THE RESISTIVE PLASMA LOADING OF A STIX COIL

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Abstract

The ratio Q of the inductive reactance of a periodic induction coil to the corresponding resistive plasma loading was evaluated by approximating the current in the coil by a stepped sheet current distribution.

The coil is assumed to be around a homogeneous cylindrical cold plasma with a constant magnetic field along the axis. The turns (or sets of turns) in the coil are spaced periodically along the axis and the current passing through two adjacent turns is equal but opposite in phase. For the case of coupling resonance, that is when the wavelength of the induction coil is equal to the wavelength of one of the modes of free oscillation for ion cyclotron waves in a cylindrical plasma, the ratio Q was plotted for turns connected both in series and in parallel as a function of the distance between two successive turns, the number of turns and the coil diameter being varied and the plasma diameter being assumed to be unity.

T.H. Stix [1] proposed generating ion cyclotron waves in a cylindrical plasma by a periodic induction coil. The coil is wrapped around the plasma in a region where the magnetic field Bo is constant and $\omega < \omega_{ci}$, ω being the frequency of the generating system and ω_{ci} the ion cyclotron frequency, so that ion cyclotron waves can propagate in the plasma along Bo out through the ends of the induction coil.

In a cylindrical plasma surrounded by vacuum there exists a set of natural modes of free oscillation for ion cyclotron waves.

To each of these modes corresponds a radial wave number ν_m and an axial wave number k_m . If the current in the coil varies periodically along the axis, one can expect a maximum of coupling provided the axial wavelength of the induction coil λ_0 is equal to the axial wavelength λ_m of a natural mode of free oscillation in the plasma.

The turns (or sets of turns) in the coil will, therefore, be spaced periodically along the axis, and the current passing in two adjacent turns will be equal but opposite in phase so that the distance 2ℓ between two successive turns will be equal to half the wavelength λ_m of one of the natural modes.

A very important question at this point is the change in the resistive impedance of the induction coil due to the generation of ion cyclotron waves in the plasma.

An expression for the ratio Q of the inductive reactance of the coil to the corresponding resistive impedance due to the plasma load has already been given by Stix. He assumes, however, that the current in the coil varies as exp i$[k_0 z - \omega t]$ and obtains an expression for Q which is not very practical for use and gives little or no information about the dependence of Q on some of the parameters involved in a real coil, such as the number and width of the turns.

In the following we shall find a more accurate expression for Q by approximating the current in the coil by a stepped sheet-current distribution which will better approximate the real current in the coil and allow us to find the Q of each turn separately. We thus obtain as well the Q of the entire

(1) T.H. Stix The Theory of Plasma Waves McGraw-Hill

coil with the turns connected both in series and in parallel, and a detailed dependence of Q on the width and number of the turns in the coil and also on the distance between two successive turns.

For an even number N = 2 M of turns we assume at a radius r=s , a sheet-current:

$$
(1)\qquad
\begin{aligned}
J_\theta^{\#} &= (-)^n J^{\#} e^{-i\omega t} && -(2n-1)\ell-\delta < z < -(2n-1)\ell+\delta \\
J_\theta^{\#} &= -(-)^n J^{\#} e^{-i\omega t} && (2n-1)\ell-\delta < z < (2n-1)\ell+\delta \\
J_\theta^{\#} &= 0 \quad \text{otherwise} && \\
& && n = 1, 2, \ldots M
\end{aligned}
$$

and for an odd number, N=2M + 1, of turns a sheet-current:

$$
(2)\qquad
\begin{aligned}
J_\theta^{\#} &= (-)^n J^{\#} e^{-i\omega t} && 2n\ell-\delta < z < 2n\ell+\delta \\
J_\theta^{\#} &= 0 \quad \text{otherwise} && \\
& && n = 0, \pm 1, \pm 2, \ldots, \pm M
\end{aligned}
$$

2δ being the width of each turn and 2ℓ the distance between two successive turns.

A solution of the Maxwell equations for a cylinder of cold plasma surrounded by vacuum with an uniform magnetic field parallel to the axis and an external sheet current of density $J_\theta^{\#} = J^{\#} \exp i\,(kz - \omega t)$ on a radius r=s is:

$$
(3)\qquad
\begin{aligned}
E_\theta^{p} &= -\frac{4\pi i\omega}{c^2} J^{\#} \frac{s}{\rho} \frac{K_1(\bar{k}s)\, J_1(\nu r)}{\bar{k} J_1(\nu\rho) K_1'(\bar{k}\rho) - \nu J_1'(\nu\rho) K_1(\bar{k}\rho)} && 0 \le r \le \rho \\
E_\theta^{v} &= \frac{4\pi i\omega}{c^2} J^{\#} s\, I_1(\bar{k}r) K_1(\bar{k}s) + E_\theta^{o} && \rho \le r \le s \\
E_\theta^{w} &= \frac{4\pi i\omega}{c^2} J^{\#} s\, I_1(\bar{k}s) K_1(\bar{k}r) + E_\theta^{o} && r \ge s
\end{aligned}
$$

$$
(4)\qquad E_\theta^{o} = \frac{4\pi i\omega}{c^2} J^{\#} s\, K_1(\bar{k}s) K_1(\bar{k}r) \frac{\nu I_1(\bar{k}\rho) J_1'(\nu\rho) - \bar{k} I_1(\bar{k}\rho) J_1(\nu\rho)}{\bar{k} J_1(\nu\rho) K_1'(\bar{k}\rho) - \nu J_1(\nu\rho) K_1(\bar{k}\rho)}
$$

With Bessel functions in the notation of G.N. Watson and the prime (') indicating differentiation with respect to the argument.

- p is the plasma radius
- ν is the radial wave number of the wave in the plasma and is given by the plasma dispersion relation

$$\bar{k} = (k^2 - \omega^2/c^2)^{1/2} \qquad -\pi/2 < \arg k < \pi/2$$

$$= (k^2 e^{-2\pi i} - \omega^2/c^2)^{1/2} \qquad \pi/2 < \arg k < 3\pi/2$$

Zero plasma temperature and infinite conductivity were assumed to find the boundary conditions:

$$E_\theta^p = E_\theta^v \; ; \quad \frac{\partial E_\theta^p}{\partial r} = \frac{\partial E_\theta^v}{\partial r} \qquad r = p$$

$$E_\theta^v = E_\theta^w \; , \quad \frac{\partial E_\theta^v}{\partial r} = \frac{\partial E_\theta^w}{\partial r} + \frac{4\pi i \omega J^\#}{c^2} \qquad r = s$$

$$E_\theta^w = 0 \qquad r = \infty$$

The solution for the natural modes of free oscillation in a plasma cylinder surrounded by vacuum corresponds to solutions with finite E_θ^p for $J^\# = 0$

They are given by the condition:

$$\nu p \frac{J_1'(\nu p)}{J_1(\nu p)} = \bar{k} p \frac{K_1'(\bar{k} p)}{K_1(\bar{k} p)}$$

The Fourier transform of the sheet-current distribution (1) is:

$$(6) \qquad J_k^\# = - \frac{J^\#}{2\pi i k} \sum_1^N (-)^n \left[e^{-ik(2n-1)\ell} - e^{ik(2n-1)\ell} \right] \left(e^{iks} - e^{-iks} \right) e^{-i\omega t}$$

and, as we know the electric field for each component of the Fourier spectrum, we can obtain by combining (3) and (6) the electric field on a radius r=s produced by the sheet-current distribution (1):

$$(7) \quad E_\theta^s = \frac{e^{-i\omega t}}{2\pi i} \sum_1^N (-)^n \int_{-\infty}^{\infty} (E_\theta)_{r=s} \left[e^{-ik(2n-1)\ell} - e^{ik(2n-1)\ell} \right] \left(e^{ik\delta} - e^{-ik\delta} \right) \frac{e^{ikz}}{k}\, dk$$

The integrals on the right side of (7) are of the type:

$$(8) \quad \int_{-\infty}^{\infty} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k}\, dk$$

where λ is real and may be positive or negative.

We shall evaluate this integral by a contour integral in the complex k plane.

$(E_\theta)_{r=s}$ has poles when the denominator in (4) is zero and a branch-cut at $|k| = \omega/c$ and along the imaginary axis for $|k| \geq \omega/c$. The contribution to the electric field due to the poles k_m on the real axis will appear in our solution as unattenuated out-going waves propagating as exp i $(k_m z - \omega t)$. It is, therefore, reasonable in the presence of a small amount of damping to displace these poles off the real k axis by a small amount. For positive k_m the poles will be just above the axis and for negative k_m just below.

In the limit $\omega/c \to 0$, which is equivalent to neglecting terms of the order $\omega s/c \ll 1$, the branch-cut of $(E_\theta)_{r=s}$ reduces to the imaginary axis and an infinitely small circle around the origin. We choose our contour integral as sketched in Fig. (a) for $\lambda > 0$ and Fig. (b) for $\lambda < 0$:

Fig (a)

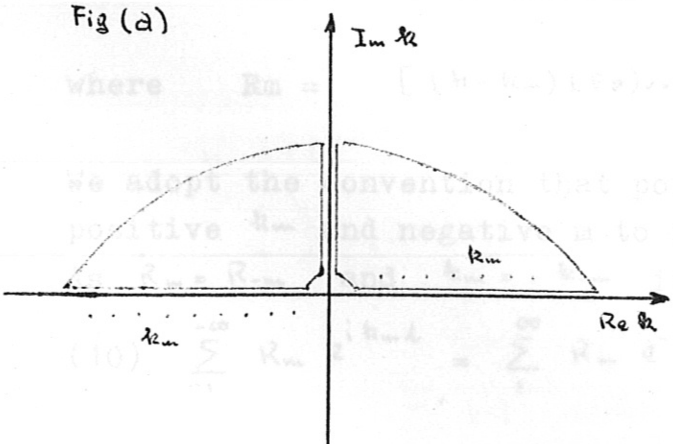
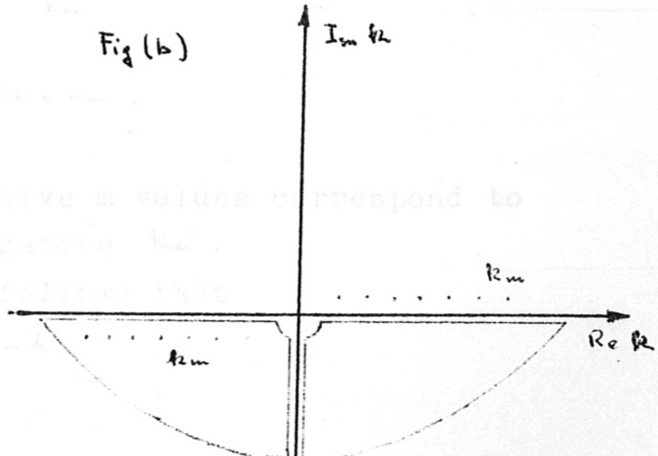


Fig (b)



$(E_\theta)_{r.s}$ is now, in each quadrant, analytic except at its poles and we have closed the coutour by an arc at infinity so that the exponential term in the integrand, $ik\lambda$, has a negative real part. In the limiting case $\omega/c \to 0$ it may be shown that, for a cold plasma, the contribution to the integral (8) made by the branch-cut along the imaginary axis and the circle around the origin gives a contribution to the electric field which is in phase with $iJ^{\#}$.
The resistive impedance of a turn is proportional to the real part of $-E_\theta \cdot (J^{\#})^*$ [$(J^{\#})^*$ is the complex conjugate of $J^{\#}$ and varies as exp $(i\omega t)$] averaged over the turn, while the inductive reactance is proportional to the imaginary part of $E_\theta \cdot (J^{\#})^*$. As we are interested in the resistive part of the impedance we need not evaluate the integral along the branch-cut, which contributes only to the imaginary part of the reactance and we are, therefore, left with the contribution due to the poles only.
According to Stix we shall neglect the residues for poles other than those just off the real k axis. These would give solutions which decay rapidly compared to the propagating wave solutions and correspond to end effects of the coil, which are of no interest to us.
The portion I of the integral (8) which we need is, therefore, given by:

$$(9)\qquad \begin{aligned} I &= 2\pi i \sum_{1}^{\infty} R_m \frac{e^{ik_m\lambda}}{k_m} & \lambda > 0 \\ &= -2\pi i \sum_{-1}^{-\infty} R_m \frac{e^{ik_m\lambda}}{k_m} & \lambda < 0 \end{aligned}$$

where $R_m = \left[(k-k_m)(E_\theta)_{r=s} \right]_{k=k_m}$.

We adopt the convention that positive m values correspond to positive k_m and negative m to negative k_m.
As $R_m = R_{-m}$ and $k_m = -k_{-m}$ it follows that

$$(10)\qquad \sum_{-1}^{-\infty} R_m e^{ik_m\lambda} = \sum_{1}^{\infty} R_m e^{-ik_m\lambda} .$$

As we are interested in the product $E_\theta \cdot (J^{\#})^*$, we must evaluate E_θ^s only in the regions where $J^{\#} = 0$ that is:

for $-(2n-1)\ell-\delta < z < -(2n-1)\ell+\delta$

and $(2n-1)\ell-\delta < z < (2n-1)\ell+\delta$

For $n = \bar{n}$ and $-(2n-1)\ell-\delta < z < -(2n-1)\ell+\delta$

setting $z = z' - (2n-1)\ell$, $|z'| < \delta$

we have

$$(11)\quad E_\theta^s = \frac{1}{2\pi i}\sum_1^N (-)^n \int_{-\infty}^{\infty} (E_\theta)_{r=s}\left[e^{-ik2(n+\bar{n}-1)\ell} - e^{ik2(n-\bar{n})\ell}\right] \times$$

$$\times\left(e^{ik\delta} - e^{-ik\delta}\right)\frac{e^{ikz'-i\omega t}}{k}\, dk =$$

$$= -\sum_1^\infty \frac{R_m}{k_m}\Big\{\sum_1^N (-)^n e^{ik_m 2(n+\bar{n}-1)\ell}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)e^{ik_m z'}$$

$$-\sum_1^{\bar{n}-1}(-)^n e^{-ik_m 2(n-\bar{n})\ell}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)e^{-ik_m z'} -$$

$$-(-)^{\bar{n}} e^{ik_m\delta}\left(e^{ik_m z'} - e^{-ik_m z'}\right) - \sum_{\bar{n}+1}^N e^{ik_m 2(n-\bar{n})\ell}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)e^{ik_m z'}\Big\}e^{-i\omega t}$$

and

$$(12)\quad \langle E_\theta^s\cdot(J^*)^*\rangle_{AV} = \frac{1}{2\delta}\int_{-\delta}^{\delta} E_\theta^s\cdot(J^*)^*\, dz' =$$

$$= \frac{(J^*)^*}{2\delta}\sum_1^\infty \frac{R_m}{i k_m^2}\Big\{-2\, e^{ik_m\delta}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right) +$$

$$+ (-)^{\bar{n}}\sum_1^N (-)^n e^{ik_m 2(n+\bar{n}-1)}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)^2 -$$

$$-(-)^{\bar{n}}\sum_1^{\bar{n}-1}(-)^n e^{ik_m 2(n-\bar{n})}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)^2 -$$

$$-(-)^{\bar{n}}\sum_{\bar{n}+1}^N (-)^n e^{ik_m 2(n-\bar{n})}\left(e^{ik_m\delta} - e^{-ik_m\delta}\right)^2\Big\} =$$

$$= 2\delta(J^*)^*\sum_1^\infty R_m\frac{\sin^2 k_m\delta}{(k_m\delta)^2}\Big\{\Big[\cot g\, k_m\delta - \sum_1^N{}_n (-)^{n+\bar{n}}\sin 2(n+\bar{n}-1)k_m\ell$$

$$+\sum_1^{\bar{n}-1}{}_n(-)^{n+\bar{n}}\sin 2(n-\bar{n})k_m\ell - \sum_{\bar{n}+1}^N{}_n \sin 2(n-\bar{n})k_m\ell\Big]$$

$$+i\Big[1 - \sum_1^N{}_n(-)^{n+\bar{n}}\cos 2(n+\bar{n}-1)k_m\ell +$$

$$+\sum_1^{\bar{n}-1}{}_n(-)^{n+\bar{n}}\cos 2(n-\bar{n})k_m\ell + \sum_{\bar{n}+1}^N{}_n(-)^{n+\bar{n}}\cos 2(n-\bar{n})k_m\ell\Big]\Big\}$$

Because all Rm are exactly 90^0 out of phase with J^* we obtain:

$$\mathrm{Re}\, \langle -E_\theta^s \cdot (J^*)^* \rangle_{Av} = -2i\,2S\,(J^*)^* \sum_1^\infty R_m \frac{\sin^2 k_m\delta}{(k_m\delta)^2} \sum_1^M {}_n (-)^{n+\bar n} \sin(2\bar n-1)k_m\ell \,\sin(2n-1)k_m\ell$$

and the same final result would be obtained for

$$(2n-1)\ell - \delta < z < (2n-1)\ell + \delta$$

The resistive impedance of the $\bar n^{th}$ turn for a single mode m is therefore proportional to:

$$(13) \quad R_{\bar n, m} = -4i\delta\,(J^*)^* \left[(k-k_m)(E_\theta)_{r=s}\right]_{k=k_m} \frac{\sin^2 k_m\delta}{(k_m\delta)^2} \sum_1^M {}_n (-)^{n+\bar n} \sin(2\bar n-1)k_m\ell \,\sin(2n-1)k_m\ell$$

In the case of a coil with an odd number, N=2M+1, of turns we can repeat the same kind of calculations as for an even number. Starting from the sheet-current distribution (2), we obtain for the resistive impedance of the $\bar n^{th}$ turn for a single mode m:

$$(14) \quad R_{\bar n, m} = 2i\delta\,(J^*) \left[(k-k_m)(E_\theta)_{r=s}\right]_{k=k_m} \frac{\sin^2 k_m\delta}{(k_m\delta)^2} \sum_{-M}^M {}_n (-)^{n+\bar n} \cos 2\bar n k_m\ell \,\cos 2n k_m\ell$$

To put this resistive plasma loading into a dimensionless and more usable form, we compare it to the inductive reactance of the induction coil.

The inductive reactance of the $\bar n^{th}$ turn is proportional to $\mathcal{L}_{\bar n}$, which is given in the Appendix by A(8).

For $\delta \ll s$ and $\ell \gtrsim s$, however, A(8) may be simplified to give:

$$\mathcal{L}_{\bar n} = \frac{4\omega S\,(J^*)(J^*)^*}{c^2} \left[\ln 4s/\delta - 1/2\right]$$

regardless of the number of turns in the coil, see A(13).

We, therefore, obtain for the $\bar n^{th}$ turn and for the mode m:

$$(15) \quad Q_{m,\bar n} = \frac{\mathcal{L}_{\bar n}}{R_{m,n}} = \frac{2\omega\,(J^*)\left[\ln 4s/\delta - 1\right]}{-i\,N c^2 \left[(k-k_m)(E_\theta)_{r=s}\right]_{k=k_m}} \frac{(k_m\delta)^2}{\sin^2(k_m\delta)} \frac{1}{S_{\bar n}(k_m\ell)}$$

$$(16) \quad \text{where} \quad S_{\bar n}(k_m\ell) = \sum_1^M {}_n (-)^{n+\bar n} \frac{\sin(2\bar n-1)k_m\ell \,\sin(2n-1)k_m\ell}{M} \qquad \text{for } N=2M$$

$$= \sum_{-M}^M {}_n (-)^{n+\bar n} \frac{\cos 2\bar n k_m\ell \,\cos 2n k_m\ell}{2M+1} \qquad \text{for } N=2M+1$$

and $S_{\bar{n}}(k_m \ell) = 1$ for $k_m \ell = \pi/2$, that is $2\ell = \lambda_m/2$

If the turns are connected in series, the Q of the entire coil is:

$$(17)\quad Q_m^{s} = \frac{\Sigma_{\bar{n}} \mathcal{L}_{\bar{n}}}{\Sigma_{\bar{n}} R_{m,\bar{n}}} = \frac{2\omega (J^{\#})^{*} \left[\ln 4s/\delta - 1/2 \right] (k_m \delta)^2}{-iNc^2 \left[(k-k_m)(E_\theta)_{r=s} \right] \sin^2(k_m \delta) \cdot S(k_m \ell)}$$

and if the turns are connected in parallel

$$(18)\quad Q_m^{p} = \left(\Sigma_{\bar{n}} \frac{1}{Q_{m,\bar{n}}} \right)^{-1} = \frac{Q_m^{s}}{N}$$

$$(19)\quad \text{where}\quad S(k_m \ell) = \left[\sum_{1}^{M} (-)^n \frac{\sin (2n-1) k_m \ell}{M} \right]^2 \qquad N = 2M$$

$$= \left[\sum_{-M}^{M} (-)^n \frac{\cos 2n k_m \ell}{2M+1} \right]^2 \qquad N = 2M+1$$

and again $S(k_m \ell) = 1$ for $k_m \ell = \pi/2$

Fig(1) gives the plot of $S(km\ell)$ for N = 2,3 and 4

The residue of $(E_\theta)_{r=s}$ at k=km may be found from (3), and with the help of (5) we obtain:

$$(20)\quad \left[(k-k_m)(E_\theta)_{r=s} \right]_{k=k_m} = -\frac{4\pi i \omega}{c^2} J^{\#} \frac{s}{\rho} \frac{K_1^2(k_m s)}{\left\{ k\rho \left(1 + \frac{\partial \nu^2}{\partial k^2}\right) K_1^2(k\rho) - \left(1 - \frac{k^2}{\nu^2}\frac{\partial \nu^2}{\partial k^2}\right) \left[k\rho K_0^2(k\rho) - 2K_0(k\rho)K_1(k\rho) \right] \right\}_{k=k_m}}$$

where k_m and ν_m must satisfy (5) and the dispersion relation for a cold plasma.
For ion cyclotron waves ($\omega \approx \omega_{ci}$, $\frac{k^2c^2}{\Pi_i^2} >> 1$) the cold plasma dispersion relation reduces to:

$$\nu^2 \simeq -k^2 \left[1 + \frac{1}{1 - \frac{\omega_{ci}^2 - \omega^2}{\omega^2} \frac{k^2 c^2}{\Pi_i^2}} \right]$$

and $$\frac{\partial \nu^2}{\partial k^2} = -1 - \left(1 + \frac{\nu^2}{k^2}\right)^2$$

where ω_{ci} is the ion cyclotron frequency
and Π_i is the ion plasma frequency

We have, therefore, for ion cyclotron waves:

(21) $$\left[(k-k_m)(E_\theta)_{r=s}\right]_{k=k_m} =$$

$$= \frac{4\pi i\omega}{c^2} J^* \frac{s}{\rho} \frac{K_1(k_m s)}{\left(1+\frac{\nu_m^2}{k_m^2}\right)^2 k_m\rho\, K_1^2(k_m\rho) + \left(3 + 2\frac{k_m^2}{\nu_m^2} + \frac{\nu_m^2}{k_m^2}\right)\left[k_m\rho\, K_0^2(k_m\rho) + 2K_0(k_m\rho)K_1(k_m\rho)\right]}$$

and the resistive plasma loading of the $\bar{n}^{th}$ turn for a single mode m will be proportional to:

(22) $$R_{m,\bar{n}} = \frac{4\pi\omega}{c^2} J^\# (J^\#)^* \frac{2s\delta}{\rho} \frac{\sin^2 k_m\delta}{(k_m\delta)^2} \times$$

$$\times \frac{N(\nu_m\rho)^2(k_m\rho)^3 K_1^2(k_m s)\, S_{\bar{n}}(k_m\ell)}{(\nu_m\rho)^2\left[(\nu_m\rho)^2+(k_m\rho)^2\right]^2 K_1^2(k_m\rho) + (k_m\rho)\left[2(k_m\rho)^4+3(k_m\rho)^2(\nu_m\rho)^2+(\nu_m\rho)^4\right]\left[(k_m\rho)K_0^2(k_m\rho)+2K_0(k_m\rho)K_1(k_m\rho)\right]}$$

At the coupling resonance, $S_{\bar{n}}(k_m\ell) = 1$ and the resistive plasma loading is the same for each turn.

Using the more correct expression A(11) for the inductive reactance of the $\bar{n}^{th}$ turn we obtain for the Q of the entire coil at the coupling resonance

(23) $$Q^s_{m(res)} = \frac{s}{\rho} \frac{\bar{\nu}_m^2(\bar{k}_m^2+\bar{\nu}_m^2)^2 K_1^2(\bar{k}_m) + \bar{k}_m\left[2\bar{k}_m^4 + 3\bar{\nu}_m^2\bar{k}_m^2 + \bar{\nu}_m^4\right]\left[\bar{k}_m K_0^2(\bar{k}_m) + 2K_0(\bar{k}_m)K_1(\bar{k}_m)\right]}{3\pi \sin^2(\pi\beta/2)\,\bar{\nu}_m^2\,\bar{k}_m\, K_1^2(k_m s/\rho)} \cdot \frac{N\mathcal{L}^* + 2(N-1)\mathcal{M}_1^*}{N^2}$$

if the turns are connected in series and:

(24) $$Q^p_{m(res)} = \frac{s}{\rho} \frac{\bar{\nu}_m^2(\bar{k}_m^2+\bar{\nu}_m^2)K_1^2(\bar{k}_m) + \bar{k}_m\left[2\bar{k}_m^4 + 3\bar{\nu}_m^2\bar{k}_m^2 + \bar{\nu}_m^4\right]\left[\bar{k}_m K_0^2(\bar{k}_m) + 2K_0(\bar{k}_m)K_1(\bar{k}_m)\right]}{3\pi \sin^2(\pi\beta/2)\,\bar{\nu}_m^2\,\bar{k}_m\, K_1(k_m s/\rho)} \cdot \frac{(\mathcal{L}^*+\mathcal{M}_1^*)(\mathcal{L}^*+2\mathcal{M}_1^*)}{N\left[N\mathcal{L}^* + (N+2)\mathcal{M}_1^*\right]}$$

if the turns are connected in parallel

$\mathcal{L}^*$ and $\mathcal{M}_1^*$ are given in the Appendix by A(9) and A(10) respectively and we have set, for simplicity, $\bar{k}_m = k_m \rho$; $\bar{\nu}_m = \nu_m \rho$ and $\rho = \delta/\ell$

Fig.(2) to Fig.(7) give some plots of $Q^p_{m(x_1)}$ and $Q^s_{m(x_1)}$ against $a/\rho = \frac{\pi}{2}\frac{1}{\bar{k}_m \rho}$ for some values of N, $^s/p$, m.

Acknowledgements

I wish to thank Dr. R. Croci and Dr. W.M. Hooke for valuable discussions.

Appendix

We shall evaluate, in the following, the inductive reactance of each turn of the coil.
The inductive reactance of the $\bar{n}^{th}$ turn is proportional to the imaginary part of $E_\theta^s \cdot (J^\#)^*$ averaged over the turn.
As in the preceding section, we must evaluate integrals of the type:

where $(E_\theta)_{r=s}$ is now the vacuum field and is given by:

$$A(1) \qquad (E_\theta)_{r=s} = \frac{4\pi i\omega}{c^2} J^\# s\, I_1(\bar{k}s)\, K_1(\bar{k}s)$$

$(E_\theta)_{r=s}$ has now no poles and again a branch-cut for $|k| = \omega/c$ and along the imaginary axis for $|k| \geq \omega/c$ which in the limit $\omega/c \to 0$ reduces to the imaginary axis and an infinitely small circle around the origin.
We shall evaluate the integral as in the preceding section and we choose the same paths of integration as in Fig.(a) and Fig.(b) for $\lambda > 0$ and $\lambda < 0$, respectively. As the integrand has now no poles, we are left with the contribution due to the branch-cut only.
We have, therefore, for $\lambda > 0$:

$$A(2) \qquad \int_{-\infty}^{\infty} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk = - \lim_{\rho\to 0} i \int_{\pi}^{\pi/2} \left[(E_\theta)_{r=s}\right]_{k=\rho e^{i\theta}} d\theta -$$

$$- \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{-\varepsilon+i\infty} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk - \lim_{\varepsilon\to 0} \int_{\varepsilon+i\infty}^{\varepsilon} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk$$

$$- \lim_{\rho\to 0} i \int_{\pi/2}^{0} \left[(E_\theta)_{r=s}\right]_{k=\rho e^{i\theta}} d\theta =$$

$$= \frac{4\pi i\omega}{c^2} J^\# s \left\{ \frac{i\pi}{2} + \int_0^\infty \left[I_1(k's e^{i\pi/2}) K_1(k's e^{i\pi/2}) - \right.\right.$$

$$\left.\left. - I_1(k's e^{-i\pi/2}) K_1(k's e^{-i\pi/2}) \right] \frac{e^{-k'\lambda}}{k'} dk' \right\} =$$

$$= \frac{4\pi i\omega}{c^2} J^\# s \left\{ \frac{i\pi}{2} - i\pi \int_0^\infty J_1^2(ks) \frac{e^{-k'\lambda}}{k'} dk' \right\}$$

and for $\lambda < 0$:

A(3) $$\int_{-\infty}^{\infty} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk = - \lim_{\rho\to 0} i \int_{\pi}^{3\pi/2} \left[(E_\theta)_{r=s} \right]_{k=\rho e^{i\theta}} d\theta -$$

$$- \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{-\varepsilon - i\infty} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk - \lim_{\varepsilon\to 0} \int_{\varepsilon - i\infty}^{\varepsilon} (E_\theta)_{r=s} \frac{e^{ik\lambda}}{k} dk$$

$$- \lim_{\rho\to 0} i \int_{-\pi/2}^{0} \left[(E_\theta)_{r=s} \right]_{k=\rho e^{i\theta}} d\theta =$$

$$= \frac{4\pi i\omega}{c^2} J^{\#} s \left\{ - \frac{i\pi}{2} - \int_0^\infty \left[I_1(k's e^{i\pi/2}) K_1(k's e^{i\pi/2}) - I_1(k's e^{-i\pi/2}) K_1(k's e^{-i\pi/2}) \right] \frac{e^{k'\lambda}}{k'} dk' \right\}$$

$$= \frac{4\pi i\omega}{c^2} J^{\#} s \left\{ - \frac{i\pi}{2} - i\pi \int_0^\infty J_1^2(ks) \frac{e^{k'\lambda}}{k'} dk' \right\}$$

The electric field produced on a radius r=s by the sheet current distribution (1) is given, as in the preceding section, by:

$$E_\theta^s = - \frac{1}{2\pi i} \sum_1^N (-)^n \int_{-\infty}^{\infty} (E_\theta)_{r=s} \left[e^{-i(2n-1)k\ell} - e^{i(2n-1)k\ell} \right] \left(e^{ik\delta} - e^{-ik\delta} \right) \frac{e^{ikz - i\omega t}}{k} dk$$

and again we must evaluate E_θ^s in the regions where $J^{\#} \neq 0$ that is for:

$$-(2n-1)\ell - \delta < z < -(2n-1)\ell + \delta$$

$$(2n-1)\ell - \delta < z < (2n-1)\ell + \delta$$

For $n=\bar{n}$ and $-(2\bar{n}-1)\ell < z < -(2\bar{n}-1)\ell + \delta$ setting $z = z' - (2\bar{n}-1)\ell$ $|z'| < \delta$ we obtain:

A(4) $$E_\theta^s = - \frac{e^{-i\omega t}}{2\pi i} \sum_1^N {}_n (-)^n \int_{-\infty}^{\infty} (E_\theta)_{r=s} \left[e^{-i2(n+\bar{n}-1)k\ell} - e^{i2(n-\bar{n})k\ell} \right] \left(e^{ik\delta} - e^{-ik\delta} \right) \frac{e^{ikz'}}{k} dk =$$

$$= \frac{e^{-i\omega t}}{2} \frac{4\pi i\omega J^{\#} s}{c^2} \left\{ (-)^{\bar{n}} - (-)^{\bar{n}} \int_0^\infty \frac{J_1^2(ks)}{k} e^{-k\delta} \left(e^{kz'} - e^{-kz'} \right) dk - \right.$$

$$- \sum_1^N {}_n (-)^n \int_0^\infty \frac{J_1^2(ks)}{k} e^{-2(n+\bar{n}-1)k\ell} \left(e^{k\delta} - e^{-k\delta} \right) e^{kz'} dk$$

$$+ \sum_1^{\bar{n}-1} {}_n (-)^n \int_0^\infty \frac{J_1^2(ks)}{k} e^{2(n-\bar{n})k\ell} \left(e^{k\delta} - e^{-k\delta} \right) e^{kz'} dk$$

$$+ \sum_{\bar{n}+1}^N {}_n (-)^n \int_0^\infty \frac{J_1^2(ks)}{k} e^{-2(n-\bar{n})k\ell} \left(e^{k\delta} - e^{-k\delta} \right) e^{kz'} dk$$

and

A(5) $$\langle E_\vartheta^S \cdot (J^\#)^* \rangle_{AV} = \frac{1}{2\delta}\int_{-\delta}^{\delta} E_\vartheta^S \, (J^\#)^* \, dz' =$$

$$= \frac{4\pi i\omega}{c^2} J^\# (J^\#)^* \frac{s}{2\delta} \Big\{ \delta + \int_0^\infty \frac{J_1^2(ks)}{k^2} (e^{-2k\delta} - 1)\, dk -$$

$$- \frac{1}{2}\sum_{1}^{M}{}_n (-)^{n+\bar n} \int_0^\infty \frac{J_1^2(ks)}{k^2} e^{-2(n+\bar n -1)k\ell} (e^{k\delta} - e^{-k\delta})^2 dk$$

$$+ \frac{1}{2}\sum_{1}^{\bar n -1}{}_n (-)^{n+\bar n} \int_0^\infty \frac{J_1^2(ks)}{k^2} e^{2(n-\bar n)k\ell} (e^{k\delta} - e^{-k\delta})^2 dk$$

$$+ \frac{1}{2}\sum_{\bar n -1}^{M}{}_n (-)^{n+\bar n} \int_0^\infty \frac{J_1^2(ks)}{k^2} e^{-2(n-\bar n)k\ell} (e^{k\delta} - e^{-k\delta})^2 dk \Big\}$$

The same final result would be obtained for $(2\bar n-1)\ell-\delta < z < (2\bar n -1)\ell+\delta$

From "Tables of Integral Transforms" Vol.I, page 47, we have:

A(6) $$\int_0^\infty \frac{J_1^2(ks)}{k^2}\, dk = \frac{4s}{3\pi}$$

and ibid. Vol.I, page 183:

A(7) $$\int_0^\infty \frac{J_1^2(ks)}{k^2} e^{-2k\lambda}\, dk = \frac{1}{2\pi}\int_0^\pi \Big\{ \sqrt{(2\lambda)^2 + 2s(1-\cos\phi)} - 2\lambda \Big\} (1-\cos\phi)\, d\phi =$$

$$= -\lambda + \frac{4s}{\pi}\sqrt{\frac{s^2+\lambda^2}{s^2}} \int_0^{\pi/2} \sqrt{1 - \frac{s^2}{s^2+\lambda^2}\sin^2\theta} \cdot \sin^2\theta \, d\theta =$$

$$= -\lambda + \frac{4s^2}{3\pi}\sqrt{\frac{s^2+\lambda^2}{s^2}} \Big\{ \frac{\lambda^2}{s^2} K\Big(\sqrt{\frac{s^2}{s^2+\lambda^2}}\Big) + \Big(1 - \frac{\lambda^2}{s^2}\Big) E\Big(\sqrt{\frac{s^2}{s^2+\lambda^2}}\Big) \Big\}$$

Where K and E are complete elliptic normal integrals,
(see also Grobner - Hofreiter - Integraltafel I - page 66)

The inductive reactance of the $\bar n^{th}$ turn will, therefore, be proportional to:

A(8) $$\mathcal{L}_{\bar n} = \frac{4\omega J^\# (J^\#)^*}{c^2} \frac{2s^2}{3\delta} \Big\{ \alpha^* + \sum_{1}^{M}{}_n (-)^{n+\bar n} \mathcal{M}^*_{n+\bar n -1}$$

$$+ \sum_{1}^{\bar n -1} (-)^{n+\bar n} \mathcal{M}^*_{\bar n - n} + \sum_{\bar n +1}^{M} (-)^{n+\bar n} \mathcal{M}^*_{n-\bar n} \Big\}$$

where:

A(9) $$\mathcal{L}^* = \sqrt{\frac{s^2+\delta^2}{s^2}}\left[\frac{\delta^2}{s^2}K\left(\sqrt{\frac{s^2}{s^2+\delta^2}}\right) + \left(1-\frac{\delta^2}{s^2}\right)E\left(\sqrt{\frac{s^2}{s^2+\delta^2}}\right)\right] - 1$$

and

A(10) $$\mathcal{M}^*_m = \frac{1}{2}\sqrt{\frac{(m\ell+\delta)^2+s^2}{s^2}}\left[\frac{(m\ell+\delta)^2}{s^2}K\left(\sqrt{\frac{s^2}{s^2+(m\ell+\delta)^2}}\right) + \left(1-\frac{(m\ell+\delta)^2}{s^2}\right)E\left(\sqrt{\frac{s^2}{s^2+(m\ell+\delta)^2}}\right)\right] +$$
$$+ \frac{1}{2}\sqrt{\frac{(m\ell-\delta)^2+s^2}{s^2}}\left[\frac{(m\ell-\delta)^2}{s^2}K\left(\sqrt{\frac{s^2}{s^2+(m\ell-\delta)^2}}\right) + \left(1-\frac{(m\ell-\delta)^2}{s^2}\right)E\left(\sqrt{\frac{s^2}{s^2+(m\ell-\delta)^2}}\right)\right]$$
$$- \sqrt{\frac{(m\ell)^2+s^2}{s^2}}\left[\frac{(m\ell)^2}{s^2}K\left(\sqrt{\frac{s^2}{s^2+(m\ell)^2}}\right) + \left(1-\frac{(m\ell)^2}{s^2}\right)E\left(\sqrt{\frac{s^2}{s^2+(m\ell)^2}}\right)\right]$$

$\mathcal{L}^*$ represents the self-inductance of the $\bar{n}^{th}$ turn, while the $\mathcal{M}^*_m$ represent the mutual inductance between the $\bar{n}^{th}$ turn and the other turns of the coil.

In practical cases the mutual inductance between two non-adjacent turns is negligible, and so only the mutual inductance between two adjacent turns, m=1, need be considered.

The inductive reactance of the $\bar{n}^{th}$ turn will, therefore, be proportional to

A(11) $$\mathcal{L}_{\bar{n}} = \frac{4\omega J^{\#}(J^{\#})^*}{c^2}\frac{2s^2}{3\delta}\left\{\mathcal{L}^* + 2\mathcal{M}^*_1\right\}$$ for each of the inside turns $\bar{m}=M$

$$= \frac{4\omega J^{\#}(J^{\#})^*}{c^2}\frac{2s^2}{3\delta}\left\{\mathcal{L}^* + \mathcal{M}^*_1\right\}$$ for the two outside turns $\bar{n}=M$

For $\delta \ll s$ and $s \leq \ell$ we have further $\mathcal{M}^*_1 \ll \mathcal{L}^*$

and:

A(12) $$\mathcal{L}^* \approx \frac{3\delta^2}{2s^2}\left[\ln 4s/\delta - 1/2\right]$$

so that:

A(13) $$\mathcal{L}_{\bar{n}} \approx \frac{4J^{\#}(J^{\#})^*\delta}{c^2}\left[\ln 4s/\delta - 1/2\right]$$

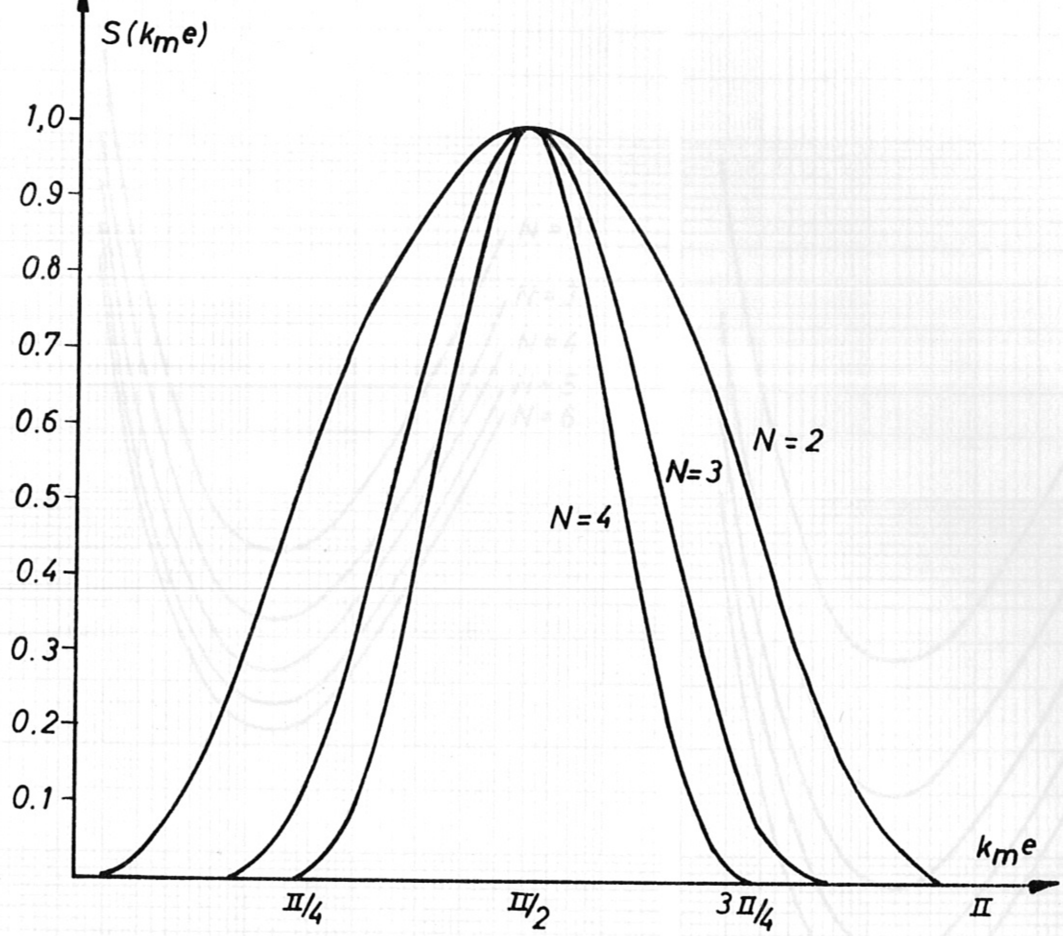


Fig. 1 Plot of $S(k_m e)$ versus $k_m e$ for N = 2, 3, 4

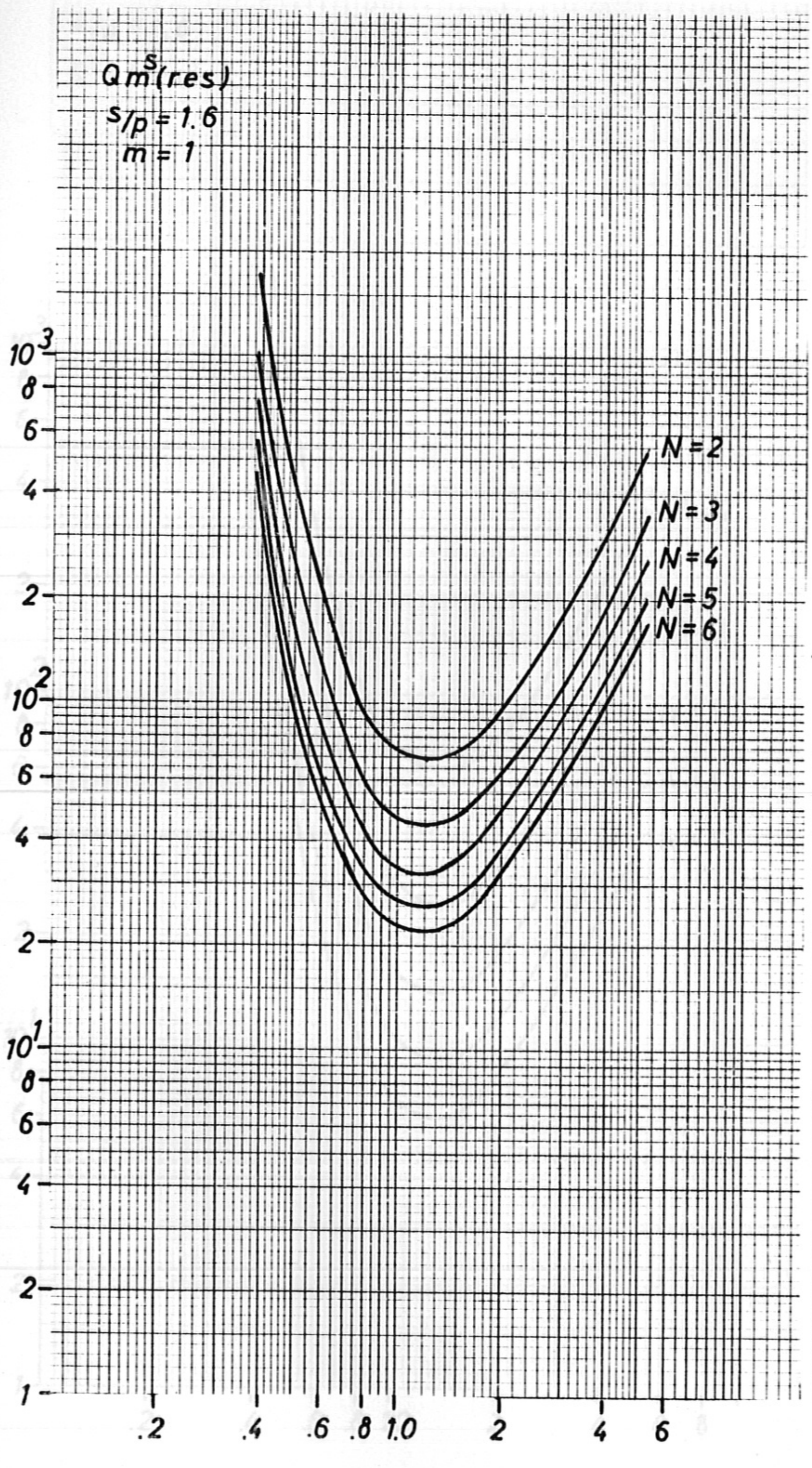


Fig. 2 Plot of Q_m^s (res) versus ℓ/p for $s/p = 1.6$, $m = 1$, and $N = 2,3,4,5,6$.

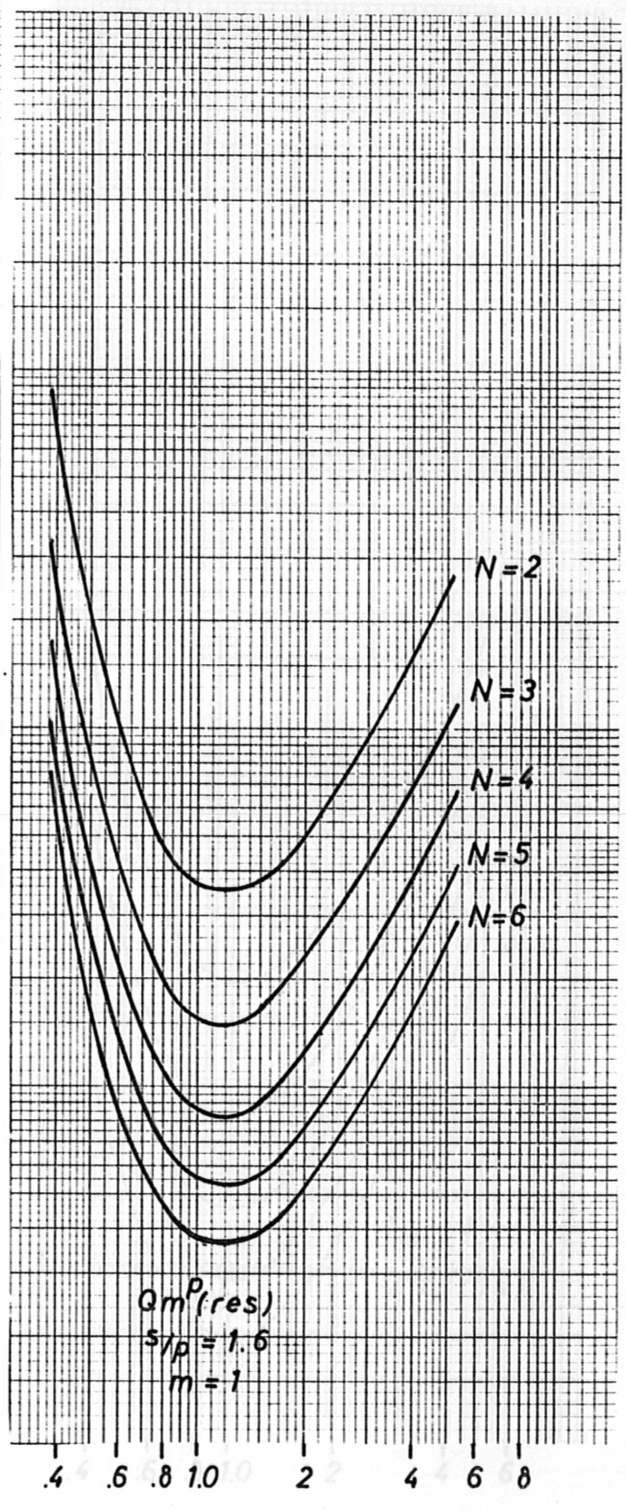


Fig. 3 Plot of Q_m^p (res) versus ℓ/p for $s/p = 1.6$, $m = 1$, and $N = 2,3,4,5,6$.

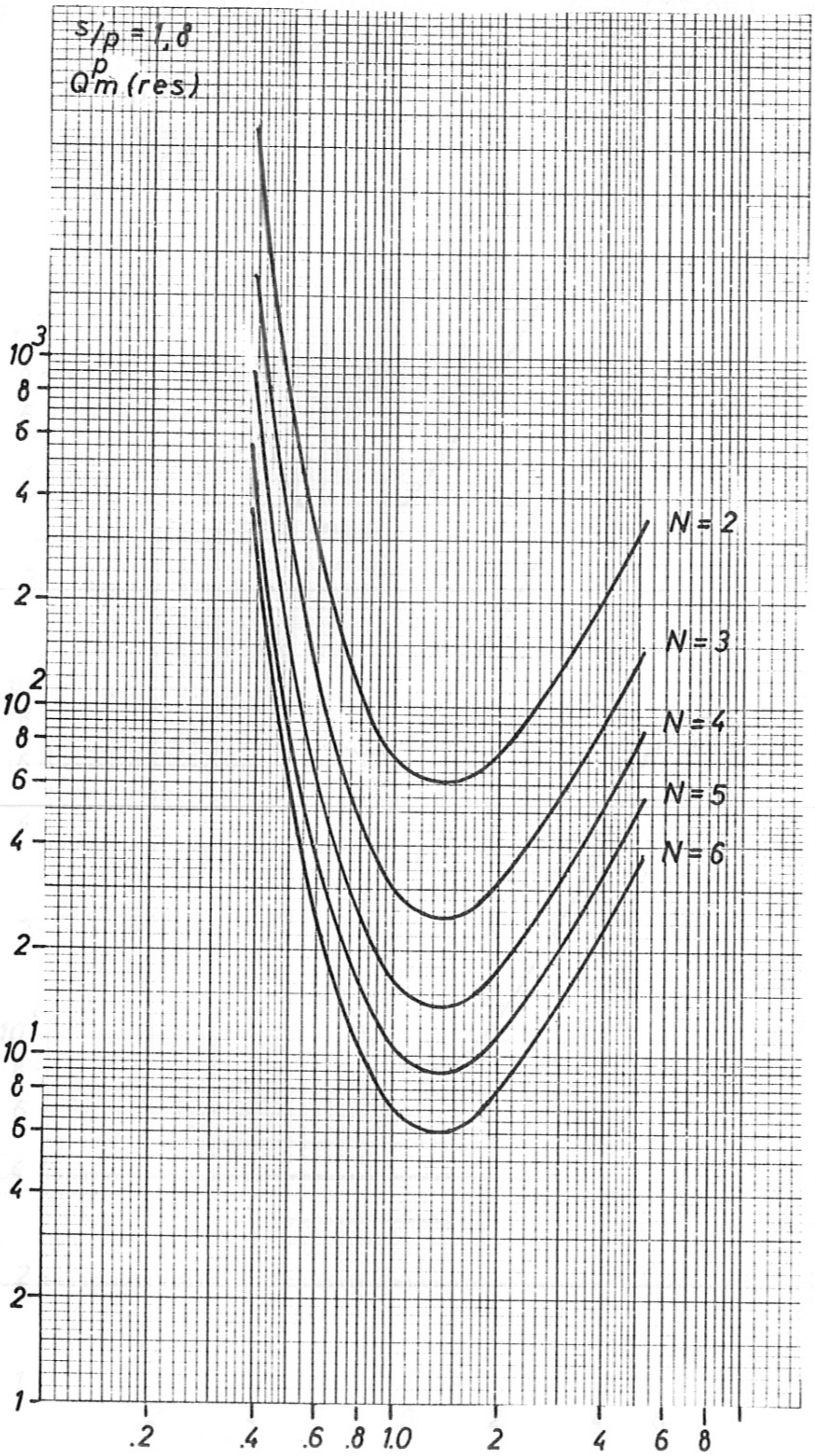


Fig. 4 Plot of Q^p_m (res) versus ℓ/p for s/p = 1.8, m = 1, and N = 2,3,4,5,6.

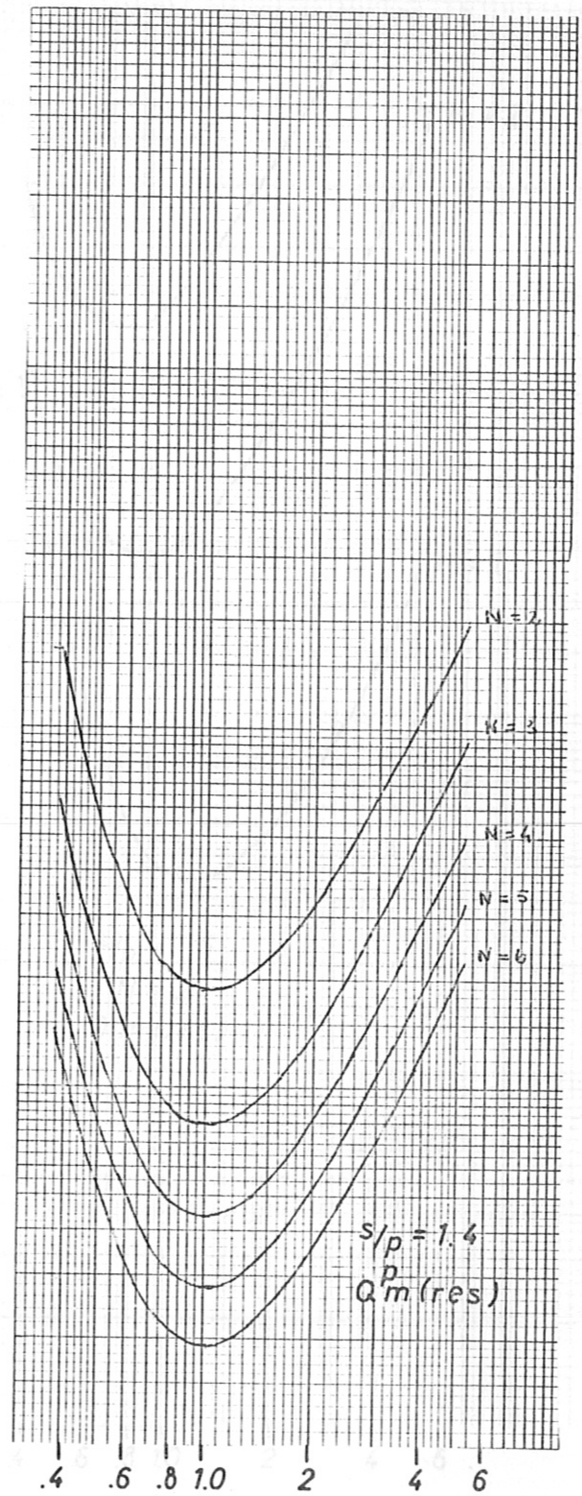


Fig. 5 Plot of Q^p_m (res) versus ℓ/p for s/p = 1.4, m = 1, and N = 2,3,4,5,6.

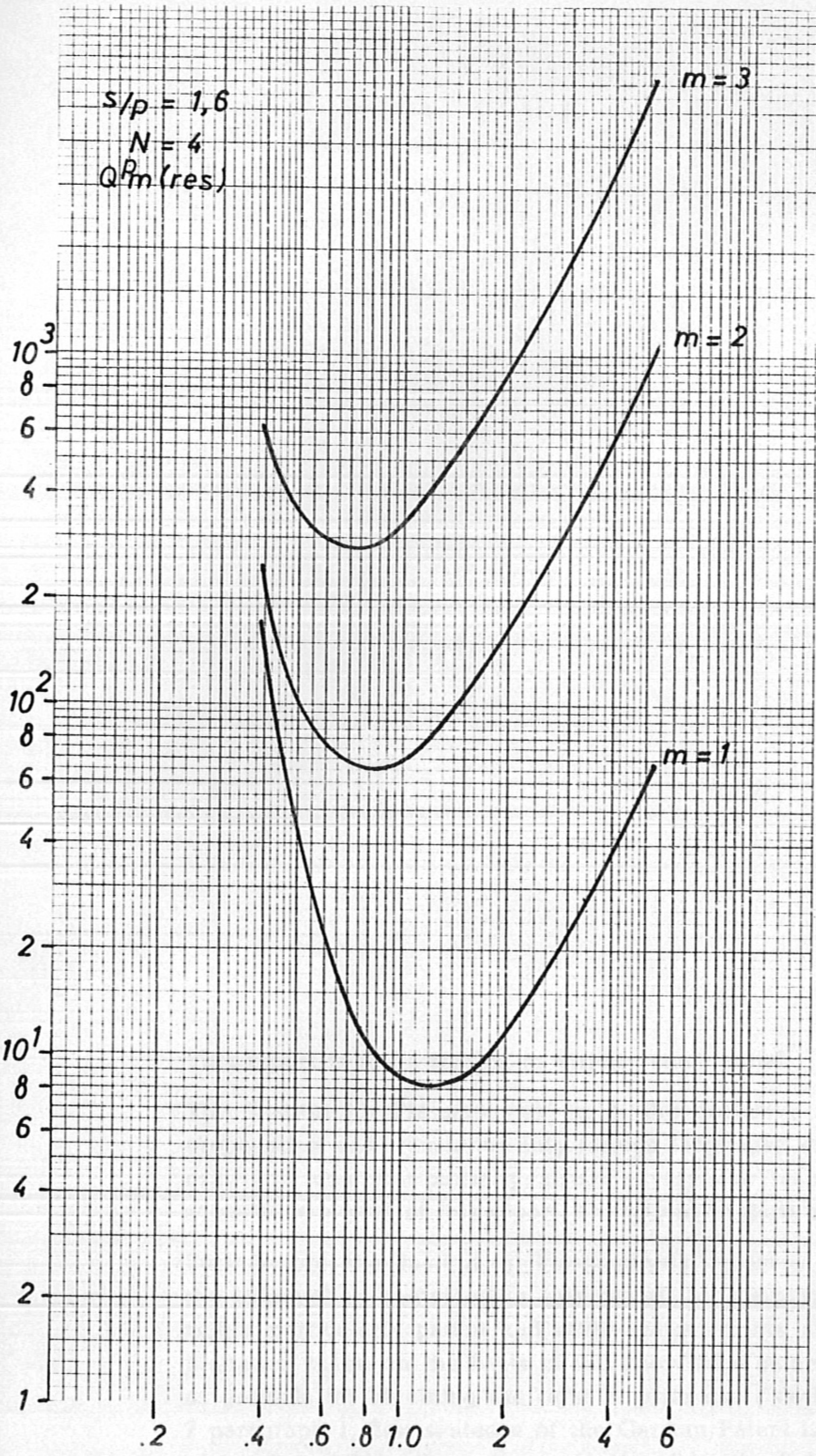


Fig. 6 Plot of Q_m^p (res) versus ℓ/p for $s/p = 1.6$, $N = 4$, and $m = 1,2,3$.

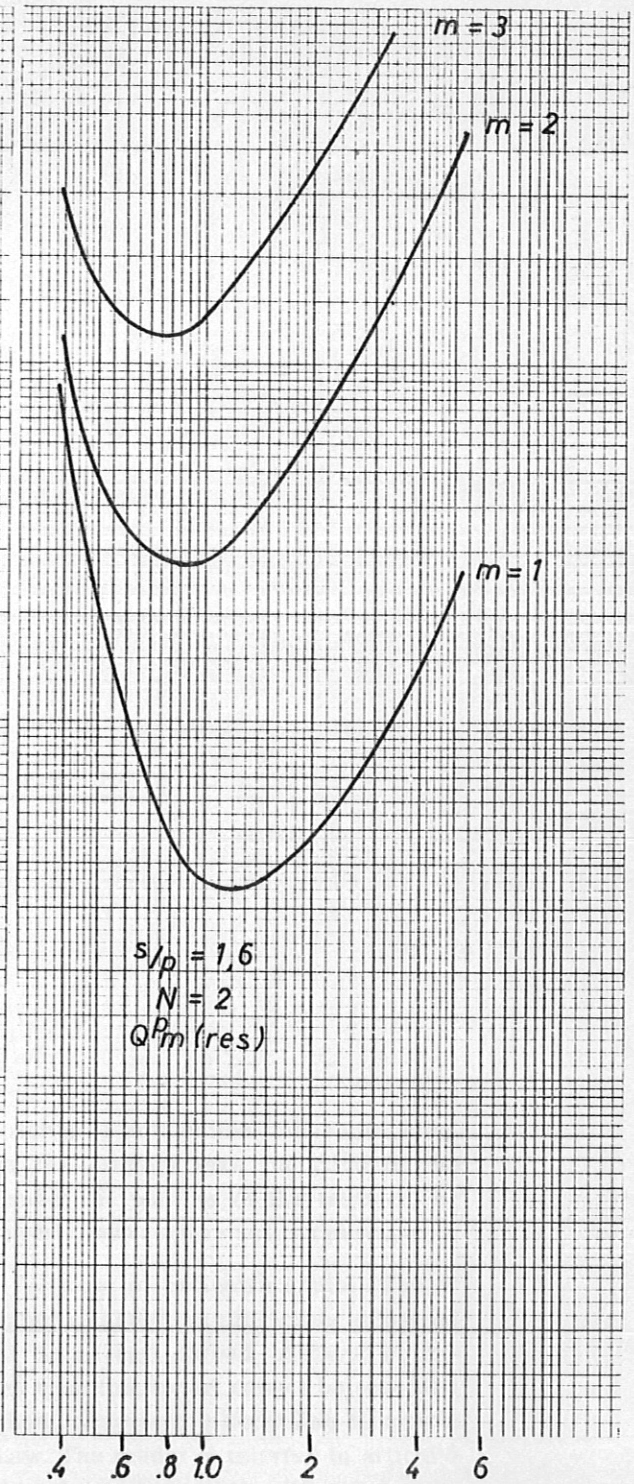


Fig. 7 Plot of Q_m^p (res) versus ℓ/p for $s/p = 1.6$, $N = 2$, and $m = 1,2,3$.