Counterterms vs. Dualities

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Abstract

We investigate and clarify the mutual compatibility of the higher order corrections arising in supergravity and string theory effective actions and the non-linear duality symmetries of these theories. Starting from a conventional tree level action leading to duality invariant equations of motion, we show how to accommodate duality invariant counterterms given as functionals of both electric and magnetic fields in a perturbative expansion, and to deduce from them a non-polynomial *bona fide* action satisfying the Gaillard–Zumino constraint. There exists a corresponding consistency constraint in the non-covariant Henneaux–Teitelboim formalism which ensures that one can always restore diffeomorphism invariance by perturbatively solving this functional identity. We illustrate how this procedure works for the $R^2 \nabla F \nabla F$ and F^4 counterterms in Maxwell theory.

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1 Introduction

Consider Einstein theory coupled to scalar fields parametrising a symmetric space $G/K(G)$ (where $K(G)$ is the maximal compact subgroup of G), and n abelian vector fields such that $G \subset Sp(2n, \mathbb{R})$ acts linearly on them and their magnetic duals. This setup is typical for the bosonic sector of various (ungauged) extended supergravity theories, and in particular for the maximally extended $\mathcal{N} = 8$ supergravity with duality group $G = E_{7(7)} \subset Sp(56, \mathbb{R})$ [1]. The purpose of this letter is to discuss the consistency of the action of this duality group, when higher order local corrections to the tree level action (of the type appearing in the string theory effective action or as counterterms in extended supergravities) are included. Accordingly, we will consider n 'electric' vector fields A_μ^m together with their 'magnetic' duals $A_\mu^{\bar{m}}$, combining them into a $2n$-plet of vectors A_μ^m with $2n$ associated field strengths $F_{\mu\nu}^m$, *viz.*

$$A_\mu^m \equiv \left(A_\mu^m, A_\mu^{\bar{m}} \right) \quad \Leftrightarrow \quad F_{\mu\nu}^m \equiv \left(F_{\mu\nu}^m, F_{\mu\nu}^{\bar{m}} \right) \tag{1.1}$$

Note that the n magnetic duals $A_\mu^{\bar{m}}$ are only defined *on shell*, as non-local functionals of the other fields of the theory. [1] Classically, this redundancy is reflected in the so-called *twisted selfduality constraint* [1] for the $2n$ field strengths $F_{\mu\nu}^m$

$$F_{\mu\nu}^m = -\frac{1}{2\sqrt{-g}} \varepsilon_{\mu\nu}{}^{\sigma\rho} J^m{}_n F_{\sigma\rho}^n \ , \tag{1.2}$$

which simultaneously halves the number of degrees of freedom and puts the theory on-shell, in such a way that the Bianchi identities for the electric vectors imply the equations of motion for the magnetic vectors, and *vice versa*. Here, $J^m{}_n$ is a 'complex structure' built from the $Sp(2n, \mathbb{R})$ invariant symplectic form Ω^{mn} and the scalar field dependent symmetric metric $G_{mn} \in G$

$$J^m{}_n \equiv \Omega^{mp} G_{pn} \quad \Rightarrow \quad J^m{}_p J^p{}_n = -\delta_n^m \tag{1.3}$$

The indices $(m, \bar{m})$ correspond to the decomposition (1.1) of the $2n$ vectors in a Darboux basis such that the symplectic form splits as

$$\Omega_{mn} = \Omega_{\bar{m}\bar{n}} = 0 \ , \qquad \Omega_{m\bar{n}} = -\Omega_{\bar{n}m} = \delta_{m\bar{n}} \ , \tag{1.4}$$

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[1] In the literature [2, 3, 4] the magnetic field strengths are often denoted by the letter $G_{\mu\nu}$, so the relation with our notation (which follows [5]) is established by making the identification $(F_{\mu\nu}^m, F_{\mu\nu}^{\bar{m}}) \equiv (F_{\mu\nu}^m, G_{\mu\nu}^m)$. Because the extension of our arguments to fermions is straightforward, we will not consider fermions in this letter, but see *e.g.* [7].

Defining $H^{\bar{m}\bar{n}}$ as the inverse of $G_{\bar{m}\bar{n}}$ one directly obtains from (1.2)

$$F_{\mu\nu}^{\bar{m}} = H^{\bar{m}\bar{n}}\left(\delta_{\bar{n}\,m}\frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}F_{\sigma\rho}^{m} - G_{\bar{n}\,m}F_{\mu\nu}^{m}\right) \quad . \tag{1.5}$$

The classical action is then re-obtained by solving the equation

$$F_{\mu\nu}^{\bar{m}} = -\delta^{\bar{m}\,m}\frac{1}{-g}\varepsilon_{\mu\nu\sigma\rho}\frac{\delta S}{\delta F_{\sigma\rho}^{m}} \quad , \tag{1.6}$$

with the result

$$S \equiv S^{(0)} = -\frac{1}{4}\int d^4x\left(\sqrt{-g}H^{\bar{m}\bar{n}}\delta_{\bar{m}\,m}\delta_{\bar{n}\,n}F^{m\,\mu\nu}F_{\mu\nu}^{n} + \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}H^{\bar{m}\bar{n}}G_{\bar{n}\,n}\delta_{\bar{m}\,m}F_{\mu\nu}^{m}F_{\sigma\rho}^{n}\right) . \tag{1.7}$$

As required, the tree level action $S^{(0)}[F_{\mu\nu}^{m}]$ depends only on the *electric* vector potentials. For more general actions S depending on the electric vector fields, the basic relations (1.6) remain the same, and are usually referred to as *constitutive relations* [3]. As shown in [2, 3] it is a general feature that the action itself is not duality invariant, but varies as

$$\delta^{\mathfrak{g}}S[F_{\mu\nu}^{m}] = \frac{1}{8}\int d^4x\left(\varepsilon^{\mu\nu\sigma\rho}X^{\bar{m}}{}_{n}\delta_{\bar{m}\,m}F_{\mu\nu}^{m}F_{\sigma\rho}^{n} - \frac{4}{-g}\varepsilon_{\mu\nu\sigma\rho}X^{m}{}_{\bar{n}}\delta^{\bar{n}\,n}\frac{\delta S}{\delta F_{\mu\nu}^{m}}\frac{\delta S}{\delta F_{\sigma\rho}^{n}}\right) \quad , \tag{1.8}$$

under the duality transformations

$$\delta^{\mathfrak{g}}G^{mn} = X^{m}{}_{p}G^{pn} + X^{n}{}_{p}G^{mp} , \qquad \delta^{\mathfrak{g}}F_{\mu\nu}^{m} = X^{m}{}_{n}F_{\mu\nu}^{n} \quad . \tag{1.9}$$

We have omitted the superscript $^{(0)}$ in (1.8) because, as shown in [2, 3], the equation (1.8) is the consistency condition for *any* action S with associated duality invariant equations of motion.[2] In particular, it must also hold for actions including non-linear deformations or higher order corrections, so that the duality transformations preserve the constitutive relations (1.6).

Suppose now that we are given a classical action $S^{(0)}$ satisfying these requirements, such as for instance the tree level action of $\mathcal{N} = 8$ supergravity, whose vector part is just given by (1.7) (for $G = E_{7(7)}$). In perturbation theory, this action will be modified by higher order counterterms and corrections whose compatibility with duality transformations and with (1.8) is not immediately obvious. The higher order corrections to the action are only defined modulo the equations of motion of the classical action $S^{(0)}$. They

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[2]Note that (1.8) is required for the duality transformations to make sense on the fields, and this is also valid when they are not symmetries of the equations of motion, but nevertheless admit a representation on the fields satisfying the equations of motion.

are generally given as functionals of the $2n$ electric *and* magnetic vectors A_μ^m, that is, in the form $\mathcal{I}^{(1)} = \mathcal{I}^{(1)}[F_{\mu\nu}^m, F_{\mu\nu}^{\bar{m}}]$.[3] In particular, the higher order counterterms in $\mathcal{N} = 8$ supergravity appear generically in this manifestly covariant form with respect to the duality group in terms of vector fields transforming in the linear **56** representation of $E_{7(7)}$ [8, 9, 10]. When trying to express the original action together with the corrections as an actual new action functional of the electric field strengths only, we are thus faced with the question what expression to substitute for the magnetic field strengths $F_{\mu\nu}^{\bar{m}}$: after all, these will be given by non-linear and possibly non-local functionals of the electric vector fields (as well as the other fields) whose form is determined precisely by the new corrected action we are looking for. A naive guess might be to substitute the tree level solution (1.5), but one quickly sees that this ansatz solves the consistency condition (1.8) only to first order in perturbation theory, and fails at higher orders. In other words, it could *a priori* appear that the corrected action functional gives rise to inconsistencies with the action of the duality transformations (1.9) [4].

2   Deformed twisted selfduality constraint

To find the right action one must therefore adopt a different strategy, taking a deformed version of the twisted selfduality constraint as the starting point. Namely, given a manifestly duality covariant counterterm correction $\mathcal{I}^{(1)}$ depending on the $2n$ field strengths $F_{\mu\nu}^m$ and their derivatives, we propose to replace (1.2) by the *deformed twisted selfduality constraint*

$$F_{\mu\nu}^m - \frac{2}{\sqrt{-g}}G^{mn}g_{\mu\sigma}g_{\nu\rho}\frac{\delta\mathcal{I}^{(1)}}{\delta F_{\sigma\rho}^n} = -\frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}J^m{}_n\left(F_{\mu\nu}^m - \frac{2}{\sqrt{-g}}G^{mn}g_{\sigma\tau}g_{\rho\omega}\frac{\delta\mathcal{I}^{(1)}}{\delta F_{\tau\omega}^n}\right) \quad (2.1)$$

or, equivalently,

$$F_{\mu\nu}^m + \frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}J^m{}_n F_{\sigma\rho}^n = \frac{2}{\sqrt{-g}}G^{mn}g_{\mu\sigma}g_{\nu\rho}\frac{\delta\mathcal{I}^{(1)}}{\delta F_{\sigma\rho}^n} + \Omega^{mn}\frac{1}{-2g}\varepsilon^{\mu\nu\sigma\rho}\frac{\delta\mathcal{I}^{(1)}}{\delta F_{\sigma\rho}^n} \; . \quad (2.2)$$

This equation is manifestly duality invariant if $\mathcal{I}^{(1)}$ is a duality invariant functional. At the same time it achieves the required halving of the number of physical degrees of freedom and imposes the (deformed) equations of motion. To reconstruct a *bona fide* action depending only on the physical fields (and only the electric vector fields, in particular) and satisfying all consistency requirements, we now have two options.

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[3]For clarity of notation, we will use the letter S only for 'true' actions defined off shell, whereas $\mathcal{I}$ denotes a general functional of both electric and magnetic fields.

- We first solve (2.1) for the magnetic field strengths $F_{\mu\nu}^{\bar{m}}$ in function of the electric field strengths $F_{\mu\nu}^{m}$ and their derivatives (as well as all other fields) as a formal power series. With the resulting expression for $F_{\mu\nu}^{\bar{m}}$ as a functional of the physical fields, we then solve (1.6) in a second step to obtain the full corrected action functional in terms of the electric vectors only. This procedure manifestly preserves four-dimensional space-time covariance.

- Alternatively, we can solve (2.1) for the time components $(F_{0\mathtt{i}}^{m}, F_{0\mathtt{i}}^{\bar{m}})$, and again reconstruct the requisite action in a second step. The resulting action depends on the *spatial* electric *and* magnetic vector components $(A_{\mathtt{i}}^{m}, A_{\mathtt{i}}^{\bar{m}})$, and therefore breaks manifest space-time covariance. Nevertheless, we will see that there is a consistency condition that guarantees full space-time covariance on-shell.

Due to the non-linear dependence of $\mathcal{I}^{(1)}$ on the magnetic field strengths $F_{\mu\nu}^{\bar{m}}$ and possibly their derivatives, the resulting corrected action will include terms of arbitrarily high order for any kind of counterterm correction, and this will be true in both approaches. In other words, the 'initial' counterterm $\mathcal{I}^{(1)}$, which is usually polynomial in the field strengths and their derivatives, must be supplemented by an infinite string of higher order terms. This completion of the 'initial' counterterm action will thus be non-polynomial, and also non-local if the initial counterterm depends on derivatives.[4] Yet, it will satisfy the consistency condition (1.8).

Let us illustrate these claims with a simple example from Maxwell theory, adopting the space-time covariant approach. For this purpose we combine the electric vector A_{μ}^{1} with its magnetic dual $A_{\mu}^{2} \equiv A_{\mu}^{\bar{1}}$ into a *complex* vector potential $A_{\mu} \equiv A_{\mu}^{1} + iA_{\mu}^{2}$, with corresponding complex field strength

$$F_{\mu\nu} = F_{\mu\nu}^{1} + iF_{\mu\nu}^{2} \quad , \tag{2.3}$$

Electromagnetic $U(1)$ duality then acts on these fields simply as a global phase rotation. It is furthermore easy to see that the original (free) Maxwell equations for A_{μ}^{1} are recovered from the twisted selfduality constraint

$$F_{\mu\nu}^{-} = 0 \quad , \tag{2.4}$$

where we define the complex selfdual and anti-selfdual field strengths as

$$F_{\mu\nu}^{\pm} := \frac{1}{2}F_{\mu\nu} \pm \frac{i}{4\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}F_{\sigma\rho} \ . \tag{2.5}$$

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[4]Note, however, that it will nevertheless remain local in a perturbative sense, *i.e.* involve only a *finite* number of derivatives at any given order in the coupling constant.

As an example of a non-trivial deformation let us consider the $U(1)$ duality invariant expression

$$\mathcal{I}^{(1)} = -\frac{1}{4} \int d^4 x \sqrt{-g}\, g^{\rho\kappa} T^{\mu\nu\sigma\lambda}\, \nabla_\mu \bar{F}_{\sigma\rho}\, \nabla_\nu F_{\lambda\kappa}\,, \qquad (2.6)$$

where $T^{\mu\nu\sigma\rho}$ is the Bel–Robinson tensor

$$T^{\mu\nu\sigma\rho} \equiv C^{\mu\kappa\sigma\lambda} C^\nu{}_\kappa{}^\rho{}_\lambda - \frac{3}{2} g^{\mu[\nu} C^{\kappa\lambda]\sigma\vartheta} C_{\kappa\lambda}{}^\rho{}_\vartheta\,, \qquad (2.7)$$

with the Weyl tensor $C_{\mu\nu\sigma\rho}$. The Bel–Robinson tensor is fully symmetric and traceless in its four indices and is conserved modulo the vacuum equations of motion.

The counterterm (2.6) is a simplified version of a typical term appearing in the supersymmetric completion of the R^4 counterterm arising at three loops in $\mathcal{N} = 8$ supergravity [11], [5] where it is proportional to (using $SL(2, \mathbb{C})$ spinor notation)

$$C^{\alpha\beta\gamma\delta} C^{\dot\alpha\dot\beta\dot\gamma\dot\delta} \nabla_{\alpha\dot\delta} F^{ij}_{\beta\gamma} \nabla_{\delta\dot\alpha} \bar{F}_{\dot\beta\dot\gamma\, ij}\,. \qquad (2.8)$$

with the $SU(8)$ field strength $F^{ij}_{\mu\nu}$. With the above notation, the deformed twisted selfduality constraint for our Maxwell example takes the form

$$F^-_{\mu\nu} + \nabla_\sigma T_{[\mu}{}^{\sigma\rho\lambda} \nabla_\rho F^+_{\nu]\lambda} = 0\,, \qquad (2.9)$$

Observe that the second term in (2.9) is complex anti-selfdual in the indices $[\mu\nu]$, as it should be, because the Bel–Robinson tensor is symmetric traceless and the torsion-free covariant derivatives preserve complex (anti)selfduality.

Let us now construct a manifestly diffeomorphism covariant Lagrangian for the deformed equations of motion in terms of the *real* Maxwell field strengths $F^1_{\mu\nu}$ only, following the above procedure. To this aim we define the differential operator

$$(\Delta(f))_{\mu\nu} \equiv \Delta_{\mu\nu}{}^{\rho\sigma} f_{\rho\sigma} := \nabla_\kappa T_{[\mu}{}^{\kappa\lambda[\sigma} \nabla_\lambda \delta^{\rho]}_{\nu]} f_{\rho\sigma}\,. \qquad (2.10)$$

acting on two-forms $f_{\mu\nu}$. This operator is self-adjoint and satisfies

$$\frac{1}{2\sqrt{-g}} \varepsilon_{\mu\nu}{}^{\kappa\lambda} \Delta_{\kappa\lambda}{}^{\sigma\rho} = -\Delta_{\mu\nu}{}^{\kappa\lambda} \frac{1}{2\sqrt{-g}} \varepsilon_{\kappa\lambda}{}^{\sigma\rho}\,, \qquad (2.11)$$

thus converting selfdual into anti-selfdual tensors, and *vice versa* (this accounts for the $\pm$ sign on $F^+_{\nu\lambda}$ in the second term of (2.9)). Decomposing (2.9) one obtains

$$\left(\delta^{\sigma\rho}_{\mu\nu} + \Delta_{\mu\nu}{}^{\sigma\rho}\right) F^2_{\sigma\rho} = \left(\delta^{\sigma\rho}_{\mu\nu} - \Delta_{\mu\nu}{}^{\sigma\rho}\right) \frac{1}{2} \varepsilon_{\sigma\rho}{}^{\kappa\lambda} F^1_{\kappa\lambda}\,, \qquad (2.12)$$

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[5]Note that the complete supersymmetry invariant is not actually duality invariant [12, 13], however its non-perturbative completion arising in string theory is believed to be $E_{7(7)}(\mathbb{Z})$ invariant (see *e.g.* [14]), and so it is important that it transforms covariantly with respect to $E_{7(7)}$.

and inverting the operator on the left-hand side we get

$$F^2_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\kappa\lambda}\left(\delta^{\sigma\rho}_{\kappa\lambda} + 2\sum_{n\geq 1}(\Delta^n)_{\kappa\lambda}{}^{\sigma\rho}\right)F^1_{\sigma\rho}\,, \tag{2.13}$$

where Δ^n is the nth power of Δ. This equation, in turn, simply follows as the Euler–Lagrange equation of the action

$$S = -\int d^4x\sqrt{-g}\left(\frac{1}{4}F^{1\,\mu\nu}F^1_{\mu\nu} + \frac{1}{2}\sum_{n=1}^{\infty}F^{1\,\mu\nu}(\Delta^n)_{\mu\nu}{}^{\sigma\rho}F^1_{\sigma\rho}\right)\,. \tag{2.14}$$

The completion of the 'initial' Bel–Robinson counterterm $\mathcal{I}^{(1)}$ from (2.6) is thus *non-polynomial* and also *non-local* (depending on arbitrarily high powers of the derivative operator ∇_μ). To check the consistency condition (1.8) we first observe that

$$\int d^4x \frac{2}{-g}\varepsilon_{\mu\nu\sigma\rho}\frac{\delta S}{\delta F^1_{\mu\nu}}\frac{\delta S}{\delta F^1_{\sigma\rho}} = \int d^4x \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\left(\delta^{\kappa\lambda}_{\mu\nu} + 2\sum_{n\geq 1}\Delta^{n\,\kappa\lambda}_{\mu\nu}\right)F^1_{\kappa\lambda}\left(\delta^{\theta\tau}_{\sigma\rho} + 2\sum_{n\geq 1}\Delta^{n\,\theta\tau}_{\sigma\rho}\right)F^1_{\theta\tau}$$

$$= \int d^4x \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F^1_{\mu\nu}\left(\delta^{\kappa\lambda}_{\sigma\rho} + 2\sum_{n\geq 1}(\text{-}\Delta)^{n\,\kappa\lambda}_{\sigma\rho}\right)\left(\delta^{\theta\tau}_{\kappa\lambda} + 2\sum_{n\geq 1}\Delta^{n\,\theta\tau}_{\kappa\lambda}\right)F^1_{\theta\tau}$$

$$= \int d^4x \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}F^1_{\mu\nu}F^1_{\sigma\rho}\quad. \tag{2.15}$$

Because $X^1{}_2 = -X^2{}_1$ for a $U(1)$ duality rotation, this means that the two terms on the right-hand side of (1.8) are the same, yielding twice the right-hand side of (2.15). Now using $\delta^{\mathsf{u}(1)}F^1_{\mu\nu} = X^1{}_2 F^2_{\mu\nu}$ together with the constitutive relations (1.6), it is straightforward to see that (1.8) is indeed satisfied for the completed action (2.14).

3   Non-covariant formulation with manifest duality

In the foregoing section we showed how to restore the full duality invariance for the corrected equations of motion. However, being on-shell, this formalism is not directly suited for quantisation because we cannot formulate the functional Ward identities for the duality symmetry in that case. For that purpose one must instead make use of a non-covariant formulation developed by Henneaux and Teitelboim [6], and worked out for $\mathcal{N} = 8$ supergravity by Hillmann [7] (see also [5]). In that formalism one takes the $2n$ spatial three-vectors A^m_{i} as the fundamental fields, while their time components are only defined on-shell. As a consequence, the action is manifestly duality invariant, but

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no longer manifestly invariant under space-time diffeomorphisms. At tree level, it takes the form

$$S_{\text{vec}}^{(0)} = \frac{1}{4}\int d^4x \left(\Omega_{mn}\varepsilon^{\mathtt{ijk}}\left(\partial_o A_{\mathtt{i}}^m + N^{\mathtt{l}}F_{\mathtt{il}}^m\right)F_{\mathtt{jk}}^n - N\sqrt{h}\,G_{mn}h^{\mathtt{ik}}h^{\mathtt{jl}}F_{\mathtt{ij}}^m F_{\mathtt{kl}}^n \right) \ . \tag{3.1}$$

It is invariant only with respect to a non-standard realisation of space-time diffeomorphisms (but, of course, still invariant under spatial diffeomorphisms). The equation of motion for the vector fields is

$$\varepsilon^{\mathtt{ijk}}\partial_{\mathtt{j}}\mathscr{E}_{\mathtt{k}}^m = 0 \ , \tag{3.2}$$

with the abbreviation

$$\mathscr{E}_{\mathtt{i}}^m \equiv \partial_o A_{\mathtt{i}}^m + N^{\mathtt{j}}F_{\mathtt{ij}}^m - \frac{N}{2\sqrt{h}}h_{\mathtt{ij}}\varepsilon^{\mathtt{jkl}}J^m{}_n F_{\mathtt{kl}}^n \ . \tag{3.3}$$

(recall that we neglect fermionic terms). It is invariant with respect to the modified diffeomorphism transformation of the vector field

$$\delta_\xi A_{\mathtt{i}}^m \equiv \xi^j F_{ji}^m + \xi^o\left(\partial_o A_{\mathtt{i}}^m - \mathscr{E}_{\mathtt{i}}^m\right) \ . \tag{3.4}$$

Although the component A_o^m of the vector field does not appear in the action, its spatial gradient can be identified from the equations of motion as

$$\partial_{\mathtt{i}}A_o^m = \mathscr{E}_{\mathtt{i}}^m \quad , \tag{3.5}$$

One then expresses the Lorentz field strength $F_{\mu\nu}^m$ via $F_{\mathtt{ij}}^m$ and

$$F_{o\mathtt{i}}^m = \partial_o A_{\mathtt{i}}^m - \mathscr{E}_{\mathtt{i}}^m \quad . \tag{3.6}$$

With this definition, one checks that the field strength $F_{\mu\nu}^m$ transforms indeed as it should with respect to diffeomorphisms modulo the equations of motion

$$\delta F_{\mu\nu}^m = \xi^\sigma \partial_\sigma F_{\mu\nu}^m - 2F_{\sigma[\mu}\partial_{\nu]}\xi^\sigma + \xi^o \mathscr{E}_{\mu\nu}^m \quad , \tag{3.7}$$

where $\mathscr{E}_{\mu\nu}^m$ is the twisted selfdual component of the equations of motion, *i.e.*

$$\mathscr{E}_{\mathtt{ij}}^m = \Omega^{mn}\varepsilon_{\mathtt{ijk}}\frac{\delta S^{(0)}}{\delta A_{\mathtt{k}}^n} \ , \quad \mathscr{E}_{o\mathtt{i}}^m = -N^{\mathtt{j}}\mathscr{E}_{\mathtt{ij}}^m + \frac{N}{2\sqrt{h}}h_{\mathtt{ij}}\varepsilon^{\mathtt{jkl}}J^m{}_n\mathscr{E}_{\mathtt{kl}}^n \quad , \tag{3.8}$$

in accord with the (undeformed) twisted selfduality constraint.

Next let us consider some higher order supersymmetric invariant $\mathcal{I}^{(1)}$ defined on-shell as a functional of $F_{\mu\nu}^m$ and the other fields of the theory, which is invariant with

respect to the ordinary action of diffeomorphisms. From this action we directly obtain the corresponding off-shell action $S^{(1)}$ by substituting (3.6) for the time-components $F_{o\mathtt{i}}^m$, *viz.*

$$S^{(1)}\left[F_{\mathtt{ij}}^m,\ F_{o\mathtt{i}}^m \equiv \partial_o A_{\mathtt{i}}^m - \mathscr{E}_{\mathtt{i}}^m\right]. \qquad (3.9)$$

Its variation under a time-like diffeomorphism with parameter ξ^o is [6]

$$\delta S^{(1)} = \int dx^4 \Omega^{mn} \varepsilon^{\mathtt{ijk}} \xi^o \frac{\delta S^{(1)}}{\delta F_{\mathtt{ij}}^m} \frac{\delta S^{(0)}}{\delta A_{\mathtt{k}}^n}. \qquad (3.10)$$

It follows that, at the same order, the action $S^{(0)} + S^{(1)}$ is invariant with respect to the modified variation

$$\delta A_{\mathtt{i}}^m = \xi^j F_{ji}^m + \xi^o \left(\partial_o A_{\mathtt{i}}^m - \mathscr{E}_{\mathtt{i}}^m - \Omega^{mn} \varepsilon^{\mathtt{ijk}} \frac{\delta S^{(1)}}{\delta F_{\mathtt{jk}}^n}\right). \qquad (3.11)$$

At this order this result is precisely the expected one: the diffeormorphism transformation of the vector field agrees with the ordinary transformation modulo the *corrected* equations of motion. Of course, in order to obtain full agreement and to establish the consistency of the deformed action one must now complete the corrected action by adding higher order terms, just like for the covariant deformed Maxwell action in the previous section. That is, we must determine the full invariant

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots \qquad (3.12)$$

with a corresponding all order corrected transformation of the vector fields. The possible obstructions in carrying out this procedure are the solutions of the diffeomorphism Wess–Zumino consistency conditions as functionals of $F_{\mu\nu}^m$ and the other fields, identified modulo the equations of motions [15]. Because the action of diffeomorphisms on $F_{\mu\nu}^m$ is identical to the conventional one modulo the equations of motion, this cohomology problem is identical to the one of identifying algebraic diffeomorphism anomalies in four dimensions. Consequently, the absence of such anomalies [16, 17] ensures the existence of a completed action S which is invariant with respect to its associated diffeomorphism action.

We will now show how to compute the complete action S perturbatively by using the invariance of the action as a first order functional derivative equation. To this aim we

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[6]The covariance under *spatial* diffeomorphisms (with parameters $\xi^{\mathtt{i}}$) is manifest.

consider the action [7]

$$S = \frac{1}{4} \int d^4 x \left(\Omega_{mn} \varepsilon^{\mathtt{ijk}} \left(\partial_o A_\mathtt{i}^m + N^\mathtt{l} F_{\mathtt{il}}^m \right) F_{\mathtt{jk}}^n \right) + I \left[F_{\mathtt{ij}}^m, \nabla_\mu F_{\mathtt{ij}}^m, \dots \right] \ , \qquad (3.13)$$

where the functional I depends on the vector fields via the *spatial* field strengths $F_{\mathtt{ij}}^m$ and their derivatives (including time derivatives). For any such I the equations of motion of the vector fields still take the form of a spatial divergence

$$\frac{\delta S}{\delta A_\mathtt{i}^m} = -\varepsilon^{\mathtt{ijk}} \partial_\mathtt{j} \left(\Omega_{mn} \left(\partial_o A_\mathtt{k}^n + N^\mathtt{l} F_{\mathtt{kl}}^n \right) - \varepsilon_{\mathtt{klh}} \frac{\delta I}{\delta F_{\mathtt{lh}}^m} \right) = 0 \ , \qquad (3.14)$$

They are thus equivalent to the first order equation

$$F_{o\mathtt{i}}^m = -N^\mathtt{j} F_{\mathtt{ij}}^m - \Omega^{mn} \varepsilon_{\mathtt{ijk}} \frac{\delta I}{\delta F_{\mathtt{jk}}^n} \ . \qquad (3.15)$$

Under the diffeomorphisms

$$\delta_\xi A_\mathtt{i}^m \equiv \xi^\mathtt{j} F_{\mathtt{ji}}^m - \xi^o \left(N^\mathtt{j} F_{\mathtt{ij}}^m + \Omega^{mn} \varepsilon_{\mathtt{ijk}} \frac{\delta I}{\delta F_{\mathtt{jk}}^n} \right) \ , \qquad (3.16)$$

the full action S transforms as

$$\delta_\xi S = \int d^4 x \xi^o \left(\frac{1}{4} \Omega_{mn} \partial_\mathtt{l} \left(N^2 h^{\mathtt{lh}} \varepsilon^{\mathtt{ijk}} F_{\mathtt{ih}}^m F_{\mathtt{jk}}^n \right) + \left(\partial_o F_{\mathtt{ij}}^m + 2 \partial_\mathtt{i} N^\mathtt{l} F_{\mathtt{jk}}^m \right) \frac{\delta I}{\delta F_{\mathtt{ij}}^m} \right) + \delta_\xi I$$

$$= \int d^4 x \xi^o \partial_\mathtt{l} \left(\frac{1}{4} \Omega_{mn} N^2 h^{\mathtt{lh}} \varepsilon^{\mathtt{ijk}} F_{\mathtt{ih}}^m F_{\mathtt{jk}}^n + \Omega^{mn} \varepsilon_{\mathtt{ijk}} \frac{\delta I}{\delta F_{\mathtt{il}}^m} \frac{\delta I}{\delta F_{\mathtt{jk}}^n} \right) + \bar{\delta}_\xi I \ , \qquad (3.17)$$

where $\bar{\delta}_\xi$ is defined to act on $F_{\mathtt{ij}}^m$ as an ordinary diffeomorphism according to (3.15)

$$\bar{\delta}_\xi F_{\mathtt{ij}}^m = \xi^\mu \partial_\mu F_{\mu\nu}^m - 2 F_{\mathtt{k[i}}^m \partial_{\mathtt{j]}} \xi^\mathtt{k} + 2 \left(N^\mathtt{k} F_{\mathtt{[i|k}}^m + \Omega^{mn} \varepsilon_{\mathtt{kl[i}} \frac{\delta I}{\delta F_{\mathtt{kl}}^n} \right) \partial_{\mathtt{j]}} \xi^o \ . \qquad (3.18)$$

The invariance of the action therefore follows from the vanishing of

$$\frac{\delta^L \bar{\delta}_\xi I}{\delta \xi^o} = \partial_\mathtt{l} \left(\frac{1}{4} \Omega_{mn} N^2 h^{\mathtt{lh}} \varepsilon^{\mathtt{ijk}} F_{\mathtt{hi}}^m F_{\mathtt{jk}}^n + \Omega^{mn} \varepsilon_{\mathtt{ijk}} \frac{\delta I}{\delta F_{\mathtt{li}}^m} \frac{\delta I}{\delta F_{\mathtt{jk}}^n} \right) \quad . \qquad (3.19)$$

---

[7]The covariant derivative $\nabla_\mu F_{\mathtt{ij}}^m$ must be defined perturbatively. At first order it is defined from the ordinary covariant derivative ∇_μ acting on $F_{\mu\nu}^m$ as defined in (3.6),

$$\nabla_\mu^{(0)} F_{\mathtt{ij}}^m = \partial_\mu F_{\mathtt{ij}}^m + 2 \Gamma_{\mu[\mathtt{i}}^\mathtt{k} F_{\mathtt{j]k}}^m - 2 \Gamma_{\mu[\mathtt{i}}^o \left(\partial_o A_{\mathtt{j]}}^m - \mathscr{E}_{\mathtt{j]}}^m \right)$$

.

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This relation can be viewed as the non-covariant analogue of the consistency condition (1.8), but now ensuring space-time covariance of our manifestly duality invariant action.

The equation (3.19) defines a functional differential equation which permits to determine I perturbatively. This equation simplifies drastically when I contains no explicit derivative terms, that is, when it is the integral of a polynomial function ('potential') V of $F_{\mathtt{ij}}^m$ and the metric, which is invariant under spatial diffeomorphisms. In that case (3.19) is no longer a functional differential equation, but reduces to the differential equation

$$\frac{1}{4}\Omega_{mn}N^2 h^{\mathtt{lh}}\varepsilon^{\mathtt{ijk}}F_{\mathtt{hi}}^m F_{\mathtt{jk}}^n = \Omega^{mn}\varepsilon_{\mathtt{ijk}}\frac{\delta I}{\delta F_{\mathtt{li}}^m}\frac{\delta I}{\delta F_{\mathtt{jk}}^n} \equiv \Omega^{mn}N^2 h\ \varepsilon_{\mathtt{ijk}}\frac{\partial V}{\partial F_{\mathtt{li}}^m}\frac{\partial V}{\partial F_{\mathtt{jk}}^n} \ . \tag{3.20}$$

4 Maxwell theory, once again

To illustrate how the procedure works in the non-covariant formulation we again study an example generalising Maxwell theory. To keep things as simple as possible we consider a modification that initially depends on the complex spatial field strength $F_{\mathtt{ij}}$ polynomially, but not on its derivatives (the inclusion of derivatives presents no problem of principle, but renders the calculations substantially more tedious). The tree level Lagrangian is now a function of the complex spatial vector field $A_{\mathtt{i}} = A_{\mathtt{i}}^{\mathit{1}} + iA_{\mathtt{i}}^{\mathit{2}}$ and reads

$$\mathcal{L} = -\frac{i}{2}\varepsilon^{\mathtt{ijk}}\big(\partial_o A_{\mathtt{i}} + N^{\mathtt{1}}F_{\mathtt{il}}\big)\bar{F}_{\mathtt{jk}} + \frac{i}{2}\varepsilon^{\mathtt{ijk}}\big(\partial_o \bar{A}_{\mathtt{i}} + N^{\mathtt{1}}\bar{F}_{\mathtt{il}}\big)F_{\mathtt{jk}} - N\sqrt{h}V[F] \ . \tag{4.1}$$

with the tree level 'potential'

$$V \equiv V^{(0)} = h^{\mathtt{ik}}h^{\mathtt{jl}}F_{\mathtt{ij}}\bar{F}_{\mathtt{kl}} = F^{ab}\bar{F}_{ab} \quad . \tag{4.2}$$

Here and in the remainder, we will mostly use flat indices

$$F_{ab} \equiv e_a^{\mathtt{i}}e_b^{\mathtt{j}}F_{\mathtt{ij}} \, , \tag{4.3}$$

where $e_a^{\mathtt{i}}$ is the inverse dreibein such that $h^{\mathtt{ij}} = \delta^{ab}e_a^{\mathtt{i}}e_b^{\mathtt{j}}$. Generalising beyond tree level, the potential V will be a more complicated function, but for any given V, the three vector transforms as

$$\delta A_{\mathtt{i}} = (\xi^{\mathtt{j}} + \xi^o N^{\mathtt{j}})F_{\mathtt{ji}} - i\xi^o \frac{N\sqrt{h}}{2}\varepsilon_{\mathtt{ijk}}\frac{\partial V}{\partial \bar{F}_{\mathtt{jk}}} \, , \tag{4.4}$$

In order to ensure full diffeomorphism invariance, V must satisfy the consistency condition (3.19) which now reads

$$\frac{\partial V}{\partial F^{a[b}}\frac{\partial V}{\partial \bar{F}^{cd]}} = \bar{F}_{a[b}F_{cd]} \ . \tag{4.5}$$

The general procedure then starts from some 'initial' corrected potential of the form $V = V^{(0)} + V^{(1)}$ and exploits (4.5) in order to complete the potential V to a more general $SO(3)$ invariant function of the spatial field strengths F_{ab} and $\bar{F}_{ab}$, such that

$$V = V^{(0)} + V^{(1)} + \ldots \tag{4.6}$$

satisfies the differential equation (3.19). As before we will thus find that, for consistency, any 'initial' counterterm $V^{(1)}$ must be supplemented by an infinite string of higher order corrections. As the simpest possible example we will consider the manifestly duality invariant expression $V^{(1)} \propto \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} F_{\dot{\alpha}\dot{\beta}} F^{\dot{\alpha}\dot{\beta}}$ obtained by squaring the complex selfdual and anti-selfdual field strengths. In the present approach this invariant can be identified with one half the duality invariant

$$Y \equiv F^{ab} F_{ab} \bar{F}^{cd} \bar{F}_{cd} \tag{4.7}$$

by using the equations of motion. Writing also

$$X \equiv F^{ab} \bar{F}_{ab} , \tag{4.8}$$

we would thus like to solve (4.5) for

$$V(X,Y) = F^{ab} \bar{F}_{ab} + \frac{1}{2} F^{ab} F_{ab} \bar{F}^{cd} \bar{F}_{cd} + \mathcal{O}(F^6) \equiv X + \frac{1}{2} Y + \mathcal{O}(F^6) \quad . \tag{4.9}$$

First of all we note that (4.5) is trivially satisfied at first order because

$$F_{a[b} F_{cd]} = 0 . \tag{4.10}$$

After some further computation it is seen that V must be of the form

$$V(X,Y) = X + \sum_{n=0}^{\infty} \frac{1}{(2+2n)!} H^{(n)}(X) Y^{1+n} \quad . \tag{4.11}$$

The condition (4.5) is satisfied provided (using $\bar{F}_{a[b} F_{cd]} = -F_{a[b} \bar{F}_{cd]}$)

$$\left(1 + \sum_{n=0}^{\infty} \frac{Y^{1+n}}{(2+2n)!} \frac{\partial H^{(n)}}{\partial X}\right)^2 = Y \left(\sum_{n=0}^{\infty} \frac{Y^n}{(1+2n)!} H^{(n)}\right)^2 + 1 \quad . \tag{4.12}$$

Observe that this ansatz is manifestly duality invariant. At first order in Y one gets

$$\frac{\partial H^{(0)}}{\partial X} = \left(H^{(0)}\right)^2 \quad , \tag{4.13}$$

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which implies (with the condition $H^{(0)}(0) = 1$) that

$$H^{(0)}(X) = \frac{1}{1-X} \quad . \tag{4.14}$$

At order Y^2 we get

$$\frac{\partial H^{(1)}}{\partial X} - \frac{4}{1-X}H^{(1)} = -\frac{3}{(1-X)^2} \quad , \tag{4.15}$$

which gives

$$H^{(1)}(X) = \frac{1}{1-X} + \frac{c^{(1)}}{(1-X)^4} \quad , \tag{4.16}$$

with an arbitrary constant $c^{(1)}$. This constant corresponds to the freedom of adding the on-shell invariant

$$\frac{c^{(1)}}{24}\left(F_{ab}F_{ab}\bar{F}_{cd}\bar{F}_{cd}\right)^2 \approx \frac{c^{(1)}}{24}\left(F_{\alpha\beta}F^{\alpha\beta}F_{\dot{\alpha}\dot{\beta}}F^{\dot{\alpha}\dot{\beta}}\right)^2 \quad , \tag{4.17}$$

to the invariant $F^2\bar{F}^2$, while preserving diffeomorphism invariance.

It is now clear how to proceed perturbatively in Y and how to determine all the functions $H^{(n)}$ by successively solving the hierarchy of first order equations [8]

$$\frac{\partial}{\partial X}\left((1-X)^{2+2n}H^{(n)}\right) =$$
$$\frac{1}{2}(1-X)^{2+2n}\left(\sum_{p=1}^{n-1}C_{2+2n}^{1+2p}H^{(p)}H^{(n-p)} - \sum_{p=0}^{n-1}C_{2+2n}^{2+2p}\frac{\partial H^{(p)}}{\partial X}\frac{\partial H^{(n-p-1)}}{\partial X}\right) \quad . \tag{4.18}$$

By construction the right-hand side is a finite Laurent series in $(1-X)$ with polynomial coefficients in $\ln(1-X)$ which can be integrated straightforwardly, modulo the definition of the homogenous solution

$$H^{(n)} = \frac{c^{(n)}}{(1-X)^{2+2n}} + \tilde{H}^{(n)} \quad , \tag{4.19}$$

$\tilde{H}^{(n)}$ being a particular solution. Clearly, the constants $c^{(n)}$ correspond to the ambiguities in defining a diffeormorphism invariant associated to the possibility of adding higher order invariants corresponding on-shell to

$$\frac{c^{(n)}}{(2n)!}\left(F_{\alpha\beta}F^{\alpha\beta}\right)^n\left(F_{\dot{\alpha}\dot{\beta}}F^{\dot{\alpha}\dot{\beta}}\right)^n \approx \frac{c^{(n)}}{(2n)!}\left(F_{ab}F_{ab}\right)^n\left(\bar{F}_{cd}\bar{F}_{cd}\right)^n \quad . \tag{4.20}$$

Setting $c^{(1)} = 5$, the potential V reads

$$V(X,Y) = X + \frac{1}{2}Y\left(1+X+X^2\right) + \frac{1}{4}Y^2 + \mathcal{O}(F^{10}) \quad . \tag{4.21}$$

---

[8]Where $C_n^p \equiv \frac{n!}{p!(n-p)!}$ are the binomial coefficients.

To establish the link of the above construction with the deformed twisted self-duality constraint (2.1), we note that, by construction, the equations of motion are invariant with respect to diffeomorphism invariance. Hence they can indeed be rewritten in this manifestly diffeomorphism covariant form (for $c^{(1)} = 5$ and higher $c^{(n)}$ chosen appropriately), *viz.*

$$F_{\mu\nu} - \frac{i}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}F_{\sigma\rho} + \frac{1}{8}\left(F_{\kappa\lambda}F^{\kappa\lambda} + \frac{i}{2\sqrt{-g}}\varepsilon^{\kappa\lambda\theta\tau}F_{\kappa\lambda}F_{\theta\tau}\right)\left(\bar{F}_{\mu\nu} - \frac{i}{2\sqrt{-g}}\varepsilon_{\mu\nu}{}^{\sigma\rho}\bar{F}_{\sigma\rho}\right) = 0 \ .$$
(4.22)

Indeed, decomposing the corresponding equations into space and time components

$$F_{oa} - \frac{i}{2}\varepsilon_{abc}F_{bc} + \frac{1}{8}\left(F_{de}F_{de} - 2F_{od}F_{od} - 2i\varepsilon_{def}F_{od}F_{ef}\right)\left(\bar{F}_{oa} - \frac{i}{2}\varepsilon_{abc}\bar{F}_{bc}\right) = 0 \ , \quad (4.23)$$

one can perturbatively solve for F_{oa} in terms of F_{ab} as

$$F_{oa} = \frac{i}{2}\varepsilon_{abc}F_{bc}\left(1 + \frac{1}{2}Y\left(1 + 2X\right)\right)$$
$$+ \frac{i}{2}\varepsilon_a{}^{bc}\bar{F}_{bc}F_{ef}F_{ef}\left(1 + X + X^2 + Y\right) + \mathcal{O}(F^{10}) \ , \quad (4.24)$$

This solution coincides with the expression following from the corrected potential V obtained above in (4.21) up to order F^{10}.

5 Conclusions

We have demonstrated for some typical examples by rather explicit computations that the higher order counterterms and corrections arising in supergravity and the effective string theory action are perfectly compatible with the full non-linear duality symmetries of these theories, provided one completes the 'initial' correction terms by solving the requisite consistency conditions. This can be done in either of two different formulations, in one of which space-time covariance is manifest but the duality symmetry is realised only on-shell, while it is the converse in the second formulation. We have exhibited the analogue of the Gaillard–Zumino constraint for the Henneaux–Teitelboim formulation, and we have furthermore shown that the two procedures give results which agree at lowest non-trivial orders in a perturbative expansion.

We conclude that the non-linear $E_{7(7)}$ symmetry is not suffcent to rule out all higher order counterterms, hence divergences, of $\mathcal{N} = 8$ supergravity. There is unfortunately no 'royal path' cutting short the explicit calculations of [18]. If $\mathcal{N} = 8$ supergravity is UV finite to all orders the reason must be sought beyond maximal supersymmetry and $E_{7(7)}$.

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