## On the Equilibrium and Stability of the Belt Pinch

by
F. Herrnegger
Max-Planck-Institut für Plasmaphysik, Euratom Association,
Garching bei München, Germany

Abstract: Exact axisymmetric toroidal equilibria are calculated. They depend on a single parameter the value of which is determined by $\beta_\varphi$ at the magnetic axis, the torus radius and the ratio of axial to radial extension of the plasma cross section. The equilibrium satisfies the necessary Mercier criterion for local stability.

Within the ideal MHD theory a class of axisymmetric toroidal equilibria has been calculated which are characterized as follows: the toroidal and the poloidal current densities vary over the plasma cross section, and the axis ratio of the elongated plasma cross section should be large (approximately 15). Thereby it is possible to increase the poloidal magnetic field considerably while the toroidal current remains below the Kruskal–Shafranov limit /1,2/.

Assuming symmetry in $\varphi$-direction and symmetry with respect to the torus mid plane ($z = 0$), the magnetic field strength $\vec{B}$ and the current density $\vec{j}$ are given by (cylindrical coordinates $r$, $\varphi$, $z$ are used)

$$\vec{B} = \frac{r_o}{r} I \, \vec{e}_\varphi + \frac{r_o}{r} (\vec{e}_\varphi \times \nabla \Psi), \quad \mu_o \vec{j} = r_o \nabla \times \left(\frac{1}{r} \vec{e}_\varphi \times \nabla \Psi\right) - \frac{r_o}{r} (\vec{e}_\varphi \times \nabla I) \quad (1)$$

where the stream function $\Psi$ satisfies the differential equation

$$\frac{r_o^2}{\mu_o r} \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Psi}{\partial r}\right) + \frac{1}{r} \frac{\partial^2 \Psi}{\partial z^2}\right] + \frac{r_o^2}{2\mu_o r^2} \frac{d}{d\Psi} I^2 + \frac{dp}{d\Psi} = 0 , \quad (2)$$

$p(\Psi)$ and $I(\Psi)$ are arbitrary functions of $\Psi$ alone, $r_o$ is the geometrical centre of the plasma cross section. Since the height and the thickness of the plasma belt are very different, one has to scale $z$ and $r$ in the following way: $z = r_o x$, $r = r_o \rho^{1/2}/\alpha$. The scaling factor $\alpha$ is a measure of the belt thickness and is given by $\alpha = \frac{1}{2} [\rho_+^{1/2} + \rho_-^{1/2}]$, where $\rho_+$, $\rho_-$, and $\rho_2$ determine the inner and outer plasma edge in the torus mid-plane and the magnetic axis. The other quantities are written in a nondimensional form as follows: $\Psi = \Psi_o \tilde{\Psi}$, $p = p_o \tilde{p}(\tilde{\Psi})$, $I = I_o \tilde{I}(\tilde{\Psi})$, $p_o$ is the plasma pressure at the magnetic axis, $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are the values of $\tilde{\Psi}$ at the plasma boundary and at the magnetic axis, resp., $\Psi_o$ and $I_o$ are related to the maximum poloidal and toroidal magnetic fields in the torus mid-plane. If one chooses $\tilde{p}(\tilde{\Psi})$ and $\tilde{I}(\tilde{\Psi})$ in the form

$$\tilde{p}(\tilde{\Psi}) = (\tilde{\Psi}^2 - \tilde{\Psi}_1^2) / (\tilde{\Psi}_2^2 - \tilde{\Psi}_1^2), \quad b = B_\varphi(x=0, \rho_2)/B_\varphi(x=0, \rho_+), \quad (3)$$

$$\tilde{I}(\tilde{\Psi}) = \left[1 - \left(1 - \frac{\rho_2}{\rho_+} b^2\right) (\tilde{\Psi}^2 - \tilde{\Psi}_1^2)/(\tilde{\Psi}_2^2 - \tilde{\Psi}_1^2)\right]^{1/2}, \quad (4)$$

then one has the situation that the toroidal current density varies across the plasma and has its maximum value at the magnetic axis, that the poloidal current density has a marked maximum nearby the plasma surface, and that the plasma pressure decreases faster to the outer edge of the torus than to the inner edge. Under these assumptions the general solution of (2) can be given immediately. A particular solution reads ($\gamma$ constant of separation related to the height of the belt)

$$\tilde{\Psi}(x, \rho) = F_o(\rho; \eta) \cos \gamma x \quad (5)$$

where $F_o(\rho; \eta)$ satisfies the Coulomb wave equation

$$F_o'' + \left(1 - \frac{2\eta}{\rho}\right) F_o = 0. \quad (6)$$

The solutions of (6) are known and depend on a parameter $\eta$ which is related to the physical quantities of the configuration by

$$\eta = \rho_+ \left(1 - \frac{\rho_2}{\rho_+} b^2\right) / 2\beta_\varphi + \gamma^2/8\alpha^2, \quad \beta_\varphi = 2\mu_o p_o / B_\varphi^2 (x=0, \rho_+). \quad (7)$$

For the Garching belt pinch experiment a typical $\eta$-value is $\eta = 10$. One can see that a particular choice of $\eta$ describes a class of equilibria all of which are produced by changing $\beta_\varphi$, $\rho_+$, $\gamma$, $\alpha$ in such a manner that $\eta$ remains unchanged. The same formula may also describe a set of equilibrium stages which occur during the experiment since the experimental conditions change slowly in time and the magnetic field at the magnetic axis increases. The numerical evaluation of (1) has been carried out for several parameter values. Since the toroidal current density is proportional to

$$j_\varphi \sim \frac{\tilde{\Psi}}{\rho^{1/2}} \left[2\eta - \frac{\gamma^2}{8\alpha^2} - \rho\right], \quad (8)$$

it changes the direction across the plasma where the bracket of (8) changes the sign. In order to avoid this one has to reduce the belt thickness for a given value of $\beta_\varphi$ or to limit $\beta_\varphi$ if the height and thickness of the belt and the torus radius are fixed.

Furthermore, it turns out that in the region near the plasma surface the pressure equilibrium is essentially determined by the poloidal currents, and in the central region by the toroidal currents. This equilibrium satisfies the necessary Mercier-criterion /3/ in the whole plasma region.

/1/ H. Zwicker; This Proceedings
/2/ H.J. Belitz, L. Janicke, P. Noll, U. Plantikow, F. Sand, J. Schlüter, F. Waelbroeck, G. Waidmann; Plasma Phys. Contr. Nucl. Fusion Res. 1971, Vol. 3, 179
/3/ C. Mercier; Nucl. Fusion 1, 47 (1960)

The author wishes to thank Professor H. Zwicker and Dr. J. Nührenberg for fruitful discussions.