psi_U = \bigotimes_{p\leq\infty} \operatorname{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_{U,p} \tag{10.104}$$

However, in the minimal representation all but the maximally degenerate Whittaker vectors on N vanish and the remaining simplify, becoming factorisable as seen for E_6,

E_7 and E_8 in the tables of example 10.27 from appendix A of [97]. This means that if a Fourier coefficient on a maximal parabolic subgroup can be expressed as a single translated maximally degenerate Whittaker vector (as in (10.101) above), then it does indeed factorise.

This is interesting since, although not much is known in general about $\mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_{U,p}$ for non-minimal parabolic subgroups, the image under the embedding

$$\pi_{\min,p} \subset \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\min,p} \hookrightarrow \mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_{U,p}\,, \tag{10.105}$$

with $\chi_{\min,p}$ spherical, has multiplicity one [107] and the corresponding spherical vectors $f^\circ_{\psi_{U,p}}$ have been computed in several cases using representation theory [90, 190, 191, 279].

Assuming the factorisation of maximal parabolic Fourier coefficients discussed above, it is possible to rederive and extend these results by considering the spherical vectors induced from $\pi_{\min}$ as coming from local factors of global Fourier coefficients.

Indeed, in [160] it was shown that the products of known spherical vectors $f^\circ_{\psi_{U,p}}$ in maximal parabolics for E_6, E_7 and E_8 give exactly the expected translated maximally degenerate Whittaker vectors in $\pi_{\min}$ giving strong support for the claim that (10.101) can be generalised to all simply-laced simple Lie groups. Let us consider the case $G = E_7$ below.

Example 10.31: E_7 spherical vectors

Let $G = E_7$ and $P_7 = LU$ be the maximal parabolic subgroup obtained by removing the simple root α_7 using the Bourbaki labelling in figure 10.1. This was one of the parabolic subgroups studied in [235]. Then U is abelian and can also be obtained from the 3-grading

$$\mathfrak{e}_7 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbf{27} \oplus (\mathfrak{e}_6 \oplus \mathbf{1}) \oplus \mathbf{27}\,, \tag{10.106}$$

with $\mathfrak{u} = \mathfrak{g}_1$.

The unique spherical vectors in $\mathrm{Ind}_{U(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi_{U,p}$ in the minimal representation have been computed at the non-archimedean places by [279] and are here shown evaluated at the identity in $G(\mathbb{Q}_p)$

$$f^\circ_{\psi_{U,p}} = \frac{1 - p^3\,|m|_p^{-3}}{1 - p^3}\,, \tag{10.107}$$

where $m \in \mathbb{Q}^\times$ is the charge of $\psi_{U,p}$ conjugated to the simple root α_7 in U.

At the archimedean place we have instead, from [90], that

$$f^\circ_{\psi_{U,\infty}} = m^{-3/2}K_{3/2}(m)\,, \tag{10.108}$$

evaluated at the identity in $G(\mathbb{R})$.

We will now rederive these results by instead viewing the spherical vectors as coming from local factors of a global Fourier coefficient F_{ψ_U} of a spherical Eisenstein series in $\pi_{\min}$. Such an Eisenstein series may be realised from a parabolically induced representation $I_{P_1}(\lambda_{\min})$ with the maximal parabolic subgroup obtained by removing α_1 and $\lambda_{\min} = 2s\Lambda_1 - \rho$ with $s = 3/2$. We consider the Fourier coefficient F_{ψ_U} with a character non-trivial only on the simple root α_1. Similar to the examples above, it simplifies to the single maximally degenerate Whittaker vector $W^{(\alpha_1)}_{\psi_N}$ charged only on the same root which, in turn, factorises.

From table A.1 of [97] we get that

$$W^{(\alpha_1)}_{\psi_N}(\lambda_{\min}, \mathbb{1}) = \frac{2}{\xi(4)} |m|^{-3/2} \sigma_3(m) K_{3/2}(m) = \frac{2}{\xi(4)} \left(\prod_{p<\infty} \frac{1 - p^3 |m|_p^{-3}}{1 - p^3} \right) \left(|m|^{-3/2} K_{3/2}(m) \right) \tag{10.109}$$

where we recognise the first parenthesis as a product of the non-archimedean spherical vectors in (10.107) and the second as the archimedean spherical vector in (10.108).

In [160], the spherical vectors are rederived in a similar way for E_6, E_7 and E_8 in both the abelian and Heisenberg realisations of the minimal representation with complete agreement.

Chapter 11

Hecke theory and automorphic L-functions

In this chapter, we outline the theory of Hecke operators and Hecke algebras. In a nutshell, Hecke operators act on the space of automorphic forms on a group G, forming a commutative ring called the Hecke algebra. The representation theory of this algebra carries a wealth of information about automorphic forms and automorphic representations that connect with many of the structures discussed in the preceding sections. We begin by outlining the Hecke theory in the case of automorphic forms on real arithmetic quotients $G(\mathbb{Z})\backslash G(\mathbb{R})$, providing detailed examples for the case of $SL(2,\mathbb{R})$. After this treatment of the classical Hecke theory, we consider the counterpart in the adelic context. The key object here is the local spherical Hecke algebra $\mathcal{H}_p^\circ$ which acts on the space of adelic automorphic forms $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We study the representation theory of the spherical Hecke algebra and show how this relates to automorphic representations via the Satake isomorphism. Our treatment is mainly done in the context of $SL(2,\mathbb{A})$ and $GL(2,\mathbb{A})$, but many results carry over to arbitrary reductive groups. In particular, in section 11.7 we give some details on the generalisation to $GL(n,\mathbb{A})$ and we make contact to the Langlands program. The starting point is the rewriting of the Casselman–Shalika formula that we encountered in section 9.7. Finally, we end this section with a brief discussion of automorphic L-functions, which form a cornerstone of the Langlands program.

11.1 Classical Hecke operators and Hecke ring: the general idea

Besides the ring of invariant differential operators there is another set of operators that act on the space $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ of automorphic functions on the group $G(\mathbb{R})$ invariant under the discrete group $G(\mathbb{Z})$. These additional operators are called *Hecke operators* and we sketch their general definition following [130]. Their power is worked out for $SL(2,\mathbb{R})$ in section 11.2 in the classical setting. Hecke operators and algebras can also be introducted in the adelic setting and this will be the topic of sections 11.3 and beyond.

Let $g \in G(\mathbb{R})$ be a fixed element *commensurable* with $G(\mathbb{Z})$, i.e., the intersection $g^{-1}G(\mathbb{Z})g \cap G(\mathbb{Z})$ has finite index in both $G(\mathbb{Z})$ and $g^{-1}G(\mathbb{Z})g$. We rewrite its double coset with respect to the discrete group $G(\mathbb{Z})$ as

$$G(\mathbb{Z})gG(\mathbb{Z}) = \bigcup_{i=1}^{d} G(\mathbb{Z})g\delta_i. \tag{11.1}$$

On the right-hand side we have written the double coset as a finite disjoint union of single cosets with representatives δ_i for $i = 1, \ldots, d$. The finiteness of this decomposition follows from the commensurability of g.

Definition 11.1. The *Hecke operator* T_g associated with a $G(\mathbb{Z})$-commensurable $g \in G(\mathbb{R})$ acting on an automorphic function φ is defined by

$$(T_g\varphi)(h) = \sum_{i=1}^{d} \varphi(g\delta_i h) \qquad \text{with } h \in G(\mathbb{R}), \tag{11.2}$$

where the δ_i are representatives of the double coset decomposition (11.1).

This operator is well-defined as a finite sum. One can check easily that T_g maps $G(\mathbb{Z})$-invariant functions to $G(\mathbb{Z})$-invariant functions.

Remark 11.2. It is often useful to take a slightly larger group than the original $G(\mathbb{R})$ if it acts on the same space. For $SL(2,\mathbb{R})$ acting on spherical automorphic functions that are defined on the upper half plane $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ one can also also consider the action of $GL(2,\mathbb{R})$ on $\mathbb{H}$ and define Hecke operators for elements $g \in GL(2,\mathbb{R})$ with respect to $SL(2,\mathbb{Z})$. This viewpoint will be useful in section 11.2 below.

Remark 11.3. The normalisation of the Hecke operators in (11.2) is not uniquely fixed. The one used there yields the Hecke ring over $\mathbb{Z}$. It can be useful to change the normalisation and then obtain a Hecke algebra over the field $\mathbb{Q}$.

The Hecke ring is formed by also allowing integer multiples mT_g of Hecke operators for $m \in \mathbb{Z}$ and defining the product of two Hecke operators T_{g_1} and T_{g_2} by representing the combined double coset $G(\mathbb{Z})g_1G(\mathbb{Z})\cdot G(\mathbb{Z})g_2G(\mathbb{Z})$ as the union of double cosets $G(\mathbb{Z})hG(\mathbb{Z})$, possibly with multiplicity. It turns out that a finite union suffices and the product of T_{g_1} and T_{g_2} is then the sum over the T_h with integer coefficients. This operation turns the set of Hecke operators into a *Hecke ring*.

The Hecke ring is usually defined together with a given choice of *semi-group* $\mathcal{S}$ of commensurable elements g. A semi-group is a set with an associative product but not all elements in $\mathcal{S}$ need to be invertible. The example to have in mind here is the set of matrices with determinant equal to some positive integer. The semi-group needs to be chosen such that $G(\mathbb{Z})$ is a (proper) subgroup of $\mathcal{S}$. Importantly, the Hecke ring (for the cases of interest here) turns out to be *commutative*. For the precise statement see [130, Thm. 3.10.10].

Furthermore, the Hecke operators also commute with the ring of differential operators. This means that we can seek common automorphic eigenfunctions of the ring of differential operators and the ring of Hecke operators. The action of the operators then puts additional constraints on the Fourier coefficients that appear in the analysis of the automorphic function and in fact captures much of the number-theoretic structure of these coefficients. An example of this is worked out in the following section for the case of $SL(2,\mathbb{R})$.

11.2 Hecke operators for $SL(2,\mathbb{R})$

In this section, we illustrate some basic features of the Hecke algebra as sketched in the previous section and the way it interacts with the Fourier expansion in the case of $SL(2,\mathbb{R})$. The presentation here is based on [2, 130].

11.2.1 Definition of Hecke operators

Let $f:\mathbb{H}\to\mathbb{R}$ be a *Maass wave form*, i.e., an $SL(2,\mathbb{Z})$ left-invariant function on the upper half plane $\mathbb{H}=SL(2,\mathbb{R})/SO(2)$ that is also an eigenfunction of the $SL(2,\mathbb{R})$ invariant Laplace operator Δ. defined in (5.38). For example f could be the non-holomorphic Eisenstein series $E(s,z)$ as considered in (5.45). As explained in section 5.1.4, Maass wave forms can also be considered spherical automorphic forms on $SL(2,\mathbb{R})$.

According to the general discussion of Hecke operators in section 11.1, we have to choose a semi-group $\mathcal{S}$ of $SL(2,\mathbb{Z})$ commensurable elements. This we do by letting $\mathcal{S}$ be the group of *diagonal* integer (2×2)-matrices with positive integer determinant n. For fixed $n>0$ let

$$g=\begin{pmatrix} m_1m_2 & 0\\ 0 & m_2\end{pmatrix} \tag{11.3}$$

be a parametrisation of such matrices. We will define a Hecke operator T_n not to a single such element but to the union of all diagonal g with determinant equal to n. This T_n can be thought of as the sum of *all* the individual T_g defined according to the formula (11.2). According to (11.1), we require the double coset decomposition into right cosets [130, Eq. (3.12.2)]

$$\bigcup_{m_1^2m_2=n} SL(2,\mathbb{Z})\begin{pmatrix} m_1m_2 & 0\\ 0 & m_2\end{pmatrix} SL(2,\mathbb{Z}) = \bigcup_{\substack{ad=n\\ 0\le b<d}} SL(2,\mathbb{Z})\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}. \tag{11.4}$$

in order to define T_n. Then to each $n>0$ we can associate a Hecke operator T_n acting on a Maass wave form $f(z)$

$$\begin{aligned}(T_nf)(z) &:= \frac{1}{\sqrt{n}}\sum_{\substack{a\ge 1\,;\,ad=n\\ 0\le b<d}} f\left(\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\cdot z\right)\\ &= \frac{1}{\sqrt{n}}\sum_{d|n}\sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right),\end{aligned} \tag{11.5}$$

which maps f to a new function $T_n f$ on the upper half plane. Here, we have slightly changed the normalisation of the operator compared to the general discussion as anticipated in remark 11.3. Note that the transformation of the argument is not in $SL(2,\mathbb{Z})$ but has determinant n. Defining the set

$$M_2(n) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a,b,d \in \mathbb{Z} \text{ with } ad = n \right\} \tag{11.6}$$

of upper triangular integer (2×2)-matrices, we can rewrite the Hecke operator also as

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{\gamma_n \in SL(2,\mathbb{Z}) \backslash M_2(n)} f(\gamma_n \cdot z). \tag{11.7}$$

The resulting function $T_n f$ is also a Maass wave form since it is (i) invariant and (ii) an eigenfunction of the Laplacian as we will now show.

(i) Invariance requires evaluating

$$(T_n f)(\gamma \cdot z) = \frac{1}{\sqrt{n}} \sum_{\gamma_n \in SL(2,\mathbb{Z}) \backslash M_2(n)} f(\gamma_n \gamma \cdot z) \tag{11.8}$$

for $\gamma \in SL(2,\mathbb{Z})$. Using

$$\gamma_n \gamma = \tilde{\gamma} \tilde{\gamma}_n \tag{11.9}$$

for some other $\tilde{\gamma} \in SL(2,\mathbb{Z})$ and $\tilde{\gamma_n} \in M_2(n)$ together with invariance of φ under $\tilde{\gamma}$ one arrives at

$$(T_n f)(\gamma \cdot z) = \frac{1}{\sqrt{n}} \sum_{\tilde{\gamma}_n \in SL(2,\mathbb{Z}) \backslash M_2(n)} f(\tilde{\gamma}_n \cdot z) = (T_n f)(z) \tag{11.10}$$

and the function $T_n f$ is $SL(2,\mathbb{Z})$ invariant for any positive n. (See also chapters 6.8 and 6.9 of [2].)

(ii) Consider the action of the Laplacian (5.38) on $T_n \varphi$. It is straight-forward to check that

$$[\Delta(T_n f)](z) = [T_n(\Delta f)](z). \tag{11.11}$$

Therefore, *the Laplacian commutes with all the Hecke operators* and if f is an eigenfunction of Δ, so is $T_n f$ and with the same eigenvalue.

Finally, we study the Fourier expansion of $T_n f$. Suppose that f has a Fourier expansion of the form (cf. (6.2))

$$f(z) = f(x+iy) = \sum_{m \in \mathbb{Z}} a_m(y) e^{2\pi i m x}. \tag{11.12}$$

Then one finds for $T_n\varphi$

$$
\begin{aligned}
(T_n f)(z) &= \frac{1}{\sqrt{n}} \sum_{d|n} \sum_{b=0}^{d-1} f\left(\frac{n}{d^2}x + \frac{b}{d} + i\frac{n}{d^2}y\right) \\
&= \frac{1}{\sqrt{n}} \sum_{d|n} \sum_{m\in\mathbb{Z}} a_m\left(\frac{n}{d^2}y\right) e^{2\pi i m n x/d^2} \sum_{b=0}^{d-1} e^{2\pi i m b/d} \\
&= \frac{1}{\sqrt{n}} \sum_{m\in\mathbb{Z}} \sum_{d|n, d|m} d\, a_m\left(\frac{n}{d^2}y\right) e^{2\pi i m n x/d^2} \\
&= \frac{1}{\sqrt{n}} \sum_{m\in\mathbb{Z}} \sum_{d|(n,m)} \frac{n}{d} a_{mn/d^2}\left(\frac{d^2}{n}y\right) e^{2\pi i m x}\,, \qquad (11.13)
\end{aligned}
$$

where we have changed the divisor sum variable from d to $\frac{n}{d}$ in the last step and have relabelled the m sum in between. The Fourier expansion of $T_n f$ is therefore

$$
(T_n f)(z) = \sum_{m\in\mathbb{Z}} \tilde{a}_m(y) e^{2\pi i m x} \quad \text{with} \quad \tilde{a}_m(y) = \frac{1}{\sqrt{n}} \sum_{d|(n,m)} \frac{n}{d} a_{mn/d^2}\left(\frac{d^2}{n}y\right)\,, \tag{11.14}
$$

where $d|(n,m)$ means that $d|n$ and $d|m$, i.e. divides the greatest common divisor of n and m which is denoted as usual by $(m,n) = \gcd(m,n)$.

11.2.2 Algebra of Hecke operators

Importantly, the Hecke operators T_n satisfy a simple algebra on the space of Maass wave forms: *they all commute*. Moreover, they satisfy the *Hecke algebra*

$$
T_m T_n = T_n T_m = \sum_{d|(m,n)} T_{mn/d^2}\,. \tag{11.15}
$$

Commutativity is manifest in this expression. To prove (11.15), one can first consider the case $(m,n) = 1$ and use the explicit definition. In the next step one can consider the case when both m and n are powers of the same prime. A proof can be found in [2, ch. 6.10] where a different normalisation is used.

11.2.3 Common eigenfunctions of T_n and Δ

Suppose $f : \mathbb{H} \to \mathbb{R}$ is an eigenfunction of all Hecke operators

$$
T_n f = c_n f \tag{11.16}
$$

for some eigenvalues c_n and at the same time a cuspidal Maass wave form with Fourier expansion

$$
f(z) = \sum_{m\neq 0} a_m(y) e^{2\pi i m x}\,. \tag{11.17}
$$

Applying T_n to f gives, with (11.14),

$$c_n f(z) = \frac{1}{\sqrt{n}} \sum_{m\neq 0} \sum_{d|(n,m)} \frac{n}{d} a_{mn/d^2}\left(\frac{d^2}{n}y\right) e^{2\pi i m x} \tag{11.18}$$

Comparing the individual Fourier modes on both sides leads to

$$c_n a_m(y) = \frac{1}{\sqrt{n}} \sum_{d|(n,m)} \frac{n}{d} a_{nm/d^2}\left(\frac{d^2}{n}y\right) \tag{11.19}$$

Setting $n = 1$ gives $c_1 a_m(y) = a_m(y)$ for all $m \neq 0$, implying $c_1 = 1$ unless f vanishes. Setting $m = 1$ implies

$$c_n a_1(y) = \sqrt{n} a_n \left(\frac{y}{n}\right). \tag{11.20}$$

If f is not constant, one has $a_1(y) \neq 0$, otherwise all the Fourier coefficients would vanish.

From solving the Laplace condition on $\Delta f = s(s-1)f$ (cf. appendix C.2) one knows that the dependence of the Fourier coefficient $a_m(y)$ on y is through the modified Bessel function as

$$a_m(y) = a_m y^{1/2} K_{s-1/2}(2\pi|m|y) \tag{11.21}$$

for some purely numerical coefficient a_m that we will now relate to the Hecke eigenvalues c_n. Rescaling f such that $a_1 = 1$ (*Hecke normalisation*) the relation (11.20) above implies that the Hecke eigenvalues equal the Fourier coefficients:

$$c_n = a_n. \tag{11.22}$$

Obtaining this simple relation was the reason for the choice of normalisation of the Hecke operator T_n. Note that the c_n are only defined for positive n but a_n for any n. The reality of f relates a_n to a_{-n}.

By virtue of the Hecke algebra (11.15) we have

$$T_m T_n f = c_m c_n f = \sum_{d|(m,n)} c_{mn/d^2} f \tag{11.23}$$

so that the Fourier coefficients of a normalized simultaneous eigenfunction satisfy

$$a_m a_n = \sum_{d|(m,n)} a_{mn/d^2}. \tag{11.24}$$

In particular, they must be *multiplicative*, i.e. for coprime m and n one has $a_m a_n = a_{mn}$. This number-theoretic property of the Fourier coefficients follows from the action of the Hecke operators and would not have been apparent from $SL(2,\mathbb{Z})$ invariance alone. Note that the constant term is not captured by these considerations.

The algebra (11.24) allows determining *all* Fourier coefficients in terms of the ones for prime numbers a_p. We note for later reference that powers of primes can be calculated recursively using the relation

$$a_{p^{k+1}} = a_{p^k} a_p - a_{p^{k-1}} \tag{11.25}$$

for $k > 1$ where again Hecke normalization $a_1 = 1$ enters. The Hecke operators T_p determine the full structure of the Hecke algebra and hence are the only relevant ones for the development of the theory. We will see soon that they fit naturally into an adelic framework.

Example 11.4: Fourier expansion of non-holomorphic Eisenstein series and Hecke algebra

We now use the Hecke algebra to rederive the Fourier expansion (1.16) of the Eisenstein series $E(s,z)$. Since the Eisenstein series $E(s,z)$ is defined as a sum over an $SL(2,\mathbb{Z})$ orbit it is easy to evaluate the Hecke operators by multiplying the acting matrices

$$\begin{aligned}(T_n E)(s,z) &= \frac{1}{2}\frac{1}{\sqrt{n}}\sum_{d|n}\sum_{b=0}^{d-1}\sum_{\gcd(p,q)=1}\left[\operatorname{Im}\left(\begin{pmatrix} n/d & b \\ 0 & d\end{pmatrix}\begin{pmatrix} * & * \\ p & q\end{pmatrix}\cdot z\right)\right]^s \\ &= \frac{1}{\sqrt{n}}\sum_{d|n}\sum_{b=0}^{d-1}\left(\frac{n}{d^2}\right)^s E(s,z) = \sum_{d|n}\left(\frac{n}{d^2}\right)^{s-1/2} E(s,z) \\ &= \underbrace{n^{s-1/2}\sigma_{1-2s}(n)}_{c_n} E(s,z)\,. \end{aligned} \tag{11.26}$$

We have used that the coset sum $B(\mathbb{Z})\backslash SL(2,\mathbb{Z})$ can be parametrised by two co-prime integers p and q and the unspecified top row corresponds to an arbitrary representative of the coset. In particular, the Eisenstein series is an eigenfunction of all Hecke operators and the relation (11.22) between the Fourier coefficients and the Hecke eigenvalues immediately implies the form (1.16) for the non-zero Fourier coefficients up to a normalization factor. The constant term is not fixed by these considerations. However, this method of deriving the Fourier modes did not require any Poisson resummation nor adelic technology.

Let us verify the relation (11.24) for the explicit example of the Eisenstein series (1.16) to check whether it is a simultaneous eigenfunction. There one has for $n > 0$

$$a_n = c_n = \sum_{d|n} d^{1-2s} n^{s-1/2} \tag{11.27}$$

where Hecke normalization was used. Let m and n be coprime, then

$$a_m a_n = \sum_{d|m}\sum_{\tilde{d}|n} d^{1-2s} m^{s-1/2} \tilde{d}^{1-2s} n^{s-1/2} = \sum_{d|mn} d^{1-2s}(mn)^{s-1/2} = a_{mn}\,. \tag{11.28}$$

The more general relation (11.24) can also be verified and the Eisenstein series is an eigenfunction of of the Hecke operators (with eigenvalues given by the Fourier coefficients).

Remark 11.5 (Hecke operators for holomorphic modular forms). For holomorphic modular forms $f : \mathbb{H} \to \mathbb{C}$ of weight k one can also define Hecke operators, see for example [2, Ch. 6]. In this case, they act by

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{bz+bd}{d^2}\right) \qquad (f \text{ holomorphic of weight } k) \tag{11.29}$$

and map holomorphic modular forms to homomorphic modular forms. Note that the normalization convention here is slightly different from the non-holomorphic case. The multiplicative law (11.15) in this case reads

$$T_m T_n = T_n T_m = \sum_{d|(m,n)} d^{k-1} T_{mn/d^2}. \qquad (\text{weight } k \text{ Hecke algebra}) \tag{11.30}$$

One can again define Hecke normalized common eigenfunctions. If f is a common eigenfunction with the Fourier expansion $f(z) = \sum_{m\geq 0} a_m q^n$ (with $q = e^{2\pi i z}$ as always), then one has again

$$T_n f = a_n f \tag{11.31}$$

when $a_1 = 1$, i.e., the modular form is Hecke normalized. In this case the Fourier coefficients satisfy

$$a_m a_n = \sum_{d|(m,n)} d^{k-1} a_{mn/d^2} \qquad (f \text{ holomorphic of weight } k) \tag{11.32}$$

because of (11.29). If f is a cusp form and its Fourier coefficients satsify the above relation (11.32) one can show that it is automatically a common eigenfunction [2, Thm. 6.15]. For non-cuspidal forms this is not guaranteed. We record the following consequence of (11.32) for later use

$$a_{p^\ell} a_p = a_{p^{\ell+1}} + p^{k-1} a_{p^{\ell-1}} \qquad (\text{Fourier coefficients of weight } k \text{ modular form}) \tag{11.33}$$

for $\ell \geq 0$. This is to be contrasted with (11.25) for non-holomorphic forms.

11.3 Hecke operators and Dirichlet series

Given the powerful applications of Hecke operators demonstrated in the previous subsections, it is natural to wonder about the action of Hecke operators in the adelic setting of automorphic forms on $SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})$. It turns out that this gives rise to an even richer structure, and provides a link to the theory of automorphic representations. In this section, we take the first steps toward such a theory by studying the Hecke operators T_p for p a prime, based on [57, 130].

Hecke's original motivation to study Hecke operators was to find a way to encode the properties of a holomorphic modular form in terms of its associated *Dirichlet series* [166, 167]. Given a weight k modular form $f(z) = \sum_{n\geq 0} a_n(f) q^n$ (with $q = e^{2\pi i z}$) one may form the series

$$L(s,f) = \sum_{n\geq 1} \frac{a_n(f)}{n^s} = \prod_{p<\infty} \sum_{\ell \geq 0} \frac{a_{p^\ell}(f)}{p^{\ell s}}, \tag{11.34}$$

which is called the Dirichlet series attached to f. The rewriting in the second step is the application of prime factorisation under the assumption of absolute convergence of the L-series. In the special case when the Fourier coefficients $a_m(f)$ are *completely multipliciative*, i.e. satisfy $a_m a_n = a_{mn}$ for any $m, n \in \mathbb{Z}$, then the L-function leads to the following Euler product via geometric series

$$L(s, f) = \prod_{p<\infty} \frac{1}{1 - a_p(f)p^{-s}}, \qquad a_m(f) \text{ completely multiplicative.} \tag{11.35}$$

This is called a degree 1 Euler product since the denominator contains at most the power p^{-s}. The prime example of a degree 1 Euler product is the Riemann zeta function $\zeta(s) = \prod_{p<\infty}(1-p^{-s})^{-1}$, corresponding to $a_m = 1$ for all $m \geq 1$, which, however, is not associated with a holomorphic modular form on $SL(2, \mathbb{R})$ but rather with $GL(1, \mathbb{A})$ as was explained in section 3.7.

Hecke showed that whenever the Fourier coefficients $a_m(f)$ are multiplicative according to (11.33) then the Dirichlet series can be written as an Euler product

$$L(s, f) = \prod_{p<\infty} L_p(s, f) = \prod_{p<\infty} \frac{1}{1 - a_p(f)p^{-s} + p^{k-1-2s}}. \tag{11.36}$$

The derivation of this formula is as follows. Let $L_p(s, f)$ be a factor in the Euler product as above. Then (11.33) implies

$$\begin{aligned} L_p(s, f) = \sum_{\ell \geq 0} \frac{a_{p^\ell}}{p^{\ell s}} &= \frac{1}{a_p}\left[p^s \sum_{\ell \geq 0} \frac{a_{p^{\ell+1}}}{p^{(\ell+1)s}} + p^{k-1-s} \sum_{\ell \geq 0} \frac{a_{p^{\ell-1}}}{p^{(\ell-1)s}}\right] \\ &= \frac{1}{a_p}\left[p^s\left(L_p(s, f) - 1\right) + p^{k-1-s} L_p(s, f)\right], \end{aligned} \tag{11.37}$$

which yields (11.36) after solving for $L_p(s, f)$. The series $L(s, f)$ is also called the *L-function* of f and $L_p(s, f)$ the *local L-factor*. The L-function $L(s, f)$ in (11.36) is of degree 2, due to the factor p^{-2s}.

Remark 11.6. In the case of non-holomorphic automorphic forms f one can go through the same derivation of an L-function. Using the normalization of Hecke operators defined in (11.5) and the Fourier coefficients a_m defined in (11.21) one obtains an L-function for a common eigenfunction f in Hecke normalization of the form

$$L(s, f) = \prod_{p<\infty} \frac{1}{1 - a_p p^{-s} + p^{-2s}}. \tag{11.38}$$

The shifted exponent on the last term in the denominator is due to the normalization of the Hecke operators. We assume for simplicity that f is even, i.e., $f(-z) = f(z)$. Then one can define a *completed L-function* via

$$L^\star(s, f) = \pi^{-s}\Gamma\left(\frac{2s + 2\nu - 1}{4}\right)\Gamma\left(\frac{2s - 2\nu + 1}{4}\right)L(s, f) \tag{11.39}$$

where ν is the eigenvalue under the Laplacian $\Delta f = \nu(\nu-1)f$. The completed L-function satisfies the simple functional relation

$$L^\star(s,f) = L^\star(1-s,f) \tag{11.40}$$

For a proof of this and extensions to odd Maass forms see [130, Prop. 3.13.5]. One should think of the normalizing factors in (11.39) as arising from the archimedean place $p=\infty$ and the completed L-function as a global one.

The L-function (11.36) attached to a modular form f therefore characterizes whether or not the Fourier coefficients exhibit a multiplicative behaviour, something which is certainly not guaranteed. When does this happen? It turns out that the Fourier coefficients of a modular form f are multiplicative if and only if f is a *Hecke eigenform*, i.e., an eigenfunction of the entire ring of Hecke operators T_n [2, Thm 6.15]. As was emphasised above, the ring of Hecke operators is generated by the T_p for p prime and we will now focus on these.

Remark 11.7. Weil [319, 320] has resolved the problem of generalising the L-function to automorphic forms for congruence subgroups $\Gamma_0(N)$ of $SL(2,\mathbb{Z})$. In this case one needs to twist the L-function by a Dirichlet character.

11.4 The spherical Hecke algebra

Recall from section 5.1 that to each modular form $f(z)$ on the upper-half plane $\mathbb{H}$ we have a corresponding automorphic form $\varphi_f(g)$ on $SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A})$. We now want to find out how the action of the Hecke operator T_p lifts to the space of automorphic forms $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$.

As for the classical case in section 11.2, the Hecke operators in the adelic context are associated with double cosets of matrices of determinant different from 1 and hence outside of $SL(2,\mathbb{Q}_p)$. For this reason, we consider the group $GL(2,\mathbb{Q}_p)$. The *convolution algebra* on $GL(2,\mathbb{Q}_p)$. This algebra is given by the space of locally constant $\mathbb{C}$-valued functions on $GL(2,\mathbb{Q}_p)$ with the (commutative) product given by convolution:

$$(\Phi_1 \star \Phi_2)(g) = \int_{GL(2,\mathbb{Q}_p)} \Phi_1(gh)\Phi_2(h^{-1})dh = \int_{GL(2,\mathbb{Q}_p)} \Phi_1(h)\Phi_2(h^{-1}g)dh, \tag{11.41}$$

where dh denotes the bi-invariant Haar measure on the uni-modular group $GL(2,\mathbb{Q}_p)$. Convolution turns the space of such functions into a ring, called the *(local) Hecke algebra*, commonly denoted by $\mathcal{H}(GL(2,\mathbb{Q}_p))$ or simply $\mathcal{H}_p$ for short. Although it is a ring it has no unit. To see the connection with the classical Hecke algebra generated by the T_p's, we now restrict to *bi-invariant functions* with respect to the maximal compact subgroup $K_p = GL(2,\mathbb{Z}_p)$, i.e. we consider functions in $\mathcal{H}_p$ that satisfy

$$\Phi(kgk') = \Phi(g), \qquad k,k' \in K_p,\ g \in GL(2,\mathbb{Q}_p). \tag{11.42}$$

We then obtain the *spherical Hecke algebra* $\mathcal{H}(GL(2,\mathbb{Q}_p))^{K_p}$ of K_p bi-invariant functions, which we denote by $\mathcal{H}_p^\circ$. It is a central result that $\mathcal{H}_p^\circ$ forms a commutative ring (see, e.g. [57]). If we fix the Haar measure on $GL(2,\mathbb{Q}_p)$ such that K_p has unit volume, then $\mathcal{H}_p^\circ$ also has a unit given by the characteristic function on K_p:

$$\mathrm{char}_{K_p}(g) = \begin{cases} 1 & g \in K_p, \\ 0 & \text{otherwise.} \end{cases} \tag{11.43}$$

To see this we calculate the convolution product of the characteristic function with any $\Phi \in \mathcal{H}_p^\circ$:

$$(\Phi \star \mathrm{char}_{K_p})(g) = \int_{SL(2,\mathbb{Q}_p)} \Phi(gh)\mathrm{char}_{K_p}(h^{-1})dh = \int_{K_p} \Phi(gh)dh = \Phi(g), \tag{11.44}$$

where in the last step we used that f is bi-invariant under K_p and K_p has unit volume. One says that the spherical Hecke algebra is *idempotented*.

The spherical Hecke algebra $\mathcal{H}_p^\circ$ acts on the space of K_p-spherical functions on $GL(2,\mathbb{Q}_p)$ via right-translation. For any $\Phi \in \mathcal{H}_p^\circ$ and K_p-spherical function φ on $GL(2,\mathbb{Q}_p)$ we define a new function on $GL(2,\mathbb{Q}_p)$ by

$$(\pi(\Phi)\varphi)(g) = \int\limits_{GL(2,\mathbb{Q}_p)} \Phi(h)\varphi(gh)dh. \tag{11.45}$$

One can check easily that this maps the right-regular action of $GL(2,\mathbb{Q}_p)$ on functions on $GL(2,\mathbb{Q}_p)$ to a representation of the spherical Hecke algebra (with convolution product (11.41)) according to

$$\pi\left(\Phi_1 \star \Phi_2\right)\varphi = \pi\left(\Phi_1\right)\left(\pi\left(\Phi_1\right)\varphi\right). \tag{11.46}$$

The space of K_p-spherical functions is therefore a representation of the spherical Hecke algebra $\mathcal{H}_p^\circ$.

Remark 11.8. By taking the restricted direct product (with respect to K_p, see section 3.5) over all the local algebras $\mathcal{H}_p^\circ$ we obtain the *global, or adelic, spherical Hecke algebra*

$$\mathcal{H}^\circ = \bigotimes_{p\le\infty}{}' \mathcal{H}_p^\circ. \tag{11.47}$$

For $p = \infty$, the spherical Hecke algebra $\mathcal{H}_\infty^\circ$ is given by $K(\mathbb{R})$-bi-finite distributions supported on $K(\mathbb{R})$ [68, Lecture 3.1]. This includes the invariant differential operators on $G(\mathbb{R})$ lying in the universal enveloping algebra $U(\mathfrak{g})$. The global Hecke algebra $\mathcal{H}^\circ$ acts on $\mathcal{A}(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}))$ by the same formula (11.45). Our main interest in the following lies with the spherical Hecke algebra $\mathcal{H}_p^\circ$ at the finite primes $p < \infty$.

We now investigate the structure of the (local) spherical Hecke algebra $\mathcal{H}_p^\circ$ in more detail. More explicitly, we define the elements $\mathbb{T}_p$ and $\mathbb{R}_p \in \mathcal{H}_p^\circ$ by the K_p-bi-invariant functions

$$\mathbb{T}_p = \mathrm{char}_{K_p\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)K_p}, \qquad \mathbb{R}_p = \mathrm{char}_{K_p\left(\begin{smallmatrix} p & \\ & p\end{smallmatrix}\right)K_p}. \tag{11.48}$$

It is an important result that $\mathbb{T}_p$, $\mathbb{R}_p$ and $\mathbb{R}_p^{-1}$ together generate the spherical Hecke algebra $\mathcal{H}_p^\circ$. A proof of this statement can be found for example in [57, Prop. 4.6.5].

On functions $\varphi : GL(2,\mathbb{Q}_p) \to \mathbb{C}$ they act according to (11.45). To ease notation we shall simply continue to call them $\mathbb{T}_p$ and $\mathbb{R}_p$ also when acting on spherical functions.

$$(\mathbb{T}_p\varphi)(g) = \int_{K_p\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)K_p} \varphi(gh)dh, \tag{11.49}$$

$$(\mathbb{R}_p\varphi)(g) = \int_{K_p\left(\begin{smallmatrix} p & \\ & p\end{smallmatrix}\right)K_p} \varphi(gh)dh. \tag{11.50}$$

Even though written in terms of integrals, they act on functions on $GL(2,\mathbb{Q}_p)$ by finite sums after performing a decomposition of the double cosets into a finite union of left cosets, similar to (11.4). This decomposition for the operator $\mathbb{T}_p$ is [57, Prop. 4.6.4]

$$K_p\begin{pmatrix} p & \\ & 1\end{pmatrix}K_p = \begin{pmatrix} 1 & \\ & p\end{pmatrix}K_p \cup \bigcup_{i=0}^{p-1}\begin{pmatrix} p & i \\ & 1\end{pmatrix}K_p, \tag{11.51}$$

such that for K_p-spherical φ

$$(\mathbb{T}_p\varphi)(g) = \varphi\left(g\left(\begin{smallmatrix} 1 & \\ & p\end{smallmatrix}\right)\right) + \sum_{i=0}^{p-1}\varphi\left(g\left(\begin{smallmatrix} p & i \\ & 1\end{smallmatrix}\right)\right). \tag{11.52}$$

The connection with the classical Hecke operators now follows from the fact that if $f : \mathbb{H} \to \mathbb{R}$ is a Maass wave form with eigenvalue a_p under T_p, then the associated adelic automorphic form $\varphi_f \in \mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ defined in section 5.1.4 is an eigenform under $\mathbb{T}_p$ with the same eigenvalue, up to a (convention-dependent) factor:

$$T_p f = a_p f \quad\longleftrightarrow\quad (\mathbb{T}_p\varphi_f)(g) = p^{1/2}a_p\varphi_f. \tag{11.53}$$

This will be verified for Eisenstein series in example 11.10 below but it is valid in general.

Remark 11.9. There is also a classical Hecke operator R_p acting on f which lifts to $\mathbb{R}_p$, but we shall not discuss this further here (see [57] for more details).

11.5 Spherical Hecke algebras and automorphic representations

This and the following sections make use of the theory of automorphic representations which was introduced in section 5.4.

Recall that $GL(2,\mathbb{A})$ acts by right-translation on $\mathcal{A}(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}))$, such that at the archimedean places it has the form of a $(\mathfrak{g}_\infty, K_\infty)$-module, while the finite places carry a representation of $GL(2,\mathbb{A}_f)$. The irreducible constituents (π, V) in the decomposition of $\mathcal{A}(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}))$ are called automorphic representations. But we have also just seen that $\mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ carries an action of the adelic spherical Hecke algebra

$\mathcal{H}^\circ$. A natural question is then: Is there a relation between these representations? Not surprisingly, the answer is yes, and we shall now sketch how to see this.

Suppose that $(\pi, V) = \otimes_{p\leq\infty}(\pi_p, V_p)$ is an unramified automorphic representation (see definition 5.24); this implies that V_p contains a spherical vector f_p° (unique up to multiplication by a complex scalar, see e.g. [57]), satisfying $\mathsf{f}_p^\circ(k) = 1$ for all $k \in K_p$. The spherical vector therefore spans the complex one-dimensional space $V_p^{K_p}$ consisting of K_p-invariant vectors in V_p.

We can for example take π_p to be the local induced representation with module

$$V_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)} \delta^{1/2}\mu, \tag{11.54}$$

where δ is the modulus character of the Borel subgroup and the quasi-character $\mu : B(\mathbb{Z}_p)\backslash B(\mathbb{Q}_p) \to \mathbb{C}^\star$ is defined by

$$\mu(b) = \mu(na) = \mu(a), \qquad n \in N(\mathbb{Q}_p), \quad a \in A(\mathbb{Q}_p). \tag{11.55}$$

In the notation of (5.94) we have therefore $\chi(g) = \mu(g)\delta^{1/2}(g)$. The explicit separation of the modulus character in (11.54) turns out to be convenient for the forthcoming analysis, and also facilitates comparison with the literature. In the notation of that section we would write $\chi(a) = a^{\lambda+\rho}$, so that $\delta^{1/2}(a) = a^\rho$ and $\mu(a) = a^\lambda$, where ρ is the Weyl vector of the Lie algebra $\mathfrak{gl}(2)$ and λ is a (complex) weight.

The spherical vector $\mathsf{f}_p^\circ \in V_p$ is the standard section defined by the extension of $\delta^{1/2}\mu$ to all of $GL(2,\mathbb{Q}_p)$ via the Iwasawa decomposition (see also section 5.5):

$$\mathsf{f}_p^\circ(g) = \mathsf{f}_p^\circ(nak) = \delta^{1/2}(a)\mu(a). \tag{11.56}$$

The local spherical Hecke algebra $\mathcal{H}_p^\circ$ acts on V_p via the action (11.45). By construction this action preserves the one-dimensional space $V_p^{K_p}$ of spherical vectors: indeed for any $\Phi \in \mathcal{H}_p^\circ$ we have for all $k \in K_p$

$$(\pi(\Phi)\mathsf{f}_p^\circ)(gk) = \int\limits_{GL(2,\mathbb{Q}_p)} \mathsf{f}_p^\circ(gkh)\Phi(h)dh = \int\limits_{GL(2,\mathbb{Q}_p)} \mathsf{f}_p^\circ(ghk)\Phi(khk^{-1})dh = (\pi(\Phi)\mathsf{f}_p^\circ)(g), \tag{11.57}$$

since f_p° is spherical and Φ K_p-bi-invariant. This implies that $V_p^{K_p}$ furnishes a representation of $\mathcal{H}_p^\circ$. Since the spherical vector f_p° spans the one-dimensional space $V_p^{K_p}$, we conclude that the action of $\mathcal{H}_p^\circ$ must give back f_p°, up to a complex scalar:

$$(\pi(\Phi)\mathsf{f}_p^\circ)(g) = \lambda_\mu(\Phi)\mathsf{f}_p^\circ(g), \tag{11.58}$$

where the eigenvalue $\lambda_\mu(\Phi)$ determines a (quasi-)character of the spherical Hecke algebra

$$\lambda_\mu : \mathcal{H}_p^\circ \longrightarrow \mathbb{C}^\times. \tag{11.59}$$

As we have indicated, this character depends on the choice of μ in (11.54).

To find an explicit description of the characters λ_μ we shall work out the action of the Hecke operator $\mathbb{T}_p$ defined in (11.48). To proceed, we parametrise the Cartan torus $A(\mathbb{Q}_p) \subset GL(2, \mathbb{Q}_p)$ by

$$a = \begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}, \qquad v_1, v_2 \in \mathbb{Q}_p^\times. \tag{11.60}$$

We can further describe the unramified character μ explicitly by

$$\mu(a) = \mu\left(\begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix}\right) = |v_1|_p^{s_1}|v_2|_p^{s_2} \tag{11.61}$$

where $s_1, s_2 \in \mathbb{C}$. Note that the parametrisation in terms of s_1 and s_2 differs from the one used in chapter 7. The reason here is to simplify some of the following expressions. The corresponding value of the modulus character on $B \subset GL(2, \mathbb{Q}_p)$ is

$$\delta\begin{pmatrix} v_1 & \\ & v_2 \end{pmatrix} = \left|\frac{v_1}{v_2}\right|_p. \tag{11.62}$$

This implies that the representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{SL(2,\mathbb{Q}_p)}\delta^{1/2}\mu$ is in fact completely determined by

$$\alpha_p \equiv p^{-s_1}, \qquad \beta_p \equiv p^{-s_2}. \tag{11.63}$$

Now we wish to compute

$$(\mathbb{T}_p \mathsf{f}_p^\circ)(g) = \int_{K_p\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)K_p} \mathsf{f}_p^\circ(gh)dh. \tag{11.64}$$

Since we know that f_p° is an eigenfunction and is normalized so that $\mathsf{f}_p^\circ(1) = 1$, it suffices to evaluate the action at the identity $1 \in GL(2, \mathbb{Q}_p)$, which then, via (11.58), directly corresponds to the value of the character λ_μ:

$$\lambda_\mu(\mathbb{T}_p) = \int_{K_p\left(\begin{smallmatrix} p & \\ & 1\end{smallmatrix}\right)K_p} \mathsf{f}_p^\circ(h)dh. \tag{11.65}$$

To evaluate this we decompose the double coset space as in (11.51). Plugging the decomposition into the integral (11.65) yields

$$\begin{aligned}
\lambda_\mu(\mathbb{T}_p) &= \sum_{i=0}^{p-1} \mathsf{f}_p^\circ\left(\begin{pmatrix} p & i \\ & 1 \end{pmatrix}\right) + \mathsf{f}_p^\circ\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix}\right) \\
&= \underbrace{(\delta^{1/2}\mu)\left(\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right) + \cdots + (\delta^{1/2}\mu)\left(\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right)}_{p \text{ terms}} + (\delta^{1/2}\mu)\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix}\right) \\
&= pp^{-1/2}p^{-s_1} + p^{1/2}p^{-s_2} \\
&= p^{1/2}(\alpha_p + \beta_p).
\end{aligned} \tag{11.66}$$

By a similar analysis one also shows that

$$\lambda_\mu(\mathbb{R}_p) = \int_{K_p \begin{pmatrix} p & \\ & p \end{pmatrix} K_p} \mathsf{f}_p^\circ(h) dh = \alpha_p \beta_p, \tag{11.67}$$

$$\lambda_\mu(\mathbb{R}_p^{-1}) = \int_{K_p \begin{pmatrix} p^{-1} & \\ & p^{-1} \end{pmatrix} K_p} \mathsf{f}_p^\circ(h) dh = \alpha_p^{-1} \beta_p^{-1}. \tag{11.68}$$

These results imply that the one-dimensional representation λ_μ of the Hecke algebra $\mathcal{H}_p^\circ$ acting on $V_p^{K_p}$ completely determines the unramified character μ, and thereby the automorphic representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)} \delta^{1/2}\mu$. It is quite remarkable that this infinite-dimensional automorphic representation can be encoded in the finite-dimensional representations of $\mathcal{H}_p^\circ$. In the next subsection we shall further investigate the consequences of this fact.

Example 11.10: Classical and p-adic Hecke operators for Eisenstein series on $SL(2)$

In this example, we come back to the mentioned relation (11.53) between the Hecke eigenvalue of a non-holomorphic Eisenstein series $E(s,z)$ on $\mathbb{H}$ under the classical T_p calculated in (11.26) and the action of $\mathbb{T}_p \in \mathcal{H}_p^\circ$ on the associated adelic Eisenstein series $E(\chi_s, g) \in \mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ defined in (5.146).

To begin with, we need to relate the parameters s_1 and s_2 of the $GL(2,\mathbb{Q}_p)$ principal series (11.61) to the parameter s occuring in $E(s,z)$ and $E(\chi_s,g)$ via (7.10). Elements of the Cartan torus in $SL(2,\mathbb{Q}_p)$ are of the form $a = \mathrm{diag}(v, v^{-1})$, so that (11.61) yields

$$\mu\left(\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}\right) = |v|_p^{s_1 - s_2}. \tag{11.69}$$

This has to be contrasted with (recall the general $\chi = \delta^{1/2}\mu$)

$$(\delta^{-1/2}\chi_s)\left(\begin{pmatrix} v & \\ & v^{-1} \end{pmatrix}\right) = |v|_p^{2s-1} \tag{11.70}$$

that follows from (7.10). For symmetry reason one therefore deduces

$$s_1 = -s_2 = s - \frac{1}{2}. \tag{11.71}$$

Plugging this into α_p and β_p in (11.63) one therefore finds from (11.66) that

$$\lambda_\mu(\mathbb{T}_p) = p^{1/2}(\alpha_p + \beta_p) = p^{1/2}\left(p^{-s+1/2} + p^{s-1/2}\right). \tag{11.72}$$

This is the eigenvalue of the adelic Eisenstein series under $\mathbb{T}_p$. From (11.26) one finds that for the classical Hecke operator T_p acting on the classical $E(s,z)$ the eigenvalues is

$$(T_p E)(s,z) = p^{s-1/2}\left(1 + p^{1-2s}\right) E(s,z) = \left(p^{s-1/2} + p^{-s+1/2}\right) E(s,z). \tag{11.73}$$

This confirms the claimed relation (11.53) that

$$\mathbb{T}_p \sim p^{1/2} T_p, \tag{11.74}$$

where we reiterate that the pre-factor is convention dependent.

11.6 The Satake isomorphism

We recall from section 4.1.1 that the Weyl group $\mathcal{W} = \mathcal{W}(\mathfrak{g})$ acts on the Cartan torus A, and consequently it also acts on the characters μ via

$$w\mu(a) = \mu(w^{-1}aw), \qquad w \in \mathcal{W}. \tag{11.75}$$

Under this action the unramified principal series remains invariant

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)} \delta^{1/2} w(\mu) \cong \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)} \delta^{1/2} \mu. \tag{11.76}$$

This is what the functional relation (8.44) for Eisenstein series expresses.

In terms of the parametrisation of μ by the complex numbers (α_p, β_p) the Weyl group $\mathcal{W} = \mathbb{Z}/2\mathbb{Z}$ simply acts by $(\alpha_p, \beta_p) \mapsto (\beta_p, \alpha_p)$. Now notice that the characters $\lambda_\mu(\mathbb{T}_p) = p^{1/2}(\alpha_p + \beta_p)$, $\lambda_\mu(\mathbb{R}_p) = \alpha_p\beta_p$ and $\lambda_\mu(\mathbb{R}_p^{-1}) = \alpha_p^{-1}\beta_p^{-1}$ are Weyl-invariant. Hence, at the level of the representations of the spherical Hecke algebra we have

$$\lambda_{w\mu} = \lambda_\mu, \qquad \forall\, w \in \mathcal{W}. \tag{11.77}$$

As a consequence, the image of the homomorphism $\mathcal{H}_p^\circ \to \mathbb{C}$ lies in the polynomial $\mathbb{C}$-ring of Weyl-invariants

$$\mathbb{C}[\alpha_p^{\pm 1}, \beta_p^{\pm 1}]^{\mathcal{W}} \cong \mathbb{C}[\alpha_p + \beta_p, \alpha_p\beta_p, \alpha_p^{-1}\beta_p^{-1}]. \tag{11.78}$$

It is an important result of Satake [278] that this homomorphism in fact yields an isomorphism between the spherical Hecke algebra and the ring of Weyl-invariant polynomials in (α_p, β_p):

$$\mathcal{H}_p^\circ \cong \mathbb{C}[\alpha_p + \beta_p, \alpha_p\beta_p, \alpha_p^{-1}\beta_p^{-1}]. \tag{11.79}$$

See [153] for a nice survey of the *Satake isomorphism* and its applications.

The key step in Satake's analysis was to introduce the *Satake transform*

$$\mathcal{S} \,:\, \mathcal{H}_p^\circ(GL(2,\mathbb{Q}_p)) \longrightarrow \mathcal{H}_p^\circ(A(\mathbb{Q}_p)) \tag{11.80}$$

from the spherical Hecke algebra of $GL(2,\mathbb{Q}_p)$ to the spherical Hecke algebra of the Cartan torus $A(\mathbb{Q}_p)$. The Satake transform is defined by

$$(\mathcal{S}\Phi)(a) = \delta^{-1/2}(a) \int_{N(\mathbb{Q}_p)} \Phi(na)dn, \qquad \Phi \in \mathcal{H}_p^\circ(GL(2,\mathbb{Q}_p)), \tag{11.81}$$

where $N(\mathbb{Q}_p)$ is the unipotent radical of the Borel subgroup $B(\mathbb{Q}_p) \subset GL(2,\mathbb{Q}_p)$. Satake then proved that the image of $\mathcal{S}$ lies in $\mathcal{H}_p^\circ(A(\mathbb{Q}_p))^{\mathcal{W}}$, the Weyl invariant elements of the spherical Hecke algebra of $A(\mathbb{Q}_p)$. To see the connection with our previous analysis, we consider again the formula for the eigenvalues λ_μ:

$$\lambda_\mu(\Phi) = \int_{GL(2,\mathbb{Q}_p)} \mathsf{f}_p^\circ(h)\Phi(h)dh, \tag{11.82}$$

which is (11.58) evaluated at the identity $g = 1$. We shall now manipulate this expression in order to elucidate the role played by the Satake transform. To the best of our knowledge this calculation was first outlined by Langlands in [220], but here we follow the more detailed exposition by Garrett [112]. We begin by splitting the integral according to the Iwasawa decomposition $GL(2,\mathbb{Q}_p) = B(\mathbb{Q}_p)K_p$:

$$\int_{GL(2,\mathbb{Q}_p)} \mathsf{f}_p^\circ(h)\Phi(h)dh = \int_{B(\mathbb{Q}_p)}\int_{K_p} \mathsf{f}_p^\circ(b^{-1}k)\Phi(b^{-1}k)dbdk, \tag{11.83}$$

where dk and db are right-invariant Haar measures on K_p and $B(\mathbb{Q}_p)$, respectively. Next, we make the change of variables $b \to b^{-1}$, which brings out a factor of δ^{-1} from the measure:

$$\int_{B(\mathbb{Q}_p)}\int_{K_p} \mathsf{f}_p^\circ(bk)\Phi(bk)\delta(b)^{-1}dbdk. \tag{11.84}$$

Using right K_p-invariance of f_p° and Φ as well as $\int_{K_p} dk = 1$ this further simplifies to

$$\int_{B(\mathbb{Q}_p)} \mathsf{f}_p^\circ(b)\Phi(b)\delta(b)^{-1}db = \int_{B(\mathbb{Q}_p)} \Phi(b)(\delta^{-1/2}\mu)(b)db, \tag{11.85}$$

where we used (11.56). To proceed we split the integral according to $B(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)$ and use the fact that $\delta^{-1/2}\mu$ is trivial on $N(\mathbb{Q}_p)$ acting on the left:

$$\int_{A(\mathbb{Q}_p)}\int_{N(\mathbb{Q}_p)} \Phi(na)(\delta^{-1/2}\mu)(na)dnda = \int_{A(\mathbb{Q}_p)}\int_{N(\mathbb{Q}_p)} \Phi(na)(\delta^{-1/2})\mu(a)dnda. \tag{11.86}$$

After reshuffling the integrand we finally arrive at the result

$$\begin{aligned} \lambda_\mu(\Phi) &= \int_{A(\mathbb{Q}_p)} \mu(a)\left[\delta^{-1/2}(a)\int_{N(\mathbb{Q}_p)} \Phi(na)dn\right]da \\ &= \int_{A(\mathbb{Q}_p)} \mu(a)(\mathcal{S}\Phi)(a)da. \end{aligned} \tag{11.87}$$

This clearly shows that the Satake transform lies at the heart of the relation between the unramified automorphic representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(2,\mathbb{Q}_p)}\delta^{1/2}\mu$ and the one-dimensional representation λ_μ of the spherical Hecke algebra $\mathcal{H}_p^\circ$, the essence of which is the Satake isomorphism (11.79).

11.7 The L-group and generalisation to $GL(n)$

It is illuminating to assemble the parameters (α_p, β_p) in a matrix

$$A_{\pi_p} = \begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix}. \tag{11.88}$$

This matrix belongs to $GL(2,\mathbb{C})$ and since conjugation $A_{\pi_p} \mapsto wA_{\pi_p}w^{-1}$ by an element $w \in \mathcal{W}$ will not alter the result (11.79) we find that the representation π_p determines a (semi-simple) *conjugacy class* $[A_{\pi_p}] \subset GL(2,\mathbb{C})$. This conjugacy class is called the *Satake parameter* of the local representation π_p.

The conclusion of the discussion in this and the previous sections is that *unramified automorphic representations* π_p *of* $GL(2,\mathbb{Q}_p)$ *are in bijection with semi-simple conjugacy classes* $[A_{\pi_p}] \subset GL(2,\mathbb{C})$. The appearance of $GL(2,\mathbb{C})$ in the context of local representations of $GL(2,\mathbb{Q}_p)$ may seem surprising, but is in fact a simple instance of a more general phenomenon envisioned by Langlands [217]. Langlands suggested that to each reductive algebraic group G over a number field $\mathbb{F}$ there exists an associated complex group ${}^LG(\mathbb{C})$, called the *L-group*, or *Langlands dual group*. We have already encountered the group ${}^LG$ briefly in section 9.7 in our discussion of the Casselman–Shalika formula but we will now put this group into a more general context.

A precise definition of ${}^LG$ can be found in [26]; we here only recall the salient features. For simple groups G the root system of ${}^LG$ is obtained from that of G by interchanging the short and long roots. In other words, the co-weight lattice $\Lambda^\vee$ of the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ is identified with the weight lattice ${}^L\Lambda$ of the dual Lie algebra ${}^L\mathfrak{g} = \mathrm{Lie}\, {}^LG$. This is captured by the isomorphism

$$\mathrm{Hom}({}^LA, U(1)) \cong \mathrm{Hom}(U(1), A), \tag{11.89}$$

between the lattice of characters on ${}^LA$ and the lattice of co-characters on A. For example, in the case of $G = GL(n,\mathbb{Q}_p)$ the L-group is $GL(n,\mathbb{C})$, and for $G = SL(n,\mathbb{Q}_p)$ we have ${}^LG = PGL(n,\mathbb{C})$. The duality is even more drastic in the case when $G = Sp(n)$ we have ${}^LG = SO(2n+1)$. See also [56, 63] for details.

Remark 11.11. The group ${}^LG$ we have introduced here is sometimes called the *connected L-group* in order to distinguish it from the L-group in the more general context of field extensions. If one considers a finite field extension $\mathbb{E}$ of $\mathbb{F} = \mathbb{Q}_p$ then the L-group ${}^LG$ is defined with the inclusion of (finite) Galois group $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ of the field extension. This more general viewpoint is relevant for the global Langlands conjectures and will be discussed in section 12.5.

The Satake parameter A_{π_p} associated with the automorphic representation π_p should thus be viewed as an element of the Cartan torus ${}^LA(\mathbb{C}) \subset GL(2,\mathbb{C})$ dual to the original Cartan torus $A(\mathbb{Q}_p)$. In fact, this holds more generally for any (split) reductive algebraic group G. From this perspective, one gets the following reformulation of the Satake isomorphism (adapted from [63]):

Theorem 11.12 (reformulated Satake isomorphism). *There is a natural bijection between the Weyl-invariant homomorphism* $\mathcal{H}_p^\circ(G) \to \mathbb{C}$ *and semi-simple* ${}^LG(\mathbb{C})$*-conjugacy classes in the dual torus* ${}^LA(\mathbb{C})$*.*

Remark 11.13. The Satake parameter $A_{\pi_p} \in {}^LA(\mathbb{C})$ already appeared in section 9.7 where it was denoted by a_λ where λ parametrises an element of the principal series representation of $G(\mathbb{Q}_p)$ which is here denoted abstractly by π_p.

Let us briefly discuss some details on the generalisation of our analysis to $G = GL(n, \mathbb{Q}_p)$. We take π_p to be the unramified principal series with module

$$V_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(n,\mathbb{Q}_p)} \delta^{1/2}\mu, \tag{11.90}$$

where the inducing character is a straightforward generalisation of (11.61):

$$\mu(a) = \mu \begin{pmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{pmatrix} = \prod_{i=1}^{n} |v_i|_p^{s_i}. \tag{11.91}$$

As before, this representation is determined by the n complex numbers:

$$\alpha_i := p^{-s_i}, \qquad i = 1, \dots, n. \tag{11.92}$$

(Note that the α_i here are for fixed prime p that we do not indicate explicitly unlike in (11.88).) Associated with the representation π_p we then have the Satake parameter

$$A_{\pi_p} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \in {}^L A(\mathbb{C}) \subset GL(n, \mathbb{C}) = {}^L GL(n, \mathbb{Q}_p), \tag{11.93}$$

on which the Weyl group $\mathcal{W}$ acts by permuting the α_i's. The generators of the spherical Hecke algebra $\mathcal{H}_p^\circ(G)$ act on elements $\varphi \in V_p$ by (11.45), *viz.*

$$\begin{aligned} (\Phi_i\varphi)(g) &= \int_{GL(n,\mathbb{Q}_p)} \varphi(gh)\mathrm{char}_{K_p\tau_iK_p}(h)dh \\ &= \int_{K_p\tau_iK_p} \varphi(gh)dh, \end{aligned} \tag{11.94}$$

where we defined [56]

$$\tau_i = \begin{pmatrix} p\mathbb{1}_i & \\ & \mathbb{1}_{n-i} \end{pmatrix}, \tag{11.95}$$

with $\mathbb{1}_r$ the $r \times r$ identity matrix. We use the convention that for $i = n$ the double coset is $K_p(p\mathbb{1}_n)K_p$. Thus, in the special case of $n = 2$ the definition (11.94) reduces to the generators in section 11.3, i.e. $\Phi_1 = \mathbb{T}_p$ and $\Phi_2 = \mathbb{R}_p$. Tamagawa has shown [303] that the operators $\Phi_1, \dots, \Phi_n$ together with Φ_n^{-1} (which is the only invertible Φ_i) generate the spherical Hecke algebra $\mathcal{H}_p^\circ$ of $GL(n, \mathbb{Q}_p)$.

As before, the one-dimensional space $V_p^{K_p} = \mathbb{C}\cdot \mathsf{f}_p^\circ$ of K_p-invariant vectors in V_p furnishes a representation of the spherical Hecke algebra, such that for any $\Phi \in \mathcal{H}_p^\circ$ and any $v^\circ \in V_p^{K_p}$ one has

$$\pi(\Phi)v^\circ = \lambda_\mu(\Phi)v^\circ, \tag{11.96}$$

where $\lambda_\mu : \mathcal{H}_p^\circ \to \mathbb{C}^\times$ is a (quasi-)character. To evaluate the eigenvalue $\lambda_\mu(\Phi)$ on all the generators Φ_i we must decompose the double cosets in (11.94). The result can be written as follows using the finite Cartan decomposition (see for instance [56] for a nice and explicit proof)

$$K_p\tau_i K_p = \bigcup_j \beta_{i,j} K_p, \tag{11.97}$$

where the matrices $\beta_{,j}$ are all integral and upper-triangular with diagonal entries are of the form p^η, where $\eta \in \{1, \dots, n\}$ and j ranges over some finite set. These generalise the matrices on the first line of (11.66) and similarly to that calculation we must evaluate the spherical vector f_p° on all $\beta_{i,j}$. Bump shows that this takes the form [56]

$$\mathsf{f}_p^\circ(\beta_{i,j}) = (\delta^{1/2}\mu)(\beta_{i,j}) = p^{-\frac{i(n+1)}{2}} \prod_{\ell=1}^{i} p^{\eta_\ell}\alpha_{\eta_\ell}, \tag{11.98}$$

where $\eta_\ell \in \{1, \dots, n\}$ are determined by j and ordered such that $\eta_1 < \eta_2 < \cdots < \eta_i$. The α_{η_ℓ} are the complex parameters (11.92) that determine the representation π_p. There are furthermore a total number of

$$p^{i(n-i-\frac{1}{2})-\sum_{\ell=1}^{i}\eta_\ell} \tag{11.99}$$

$\beta_{i,j}$ for each $i \in \{1, \dots, n\}$. Combining everything we find that the eigenvalue of the Hecke operator Φ_i is given by

$$\begin{aligned}
\lambda_\mu(\Phi_i) &= \int_{K_p\tau_i K_p} \mathsf{f}_p^\circ(h)dh = \sum_{\beta_i\in\Lambda_i} \mathsf{f}_p^\circ(\beta_i) \\
&= \sum_{\eta_1<\cdots<\eta_i} p^{i(n-i-\frac{1}{2})-\sum_{j=1}^{i}\eta_j} p^{-\frac{i(n+1)}{2}} \prod_{\ell=1}^{i} p^{\eta_\ell}\alpha_{\eta_\ell} \\
&= p^{i(n-i)/2} \sum_{\eta_1<\cdots<\eta_i} \alpha_{\eta_1}\cdots\alpha_{\eta_n} \\
&= p^{i(n-i)/2} e_i(\alpha_1, \dots, \alpha_n),
\end{aligned} \tag{11.100}$$

where $e_i(\alpha_1, \dots, \alpha_n)$ is the ith elementary symmetric polynomial in n variables. In fact, the Satake isomorphism can be written in terms of these elementary symmetric polynomials

$$\mathcal{H}_p^\circ(GL(n, \mathbb{Q}_p)) \cong \mathbb{C}[e_1(\alpha_1, \dots, \alpha_n), \dots, e_n(\alpha_1, \dots, \alpha_n), e_n(\alpha_1, \dots, \alpha_n)^{-1}], \tag{11.101}$$

corresponding to the values on the generators $\Phi_1, \dots, \Phi_n$ and Φ_n^{-1} of the spherical Hecke algebra of $GL(n, \mathbb{Q}_p)$. Indeed, for $n = 2$ we have

$$e_1(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2, \qquad e_2(\alpha_1, \alpha_2) = \alpha_1\alpha_2, \tag{11.102}$$

thus recovering (11.79).

Let us end this section with a comment on how these results fit into the general theory of automorphic forms. Recall from definition 5.6 that an automorphic form φ on the adelic quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is required to be $\mathcal{Z}(\mathfrak{g})$-finite, i.e. that φ is an eigenfunction of the entire ring of invariant differential operators in the center of $U(\mathfrak{g})$. This can be viewed as a statement about the behavior of φ under the action of differential operators in the *real* group $G_\infty = G(\mathbb{R})$. For the case of automorphic forms attached to unramified automorphic representations $\pi = \pi_\infty \otimes \bigotimes_{p<\infty} \pi_p$ the spherical Hecke algebra provides the non-archimedean analogue of this: for each finite place p, φ is an eigenfunction of the ring of Hecke operators generated by $\Phi_i \in \mathcal{H}_p^\circ$. These statements combine together in the global Hecke algebra as mentioned in remark 11.8.

11.8 The Casselman–Shalika formula revisited

There is a close relation between the discussion above and the Casselman–Shalika formula for the p-adic spherical Whittaker vector W_ψ°. Spherical Whittaker vectors were the central objects in chapter 9 and a glimpse of the relation between them and representation theory was already visible in section 9.7. Here, we recall and extend some of the notions in a more general context. For an unramified character $\psi : N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p) \to U(1)$ we have an embedding

$$W_\psi : \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL(n,\mathbb{Q}_p)} \delta^{1/2}\mu \longrightarrow \mathrm{Ind}_{N(\mathbb{Q}_p)}^{GL(n,\mathbb{Q}_p)} \psi \tag{11.103}$$

of the unramified principal series into the space of functions $W : GL(n, \mathbb{Q}_p) \to \mathbb{C}$ satisfying

$$W_\psi(\delta^{1/2}\mu, ng) = \psi(n) W_\psi(\delta^{1/2}\mu, g), \qquad \forall n \in N(\mathbb{Q}_p), \tag{11.104}$$

where, as in chapter 9, the first argument indicates the dependence on the inducing character μ in the unramified principal series that was written there in terms of $\chi = \delta^{1/2}\mu$. The image of the space $V_p^{K_p}$ of K_p-fixed vectors in V_p is a one-dimensional space of *spherical Whittaker vectors*. In particular, for the generator $\mathsf{f}_p^\circ \in V_p^{K_p}$ we obtain a canonical spherical Whittaker vector via the explicit *Jacquet integral* (see chapter 9 for details)

$$W_\psi^\circ(\delta^{1/2}\mu, g) = \int_{N(\mathbb{Q}_p)} \mathsf{f}_p^\circ(w_0 n g)\overline{\psi(n)} dn, \tag{11.105}$$

where we used $\mathsf{f}_p^\circ = \delta^{1/2}\mu$. This satisfies

$$W_\psi^\circ(\delta^{1/2}\mu, nak) = \psi(n) W_\psi(\delta^{1/2}\mu, a), \tag{11.106}$$

and so is completely determined by its restriction to the Cartan torus $A(\mathbb{Q}_p)$. For $GL(n, \mathbb{Q}_p)$ the vanishing properties of W_ψ° analyzed in section 9.2.2 can be simplified as follows. Parametrising a according to

$$a = \varpi^J := \begin{pmatrix} p^{j_1} & & \\ & \ddots & \\ & & p^{j_n} \end{pmatrix} \in A(\mathbb{Q}_p)/A(\mathbb{Z}_p), \tag{11.107}$$

with $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, one finds that (see, e.g., [68])

$$W_\psi^\circ(\delta^{1/2}\mu, a) = 0. \qquad \text{unless} \quad j_1 \geq j_2 \geq \cdots \geq j_n. \tag{11.108}$$

The map (11.103) commutes with the Hecke action and therefore the spherical Whittaker vector is an eigenfunction of all the Hecke operators with the same eigenvalue (11.100) as before:

$$\Phi_i W_\psi^\circ(\delta^{1/2}\mu, a) = \lambda_\mu(\Phi_i) W_\psi^\circ(\delta^{1/2}\mu, a). \tag{11.109}$$

This fact can be used to derive a recursive formula for the value $W_\psi^\circ(\delta^{1/2}\mu, a)$ as we will now show. This will give the connection with the Casselman–Shalika formula that we are after.

The main difference with the calculation (11.100) is of course that *a priori* we do not know the explicit value of W_ψ° on $A(\mathbb{Q}_p)$, in contrast to the case of the original spherical vector f_p° where we had the formula (11.56) at hand. The key is that we should parametrise the decomposition of the cosets $K_p\tau_iK_p$ in such a way that we can make use of the defining relation (11.106). Such a parametrisation was given by Shintani [298]; here we follow the treatment by Cogdell [68], which reads

$$K_p\tau_iK_p = \bigcup_{\epsilon\in I_i}\bigcup_{n\in N_\epsilon} n\varpi^\epsilon K_p, \tag{11.110}$$

where the set I_i is defined as

$$I_i = \{\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}^n \,|\, \epsilon_j \in \{0,1\}, \sum_{j=1}^n \epsilon_j = i\}, \tag{11.111}$$

and

$$N_\epsilon = N(\mathbb{Z}_p)/(N(\mathbb{Z}_p)\cap \varpi^\epsilon K_p \varpi^{-\epsilon}). \tag{11.112}$$

Using this result we can compute the left hand side of (11.109) explicitly:

$$\begin{aligned}
\int_{K_p\tau_iK_p} W_\psi^\circ(\delta^{1/2}\mu, \varpi^J h)dh &= \sum_{\epsilon\in I_i}\sum_{n\in N_\epsilon} W_\psi^\circ(\delta^{1/2}\mu, \varpi^J n\varpi^\epsilon) \\
&= \sum_{\epsilon\in I_i}\sum_{n\in N_\epsilon} W_\psi^\circ(\delta^{1/2}\mu, \varpi^J n\varpi^{-J}\varpi^J\varpi^\epsilon) \\
&= \sum_{\epsilon\in I_i}\sum_{n\in N_\epsilon} \psi(\varpi^J n\varpi^{-J}) W_\psi^\circ(\delta^{1/2}\mu, \varpi^J\varpi^\epsilon),
\end{aligned} \tag{11.113}$$

where we used that $\varpi^J n\varpi^{-J} \in N(\mathbb{Q}_p)$ combined with (11.106). In fact, because of the constraint (11.108), which requires $j_1 \geq \cdots \geq j_n$, we have that $\varpi^J n\varpi^{-J} \in N(\mathbb{Z}_p)$ and consequently $\psi(\varpi^J n\varpi^{-J}) = 1$. The summand is therefore independent of n and the sum yields only a factor corresponding to the size of the coset space (11.112). Cogdell shows that [68]

$$|N_\epsilon| = p^{i(n-i)/2}\delta^{-1/2}(\varpi^\epsilon), \tag{11.114}$$

so we obtain for all i [68, Prop. 7.3]

$$\lambda_\mu(\Phi_i) W_\psi^\circ(\delta^{1/2}\mu, \varpi^J) = \sum_{\epsilon\in I_i} p^{i(n-i)/2}\delta^{-1/2}(\varpi^\epsilon) W_\psi^\circ(\delta^{1/2}\mu, \varpi^{J+\epsilon}). \tag{11.115}$$

This is a recursive formula for the spherical Whittaker vector $W_\psi^\circ(\delta^{1/2}\mu, \varpi^J)$! We recall that all the Hecke eigenvalues $\lambda_\mu(\Phi_i)$ are known from (11.100).

Example 11.14: Unramified Whittaker vectors for $GL(2, \mathbb{Q}_p)$

Let us determine some unramified spherical Whittaker vectors for $GL(2, \mathbb{Q}_p)$ using the recursion relation (11.115), starting from $J = (0, 0)$. The recursion relation then reads for the two values $i = 1, 2$

$$p^{1/2}(\alpha_p + \beta_p) W^{(0,0)} = p^{1/2}\delta^{-1/2}\left(\varpi^{(0,1)}\right) W^{(0,1)} + p^{1/2}\delta^{-1/2}\left(\varpi^{(1,0)}\right) W^{(1,0)}, \tag{11.116}$$

$$\alpha_p \beta_p W^{(0,0)} = \delta^{-1/2}\left(\varpi^{(1,1)}\right) W^{(1,1)}, \tag{11.117}$$

where (11.66) and (11.67) were used and we have introduced the short-hand notations

$$W^{(j_1, j_2)} \equiv W_\psi^\circ(\delta^{1/2}\mu, \varpi^J) \quad \text{and} \quad \varpi^{(j_1,j_2)} \equiv \varpi^J. \tag{11.118}$$

Since $W^{(0,1)} = 0$ according to (11.108) we can solve for $W^{(1,0)}$ and $W^{(1,1)}$ in terms of $W^{(0,0)}$ to obtain

$$W^{(1,0)} = W^{(0,0)}\delta^{1/2}\left(\varpi^{(1,0)}\right)(\alpha_p + \beta_p), \quad W^{(1,1)} = W^{(0,0)}\delta^{1/2}\left(\varpi^{(1,1)}\right)\alpha_p\beta_p. \tag{11.119}$$

We note that

$$\alpha_p + \beta_p = \operatorname{Tr}\begin{pmatrix}\alpha_p & \\ & \beta_p\end{pmatrix} = \operatorname{Tr}_{(1,0)}(A_{\pi_p}) \quad \text{and} \quad \alpha_p\beta_p = \operatorname{Tr}\left(\alpha_p\beta_p\right) = \operatorname{Tr}_{(1,1)}(A_{\pi_p}) \tag{11.120}$$

are the characters of the Satake parameter A_{π_p} in the two- and one-dimensional representations of $GL(2, \mathbb{C}) = {}^L GL(2, \mathbb{Q}_p)$, respectively, that are labelled here by their Young tableaux indexed by $J = (1, 0)$ and $J = (1, 1)$. The translation from non-increasing tuples $(j_1, \ldots, j_n)$ to a *Young tableau* is such that the ith row has j_i boxes. Therefore, we have

$$J = (1,0) \longleftrightarrow \begin{array}{|c|}\hline \\ \hline \end{array} \qquad \text{and} \qquad J = (1,1) \longleftrightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}\,,$$

such that $(1, 0)$ corresponds to the fundamental two-dimensional representation and $(1, 1)$ to the one-dimensional $GL(2, \mathbb{C})$ representation of weight 1 (tensor density). The relation between spherical Whittaker vectors and characters is no coincidence as we explain in the text.

The key to solving the recursion relation (11.115) for $GL(n, \mathbb{Q}_p)$ is is to note that the set of integers $J = (j_1, \ldots, j_n)$, subject to the condition $j_1 \geq j_2 \geq \cdots \geq j_n$, is well-known to parametrise the highest weights of irreducible representations V_J of $GL(n, \mathbb{C})$, which, we recall, is the L-group ${}^L G$ of $GL(n, \mathbb{Q}_p)$. But the analogy goes even further than that. Let $\chi_J = \operatorname{Tr}_{V_J}$ be the character of the representation V_J. This is a *class function*, meaning that it is invariant under conjugation

$$\operatorname{Tr}_{V_J}(g) = \operatorname{Tr}_{V_J}(hgh^{-1}), \qquad g, h \in GL(n, \mathbb{C}), \tag{11.121}$$

and so only depends on the conjugacy class of V_J. If we take V_J to be the fundamental n-dimensional representation of $GL(n, \mathbb{C})$ then we already have a conjugacy class at hand, namely the Satake parameter $A_{\pi_p} \in {}^L A(\mathbb{C})$ (11.93) of π_p. From this perspective J is a dominant weight in the weight lattice $\Lambda^\vee$ of ${}^L\mathfrak{g}$, which is the co-weight lattice of $\mathfrak{g}$. One can then solve the recursion (11.115) in terms of the characters χ_J with the result [68]

$$W_\psi^\circ(\delta^{1/2}\mu, \varpi^J) = \begin{cases} \text{const} \times \delta^{1/2}(\varpi^J)\chi_J(A_{\pi_p}) & \text{if } J \in \Lambda^\vee \text{ is dominant} \\ 0 & \text{otherwise.} \end{cases} \tag{11.122}$$

We note that the recursion relation only determines the spherical Whittaker vector up to a constant. At first sight this looks very different from the Casselman–Shalika formula (9.21) we derived in chapter 9. To see that they indeed coincide we shall rewrite the formula given there in a way similar to what was done in section 9.7. Setting $a = \varpi^J$ in (9.21) and doing some reshuffling we arrive at

$$\begin{aligned}\frac{1}{\zeta(\lambda)}\sum_{w\in\mathcal{W}}\epsilon(w\lambda)|a^{w\lambda+\rho}| &= \frac{1}{\zeta(\delta^{1/2}\mu)}a^{\rho}\sum_{w\in\mathcal{W}} w\left[\frac{a^{\lambda}}{\prod_{\alpha>0}(1-p^{\langle\lambda|\alpha\rangle})}\right]\\ &= \frac{1}{\zeta(\delta^{1/2}\mu)}\delta^{1/2}(\varpi^J)\sum_{w\in\mathcal{W}} w\left[\frac{\mu(\varpi^J)}{\prod_{\alpha>0}(1-\mu(\varpi^{-\alpha}))}\right],\end{aligned} \tag{11.123}$$

where we rewrote the arguments as follows

$$a^{\rho} = e^{\left\langle\rho|H(\varpi^J)\right\rangle} = \delta^{1/2}(\varpi^J), \tag{11.124a}$$

$$a^{\lambda} = e^{\left\langle\lambda|H(\varpi^J)\right\rangle} = p^{-\langle\lambda|J\rangle} = \mu(\varpi^J), \tag{11.124b}$$

$$p^{\langle\lambda|\alpha\rangle} = \mu(\varpi^{-\alpha}), \tag{11.124c}$$

To interpret the new form (11.123) of the Casselman–Shalika formula we recall that the weight lattice of the L-group ${}^LG(\mathbb{C})$ is $\Lambda^\vee$, the co-weight lattice of G. We now identify this with the character lattice $X^*({}^LA)$ according to

$$\Lambda^\vee \cong X^*({}^LA) \cong \mathrm{Hom}({}^LA, U(1)) \cong \mathbb{Z}^n. \tag{11.125}$$

Under this identification a weight $J = (j_1,\dots,j_n) \in \Lambda^\vee$ can be interpreted as a character $J : {}^LA(\mathbb{C}) \to U(1)$. We can in particular evaluate this character on the Satake parameter $A_{\pi_p} \in {}^LA(\mathbb{C})$ with the result

$$A_{\pi_p}^J = J(A_{\pi_p}) = J\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} = \prod_{i=1}^n \alpha_i^{j_i}, \tag{11.126}$$

which further implies the equality

$$A_{\pi_p}^J = \mu(\varpi^J). \tag{11.127}$$

Remark 11.15. The standard notation being used here might be the source for confusions: In general we denote the value of the character J on $a \in {}^LA$ by a^J or $J(a)$ as in (11.126); however this should *not* be confused with the *matrix* ϖ^J, which is defined in (11.107). We trust that this will not cause any trouble since it should be clear from the context which definition is referred to.

Next we compare (11.123) with the Weyl character formula for a representation V_J of a Lie group G with highest weight J. According to (4.27), the character χ_J evaluated at $z \in A$ is explicitly given by

$$\chi_J(z) = \sum_{w\in\mathcal{W}} w\left[\frac{z^J}{\prod_{\alpha>0}(1-z^{-\alpha})}\right], \qquad z\in A. \tag{11.128}$$

We can therefore rewrite (11.123) as

$$\frac{1}{\zeta(\delta^{1/2}\mu)}\delta^{1/2}(\varpi^J)\sum_{w\in\mathcal{W}} w\left[\frac{\mu(\varpi^J)}{\prod_{\alpha>0}(1-\mu(\varpi^{-\alpha}))}\right] = \frac{1}{\zeta(\delta^{1/2}\mu)}\delta^{1/2}(\varpi^J)\chi_J(A_{\pi_p}). \tag{11.129}$$

Comparing this with (11.122) we indeed find perfect agreement, provided that we fix the overall constant there to be $\zeta(\mu)^{-1}$. We conclude that the Casselman–Shalika formula for the spherical Whittaker vector $W^\circ_\psi \in \left(\mathrm{Ind}_{N(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\psi\right)^{K_p}$ can be written in terms of the Weyl character formula for an irreducible representation V_J of the Langlands dual group ${}^LG(\mathbb{C})$:

$$W^\circ_\psi(\delta^{1/2}\mu, A_{\pi_p}) = \frac{1}{\zeta(\delta^{1/2}\mu)}\delta^{1/2}(\varpi^J)\chi_J(A_{\pi_p}). \tag{11.130}$$

For $GL(n,\mathbb{C})$ the characters of V_J is well-known to be given by symmetric polynomials that can be expressed in the basis of Schur polynomials. Examples for $GL(2)$ can be found in 11.14.

11.9 Automorphic L-functions

Equipped with the adelic Hecke technology of the previous sections we shall now revisit the discussion of Dirichlet series of section 11.3 in the more general context of $GL(n,\mathbb{A})$.

Suppose first that f is a Maass form on the upper-half plane $\mathbb{H}$ which is an eigenfunction of the classical Hecke operator T_p with eigenvalue a_p. For instance, f could be a non-holomorphic Eisenstein series. This lifts to an automorphic form $\varphi_f \in \mathcal{A}(SL(2,\mathbb{Q})\backslash SL(2,\mathbb{A}))$ which is an eigenfunction of $\mathbb{T}_p$ with eigenvalue $\lambda_\mu(\mathbb{T}_p) = p^{1/2}(\alpha_p+\beta_p)$ as we found in (11.66). According to (11.53) the relation between the eigenvalues is thus

$$a_p = \alpha_p + \beta_p. \tag{11.131}$$

This implies that we can rewrite the local factor in the Dirichlet series (11.38) as follows

$$(1-a_pp^{-s}+p^{-2s})^{-1} = \left[(1-\alpha_pp^{-s})(1-\beta_pp^{-s})\right]^{-1} = \det\left(\mathbb{1}-A_{\pi_p}p^{-s}\right)^{-1}, \tag{11.132}$$

where A_{π_p} is the semi-simple Satake parameter (11.88) in the fundamental matrix representation.

The relation (11.131) has a natural generalisation to higher rank groups. Suppose $\varphi \in \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K_{\mathbb{A}}}$, i.e., φ is a spherical automorphic form, is attached to an unramified automorphic representation π. Suppose also that φ is an eigenfunction of the spherical Hecke algebras $\mathcal{H}^\circ_p = \mathcal{H}(\mathbb{Q}_p)^{K_p}$. This implies that for $\Phi \in \mathcal{H}^\circ_p$ we have $\pi(\Phi)\varphi = \lambda_\pi(\Phi)\varphi$. In this situation there exists a unique Satake class $[A_{\pi_p}] \subset {}^LG(\mathbb{C})$ such that

$$\lambda_\pi(\Phi) = p^\sharp\mathrm{Tr}_\pi(A_{\pi_p}), \tag{11.133}$$

where the prefactor is some power of the prime p. In particular, for $G = GL(n)$ we see from (11.100) that

$$\lambda_\pi(\Phi_1) = \lambda_\mu(\Phi_1) = p^{(n-1)/2}(\alpha_1+\cdots+\alpha_n) = p^{(n-1)/2}\mathrm{Tr}_\pi(A_{\pi_p}), \tag{11.134}$$

where the semi-simple conjugacy class A_{π_p} is given in (11.93).

We can now generalise the construction of the Dirichlet series to $GL(n,\mathbb{A})$. To this end let $\pi = \bigotimes_{p\leq\infty}\pi_p$ be the unramified principal series $\mathrm{Ind}_{B(\mathbb{A})}^{GL(n,\mathbb{A})}\delta^{1/2}\mu$ and A_{π_p} be the corresponding Satake parameter associated with each local factor π_p. To this data we attach the following *local L-factor*:

$$L_p(\pi_p, s) = \det\left(\mathbb{1} - A_{\pi_p}p^{-s}\right)^{-1}, \tag{11.135}$$

and we define the *standard L-function* as

$$L(\pi, s) = \prod_{p<\infty} L_p(\pi_p, s). \tag{11.136}$$

Langlands has proven [218] that this can be completed by adding a certain factor for the prime at infinity

$$L^\star(\pi, s) = L_\infty(\pi_\infty, s)\prod_{p<\infty} L_p(\pi_p, s), \tag{11.137}$$

which has an analytic continuation to a meromorphic function in the entire complex s-plane, and satisfying a functional equation. This is a vast generalisation of the completed Riemann zeta-function $\xi(s) = \xi_\infty(s)\prod_{p<\infty}(1-p^{-s})^{-1}$, where the prime at infinity corresponds to the Gamma-factor $\xi_\infty(s) = \pi^{-s/2}\Gamma(s/2)$. For Maass wave forms on $\mathbb{H}$ the factors at infinity were given in (11.39).

But Langlands suggested to generalise this even further. Suppose G is a reductive algebraic group over $\mathbb{Q}_p$ and π_p is an unramified automorphic representation of $G(\mathbb{Q}_p)$. Let A_{π_p} be the associated Satake parameter, giving a semi-simple conjugacy class $[A_{\pi_p}] \subset {}^LG(\mathbb{C})$. Let further

$$\rho\,:\,{}^LG(\mathbb{C}) \longrightarrow GL(n,\mathbb{C}) = \mathrm{Aut}(\mathbb{C}^n) \tag{11.138}$$

be an n-dimensional representation of the L-group. Note that the representation does *not* depend on the prime p. In the case of $G = GL(n,\mathbb{Q}_p)$ and ρ the fundamental representation, $\rho(A_{\pi_p})$ will just be the diagonal matrix (11.93), but in general this need not be the case.

Moreover, in general one has that for an unramified global representation π of $G(\mathbb{A})$, only for *all but finitely many* p the local representations π_p are spherical, i.e. contain vectors f_p° fixed under $K_p = G(\mathbb{Z}_p)$. To take care of this complication we let S be a finite set of places such that if $p\notin S$, π_p is spherical. The set S always includes the archimedean place $p=\infty$. For this data we now construct the *Langlands L-function*

$$L_S(\pi, s, \rho) = \prod_{p\notin S}\frac{1}{\det\left(\mathbb{1} - \rho(A_{\pi_p})p^{-s}\right)}. \tag{11.139}$$

In this situation the analytic continuation is more involved but Langlands has conjectured that $L_S(\pi, s, \rho)$ can be completed at the unramified places S to obtain a meromorphic function $L^\star(\pi, s, \rho)$ of s, called the *global Langlands L-function*.

Example 11.16: L-function for $G = GL(2, \mathbb{Q}_p)$

To give a simple example of how such an L-function would look like, let us consider $G = GL(2, \mathbb{Q}_p)$ but now take ρ to be the k:th symmetric power $Sym^k(\mathbb{C}^2)$ of the fundamental representation $\mathbb{C}^2$ of $GL(2, \mathbb{C})$ (see, e.g., [118] for a nice discussion of this and other examples). The resulting L-function reads

$$L(\pi, s, Sym^k) = \prod_{p<\infty} \frac{1}{\det(\mathbb{1} - \rho(A_{\pi_p})p^{-s})} = \prod_{p<\infty} \prod_{j=0}^{k} \frac{1}{1 - \alpha_p^j \beta_p^{k-j} p^{-s}}. \tag{11.140}$$

Using the formalism outlined above, Langlands thus provided a systematic procedure for attaching L-functions to automorphic forms, a task that had previously only been understood in special cases. The relation between automorphic forms on G, the L-group ${}^L G$ and automorphic L-functions provides the cornerstone behind the Langlands program, which are a set of far-reaching conjectures put forward by Langlands, of which only a tiny fraction have been proven. In section 12.5 we briefly discuss some of the ideas in the Langlands program, and how they relate and extend the theory we have presented in this work.

11.10 The Langlands–Shahidi method*

It is important to study the functional properties of L-functions such as (11.140) since these can be used to give estimates on Hecke eigenvalues (or Fourier coefficients) of cusp forms. This application to number theory is reviewed for example in [118, 292, 294]; we will content ourselves here with explaining the basic construction and its relation to Eisenstein series and Whittaker vectors.

The starting point is the knowledge of the functional equation (8.44) for Eisenstein series on G induced from a representation of the Levi subgroup L of some parabolic subgroup $P = LU$ of G. From this functional equation and the knowledge how the L-function of interest arises in the Fourier expansion one can then deduce properties of the L-function. This method was suggested by Langlands in [218, 219] and then developed in detail by Shahidi [288–290, 292, 294].

To motivate the procedure, we look at the Fourier expansion of the $SL(2, \mathbb{R})$ Eisenstein series (cf. (1.16))

$$E(s, \tau) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{n \neq 0} |n|^{s-1/2} \sigma_{1-2s}(n) K_{s-1/2}(2\pi|n|y) e^{2\pi i n x}. \tag{11.141}$$

The Eisenstein series satisfies the functional equation (cf. (7.13))

$$E(1-s, \tau) = \frac{\xi(2s)}{\xi(2s-1)} E(s, \tau) \tag{11.142}$$

and the L-function whose properties one is interested in is the completed Riemann zeta function $\xi(k)$. As we have seen in chapters 7 and 8, this functional equation can be read off from the constant terms of the Eisenstein series and does not require the knowledge of the completed Riemann zeta function beyond its definition in terms of an Euler product.

Additional properties of $\xi(s)$ can be inferred from the first Fourier coefficient ($n = 1$). The functional relation (11.142) for this Fourier coefficient reads

$$\frac{1}{\xi(2(1-s))} K_{1/2-s}(2\pi y) = \frac{\xi(2s)}{\xi(2s-1)} \frac{1}{\xi(2s)} K_{s-1/2}(2\pi y). \tag{11.143}$$

Using the property $K_t(x) = K_{-t}(x)$ of the modified Bessel function, one deduces that

$$\xi(2s-1) = \xi(2-2s) \quad \Leftrightarrow \quad \xi(k) = \xi(1-k). \tag{11.144}$$

Thus, the functional equation of the completed Riemann zeta function $\xi(s)$ is a consequence of the functional equation of Eisenstein series. One can also deduce the non-vanishing of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$ from the holomorphy (in s) of $E(s,\tau)$ on the line $\mathrm{Re}(s) = 0$ and further properties of $\zeta(s)$ from the study of $E(s,\tau)$ [118]. (The higher Fourier coefficients $n > 1$ provide no additional information.)

The more general realisation of this method relies on Eisenstein series on G induced from a cuspidal automorphic representation π_L of the Levi factor L of a maximal parabolic subgroup $P = LU \subset G$. We assume that the representation π_L is spherical at almost all places p.

As before, we have that at the spherical finite places p one can characterize the representation by means of its Satake parameter $A_{\pi_p} \in {}^L A$. Let also S be a set of places that includes all the non-spherical places and the archimedean one. In the everywhere-unramified case one would have $S = \{\infty\}$. Since ${}^L L$ is a complex linear group, it admits standard finite-dimensional complex representations $\rho_L : {}^L L \to GL(n, \mathbb{C})$ where n is the dimension of the representation. For any such pair (π_L, ρ_L), the *partial Langlands L-function* is given by

$$L_S(s, \pi_L, \rho_L) = \prod_{p \notin S} L_p(s, \pi_L, \rho_L) = \prod_{p \notin S} \frac{1}{\det(\mathbb{1} - \rho_L(A_{\pi_p}) p^{-s})}, \tag{11.145}$$

where the determinant is taken in the representation associated with ρ_L. Formally, this is the same as the definition (11.139) above but this time we have emphasised that this is for the Levi part L of a parabolic subgroup P of G. The *global Langlands L-function* requires the definition of factors for the places S that is less uniform and not known in full generality. Important progress for the global L-functions for $GL(n)$ and $SO(2n+1)$ can be found in [163, 164, 169, 179].

The virtue of these L-functions is that they arise in the Fourier expansion of Eisenstein series induced from a *cuspidal* representation π_L of L. For an automorphic form $\phi \in \pi_L$ we let

$$E(s, \phi, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \phi(\gamma g) \delta_P(\gamma g)^s \tag{11.146}$$

be the *Eisenstein series on G induced from $\phi \in \pi_L$*. $\delta_P(g)$ is here (the trivial extension to G of) the modulus character on $P \subset G$ defined by

$$d(lul^{-1}) = \delta_P(l)du \tag{11.147}$$

and $\delta_P(ulk) = \delta_P(l)$. It can be given explicitly by $\delta_P(l) = l^{2\rho_P}$, where ρ_P is half the sum of the positive roots contained in U. In the discussion above in section 5.6, we had taken $\phi = 1$ in the non-cuspidal trivial representation.

The Eisenstein series $E(s, \phi, g)$ on G has a Fourier expansion with respect to the unipotent U that is simpler due to the fact that ϕ is taken from a *cuspidal* representation of the Levi factor L. This arises because in the Bruhat decomposition of G most classes have a vanishing contribution as ϕ is cuspidal. This is a collapse mechanism not unsimilar to the one discussed for constant terms in section 10.3 and for Whittaker vectors in section 9.5.

Langlands showed [219] that the constant term of $E(s, \phi, g)$ along P' (the opposite of P) is controlled by partial L-functions (11.145) and Shahidi extended this to non-trivial Fourier coefficients [288–290, 292, 294]. Shahidi's work relies also on the Casselman–Shalika formula for (generic) Whittaker vectors of an Eisenstein series $E(\lambda, g)$ at unramified places (11.130).

We first explain why Langlands L-functions arise from formula (11.130). If (11.130) is evaluated for the special case of $A_{\pi_p} = \mathbb{1}$, corresponding to the trivial representation, one obtains

$$W^\circ(\delta^{1/2}\mu, \mathbb{1}) = \frac{1}{\zeta(\delta^{1/2}\mu)} = \prod_{\alpha^\vee > 0} (1 - p^{-1-\langle\lambda|\alpha\rangle}) = \prod_{\alpha^\vee > 0} (1 - p^{-1}\mu(\varpi^{\alpha^\vee})) \tag{11.148}$$

Now each $\mu(\varpi^{\alpha^\vee})$ corresponds to the adjoint action of the Satake parameter A_{π_p} on the root space of $\alpha^\vee$ which is nothing but the representation of the split torus ${}^LA$ on the Lie algebra ${}^L\mathfrak{n}$. Denoting this action by $\rho : {}^LA \to \mathrm{End}({}^L\mathfrak{n})$, we have that

$$\prod_{\alpha^\vee > 0} (1 - p^{-1}\mu(\varpi^{\alpha^\vee}) = \det(\mathbb{1} - \rho(A_{\pi_p})p^{-1}) \tag{11.149}$$

since the representation ρ of ${}^LA$ decomposes into the direct sum of one-dimensional representations labelled by the positive roots (and the determinant therefore factorises). Hence

$$W^\circ(\lambda, \mathbb{1}) = \int\limits_{N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p)} E(\lambda, n)\overline{\psi(n)}dn = \frac{1}{L_p(1, \lambda, \rho)}, \tag{11.150}$$

i.e., an L-function of the type (11.145). Here, we have labelled the representation π_L of the Levi ${}^LA$ of the minimal parabolic (Borel) ${}^LB$ by its quasi-character λ.

In the more general case of the Eisenstein series $E(s, \phi, g)$ induced from a cuspidal representation π_L of the Levi subgroup of a maximal parabolic $P = UL \subset G$ one has to consider the adjoint action ρ of ${}^LL$ on the Lie algebra ${}^L\mathfrak{u}$ of the unipotent ${}^LU$. Under this action, ${}^L\mathfrak{u}$ decomposes into a finite number of irreducible representations according to

$$\rho = \bigoplus_{j=1}^{m} \rho_j, \tag{11.151}$$

where m is the maximum coefficient (among all roots of G) of the simple root defining the maximal parabolic subgroup $P \subset G$. Shahidi showed [289, 294] that it is possible to choose $\phi \in \pi_L$ such that the generic Fourier coefficient of $E(s,\phi,g)$ at $g = \mathbb{1}$ is given (for an unramified place p) by

$$\int\limits_{U(\mathbb{Z}_p)\backslash U(\mathbb{Q}_p)} E(s,\phi,u)\overline{\psi(u)}du = \prod_{j=1}^{m} \frac{1}{L_p(1+ajs,\pi_L,\rho_j)}, \tag{11.152}$$

where a is a fixed number that depends on the choice of parabolic subgroup. The shifts by s in the argument of the L-function (compared to (11.150)) is due to the factor $\delta_P(\gamma g)^s$ in the definition of the Eisenstein series.

In the constant term, the same L-functions appear, cf. the intertwining factors $M(w,\lambda)$ in (8.42). Langlands showed [219] that the intertwiner appearing in the constant term for a place $p \notin S$ is

$$\prod_{j=1}^{m} \frac{L_p(ajs,\pi_L,\rho_j)}{L_p(1+ajs,\pi_L,\rho_j)}. \tag{11.153}$$

Due to the cuspidality of π_L this is the only non-trivial coefficient appearing in the constant term and it plays the role of the coefficient of y^{1-s} in (11.141) above. It also appears in the functional equation satisfied by $E(s,\phi,g)$ and allows one to deduce a functional equation for the partial L-function L_S obtained from all the places $p \in S$:

$$\prod_{j=1}^{m} L_S(ajs,\pi_L,r_j) = \prod_{j=1}^{m} L_S(1-ajs,\pi_L,\tilde{r}_j) \cdot \prod_{p\in S} C(s,\tilde{\pi}_v), \tag{11.154}$$

which is called the *crude functional equation* [288, 289]. The factors $C(s,\tilde{\pi}_v)$ appearing in this functional relation are called *local factors* and they can be determined from the study of the intertwining operator for $p \in S$ [288, 289]. The tildes in the above formula refer to the parabolic subgroup $\tilde{P}$ opposite to P. From this identification of the product of m partial L-functions as a Fourier coefficient of an Eisenstein series one can also deduce that the product extends to a meromorphic function (in s) and does not vanish on the imaginary axis [294]. Moreover, it is possible to perform induction on m to deduce the same statements for each of the individual factors. This produces a host of non-trivial results for generalised L-functions in various representations r_i that arise from all maximal parabolic subgroups [291]. The results described here for split groups can also be extended to so-called *quasi-split groups* [210, 294].

Remark 11.17. Besides the Langlands–Shahidi method just outlined, L-functions have also been studied using *converse theorems*, most notably those of Cogdell and Piatetski-Shapiro for the general linear group [69, 70]. The virtue of these converse theorems is that they allow to conclude that an L-function satisfying certain technical conditions must be a global L-function arising from an automorphic form on the general linear group. Such converse theorems can be seen as the extensions of Hecke's results for L-functions that were discussed at the end of section 11.3.

Converse theorems make it possible to deduce *Langlands functoriality* in some examples. As will be discussed more in section 12.5, Langlands functoriality deals with the question of transferring automorphic forms from a group G to a group G' in which G is a subgroup. Concretely, one starts from an L-function that is tentatively associated with the group G' and takes the *Rankin–Selberg product* with automorphic L-functions on subgroups $G \subset G'$. If certain technical conditions are fulfilled, one can conclude that there must be a (cuspidal) automorphic representation of G' whose L-function is the one under study, thereby lifting the representations from G to G'. For a nice discussion of this we refer to [118].

Chapter 12

Outlook

It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates.
To see it whole is certainly a daunting, for the moment even impossible, task.

— Robert P. Langlands[§]

In this concluding chapter, we collect various topics and further directions that we decided not to include in full detail but that are active fields of research providing interesting context for the study of automorphic functions. The emphasis of the first topics discussed here is mainly on theoretical physics, then we move on to more mathematical areas. We will be much more sketchy in this chapter and refer to the cited literature for additional details.

12.1 String scattering amplitudes and automorphic forms

This section is a continuation of the discussion of string theory and automorphic forms in chapter 2 which will elaborate on recent research on the topic. First, we will summarize some of the results from chapter 2. In section 12.1.1 we will discuss how automorphic representations are used to specify the coefficients $\mathcal{E}^{(D)}_{(p,q)}$ and what this tells us about their Fourier coefficients. Section 12.1.2 treats the D^6R^4 term which, as seen in (2.12c), has an extra source term, and hints at a theory beyond the automorphic forms discussed here.

[§]A review of Haruzo Hida's p-adic automorphic forms on Shimura varieties.

As discussed in chapter 2, the four-graviton scattering amplitude can be expanded in α' as (2.9) reproduced here for convenience

$$\mathcal{A}^{(D)}(s,t,u,\epsilon_i;g) = \left[\mathcal{E}^{(D)}_{(0,-1)}(g)\frac{1}{\sigma_3} + \sum_{p\geq 0}\sum_{q\geq 0}\mathcal{E}^{(D)}_{(p,q)}(g)\sigma_2^p\sigma_3^q\right]\mathcal{R}^4\,. \tag{12.1}$$

where $\sigma^2 = s^2+t^2+u^2$ and $\sigma^3 = s^3+t^3+u^3$.

The coefficients $\mathcal{E}^{(D)}_{(p,q)}(g)$ are functions on $\mathcal{M} = G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$, where $G(\mathbb{R})$ is the classical symmetry group, $K(\mathbb{R})$ its maximal compact subgroup, and $G(\mathbb{Z})$ the discrete U-duality subgroup shown in table 2.1. They satisfy the differential equations (2.12)

$$R^4: \qquad \left(\Delta_{G/K} - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)}(g) = 6\pi\delta_{D,8}, \tag{12.2a}$$

$$D^4R^4: \qquad \left(\Delta_{G/K} - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)}(g) = 40\zeta(2)\delta_{D,7}, \tag{12.2b}$$

$$D^6R^4: \qquad \left(\Delta_{G/K} - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)}(g) = 40\zeta(3)\delta_{D,6} - (\mathcal{E}^{(D)}_{(0,0)}(g))^2, \tag{12.2c}$$

where $\Delta_{G/K}$ is the Laplace-Beltrami operator on G/K, and are well behaved in the limits corresponding to cusps in G/K.

As was also covered in chapter 2, there is strong evidence from various consistency checks that they are given by (combinations of) maximal parabolic Eisenstein series defined in section 5.6 and in particular in example 5.32. Specifically, for dimensions $D = 5,4,3$ corresponding to tori T^d with $d = 5,6,7$ in table 2.1, if one considers the maximal parabolic subgroups P of E_{d+1} that have semi-simple Levi parts $SO(d,d)$, then the solutions (2.13)

$$R^4: \qquad \mathcal{E}^{(D)}_{(0,0)}(g) = 2\zeta(3)E(\lambda_{3/2},P,g), \tag{12.3a}$$

$$D^4R^4: \qquad \mathcal{E}^{(D)}_{(1,0)}(g) = \zeta(5)E(\lambda_{5/2},P,g). \tag{12.3b}$$

to equations (12.2a) and (12.2b) are the conjectured coefficient functions appearing in the four graviton scattering amplitude and have been subjected to numerous consistency checks [144, 146, 263].

The character defining the Eisenstein series is given by the weight (2.14)

$$\lambda_s = 2s\Lambda_P - \rho, \tag{12.4}$$

where Λ_P denotes the fundamental weight orthogonal to the Levi subgroup L of $P = LU$.

12.1.1 Small representations and string amplitudes

As automorphic representations the functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ appearing in (12.3) are attached to so-called *small representations*. According to (5.195), the functional dimension of the automorphic representation induced from a parabolic subgroup $P(\mathbb{A}) \subset G(\mathbb{A})$ is given by

$$\mathrm{GKdim} I_P(\chi) = \dim G - \dim P = \dim U, \tag{12.5}$$

where χ corresponds to a generic character on the parabolic subgroup $P(\mathbb{A}) = L(\mathbb{A})U(\mathbb{A})$. It turns out that for very special choices of the inducing character χ there may exist unitarizable submodules of $I_P(\chi)$ with smaller functional dimension. For example, it is well-known that for any real semi-simple Lie group $G(\mathbb{R})$ there exists a *minimal unitary representation* which has the smallest non-trivial functional dimension among all G-representations [181, 182].

The notion of a minimal representation also extends to p-adic groups $G(\mathbb{Q}_p)$ [107] and globally one says that an automorphic representation $\pi = \otimes_p \pi_p$ of an adelic group $G(\mathbb{A})$ is minimal if at least one local component π_p has smallest non-trivial functional dimension [126].

Minimal representations of a group G are closely related to minimal nilpotent G-orbits. Specifically, via *Kirillov's 'orbit method'* one can obtain π_{min} through the geometric quantisation of the minimal nilpotent orbit $\mathcal{O}_{min}$ [55]. This implies that there exists a sequence of small G-representations with increasing functional dimension associated with nilpotent orbits of smaller dimension than than the regular orbit. See for example [154] for an analysis pertaining to exceptional Lie groups of real rank 4.

Automorphic forms attached to small representations π are interesting both from a mathematical and a physical perspective. It was shown in the seminal paper by Ginzburg–Rallis–Soudry [126] that automorphic forms in the minimal representation π_{min} have very few non-vanishing Fourier coefficients, a fact that has far-reaching consequences. In particular, it allows to describe the complete Fourier expansion very explicitly, a task which is generally very difficult for Lie groups beyond $SL(2)$. One of the main applications of the theory of small representations has been in the context of the so called *theta correspondence* which is a method of lifting automorphic representations from one group G to another G'. This lifting can be realized as an integral transform where the minimal representation appears as the kernel. Ginzburg has also developed a method which uses small automorphic representations for constructing new automorphic L-functions (see [125] for a survey).

The functions (12.3) arising in string scattering amplitudes at lowest order in the derivative expansion are associated with small representations. The physical reason is that, as mentioned in chapter 2, the sum over characters ψ_U in the Fourier expansion of some automorphic form with respect to a unipotent radical $U \subset G$ may be viewed as a sum over instanton charges. For certain special physical quantities preserving some supersymmetry, there are constraints that forces many of the instanton configurations to be trivially realised, implying that the entire Fourier expansion has support on a smaller set of charges. This happens for instance in the case of four-gravitational scattering amplitudes in string theory compactificed on a torus T^d for the R^4 and D^4R^4 couplings which is the case discussed in section 12.1. For R^4, the physical constraints go by the name $\frac{1}{2}$-*BPS* and have been shown to be precisely those arising from the minimal representations of the exceptional Lie groups E_{n+1} [146, 235, 263] and indeed the function (12.3a) is known to be associated with the minimal representation [126]. This means in particular that its wavefront set (cf. section 6.4) is associated with the *minimal nilpotent orbit* of Bala–Carter type A_1. Similarly, the function (12.3b) corresponding to the D^4R^4 coupling of $\frac{1}{4}$-*BPS type* has a wavefront set associated with the *next-to-minimal nilpotent orbit* of type $2A_1$ [146, 263].

Both the minimal and next-to-minimal orbit are *special* orbits and the automorphic representation of *special unipotent* type. This connection with scattering amplitudes in string theory has also spawned new developments in mathematics. In particular, the paper [235] by Miller and Sahi, classifying *character variety orbits* of all classical and exceptional Lie groups, was in part motivated by these developments in string theory.

12.1.2 D^6R^4-amplitudes and new automorphic forms

The inhomogeneous Laplace equation (12.2c) for the D^6R^4 coupling does not represent a typical $\mathcal{Z}(\mathfrak{g})$-finiteness condition and therefore the coefficient function $\mathcal{E}^{(D)}_{(0,1)}(g)$ is not expected to be an automorphic form in the strict sense of definition 5.6. Its solutions have nevertheless been investigated recently in detail by Green, Miller and Vanhove in [145] (see [33, 34, 84, 152] for earlier and related work). An $SL(2,\mathbb{Z})$-invariant solution was found and its Fourier expansion has been studied.

Green, Miller and Vanhove have also succeeded in expressing the solution as a sum over $G(\mathbb{Z})$-orbits similar to the standard form of Langlands–Eisenstein series [145]

$$\mathcal{E}^{(10)}_{(0,1)}(g) = \sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} \Phi(\gamma g), \tag{12.6}$$

where $\Phi : G \to \mathbb{R}$ is a right $K = SO(2,\mathbb{R})$ invariant function and hence can be interpreted as a function on $B(\mathbb{R})$. It is furthermore invariant under $B(\mathbb{Z})$. However, unlike the case of Eisenstein series, the function Φ is *not* a character on the Borel subgroup $B(\mathbb{R})$ but rather a highly non-trivial function.

A proper framework for $G(\mathbb{Z})$-invariant functions that satisfy differential equations of the type (12.2c) appears to be required in string theory. The class of functions extends the notion of automorphic form discussed elsewhere here. The analysis in [145] points in the direction of a relation to *automorphic distributions* [236–238, 281, 312].

The function $\mathcal{E}^{(10)}_{(0,1)}(g)$ has the following constant terms [145, Eq. (2.25)]

$$\int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \mathcal{E}^{(10)}_{(0,1)}(ng)dn = \frac{2\zeta(3)^2}{3}y^3 + \frac{4\zeta(2)\zeta(3)}{3}y + \frac{4\zeta(4)}{y} + \frac{4\zeta(6)}{27y^3} + \text{non-poly. terms in } y. \tag{12.7}$$

Here, we have used the usual coordinates from appendix A on $SL(2,\mathbb{R})$. The non-polynomial terms are of the form $\sum_{n>0} a_n e^{-4\pi n y}/y^2$ and do not have an expansion around weak coupling $y \to \infty$. These terms have an interpretation as instanton/anti-instanton bound states. We see that the structure of constant terms is quite different from that of Eisenstein series where, according to the Langlands constant term formula of theorem 8.1, one has a sum of polynomial terms in y only and the number is bounded from above by the order of the Weyl group $\mathcal{W}$ which would be $|\mathcal{W}| = 2$ here.

In terms of string perturbation theory, the four polynomial terms in (12.7) correspond to contributions from string world-sheets of genus $h = 0, 1, 2, 3$. We recognise the genus $h = 0$ contribution from (2.31). The genus $h = 2$ contribution predicted here was recently compared to a first principles string theory calculation and found to agree [83, 84], where also remarkably a connection to the so-called *Zhang–Kawazumi invariant* on the moduli space of genus $h = 2$ Riemann surfaces was found [189, 325]. As a consequence of equation (12.2c), [83, 84] discovered that the Zhang–Kawazumi invariant must satisfy a simple Laplace eigenvalue equation on the moduli space of Riemann surfaces. The genus $h = 3$ term in (12.7) has been verified directly from a string perturbation calculation very recently in [133] in the pure spinor formalism.

In terms of wavefront sets and automorphic representations it seems natural to associate the D^6R^4 coupling to the (special) nilpotent orbits of type $3A_1$ and A_2 [34]. A proper interpretation of these wavefront sets for $SL(2,\mathbb{R})$ is missing since the largest nilpotent orbit is the regular A_1-type orbit. D^6R^4 correction terms have been analysed recently in various dimensions by different methods [16, 17, 31, 35, 84, 140, 147, 148, 265].

12.1.3 Wavefront sets of curvature corrections and their reduction

In this section, we would like to collect and systematize some of the remarks on wavefront sets and curvature corrections that have been made in the preceding discussion. We will do this for the case $G = E_7(\mathbb{R})$ that is relevant for $D = 4$ space-time dimensions and maximal supersymmetry. The *closure diagram* of nilpotent orbits of $\mathfrak{e}_7(\mathbb{C})$ can be found for example in [300] and that of $\mathfrak{e}_7(\mathbb{R})$ in [255]. We display the closure (or Hasse) diagram of the smallest nilpotent orbits of $\mathfrak{e}_7(\mathbb{C})$ in figure 12.1.

In the figure, we have also shown the wavefront sets of the various types of curvature corrections $D^{2k}R^4$ following [31, 32, 34, 146, 263]. What is noticeable is that the wavefront sets appear to be associated only with *special* orbits [32]. Preliminary investigations of higher derivative terms in [31] suggest that correction terms with more than six derivatives acting on R^4 will generically have contributions from the orbit $(A_3 + A_1)''$. The expansion in increasing orders of derivatives seems to be related to an expansion in terms of size of the associated wavefront set (with only special orbits as maximal orbits).

We note also that there can be several maximal nilpotent orbits contributing to a given curvature correction, as in the case of D^6R^4. This is related to the fact the U-duality invariant functions $\mathcal{E}^{(D)}_{(p,q)}$ that arise are not necessarily automorphic functions of the standard type but more general as discussed in section 12.1.2. The branching of the diagram is associated in physics with the existence of independent (linearised) supersymmetry invariants [35].

Let us also relate this discussion back to the analysis of small representations of sections 10.4 and 12.1.1. In the case of $E_7(\mathbb{R})$ one has a degenerate principal series representation of functional dimension 33 that can be realised with Eisenstein series by choosing the weight $\lambda = 2s\Lambda_1 - \rho$ and we write the associated Eisenstein series as

$$E\left(\left[\,{}_{s\,0\,\overset{0}{0}\,0\,0\,0}\,\right]\right). \tag{12.8}$$

The wavefront set in the case of generic s is of type A_2 of dimension 66. This is twice the dimension of the coset $P_1\backslash E_7$ where P_1 is the maximal parabolic subgroup associated with

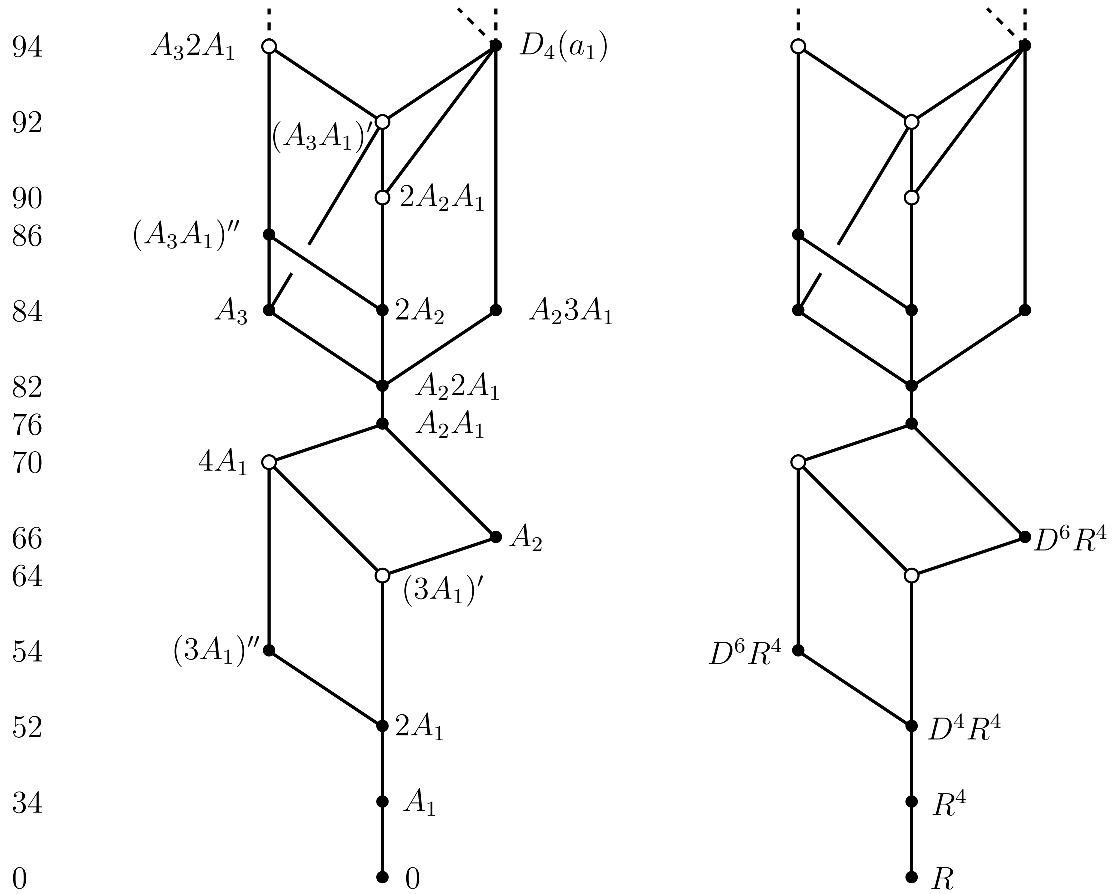


Figure 12.1: *The smallest nilpotent orbits of $\mathfrak{e}_7(\mathbb{C})$ and their closure ordering. The vertical axis is the dimension of the orbit and on the left they are labelled according to the Bala–Carter classification where we have denoted $2A_2 + A_1 \equiv 2A_2A_1$ etc. for brevity. The open circles indicate orbits that are not special. The figure is adapted from [31,34,300]. On the right, the wavefront sets of the various curvature terms appearing in the four-graviton scattering amplitude are shown on the same kind of diagram.*

node 1 in the Dynkin diagram of figure 2.2. Reductions occur in this case for the values $s = \frac{5}{2}$ and $s = \frac{3}{2}$ when the wavefront set collapses to the $2A_1$ and A_1, respectively. The reduction can be analysed using theorem 9.5 as discussed in section 10.4. The two cases where the wavefront set reduces corresponds to the R^4 and D^4R^4 curvature correction. There is a contribution of this function to the D^6R^4 correction for a non-special value of s.

Remark 12.1. The most well-studied case of curvature corrections is that of $D = 10$ type IIB superstring theory where the symmetry group is $SL(2,\mathbb{R})$ with U-duality group $SL(2,\mathbb{Z})$. The set of nilpotent orbits of $\mathfrak{sl}(2,\mathbb{C})$ is very degenerate and consists only of either the trivial or the regular (A_1-type) orbit. Nevertheless, the various curvature corrections of type $D^{2k}R^4$ come with very specific orders s of the non-holomorphic Eisenstein series $E(s,z)$. It is an open problem to understand the specific values that appear, notably

$s = \frac{3}{2}$ and $s = \frac{5}{2}$, from a mathematical point of view as there seems to be nothing special happening for the automorphic representation for these values.

12.2 Automorphic functions and lattice sums

As discussed in the introduction, the non-holomorphic Eisenstein series $E(\chi_s, z)$ of $SL(2,\mathbb{R})$ (cf. (1.11)) can be equivalently written in terms of a sum over an integral lattice:

$$E(\chi_s, z) = \sum_{\gamma \in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \chi_s(\gamma \cdot z) = \frac{1}{\zeta(2s)} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^2}{|cz+d|^{2s}}. \tag{12.9}$$

In physics, writing the Eisenstein as a *lattice sum* can sometimes be interpreted as the sum over the lattice of all possible charges that define the U-duality group (cf. chapter 2). The sum over the group coset $B(\mathbb{Z})\backslash G(\mathbb{Z})$, on the other hand, can be interpreted as the contribution from a single U-duality orbit, if $G(\mathbb{Z})$ is the U-duality group $G(\mathbb{Z})$ of table 2.1 in chapter 2. From the latter point of view, the functions discussed in (12.3) represent simply the U-duality orbit of the perturbative tree level scattering amplitude, whereas the function in (12.6) is the U-duality orbit of a finite number of perturbative terms and an infinite number of non-perturbative terms.

Having a representation of an automorphic function as lattice sum can be physically intuitive and it certainly opens up the possibility of employing Poisson resummation for performing the Fourier expansion of the function, as is done in the $SL(2,\mathbb{R})$ example in appendix B.

Lattice sums for more general groups G were considered by Obers and Pioline in [252]. They write the group element $g \in G(\mathbb{R})$ in some linear finite-dimensional representation $\mathcal{R}$. In the same representation, a lattice $\Lambda_{\mathcal{R}}$ is embedded that is preserved by the action of $G(\mathbb{Z})$. This can be constructed for example by starting from the highest weight vector in the representation $\mathcal{R}$. One can form a scalar invariant by considering

$$||g^{-1}\omega||^2, \tag{12.10}$$

where $\omega \in \Lambda_{\mathcal{R}}$ and the norm is computed using the $K(\mathbb{R})$-invariant inner product on $\mathcal{R}$. In the example (12.9) above, this is realised by working in the two-dimensional representation, letting $\omega = \begin{pmatrix} -d \\ c \end{pmatrix} \in \mathbb{Z}^2$ and using the Euclidean norm. Then

$$\sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)\neq(0,0)}} \frac{y^2}{|cz+d|^{2s}} = \sum_{0\neq\omega\in\Lambda_{\mathcal{R}}} ||g^{-1}\omega||^{-2s}. \tag{12.11}$$

The quantity (12.10) is well-defined by construction on G/K for any $\omega \in \Lambda_{\mathcal{R}}$ and one can form a $G(\mathbb{Z})$-invariant function very generally by letting

$$\tilde{\mathcal{E}}_{\mathcal{R},s}(g) = \sum_{0\neq\omega\in\Lambda_{\mathcal{R}}} ||g^{-1}\omega||^{-2s}. \tag{12.12}$$

This function is K-finite, of moderate growth and $G(\mathbb{Z})$ invariant. Moreover, it is directly amenable to Poisson resummation on the lattice $\Lambda_{\mathcal{R}}$ and this has been exploited widely to obtain results about the constant terms and also partly the non-constant terms of $\tilde{\mathcal{E}}_{\mathcal{R},s}(g)$ [10, 157, 211, 212].

For some groups $G(\mathbb{R})$ and some representations $\mathcal{R}$ it can happen that the function $\tilde{\mathcal{E}}_{\mathcal{R},s}(g)$ is proportional to a (maximal parabolic) Eisenstein series as defined in sections 5.3 and 5.6, and as is the case in the $SL(2,\mathbb{R})$ example in (12.9) above. However, as was already emphasised in [252], the function $\tilde{\mathcal{E}}_{\mathcal{R},s}(g)$ will in general *not* be an eigenfunction of the ring of invariant differential operators, i.e., it will not be $\mathcal{Z}(\mathfrak{g})$-finite and hence not a proper automorphic form.

The failure of being an automorphic can be remedied by restricting the lattice sum over $\mathcal{R}$ to an appropriate $G(\mathbb{Z})$-invariant subset. Such a subset can be found for example by considering the symmetric tensor product $\mathcal{R} \otimes \mathcal{R}$ and then projecting to the largest invariant subspace in there [252]. The symmetric tensor product arises because (12.10) is computing a symmetric quantity in the $\mathcal{R}$-valued ω. An automorphic form is then given by

$$\mathcal{E}_{\mathcal{R},s}(g) = \sum_{0 \neq \omega \in \Lambda_{\mathcal{R}}} \delta(\omega \otimes \omega) ||g^{-1}\omega||^{-2s}, \tag{12.13}$$

where $\delta(\omega \otimes \omega)$ projects on the invariant subspace in $\mathcal{R} \otimes \mathcal{R}$ defined above. In physical applications, this projection has the interpretation of implementing certain conditions that are called *BPS conditions* and correspond to considering contributions only from a subset of all instantonic states. The presence of the projection $\delta(\omega \otimes \omega)$ in the sum often makes the direct application of Poisson resummation impossible and renders the Fourier expansion much more difficult. Examples where the full Fourier expansion of a constrained sum was carried out can be found in [12, 13].

Another way of turning $\tilde{\mathcal{E}}_{\mathcal{R},s}(g)$ into an automorphic form is by restricting to a single $G(\mathbb{Z})$-orbit within $\Lambda_{\mathcal{R}}$ and this leads back to Langlands' definition.

12.3 Asymptotics of Fourier coefficients

In applications to physics one is often interested in extracting the asymptotic behavior of Fourier coefficients. Asymptotic here refers to a chosen direction on the symmetric space $G(\mathbb{R})/K(\mathbb{R})$ on which the Eisenstein series and their Fourier coefficients are defined. The symmetric space often has the interpretation as the *moduli space*, a point of which corresponds to the *vacuum expectation values* of some fields in a physical theory, cf. the discussion in chapter 2. In an effective description these values are turned into coupling constants and similar parameters of the theory and therefore sending a certain coupling constant to zero corresponds to a limit on the symmetric space $G(\mathbb{R})/K(\mathbb{R})$. The asymptotic behaviour then reveals the *instanton action* of a particular non-perturbative object in the theory. In the case of $SL(2,\mathbb{R})$ this was explained in the introduction in (2.19) and in general one would like to know how the real part of the action behaves asymptotically as a function of the coupling constants. String theory makes definite predictions for these coupling constants that were formalised as conjectures in [110, Appendix A.6]. In this section we prove these conjectures by solving the relevant Laplace equations asymptotically.

The interesting coupling constants are associated with directions in the non-compact abelian subgroup $A(\mathbb{R})$. Choosing one direction in this space is tantamount to picking a certain weight, or, equivalently, one chooses a maximal parabolic subgroup $P(\mathbb{R}) \subset G(\mathbb{R})$. In order to find the asymptotic behavior of an automorphic function $\varphi(g)$ and its Fourier coefficients in this limit one can analyse the Laplace differential equation that it satisfies. This equation reads

$$\Delta_{G/K}\varphi(g) = \mu\varphi(g) \tag{12.14}$$

for some eigenvalue μ.

For a maximal parabolic $P(\mathbb{R})$ we know from (4.40) that there is a $GL(1,\mathbb{R})$ subgroup in the Levi factor $L(\mathbb{R})$ and we denote its Cartan subalgebra generator by $d \in \mathfrak{h}$. This element can be used to introduce a grading of $\mathfrak{p}(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) \oplus d\mathbb{R} \oplus \mathfrak{u}(\mathbb{R})$ with the properties

$$[d, m] = 0 \qquad \text{for } m \in \mathfrak{m} = \mathrm{Lie}(M), \tag{12.15a}$$

$$[d, u_\ell] = \ell u_\ell \qquad \text{for } u_\ell \in \mathfrak{u}_\ell \text{ with } \mathfrak{u} = \bigoplus_\ell \mathfrak{u}_\ell. \tag{12.15b}$$

(The decomposition of $\mathfrak{u}$ is the same that arose in (11.151).) To now consider a 'pure instanton' at degree ℓ means that we are interested in group elements g of the form $g = e^{\phi d} e^{\chi E_\ell}$ where $E_\ell \in \mathfrak{u}_\ell$ is a chosen fixed generator. Treating the expectation value of e^ϕ as the coupling constant, the *weak coupling limit* then corresponds to $\phi \to -\infty$. The relevant part of the Laplacian for the directions ϕ and χ is

$$\Delta_{G/K} \propto \partial_\phi^2 + \beta\partial_\phi + e^{-2\ell\phi}\partial_\chi^2 + \ldots, \tag{12.16}$$

where $\beta = \sum_\ell \ell \dim(\mathfrak{u}_\ell)$ and we have not fixed the normalization of the Laplace operator as it can be absorbed into the eigenvalue μ. For an instanton of charge m we now make the ansatz for the automorphic function that asymptotically for $\phi \to -\infty$

$$\varphi_m(\phi, \chi) = e^{-ae^{-b\phi} + 2\pi i m\chi}\left(1 + O(e^\phi)\right). \tag{12.17}$$

This correspond to a Fourier coefficient for a character $\psi\left(e^{\chi E_\ell}\right) = e^{2\pi i m\chi}$.

Acting with the relevant part (12.16) of the Laplace operator on this ansatz shows that it can only be an eigenfunction (asymptotically) if

$$a = 2\pi|m| \quad \text{and} \quad b = \ell. \tag{12.18}$$

Note that this reasoning is independent of the eigenvalue μ and of whether φ is an Eisenstein series or any other automorphic function. The important point about (12.18) is that it shows that the leading part of the *instanton action* is

$$S_{\mathrm{E}}(\phi, \chi) = \log\varphi_m(\phi, \chi) = 2\pi|m|e^{-\ell\phi} + 2\pi i m\chi + \text{sub-leading in } e^\phi. \tag{12.19}$$

This is the typical of type ℓ instanton where 'type' here refers to the degree in $\mathfrak{u}(\mathbb{R})$. Making the link to the non-abelian Fourier expansion of section 6.2 shows that the more non-abelian a Fourier coefficient is the faster its fall-off in the corresponding weak coupling expansion.

The typical cases encountered in string theory are when $e^{\phi} = g_s$ is the *string coupling*. Instantons with $\ell = 1$ are then D-instantons and those with $\ell = 2$ are NS-instantons [12, 13, 18, 138, 139, 267]. In low space-time dimensions one expects also instantons with $\ell > 2$ [92, 253].

12.4 Black hole counting and automorphic representations

As explained in chapter 2, string theory compactified on a compact six-dimensional manifold X gives rise to an effective supersymmetric gravitational theory in 4 dimensions. The number of preserved supersymmetries, usually denoted by $\mathcal{N}$, depends on the properties of X. Previously we have mainly discussed the case of $X = T^6$, but there are other interesting and relevant manifolds, such as $X = K3 \times T^2$ and when X is a Calabi-Yau threefold. The resulting theory has black hole solutions carrying electric and magnetic charges taking values in a lattice Γ.

An important observable is the *BPS-index* $\Omega(\gamma)$ which is a function $\Omega : \Gamma \to \mathbb{Z}$ that counts the (signed) degeneracies of a certain class of black holes (called BPS-black holes) with charge vector $\gamma \in \Gamma$. This index then provides a microscopic description of the black hole entropy $S(\gamma)$ via Boltzmann's formula $S(\gamma) = \log \Omega(\gamma) + \cdots$, where the ellipsis denote subleading corrections.

The BPS-index $\Omega(\gamma)$ holds the key to many interesting connections between string theory and mathematics. The charge lattice Γ is nothing but the cohomology lattice $H^*(X, \mathbb{Z})$ of the compact manifold X and the index $\Omega(\gamma)$ can roughly be thought of as counting certain submanifolds of X in the cohomology class $[\gamma] \subset H^*(X, \mathbb{Z})$. It is therefore naturally related to the enumerative geometry of X. Remarkably, the index also provides a link to automorphic forms. To illustrate these statements we shall now consider a few examples.

12.4.1 $\mathcal{N} = 8$ supersymmetry

Let us first consider the case when $X = T^6$, the real six-dimensional torus. This leads to $\mathcal{N} = 8$ supersymmetry and is the case discussed in section 12.1. In four dimensions with electric and magnetic charges taking values in a lattice $\Gamma \cong \mathbb{Z}^{56}$. As reviewed in chapter 2 and section 12.1, this theory exhibits a classical $E_7(\mathbb{R})$-symmetry which is broken in the quantum theory to the $E_7(\mathbb{Z}) = \{g \in E_7(\mathbb{R}) \,|\, g\Gamma = \Gamma\}$. This implies that the weighted degeneracy $\Omega(\gamma)$ of BPS-black holes of charge $\gamma \in \Gamma$ must be invariant under $E_7(\mathbb{Z})$.

However, not all black holes have charges supported on the entire lattice Γ. For example, the $\frac{1}{2}$-BPS black holes preserve half of the supersymmetries of the theory and can only have charges supported on a 28-dimensional (Lagrangian) subspace $\mathcal{C}_{1/2} \subset \Gamma$. Similarly, $\frac{1}{4}$-BPS black holes have support on a 45-dimensional subspace $\mathcal{C}_{1/4} \subset \Gamma$.

Now denote by $\Omega_{1/A}(\gamma)$ the index counting $\frac{1}{A}$-BPS-black holes ($A = 2, 4$). Due to the $E_7(\mathbb{Z})$-invariance it is natural to suspect that the index arises as the Fourier coefficient of an automorphic form, constrained so that $\Omega_{1/A}(\gamma)$ is non-vanishing only when $\gamma \in \mathcal{C}_A$.

Let us consider the $A = 2$ case for illustration. It turns out that all the expected properties are fulfilled by an automorphic form φ_{min} on $E_8(\mathbb{Z})\backslash E_8(\mathbb{R})$ attached to the minimal representation π_{min} of $E_8(\mathbb{R})$ [159, 262, 264]. This representation has Gelfand-Kirillov dimension 29 and can thus be realised as the unitary action of E_8 on a space of functions of 29 variables [158], say $(p, k) \in \mathbb{Z}^{28} \times \mathbb{Z}$. These integers parametrise characters on the Heisenberg unipotent radical $U_{Heis} \subset E_8$, which has an associated Levi factor $L_{Heis} = E_7 \times \mathbb{R}$. The centre $Z = [U_{Heis}, U_{Heis}]$ is one-dimensional and the integer k parametrises a unitary character $\psi_Z : Z(\mathbb{Z})\backslash \mathbb{Z}(\mathbb{R}) \to U(1)$, trivial on the abelianization $Z\backslash U_{Heis}$. On the other hand the electric and magnetic charges $\gamma = (p, q) \in \Gamma \cong \mathbb{Z}^{56}$ parametrise characters $\psi : U_{Heis}(\mathbb{Z})\backslash U_{Heis}(\mathbb{R}) \to U(1)$, trivial on $Z(\mathbb{R})$. Consider the constant term of φ_{min} with respect to Z:

$$\varphi_{Z,min} = \int_{Z(\mathbb{Z})\backslash Z(\mathbb{R})} \varphi_{min}(zg)dz. \tag{12.20}$$

This is a function $\varphi_{Z,min} : E_7(\mathbb{R}) \to \mathbb{C}$ invariant under $E_7(\mathbb{Z})$. By taking the constant term with respect to Z we have effectively removed the dependence on the variable k. The function $\varphi_{Z,min}$ can be expanded further (see section 6)

$$\varphi_{Z,min}(g) = \varphi_{U_{Heis}} + \sum_{\psi \neq 1} F_\psi(\varphi_{min}, g), \tag{12.21}$$

where

$$F_\psi(\varphi_{min}, g) = \int_{U_{Heis}(\mathbb{Z})\backslash U_{Heis}(\mathbb{R})} \varphi(ug)\overline{\psi(u)}du. \tag{12.22}$$

In general, such a Fourier coefficient might not be Eulerian (i.e. have an Euler product factorisation); however, as we explained in section 10.4.4, for the minimal representation that turns out to be the case:

$$F_\psi(\varphi_{min}, g) = F_{\psi_\infty}(\varphi_{min}, g_\infty) \times \prod_{p<\infty} F_{\psi_p}(\varphi_{min}, g_p). \tag{12.23}$$

It was shown in [190, 191] that these Fourier coefficients indeed have support on the Lagrangian subspace $\mathcal{C}_{1/2}$. We can now state the relation to 1/2 BPS black holes:

Conjecture 12.2 ([159, 262, 264]). *The index $\Omega_{1/2}(p, q)$ counting charged 1/2 BPS black holes in four-dimensional, $\mathcal{N} = 8$ supergravity is given by*

$$\Omega_{1/2}(p, q) = \prod_{p<\infty} F_{\psi_p}(\varphi_{min}, 1), \tag{12.24}$$

where $F_{\psi_p}(\varphi_{min}, 1)$ is the p-adic spherical vector in the minimal representation π_{min} of E_8 (obtained in [190]) and the electric-magnetic charges (p, q) parametrises the character ψ_p.

Similarly, for the 1/4 BPS black holes we have:

Conjecture 12.3 ([159, 262, 264]). *The index $\Omega_{1/4}(p,q)$ counting charged 1/4 BPS black holes is given by*

$$\Omega_{1/4}(p,q) = \prod_{p<\infty} F_{\psi_p}(\varphi_{ntm}, 1), \tag{12.25}$$

where φ_{ntm} is an automorphic form in the next-to-minimal representation π_{ntm} of E_8.

12.4.2 $\mathcal{N} = 4$ supersymmetry

Let us now take $X = K3 \times T^2$, where the first factor is a compact $K3$-surface. This yields $\mathcal{N} = 4$ supersymmetry in 4 dimensions, which admits $\frac{1}{2}$- and $\frac{1}{4}$-BPS-black holes with electric magnetic charges $\gamma = (p,q)$ taking values in $\Gamma = H^*(X,\mathbb{Z})$. The quantum symmetry of this theory is $SL(2,\mathbb{Z}) \times SO(6,22;\mathbb{Z})$ [282, 285, 322] and we are interestested in finding invariant BPS-indices $\Omega_{1/2}(p,q)$ and $\Omega_{1/4}(q,p)$. Mathematically, these indices are counting special Lagrangian submanifolds of X in the class $[\gamma] \subset H^*(X,\mathbb{Z})$. As we shall see the counting works quite differently in this case compared to the $\mathcal{N} = 8$ theory considered above.

The 1/2 BPS-states are purely electric $\gamma = (0,q)$ or purely magnetic $\gamma = (p,0)$ and they are known to be exactly counted by [74, 76]

$$\Omega_{1/2}(q,0) = d(q^2/2), \tag{12.26}$$

where $d(n)$ are the Fourier coefficients of the discriminant function ($\tau \in \mathbb{H}$):

$$\Delta(\tau) = \frac{1}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} d(n) e^{2\pi i n\tau}, \tag{12.27}$$

which is a cusp form of weight 12 for $SL(2,\mathbb{Z})$. Note that the index is automatically invariant under $SO(6,22;\mathbb{Z})$ since it only depends on the invariant square $q^2 = q \cdot q$ of the charge vector q. On the other hand the $SL(2,\mathbb{Z})$-part of the quantum symmetry is broken since (p,q) transforms in a doublet. Thus, in order to preserve the full symmetry group we must consider both electric and magnetic charges, as is the case for the $\frac{1}{4}$-BPS-index $\Omega_{1/4}(p,q)$. Moreover, in order to preserve $SO(6,22;\mathbb{Z})$ this can only depend on the invariant combinations $q^2, p^2, p \cdot q$. The answer is that $\Omega_{1/4}(p,q)$ is the Fourier coefficient of the unique weight 10 cusp form for $Sp(4;\mathbb{Z})$, known as the *Igusa cusp form* and usually denoted by Φ_{10}. The precise statement is [87, 297]:

$$\Omega_{1/4}(p,q) = D(q^2/2, p^2/2, p \cdot q), \tag{12.28}$$

where the numbers $D(m,n,\ell)$ are extracted from the expansion of the inverse of the Igusa cusp form:

$$\frac{1}{\Phi_{10}(\rho,\sigma,\tau)} = \sum_{m,n,\ell} D(m,n,\ell) e^{2\pi i m\sigma} e^{2\pi i n\tau} e^{2\pi i \ell\rho}, \tag{12.29}$$

where (ρ,σ,τ) are complex variables parametrising the *Siegel upper half plane*. This can be generalised to orbifolds of $X = K3 \times T^2$ by some discrete subgroup $\mathbb{Z}_N$, in which case the counting is given by *Siegel modular forms* for paramodular groups (see [286] for a review and further references).

12.4.3 $\mathcal{N} = 2$ supersymmetry

Finally, we consider the case when X is a *Calabi–Yau 3-fold*. This gives rise to $\mathcal{N} = 2$ supersymmetry in 4 spacetime dimensions. The lattice Γ of electric and magnetic charges is either $H^{\text{even}}(X, \mathbb{Z})$ or $H^3(X, \mathbb{Z})$ depending on whether we consider type IIA or type IIB string theory. According to Kontsevich's *homological mirror symmetry conjecture* [203] a BPS black hole with charge $\gamma \in H^{\text{even}}(X, \mathbb{Z})$ can be viewed as a (semi-)stable object in the (bounded) *derived category of coherent sheaves* $\mathrm{D^bCoh}(X)$, while black holes with charges $\gamma \in H^3(X, \mathbb{X})$ correspond to (semi-)stable objects (special Lagrangians) in the *derived Fukaya category* $\mathrm{D^bFuk}(X)$. The BPS-index $\Omega : \Gamma \to \mathbb{Z}$ should then be identified with the *generalised Donaldson–Thomas invariants* of X [104, 184, 202]. String theory predicts that there should be an action of a discrete Lie group $G(\mathbb{Z})$ on the categories $\mathrm{D^bCoh}(X)$ and $\mathrm{D^bFuk}(X)$, which is very unexpected from a purely mathematical viewpoint. In general it is not known what the group $G(\mathbb{Z})$ should be but it must at least contain the "S-duality" group $SL(2, \mathbb{Z})$ (see, e.g., [75, 81]). For certain choices of X there are, however, precise conjectures regarding the nature of $G(\mathbb{Z})$.

Let X be a rigid Calabi-Yau 3-fold ($h_{2,1}(X) = 0$) of CM-type, i.e. admitting complex multiplication by the ring of algebraic integers $\mathcal{O}_d$ in the quadratic number field $\mathbb{Q}(\sqrt{-d})$. In this case the intermediate Jacobian of X is an elliptic curve:

$$H^3(X, \mathbb{R})/H^3(X, \mathbb{Z}) \cong \mathbb{C}/\mathcal{O}_d. \tag{12.30}$$

We then have:

Conjecture 12.4 ([12, 13]). *For type IIB string theory compactified on a rigid Calabi-Yau 3-fold X of CM-type the "U-duality group" $G(\mathbb{Z})$ is the Picard modular group*

$$SU(2, 1; \mathcal{O}_d) := SU(2, 1) \cap GL(3, \mathcal{O}_d). \tag{12.31}$$

In particular, this group acts on the charge lattice $H^3(X, \mathbb{Z})$ and consequently on $\mathrm{D^bFuk(X)}$.

If correct, this suggests that the BPS-index $\Omega(\gamma)$ should arise as the Fourier coefficient of an automorphic form on $SU(2, 1)$ in a similar vein as for $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity discussed above. Constraints from supersymmetry further imply that there exists a class of 1/2 BPS-states that have support only on charges γ such that $\mathcal{Q}_4(\gamma) \geq 0$, where $\mathcal{Q}(\gamma)$ is a quartic polynomial in the charge vector γ. In other words, the BPS-index is constrained such that

$$\Omega(\gamma) = \begin{cases} n \neq 0 & \mathcal{Q}_4(\gamma) \geq 0 \\ 0 & \mathcal{Q}_4(\gamma) < 0. \end{cases} \tag{12.32}$$

It turns out that this constraint is precisely satisfied for Fourier coefficients of automorphic forms attached to the quaternionic discrete series of Lie groups G in their quaternionic real form [154, 155, 317]. This leads to the following:

Conjecture 12.5 ([12, 13, 259]). *The generalised Donaldson-Thomas invariants $\Omega(\gamma)$ of a CM-type rigid Calabi-Yau threefold X are captured by the Fourier coefficients of an automorphic form attached to the quaternionic discrete series of $SU(2, 1)$.*

Another interesting case is when X is a Calabi-Yau threefold with $h_{1,1} = 1$. One then expects that the U-duality group is an arithmetic subgroup $G_2(\mathbb{Z})$ of the split real form $G_2(\mathbb{R})$. Automorphic forms on G_2 associated with the quaternionic discrete series have been analysed in detail by Gan, Gross, Savin [106], and one has:

Conjecture 12.6 ([259, 267]). *There exists Calabi-Yau 3-folds X with $h_{1,1} = 1$ whose Donaldson-Thomas invariants $\Omega(\gamma)$ are captured by automorphic forms attached to the quaternionic discrete series of G_2, as analysed by Gan-Gross-Savin.*

Remark 12.7. For large values of the charges the index should reproduce the macroscopic entropy of the black hole which is known to be given by $S(\gamma) = \pi\sqrt{\mathcal{Q}_4(\gamma)}$. Translated into mathematics this implies that the Fourier coefficient should have an asymptotic growth given by

$$\Omega(\gamma) \sim e^{\pi\sqrt{\mathcal{Q}_4(\gamma)}} \quad \text{as} \quad \gamma \to \infty. \tag{12.33}$$

This gives rise to the following interesting puzzle. In general, Hecke eigenforms always give rise to Fourier coefficients that grow polynomially, and hence the growth in (12.33) does not seem to be compatible with the fact that the automorphic forms of Gan-Gross-Savin are indeed Hecke eigenforms. One possible resolution to this problem is that one should not consider honest automorphic forms in the quaternionic discrete series but rather some analogue of mock modular forms for G_2, a possibility suggested by Stephen D. Miller. This might also be consistent with the fact that the BPS-index $\Omega(\gamma)$ jumps discontinuously at certain co-dimension one walls in parameter space (known as *wall-crossing*) and this phenomenon is closely related to mock modularity (see, e.g., [77, 227, 228]).

12.5 The Langlands program

Any survey on automorphic forms would be incomplete without at least mentioning some of the key ideas involved in the *Langlands program*, the collective name given to the visionary conjectures outlined by Langlands in his letter to Weil in 1967 [216], and later expanded upon in the lecture notes "Problems in the theory of automorphic forms" [217]. To give a complete account of these conjectures goes far beyond the scope of this survey. However, we would like to give a heuristic discussion of some of the ingredients and their implications. This section leans on the discussions in sections 11.7 to 11.9. We will also make a few remarks on the geometric version of the Langlands program along with some speculative remarks on the connection with physics.

12.5.1 The classical version

The context of Langlands' letter to Weil was reductive groups G defined over an arbitrary number field $\mathbb{F}$ that can be either local (like $\mathbb{Q}_p$) or global (like $\mathbb{Q}$). Let us focus on the global situation. As usual we restrict our treatment to $\mathbb{F} = \mathbb{Q}$, and we let G be a split group over $\mathbb{Q}$; for example $GL(n, \mathbb{Q})$. Recall that being split over $\mathbb{Q}$ means that there exists a maximal torus which is a product of $GL(1, \mathbb{Q})$s. However, Langlands also considered groups G that were only *quasi-split*, meaning that they contain a Borel subgroup which is defined over $\mathbb{Q}$. Equivalently, a quasi-split group is split over an *unramified finite extension* $\mathbb{E}/\mathbb{F}$. We recall that finite extension of a field $\mathbb{F}$ is another field $\mathbb{E}$ that contains $\mathbb{F}$ and which has finite dimension as a vector space over $\mathbb{F}$, so in this case it is a finite-dimensional vector space over $\mathbb{Q}$. The group of automorphisms of the extension $\mathbb{E}$ is called the *Galois group* and denoted by $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$. In this more general context the L-group of $G(\mathbb{Q})$ is really defined as the semi-direct product

$$^{L}G = \widehat{G}(\mathbb{C}) \ltimes \mathrm{Gal}(\mathbb{E}/\mathbb{F}), \tag{12.34}$$

where the first factor is the complex group that we discussed in section 11.7. In the case when G is split over $\mathbb{F} = \mathbb{Q}$, like for $GL(n, \mathbb{Q})$, the Galois group acts trivially and the L-group becomes a direct product $^{L}G = \widehat{G}(\mathbb{C}) \times \mathrm{Gal}(E/\mathbb{Q})$. In this situation one can take the representation $\rho : {}^{L}G \to GL(n, \mathbb{C})$ that enters in the construction of L-functions $L(\pi, s, \rho)$ defined in (11.139), to have a trivial projection on the second Galois factor in ^{L}G, and we therefore recover the description of L-functions in section 11.9 where we had simply assumed $^{L}G = \widehat{G}(\mathbb{C})$, see also remark 11.11.

One of the main parts of Langlands conjectures is the *principle of functoriality*. To state it, let G and G' be reductive groups over $\mathbb{Q}$. The principle of functoriality asserts that whenever we have a group homomorphism between the associated L-groups

$$\Psi \,:\, {}^{L}G \longrightarrow {}^{L}G', \tag{12.35}$$

there should be a close relation between the associated automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ and $G(\mathbb{Q})\backslash G(\mathbb{A})$. What does "close relationship" mean? Suppose π is an automorphic representation of G associated with a Satake class $[A_\pi]$ in the dual group ^{L}G. Functoriality implies that there exists an automorphic representation π' of G' with Satake class $[A_{\pi'}] \subset {}^{L}G'$, such that

$$[A_{\pi'}] \cong [\Psi(A_\pi)]. \tag{12.36}$$

It turns out that this has far-reaching consequences even for the case when the first group is taken to be trivial. Suppose for example that $G = \{1\}$ and $G' = GL(n)$. In this situation the dual group of G is simply the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, where the extension is $\mathbb{E} = \bar{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. (Recall that algebraic closure $\bar{\mathbb{F}}$ of a number field $\mathbb{F}$ is obtaining by adjoining to $\mathbb{F}$ all roots of all polynomials over $\mathbb{F}$. This not a finite extension and so generalises the discussion above.) The L-dual group of G' is the direct product $^{L}G' = GL(n, \mathbb{C}) \ltimes \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The map Ψ then yields a homomorphism

$$\Psi \,:\, \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(n, \mathbb{C}). \tag{12.37}$$

This has the remarkable consequence that to each automorphic representation π of $GL(n, \mathbb{A})$ there should exist an associated n-dimensional representation R of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that

$$L(s, \pi) = L_A(s, R), \tag{12.38}$$

where the object on the left is the standard L-function of π discussed in section 11.9 (i.e. corresponding to ρ in section 11.9 being the fundamental representation of $GL(n, \mathbb{C})$) and the object on the right is the so called *Artin L-function* of the Galois representation R. We shall not go into the details of Artin L-functions but rather refer to [215] for a nice discussion of the two sides of (12.38) and also to remark 11.17.

There are numerous sources which give overviews of various aspects of the Langlands program; we would like to especially mention [4, 63, 116, 196, 215]. See also the two papers by Knapp [197, 198] which summarizes the key references in the field.

12.5.2 The Langlands program and physics

As we have indicated at several occasions in this treatise, automorphic forms occur in abundance in string theory (see [12, 13, 138–141, 195, 252, 259, 267] for a sample). Despite this fact the physical role of the classical Langlands program remains unclear. We know that automorphic representations play a role in understanding BPS-states and instantons in string theory, but we have no clue as to what is the physical interpretation of the dual side, involving representations of the Galois group. It would be very interesting to find out whether such an interpretation exists. Given that automorphic L-functions lie at the heart of the Langlands program, a very natural question, posed by Moore in [246], is the following:

Open question: *Is there a natural role for L-functions in BPS-state counting problems?*

For some speculations on this and related issues, see [234, 247], and for a conjectured connection between BPS-states in string theory and Galois representations, see [316].

12.5.3 The geometric version

We should also mention that there exists a version of the Langlands program which does not have its roots in number theory, but rather in the geometry of Riemann surfaces. This is commonly referred to as the *geometric Langlands program* (for a nice survey see the lectures notes by Frenkel [98]). To each object in the original (or, "classical") Langlands program there exists geometric counterparts; for instance, the role of the Galois group is played by the fundamental group of the Riemann surface, while automorphic forms are replaced by certain "automorphic sheaves" on the moduli space of principal bundles on the Riemann surface. Remarkably, Kapustin and Witten have shown [187] that the geometric Langlands program can be naturally understood in the context of quantum field theory (more precisely, a twisted version of $\mathcal{N} = 4$ supersymmetric quantum field theory in four spacetime dimension). In this context the analogue of the "Langlands duality" (12.37) corresponds to a variant of (homological) mirror symmetry.

12.6 Whittaker vectors, multiple Dirichlet series and statistical physics

In this section, we discuss some issues related to a fascinating connection between Whittaker vectors and statistical mechanics. Starting from a rewriting of the Casselman–Shalika formula, generalisations of Whittaker vectors to metaplectic groups will be given. Their relation to Weyl group multiple Dirichlet series will be discussed and an alternative interpretation in terms of lattice models. This is an active area of research that has received a lot of momentum through the work of Brubaker, Bump, Chinta, Friedberg, Gunnells, Hoffstein and many others. We rely in our exposition mainly on the collection [58] and on [50] and refer the reader also to [53] for an overview.

12.6.1 Generalisations of the Weyl character formula

The Casselman–Shalika vector for spherical Whittaker vectors $W^\circ(\lambda, a)$ on the group $G(\mathbb{Q}_p)$ was discussed in detail in chapter 9 and given an interpretation in terms of characters ch_Λ of the Langlands dual group ${}^LG$ in equation (9.119). This formula can actually be inverted to give an alternative formula for highest weight characters of ${}^LG$ through

$$\mathrm{ch}_\Lambda(a_\lambda) = \frac{W^\circ(\lambda, a_\Lambda)\delta^{-1/2}(a_\Lambda)}{\prod_{\alpha>0}(1-p^{-1}a_\lambda^\alpha)}. \tag{12.39}$$

Here, λ is a weight of the original group G parametrising the principal series representation and a_λ and a_Λ distinguished elements of A and ${}^LA$, respectively. These distinguished elements were defined in section 9.7.

Formula (12.39) resembles the standard Weyl character formula (4.27), in particular the denominator. Independent of Whittaker vectors, Tokuyama [310] considered a one-parameter family of deformations of the Weyl character formula that can be written as

$$\mathrm{ch}_\Lambda(a_\lambda) = \frac{\sum_{v\in\mathcal{B}_{\Lambda+\rho}} G(v,t)a_\lambda^{\mathrm{wt}(v)+\rho}}{\prod_{\alpha>0}(1+ta_\lambda^\alpha)} \tag{12.40}$$

where $t \in \mathbb{C}$. In this expression, all quantities refer to the Langlands dual group ${}^LG$. The sum here is over all v in the *crystal* $\mathcal{B}_{\Lambda+\rho}$. The crystal $\mathcal{B}_{\Lambda+\rho}$ is a directed graph with vertices v given by all the weights (with multiplicity) of the irreducible highest weight representation $V_{\Lambda+\rho}$ of ${}^LG$, where the ρ shift is important, and the edges labelled by simple roots. The map $\mathrm{wt} : \mathcal{B}_{\Lambda+\rho} \to \mathfrak{h}$ identifies the vertices with points in the weight lattice of ${}^LG$. Crystals were introduced by Kashiwara [188] in his study of the *quantum deformed universal enveloping algebra* $U_q({}^L\mathfrak{g})$ (closely related to *quantum groups* [226]) and possess a *canonical basis* in the sense of Kashiwara and Lusztig [188, 225]. The operators corresponding to the edges are the simple step operators f_i in the limit $q \to 0$. Kashiwara introduced also the crystal $\mathcal{B}_\infty$ that is modeled on the canonical (free) *Verma module* of $U({}^L\mathfrak{n}_-)$.

Following [52], we will call the complex function $G(v,t)$ a *Tokuyama function* and it is the main object of interest in this expression. In the original paper [310], the numerator was not written in terms of the crystal $\mathcal{B}_{\Lambda+\rho}$ but in terms of *Gelfand–Tsetlin patterns* [119] with top row $\Lambda+\rho$ and the analysis restricted to the special linear group. We will comment later on the status for other groups.

Interesting special cases of the deformed character formula (12.40) are

- $t=-1$. This is the value for the standard character formula (4.27). In this case the sum over the crystal collapses to a sum over the Weyl orbit of the shifted highest weight $\Lambda+\rho$. In other words, $G(v,-1)=0$ unless $\mathrm{wt}(v)=w(\Lambda+\rho)$ for some $w\in\mathcal{W}$ and in that case $G(v,-1)=\epsilon(ww_{\mathrm{long}})$.

- $t=1$. In this case one obtains a relation to the original formulation of Gelfand-Tsetlin patterns.

- $t=0$. The denominator trivialises and Tokuyama use this case to recover a relation of Stanley's [301] between Gelfand-Tsetlin patterns and 'most singular' values of Hall–Littlewood polynomials. In the crystal formulation, the only contributing terms arise from the embedding of $\mathcal{B}_\Lambda\to\mathcal{B}_{\Lambda+\rho}$ [58] and the sum then becomes the character in the form (4.22).

- $t=-p^{-1}$. This is the case relevant for the Casselman–Shalika formula and will be discussed in more detail below.

The first three cases were originally studied by Tokuyama [310]. The last case in relation to Whittaker vectors was first explored by Bump, Brubaker, Bump, Friedberg and Hoffstein [43], see also [161] for combinatorial aspects.

In the case $t=-p^{-1}$, we can compare (12.40) and (12.39) to deduce that we have an alternative description of Whittaker vectors in terms of a sum over a crystal with a Tokuyama function $G(v,-p^{-1})$:

$$W^\circ(\lambda,a_\Lambda)=\delta^{1/2}(a_\Lambda)\sum_{v\in\mathcal{B}_{\Lambda+\rho}}G(v,-p^{-1})a_\lambda^{\mathrm{wt}(v)+\rho}. \tag{12.41}$$

In order to ease notation, we will suppress the t-value in the Tokuyama function in the sequel and will simply write $G(v)$ instead of $G(v,-p^{-1})$.

The identity (12.41) in some sense defines the Tokuyama function $G(v)$ given the spherical Whittaker. But it is desirable to have an independent description of the function $G(v)$. This was achieved in crystal form in [50] and can be given in terms of so-called decorated *Berenstein–Zelevinsky–Littelmann paths* (BZL paths) in the crystal [21, 223]. A BZL path of a vertex $v\in\mathcal{B}_{\Lambda+\rho}$ is given by first fixing a choice of a reduced expression of the longest Weyl word w_{long}:

$$w_{\mathrm{long}}=w_{i_1}\cdots w_{i_\ell}, \tag{12.42}$$

where $\ell=\ell(w_{\mathrm{long}})$ is the length of the longest Weyl word and w_i is the i-th fundamental reflection. The BZL path $\mathrm{BZL}(v)$ of a crystal vertex $v\in\mathcal{B}_{\Lambda+\rho}$ is then obtained by following the simple lowering operators f_i as far as possible through the crystal in the order given in the reduced expression of w_{long}. Let b_1 be the largest integer such that $f_{i_1}^{b_1} v \neq 0$, that is, b_1 is the maximum number of steps in the direction of f_{i_1} one can take in the crystal without leaving it. Starting from the point obtained in this way one then constructs b_2 as the maximum number of steps in the f_{i_2} direction and so. This yields a sequence of non-negative integers

$$\text{BZL}(v) = (b_1, b_2, \ldots, b_\ell) \tag{12.43}$$

and the endpoint of the crystal always corresponds to the 'lowest weight' $v_- \in \mathcal{B}_{\Lambda+\rho}$ with $\text{wt}(v_-) = w_{\text{long}}(\Lambda + \rho)$. A vertex v is uniquely characterised by its string $\text{BZL}(v)$ (for a fixed choice of reduced expression (12.42)).

For determining the function $G(v)$, the BZL string $(b_1, \ldots, b_\ell)$ needs to be decorated further. In the case of $G = GL(r+1)$ of rank r this is described in [50] for two choices of reduced w_{long} words. We will give here the version for

$$w_{\text{long}} = w_1 w_2 w_1 w_3 w_2 w_1 \cdots w_r w_{r-1} \cdots w_2 w_1 \tag{12.44}$$

and note that $\ell = \frac{1}{2} r(r+1)$.

Remark 12.8. There is another common choice of reduced expression $w_{\text{long}} = w_r w_{r-1} w_r w_{r-1} w_{r-2} \cdots w_r \cdots w_2 w_1$ that is obtained by starting at the other end of the Dynkin diagram [51]. We will not use it here—it corresponds to Δ-ice in a statistical mechanics interpretation whereas the choice here corresponds to Γ-ice.

The numbers in the BZL string $\text{BZL}(v)$ of (12.43) are then arranged in a triangular (Gelfand–Tsetlin-like) pattern according to

$$\text{BZL}(v) = \left\{ \begin{array}{ccc} \cdots & \cdots & \cdots \\ b_3 & b_2 & \\ b_1 & & \end{array} \right\}, \tag{12.45}$$

such that the i-th column contains all numbers associated with the w_i fundamental reflection. Littelmann proved that the numbers along a fixed row are weakly increasing [223]. Entries in this tableaux now get circle or box decorations according to the following rules: (i) if an entry b_k is equal to its left neighbour (or equal to 0 if it does not have one) it is circled; (ii) if the crystal point $f_{i_{k-1}}^{b_{k-1}} \cdots f_{i_1}^{b_1} v$ does not have a neighbour in the e_{i_k} direction, i.e. it sits on the boundary, then b_k is boxed. An example of this description is given in figure 12.2 for the case $r = 3$. The boxing and circling rules can be given a geometrical interpretation in terms of the embedding of $\mathcal{B}_{\Lambda+\rho}$ into $\mathcal{B}_\infty$ [59].

The decorated BZL string can then be used to define the Tokuyama function $G(v)$ via [58]

$$G(v) = \prod_{b_k \in \text{BZL}(v)} \left\{ \begin{array}{cl} 1 & \text{if } b_k \text{ is circled but not boxed} \\ -p^{-1} & \text{if } b_k \text{ is boxed but not circled} \\ 1 - p^{-1} & \text{if } b_k \text{ is neither boxed nor circled} \\ 0 & \text{if } b_k \text{ is boxed and circled} \end{array} \right. \tag{12.46}$$

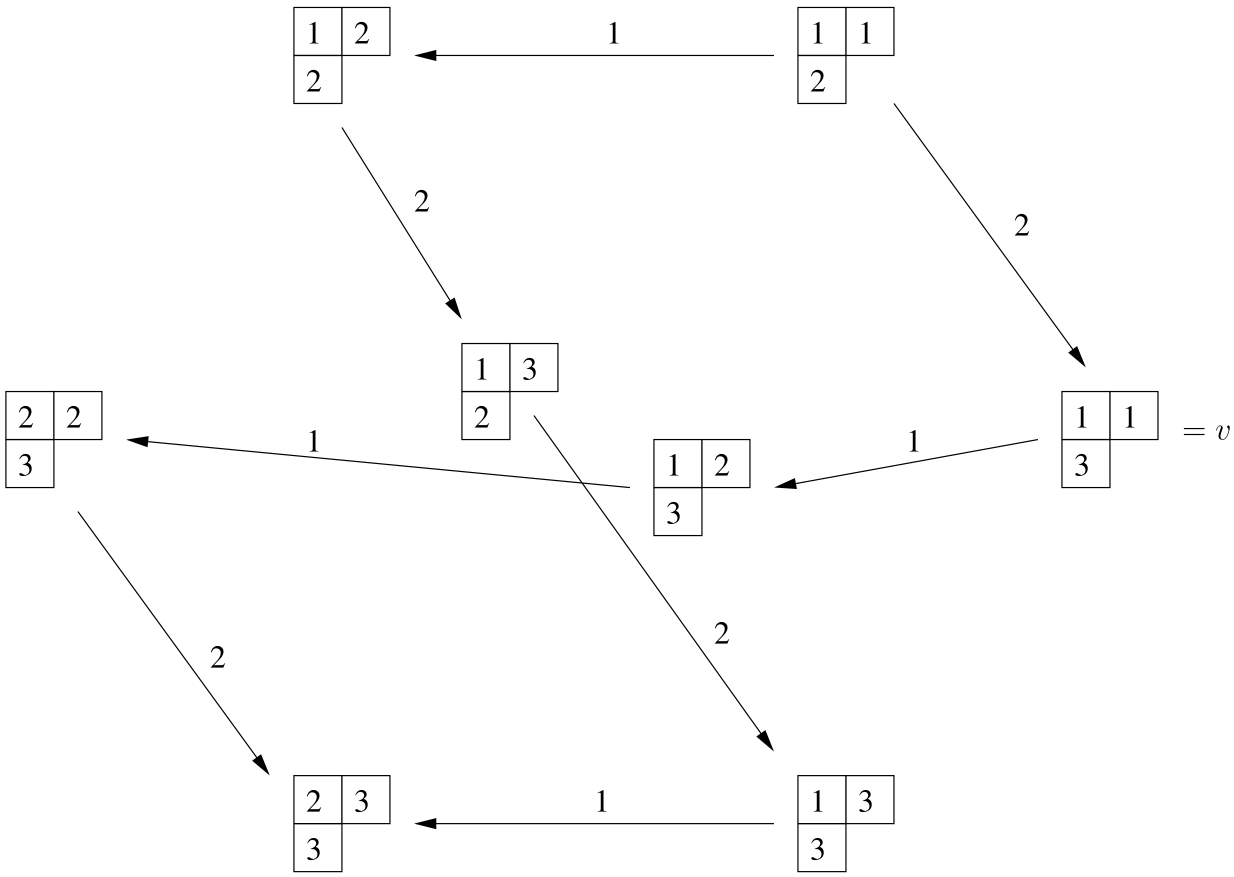


Figure 12.2: *The crystal* $\mathcal{B}_{\Lambda+\rho}$ *for* $SL(3)$ *and* $\Lambda = 0$. *The* ρ*-shift turns this into the weight diagram of the adjoint representation and we label the different vertices of the crystal by filled Young tableaux. The arrows with numbers indicate the action of the operators* f_1 *and* f_2. *The two tableaux in the center correspond to the multiplicity two weight space of the adjoint representation associated with the two-dimensional Cartan subalgebra. For the choice of vertex* $v = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$, *and the reduced expression* $w_{long} = w_1 w_2 w_1$, *the BZL path is* $\mathrm{BZL}(v) = (2, 1, 0) = \left\{ \begin{array}{cc} ⓪ & \boxed{1} \\ \boxed{2} & \end{array} \right\}$, *where we have circled and boxed the entries according to the rules described in the text.*

More complicated versions of this rule exist for other values of the Tokuyama deformation parameter t [50]. An equivalent description of $G(v,t)$ directly in terms of Gelfand-Tsetlin patterns was given in [43, 48, 51].

Example 12.9: Crystal description of $SL(2,\mathbb{Q}_p)$ Whittaker vector

We consider the case $G = SL(2,\mathbb{Q}_p)$ and verify formulas (12.41) and (12.46). The Whittaker vector $W^\circ(\lambda, a_\Lambda)$ is (cf. (9.125))

$$W^\circ(\lambda, a_\Lambda) = (1-p^{-2s})\frac{p^{sN-N} - p^{-2s+1-sN}}{1-p^{-2s+1}}. \tag{12.47}$$

To work out the crystal sum we fix the longest Weyl word as $w_{\text{long}} = w_1$ and the highest weight is $\Lambda + \rho = (N+1)\rho$ where we recall that everything refers to Langlands dual group of $SL(2,\mathbb{Q}_p)$. Then the crystal $\mathcal{B}_{\Lambda+\rho}$ consists of the vertices $v \in \{(N+1)\rho, (N-1)\rho, \ldots, -(N+1)\rho\}$ that we label $v_k = (N+1-2k)\rho$ for $k = 0, \ldots, N+1$. The highest weight representation $V_{\Lambda+\rho}$ is of dimension $N+2$. The BZL path of a vertex v_k is

$$\text{BZL}(v_k) = (N+1-k) \tag{12.48}$$

and its single entry is circled for $k = N+1$ and boxed for $k = 0$, otherwise it is undecorated. Therefore

$$G(v_k) = \begin{cases} -p^{-1} & \text{for } k = 0 \\ 1 & \text{for } k = N+1 \\ 1-p^{-1} & \text{otherwise} \end{cases} \tag{12.49}$$

The right-hand side of equation (12.41) becomes therefore ($a_\lambda^\rho = p^{-(2s-1)/2}$)

$$\begin{aligned} p^{-N/2}\sum_{k=0}^{N+1} G(v_k) a_\lambda^{(N+2-2k)\rho} &= p^{-sN-2s+1}\left(-p^{-1} + (1-p^{-1})\sum_{k=1}^{N} p^{k(2s-1)} + p^{(N+1)(2s-1)}\right) \\ &= p^{-sN-2s+1}\left(-1 + (1-p^{-1})\frac{1-p^{(N+1)(2s-1)}}{1-p^{2s-1}} + p^{(N+1)(2s-1)}\right) \\ &= p^{-sN-2s}\frac{-1+p^{2s}+p^{(N+1)(2s-1)} - p^{(N+1)(2s-1)+2s}}{1-p^{2s-1}} \\ &= p^{-(N+2)s}(1-p^{2s})\frac{1-p^{(N+1)(2s-1)}}{p^{2s-1}-1} \\ &= p^{-sN}(1-p^{-2s})\frac{p^{N(2s-1)} - p^{-2s+1}}{1-p^{-2s+1}}, \end{aligned} \tag{12.50}$$

which agrees with (12.47).

Tokuyama's formula for the Tokuyama function $G(v)$ of (12.46) gives a purely combinatorial description of the spherical Whittaker vector $W^\circ(\lambda, a_\Lambda)$ for the case $G = GL(r+1,\mathbb{Q}_p)$. One may wonder whether other choices of Tokuyama function $G(v)$ also correspond to objects related to automorphic forms. An affirmative answer to this was provided by Bump, Brubaker and Friedberg and we will discuss this now in a broader context [58].

12.6.2 Weyl group multiple Dirichlet series

In sections 11.3 and 11.9 we introduced Dirichlet series and automorphic L-functions. Both are meromorphic functions of a single complex variable s, satisfy functional equations for $s \leftrightarrow 1-s$ and have an Euler product form. They correspond to multiplicative sequences a_n of numbers, in the simplest case of an $SL(2,\mathbb{Z})$ cuspidal Hecke eigenform f these are just the Fourier coefficients of f, cf. (11.34), so there is a close connection between Fourier expansions of automorphic forms and Dirichlet series, see sections 11.9 and 11.10 for more details.

It is natural to wonder whether these concepts can be generalised to functions of *several* complex variables $s_1, \dots, s_r$. This is a non-trivial problem and it turns out that multiplicativity of the coefficients cannot be maintained, see [58] that also discusses the history of the subject. One way of constructing such multiple Dirichlet series is as so-called *Weyl group multiple Dirichlet series* [52].

To introduce them we again restrict to $G = GL(r+1)$ of rank r and introduce the following additional definitions [46]. Let $\mathbb{F}$ be a number field that contains the group μ_{2n} of $2n$-th roots of unity. Let S be a finite set of places of $\mathbb{F}$ such that S includes all archimedean places (e.g. $p = \infty$) and all divisors of n. We denote by $\mathbb{F}_p$ the completion of $\mathbb{F}$ at a place p and by $\mathfrak{o}_p$ the corresponding integers for p non-archimedean. The ring of S-integers $\mathfrak{o}_S$ in $\mathbb{F}$ are those $x \in \mathbb{F}$ whose component x_p is in $\mathfrak{o}_p$ for all $p \notin S$. We can enlarge, if necessary, S such that the S-integers $\mathfrak{o}_S$ are a principal ideal domain. We denote $\mathbb{F}_S = \prod_{p\in S} \mathbb{F}_p$.

A general form for a multiple Dirichlet series is then given by

$$Z_\Psi(\mathbf{s},\mathbf{m}) = \sum_{\text{ideals}(C_i)} \Psi(C_1,\dots,C_r) H(C_1,\dots,C_r; m_1,\dots,m_r) |C_1|^{-2s_1} \cdots |C_r|^{-2s_r} \quad (12.51)$$

for $\mathbf{s} = (s_1,\dots,s_r) \in \mathbb{C}^r$ and $\mathbf{m} = (m_1,\dots,m_r) \in (\mathfrak{o}_S)^r$. Here, $\Psi : (\mathbb{F}_S^\times)^r \to \mathbb{C}$ and $H : (\mathbb{F}_S^\times)^r \times (\mathfrak{o}_S)^r \to \mathbb{C}$ are functions with multiplicativity properties that ensure that the sum over ideals in the principal ideal domain $\mathfrak{o}_S$ is well-defined. These properties rely on the properties of the *n-order Hilbert symbol* and on n-th order reciprocity. We will make no further use of the precise conditions here and refer the reader to [46] for the details. We note, however, that the conditions make Ψ a member of a finite-dimensional vector space $\mathcal{M}$ and that this space carries an action of the Weyl group $\mathcal{W}$ [46].

The function H satisfies an additional multiplicative property, called *twisted multiplicativity*, that ensures that it is completely determined by its values on prime powers $H(p^{k_1},\dots,p^{k_r}; p^{l_1},\dots,p^{l_r})$. The parameters $\mathbf{m}$ appearing in (12.51) are called the twisting parameters and they enter crucially in the twisted multiplicativity relation. The problem of finding interesting multiple Dirichlet series is then reduced to specifying the $H(p^{k_1},\dots,p^{k_r}; p^{l_1},\dots,p^{l_r})$. One requirement one would naturally impose on them is that, when viewed as a function of one s_i alone, one obtains sums of single Dirichlet functions with standard functional relations. (Since we are working over a field that contains μ_{2n} these will actually be so-called *Kubota Dirichlet series* [208] and we refer again to [46] for the details.)

This requirement will ensure that the multiple Dirichlet series $Z_\Psi(\mathbf{s},\mathbf{m})$ will satisfy a functional relation under the fundamental Weyl reflection w_i of the form

$$Z_\Psi(w_i\mathbf{s},\mathbf{m}) = Z_{\Psi'}(\mathbf{s},\mathbf{m}), \tag{12.52}$$

where Ψ' is some other element of the finite-dimensional space $\mathcal{M}$. By choosing a suitable normalisation

$$Z^\star_\Psi(\mathbf{s},\mathbf{m}) = \left(\prod_{\alpha>0}\zeta_\alpha(\mathbf{s})G_\alpha(\mathbf{s})\right) Z_\Psi(\mathbf{s},\mathbf{m}) \tag{12.53}$$

in terms of factors of the Dedekind zeta function of $\mathbb{F}$ and appropriate Γ-function factors evaluated at places parametrised by the positive roots α and $\mathbf{s}$ one can bring this functional relation into the nicer form

$$Z^\star_{w\Psi}(w\mathbf{s},\mathbf{m}) = Z^\star_\Psi(\mathbf{s},\mathbf{m}) \tag{12.54}$$

by using the action of $\mathcal{W}$ on $\mathcal{M}$. A functional equation of this type allows meromorphic continuation of the multiple Dirichlet series from the domain of convergence of (12.51) to $\mathbb{C}^r$ by means of a variant of Bochner's theorem about complex functions in tube domains [52].

Finding $H(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r})$ that satisfy this requirement is a non-trivial combinatorial problem. Essential information on the multiple Dirichlet series is contained in the so-called *p-part* of Z_Ψ that is defined by suppressing Ψ:

$$\sum_{k_i=0}^{\infty} H(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r})|p|^{-2k_1s_1-\ldots-2k_rs_r}. \tag{12.55}$$

The p-part depends on the s_i and on the twisting parameters that are now given in terms of the integers l_i. We would like to interpret the p-part as an expression evaluated on a crystal of ${}^LG$ evaluated at a special point a_λ as in (12.41). To this end, we consider the case when the non-negative integers $(k_1,\ldots,k_r)$ correspond to a vector κ linking a (strongly dominant) highest weight $\Lambda+\rho$ to one of its Weyl images, i.e.,

$$\kappa = \sum_i k_i\alpha_i = \Lambda+\rho - w(\Lambda+\rho) \tag{12.56}$$

for some $w\in\mathcal{W}$, such that the second term in (12.55) is basically $\delta^{1/2}(a_\Lambda)a_\lambda^{\kappa+\rho}$. The highest weight Λ here is determined by the integers l_i through $\Lambda=\sum_i l_i\varpi_i$, where ϖ_i are the fundamental weights of ${}^LG$. Suppose now that the twisting parameters l_i and hence Λ are fixed, then one can evaluate for $H(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r})$ for those k_i for which (12.56) is satisfied as finite product of Gauss sums [47]

$$H(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r}) = \prod_{\substack{\alpha>0\\ w\alpha<0}} g_2(p^{\langle\Lambda+\rho|\alpha\rangle-1},p^{\langle\Lambda+\rho|\alpha\rangle}), \tag{12.57}$$

where all elements now refer to the Langlands dual group and g_2 is a certain Gauss sum [47]. The points k_i for which one thus has a relatively simple formula for the value of H on the

Weyl orbit of the highest weight $\Lambda+\rho$. It is an important observation that the only other values of k_i for which $H(p^{k_1},\dots,p^{k_r};p^{l_1},\dots,p^{l_r})$ can be non-zero are the other points of the crystal $\mathcal{B}_{\Lambda+\rho}$. We are thus in a very similar situation to the discussion of the Weyl character formula and its generalisations above. For the strict Weyl character of the highest weight representation $V_{\Lambda+\rho}$ the crystal sum (12.40) only had support on the Weyl images of $\Lambda+\rho$, but for the spherical Whittaker vector one needed to consider also the other points of $\mathcal{B}_{\Lambda+\rho}$, in particular those in its interior.

Determining $H(p^{k_1},\dots,p^{k_r};p^{l_1},\dots,p^{l_r})$ at the other points of $\mathcal{B}_{\Lambda+\rho}$ is non-trivial and a number of approaches to this problem exist.

- One approach is called the *averaging method* of Chinta–Gunnells [58, 65, 66] that employs a deformed character constructed using the averaged Weyl group action on the field of rational functions in several variables. This approach works uniformly for any type of root system. It has also been extended to the affine case in [221].

- The approach by Bump, Brubaker and Friedberg [47, 50] starts from the just mentioned analogy with the crystal sum and finds rules for computing the Tokuyama function $G(v)$ on the crystal. These resemble the rules (12.46) above but instead one gives different weights to the various parts of the BZL path. These weights are not simple powers of p but instead involve n-th order Gauss sums. In this approach one has to treat each type of root system separately. For the various classical and some exceptional types we refer the reader to [19, 20, 45, 67, 100–102].

- Given a Tokuyama function $G(v)$ on the crystal one might wonder, in view of (12.41), whether the crystal sum can be interpreted as the Whittaker vector of some Eisenstein series. It turns out that in order for this to be true one needs to consider *metaplectic Eisenstein series*. These are functions defined over the group $G(\mathbb{F})$ where $\mathbb{F}$ is the number field that contains the roots of unity μ_{2n}. One can define Eisenstein series over $G(\mathbb{F})$ and consider their Whittaker vectors in the same way as we did for $G(\mathbb{Q}_p)$. However, it turns out that for $n>1$ one no longer has a multiplicity one theorem for Whittaker vectors and that over a global field the Eulerian property is similarly no longer guaranteed. The multiplicative property of standard (non-metaplectic) Whittaker vectors is replaced by the *twisted multiplicativity* that we have already encountered above. The *metaplectic Whittaker vectors* are sources for the coefficients of multiple Dirichlet series as shown by Brubaker, Bump and Friedberg [44, 50, 58, 231, 232]. This approach works for all types of root system.

- Finally, one can interpret the crystal sum as the partition function of a statistical mechanical mode, as done by Brubaker, Bump and Friedberg in [49, 58]. In the case $n=1$, this opens up new tools from the theory of integrable systems, most notably the Yang–Baxter equation. This goes back to work of Kostant on the (quantum) Toda lattice and representation theory [206]. In this approach one has to treat different types of root system differently.

The equivalence of these different approaches has been shown in many cases and we refer to [58] for an overview.

12.7 Extension to Kac–Moody groups

In this article we have concentrated on the study of automorphic forms and in particular Eisenstein series defined on finite-dimensional Lie groups, as categorised in the Cartan classification. In this section we give a short summary of what happens as one makes the extension to infinite-dimensional *Kac–Moody groups*, which are generated by an infinite number of raising and lowering operators. A complete classification of Kac–Moody groups is at present not known and we will restrict our attention here mainly to Eisenstein series defined on *affine, hyperbolic and Lorentzian Kac–Moody groups*. Full accounts of Kac–Moody algebras can be found in the books [128, 186, 209, 245, 315]. The motivation for studying Eisenstein series defined on infinite-dimensional Kac–Moody groups is twofold.

12.7.1 String theory motivation: infinite-dimensional U-duality

In string theory, Kac–Moody groups appear, for example, in the list of U-duality groups encoding discrete symmetries of type II string theory compactified on a $(10-D)$-dimensional torus from ten down to D space-time dimensions. The list of these groups was given in table 2.1 in chapter 2 and consists of the groups in the exceptional series of the Cartan classification, where in $D \geq 3$ dimensions the respective U-duality group is given by the finite-dimensional and discrete group $E_{11-D}(\mathbb{Z})$. In $D = 2, 1$ and 0 dimensions, however, the corresponding U-duality groups are infinite-dimensional and are conjectured to be given by the affine, hyperbolic and Lorentzian Kac–Moody groups $E_9(\mathbb{Z})$, $E_{10}(\mathbb{Z})$ and $E_{11}(\mathbb{Z})$, respectively [171]. In particular, the groups E_{10} and E_{11} are of special relevance [78, 321], since they have been conjectured as fundamental symmetries of *M theory* [272, 322], a theory whose low-energy limit is eleven-dimensional supergravity and from which the five different known types of string theories can be derived as particular limits.

12.7.2 Mathematical motivation: new automorphic L-functions

From a mathematician's perspective one motivation to study Kac–Moody Eisenstein series is to consider them as a potential source for deriving new L-functions in a Fourier through a Fourier expansion of the series. It is however not precisely clear if this extension of the theory of Eisenstein series will yield necessarily to new types of L-functions and the focus of the discussion has so far been on series defined on affine groups. In fact in [293] an argument was provided that no new functions will be found, while in [111] a new method for obtaining such functions was devised. This new method relies on an expansion of the series with respect to "lower triangular parabolics", instead of only "upper triangular parabolics".

In recent years there has been some work, developing the theory of Eisenstein series for Kac–Moody groups. The most well developed part is that of Eisenstein series defined on affine Kac–Moody groups which was started by Garland [108, 109]. While for the finite-dimensional groups convergence with respect to the (complex) defining weight λ was proven over almost all of the complex plane, c.f. (5.100), for the infinite-dimensional Kac–Moody groups convergence is restricted and the defining weight has lie inside the Tits cone [186].

Furthermore, a restriction on the group element forming the argument of the series has to be imposed [109]. First steps towards a definition of Eisenstein series on hyperbolic Kac–Moody groups have been made in [61], where the case of rank 2 hyperbolics was considered. Furtheromore, in [95, 97] Eisenstein series defined on the hyperbolic E_{10} group (along with E_9 and E_{11}), were discussed. A general proof of convergence of Eisenstein series on general hyperbolics remains to be developed, however.

12.7.3 Fourier coefficients and small representations

Despite the absence of a mathematically rigorous definition of Kac–Moody Eisenstein series, quite a bit can be said about the Fourier expansion of these series. The foundation for this work was laid in [108], where the analogue of Langlands' formula (8.41) for the constant term, was developed for the case of affine Kac–Moody Eisenstein series. While for Eisenstein series on finite-dimensional groups we have explained how to evaluate Langlands' formula in section 10.3, it is not clear how to apply this formula in the case of affine Kac–Moody Eisenstein series. The reason for this is that the sum over Weyl words appearing in the formula is an infinite sum due to the infinite-dimensional nature of affine groups and their associated Weyl groups. The same problem of course also appears when considering the extension of Langlands' formula for the cases of hyperbolic and Lorentzian Kac–Moody groups. This question was taken up in [95], see also [96] for a summary of this work, where it was shown that for special types of Kac–Moody Eisenstein series, the naively infinite sum 'collapses' to finite sum and can be explicitly *computed*. On a more technical level, to evaluate Langlands' formula, one proceeds just as in the case of a finite-dimensional group and one successively constructs Weyl words in the set $\mathcal{C}(\lambda)$ by the orbit method, c.f. section 10.3. It can then be shown that for particular types of Eisenstein series, which we will discuss in a moment, only the coefficients $M(w, \lambda)$ associated with the first few Weyl words w in the carefully constructed orbit, are non-zero. All other coefficients associated with the infinite number of Weyl words that follow in the orbit are however zero and therefore do not contribute to the constant term.

The Eisenstein series for which this collapse of the constant term happens are of the general form (5.196) defined on the groups E_9, E_{10} and E_{11}. More specifically, the defining weight λ is of the form $\lambda = 2s\Lambda_1 - \rho$, such that the series is defined with respect to the maximal parabolic subgroup P_1 associated with the first node of the E_{11-D} Dynkin diagram in Bourbaki labelling. In order to observe collapse, the generically complex parameter s has to take real values of $s = \frac{3}{2}, \frac{5}{2}, \ldots$, see [95] for an extended list.

Remark 12.10. Let us mention as an aside that these particular Eisenstein series appear as the automorphic couplings of the R^4 and D^4R^4 curvature correction terms in the low-energy effective action of type IIB string theory in $D = 2, 1$ and 0 dimensions, in line with the discussion in section 12.1. Since the R^4 and D^4R^4 term are $\frac{1}{2}$- and $\frac{1}{4}$-BPS protected terms, it is a reassuring confirmation that the corresponding constant terms only contain a finite number of perturbative contributions. Furthermore, these Eisenstein series are associated with small representations which we discuss in some detail in section 12.1.1.

Developing an understanding of the structure of Fourier coefficients of Kac–Moody Eisenstein series is an open problem and part of ongoing work. However, it is possible to apply formula (9.59) for the degenerate Whittaker vectors also to Kac–Moody Eisenstein series (with slight modifications in the affine case) and use this to make some statements about the Fourier coefficients. This was done in [97], where the formula for the degenerate Whittaker vectors (9.59) was derived, applied to the cases of the particular maximal parabolic Eisenstein series just mentioned above and explicit expressions for the Whittaker vectors were computed. In particular it was found that in the case of $s = \frac{3}{2}$ the (abelian) Fourier coefficients are completely determined by maximally degenerate Whittaker vectors. The collapse property discussed for the case of the constant term above, also plays a central role in computing these Whittaker vectors. Related work, with a focus on the rank 2 affine case, can be found in [224]. We formalise the observations of [95, 97] as:

Conjecture 12.11 (Small representations for Kac–Moody groups). *Kac–Moody groups possess a minimal unitary representation that can be realised automorphically. In the case of $E_n(\mathbb{R})$ (for $n \geq 9$) this can be achieved by inducing from the maximal parabolic subgroup of with semi-simple Levi group of type D_{n-1} that is obtained by deleting the first node of the E_n Dynkin diagram. The canonically associated Eisenstein series for $s = \frac{3}{2}$ (obtained by analytic continuation) is the spherical vector in the minimal automorphic representation. The wave-front set is of Bala–Carter type A_1.*

A similar next-to-minimal representation is obtained for $s = \frac{5}{2}$ and its wave-front set is of Bala–Carter type $2A_1$.

12.7.4 Langlands program for Kac–Moody groups

Braverman and Kazhdan have also started to develop the local theory for affine Kac–Moody groups (see [37] for a survey). In particular, they have constructed the local spherical Hecke algebra [39], as have Gaussent and Rousseau [115]. With Patnaik they have proven an affine version of the Satake isomorphism [38]. These results were recently used in [36] to prove a Gindikin–Karpelevich formula for affine Kac–Moody groups and in [193, 257] the Casselman–Shalika formula has been generalised to the affine setting.

As formulated in [37], *the dream* is to have a fully developed representation theory and an associated Langlands correspondence for any (symmetrizable) Kac–Moody group. Although at present this remains a dream, the recent developments reviewed above certainly provides hope that such a theory is within reach.

Appendix A

$SL(2,\mathbb{R})$, $\mathbb{H}$ and $SL(2,\mathbb{Z})$

This appendix serves as a reference for our conventions on notation related to $SL(2,\mathbb{R})$.

A.1 $SL(2,\mathbb{R})$ Lie group and $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

We take $SL(2,\mathbb{R})$ to be the real Lie group defined (in its fundamental representation) by

$$SL(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \, \text{and } \det(g) = ad - bc = 1 \right\}. \tag{A.1}$$

The maximal compact subgroup is $K = SO(2,\mathbb{R})$ corresponding to the orthogonal matrices within $SL(2,\mathbb{R})$.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ has the standard Chevalley basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.2}$$

with commutation relations

$$[h,e] = 2e\,, \quad [h,f] = -2f\,, \quad [e,f] = h\,. \tag{A.3}$$

The generator h acts diagonally and is called the Cartan generator; e is a positive step operator and f a negative step operator. The compact subgroup $SO(2,\mathbb{R})$ is generated by the combination $e - f$.

The universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2,\mathbb{R}))$ has a distinguished second order element, called the *Casimir operator* and that we define by

$$\Omega = \frac{1}{4}h^2 + \frac{1}{2}ef + \frac{1}{2}fe = \frac{1}{4}h^2 - \frac{1}{2}h + ef. \tag{A.4}$$

This definition is unique up to normalisation. The Casimir operator commutes with all Lie algebra elements.

The Iwasawa decomposition of $SL(2,\mathbb{R})$ can be chosen in the form $SL(2,\mathbb{R}) = NAK$; where N is in the image of the exponential map exp applied to e; the maximal torus is in the image of exp applied to h and K is the compact subgroup $SO(2,\mathbb{R})$ whose identity component is the exponential of $e-f$. Concretely that means that we can write any element g of $SL(2,\mathbb{R})$ as

$$\begin{aligned} g = nak &= \exp(xe)\exp\left(\frac{1}{2}\log(y)h\right)\exp(\theta(e-f)) \\ &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned} \tag{A.5}$$

with $k \in K = SO(2,\mathbb{R})$ and $y > 0$.

A.2 The upper half plane $\mathbb{H}$ and $SL(2,\mathbb{Z})$

A main object of interest to us is the two-dimensional coset $G/K = SL(2,\mathbb{R})/SO(2,\mathbb{R})$; a representative for any point of this space is given by the first two factors in (A.5). The coset space can therefore be parametrised by elements of the *upper half plane*

$$\mathbb{H} = \{z = x+iy \mid x, y \in \mathbb{R} \text{ and } y > 0\} \cong G/K. \tag{A.6}$$

The coset space G/K (or, equivalently, the upper half plane $\mathbb{H}$) carries an action of $SL(2,\mathbb{R})$ by left multiplication: An element $\gamma \in G$ transforms a g into $g' = \gamma g$. The action on the explicit parameters $z \in \mathbb{H}$ can be read off from writing the new element in Iwasawa form $g' = n'a'k'$. Performing this calculation one finds

$$z' = \gamma \cdot z = \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}). \tag{A.7}$$

Using the Iwasawa decomposition (A.5), we see that the point i is left invariant by the maximal compact subgroup $K = SO(2,\mathbb{R})$ and that

$$g \cdot i = x + iy = z. \tag{A.8}$$

For $SL(2,\mathbb{R})$ automorphic forms one requires functions $f(g)$ that are invariant under the action of a discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$. Taking $\Gamma = SL(2,\mathbb{Z})$ to consist of the $SL(2,\mathbb{R})$ matrices with integral entries, the invariance $f(\gamma g) = f(g)$ for all $\gamma \in SL(2,\mathbb{Z})$ means we requires the double quotient $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2,\mathbb{R})$ where $SL(2,\mathbb{Z})$-equivalent points are identified. Using the upper half plane $\mathbb{H}$ presentation of $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ one can give a very explicit description of the double quotient.

The group $SL(2,\mathbb{Z})$ is well-known to be generated by [86].

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.9}$$

When acting on $z \in \mathbb{H}$, they generate

$$T \cdot z = z+1, \quad S \cdot z = -\frac{1}{z}. \tag{A.10}$$

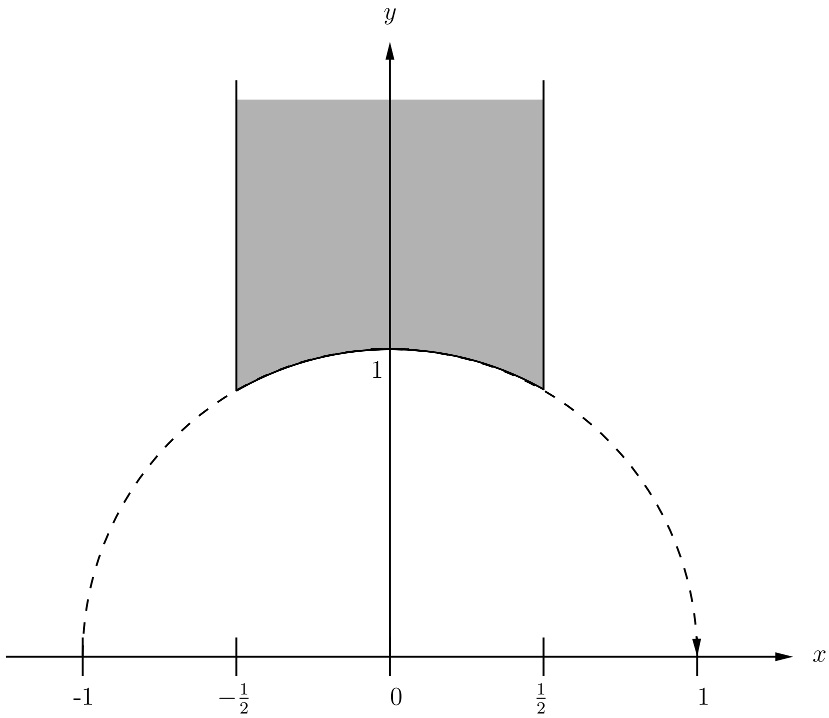


Figure A.1: *A fundamental domain for the action of $SL(2,\mathbb{Z})$ acting on the upper half plane (grey region). The cusp is at $y \to \infty$.*

Therefore, T is a translation by one unit and S is inversion in the unit circle combined with a reflection in the y-axis. A fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half plane is depicted in figure A.1. The fundamental domain clearly displays a single *cusp* where it touches the boundary of the space. This cusp corresponds to the limit $y \to \infty$. For discrete groups Γ different from $SL(2,\mathbb{Z})$ there can be multiple cusps [86].

Remark A.1. What we are dealing with is effectively $PSL(2,\mathbb{Z})$ rather than $SL(2,\mathbb{Z})$, where the 'P' indicates that a matrix has to be identified with minus itself. The reason is that the two matrices have identical action on the upper half plane as easily verified from (A.7).

A.3 Action of $SL(2,\mathbb{R})$ on smooth functions on $SL(2,\mathbb{R})$

The group $SL(2,\mathbb{R})$ acts on functions on $SL(2,\mathbb{R})$ via the *right-regular action*. Let g' be an element of $SL(2,\mathbb{R})$ and $\varphi(g)$ a function on $SL(2,\mathbb{R})$. The right-regular action is defined by:

$$\Big(\pi(g')\varphi\Big)(g) = \varphi(gg'). \tag{A.11}$$

The action of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is then given by differential operators acting on *smooth* functions. Using (A.11) one finds the following differential operators corresponding to the Chevalley basis generators:

$$\begin{aligned} h &= -2\sin(2\theta)y\partial_x + 2\cos(2\theta)y\partial_y + \sin(2\theta)\partial_\theta, \\ e &= \cos(2\theta)y\partial_x + \sin(2\theta)y\partial_y + \sin^2\theta\partial_\theta, \\ f &= \cos(2\theta)y\partial_x + \sin(2\theta)y\partial_y - \cos^2\theta\partial_\theta. \end{aligned} \tag{A.12a}$$

The compact generator $e-f$ of $SO(2,\mathbb{R})$ acts by ∂_θ. We record also the inverse relations

$$y\partial_x = \frac{1}{2}((e+f)\cos(2\theta) + e - f - h\sin(2\theta)), \tag{A.13a}$$

$$y\partial_y = \frac{1}{2}((e+f)\sin(2\theta) + h\cos(2\theta)), \tag{A.13b}$$

$$\partial_\theta = e - f. \tag{A.13c}$$

The Casimir operator (A.4) then becomes a second order differential operator, namely the Laplacian

$$\Delta = y^2\left(\partial_x^2 + \partial_y^2\right) - y\partial_x\partial_\theta. \tag{A.14}$$

In section 5.1 of the main text, the so-called compact basis for $\mathfrak{sl}(2,\mathbb{R})$ is discussed as well. This is a representation of $\mathfrak{sl}(2,\mathbb{R})$ in terms of (2×2)-matrices different from (A.2) and given explicitly by

$$H = -i(e-f), \quad E = \frac{1}{2}\left(h + i(e+f)\right), \quad F = \frac{1}{2}\left(h - i(e+f)\right), \tag{A.15}$$

that is,

$$H = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{A.16}$$

The generators satisfy the standard $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H. \tag{A.17}$$

The Cartan generator H is Hermitian in this basis and this is the reason for the name compact basis.

The representation (A.16) of $\mathfrak{sl}(2,\mathbb{R})$ is unitarily equivalent to the standard representation (A.2) through the transformation

$$UHU^\dagger = h, \text{etc.} \qquad \text{for} \quad U = \frac{1}{2}\begin{pmatrix} -1+i & 1+i \\ -1+i & -1-i \end{pmatrix}. \tag{A.18}$$

The differential operators associated with this basis are then given by

$$H = -i\partial_\theta, \tag{A.19a}$$

$$E = 2ie^{2i\theta}\left(y\partial_z - \frac{1}{4}\partial_\theta\right), \tag{A.19b}$$

$$F = -2ie^{-2i\theta}\left(y\partial_{\bar{z}} - \frac{1}{4}\partial_\theta\right), \tag{A.19c}$$

where we have used standard holomorphic and antiholomorphic derivatives:

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right). \tag{A.20}$$

Because the compact basis is unitarily equivalent, the Casimir operator does not change.

Remark A.2. The change of basis is basically that induced by the *Sekiguchi isomorphism* [204, 283] that enters in the description of real nilpotent orbits.

Appendix B

Fourier expansion of $SL(2,\mathbb{R})$ series by Poisson resummation

In this appendix we perform the Fourier expansion of the series (1.1)

$$f_s(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{z_2^s}{|cz+d|^{2s}} \tag{B.1}$$

that is related to the standard $SL(2,\mathbb{R})$ Eisenstein series through $f_s(z) = 2\zeta(2s)E(s,z)$, cf. (1.10). Here, $z = x+iy$ lies on the upper half plane $\mathbb{H}$ as defined in appendix A.

The invariance of $f_s(z)$ under shifts $z \to z+1$ implies that it should have a Fourier expansion

$$f_s(z) = C(y) + \sum_{m\neq 0} a_m(y) e^{2\pi i m x}\,. \tag{B.2}$$

The 'constant term(s)' $C(y)$ and the non-zero Fourier coefficients $a_m(t)$ are determined in the following. We suppress the label s on the constant terms and Fourier coefficients for ease of notation.

The technique to be used rests on Poisson resummation whose fundamental equation here is (cf. [272, Eqn. (8.2.210])

$$\sum_{m\in\mathbb{Z}} \exp(-\pi a m^2 + 2\pi i b m) = a^{-1/2} \sum_{\tilde{m}\in\mathbb{Z}} \exp\left(-\frac{\pi(\tilde{m}-b)^2}{a}\right)\,. \tag{B.3}$$

Another useful form of this same formula is

$$\sum_{m\in\mathbb{Z}} \exp\left(-\frac{\pi}{t}(m+nx)^2\right) = t^{1/2} \sum_{\tilde{m}\in\mathbb{Z}} \exp\left(-\pi t \tilde{m}^2 - 2\pi i \tilde{m} n x\right)\,. \tag{B.4}$$

Note that the sums are over all integers and not constrained to a single $SL(2,\mathbb{Z})$-orbit.

We will also use the following representation of powers for $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(M) > 0$

$$M^{-s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t}M}\,. \tag{B.5}$$

Finally, we require the following integral representation of the modified Bessel function for real $a, b \neq 0$

$$\int_0^\infty \frac{dt}{t^{s+1}} e^{-\pi t a^2 - \frac{\pi}{t} b^2} = 2 \left| \frac{a}{b} \right|^s K_s(2\pi |ab|) \,. \tag{B.6}$$

B.1 Constant term(s)

First extract the term $c = 0$ from (B.1). Then $d \neq 0$ and

$$f_s(z) = y^s \sum_{d \neq 0} |d|^{-2s} + \underbrace{y^s \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} |cz + d|^{-2s}}_{f_s^{(1)}(z)} = 2\zeta(2s) y^s + f_s^{(1)}(z) \,. \tag{B.7}$$

The power $|cz + d|^{-2s}$ appearing in the second term can be rewritten as an integral using (B.5). Then one can Poisson resum over $d \in \mathbb{Z}$ using (B.4):

$$\begin{aligned} f_s^{(1)}(z) &= \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} \int_0^\infty \frac{dt}{t^{s+1}} \exp\left(-\frac{\pi}{t} |cz + d|^2 \right) \\ &= \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} \int_0^\infty \frac{dt}{t^{s+1}} \exp\left(-\frac{\pi}{t} ((cx + d)^2 + (cy)^2) \right) \\ &= \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \sum_{\tilde{d} \in \mathbb{Z}} \int_0^\infty \frac{dt}{t^{s+1}} t^{1/2} \exp\left(-\pi t \tilde{d}^2 - 2\pi i \tilde{d} c x - \frac{\pi}{t} (cy)^2 \right) . \end{aligned} \tag{B.8}$$

In the final line of (B.8) one can separate out the term with $\tilde{d} = 0$ by

$$f_s^{(1)}(z) = \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \int_0^\infty \frac{dt}{t^{s+1/2}} \exp\left(-\frac{\pi}{t} (cy)^2 \right) + f_s^{(2)}(z) \tag{B.9}$$

since it does not have any x dependence and where $f_s^{(2)}$ are the terms with $\tilde{d} \neq 0$:

$$f_s^{(2)}(z) = \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \sum_{\tilde{d} \neq 0} \int_0^\infty \frac{dt}{t^{s+1/2}} \exp\left(-\pi t \tilde{d}^2 - 2\pi i \tilde{d} c x - \frac{\pi}{t} (cy)^2 \right) . \tag{B.10}$$

The integral in the term with $\tilde{d} = 0$ can be undone using (B.5) and the sum over $c \neq 0$ can be carried out afterwards. Hence the first term in (B.9) becomes

$$\begin{aligned} \frac{\pi^s}{\Gamma(s)} y^s \sum_{c \neq 0} \int_0^\infty \frac{dt}{t^{s+1/2}} \exp\left(-\frac{\pi}{t} (cy)^2 \right) \quad &= \frac{\pi^s}{\Gamma(s)} \frac{\Gamma(s - 1/2)}{\pi^{s-1/2}} y^{s-2(s-1/2)} \sum_{c \neq 0} c^{-2(s-1/2)} \\ = 2\zeta(2s) \frac{\pi^{-(s-1/2)} \Gamma(s - 1/2) \zeta(2s - 1)}{\pi^{-s} \Gamma(s) \zeta(2s)} y^{1-s} \quad &= 2\zeta(2s) \frac{\xi(2s - 1)}{\xi(2s)} y^{1-s} \,, \end{aligned} \tag{B.11}$$

where we have pulled out the same overall factor as in (B.7) and regrouped the π-factors to use the definition of the completed Riemann zeta function $\xi(k) = \pi^{-k/2} \Gamma(k/2) \zeta(k)$.

B.2 Non-zero Fourier modes

The current status of the Fourier expansion is then

$$f_s(z) = 2\zeta(2s)\left(y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s}\right) + f_s^{(2)}(z)\,, \tag{B.12}$$

with the non-zero mode part $f_s^{(2)}$ given by (B.10). The t-integral appearing in that expression is a Bessel integral and can be evaluated using (B.6) as

$$\begin{aligned} f_s^{(2)}(z) &= \frac{2\pi^s}{\Gamma(s)}y^s \sum_{c\neq 0}\sum_{\tilde{d}\neq 0}\left|\frac{\tilde{d}}{ny}\right|^{s-1/2} K_{s-1/2}(2\pi|\tilde{d}c|y)e^{-2\pi i\tilde{d}cx} \\ &= \frac{2\pi^s}{\Gamma(s)}y^{1/2} \sum_{c\neq 0}\sum_{\tilde{d}\neq 0}\left|\frac{\tilde{d}}{c}\right|^{s-1/2} K_{s-1/2}(2\pi|\tilde{d}c|y)e^{-2\pi i\tilde{d}cx}\,. \end{aligned} \tag{B.13}$$

To find the Fourier coefficient $a_m(y)$ of a mode $e^{2\pi imx}$ we transform the double summation to one over $m \neq 0$ and the (positive) divisors $d|m$. Then

$$\begin{aligned} f_s^{(2)}(z) &= \frac{4\pi^s}{\Gamma(s)}y^{1/2}\sum_{m\neq 0}\sum_{d|m} d^{1-2s}|m|^{s-1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi imx} \\ &= 2\zeta(2s)\frac{2y^{1/2}}{\xi(2s)}\sum_{m\neq 0}|m|^{1/2-s}\sigma_{2s-1}(m)K_{s-1/2}(2\pi|m|y)e^{2\pi imx}\,, \end{aligned} \tag{B.14}$$

again pulling out the same overall factor $2\zeta(2s)$ and using the divisor sum

$$\sigma_s(m) = \sum_{d|m} d^s \tag{B.15}$$

where only positive divisors are included.

The full Fourier expansion is therefore given by

$$f_s(z) = 2\zeta(2s)\left[y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s} + \frac{2y^{1/2}}{\xi(2s)}\sum_{\neq 0}|m|^{1/2-s}\sigma_{2s-1}(m)K_{s-1/2}(2\pi|m|y)e^{2\pi imx}\right]. \tag{B.16}$$

The term in the square brackets is the full expansion of the Eisenstein series $E(s,\tau)$ for $SL(2,\mathbb{R})$. This agrees with the adelic derivation of theorem 7.1.

Appendix C

Laplace operators on G/K and automorphic forms

In this appendix, we briefly review the connection between the *scalar Laplace operator* on the symmetric space $G(\mathbb{R})/K(\mathbb{R})$ and the quadratic Casimir. We do this first for a general simple, simply-laced split group $G(\mathbb{R})$ and then give a very explicit analysis for $G = SL(2,\mathbb{R})$.

C.1 Scalar Laplace operator and quadratic Casimir

For a simple, simply-laced split $G(\mathbb{R})$ we denote by $\mathfrak{h}$ a fixed Cartan subalgebra of the Lie algebra $\mathfrak{g}(\mathbb{R})$ of $G(\mathbb{R})$. With respect to $\mathfrak{h}$ and a choice of simple roots α_i ($i = 1,\dots,r$ with $r = \dim_{\mathbb{R}}(\mathfrak{h})$) the remaining generators arrange into positive and negative step operators, cf. (4.15). We denote by E_α the step operator of a given root α. In *Iwasawa gauge* we choose to write an arbitrary element $g \in G(\mathbb{R})/K(\mathbb{R})$ as

$$g = na = \exp\left(\sum_{\alpha>0} u_\alpha E_\alpha\right) \prod_{i=1}^{r} v_i^{h_i}, \tag{C.1}$$

where h_i are the Cartan generators associated with the choice of simple roots cf. (4.17). The variables v_i (for $= 1,\dots,r$) and u_α (for $\alpha \in \Delta_+$) are coordinates on the symmetric space $G(\mathbb{R})/K(\mathbb{R})$.

The $G(\mathbb{R})$-invariant metric on the symmetric space can be constructed from

$$ds^2_{G/K} = 2\langle \mathcal{P} | \mathcal{P} \rangle, \tag{C.2}$$

where we chose a convenient normalisation and

$$\mathcal{P} = \frac{1}{2}\left(g^{-1}dg - \theta(g^{-1}dg)\right) \tag{C.3}$$

is the coset projection of the *Maurer–Cartan form* $g^{-1}dg$ associated with the vector space decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. Here, $\mathfrak{k}$ is the Lie algebra of K. The *(Cartan) involution* θ

leaving $\mathfrak{k}$ fixed can be defined by

$$\theta(E_\alpha) = -E_{-\alpha}, \quad \theta(h_i) = -h_i. \tag{C.4}$$

With this convention, $\mathfrak{k}$ and $\mathfrak{p}$ have the bases

$$\begin{aligned} \mathfrak{k} &= \langle E_\alpha - E_{-\alpha} \,|\, \alpha > 0 \rangle, \\ \mathfrak{p} &= \langle E_\alpha + E_{-\alpha} \,|\, \alpha > 0 \rangle \oplus \langle h_i \,|\, i = 1, \ldots, r \rangle. \end{aligned} \tag{C.5}$$

We further choose the normalisation (A_{ij} is the Cartan matrix (4.18) of the simply-laced $\mathfrak{g}(\mathbb{R})$)

$$\langle E_\alpha | E_{-\beta} \rangle = \delta_{\alpha,\beta}, \quad \langle h_i | h_j \rangle = A_{ij}. \tag{C.6}$$

Working out the Maurer–Cartan form for the element (C.1) one finds

$$\begin{aligned} g^{-1}dg &= \sum_{i=1}^{r} v_i^{-1} dv_i h_i + a^{-1} \left(\sum_{\alpha>0} Du_\alpha E_\alpha \right) a \\ &= \sum_{i=1}^{r} v_i^{-1} dv_i h_i + \sum_{\alpha>0} a^{-\alpha} Du_\alpha E_\alpha \end{aligned} \tag{C.7}$$

where $Du_\alpha = du_\alpha + \ldots$ and the dots represent finitely many terms coming from commutator terms when expanding out the *Baker–Campbell–Hausdorff identity*

$$e^{-X} d(e^X) = dX - \frac{1}{2!}[X, dX] + \frac{1}{3!}[X, [X, dX]] + \ldots \tag{C.8}$$

for the nilpotent E_α. The expression (C.7) together with (C.6) leads to a block-diagonal metric of the form

$$ds^2_{G/K} = g_{\mu\nu} dx^\mu dx^\nu = 2 \sum_{i,j=1}^{r} v_i^{-1} v_j^{-1} dv_i dv_j A_{ij} + \sum_{\alpha>0} a^{-2\alpha} (Du_\alpha)^2. \tag{C.9}$$

The scalar Laplacian associated with this metric is ($\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\sqrt{g} = \sqrt{\det(g_{\mu\nu})}$)

$$\begin{aligned} \Delta_{G/K} &= \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \right) \\ &= \frac{1}{2} \sum_{i,j=1}^{r} (A^{-1})^{ij} a^{2\rho} v_i \partial_i \left(a^{-2\rho} v_j \partial_j \right) + \sum_{\alpha>0} a^{2\alpha} \partial_\alpha^2 + \ldots, \end{aligned} \tag{C.10}$$

where the dots come from inverting the metric in the $du_\alpha du_\beta$ sector and $(A^{-1})^{ij}$ is the inverse of the Cartan matrix. We have used the relation $\sum_{\alpha>0} \alpha = 2\rho$ for the Weyl vector, cf. (4.5). The Laplace operator (C.10) is $G(\mathbb{R})$-invariant since the Maurer–Cartan form trivially is: The transformation of $g \in G(\mathbb{R})/K(\mathbb{R})$ is given by $g \to g_0 g k^{-1}$ with constant $g_0 \in G(\mathbb{R})$ and $k \in K(\mathbb{R})$ such that $g^{-1}dg \to k(g^{-1}dg)k - dkk^{-1}$ is independent of g_0.

We can evaluate the eigenvalue of the Laplacian (C.10) when acting on an Eisenstein series $E(\lambda, g)$ as defined in (5.99). Due to the invariance of the Laplacian, it suffices to evaluate it on the summand $\chi(g) = \chi(a) = a^{\lambda+\rho}$, corresponding to $\gamma = \mathbb{1}$. For this term, the derivatives ∂_α with respect to the coordinates u^α vanish and one finds

$$\begin{aligned}\Delta_{G/K} a^{\lambda+\rho} = \Delta_{G/K} \prod_{i=1}^{r} v_i^{2s_i} &= \frac{1}{2} \sum_{i,j=1}^{r} (A^{-1})^{ij} 2s_i(2s_j - 2) a^{\lambda+\rho} \\ &= \frac{1}{2} \left(\langle\lambda|\lambda\rangle - \langle\rho|\rho\rangle\right) a^{\lambda+\rho},\end{aligned} \tag{C.11}$$

where we stress that we assumed $\mathfrak{g}$ to be simply-laced. As already mentioned, $G(\mathbb{R})$-invariance implies that this is also the eigenvalue for the full Eisenstein series:

$$\Delta_{G/K} E(\lambda, g) = \frac{1}{2} \left(\langle\lambda|\lambda\rangle - \langle\rho|\rho\rangle\right) E(\lambda, g). \tag{C.12}$$

This agrees up to a factor with the standard quadratic Casimir evaluated on a lowest weight representation with lowest weight $\Lambda = \lambda + \rho$ [186].

C.2 Automorphic forms on $SL(2, \mathbb{R})$ as Laplace eigenfunctions

For the case of $SL(2, \mathbb{R})$ we can give fully explicit expressions. Using

$$g = na = \exp(ue) v^h \tag{C.13}$$

one finds from (C.2)

$$ds^2_{G/K} = 4v^{-2} dv^2 + e^{-4v} du^2 \quad \Rightarrow \quad \Delta_{G/K} = \frac{1}{4} e^{2v} v\partial_v \left(e^{-2v} v\partial_v\right) + e^{4v} \partial_u^2. \tag{C.14}$$

This can be brought into a more familiar form by using $v = y^{1/2}$ and $u = x$, cf. (A.5). This leads to

$$\Delta_{G/K} = y^2 \left(\partial_x^2 + \partial_y^2\right), \tag{C.15}$$

which agrees with the Laplacian on the upper half plane $\mathbb{H}$ given in (A.14).

Consider now a real eigenfunction $\varphi(z)$ of the Laplace operator with eigenvalue $s(s-1)$:

$$\Delta_{G/K} \varphi(z) = s(s-1)\varphi(z). \tag{C.16}$$

If the function is furthermore invariant under $SL(2, \mathbb{Z})$, this implies $\varphi(z) = \varphi(z+1)$ and one has a Fourier expansion of the form

$$\varphi(z) = \sum_{m \in \mathbb{Z}} a_m(y) e^{2\pi i m x} \tag{C.17}$$

where $m \in \mathbb{Z}$ denotes the 'instanton charge' of the character in the terminology of section 6.2 and $a_0(y)$ is the constant term. This Fourier expansion is due to the translations $x \to x+1$ contained in the action of $SL(2,\mathbb{Z})$ acting on $SL(2,\mathbb{R})$. Reality of $\varphi(z)$ implies that $a_m(y) = a_{-m}(y)$ for all $m > 0$. We therefore restrict to $m \geq 0$.

Plugging the Fourier expansion (C.17) into the Laplace equation (C.16) and analysing each mode individually leads to the following equations

$$m = 0: \qquad y^2 \partial_y^2 a_0(y) = s(s-1)a_0(y), \tag{C.18a}$$

$$m \neq 0: \qquad y^2 \left(\partial_y^2 - 4\pi^2 m^2\right) a_m(y) = s(s-1)a_m(y). \tag{C.18b}$$

The equation (C.18a) for the constant term has two linearly independent solutions

$$s \neq \frac{1}{2}: \qquad a_0(y) = y^s \qquad \text{or} \qquad a_0(y) = y^{1-s}, \tag{C.19a}$$

$$s = \frac{1}{2}: \qquad a_0(y) = y^{1/2} \qquad \text{or} \qquad a_0(y) = y^{1/2}\log y. \tag{C.19b}$$

All these solutions are at most power laws when y approaches any cusp, e.g. $y \to \infty$.

Equation (C.18b) for the non-zero modes becomes more familiar when one uses $a_m(y) = y^{1/2} b_m(y)$ which leads to

$$y^2 \partial_y^2 b_m(y) + y\partial_y b_m(y) - \left(4\pi m^2 y^2 + \left(s - \tfrac{1}{2}\right)^2\right) b_m(y) = 0. \tag{C.20}$$

After a rescaling of the y coordinate this becomes the modified Bessel equation with the two modified Bessel functions $K_{s-1/2}$ and $I_{s-1/2}$ as linearly independent solutions. Translated back to $a_m(y)$ these are

$$a_m(y) = y^{1/2} K_{s-1/2}(2\pi|m|y) \quad \text{or} \quad a_m(y) = y^{1/2} I_{s-1/2}(2\pi|m|y). \tag{C.21}$$

If one insists on at most power law growth near the cusp $y \to \infty$ the solution involving the function $I_{s-1/2}$ is disallowed. This is an instance of the 'multiplicity one theorem' mentioned in chapter 6.

Putting everything together, we see that any real function $\varphi(g)$ on $SL(2,\mathbb{R})$ that is right-invariant under $SO(2,\mathbb{R})$ and satisfies the three conditions stated for automorphic forms in the introduction can be expanded as

$$\varphi(z) = a_0^{(s)} y^s + a_0^{(1-s)} y^{1-s} + y^{1/2} \sum_{m \neq 0} a_m K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{C.22}$$

with $a_m = a_{-m}$ and these are purely numerical coefficients. For cusp forms one has that the numerical coefficients $a_0^{(s)}$ and $a_0^{(1-s)}$ vanish identically. The above expansion is valid for $s \neq \frac{1}{2}$; for $s = \frac{1}{2}$ one has to replace the constant terms by the solutions of (C.19b).

As shown in section 11.2, the coefficients a_m can also be determined for cusp forms if one demands in addition to the Laplace condition that $\varphi(z)$ is also an eigenfunction of all the Hecke operators. These can be thought of as the analogues of the Laplace operator for finite $p < \infty$ and therefore an automorphic function that obeys simple equations for all $p \leq \infty$ is uniquely fixed (up to an overall normalisation), cf. remark 11.8 and example 11.10.

Appendix D

Local-to-global principle

In this appendix we provide some background on the local-to-global principle, also known as Hasse's principle. This principle forms the basis of a powerful approach to problems in arithmetic which we are going to illustrate in the following.

The local-to-global principle is nicely motivated by the study of algebraic equations. For instance, consider the polynomial

$$f(x) = 7x^3 - 2x + 2 \tag{D.1}$$

which has integer coefficients. Then the question of whether the equation $f(x) = 0$ has any integer solutions can be answered by considering a reduction of the polynomial's coefficients modulo 3 leaving us with

$$\tilde{f}(x) = x^3 + x + 2\,. \tag{D.2}$$

Since it is only necessary to test three integers, one quickly verifies that $\tilde{f}(x) = 0$ possesses no solution in $\mathbb{Z}/3\mathbb{Z}$. Now, since $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a ring homomorphism, any solution of $f(x) = 0$ in $\mathbb{Z}$ is mapped to a solution of $\tilde{f}(x) = 0$ in $\mathbb{Z}/3\mathbb{Z}$ we have also shown that $f(x) = 0$ has no integer solutions.

Considering an algebraic equation over $\mathbb{Z}$, such as the one above, modulo a prime number p is referred to as seeking solutions *locally* in other words in $\mathbb{Z}/p\mathbb{Z}$. If no solution is found locally it is then possible to deduce also that no solution exists *globally* namely in $\mathbb{Z}$. The approach presented above provides a simple example for what is known as the local-to-global principle. Although the method has worked nicely in this simple example it actually has limited applicability. In order to develop a more powerful method that realises the local-to-global principle it is useful to introduce p-adic numbers.

To this end consider for instance the quartic equation

$$7y^4 = 2^2x^6 + 2 \cdot 7^3x^2 + 7^4 \tag{D.3}$$

and we ask the question whether this equation possesses any non trivial solutions $(x, y) \in \mathbb{Q}^2$. In a first attempt we may try to proceed in an analogous way to above and reduce the equation modulo 2 or modulo 7. In the first case this leaves us with

$$y^4 = 1 \tag{D.4}$$

which implies that $(x, \pm 1)$ is a solution for all $x \in \mathbb{Q}$. From the second reduction we obtain

$$4x^6 = 0 \tag{D.5}$$

implying as solution $(0, y)$ for all $y \in \mathbb{Q}$. Using the simple method of reducing the equation with respect to prime numbers thus does not help us in answering the question posed. Instead we will now use p-adic numbers to show that no solution to the equation exists in $\mathbb{Q}_p$ and since $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ also no solution can exist in $\mathbb{Q}$.

Specifically we will work with $p = 7$ and define the 7-adic valuations of the variables x and y as $\nu_7(x) = n$ and $\nu(y)_7 = m$ with $m, n \in \mathbb{Z}$. Taking the 7-adic valuation of the left-and right-hand-sides of our algebraic equation we thus find

$$\nu_7(\text{l.h.s.}) \equiv \nu_7(7y^4) = 1 + 4m \tag{D.6}$$

and from the right-hand-side

$$\nu_7(\text{r.h.s.}) \equiv \nu_7(2^2x^6 + 2 \cdot 7^3x^2 + 7^4) \geq \min(4, 3 + 2n, 6n)\,, \tag{D.7}$$

where we have used property (3.8) of the p-adic valuation. In fact, since none of the arguments of the minimum function are equal, the inequality sharpens to an equality, $\nu_7(\text{r.h.s.}) = \min(4, 3 + 2n, 6n)$. The value of the 7-adic valuation of the left-hand-side $\nu_7(\text{l.h.s.})$ is odd and we would thus require that $\nu_7(\text{r.h.s.}) = 3 + 2n$. However note that for $n \geq 1$ we have that $\nu(\text{r.h.s.}) = 4$ and for $n < 1$ we have $\nu_7(\text{r.h.s.}) = 6n$. Either case is in contradiction with the value of the 7-adic valuation of the left-hand-side and we thus conclude that no solution in $\mathbb{Q}_7$ exists. As a consequence also no solution in $\mathbb{Q}$ exists, providing another realisation of the local-to-global principle.

Even though the introduction of p-adic numbers improves our ability to analyse algebraic equations, there is still a major limitation to our analysis. In particular neither of the above methods is able to prove the existence of a global solution, meaning a solution in $\mathbb{Q}$. In other words, if we find a solution of an algebraic equation locally in $\mathbb{Q}_p$ for some prime p, this does not imply that a solution in $\mathbb{Q}$ exists. In fact one can give examples of algebraic equations where a local solution exists for every prime p, but no global solution exists. See for instance [284].

Nevertheless, in some cases one can go further and prove theorems which provide information about the existence of global solutions from the local analysis. An example of such a case is the Hasse–Minkowski theorem for quadratic forms. Although the theorem holds for general number fields K we will content ourselves to stating the theorem for quadratic forms over the rationals which we also define for completeness.

Definition D.1 (Quadratic form). A quadratic form f over $\mathbb{Q}$ is a polynomial of degree two in the variables $x_i \in K$ with $i = 1, ..., n$, where n is called the rank of the form. Given some $y \in \mathbb{Q}$, we say that the quadratic form f represents y if there exists a solution $(X_1, ..., X_n) \in \mathbb{Q}^n$ with $(X_1, ..., X_n) \neq (0, ..., 0)$, such that

$$y = f(X_1, ..., X_n)\,. \tag{D.8}$$

With this definition we can then state the Hasse-Minkowski theorem which applies to non-degenerate quadratic forms.

Theorem D.2 (Hasse–Minkowski). *Let f be a quadratic form over $\mathbb{Q}$ and for a prime number p let f_p be the form over $\mathbb{Q}_p$. Then f represents zero if and only if f_p represents zero for all prime numbers p including the prime at infinity.*

Put differently, the theorem states that for f to have a global zero it is necessary and sufficient for f to have a local zero at all places. We refer the reader to [287] for more details and a proof of this theorem.

References

[1] T. Apostol, *Introduction to Analytic Number Theory*. Springer International Student Edition. Springer, 1976.

[2] T. M. Apostol, *Modular functions and Dirichlet series in number theory*. Graduate Texts in Mathematics. Springer London, 1997.

[3] J. Arthur, "Unipotent automorphic representations: conjectures," 1989.

[4] J. Arthur, "The principle of functoriality," *Bull. Amer. Math. Soc.* **40** no. 1, (2002) 39–53.

[5] J. Arthur, "An introduction to the trace formula," *Harmonic analysis, the trace formula, and Shimura varieties* **4** (2005) 1–263.

[6] R. Aurich, S. Lustig, F. Steiner, and H. Then, "Can one hear the shape of the Universe?," *Phys. Rev. Lett.* **94** (2005) 021301, arXiv:astro-ph/0412407 [astro-ph].

[7] P. Bala and R. W. Carter, "Classes of unipotent elements in simple algebraic groups. I," *Math. Proc. Cambridge Philos. Soc.* **79** no. 3, (1976) 401–425.

[8] P. Bala and R. W. Carter, "Classes of unipotent elements in simple algebraic groups. II," *Math. Proc. Cambridge Philos. Soc.* **80** no. 1, (1976) 1–17.

[9] L. Bao, J. Bielecki, M. Cederwall, B. E. Nilsson, and D. Persson, "U-Duality and the Compactified Gauss-Bonnet Term," *JHEP* **0807** (2008) 048, arXiv:0710.4907 [hep-th].

[10] L. Bao and L. Carbone, "Integral forms of Kac-Moody groups and Eisenstein series in low dimensional supergravity theories," arXiv:1308.6194 [hep-th].

[11] L. Bao, M. Cederwall, and B. E. Nilsson, "Aspects of higher curvature terms and U-duality," *Class.Quant.Grav.* **25** (2008) 095001, arXiv:0706.1183 [hep-th].

[12] L. Bao, A. Kleinschmidt, B. E. Nilsson, D. Persson, and B. Pioline, "Instanton Corrections to the Universal Hypermultiplet and Automorphic Forms on SU(2,1)," *Commun.Num.Theor.Phys.* **4** (2010) 187–266, arXiv:0909.4299 [hep-th].

[13] L. Bao, A. Kleinschmidt, B. E. Nilsson, D. Persson, and B. Pioline, "Rigid Calabi-Yau threefolds, Picard Eisenstein series and instantons," arXiv:1005.4848 [hep-th].

[14] A. Basu, "The D**10 R**4 term in type IIB string theory," *Phys.Lett.* **B648** (2007) 378–382, arXiv:hep-th/0610335.

[15] A. Basu, "The D**4 R**4 term in type IIB string theory on T**2 and U-duality," *Phys.Rev.* **D77** (2008) 106003, arXiv:0708.2950 [hep-th].

[16] A. Basu, "The D**6 R**4 term in type IIB string theory on T**2 and U-duality," *Phys. Rev.* **D77** (2008) 106004, arXiv:0712.1252 [hep-th].

[17] A. Basu, "The D^6R^4 term from three loop maximal supergravity," *Class. Quant. Grav.* **31** no. 24, (2014) 245002, arXiv:1407.0535 [hep-th].

[18] K. Becker, M. Becker, and A. Strominger, "Five-branes, membranes and nonperturbative string theory," *Nucl.Phys.* **B456** (1995) 130–152, arXiv:hep-th/9507158.

[19] J. Beineke, B. Brubaker, and S. Frechette, "A crystal definition for symplectic multiple Dirichlet series," in *Multiple Dirichlet series, L-functions and automorphic forms*, vol. 300 of *Progr. Math.*, pp. 37–63. Birkhäuser/Springer, New York, 2012. http://dx.doi.org/10.1007/978-0-8176-8334-4_2.

[20] J. Beineke, B. Brubaker, and S. Frechette, "Weyl group multiple Dirichlet series of type C," *Pacific J. Math.* **254** no. 1, (2011) 11–46. http://dx.doi.org/10.2140/pjm.2011.254.11.

[21] A. Berenstein and A. Zelevinsky, "Tensor product multiplicities, canonical bases and totally positive varieties," *Invent. Math.* **143** no. 1, (2001) 77–128. http://dx.doi.org/10.1007/s002220000102.

[22] M. Bhargava, "Higher composition laws. I. A new view on Gauss composition, and quadratic generalizations," *Ann. of Math. (2)* **159** no. 1, (2004) 217–250. http://dx.doi.org/10.4007/annals.2004.159.217.

[23] D. Blasius, "On multiplicities for SL(n)," *Israel J. Math.* **88** no. 1-3, (1994) 237–251. http://dx.doi.org/10.1007/BF02937513.

[24] R. Blumenhagen, D. Lst, and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013. http://www.springer.com/physics/theoretical%2C+mathematical+%26+computational+physics/book/978-3-642-29496-9.

[25] E. B. Bogomolny, B. Georgeot, M. J. Giannoni, and C. Schmit, "Arithmetical chaos," *Phys. Rept.* **291** (1997) 219–324.

[26] A. Borel, "Automorphic L-functions," in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pp. 27–61. Amer. Math. Soc., Providence, R.I., 1979.

[27] A. Borel, *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer-Verlag, 1991.

[28] A. Borel, *Automorphic Forms on $SL(2,\mathbb{R})$*, vol. 130 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1997.

[29] A. Borel, "Introduction to automorphic forms," in *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pp. 199–210. Amer. Math. Soc., Providence, R.I., 1966.

[30] W. Borho and J.-L. Brylinski, "Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules," *Invent. Math.* **69** no. 3, (1982) 437–476. http://dx.doi.org/10.1007/BF01389364.

[31] G. Bossard and A. Kleinschmidt, "Loops in exceptional field theory," arXiv:1510.07859 [hep-th].

[32] G. Bossard and A. Kleinschmidt, "Supergravity divergences, supersymmetry and automorphic forms," *JHEP* **08** (2015) 102, arXiv:1506.00657 [hep-th].

[33] G. Bossard and V. Verschinin, "$\mathcal{E}\nabla^4 R^4$ type invariants and their gradient expansion," arXiv:1411.3373 [hep-th].

[34] G. Bossard and V. Verschinin, "Minimal unitary representations from supersymmetry," *JHEP* **1410** (2014) 008, arXiv:1406.5527 [hep-th].

[35] G. Bossard and V. Verschinin, "The two $\nabla^6 R^4$ type invariants and their higher order generalisation," *JHEP* **07** (2015) 154, arXiv:1503.04230 [hep-th].

[36] A. Braverman, H. Garland, D. Kazhdan, and M. Patnaik, "An affine Gindikin-Karpelevich formula," arXiv:1212.6473 [math.RT].

[37] A. Braverman and D. Kazhdan, "Representations of affine Kac-Moody groups over local and global fields: a survey of some recent results," arXiv:1205.0870 [math.RT].

[38] A. Braverman, D. Kazhdan, and M. Patnaik, "Iwahori-Hecke algebras for p-adic loop groups," arXiv:1403.0602 [math.RT].

[39] A. Braverman and D. Kazhdan, "The spherical Hecke algebra for affine Kac-Moody groups I," *Ann. of Math. (2)* **174** no. 3, (2011) 1603–1642. http://dx.doi.org/10.4007/annals.2011.174.3.5.

[40] L. Brekke and P. Freund, "p-adic numbers in physics," *Phys.Rept.* **233** (1993) 1–66.

[41] J. Broedel, C. R. Mafra, N. Matthes, and O. Schlotterer, "Elliptic multiple zeta values and one-loop superstring amplitudes," *JHEP* **07** (2015) 112, arXiv:1412.5535 [hep-th].

[42] J. Broedel, O. Schlotterer, S. Stieberger, and T. Terasoma, "All order α'-expansion of superstring trees from the Drinfeld associator," *Phys. Rev.* **D89** no. 6, (2014) 066014, arXiv:1304.7304 [hep-th].

[43] B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein, "Weyl group multiple Dirichlet series. III. Eisenstein series and twisted unstable A_r," *Ann. of Math. (2)* **166** no. 1, (2007) 293–316. http://dx.doi.org/10.4007/annals.2007.166.293.

[44] B. Brubaker and S. Friedberg, "Whittaker Coefficients of Metaplectic Eisenstein Series," arXiv:1403.6055 [math.NT].

[45] B. Brubaker, D. Bump, G. Chinta, and P. E. Gunnells, "Metaplectic Whittaker functions and crystals of type B," in *Multiple Dirichlet series, L-functions and automorphic forms*, vol. 300 of *Progr. Math.*, pp. 93–118. Birkhäuser/Springer, New York, 2012. http://dx.doi.org/10.1007/978-0-8176-8334-4_4.

[46] B. Brubaker, D. Bump, and S. Friedberg, "Weyl group multiple Dirichlet series. II. The stable case," *Invent. Math.* **165** no. 2, (2006) 325–355. http://dx.doi.org/10.1007/s00222-005-0496-2.

[47] B. Brubaker, D. Bump, and S. Friedberg, "Twisted Weyl group multiple Dirichlet series: the stable case," in *Eisenstein series and applications*, vol. 258 of *Progr. Math.*, pp. 1–26. Birkhäuser Boston, Boston, MA, 2008. http://dx.doi.org/10.1007/978-0-8176-4639-4_1.

[48] B. Brubaker, D. Bump, and S. Friedberg, "Gauss sum combinatorics and metaplectic Eisenstein series," in *Automorphic forms and L-functions I. Global aspects*, vol. 488 of *Contemp. Math.*, pp. 61–81. Amer. Math. Soc., Providence, RI, 2009. http://dx.doi.org/10.1090/conm/488/09564.

[49] B. Brubaker, D. Bump, and S. Friedberg, "Schur polynomials and the Yang-Baxter equation," *Comm. Math. Phys.* **308** no. 2, (2011) 281–301. http://dx.doi.org/10.1007/s00220-011-1345-3.

[50] B. Brubaker, D. Bump, and S. Friedberg, "Weyl group multiple Dirichlet series, Eisenstein series and crystal bases," *Ann. of Math. (2)* **173** no. 2, (2011) 1081–1120. http://dx.doi.org/10.4007/annals.2011.173.2.13.

[51] B. Brubaker, D. Bump, and S. Friedberg, *Weyl group multiple Dirichlet series: type A combinatorial theory*, vol. 175 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2011.

[52] B. Brubaker, D. Bump, G. Chinta, S. Friedberg, and J. Hoffstein, "Weyl group multiple Dirichlet series. I," in *Multiple Dirichlet series, automorphic forms, and analytic number theory*, vol. 75 of *Proc. Sympos. Pure Math.*, pp. 91–114. Amer. Math. Soc., Providence, RI, 2006. http://dx.doi.org/10.1090/pspum/075/2279932.

[53] B. Brubaker, D. Bump, and S. Friedberg, "Eisenstein series, crystals, and ice," *Notices Amer. Math. Soc.* **58** no. 11, (2011) 1563–1571.

[54] J. Bruinier, ed., *The 1-2-3 of Modular Forms: Lectures at a Summer School in Nordfjordeid, Norway*, ch. Siegel Modular Forms and Their Applications, p. 181. Universitext (En ligne). Springer-Verlag, 2008.

[55] R. Brylinski and B. Kostant, "Minimal representations, geometric quantization, and unitarity," *Proc. Nat. Acad. Sci. U.S.A.* **91** no. 13, (1994) 6026–6029. http://dx.doi.org/10.1073/pnas.91.13.6026.

[56] D. Bump, "Hecke algebras." Available online at http://sporadic.stanford.edu/bump/math263/hecke.pdf. [Accessed on 5. February 2015].

[57] D. Bump, *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[58] D. Bump, S. Friedberg, and D. Goldfeld, eds., *Multiple Dirichlet series, L-functions and automorphic forms*, vol. 300 of *Progress in Mathematics*. Birkhäuser/Springer, New York, 2012. http://dx.doi.org/10.1007/978-0-8176-8334-4.

[59] D. Bump and M. Nakasuji, "Integration on p-adic groups and crystal bases," *Proc. Amer. Math. Soc.* **138** no. 5, (2010) 1595–1605. http://dx.doi.org/10.1090/S0002-9939-09-10206-X.

[60] D. W. Bump, *Automorphic Forms on GL(3,R)*. ProQuest LLC, Ann Arbor, MI, 1982. http://search.proquest.com/docview/303092228. Thesis (Ph.D.)–The University of Chicago.

[61] L. Carbone, K.-H. Lee, and D. Liu, "Eisenstein series on rank 2 hyperbolic Kac–Moody groups," arXiv:1306.3280 [math.RT].

[62] R. W. Carter, *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[63] B. Casselman, "The L-group." Available online at http://www.math.ubc.ca/~cass/research/pdf/miyake.pdf. [Accessed on 5. February 2015].

[64] W. Casselman and J. Shalika, "The unramified principal series of p-adic groups. ii. the whittaker function," *Compositio Mathematica* **41** no. 2, (1980) 207–231. http://eudml.org/doc/89456.

[65] G. Chinta and P. E. Gunnells, "Weyl group multiple Dirichlet series constructed from quadratic characters," *Invent. Math.* **167** no. 2, (2007) 327–353. http://dx.doi.org/10.1007/s00222-006-0014-1.

[66] G. Chinta and P. E. Gunnells, "Constructing Weyl group multiple Dirichlet series," *J. Amer. Math. Soc.* **23** no. 1, (2010) 189–215. http://dx.doi.org/10.1090/S0894-0347-09-00641-9.

[67] G. Chinta and P. E. Gunnells, "Littelmann patterns and Weyl group multiple Dirichlet series of type D," in *Multiple Dirichlet series, L-functions and automorphic forms*, vol. 300 of *Progr. Math.*, pp. 119–130. Birkhäuser/Springer, New York, 2012. http://dx.doi.org/10.1007/978-0-8176-8334-4_5.

[68] J. W. Cogdell, *Lectures on L-functions, Converse Theorems, and Functoriality for $GL(n)$*.

[69] J. W. Cogdell and I. I. Piatetski-Shapiro, "Converse theorems for GL_n," *Inst. Hautes Études Sci. Publ. Math.* no. 79, (1994) 157–214. http://www.numdam.org/item?id=PMIHES_1994__79__157_0.

[70] J. W. Cogdell and I. I. Piatetski-Shapiro, "Converse theorems for GL_n. II," *J. Reine Angew. Math.* **507** (1999) 165–188. http://dx.doi.org/10.1515/crll.1999.507.165.

[71] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[72] E. Cremmer and B. Julia, "The N=8 Supergravity Theory. 1. The Lagrangian," *Phys.Lett.* **B80** (1978) 48.

[73] E. Cremmer, B. Julia, H. Lu, and C. Pope, "Dualization of dualities. 1.," *Nucl.Phys.* **B523** (1998) 73–144, arXiv:hep-th/9710119.

[74] A. Dabholkar, "Exact counting of black hole microstates," *Phys.Rev.Lett.* **94** (2005) 241301, arXiv:hep-th/0409148.

[75] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Precision counting of small black holes," *JHEP* **0510** (2005) 096, arXiv:hep-th/0507014.

[76] A. Dabholkar and J. A. Harvey, "Nonrenormalization of the Superstring Tension," *Phys.Rev.Lett.* **63** (1989) 478.

[77] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," arXiv:1208.4074 [hep-th].

[78] T. Damour, M. Henneaux, and H. Nicolai, "E(10) and a 'small tension expansion' of M theory," *Phys.Rev.Lett.* **89** (2002) 221601, arXiv:hep-th/0207267 [hep-th].

[79] W. A. de Graaf, "Computing representatives of nilpotent orbits of θ-groups," *J. Symbolic Comput.* **46** no. 4, (2011) 438–458. http://dx.doi.org/10.1016/j.jsc.2010.10.015.

[80] A. Deitmar, *Automorphic Forms*. Universitext - Springer-Verlag. Springer London, 2012.

[81] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," *JHEP* **1111** (2011) 129, arXiv:hep-th/0702146.

[82] E. D'Hoker, "Perturbative string theory," in *Quantum fields and strings : a course for mathematicians*. American Mathematical Society, 1999. https://www.math.ias.edu/qft.

[83] E. D'Hoker and M. B. Green, "Zhang-Kawazumi Invariants and Superstring Amplitudes," arXiv:1308.4597 [hep-th].

[84] E. D'Hoker, M. B. Green, B. Pioline, and R. Russo, "Matching the D^6R^4 interaction at two-loops," *JHEP* **01** (2015) 031, arXiv:1405.6226 [hep-th].

[85] E. D'Hoker and D. Phong, "Lectures on two loop superstrings," *Conf.Proc.* **C0208124** (2002) 85–123, arXiv:hep-th/0211111.

[86] F. Diamond and J. Shurman, *A first course in modular forms*, vol. 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[87] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, "Counting dyons in N=4 string theory," *Nucl.Phys.* **B484** (1997) 543–561, arXiv:hep-th/9607026.

[88] P. A. Dirac, "Quantized Singularities in the Electromagnetic Field," *Proc.Roy.Soc.Lond.* **A133** (1931) 60–72.

[89] R. Donagi and E. Witten, "Supermoduli Space Is Not Projected," in *Proceedings, String-Math 2012, Bonn, Germany, July 16-21, 2012*, pp. 19–72. 2013. arXiv:1304.7798 [hep-th].

[90] A. Dvorsky and S. Sahi, "Explicit Hilbert spaces for certain nilpotent representations II," *Invent. Math.* **138** no. 1, (1999) 203–224.

[91] E. B. Dynkin, "Semisimple subalgebras of semisimple Lie algebras," *Mat. Sbornik N.S.* **30(72)** (1952) 349–462 (3 plates).

[92] F. Englert, L. Houart, A. Kleinschmidt, H. Nicolai, and N. Tabti, "An E(9) multiplet of BPS states," *JHEP* **0705** (2007) 065, `arXiv:hep-th/0703285`.

[93] B. Enriquez, "Elliptic associators," *Selecta Math. (N.S.)* **20** no. 2, (2014) 491–584. `http://dx.doi.org/10.1007/s00029-013-0137-3`.

[94] D. Flath, *Decomposition of representations into tensor products*, p. 179. No. pt. 2 in Automorphic Forms, Representations and L-functions: Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society Held at Oregon State University, Corvallis, Oregon, July 11-August 5, 1977. American Mathematical Society, 1979.

[95] P. Fleig and A. Kleinschmidt, "Eisenstein series for infinite-dimensional U-duality groups," *JHEP* **1206** (2012) 054, `arXiv:1204.3043 [hep-th]`.

[96] P. Fleig and A. Kleinschmidt, "Perturbative terms of Kac-Moody-Eisenstein series," in *Proceedings, 3Quantum: Algebra Geometry Information (QQQ Conference 2012)*. 2012. `arXiv:1211.5296 [hep-th]`.

[97] P. Fleig, A. Kleinschmidt, and D. Persson, "Fourier expansions of Kac-Moody Eisenstein series and degenerate Whittaker vectors," *Commun.Num.TheorPhys.* **08** (2014) 41–100, `arXiv:1312.3643 [hep-th]`.

[98] E. Frenkel, "Lecture notes on the Langlands Program and Conformal Field Theory.".

[99] D. Friedan, E. J. Martinec, and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," *Nucl.Phys.* **B271** (1986) 93.

[100] S. Friedberg and L. Zhang, "Eisenstein Series on Covers of Odd Orthogonal Groups," `arXiv:1301.3026 [math.NT]`.

[101] S. Friedberg and L. Zhang, "Tokuyama-type formulas for type B," `arXiv:1409.0464 [math.CO]`.

[102] H. Friedlander, L. Gaudet, and P. E. Gunnells, "Crystal graphs, Tokuyama's theorem, and the Gindikin–Karpelevic formula for G_2," `arXiv:1402.0411 [math.CO]`.

[103] W. Fulton and J. Harris, *Representation Theory: A first course*. Graduate Texts in Mathematics. Springer London, 2008.

[104] D. Gaiotto, G. W. Moore, and A. Neitzke, "Four-dimensional wall-crossing via three-dimensional field theory," *Commun.Math.Phys.* **299** (2010) 163–224, `arXiv:0807.4723 [hep-th]`.

[105] W. T. Gan, "Lecture slides on automorphic forms and representations." Lecture slides from a course in Hangzhou, China, (2004), online available at `http://www.math.nus.edu.sg/~matgwt/`. [Accessed 2 August 2013].

[106] W. T. Gan, B. Gross, and G. Savin, "Fourier coefficients of modular forms on G_2," *Duke Math. J.* **115** no. 1, (2002) 105–169. `http://dx.doi.org/10.1215/S0012-7094-02-11514-2`.

[107] W. T. Gan and G. Savin, "On minimal representations definitions and properties," *Represent. Theory* **9** (2005) 46–93 (electronic). `http://dx.doi.org/10.1090/S1088-4165-05-00191-3`.

[108] H. Garland, "Certain Eisenstein series on loop groups: convergence and the constant term," in *Proceedings of the International Conference on Algebraic Groups and Arithmetic (in honor of M.S. Raghunathan)*, pp. 275–319. Tata Institute of Fundamental Research, Mumbai, 2001.

[109] H. Garland, "Absolute convergence of Eisenstein series on loop groups," *Duke Math. J.* **135** (2006) 203–260.

[110] H. Garland, S. D. Miller, and M. M. Patnaik, "Entirety of cuspidal Eisenstein series on loop groups," `arXiv:1304.4913 [math.NT]`.

[111] H. Garland, "On extending the Langlands-Shahidi method to arithmetic quotients of loop groups," in *Representation theory and mathematical physics*, vol. 557 of *Contemp. Math.*, pp. 151–167. Amer. Math. Soc., Providence, RI, 2011. `http://dx.doi.org/10.1090/conm/557/11030`.

[112] P. Garrett, "Satake parameters versus unramified principal series," 1999.

[113] P. Garrett, "Transition: Eisenstein series on adele groups." Available online at `http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/12_2_transition_Eis.pdf`, 2014. [Accessed on 4. November 2015].

[114] V. Gatti and E. Viniberghi, "Spinors of 13-dimensional space," *Adv. in Math.* **30** no. 2, (1978) 137–155. `http://dx.doi.org/10.1016/0001-8708(78)90034-8`.

[115] S. Gaussent and G. Rousseau, "Spherical Hecke algebras for Kac-Moody groups over local fields," *Ann. of Math. (2)* **180** no. 3, (2014) 1051–1087. `http://dx.doi.org/10.4007/annals.2014.180.3.5`.

[116] S. Gelbart, "An elementary introduction to the Langlands program," *Bull. Amer. Math. Soc.* **10** no. 2, (1984) 177–219.

[117] S. S. Gelbart, *Automorphic Forms on Adele Groups*. Annals of Mathematics Studies. Princeton University Press, 1975.

[118] S. S. Gelbart and S. D. Miller, "Riemann's zeta function and beyond," *Bull. Amer. Math. Soc.* **41** no. 1, (2003) 59–112.

[119] I. M. Gel′fand and M. L. Cetlin, "Finite-dimensional representations of the group of unimodular matrices," *Doklady Akad. Nauk SSSR (N.S.)* **71** (1950) 825–828.

[120] I. Gelfand, M. Graev, and I. Piatetski-Shapiro, *Representation theory and automorphic functions*. Saunders Mathematics Books. Saunders, 1968.

[121] D. Ginzburg and J. Hundley, "Constructions of global integrals in the exceptional groups," `arXiv:1108.1401 [math.RT]`.

[122] D. Ginzburg, S. Rallis, and D. Soudry, "On Fourier coefficients of automorphic forms of symplectic groups," *Manuscripta Math.* **111** no. 1, (2003) 1–16. http://dx.doi.org/10.1007/s00229-003-0355-7.

[123] D. Ginzburg, "Certain conjectures relating unipotent orbits to automorphic representations," *Israel J. Math.* **151** (2006) 323–355. http://dx.doi.org/10.1007/BF02777366.

[124] D. Ginzburg, "Towards a classification of global integral constructions and functorial liftings using the small representations method," *Adv. Math.* **254** (2014) 157–186. http://dx.doi.org/10.1016/j.aim.2013.12.007.

[125] D. Ginzburg, "Towards a classification of global integral constructions and functorial liftings using the small representations method," *Adv. Math.* **254** (2014) 157–186. http://dx.doi.org/10.1016/j.aim.2013.12.007.

[126] D. Ginzburg, S. Rallis, and D. Soudry, "On the automorphic theta representation for simply laced groups," *Israel J. Math.* **100** (1997) 61–116.

[127] P. Goddard, J. Nuyts, and D. I. Olive, "Gauge Theories and Magnetic Charge," *Nucl.Phys.* **B125** (1977) 1.

[128] P. Goddard and D. Olive, "Kac-Moody and Virasoro algebras in relation to quantum physics," *Internat. J. Modern Phys. A* **1** no. 2, (1986) 303–414. http://dx.doi.org/10.1142/S0217751X86000149.

[129] R. Godement, "Domaines fondamentaux des groupes arithmétiques," in *Séminaire Bourbaki, Vol. 8, Exp. No. 257*, pp. 201–225. 1962.

[130] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n,\mathbb{R})$*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.

[131] D. Goldfeld and J. Hundley, *Automorphic representations and L-functions for the general linear group. Volume I*, vol. 129 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011. http://dx.doi.org/10.1017/CBO9780511973628. With exercises and a preface by Xander Faber.

[132] D. Goldfeld and J. Hundley, *Automorphic representations and L-functions for the general linear group. Volume II*, vol. 130 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011. http://dx.doi.org/10.1017/CBO9780511910531. With exercises and a preface by Xander Faber.

[133] H. Gomez and C. R. Mafra, "The closed-string 3-loop amplitude and S-duality," *JHEP* **1310** (2013) 217, arXiv:1308.6567 [hep-th].

[134] R. Gomez, D. Gourevitch, and S. Sahi, "Generalized and degenerate Whittaker models," arXiv:1502.06483 [math.RT].

[135] D. Gourevitch and S. Sahi, "Degenerate Whittaker functionals for real reductive groups," arXiv:1210.4064 [math.RT].

[136] D. Gourevitch and S. Sahi, "Annihilator varieties, adduced representations, Whittaker functionals, and rank for unitary representations of GL(n)," *Selecta Math. (N.S.)* **19** no. 1, (2013) 141–172. http://dx.doi.org/10.1007/s00029-012-0100-8.

[137] D. Gourevitch and S. Sahi, "Degenerate Whittaker functionals for real reductive groups," *Amer. J. Math.* **137** no. 2, (2015) 439–472. http://dx.doi.org/10.1353/ajm.2015.0008.

[138] M. B. Green and M. Gutperle, "Effects of D-instantons," *Nuclear Physics B* **498** (Feb., 1997) 195–227, arXiv:hep-th/9701093.

[139] M. B. Green and M. Gutperle, "D-particle bound states and the D-instanton measure," *Journal of High Energy Physics* **1** (Jan., 1998) 5, arXiv:hep-th/9711107.

[140] M. B. Green, J. G. Russo, and P. Vanhove, "Automorphic properties of low energy string amplitudes in various dimensions," *Phys. Rev.* **81** no. 8, (Apr., 2010) 086008, arXiv:1001.2535 [hep-th].

[141] M. B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity," *Phys. Rev. D* **59** no. 4, (Feb., 1999) 046006, arXiv:hep-th/9808061.

[142] M. B. Green, "A Gas of D instantons," *Phys. Lett.* **B354** (1995) 271–278, arXiv:hep-th/9504108 [hep-th].

[143] M. B. Green, H.-h. Kwon, and P. Vanhove, "Two loops in eleven-dimensions," *Phys. Rev.* **D61** (2000) 104010, arXiv:hep-th/9910055 [hep-th].

[144] M. B. Green, S. D. Miller, J. G. Russo, and P. Vanhove, "Eisenstein series for higher-rank groups and string theory amplitudes," *Commun. Num. Theor. Phys.* **4** (2010) 551–596, arXiv:1004.0163 [hep-th].

[145] M. B. Green, S. D. Miller, and P. Vanhove, "$SL(2,\mathbb{Z})$-invariance and D-instanton contributions to the D^6R^4 interaction," *Commun. Num. Theor. Phys.* **09** (2015) 307–344, arXiv:1404.2192 [hep-th].

[146] M. B. Green, S. D. Miller, and P. Vanhove, "Small representations, string instantons, and Fourier modes of Eisenstein series (with an appendix by D. Ciubotaru and P. Trapa)," *J. Number Theory* **146** (2015) 187–309, arXiv:1111.2983 [hep-th].

[147] M. B. Green, J. G. Russo, and P. Vanhove, "Low energy expansion of the four-particle genus-one amplitude in type II superstring theory," *JHEP* **02** (2008) 020, arXiv:0801.0322 [hep-th].

[148] M. B. Green, J. G. Russo, and P. Vanhove, "Modular properties of two-loop maximal supergravity and connections with string theory," *JHEP* **07** (2008) 126, arXiv:0807.0389 [hep-th].

[149] M. B. Green, J. G. Russo, and P. Vanhove, "String theory dualities and supergravity divergences," *JHEP* **1006** (2010) 075, arXiv:1002.3805 [hep-th].

[150] M. B. Green, J. Schwarz, and E. Witten, *Superstring theory. Vol. I & II.* Cambridge University Press, 1987.

[151] M. B. Green and P. Vanhove, "D instantons, strings and M theory," *Phys.Lett.* **B408** (1997) 122–134, arXiv:hep-th/9704145.

[152] M. B. Green and P. Vanhove, "Duality and higher derivative terms in M theory," *JHEP* **0601** (2006) 093, arXiv:hep-th/0510027.

[153] B. H. Gross, "On the Satake isomorphism." Available online at http://www.math.harvard.edu/~gross/preprints/sat.pdf. [Accessed on 5. February 2015].

[154] B. H. Gross and N. R. Wallach, "A distinguished family of unitary representations for the exceptional groups of real rank = 4," in *Lie theory and geometry*, vol. 123 of *Progr. Math.*, pp. 289–304. Birkhäuser Boston, Boston, MA, 1994.

[155] B. H. Gross and N. R. Wallach, "On quaternionic discrete series representations, and their continuations," *J. Reine Angew. Math.* **481** (1996) 73–123. http://dx.doi.org/10.1515/crll.1996.481.73.

[156] D. J. Gross and E. Witten, "Superstring Modifications of Einstein's Equations," *Nucl.Phys.* **B277** (1986) 1.

[157] F. Gubay, N. Lambert, and P. West, "Constraints on Automorphic Forms of Higher Derivative Terms from Compactification," *JHEP* **1008** (2010) 028, arXiv:1002.1068 [hep-th].

[158] M. Gunaydin, K. Koepsell, and H. Nicolai, "The Minimal unitary representation of E(8(8))," *Adv. Theor. Math. Phys.* **5** (2002) 923–946, arXiv:hep-th/0109005 [hep-th].

[159] M. Gunaydin, A. Neitzke, B. Pioline, and A. Waldron, "BPS black holes, quantum attractor flows, and automorphic forms," *Phys. Rev. D* **73** no. 8, (Apr., 2006) 084019, arXiv:hep-th/0512296.

[160] H. P. A. Gustafsson, A. Kleinschmidt, and D. Persson, "Small automorphic representations and degenerate Whittaker vectors," arXiv:1412.5625 [math.NT]. (Updated version to appear soon).

[161] A. M. Hamel and R. C. King, "Symplectic shifted tableaux and deformations of Weyl's denominator formula for sp($2n$)," *J. Algebraic Combin.* **16** no. 3, (2002) 269–300 (2003). http://dx.doi.org/10.1023/A:1021804505786.

[162] Harish-Chandra, "Admissible invariant distributions on reductive p-adic groups," in *Lie theories and their applications (Proc. Ann. Sem. Canad. Math. Congr., Queen's Univ., Kingston, Ont., 1977)*, pp. 281–347. Queen's Papers in Pure Appl. Math., No. 48. Queen's Univ., Kingston, Ont., 1978.

[163] M. Harris, "On the local Langlands correspondence," in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pp. 583–597. Higher Ed. Press, Beijing, 2002.

[164] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, vol. 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[165] M. Hashizume, "Whittaker functions on semisimple Lie groups," *Hiroshima Math. J.* **12** no. 2, (1982) 259–293. http://projecteuclid.org/euclid.hmj/1206133751.

[166] E. Hecke, "Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I," *Math. Ann.* **114** no. 1, (1937) 1–28. http://dx.doi.org/10.1007/BF01594160.

[167] E. Hecke, "Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. II," *Math. Ann.* **114** no. 1, (1937) 316–351. http://dx.doi.org/10.1007/BF01594180.

[168] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*. Graduate Studies in Mathematics. American Mathematical Society, 2001.

[169] G. Henniart, "Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique," *Invent. Math.* **139** no. 2, (2000) 439–455. http://dx.doi.org/10.1007/s002220050012.

[170] R. Howe, "The Fourier transform and germs of characters (case of Gl_n over a p-adic field)," *Math. Ann.* **208** (1974) 305–322.

[171] C. Hull and P. Townsend, "Unity of superstring dualities," *Nucl.Phys.* **B438** (1995) 109–137, arXiv:hep-th/9410167 [hep-th].

[172] J. E. Humphreys, *Introduction to Lie algebras and Representation Theory*. Graduate Texts in Mathematics. Springer London, 1997.

[173] Y.-H. Ishikawa, "The generalized whittaker functions for su(2, 1) and the fourier expansion of automorphic forms," *J. Math. Sci. Univ. Tokyo* **6** (1999) 477–526.

[174] H. Jacquet, "Fonctions de whittaker associées aux groupes de chevalley," *Bulletin de la Société Mathématique de France* **95** (1967) 243–309.

[175] H. Jacquet, "On the residual spectrum of GL(n)," in *Lie group representations, II (College Park, Md., 1982/1983)*, vol. 1041 of *Lecture Notes in Math.*, pp. 185–208. Springer, Berlin, 1984. http://dx.doi.org/10.1007/BFb0073148.

[176] H. Jacquet and R. P. Langlands, "Automorphic forms on $gl(2)$," in *LECTURE NOTES 114*. Springer, 1970.

[177] D. Jiang, B. Liu, and G. Savin, "Raising nilpotent orbits in wave-front sets," arXiv:1412.8742 [math.NT].

[178] D. Jiang and S. Rallis, "Fourier coefficients of Eisenstein series of the exceptional group of type G_2," *Pacific J. Math.* **181** no. 2, (1997) 281–314. http://dx.doi.org/10.2140/pjm.1997.181.281.

[179] D. Jiang and D. Soudry, "Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}," in *Contributions to automorphic forms, geometry, and number theory*, pp. 457–519. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[180] G. Jones and J. Jones, *Elementary Number Theory*. Springer Undergraduate Mathematics Series. Springer Verlag, 1998.

[181] A. Joseph, "Minimal realizations and spectrum generating algebras," *Comm. Math. Phys.* **36** (1974) 325–338.

[182] A. Joseph, "The minimal orbit in a simple Lie algebra and its associated maximal ideal," *Ann. Sci. École Norm. Sup. (4)* **9** no. 1, (1976) 1–29.

[183] A. Joseph, "On the associated variety of a primitive ideal," *J. Algebra* **93** no. 2, (1985) 509–523. http://dx.doi.org/10.1016/0021-8693(85)90172-3.

[184] D. Joyce and Y. Song, "A theory of generalized Donaldson-Thomas invariants," *Mem. Amer. Math. Soc.* **217** no. 1020, (2012) iv+199. http://dx.doi.org/10.1090/S0065-9266-2011-00630-1.

[185] B. Julia, "Group disintegrations," *Conf.Proc.* **C8006162** (1980) 331–350.

[186] V. G. Kac, *Infinite Dimensional Lie Algebras (3rd ed.)*. Cambridge University Press, 1990.

[187] A. Kapustin and E. Witten, "Electric-Magnetic Duality And The Geometric Langlands Program," *Commun. Num. Theor. Phys.* **1** (2007) 1–236.

[188] M. Kashiwara, "Crystalizing the q-analogue of universal enveloping algebras," *Comm. Math. Phys.* **133** no. 2, (1990) 249–260. http://projecteuclid.org/euclid.cmp/1104201397.

[189] N. Kawazumi, "Johnson's homomorphisms and the Arakelov-Green function," arXiv:0801.4218 [math.GT].

[190] D. Kazhdan and A. Polishchuk, "Minimal representations: spherical vectors and automorphic functionals," in *Algebraic groups and arithmetic*, pp. 127–198. Tata Inst. Fund. Res., Mumbai, 2004. arXiv:math/0209315.

[191] D. Kazhdan, B. Pioline, and A. Waldron, "Minimal representations, spherical vectors, and exceptional theta series," *Commun.Math.Phys.* **226** (2002) 1–40, arXiv:hep-th/0107222.

[192] D. Kazhdan and G. Savin, "The smallest representation of simply laced groups," in *Festschrift in honor of II Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989)*, vol. 2, pp. 209–223. 1990.

[193] H. H. Kim and K.-H. Lee, "Quantum affine algebras, canonical bases and q-deformation of arithmetical functions," arXiv:1101.4912 [math.RT].

[194] H. H. Kim, "The residual spectrum of G_2," *Canad. J. Math.* **48** no. 6, (1996) 1245–1272. http://dx.doi.org/10.4153/CJM-1996-066-3.

[195] E. Kiritsis and B. Pioline, "On R^4 threshold corrections in type IIB string theory and (p,q)-string instantons," *Nuclear Physics B* **508** (Feb., 1997) 509–534, arXiv:hep-th/9707018.

[196] A. W. Knapp, "Introduction to the Langlands Program," *Proc. Symp. Pure Math.* **61** (1997) 245–302.

[197] A. W. Knapp, "First Steps with the Langlands Program," in *Automorphic Forms and the Langlands Program*, Advanced Lectures in Mathematics 9, pp. 10–20. International Press, 2009.

[198] A. W. Knapp, "Prerequisites for the Langlands Program," in *Automorphic Forms and the Langlands Program*, Advanced Lectures in Mathematics 9, pp. 1–9. International Press, 2009.

[199] T. Kobayashi and G. Savin, "Global uniqueness of small representations," arXiv:1412.8019 [math.RT].

[200] N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*. Springer New York, 1984. http://dx.doi.org/10.1007/978-1-4612-1112-9.

[201] N. Koblitz, *Introduction to elliptic curves and modular forms*, vol. 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second ed., 1993. http://dx.doi.org/10.1007/978-1-4612-0909-6.

[202] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," arXiv:0811.2435 [math.AG].

[203] M. Kontsevich, "Homological algebra of mirror symmetry," in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pp. 120–139. Birkhäuser, Basel, 1995.

[204] B. Kostant and S. Rallis, "Orbits and representations associated with symmetric spaces," *Amer. J. Math.* **93** (1971) 753–809.

[205] B. Kostant, "Lie group representations on polynomial rings," *Amer. J. Math.* **85** (1963) 327–404.

[206] B. Kostant, "On Whittaker vectors and representation theory," *Invent. Math.* **48** no. 2, (1978) 101–184. http://dx.doi.org/10.1007/BF01390249.

[207] S. Krutelevich, "Jordan algebras, exceptional groups, and Bhargava composition," *J. Algebra* **314** no. 2, (2007) 924–977. http://dx.doi.org/10.1016/j.jalgebra.2007.02.060.

[208] T. Kubota, *On automorphic functions and the reciprocity law in a number field.* Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 2. Kinokuniya Book-Store Co., Ltd., Tokyo, 1969.

[209] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, vol. 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002. http://dx.doi.org/10.1007/978-1-4612-0105-2.

[210] K. F. Lai, "On the Tamagawa number of quasi-split groups," *Bull. Amer. Math. Soc.* **82** no. 2, (1976) 300–302.

[211] N. Lambert and P. West, "Perturbation Theory From Automorphic Forms," *JHEP* **1005** (2010) 098, arXiv:1001.3284 [hep-th].

[212] N. Lambert and P. C. West, "Duality Groups, Automorphic Forms and Higher Derivative Corrections," *Phys.Rev.* **D75** (2007) 066002, arXiv:hep-th/0611318.

[213] S. Lang, $SL(2,\mathbb{R})$, vol. 105 of *Graduate Texts in Mathematics*. Springer, 1975.

[214] R. P. Langlands, "Eisenstein series," in *Algebraic Groups and Discontinuous Subgroups*, A. Borel and G. D. Mostow, eds., vol. 9, pp. 235–252, Proc. Sympos. Pure Math., Boulder, Colo., 1965. Amer. Math. Soc., Providence, R.I., 1966. Available online at http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/eisen2-ps.pdf [Accessed on 2. August 2013].

[215] R. P. Langlands, "L-functions and automorphic representations," in *Proc. Int. Congr. Math.*, Helsinki, pp. 165–175. 1967.

[216] R. P. Langlands, "Letter to André Weil." Available online at http://publications.ias.edu/sites/default/files/ltw_1.pdf, 1967. [Accessed on 5. February 2015].

[217] R. P. Langlands, "Problems in the theory of automorphic forms," *Lectures in modern analysis and applications III, Lecture Notes in Mathematics* **170** (1970) .

[218] R. P. Langlands, *On the Functional Equations Satisfied by Eisenstein Series*, vol. 544 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, Berlin-Heidelberg, New York, 1976.

[219] R. Langlands, *Euler Products*. James K. Whittemore lectures in mathematics given at Yale University. Yale University, Department of Mathematics, 1967. Available online at http://sunsite.ubc.ca/DigitalMathArchive/Langlands/pdf/ep-ps.pdf [Accessed on 2. August 2013].

[220] R. Langlands, "Letter to Godement." Available online at http://publications.ias.edu/sites/default/files/godement-ps.pdf, 1967. [Accessed on 5. February 2015].

[221] K.-H. Lee and Y. Zhang, "Weyl group multiple Dirichlet series for symmetrizable Kac-Moody root systems," *Trans. Amer. Math. Soc.* **367** no. 1, (2015) 597–625. http://dx.doi.org/10.1090/S0002-9947-2014-06159-X.

[222] P. Littelmann, "An effective method to classify nilpotent orbits," in *Algorithms in algebraic geometry and applications (Santander, 1994)*, vol. 143 of *Progr. Math.*, pp. 255–269. Birkhäuser, Basel, 1996.

[223] P. Littelmann, "Cones, crystals, and patterns," *Transform. Groups* **3** no. 2, (1998) 145–179. http://dx.doi.org/10.1007/BF01236431.

[224] D. Liu, "Eisenstein series on loop groups," *Trans. Amer. Math. Soc.* **367** no. 3, (2015) 2079–2135. http://dx.doi.org/10.1090/S0002-9947-2014-06220-X.

[225] G. Lusztig, "Canonical bases arising from quantized enveloping algebras," *J. Amer. Math. Soc.* **3** no. 2, (1990) 447–498. http://dx.doi.org/10.2307/1990961.

[226] G. Lusztig, *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. http://dx.doi.org/10.1007/978-0-8176-4717-9. Reprint of the 1994 edition.

[227] J. Manschot, "Stability and duality in N=2 supergravity," *Commun.Math.Phys.* **299** (2010) 651–676, arXiv:0906.1767 [hep-th].

[228] J. Manschot and G. W. Moore, "A Modern Farey Tail," *Commun.Num.Theor.Phys.* **4** (2010) 103–159, arXiv:0712.0573 [hep-th].

[229] P. Marios Petropoulos and P. Vanhove, "Gravity, strings, modular and quasimodular forms," arXiv:1206.0571 [math-ph].

[230] H. Matumoto, "Whittaker vectors and associated varieties," *Invent. Math.* **89** no. 1, (1987) 219–224. http://dx.doi.org/10.1007/BF01404678.

[231] P. J. McNamara, "Metaplectic Whittaker functions and crystal bases," *Duke Math. J.* **156** no. 1, (2011) 1–31. http://dx.doi.org/10.1215/00127094-2010-064.

[232] P. J. McNamara, "Principal series representations of metaplectic groups over local fields," in *Multiple Dirichlet series, L-functions and automorphic forms*, vol. 300 of *Progr. Math.*, pp. 299–327. Birkhäuser/Springer, New York, 2012. http://dx.doi.org/10.1007/978-0-8176-8334-4_13.

[233] S. D. Miller, "Residual automorphic forms and spherical unitary representations of exceptional groups," *Ann. of Math. (2)* **177** no. 3, (2013) 1169–1179. http://dx.doi.org/10.4007/annals.2013.177.3.9.

[234] S. D. Miller and G. W. Moore, "Landau-Siegel zeroes and black hole entropy," arXiv:hep-th/9903267 [hep-th].

[235] S. D. Miller and S. Sahi, "Fourier coefficients of automorphic forms, character variety orbits, and small representations," *J. Number Theory* **132** no. 12, (2012) 3070–3108. http://dx.doi.org/10.1016/j.jnt.2012.05.032.

[236] S. D. Miller and W. Schmid, "The highly oscillatory behavior of automorphic distributions for SL(2)," *Lett. Math. Phys.* **69** (2004) 265–286. http://dx.doi.org/10.1007/s11005-004-0470-8.

[237] S. D. Miller and W. Schmid, "Summation formulas, from Poisson and Voronoi to the present," in *Noncommutative harmonic analysis*, vol. 220 of *Progr. Math.*, pp. 419–440. Birkhäuser Boston, Boston, MA, 2004.

[238] S. D. Miller and W. Schmid, "The Rankin-Selberg method for automorphic distributions," in *Representation theory and automorphic forms*, vol. 255 of *Progr. Math.*, pp. 111–150. Birkhäuser Boston, Boston, MA, 2008. http://dx.doi.org/10.1007/978-0-8176-4646-2_4.

[239] C. Mœglin, "Front d'onde des représentations des groupes classiques p-adiques," *Amer. J. Math.* **118** no. 6, (1996) 1313–1346. http://muse.jhu.edu/journals/american_journal_of_mathematics/v118/118.6moe_glin.pdf.

[240] C. Mœglin, "Correspondance de Howe et front d'onde," *Adv. Math.* **133** no. 2, (1998) 224–285. http://dx.doi.org/10.1006/aima.1997.1689.

[241] C. Mœglin and J.-L. Waldspurger, "Modèles de Whittaker dégénérés pour des groupes p-adiques," *Math. Z.* **196** no. 3, (1987) 427–452. http://dx.doi.org/10.1007/BF01200363.

[242] C. Mœglin and J.-L. Waldspurger, "Le spectre résiduel de $\mathrm{GL}(n)$," *Ann. Sci. École Norm. Sup. (4)* **22** no. 4, (1989) 605–674. http://www.numdam.org/item?id=ASENS_1989_4_22_4_605_0.

[243] C. Mœglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*. Cambridge University Press, 1995.

[244] C. Mœglin, "Représentations unipotentes et formes automorphes de carré intégrable," *Forum Math.* **6** no. 6, (1994) 651–744. http://dx.doi.org/10.1515/form.1994.6.651.

[245] R. V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1995. A Wiley-Interscience Publication.

[246] G. W. Moore, "Physical mathematics and the future." Available online at http://www.physics.rutgers.edu/ gmoore/PhysicalMathematicsAndFuture.pdf, 2014.

[247] G. W. Moore, "Arithmetic and attractors," arXiv:hep-th/9807087 [hep-th].

[248] G. W. Moore, N. Nekrasov, and S. Shatashvili, "D particle bound states and generalized instantons," *Commun. Math. Phys.* **209** (2000) 77–95, arXiv:hep-th/9803265 [hep-th].

[249] H. Narita, "Fourier-jacobi expansion of automorphic forms on sp(1, q) generating quaternionic discrete series," *J. Funct. Anal.* **239** (2006) 638–682.

[250] R. I. Nepomechie, "Magnetic Monopoles from Antisymmetric Tensor Gauge Fields," *Phys.Rev.* **D31** (1985) 1921.

[251] J. Neukirch, *Algebraische Zahlentheorie*. Springer, 2006.

[252] N. A. Obers and B. Pioline, "Eisenstein Series and String Thresholds," *Communications in Mathematical Physics* **209** (2000) 275–324, arXiv:hep-th/9903113.

[253] N. Obers and B. Pioline, "U duality and M theory," *Phys.Rept.* **318** (1999) 113–225, arXiv:hep-th/9809039 [hep-th].

[254] N. A. Obers and B. Pioline, "Eisenstein series in string theory," *Class.Quant.Grav.* **17** (2000) 1215–1224, arXiv:hep-th/9910115.

[255] D. Ž. Đoković, "The closure diagram for nilpotent orbits of the split real form of E_7," *Represent. Theory* **5** (2001) 284–316 (electronic). http://dx.doi.org/10.1090/S1088-4165-01-00124-8.

[256] D. Ž. Đoković, "The closure diagram for nilpotent orbits of the split real form of E_8," *Cent. Eur. J. Math.* **1** no. 4, (2003) 573–643 (electronic). http://dx.doi.org/10.2478/BF02475183.

[257] M. M. Patnaik, "Unramified Whittaker Functions on p-adic Loop Groups," arXiv:1407.8072 [math.RT].

[258] D. Persson, *Arithmetic and Hyperbolic Structures in String Theory*. PhD thesis, Zurich, ETH, 2010. arXiv:1001.3154 [hep-th].

[259] D. Persson, "Automorphic Instanton Partition Functions on Calabi-Yau Threefolds," *J. Phys. Conf. Ser.* **346** (2012) .

[260] I. I. Piatetski-Shapiro, "Multiplicity one theorems," in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pp. 209–212. Amer. Math. Soc., Providence, R.I., 1979.

[261] B. Pioline, "A Note on nonperturbative R**4 couplings," *Phys.Lett.* **B431** (1998) 73–76, arXiv:hep-th/9804023 [hep-th].

[262] B. Pioline, "BPS black hole degeneracies and minimal automorphic representations," *Journal of High Energy Physics* **8** (Aug., 2005) 71, arXiv:hep-th/0506228.

[263] B. Pioline, "R**4 couplings and automorphic unipotent representations," *Journal of High Energy Physics* **3** (Mar., 2010) 116, arXiv:1001.3647 [hep-th].

[264] B. Pioline, "Lectures on black holes, topological strings and quantum attractors," *Class.Quant.Grav.* **23** (2006) S981, arXiv:hep-th/0607227.

[265] B. Pioline, "D^6R^4 amplitudes in various dimensions," *JHEP* **04** (2015) 057, arXiv:1502.03377 [hep-th].

[266] B. Pioline, H. Nicolai, J. Plefka, and A. Waldron, "R**4 couplings, the fundamental membrane and exceptional theta correspondences," *JHEP* **0103** (2001) 036, arXiv:hep-th/0102123.

[267] B. Pioline and D. Persson, "The Automorphic NS5-brane," *Commun. Num. Theor. Phys.* **3** (2009) 697–754, arXiv:0902.3274 [hep-th].

[268] B. Pioline and S. Vandoren, "Large D-instanton effects in string theory," *JHEP* **0907** (2009) 008, arXiv:0904.2303 [hep-th].

[269] B. Pioline and A. Waldron, "Automorphic forms: A Physicist's survey," in *Frontiers in number theory, physics, and geometry 1: On random matrices, zeta functions and dynamical systems. Proceedings, Meeting, Les Houches, France, March 9-21, 2003*. 2003. arXiv:hep-th/0312068 [hep-th].

[270] B. Pioline and A. Waldron, "The Automorphic membrane," *JHEP* **0406** (2004) 009, arXiv:hep-th/0404018.

[271] V. Platonov, A. Rapinchuk, and R. Rowen, *Algebraic Groups and Number Theory*. Pure and Applied Mathematics. Elsevier Science, 1993.

[272] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string & Vol. 2: Superstring theory and beyond*. Cambridge University Press, 2007.

[273] G. Prasad, "Strong approximation for semi-simple groups over function fields," *Annals of Mathematics* **105** no. 3, (1977) pp. 553–572.

[274] N. V. Proskurin, "Expansion of automorphic functions," *Journal of Soviet Mathematics* **26** no. 3, (Aug, 1984) 1908–1921. `http://dx.doi.org/10.1007/bf01670579`.

[275] B. Riemann, "Über die Anzahl der Primzahlen unter einer gegebenen Größe," *Monatsber. Berliner Akademie* (1859) 671–680.

[276] D. Robles-Llana, M. Rocek, F. Saueressig, U. Theis, and S. Vandoren, "Nonperturbative corrections to 4D string theory effective actions from SL(2,Z) duality and supersymmetry," *Phys. Rev. Lett.* **98** (2007) 211602, `arXiv:hep-th/0612027 [hep-th]`.

[277] F. Rodier, "Modèle de Whittaker et caractères de représentations," in *Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1974)*, pp. 151–171. Lecture Notes in Math., Vol. 466. Springer, Berlin, 1975.

[278] I. Satake, "Theory of spherical functions on reductive algebraic groups over p-adic fields," *Publ. I. H. E. S.* **18** (1963) 1–69.

[279] G. Savin and M. Woodbury, "Structure of internal modules and a formula for the spherical vector of minimal representations," *J. Algebra* **312** no. 2, (2007) 755–772. `http://dx.doi.org/10.1016/j.jalgebra.2007.01.014`.

[280] O. Schlotterer and S. Stieberger, "Motivic Multiple Zeta Values and Superstring Amplitudes," *J. Phys.* **A46** (2013) 475401, `arXiv:1205.1516 [hep-th]`.

[281] W. Schmid, "Automorphic distributions for $\mathrm{SL}(2,\mathbb{R})$," in *Conférence Moshé Flato 1999, Vol. I (Dijon)*, vol. 21 of *Math. Phys. Stud.*, pp. 345–387. Kluwer Acad. Publ., Dordrecht, 2000.

[282] J. H. Schwarz and A. Sen, "Duality symmetries of 4-D heterotic strings," *Phys.Lett.* **B312** (1993) 105–114, `arXiv:hep-th/9305185 [hep-th]`.

[283] J. Sekiguchi, "Remarks on real nilpotent orbits of a symmetric pair," *J. Math. Soc. Japan* **39** no. 1, (1987) 127–138. `http://dx.doi.org/10.2969/jmsj/03910127`.

[284] E. Selmer, "The diophantine equation $ax^3 + by^3 + cz^3 = 0.$," *Acta Mathematica* **85** no. 1, (1951) 203–362. `http://dx.doi.org/10.1007/BF02395746`.

[285] A. Sen, "Strong - weak coupling duality in four-dimensional string theory," *Int.J.Mod.Phys.* **A9** (1994) 3707–3750, `arXiv:hep-th/9402002 [hep-th]`.

[286] A. Sen, "Black Hole Entropy Function, Attractors and Precision Counting of Microstates," *Gen.Rel.Grav.* **40** (2008) 2249–2431, `arXiv:0708.1270 [hep-th]`.

[287] J.-P. Serre, *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[288] F. Shahidi, "Functional equation satisfied by certain L-functions," *Compositio Math.* **37** no. 2, (1978) 171–207.

[289] F. Shahidi, "On certain L-functions," *Amer. J. Math.* **103** no. 2, (1981) 297–355. http://dx.doi.org/10.2307/2374219.

[290] F. Shahidi, "Local coefficients as Artin factors for real groups," *Duke Math. J.* **52** no. 4, (1985) 973–1007. http://dx.doi.org/10.1215/S0012-7094-85-05252-4.

[291] F. Shahidi, "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups," *Ann. of Math. (2)* **132** no. 2, (1990) 273–330. http://dx.doi.org/10.2307/1971524.

[292] F. Shahidi, "Intertwining operators, l-functions, and representation theory," in *Lecture Notes of the Eleventh KAIST Mathematics Workshop 1996, Taejon, Korea*. 1996. http://www.math.rutgers.edu/~sdmiller/l-functions/shahidi-korea.pdf.

[293] F. Shahidi, "Infinite dimensional groups and automorphic L-functions," *Pure Appl. Math. Q.* **1** no. 3, part 2, (2005) 683–699. http://dx.doi.org/10.4310/PAMQ.2005.v1.n3.a8.

[294] F. Shahidi, *Eisenstein series and automorphic L-functions*, vol. 58 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010.

[295] J. A. Shalika, "The multiplicity one theorem for GL_n," *Ann. of Math. (2)* **100** (1974) 171–193.

[296] S. H. Shenker, "The Strength of nonperturbative effects in string theory," in *The large N expansion of quantum field theory and statistical physics*, E. Brezin and S. Wadia, eds., pp. 191–200. 1990.

[297] D. Shih, A. Strominger, and X. Yin, "Recounting dyons in N=4 string theory," *JHEP* **0610** (2006) 087, arXiv:hep-th/0505094.

[298] T. Shintani, "On an explicit formula for class-1 "whittaker functions" on $gl(n)$ over p-adic fields," *Proc. Japan. Acad.* **52** (1976) 180–182.

[299] C. Soulé, "An introduction to arithmetic groups," in *Frontiers in Number Theory, Physics, and Geometry II*, pp. 247–276. Springer, Berlin, 2007. http://dx.doi.org/10.1007/978-3-540-30308-4_6.

[300] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, vol. 946 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982.

[301] R. P. Stanley, "A baker's dozen of conjectures concerning plane partitions," in *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, vol. 1234 of *Lecture Notes in Math.*, pp. 285–293. Springer, Berlin, 1986. http://dx.doi.org/10.1007/BFb0072521.

[302] L. A. Takhtajan, "A simple example of modular forms as tau-functions for integrable equations," *Theoretical and Mathematical Physics* **93** (1992) 1308–1317.

[303] T. Tamagawa, "On the ζ-functions of a division algebra," *Ann. of Math. (2)* **77** (1963) 387–405.

[304] T. Tao, "Tates proof of the functional equation." Available online at https://terrytao.wordpress.com/2008/07/27/tates-proof-of-the-functional-equation/, 2008.

[305] J. T. Tate, "Fourier analysis in number fields, and Hecke's zeta-functions," in *Algebraic Number Theory*, pp. 305–347. Thompson, Washington, D.C., Proc. Instructional Conf., Brighton, 1965, 1967.

[306] C. Teitelboim, "Gauge Invariance for Extended Objects," *Phys.Lett.* **B167** (1986) 63–68.

[307] C. Teitelboim, "Monopoles of Higher Rank," *Phys.Lett.* **B167** (1986) 69–72.

[308] A. Terras, *Harmonic analysis on symmetric spaces and applications. I*. Springer-Verlag, New York, 1985. http://dx.doi.org/10.1007/978-1-4612-5128-6.

[309] A. Terras, *Harmonic analysis on symmetric spaces and applications. II*. Springer-Verlag, Berlin, 1988. http://dx.doi.org/10.1007/978-1-4612-3820-1.

[310] T. Tokuyama, "A generating function of strict Gel′fand patterns and some formulas on characters of general linear groups," *J. Math. Soc. Japan* **40** no. 4, (1988) 671–685. http://dx.doi.org/10.2969/jmsj/04040671.

[311] D. Tong, "String Theory," arXiv:0908.0333 [hep-th].

[312] A. Unterberger, *Pseudodifferential analysis, automorphic distributions in the plane and modular forms*, vol. 8 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011. http://dx.doi.org/10.1007/978-3-0348-0166-9.

[313] È. B. Vinberg, "The classification of nilpotent elements of graded Lie algebras," *Dokl. Akad. Nauk SSSR* **225** no. 4, (1975) 745–748.

[314] A. Vinogradov and L. Takhtadžjan, "Theory of the eisenstein series for the group sl(3, r) and its application to a binary problem. i. fourier expansion of the highest eisenstein series," *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **76** (1978) 5–52.

[315] M. Wakimoto, *Lectures on infinite-dimensional Lie algebra*. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. http://dx.doi.org/10.1142/9789812810700.

[316] J. Walcher, "On the Arithmetic of D-brane Superpotentials: Lines and Conics on the Mirror Quintic," *Commun. Num. Theor. Phys.* **6** (2012) 279–337, arXiv:1201.6427 [hep-th].

[317] N. R. Wallach, "Generalized Whittaker vectors for holomorphic and quaternionic representations," *Comment. Math. Helv.* **78** no. 2, (2003) 266–307. http://dx.doi.org/10.1007/s000140300012.

[318] H. Weber and R. Dedekind, eds., *Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlass: The collected works of Bernhard Riemann*, ch. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse, p. 136. Gesammelte mathematische Werke und wissenschaftlicher Nachlass / Bernhard Riemann. Dover, 1902.

[319] A. Weil, "Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen," *Math. Ann.* **168** (1967) 149–156.

[320] A. Weil, "Séries de Dirichlet et fonctions automorphes," in *Séminaire Bourbaki, Vol. 10*, pp. Exp. No. 346, 547–552. Soc. Math. France, Paris, 1995.

[321] P. C. West, "E(11) and M theory," *Class.Quant.Grav.* **18** (2001) 4443–4460, `arXiv:hep-th/0104081 [hep-th]`.

[322] E. Witten, "String theory dynamics in various dimensions," *Nucl.Phys.* **B443** (1995) 85–126, `arXiv:hep-th/9503124 [hep-th]`.

[323] D. Zagier, "Elliptic modular forms and their applications," in *The 1-2-3 of modular forms*, Universitext, pp. 1–103. Springer, Berlin, 2008. `http://dx.doi.org/10.1007/978-3-540-74119-0_1`.

[324] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of GL(n)," *Ann. Sci. École Norm. Sup. (4)* **13** no. 2, (1980) 165–210. `http://www.numdam.org/item?id=ASENS_1980_4_13_2_165_0`.

[325] S.-W. Zhang, "Gross-Schoen cycles and dualising sheaves," *Invent. Math.* **179** no. 1, (2010) 1–73. `http://dx.doi.org/10.1007/s00222-009-0209-3`.

[326] B. Zwiebach, *A first course in string theory*. Cambridge University Press, 2004.

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