SCALING OF A WALL-STABILIZED HIGH-BETA STELLARATOR

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Abstract:

On the basis of the surface current model, the scaling of an m=1 wall-stabilized high-ß stellarator is discussed in terms of the compression ratio $\varkappa$, aspect ratio A, helical distortions δ_o, δ_1, δ_2, and ß. The m=1, k= 0 mode can be wall-stabilized for $\varkappa \simeq 2$, $\delta_1 \simeq 1$, $A \simeq 200$, $\beta \simeq 0.83$, $\delta_o \simeq 0.21$, $\delta_2 \simeq 0.07$. The number of periods around the torus is then $n_0 \simeq 0.6\ A/\varkappa^2$.

I. INTRODUCTION

In the ISAR T 1 experiment, the toroidal finite-ß MHD equilibrium characterized by zero toroidal net current is produced by superimposing stellarator fields of different ℓ-numbers on the main (toroidal) magnetic field /2,4/. The combination of an ℓ-fold helical field with lower amplitude $\ell \pm 1$ helical fields of proper period and phase compensates the outward toroidal drift force acting on the plasma column. Equilibria of this type can be regarded as a generalized class of M & S-type equilibria with non-planar magnetic axis.

The stability calculations of the m=1 mode /1,4,5/ show that, in leading order, the ℓ=1 helical field is the most favorable one for creating a high-ß equilibrium which is stable to the m=1, k=0 mode. Therefore, magnetic field configurations with ℓ=1 corrugation fields in leading order and lower-amplitude ℓ=0 and/or ℓ=2 have been investigated in more detail with respect to the stability behaviour/3,5/. The available theories based on the surface-current model show that the m=1 mode (rigid gross displacement) can be stabilized if the stabilizing effect of the conducting wall is taken into account. In the following note, we will discuss the scaling of experiments in the high-ß stellarator geometry which are wall-stabilized to m=1, k=0 modes /6/ and we will estimate the optimum regimes using those formulas for the growth rate of the m=1, k=0 mode in an ℓ=0,1,2 system which have been derived from the surface current model /1,2,3,5/.

The small parameter, used in the theories just cited as an expansion parameter, is $\epsilon := ha$ (a mean plasma radius, $h = 2\pi/L$ periodicity number of the $\ell = 1$ helical field). In addition to ϵ, we use the following parameters: the compression ratio $\varkappa = b/a$ (b mean wall radius) and the aspect ratio $A = R_o/a$ related to the plasma radius (R_o major torus radius). The toroidal equilibrium configuration has n periods around the major circumference. The parameter ϵ is then related to the aspect ratio by $\epsilon = n/A$. The only parameter relating the quantities inside the plasma to those outside is $\beta = 2p/B_o^2$ (kinetic plasma pressure over magnetic pressure of the outside main magnetic field B_o). The superposition of the helical ℓ=1 and of lower-amplitude ℓ=0,2 fields produces a deformation of the plasma surface of the form

$$r = a \{ 1 + \sum \delta_\ell \cos (\ell\theta - hz)\}, \ \ell = 0,1,2 \tag{1}$$

where δ_ℓ are the deformations in units of the mean plasma radius.

The marginal limit of the m=1 mode can be expressed in six dimensionless parameters as a condition (see below) :

$$F(\delta_1, A, \varkappa; \alpha, n, \beta) = 0 .$$

The search for an optimal working range under this condition has to include technical and other arguments which will be briefly listed for the parameters δ_1, A and $\varkappa$ in the following.

Firstly, δ_1 has to be small. The need for a small helical distortion δ_1 is imposed mainly for technical reasons. But the limit of the validity of the formulas for equilibrium and stability is also reached when δ_1 becomes too large. The aspect ratio A has to be not to large; this is a condition from the view of reactor considerations. For the compression ratio $\varkappa$ one has to look for a large value to reduce the problems of shock heating. It turned out that optimization of these three quantities ($\delta_1 \rightarrow$ min!, A$\rightarrow$ min !, $\varkappa \rightarrow$ max!) with respect to the remaining quantities α, n and ß leads in every case to the identical set of values α_o, n_o, and β_o. So the requirements imposed on δ_1, A, and $\varkappa$ can be met at the same time and we can restrict ourselves in the following to the minimization of δ_1.

The optimum values of the distortions δ_ℓ will be determined for systems with $\ell = 0, 1$ and with $\ell = 0, 1, 2$ corrugation fields. The amplitudes of the distortions of the plasma surface are given by the equilibrium conditions

$$\delta_o \delta_1 = \frac{2A\alpha}{n^2\,(3-2\beta)} \qquad (2)$$

$$\delta_2 \delta_1 = (1-\alpha)\,\frac{2A}{n^2(2-\beta)} . \qquad (3)$$

The parameter α varies between 0 and 1 ($0 < \alpha \leq 1$) and parameterizes the relative contribution of the $\ell = 0$ and the $\ell = 2$ corrugation fields. An optimum value of α will be derived lateron which depends only on ß.

II. SCALING OF THE $\ell = 0,1,2$ CORRUGATION FIELDS

For the case of superimposed $\ell = 0,1,2$ corrugation fields, the growth rate γ of the $m = 1$, $k = 0$ mode (k wave number in longitudinal direction) is given by /3/

$$\gamma^2 = \frac{V^2 n^2}{A^2 a^2} \frac{\beta}{(1-\beta)} \left[G_0 \delta_0^2 + \frac{n^2}{A^2} G_1 \delta_1^2 + G_2 \delta_2^2 + G_3 \delta_1^4 - \frac{\beta}{\varkappa^4} \delta_1^2 \right] \quad (4)$$

where $V = B_i / (\mu_0 \rho)^{1/2}$ is the Alfvén velocity and $B_0^2 / \mu_0 \rho = \frac{V^2}{(1-\beta)}$.

The functions G_ℓ, $\ell = 0,1,2$ have been introduced in /1/ (without a factor of 2):

$$G_0 = \frac{(3-2\beta)(1-\beta)}{(2-\beta)} \quad (5a)$$

$$G_1 = \frac{(4-3\beta)(2-\beta)}{8(1-\beta)} \quad (5b)$$

$$G_2 = \frac{1}{2}(2-\beta) \quad (5c)$$

$$G_3 = \frac{\beta^4}{32(2-\beta)} . \quad (5d)$$

The last term in Eq.(4) is the stabilizing wall term proportional to $\varkappa^{-4}$. Therefore, $\varkappa$ should be small (e.g. 2) in order to achieve effective wall stabilization. The stabilizing effect arises from the induced helical dipole current. The unstable term G_3 is neglected because it is small compared to G_1 and it can be fully compensated when an elliptical deformation is introduced /5/.

The two essential points in using the $\ell=1$ field in leading order (in contrast to the classical M & S configuration /7/) are

a) there is no unstable term of order $O(n^2 \delta_1^2 / A^2)$,

b) there exists a strong stabilizing term of order $O(n^2 \delta_1^2 / A^2 \varkappa^4)$

of the same order as the unstable terms if $\varkappa^4$ is small enough.

The growth rate of the m = 1, $k \neq 0$ mode is smaller than that for k = 0; therefore, the case $k \neq 0$ will not be investigated.

With the help of (2) and (3) we eliminate δ_o and δ_2 in Eq.(4) and carry out the optimization for marginal stability ($\gamma^2 = 0$). In this case Eq.(4) can be explicitly solved for δ_1

$$\delta_1^4 = \frac{4\alpha^2(1-\beta)/(3-2\beta) + 2(1-\alpha)^2}{(2-\beta)\left[\beta n^4/(\varkappa^4 A^2) - G_1 n^6/A^4\right]} . \qquad (6)$$

We shall now minimize δ_1 with respect to n, α, and β in that order.

The function δ_1^4 has local minima with respect to n, α, and β. After minimizing with respect to n, one sees that the A-and $\varkappa$-dependences are simply given by the factor $\varkappa^{12}/A^2$ in front of δ_1^4.

Furthermore, the three parameters A, n and $\varkappa$ occur in Eq.(6) as the two combinations $n/(\varkappa A^{1/2})$ and $n/A^{2/3}$. Now the quantity δ_1^4 has a minimum with respect to n for $n=n_1$, where n_1 is given by

$$n_1(\beta) = N_1 A/\varkappa^2 , \qquad N_1 = (2\beta/3G_1)^{1/2} . \qquad (7)$$

There is precisely one solution n_1 for the number of periods and this solution is always positive.

II. 1. $\ell = 0,1$ - system ($\alpha = 1$)

With Eqs.(6) and (7) δ_1 was calculated for $\alpha = 1$:

$$\delta_{1,n} = D_n \varkappa^3/A^{1/2}$$

with $D_n(\beta) = \left[27(4-3\beta)^2(2-\beta)/64\beta^3(1-\beta)(3-2\beta)\right]^{1/4}$

(the index n indicates minimization with respect to n).

The function $D_n(\beta)$ varies weakly in the region $0.5 \lesssim \beta \leq 0.9$, assumes the minimum value of 1.71 for $\beta = \beta_2 = 0.86$ (see Fig. 1), and is infinite for $\beta = 0$ and 1. The corresponding $\ell = 0$ deformation of the plasma surface is calculated from Eq.(2):

$$(\delta_0 \delta_1)_n = D_0 \varkappa^4 / A$$

with

$$D_0 = \frac{3(4-3\beta)(2-\beta)}{8\beta(1-\beta)(3-2\beta)} . \tag{9}$$

In the special case where $\beta = \beta_2$ minimizes D_n above one obtains $D_0(\beta_2) = 3.94$, which does not coincide with the minimum value of D_0. The β-dependence of the $\ell=0$ distortion after minimizing δ_1^4 with respect to n is given by $D_0(\beta)$. The function D_0 is shown in Fig.2 (dashed curve).

Finally, after minimization of δ_1 with respect to n and β one gets:

$$\delta_{1,n\beta} = 1.71\, \varkappa^3 / A^{1/2} . \tag{10}$$

Fig.3 shows the curves $\varkappa = \varkappa(A; \delta_{1,n\beta})$ in a logarithmic A, $\varkappa$ - diagram (dashed curves); the parameter of these curves is $\delta_{1,n\beta}$ and assumes the values 0.25, 0.50, 1, 2,3. The region above the curves is the unstable region.

<u>II.2. $\ell = 0,1,2$ - system ($\alpha \neq 1$)</u>

Next we minimize δ_1^4, Eq.(6), with respect to α and with respect to n. Consequently, with Eq.(7) we obtain from Eq.(6) the minimizing α-value, α_1, as a function of β:

$$\alpha_1 = \frac{(3-2\beta)}{(5-4\beta)} \tag{11}$$

which increases monotonically: $3/5 \leq \alpha_1 \leq 1$; α_1 depends only on β and does not depend on the geometric parameters of the configuration. It follows from Eq.(11) that the minimum value of α is at $\beta = 0$ and is given by 0.6.

Consequently, a non-vanishing $\ell = 0$ distortion of the plasma surface is needed in the case of minimizing δ_1^4, Eq.(6), with respect to n and α at arbitrary ß. Using the minimizing functions α_1 and n_1, the ratio of the $\ell = 2$ distortion to the $\ell = 0$ distortion is then

$$\delta_2/\delta_0 = 2(1-\beta)/(2-\beta) .$$

We see from this formula that the $\ell = 0,2$ distortions are approximately of the same magnitude for small ß (e.g. for $\beta = 0.1$: $\delta_2/\delta_0 = 0.94$), and that the ratio δ_2/δ_0 decreases monotonically to small values with increasing ß (e.g. for $\beta = 0.9$: $\delta_2/\delta_0 = 0.18$).

Inserting in Eq.(6) the minimizing functions n_1 and α_1, one gets for δ_1

$$\delta_{1,n\alpha} = D_{n\alpha}\, \varkappa^3/A^{1/2} \tag{12}$$

$$D_{n\alpha} = D_n\, \alpha_1^{1/4} \tag{13}$$

(the index $n\alpha$ indicates minimization with respect to n and α).

The function $D_{n\alpha}$ is infinite for $\beta = 0, 1$ and has a minimum value of 1.62 for $\beta = 0.83 =: \beta_0$.

δ_2/δ_0 for this minimizing β_0- value is 0.31. The minimum values of the functions $D_n(\alpha = 1)$ and $D_{n\alpha}$ $(\alpha = \alpha_1)$ differ little from each other (Fig.1). The fact that $D_{n\alpha}$ varies slowly below $\beta_0 = 0.83$ is important; therefore, the final scaling law (shown in Fig.3) hardly changes if a ß-value different from the optimum value β_0 is used : $\alpha_1(\beta_0) = 0.8 =: \alpha_0$; $n_1(\beta_0) = 0.66\, A/\varkappa^2 =: n_0$.

The final scaling of δ_1 after minimization with respect to n, α, ß is

$$\delta_{1,n\alpha\beta} = 1.62\ \varkappa^3/A^{1/2} . \tag{14}$$

Let us now express $\varkappa = \varkappa(A; \delta_{1,n\alpha\beta})$. Fig.3 then represents $\varkappa = \varkappa(A)$ for different choises of $\delta_{1,n\alpha\beta}$. The stable $A,\varkappa$-region for the m=1, k=0 mode lies below these curves.

The ℓ = 0,2 distortions have to be calculated from (2), (3), and are given by

$$(\delta_0 \delta_1)_{n\alpha} = d_0 \varkappa^4/A , \quad d_0(\beta) = D_0 \alpha_1 , \quad d_0(0.83) = 2.79 ,$$

$$(\delta_2 \delta_1)_{n\alpha} = D_2 \varkappa^4/A , \quad D_2(\beta) = \frac{3(4-3\beta)}{4\beta(5-4\beta)} , \quad D_2(0.83) = 0.81 .$$

The functions d_0 and D_2 are shown in Fig.2. For the optimum case of $\beta_0 = 0.83$, the ℓ = 0 distortion is roughly three times as large as the ℓ = 2 distortion. If the ß-value is increased above the optimum value β_0, the ℓ = 0 distortion increases, while the ℓ = 2 distortion decreases. If the ß-value is decreased below the optimum β_0, $0.5 \lesssim \beta \leq \beta_0$, the ℓ = 0 distortion decreases and the ℓ = 2 distortion increases, both slowly. The ℓ = 2 distortion is always smaller than the ℓ = 0 distortion.

II.3. Limiting curves for non-shaped vessel

Up to now we have minimized the ℓ = 1 distortion of the plasma surface and have found the $(\delta_1, A, \varkappa)$ triplets for the marginal stability of the m=1, k=0 mode. We now describe the conditions under which a smooth toroidal vacuum vessel or a helically shaped vessel is needed in order to produce a toroidal high-ß equilibrium with superimposed ℓ = 0,1,2 fields which is wall-stabilized with respect to the m=1, k=0 mode.

The limiting curve in the A, $\varkappa$-diagram is to be calculated from the assumption that the distorted plasma column should lie within a non-shaped vessel, this fact being stated by

$$a + \Delta a \leq r_v \tag{15}$$

where r_v is the mean radius of the vessel and Δa is the total plasma distortion (for the estimate it is enough to take into account the ℓ=1 distortion). Inserting the non-dimensional parameters we get the relation

$$\delta_1 \leq \varkappa \nu - 1 \tag{16}$$

where $\nu = r_v/b$ is a measure of the wall thickness of the vessel; Eq.(16) gives the admissible δ_1 values as functions of $\varkappa$ for the case of a non-shaped vessel. Using Eq.(14)(i.e. $\alpha = \alpha_0$) we get the limiting curve in the A,$\varkappa$ - diagram:

$$A_1 = \frac{\varkappa^6 D^2}{(\nu\varkappa - 1)^2} \qquad (17)$$

with D = 1.62 .

The limiting curve is shown in Fig.3: the curve (i) represents $\nu = 1$, $\alpha = \alpha_o$; the curve (ii) represents $\nu = 0.8$, $\alpha = \alpha_o$ which is more realistic.

The region to the right-hand side of the limiting curve is admissible if a non-shaped vessel is used. A_1 depends strongly on ν.

III. CONCLUSIONS

The scaling of an m = 1 wall-stabilized high-ß stellarator is discussed on the basis of the surface-current model. Scaling laws and curves in terms of the aspect ratio A, compression ratio $\varkappa$, helical distortions δ_ℓ (ℓ = 0, 1, 2), and ß are given. The minimization of the ℓ = 1 distortion of the plasma surface in the case of marginal stability was performed with respect to the number of periods (n) around the torus, the amplitudes of the helical ℓ = 0 and ℓ = 2 corrugation fields (δ_0, δ_2), and ß. After minimization a possible choice of these parameters is $\varkappa = 2$, $\delta_1 = 1$, A = 200, ß = 0.83 : $\delta_o = 0.21$, $\delta_2 = 0.07$, where wall stabilization of the m=1, k=0 mode is effective.

In the case of no ℓ = 2 distortion ($\alpha = 1$), the optimum ß-value is ß = 0.86 and the resulting scaling law is $\delta_{1,n\beta} = 1.71\,\varkappa^3/A^{1/2}$, the number of periods is $n_1 = 0.63\,A/\varkappa^2$.

In the case of $\alpha = \alpha_o$ (ℓ = 0,1,2 corrugation fields), the optimum ß-value is $\beta_o = 0.83$ and the resulting scaling law is $\delta_{1,n\beta} = 1.62\,\varkappa^3/A^{1/2}$ ($\alpha_0 = 0.8$). The corresponding number of periods n is $n_o = 0.66\,A/\varkappa^2$.

In all cases of practical interest (A not to large) it is necessary to use a vacuum vessel which has a helical-toroidal shape.

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Figure Captions

Fig. 1 : a) Function $D_n = [27(4-3\beta)^2 (2-\beta)/64\beta^3 (1-\beta)(3-2\beta)]^{1/4}$ describing the $\ell=1$ ($\alpha = 1$) distortion $\delta_{1,n} = (\varkappa^3/\sqrt{A})\, D_n$.

b) Function $D_{n\alpha} = D_n\, [(3-2\beta)/(5-4\beta)]^{1/4}$ describing the $\ell = 1$ ($\alpha = \alpha_1$) distortion $\delta_{1,n\alpha} = (\varkappa^3/\sqrt{A})\, D_{n\alpha}$.

Fig.2: a) Function $D_o = 3(4-3\beta)(2-\beta)/8\beta(1-\beta)(3-2\beta)$ describing the $\ell = 0$ distortion in the case of $\alpha = 1$: $(\delta_o \delta_1)_n = (\varkappa^4/A)\, D_o$ (dashed curve optimized with respect to n).

b) Function $d_o = D_o(3-2\beta)/(5-4\beta)$ describing the $\ell = 0$ distortion in case of $\alpha = \alpha_1$: $(\delta_o \delta_1)_{n\alpha} = (\varkappa^4/A)\, d_o$ (slight curve; optimized with respect to n, α).

c) Function $D_2 = 3(4-3\beta)/4\beta(5-4\beta)$ describing the $\ell = 2$ distortion in case of $\alpha = \alpha_1$: $(\delta_2 \delta_1)_{n\alpha} = (\varkappa^4/A)\, D_2$ (bold curve; optimized with respect to n, α). In all three cases δ_1 is parameter.

Fig.3: Scaling law for marginal stability of the m=1, k=0 mode (rigid displacement; A aspect ratio with respect to plasma radius; δ_1 optimum $\ell=1$ distortion).

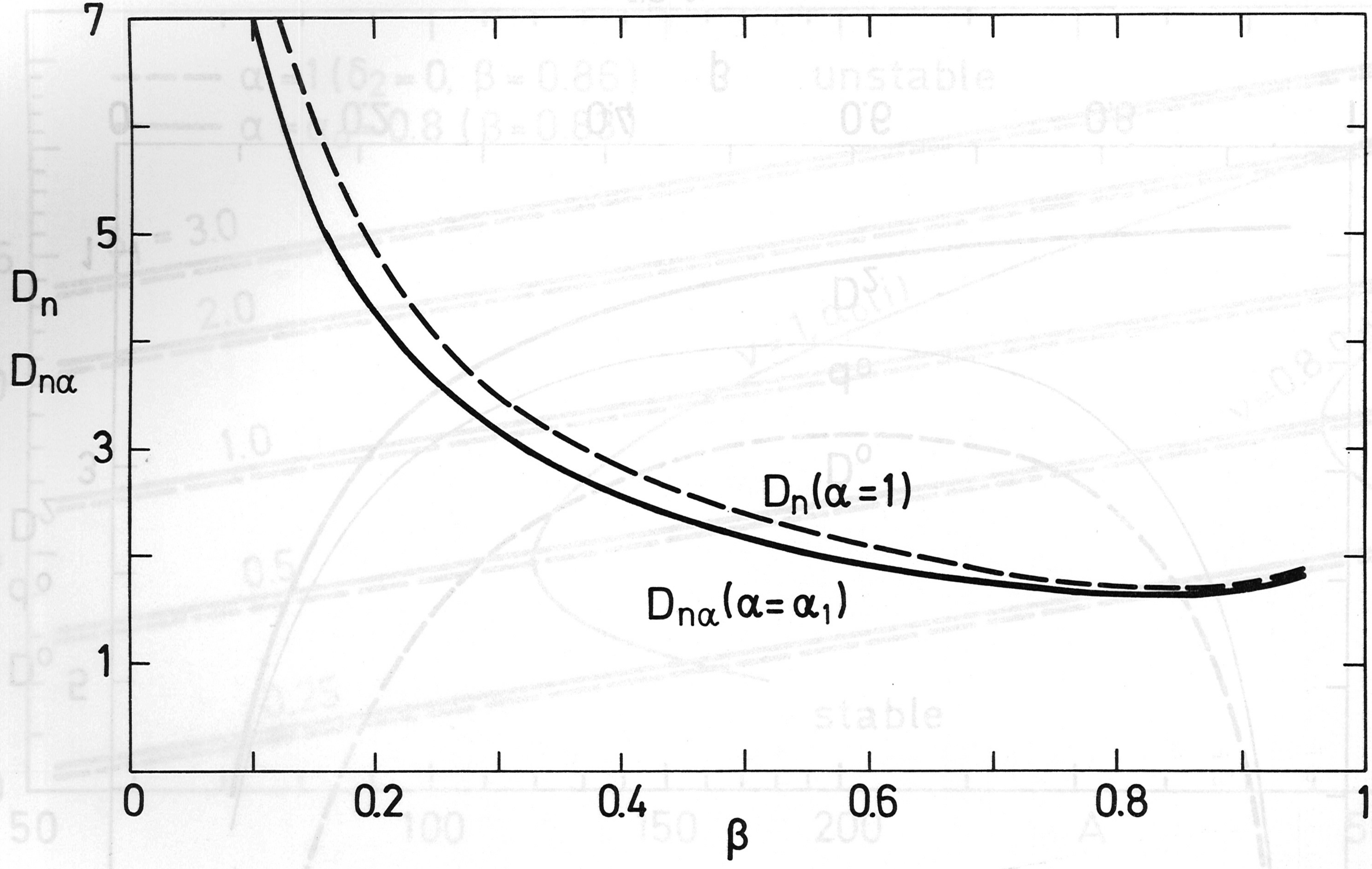

FIG. 1

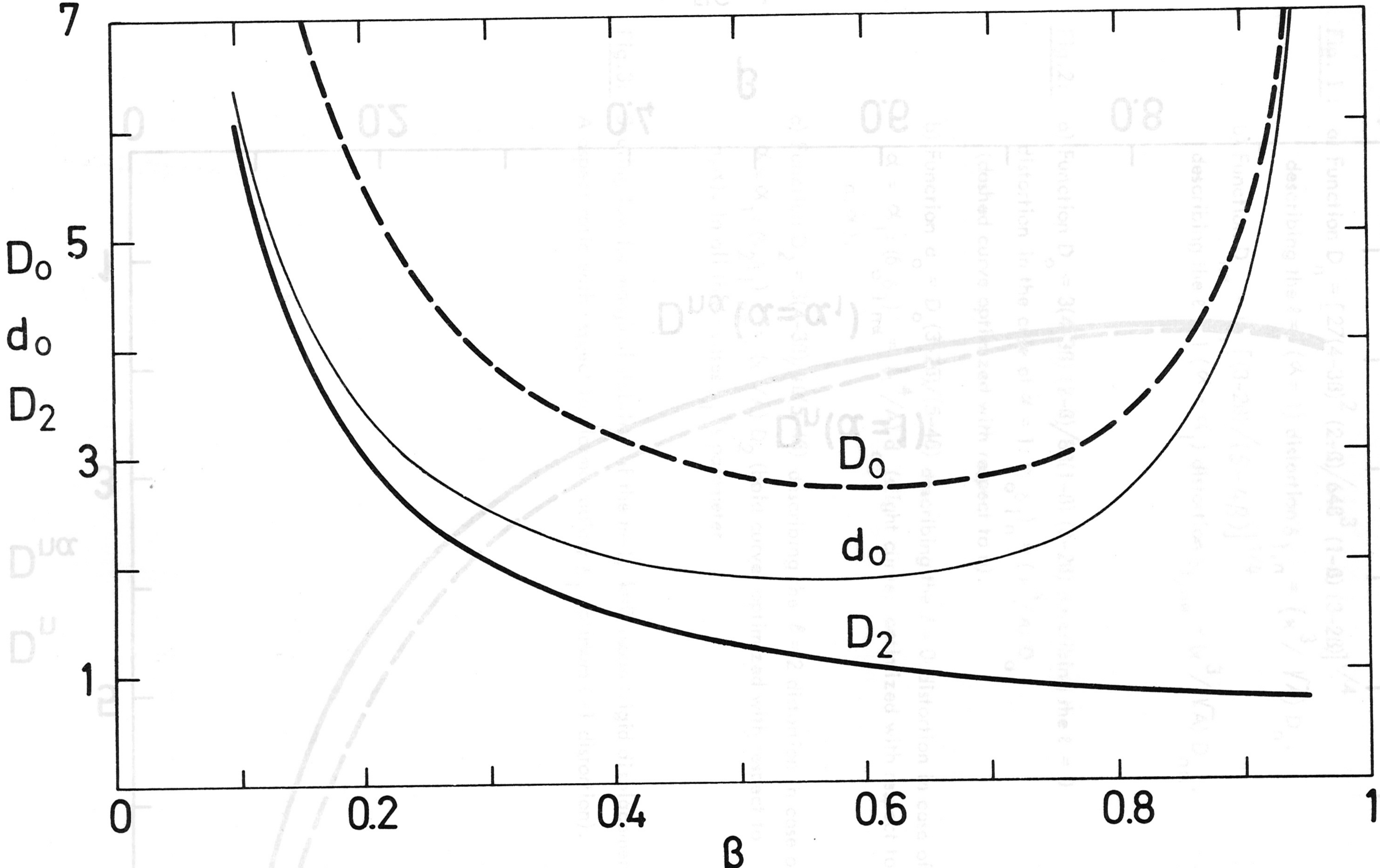

FIG. 2

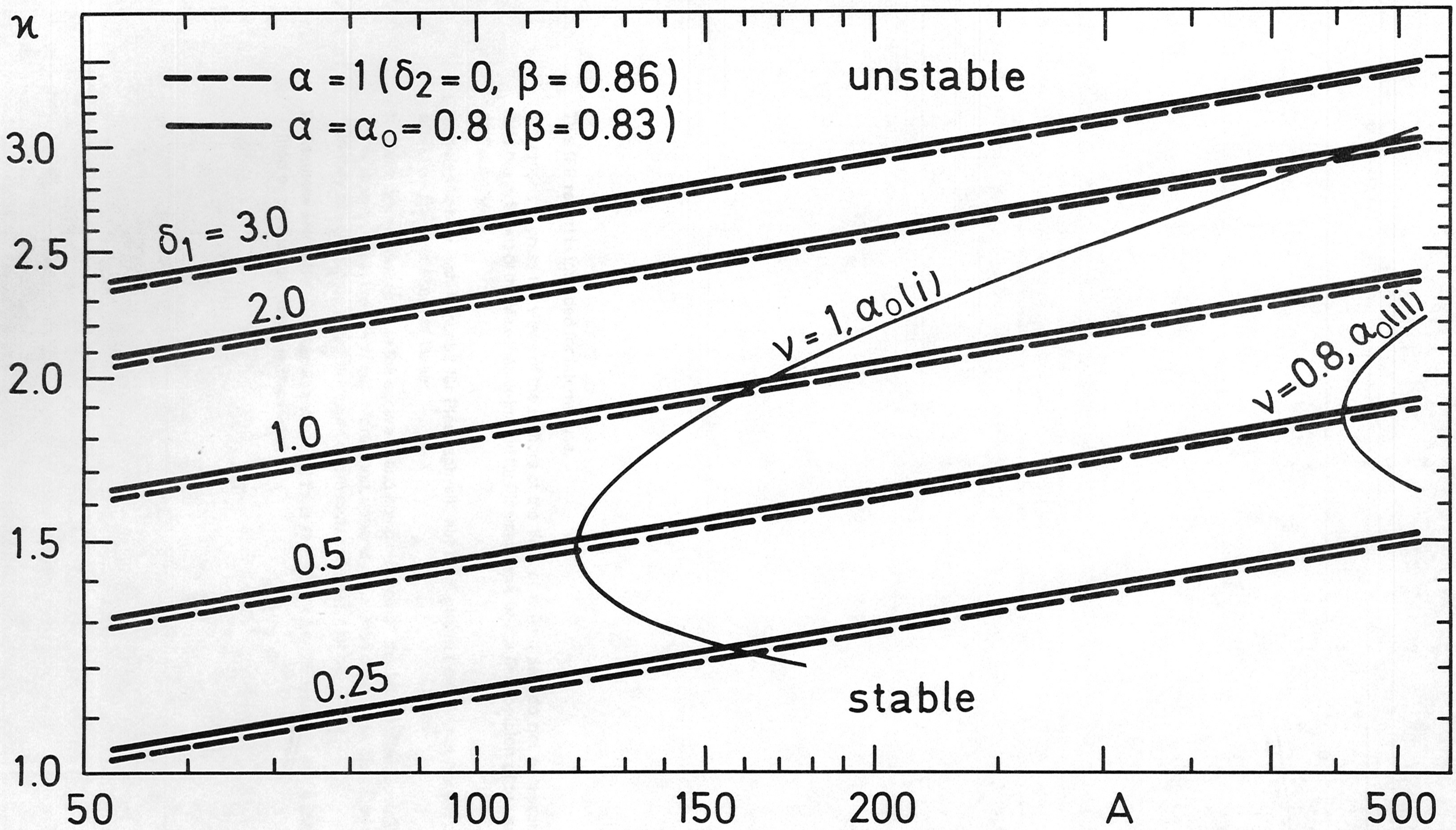
ϰ
α = 1 (δ2 = 0, β = 0.86)
α = α0 = 0.8 (β = 0.83)
unstable
stable
δ1 = 3.0
2.0
1.0
0.5
0.25
ν = 1, α0(i)
ν = 0.8, α0(ii)
3.0
2.5
2.0
1.5
1.0
50
100
150
200
A
500

FIG. 3