Multivariate Fine-Grained Complexity of Longest Common Subsequence*

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Abstract

We revisit the classic combinatorial pattern matching problem of finding a longest common subsequence (LCS). For strings x and y of length n, a textbook algorithm solves LCS in time $O(n^2)$, but although much effort has been spent, no $O(n^{2-\varepsilon})$ -time algorithm is known. Recent work indeed shows that such an algorithm would refute the Strong Exponential Time Hypothesis (SETH) [Abboud, Backurs, Vassilevska Williams FOCS'15; Bringmann, Künnemann FOCS'15].

Despite the quadratic-time barrier, for over 40 years an enduring scientific interest continued to produce fast algorithms for LCS and its variations. Particular attention was put into identifying and exploiting input parameters that yield strongly subquadratic time algorithms for special cases of interest, e.g., differential file comparison. This line of research was successfully pursued until 1990, at which time significant improvements came to a halt. In this paper, using the lens of fine-grained complexity, our goal is to (1) justify the lack of further improvements and (2) determine whether some special cases of LCS admit faster algorithms than currently known.

To this end, we provide a systematic study of the multivariate complexity of LCS, taking into account all parameters previously discussed in the literature: the input size $n := \max\{|x|, |y|\}$, the length of the shorter string $m := \min\{|x|, |y|\}$, the length L of an LCS of x and y, the numbers of deletions $\delta := m - L$ and $\Delta := n - L$, the alphabet size, as well as the numbers of matching pairs M and dominant pairs d. For any class of instances defined by fixing each parameter individually to a polynomial in terms of the input size, we prove a SETH-based lower bound matching one of three known algorithms (up to lower order factors of the form $n^{o(1)}$). Specifically, we determine the optimal running time for LCS under SETH as $(n + \min\{d, \delta\Delta, \delta m\})^{1\pm o(1)}$. Polynomial improvements over this running time must necessarily refute SETH or exploit novel input parameters. We establish the same lower bound for any constant alphabet of size at least 3. For binary alphabet, we show a SETH-based lower bound of $(n + \min\{d, \delta\Delta, \delta M/n\})^{1-o(1)}$ and, motivated by difficulties to improve this lower bound, we design an $\mathcal{O}(n + \delta M/n)$ -time algorithm, yielding again a matching bound.

We feel that our systematic approach yields a comprehensive perspective on the well-studied multivariate complexity of LCS, and we hope to inspire similar studies of multivariate complexity landscapes for further polynomial-time problems.

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1 Introduction

String comparison is one of the central tasks in combinatorial pattern matching, with various applications such as spelling correction [68, 84], DNA sequence comparison [8], and differential file comparison [46, 66]. Perhaps the best-known measure of string similarity is the length of the longest common subsequence (LCS). A textbook dynamic programming algorithm computes the LCS of given strings x, y of length n in time $\mathcal{O}(n^2)$, and in the worst case only an improvement by logarithmic factors is known [65]. In fact, recent results show that improvements by polynomial factors would refute the Strong Exponential Time Hypothesis (SETH) [1, 28] (see Section 2.2 for a definition).

Despite the quadratic-time barrier, the literature on LCS has been steadily growing, with a changing focus on different aspects of the problem over time (see Section 1.2 for an overview). Spurred by an interest in practical applications, a particular focus has been the design of LCS algorithms for strings that exhibit certain structural properties. This is most prominently witnessed by the UNIX diff utility, which quickly compares large, similar files by solving an underlying LCS problem. A practically satisfying solution to this special case was enabled by theoretical advances exploiting the fact that in such instances the LCS differs from the input strings at only few positions (e.g., [66, 70]). In fact, since Wagner and Fischer introduced the LCS problem in 1974 [84], identifying and exploiting structural parameters to obtain faster algorithms has been a decades-long effort [13, 14, 37, 45, 47, 50, 70, 71, 87].

Parameters that are studied in the literature are, besides the input size $n := \max\{|x|, |y|\}$, the length $m := \min\{|x|, |y|\}$ of the shorter string, the size of the alphabet Σ that x and y are defined on, the length L of a longest common subsequence of x and y, the number $\Delta = n - L$ of deleted symbols in the longer string, the number $\delta = m - L$ of deleted symbols in the shorter string, the number of matching pairs M, and the number of dominant pairs d (see Section 2.1 for definitions). Among the fastest currently known algorithms are an $\tilde{\mathcal{O}}(n+\delta L)$ -algorithm due to Hirschberg [45], an $\tilde{\mathcal{O}}(n+\delta \Delta)$ -algorithm due to Wu, Manbers, Myers, and Miller [87], and an $\tilde{\mathcal{O}}(n+d)$ -algorithm due to Apostolico [13] (with log-factor improvements by Eppstein, Galil, Giancarlo, and Italiano [37]). In the remainder, we refer to such algorithms, whose running time is stated in more parameters than just the problem size n, as multivariate algorithms. See Table 1 on page 3 for a non-exhaustive survey containing the asymptotically fastest multivariate LCS algorithms.

The main question we aim to answer in this work is: Are there significantly faster multivariate LCS algorithms than currently known? E.g., can ideas underlying the fastest known algorithms be combined to design an algorithm that is much faster than all of them?

1.1 Our Approach and Informal Results

We systematically study special cases of LCS that arise from polynomial restrictions of any of the previously studied input parameters. Informally, we define a parameter setting (or polynomial restriction of the parameters) as the subset of all LCS instances where each input parameter is individually bound to a polynomial relation with the input size n, i.e., for each parameter p we fix a constant α_p and restrict the instances such that p attains a value $\Theta(n^{\alpha_p})$. An algorithm for a specific parameter setting of LCS receives as input two strings x, y guaranteed to satisfy the parameter setting and outputs (the length of) an LCS of x and y. We call a parameter setting trivial if it is satisfied by at most a finite number of instances; this happens if the restrictions on different parameters are contradictory. For each non-trivial parameter setting, we construct a family of hard instances via a reduction from satisfiability, thus obtaining a conditional lower bound. This greatly extends the construction of hard instances for the $n^{2-o(1)}$ lower bound [1, 28].

Results for large alphabets. Since we only consider exact algorithms, any algorithm for LCS takes time $\Omega(n)$. Beyond this trivial bound, for any non-trivial parameter setting we obtain a SETH-based lower bound of

 $\min \{d, \delta \Delta, \delta m\}^{1-o(1)}.$

Note that this bound is matched by the known algorithms with running times $\tilde{\mathcal{O}}(n+d)$, $\tilde{\mathcal{O}}(n+\delta L)^1$, and $\tilde{\mathcal{O}}(n+\delta\Delta)$. Thus, our lower bound very well explains the lack of progress since the discovery of these three algorithms (apart from lower-order factors).

Results for constant alphabet size. For the alphabet size $|\Sigma|$, we do not only consider the case of a polynomial relation with n, but also the important special cases of $|\Sigma|$ being any fixed constant. We show that our conditional lower bound for polynomial alphabet size also holds for any constant $|\Sigma| \geq 3$. For $|\Sigma| = 2$, we instead obtain a SETH-based lower bound of

$$\min \left\{ d, \, \delta \Delta, \, \delta M/n \right\}^{1-o(1)}$$
.

This lower bound is weaker than the lower bound for $|\Sigma| \geq 3$ (as the term $\delta M/n$ is at most δm by the trivial bound $M \leq mn$; see Section 2.1 for the definition of M). Surprisingly, a stronger lower bound is impossible (assuming SETH): Motivated by the difficulties to obtain the same lower bound as for $|\Sigma| \geq 3$, we discovered an algorithm with running time $\mathcal{O}(n + \delta M/n)$ for $|\Sigma| = 2$, thus matching our conditional lower bound. To the best of our knowledge, this algorithm provides the first polynomial improvement for a special case of LCS since 1990, so while its practical relevance is unclear, we succeeded in uncovering a tractable special case. Interestingly, our algorithm and lower bounds show that the multivariate fine-grained complexity of LCS differs polynomially between $|\Sigma| = 2$ and $|\Sigma| \geq 3$. So far, the running time of the fastest known algorithms for varying alphabet size differed at most by a logarithmic factor in $|\Sigma|$.

We find it surprising that the hardness assumption SETH is not only sufficient to prove a worst-case quadratic lower bound for LCS, but extends to the *complete spectrum* of multivariate algorithms using the previously used 7 parameters, thus proving an optimal running time bound which was implicitly discovered by the computer science community within the first 25 years of research on LCS (except for the case of $\Sigma = \{0, 1\}$, for which we provide a missing algorithm).

1.2 Related Work on LCS

Table 1 on the next page gives a non-comprehensive overview of progress on multivariate LCS, including the asymptotically fastest known algorithms. Note that the most recent polynomial factor improvement for multivariate LCS was found in 1990 [87]. Further progress on multivariate LCS was confined to log-factor improvements (e.g., [37, 50]). Therefore, the majority of later works on LCS focused on transferring the early successes and techniques to more complicated problems, such as longest common increasing subsequence [33, 58, 67, 88], tree LCS [69], and many more generalizations and variants of LCS, see, e.g., [6, 7, 9, 10, 20, 21, 24, 32, 36, 38, 41, 43, 48, 49, 53–55, 57, 60–62, 74, 78, 79, 81, 85]. One branch of generalizations considered the LCS of more

¹Note that $L \leq m$. At first sight it might seem as if the $\tilde{\mathcal{O}}(n + \delta L)$ algorithm could be faster than our lower bound, however, for $L \geq m/2$ we have $\delta L = \Theta(\delta m)$, which appears in our lower bound, and for $L \leq m/2$ we have $\delta = m - L = \Theta(m)$ and thus $\delta L = \Theta(Lm)$ which is $\Omega(d)$ by Lemma 6.3, and d appears in our lower bound.

²See [23] for how to extend the Masek-Paterson algorithm to non-constant alphabets.

³Wu et al. state their running time as $\mathcal{O}(n\delta)$ in the worst case and $\mathcal{O}(n+\delta\Delta)$ in expectation for random strings. However, Myers worst-case variation trick [70, Section 4c] applies and yields the claimed time bound $\mathcal{O}(n\log n + \delta\Delta)$. The additional $\mathcal{O}(n\log n)$ comes from building a suffix tree.

Reference	Running Time	
Wagner and Fischer [84]	$\mathcal{O}(mn)$	
Hunt and Szymanski [47]	$\mathcal{O}((n+M)\log n)$	
Hirschberg [45]	$\mathcal{O}(n\log n + Ln)$	
Hirschberg [45]	$\mathcal{O}(n\log n + L\delta\log n)$	
Masek and Paterson [65]	$\mathcal{O}(n + nm/\log^2 n)$ assuming $ \Sigma = \mathcal{O}(1)$	
	$\mathcal{O}(n + nm \cdot (\frac{\log \log n}{\log n})^2)^{-2}$	
Nakatsu, Kambayashi and Yajima [71]	$\mathcal{O}(n\delta)$	
Apostolico [13]	$\mathcal{O}(n\log n + d\log(mn/d))$	
Myers [70]	$\mathcal{O}(n\log n + \Delta^2)$	
Apostolico and Guerra [14]	$\mathcal{O}(n\log n + Lm\min\{\log m, \log(n/m)\})$	
Wu, Manbers, Myers and Miller [87]	$\mathcal{O}(n\log n + \delta\Delta)^{3}$	
Eppstein, Galil, Giancarlo and Italiano [37]	$\mathcal{O}(n\log n + d\log\log\min\{d, nm/d\})$	
Iliopoulos and Rahman [50]	$\mathcal{O}(n + M \log \log n)$	

Table 1: Short survey of LCS algorithms. See Section 2.1 for definitions of the parameters. When stating the running times, every factor possibly attaining non-positive values (such as δ , $\log(n/m)$, etc.) is to be read as $\max\{\cdot,1\}$. For simplicity, $\log(\Sigma)$ -factors have been bounded from above by $\log n$ (see [72] for details on the case of constant alphabet size).

than two strings (e.g., [1, 25]), with variations such as string consensus (e.g., [11, 12]) and more (e.g., [9, 19, 35, 41, 42, 61]). Since natural language texts are well compressible, researchers also considered solving LCS directly on compressed strings, using either run-length encoding (e.g., [15, 31, 34, 57]) or straight-line programs and other Lempel-Ziv-like compression schemes (e.g., [40, 44, 63, 80]). Further research directions include approximation algorithms for LCS and its variants (e.g., [42, 43, 59]), as well as the LCS length of random strings [18, 64]. For brevity, here we ignore the equally vast literature on the closely related edit distance. Furthermore, we solely regard the time complexity of computing the length of an LCS and hence omit all results concerning space usage or finding an LCS. See, e.g., [22, 72] for these and other aspects of LCS (including empirical evaluations).

1.3 (Multivariate) Hardness in P

After the early success of 3SUM-hardness in computational geometry [39], recent years have brought a wealth of novel conditional lower bounds for polynomial time problems, see, e.g., [1–4, 16, 17, 26–29, 56, 75, 83, 86] and the recent survey [82]. In particular, our work extends the recent successful line of research proving SETH-based lower bounds for a number problems with efficient dynamic programming solutions such as Fréchet distance [26, 29], edit distance [16, 28], LCS and dynamic time warping [1, 28]. Beyond worst-case conditional lower bounds of the form $n^{c-o(1)}$, recently also more detailed lower bounds targeting additional input restrictions have gained interest. Such results come in different flavors, as follows.

Input parameters, polynomial dependence. Consider one or more parameters in addition to the input size n, where the optimal time complexity of the studied problem depends polynomially on n and the parameters. This is the situation in this paper as well as several previous studies, e.g., [4, 26, 56]. To the best of our knowledge, our work is the first of this kind to study combinations of more than two parameters that adhere to a complex set of parameter relations – for previous results, typically the set of non-trivial parameter settings was obvious and simultaneously controlling all

parameters was less complex.

Input parameters, superpolynomial dependence. Related to the above setting, parameters have been studied where the time complexity depends polynomially on n and superpolynomially on the parameters. If the studied problem is NP-hard then this is known as fixed-parameter tractability (FPT). However, here we focus on problems in P, in which case this situation is known as "FPT in P". Hardness results in this area were initiated by Abboud, Vassilevska Williams, and Wang [4].

A finite/discrete number of special cases. Some input restrictions yield a discrete or even finite set of special cases. For example, Backurs and Indyk [17] and later Bringmann et al. [27] studied special cases of regular expression pattern matching by restricting the input to certain "types" of regular expressions. The set of types is discrete and infinite, however, there are only finitely many tractable types, and finitely many minimal hardness results. Their approach is similarly systematic to ours, as they classify the complexity of pattern matching for any type of regular expressions. The major difference is that our parameters are "continuous", specifically our parameter exponents α_p are continuous, and thus our algorithms and lower bounds trace a continuous tradeoff.

While in all of the above settings the design of fast multivariate algorithms is well established, tools for proving matching conditional lower bounds have been developed only recently. In particular, the systematic approach to multivariate lower bounds pursued in this paper provides an effective complement to multivariate algorithmic studies in P, since it establishes (near-)optimality and may uncover tractable special cases for which improved algorithms can be found.

Beyond SETH. Motivated in part to find barriers even for polylogarithmic improvements on LCS, a surprising result of Abboud et al. [2] strengthens the conditional quadratic-time hardness of LCS substantially. More precisely, they show that a strongly subquadratic-time algorithm for LCS would even refute a natural, weaker variant of SETH on *branching programs*. In Section 12, we survey their result and show that the conditional lower bounds we derive in this paper also hold under this weaker assumption.

2 Preliminaries

We write $[n] := \{1, \ldots, n\}$. For a string x, we denote its length by |x|, the symbol at its i-th position by x[i], and the substring from position i to position j by x[i...j]. If string x is defined over alphabet Σ , we denote the number of occurrences of symbol $\sigma \in \Sigma$ in x by $\#_{\sigma}(x)$. In running time bounds we write Σ instead of $|\Sigma|$ for readability. For two strings x, y, we denote their concatenation by $x \circ y = xy$ and define, for any $\ell \geq 0$, the ℓ -fold repetition $x^{\ell} := \bigcap_{i=1}^{\ell} x$. For any strings x, y we let LCS(x, y) be any longest common subsequence of x and y, i.e., a string z = z[1..L] of maximum length L such that there are $i_1 < \ldots < i_L$ with $x[i_k] = z[k]$ for all $1 \leq k \leq L$ and there are $j_1 < \ldots < j_L$ with $y[j_k] = z[k]$ for all $1 \leq k \leq L$. For a string x of length n, let $rev(x) := x[n]x[n-1]\ldots x[1]$ denote its reverse.

2.1 Parameter Definitions

We survey parameters that have been used in the analysis of the LCS problem (see also [22, 72]). Let x, y be any strings. By possibly swapping x and y, we can assume that x is the longer of the two strings, so that n = n(x, y) := |x| is the input size (up to a factor of two). Then m = m(x, y) := |y| is the length of the shorter of the two strings. Another natural parameter is the solution size, i.e., the length of any LCS, L = L(x, y) := |LCS(x, y)|.

Since any symbol not contained in x or in y cannot be contained in a LCS, we can ensure the following using a (near-)linear-time preprocessing.

```
n(x,y) := |x|, \quad m(x,y) := |y|
      dabccbd
                                       L(x,y) := |LCS(x,y)|
     1 1 1 1 1 1 1
d
                                       \delta(x, y) := |y| - L(x, y), \quad \Delta(x, y) := |x| - L(x, y)
\mathbf{c}
                                      \Sigma(x,y) := \#(\{x[i] \mid 1 \le i \le |x|\} \cap \{y[j] \mid 1 \le j \le |y|\})
     1 1 2 2 2 3 3 3 1 2 2 2 2 3 4 1 2 2 3 3 3 4
b
                                      M(x,y) := \#\{(i,j) \mid x[i] = y[j]\}
\mathbf{a}
\mathrm{d}
                                       d(x,y) := \#\{(i,j) \mid L[i,j] > L[i-1,j] \text{ and } L[i,j] > L[i,j-1]\},
                                                     where L[i, j] := |LCS(x[1..i], y[1..j])|.
             (a)
                                                                               (b)
```

Figure 1: (a) Illustration of the L-table, matching pairs and dominant pairs. Entries marked in orange color and bold letters correspond to dominant pairs (which by definition are also matching pairs), while entries marked in blue are matching pairs only. (b) Summary of all input parameters.

Assumption 2.1. Every symbol $\sigma \in \Sigma$ occurs at least once in x and in y, i.e., $\#_{\sigma}(x), \#_{\sigma}(y) \geq 1$.

Consider the alphabet induced by x and y after ensuring Assumption 2.1, namely $\Sigma = \{x[i] \mid 1 \le i \le |x|\} \cap \{y[j] \mid 1 \le j \le |y|\}$. Its size $\Sigma(x,y) := |\Sigma|$ is a natural parameter.

Beyond these standard parameters $n, m, L, |\Sigma|$ (applicable for any optimization problem on strings), popular structural parameters measure the *similarity* and *sparsity* of the strings. These notions are more specific to LCS and are especially relevant in practical applications such as, e.g., the diff file comparison utility, where symbols in x and y correspond to lines in the input files.

Notions of similarity. To obtain an LCS, we have to delete $\Delta = \Delta(x, y) := n - L$ symbols from x or $\delta = \delta(x, y) := m - L$ symbols from y. Hence for very similar strings, which is the typical kind of input for file comparisons, we expect δ and Δ to be small. This is exploited by algorithms running in time, e.g., $\tilde{\mathcal{O}}(n + \delta \Delta)$ [87] or $\tilde{\mathcal{O}}(n + \delta L)$ [45].

Notions of sparsity. Based on the observation that the dynamic programming table typically stores a large amount of redundant information (suggested, e.g., by the fact that an LCS itself can be reconstructed examining only O(n) entries), algorithms have been studied that consider only the most relevant entries in the table. The simplest measure of such entries is the number of matching pairs $M = M(x, y) := \#\{(i, j) \mid x[i] = y[j]\}$. Especially for inputs with a large alphabet, this parameter potentially significantly restricts the number of candidate pairs considered by LCS algorithms, e.g., for files where almost all lines occur only once. Moreover, in the special case where x and y are permutations of Σ we have M = n = m, and thus algorithms in time $\tilde{\mathcal{O}}(n+M)$ [46, 47, 50] recover the near-linear time solution for LCS of permutations [77].

One can refine this notion to obtain the dominant pairs. A pair (i,j) dominates a pair (i',j') if we have $i \leq i'$ and $j \leq j'$. A k-dominant pair is a pair (i,j) such that L(x[1..i],y[1..j]) = k and no other pair (i',j') with L(x[1..i'],y[1..j']) = k dominates (i,j). By defining L[i,j] := L(x[1..i],y[1..j]) and using the well known recursion $L[i,j] = \max\{L[i-1,j],L[i,j-1],L[i-1,j-1]+\mathbf{1}_{x[i]=y[j]}\}$, we observe that (i,j) is a k-dominant pair if and only if L[i,j] = k and L[i-1,j] = L[i,j-1] = k-1. Denoting the set of all k-dominant pairs by D_k , the set of dominant pairs of x,y is $\bigcup_{k\geq 1} D_k$, and we let d = d(x,y) denote the number of dominant pairs. Algorithms running in time $\tilde{\mathcal{O}}(n+d)$

exploit a small number of dominant pairs [13, 37]. Figure 1a illustrates matching and dominant pairs.

While at first sight the definition of dominant pairs might not seem like the most natural parameter, it plays an important role in analyzing LCS: First, from the set of dominant pairs alone one can reconstruct the L-table that underlies the basic dynamic programming algorithm. Second, the parameter d precisely describes the complexity of one of the fastest known (multivariate) algorithms for LCS. Finally, LCS with parameter d is one of the first instances of the paradigm of sparse dynamic programming (see, e.g., [37]).

On practical instances, exploiting similarity notions seems to typically outperform algorithms based on sparsity measures (see [66] for a classical comparison to an algorithm based on the number of matching pairs M [46, 47]). To the best of our knowledge, Figure 1b summarizes all parameters which have been exploited to obtain multivariate algorithms for LCS.

2.2 Hardness Hypotheses

Strong Exponential Time Hypothesis (SETH): For any $\varepsilon > 0$ there is a $k \geq 3$ such that k-SAT on n variables cannot be solved in time $\mathcal{O}((2-\varepsilon)^n)$.

SETH was introduced by Impagliazzo, Paturi, and Zane [51] and essentially asserts that satisfiability has no algorithms that are much faster than exhaustive search. It forms the basis of many conditional lower bounds for NP-hard as well as polynomial-time problems.

Effectively all known SETH-based lower bounds for polynomial-time problems use reductions via the Orthogonal Vectors problem (OV): Given sets \mathcal{A} , $\mathcal{B} \subseteq \{0,1\}^D$ of size $|\mathcal{A}| = n$, $|\mathcal{B}| = m$, determine whether there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$ with $\sum_{i=1}^{D} a[i] \cdot b[i] = 0$ (which we denote by $\langle a,b \rangle = 0$). Simple algorithms solve OV in time $\mathcal{O}(2^D(n+m))$ and $\mathcal{O}(nmD)$. The fastest known algorithm for $D = c(n) \log n$ runs in time $n^{2-1/\mathcal{O}(\log c(n))}$ (when n = m) [5], which is only slightly subquadratic for $D \gg \log n$. This has led to the following reasonable hypothesis.

Orthogonal Vectors Hypothesis (OVH): OV restricted to $n = |\mathcal{A}| = |\mathcal{B}|$ and $D = n^{o(1)}$ requires time $n^{2-o(1)}$.

A well-known reduction by Williams [86] shows that SETH implies OVH. Thus, OVH is the weaker assumption and any OVH-based lower bound also implies a SETH-based lower bound. The results in this paper do not only hold assuming SETH, but even assuming the weaker OVH.

3 Formal Statement of Results

Recall that n is the input size and $\mathcal{P} := \{m, L, \delta, \Delta, |\Sigma|, M, d\}$ is the set of parameters that were previously studied in the literature. We let $\mathcal{P}^* := \mathcal{P} \cup \{n\}$. A parameter setting fixes a polynomial relation between any parameter and n. To formalize this, we call a vector $\boldsymbol{\alpha} = (\alpha_p)_{p \in \mathcal{P}}$ with $\alpha_p \in \mathbb{R}_{\geq 0}$ a parameter setting, and an LCS instance x, y satisfies the parameter setting $\boldsymbol{\alpha}$ if each parameter p attains a value $p(x, y) = \Theta(n^{\alpha_p})$. This yields a subproblem of LCS consisting of all instances that satisfy the parameter setting. We sometimes use the notation $\alpha_n = 1$.

For our running time bounds, for each parameter $p \in \mathcal{P}$ except for $|\Sigma|$ we can assume $\alpha_p > 0$, since otherwise one of the known algorithms runs in time $\tilde{\mathcal{O}}(n)$ and there is nothing to show. Similarly, for $\alpha_d \leq 1$ there is an $\tilde{\mathcal{O}}(n)$ algorithm and there is nothing to show. For Σ , however, the case $\alpha_{\Sigma} = 0$, i.e., $|\Sigma| = \Theta(1)$, is an important special case. We study this case more closely by also considering parameter settings that fix $|\Sigma|$ to any specific constant greater than 1.

Parameter	Restriction		
m	$0 \le \alpha_m \le 1$		
L	$0 \le \alpha_L \le \alpha_m$		
δ	$\begin{cases} 0 \le \alpha_{\delta} \le \alpha_m & \text{if } \alpha_L = \alpha_m \\ \alpha_{\delta} = \alpha_m & \text{otherwise} \end{cases}$		
	$\alpha_{\delta} = \alpha_m$ otherwise		
Λ	$\int \alpha_{\delta} \le \alpha_{\Delta} \le 1 \text{if } \alpha_{L} = \alpha_{m} = 1$		
<u> </u>	$\alpha_{\Delta} = 1$ otherwise		
$ \Sigma $	$0 \le \alpha_{\Sigma} \le \alpha_m$		
d	$\max\{\alpha_L, \alpha_\Sigma\} \le \alpha_d \le \min\{2\alpha_L + \alpha_\Sigma, \alpha_L + \alpha_m, \alpha_L + \alpha_\Delta\}$		
M	$\max\{1, \alpha_d, 2\alpha_L - \alpha_{\Sigma}\} \le \alpha_M \le \alpha_L + 1$ if $ \Sigma = 2$: $\alpha_M \ge \max\{\alpha_L + \alpha_m, 1 + \alpha_d - \alpha_L\}$ if $ \Sigma = 3$: $\alpha_M \ge \alpha_m + \alpha_d - \alpha_L$		

Table 2: Complete set of restrictions for non-trivial parameter settings.

Definition 3.1 (Parameter Setting). Fix $\gamma \geq 1$. Let $\alpha = (\alpha_p)_{p \in \mathcal{P}}$ with $\alpha_p \in \mathbb{R}_{\geq 0}$. We define $LCS^{\gamma}(\alpha)$ as the problem of computing the length of an LCS of two given strings x, y satisfying $n^{\alpha_p}/\gamma \leq p(x,y) \leq n^{\alpha_p} \cdot \gamma$ for every parameter $p \in \mathcal{P}$, where n = |x|, and $|x| \geq |y|$. We call α and $LCS^{\gamma}(\alpha)$ parameter settings. In some statements we simply write $LCS(\alpha)$ to abbreviate that there exists a $\gamma \geq 1$ such that the statement holds for $LCS^{\gamma}(\alpha)$.

For any fixed alphabet Σ , constant $\gamma \geq 1$, and parameter setting α with $\alpha_{\Sigma} = 0$, we also define the problem $LCS^{\gamma}(\alpha, \Sigma)$, where additionally the alphabet of x, y is fixed to be Σ . We again call (α, Σ) and $LCS^{\gamma}(\alpha, \Sigma)$ parameter settings.

We call a parameter setting α or (α, Σ) trivial if for all $\gamma \geq 1$ the problem $LCS^{\gamma}(\alpha)$ or $LCS^{\gamma}(\alpha, \Sigma)$, respectively, has only finitely many instances.

As our goal is to prove hardness for any non-trivial parameter setting, for each parameter setting we either need to construct hard instances or verify that it is trivial. That is, in one way or the other we need a complete *classification* of parameter settings into trivial and non-trivial ones. To this end, we need to understand all interactions among our parameters that hold up to constant factors, which is an interesting question on its own, as it yields insight into the structure of strings from the perspective of the LCS problem. For our seven parameters, determining all interactions is a complex task. This is one of the major differences to previous multivariate fine-grained complexity results, where the number of parameters was one, or in rare cases two, limiting the interaction among parameters to a simple level.

Theorem 3.2 (Classification of non-trivial parameter settings). A parameter setting α or (α, Σ) is non-trivial if and only if it satisfies all restrictions in Table 2.

Note that the restrictions in Table 2 consist mostly of linear inequalities, and that for small alphabet sizes $|\Sigma| \in \{2,3\}$ additional parameter relations hold. The proof of this and the following results will be outlined in Section 4. We are now ready to state our main lower bound.

Theorem 3.3 (Hardness for Large Alphabet). For any non-trivial parameter setting α , there is a constant $\gamma \geq 1$ such that $LCS^{\gamma}(\alpha)$ requires time min $\{d, \delta\Delta, \delta m\}^{1-o(1)}$, unless OVH fails.

In the case of constant alphabet size, the (conditional) complexity differs between $|\Sigma| = 2$ and $|\Sigma| \geq 3$. Note that $|\Sigma| = 1$ makes LCS trivial.

Relation	Restriction	Reference
$L \le m \le n$		trivial
$L \le d \le M$		trivial
$\Delta \le n$		trivial
$\delta \leq m$		trivial
$\delta \leq \Delta$		trivial
$\delta = m - L$		by definition
$\Delta = n - L$		by definition
$ \Sigma \le m$		Assumption 2.1
$n \leq M$		Assumption 2.1
$d \le Lm$		Lemma 6.3
$d \le L^2 \Sigma $		Lemma 6.3
$d \le 2L(\Delta + 1)$		Lemma 6.4
$ \Sigma \le d$		Lemma 6.5
$\frac{L^2}{ \Sigma } \le M \le 2Ln$		Lemma 6.6
$M \ge Lm/4$	if $ \Sigma = 2$	Lemma 6.7
$M \ge nd/(5L)$	if $ \Sigma =2$	Lemma 6.9
$M \ge md/(80L)$	if $ \Sigma = 3$	Lemma 6.10

Table 3: Relations between the parameters.

Theorem 3.4 (Hardness for Small Alphabet). For any non-trivial parameter setting (α, Σ) , there is a constant $\gamma \geq 1$ such that, unless OVH fails, LCS $^{\gamma}(\alpha, \Sigma)$ requires time

- $\min \{d, \delta \Delta, \delta m\}^{1-o(1)} \text{ if } |\Sigma| \ge 3,$
- $\min \{d, \delta \Delta, \delta M/n\}^{1-o(1)}$ if $|\Sigma| = 2$.

Finally, we prove the following algorithmic result, handling binary alphabets faster if M and δ are sufficiently small. This yields matching upper and lower bounds also for $|\Sigma| = 2$.

Theorem 3.5 (Section 11). For $|\Sigma| = 2$, LCS can be solved in time $\mathcal{O}(n + \delta M/n)$.

4 Hardness Proof Overview

In this section we present an overview of the proofs of our main results. We first focus on the large alphabet case, i.e., parameter settings α , and discuss small constant alphabets in Section 4.4.

4.1 Classification of Non-trivial Parameter Settings

The only-if-direction of Theorem 3.2 follows from proving inequalities among the parameters that hold for all strings, and then converting them to inequalities among the α_p 's, as follows.

Lemma 4.1 (Parameter Relations, Section 6). For any strings x, y the parameter values \mathcal{P}^* satisfy the relations in Table 3. Thus, any non-trivial parameter setting α or (α, Σ) satisfies Table 2.

Proof Sketch. The full proof is deferred to Section 6. Some parameter relations follow trivially from the parameter definitions, like $L \leq m \leq n$. Since by Assumption 2.1 every symbol in Σ appears in

x and y, we obtain parameter relations like $|\Sigma| \leq m$. Other parameter relations need a non-trivial proof, like $M \geq md/(80L)$ if $|\Sigma| = 3$.

From a relation like $L \leq m$ we infer that if $\alpha_L > \alpha_m$ then for sufficiently large n no strings x, y have $L(x, y) = \Theta(n^{\alpha_L})$ and $m(x, y) = \Theta(n^{\alpha_m})$, and thus $LCS^{\gamma}(\alpha)$ is finite for any $\gamma > 0$. This argument converts Table 3 to Table 2.

For the if-direction of Theorem 3.2, the task is to show that any parameter setting satisfying Table 2 is non-trivial, i.e., to construct infinitely many strings in the parameter setting. We start with a construction that sets a single parameter p as specified by α , and all others not too large.

Lemma 4.2 (Paddings, Section 8). Let α be a parameter setting satisfying Table 2. For any parameter $p \in \mathcal{P}^*$ and any $n \geq 1$ we can construct strings x_p, y_p such that (1) $p(x_p, y_p) = \Theta(n^{\alpha_p})$, and (2) for all $q \in \mathcal{P}^*$ we have $q(x_p, y_p) = O(n^{\alpha_q})$. Moreover, given n we can compute x_p, y_p , and $L(x_p, y_p)$ in time $\mathcal{O}(n)$.

Note that although for Theorem 3.2 the *existence* of infinitely many strings would suffice, we even show that they can be *computed* very efficiently. We will use this additional fact in Section 4.2.

Proof Sketch. We defer the full proof to Section 8 and here only sketch the proof for the parameter $|\Sigma|$. Let $w:=12\dots t$ be the concatenation of $t:=\lceil n^{\alpha_{\Sigma}}\rceil$ unique symbols. We argue that the strings w,w or the strings w, rev(w) prove Lemma 4.2 for parameter $p=|\Sigma|$, depending on the parameter setting α . Clearly, both pairs of strings realize an alphabet of size $t=\Theta(n^{\alpha_{\Sigma}})$, showing (1). By Table 2, we have $\alpha_L=\alpha_m$ or $\alpha_\delta=\alpha_m$. In the first case, we use $L(w,w)=t=O(n^{\alpha_{\Sigma}})$ together with $\alpha_{\Sigma}\leq\alpha_m=\alpha_L$, as well as $\delta(w,w)=\Delta(w,w)=0\leq n^{\alpha_\delta}\leq n^{\alpha_\Delta}$, to show (2) for the parameters L,δ,Δ . In the second case, we similarly have $L(w,\text{rev}(w))=1\leq n^{\alpha_L}$ and $\delta(w,\text{rev}(w))=\Delta(w,\text{rev}(w))=t-1=O(n^{\alpha_{\Sigma}})$ and $\alpha_{\Sigma}\leq\alpha_m=\alpha_{\delta}\leq\alpha_{\Delta}$.

The remaining parameters are straight-forward. Let $(x,y) \in \{(w,w),(w,\operatorname{rev}(w))\}$. We have $n(x,y) = m(x,y) = t = O(n^{\alpha_{\Sigma}}) = O(n^{\alpha_m}) = O(n)$. Moreover, $d(x,y) \leq M(x,y) = t = O(n^{\alpha_{\Sigma}}) = O(n^{\alpha_d}) = O(n^{\alpha_d})$. Clearly, the strings and their LCS length can be computed in time $\mathcal{O}(n)$.

To combine the paddings for different parameters, we need the useful property that all studied parameters sum up if we concatenate strings over disjoint alphabets.

Lemma 4.3 (Disjoint Alphabets). Let $\Sigma_1, \ldots, \Sigma_k$ be disjoint alphabets and let x_i, y_i be strings over alphabet Σ_i with $|x_i| \geq |y_i|$ for all i. Consider $x := x_1 \ldots x_k$ and $y := y_1 \ldots y_k$. Then for any parameter $p \in \mathcal{P}^*$, we have $p(x, y) = \sum_{i=1}^k p(x_i, y_i)$.

Proof. The statement is trivial for the string lengths n, m, alphabet size $|\Sigma|$, and number of matching pairs M. For the LCS length L we observe that any common subsequence z can be decomposed into $z_1 \ldots z_k$ with z_i using only symbols from Σ_i , so that $|z_i| \leq L(x_i, y_i)$ and thus $L(x, y) \leq \sum_{i=1}^k L(x_i, y_i)$. Concatenating longest common subsequences of x_i, y_i , we obtain equality. Using $\delta = m - L$ and $\Delta = n - L$, the claim follows also for δ and Δ .

Since every dominant pair is also a matching pair, every dominant pair of x, y stems from prefixes $x_1
ldots x_j x'$ and $y_1
ldots y_j y'$, with x' being a prefix of x_{j+1} and y' being a prefix of y_{j+1} for some j. Since $L(x_1
ldots x_j x', y_1
ldots y_j y') = <math>\sum_{i=1}^j L(x_i, y_i) + L(x', y')$, where the first summand does not depend on x', y', the dominant pairs of x, y of the form $x_1
ldots x_j x', y_1
ldots y_j y'$ are in one-to-one correspondence with the dominant pairs of x_{j+1}, y_{j+1} . This yields the claim for parameter d. \square

With these preparations we can finish our classification.

Proof of Theorem 3.2 for large alphabet. One direction follows from Lemma 4.1. For the other direction, let α be a parameter setting satisfying Table 2. For any $n \geq 1$ consider the instances x_p, y_p constructed in Lemma 4.2, and let them use disjoint alphabets for different $p \in \mathcal{P}^*$. Then the concatenations $x := \bigcirc_{p \in \mathcal{P}^*} x_p$ and $y := \bigcirc_{p \in \mathcal{P}^*} x_p$ form an instance of LCS(α), since for any parameter $p \in \mathcal{P}^*$ we have $p(x_p, y_p) = \Theta(n^{\alpha_p})$, and for all other instances $x_{p'}, y_{p'}$ the parameter p is $\mathcal{O}(n^{\alpha_p})$, and thus $p(x, y) = \Theta(n^{\alpha_p})$ by the Disjoint Alphabets Lemma. Thus, we constructed instances of LCS(α) of size $\Theta(n)$ for any $n \geq 1$, so the parameter setting α is non-trivial. \square

We highlight two major hurdles we had to be overcome to obtain this classification result:

- Some of the parameter relations of Table 3 are scattered through the LCS literature, e.g., the inequality $d \le Lm$ is mentioned in [14]. In fact, proving any single one of these inequalities is not very hard the main issue was to find a *complete* set of parameter relations. The authors had to perform many iterations of going back and forth between searching for new parameter relations (i.e., extending Lemma 4.1) and constructing strings satisfying specific parameter relations (i.e., extending Lemma 4.2), until finally coming up with a complete list.
- The dependency of d on the other parameters is quite complicated. Indeed, eight of the parameter relations of Table 3 involve dominant pairs. Apostolico [13] introduced the parameter under the initial impression that "it seems that whenever [M] gets too close to mn, then this forces d to be linear in m". While we show that this intuition is somewhat misleading by constructing instances with high values of both M and d, it is a rather complex task to generate a desired number of dominant pairs while respecting given bounds on all other parameters. Intuitively, handling dominant pairs is hard since they involve restrictions on each pair of prefixes of x and y. For Lemma 4.2, we end up using the strings $(01)^{R+S}$, $0^R(01)^S$ as well as $((1 \circ \ldots \circ t) \circ (t' \circ \ldots \circ 1))^R (1 \circ \ldots \circ t)^{S-R}$, $(1 \circ \ldots \circ t)^S$ for different values of R, S, t, t'.

4.2 Monotonicity of Time Complexity

It might be tempting to assume that the optimal running time for solving LCS is monotone in the problem size n and the parameters \mathcal{P} (say up to constant factors, as long as all considered parameters settings are non-trivial). However, since the parameters have complex interactions (see Table 3) it is far from obvious whether this intuition is correct. In fact, the intuition fails for $|\Sigma| = 2$, where the running time $\mathcal{O}(n + \delta M/n)$ of our new algorithm is not monotone, and thus also the tight time bound $(n + \min\{d, \delta \Delta, \delta M/n\})^{1 \pm o(1)}$ is not monotone.

Nevertheless, we will prove monotonicity for any parameter setting α , i.e., when the alphabet size can be assumed to be at least a sufficiently large constant. To formalize monotonicity, we define problem $LCS_{\leq}(\alpha)$ consisting of all instances of LCS with all parameters at most as in $LCS(\alpha)$.

Definition 4.4 (Downward Closure of Parameter Setting). Fix $\gamma \geq 1$ and let α be a parameter setting. We define the downward closure $LCS_{\leq}^{\gamma}(\alpha)$ as follows. An instance of this problem is a triple (n, x, y), where $p(x, y) \leq \gamma \cdot n^{\alpha_p}$ for any $p \in \mathcal{P}^*$, and the task is to compute the length of an LCS of x, y. In some statements, we simply write $LCS_{\leq}(\alpha)$ to abbreviate that there exists a $\gamma \geq 1$ such that the statement holds for $LCS_{\leq}^{\gamma}(\alpha)$.

Similarly, for any fixed alphabet Σ we consider the downward closure $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}, \Sigma)$ with instances (n, x, y), where x, y are strings over alphabet Σ and $p(x, y) \leq \gamma \cdot n^{\alpha_p}$ for any $p \in \mathcal{P}^*$.

Lemma 4.5 (Monotonicity). For any non-trivial parameter setting α and $\beta \geq 1$, $LCS^{\gamma}(\alpha)$ has an $\mathcal{O}(n^{\beta})$ -time algorithm for all γ if and only if $LCS^{\gamma}(\alpha)$ has an $\mathcal{O}(n^{\beta})$ -time algorithm for all γ .

Proof of Lemma 4.5. The if-direction follows from the fact that if (x, y) is an instance of LCS $^{\gamma}(\alpha)$ then (|x|, x, y) is an instance of LCS $^{\gamma}(\alpha)$.

For the other direction, let (n, x, y) be an instance of $LCS_{\leq}(\alpha)$. Since α is non-trivial, it satisfies Table 2, by Theorem 3.2. Lemma 4.2 thus allows us to construct paddings x_p, y_p for any $p \in \mathcal{P}^*$ such that (1) $p(x_p, y_p) = \Theta(n^{\alpha_p})$ and (2) (n, x_p, y_p) is an instance of $LCS_{\leq}(\alpha)$. We construct these paddings over disjoint alphabets for different parameters and consider the concatenations $x' := x \circ \bigcirc_{p \in \mathcal{P}} x_p$ and $y' := y \circ \bigcirc_{p \in \mathcal{P}} y_p$. Then (1), (2), and the Disjoint Alphabets Lemma imply that $p(x', y') = \Theta(n^{\alpha_p})$ for any $p \in \mathcal{P}^*$, so that (x', y') is an instance of $LCS(\alpha)$. By assumption, we can thus compute L(x', y') in time $O(n^{\beta})$. By the Disjoint Alphabets Lemma, we have $L(x, y) = L(x', y') - \sum_{p \in \mathcal{P}} L(x_p, y_p)$, and each $L(x_p, y_p)$ can be computed in time O(n) by Lemma 4.2, which yields L(x, y) and thus solves the given instance (n, x, y). We finish the proof by observing that the time to construct x', y' is bounded by $\mathcal{O}(n)$.

Note that this proof was surprisingly simple, considering that monotonicity fails for $|\Sigma|=2$.

4.3 Hardness for Large Alphabet

Since we established monotonicity for parameter settings α , it suffices to prove hardness for $LCS_{\leq}(\alpha)$ instead of $LCS(\alpha)$. This makes the task of constructing hard strings considerably easier, since we only have to satisfy upper bounds on the parameters. Note that our main result Theorem 3.3 follows from Lemma 4.5 and Theorem 4.6 below.

Theorem 4.6 (Hardness for Large Alphabet, Section 9). For any non-trivial parameter setting α , there exists $\gamma \geq 1$ such that $LCS_{\leq}^{\gamma}(\alpha)$ requires time min $\{d, \delta m, \delta \Delta\}^{1-o(1)}$, unless OVH fails.

Proof Sketch. The full proof is deferred to Section 9. We provide different reductions for the cases $\alpha_{\delta} = \alpha_{m}$ and $\alpha_{L} = \alpha_{m}$. Intuitively, this case distinction is natural, since after this choice all remaining restrictions from Table 2 are of an easy form: they are linear inequalities.

In the case $\alpha_{\delta} = \alpha_{m}$ the complexity $\min\{d, \delta\Delta, \delta m\}^{1\pm o(1)}$ simplifies to $d^{1\pm o(1)}$ (since $\alpha_{d} \leq \alpha_{L} + \alpha_{m} \leq 2\alpha_{m} = \alpha_{\delta} + \alpha_{m}$ and similarly $\alpha_{d} \leq \alpha_{\delta} + \alpha_{m} = 2\alpha_{\delta} \leq \alpha_{\delta} + \alpha_{\Delta}$, see Table 2). This simplification makes this case much easier. For constant alphabet, instantiating the known reduction from OV to LCS [28] such that x chooses one of $\approx L$ vectors and y chooses one of $\approx d/L$ vectors yields the claim. For larger alphabet, the right-hand side of the parameter relation $d \leq L^{2}|\Sigma|$ increases and allows for potentially more dominant pairs. In this case, the second set of vectors would increase to a size of $\approx d/L = \omega(L)$, and the length of an LCS of this construction becomes too large. We thus adapt the reduction by using the construction for constant alphabet multiple times over disjoint alphabets and concatenating the results (reversing the order in one).

The case $\alpha_L = \alpha_m$ is harder, since all three terms of the complexity $\min\{d, \delta\Delta, \delta m\}^{1\pm o(1)}$ are relevant. The known reduction [28] fails fundamentally in this case, roughly speaking since the resulting δ is always as large as the number of vectors encoded by any of x and y. Hence, we go back to the "normalized vector gadgets" from the known reduction [28], which encode vectors a, b by strings NVG(a), NVG(b) whose LCS length only depends on whether a, b are orthognal. We then carefully embed these gadgets into strings that satisfy any given parameter setting. A crucial trick is to pad each gadget to $\text{NVG}'(a) := 0^{\alpha} 1^{\beta} (01)^{\gamma} \text{NVG}(a) 1^{\gamma}$ for appropriate lengths α, β, γ . It is easy to see that this constructions ensures the following:

(1vs1) The LCS length of NVG'(a), NVG'(b) only depends on whether a, b are orthognal, and (2vs1) NVG'(b) is a subsequence of $NVG'(a) \circ NVG'(a')$ for any vectors a, a', b.

In particular, for any vectors $a^{(1)}, \ldots, a^{(2k-1)}$ and $b^{(1)}, \ldots, b^{(k)}$, on the strings $x = \bigcirc_{i=1}^{2k-1} \text{NVG}'(a^{(i)})$ and $y = \bigcirc_{j=1}^k \text{NVG}'(b^{(j)})$ we show that any LCS consists of k-1 matchings of type 2vs1 and one matching of type 1vs1 (between NVG' $(b^{(j)})$) and NVG' $(a^{(2j-1)})$ for some j). Thus, the LCS length of x and y only depends on whether there exists a j such that $a^{(2j-1)}, b^{(j)}$ are orthogonal. Moreover, since most of y is matched by type 2vs1 and thus completely contained in x, the parameter $\delta(x,y)$ is extremely small compared to the lengths of x and y – which is not achievable with the known reduction [28]. Our proof uses an extension of the above construction, which allows us to have more than one matching of type 1vs1. We think that this 1vs1/2vs1-construction is our main contribution to specific proof techniques and will find more applications.

4.4 Small Alphabet

Proving our results for small constant alphabets poses additional challenges. For instance, our proof of Lemma 4.5 fails for parameter settings (α, Σ) if $|\Sigma|$ is too small, since the padding over disjoint alphabets produces strings over alphabet size at least $|\mathcal{P}| = 7$. In particular, for $|\Sigma| = 2$ we may not use the Disjoint Alphabets Lemma at all, rendering Lemma 4.2 completely useless. However, the classification Theorem 3.2 still holds for parameter settings (α, Σ) . A proof is implicit in Section 10, as we construct (infinitely many) hard instances for all parameter settings (α, Σ) satisfying Table 2.

As mentioned above, the Monotonicity Lemma (Lemma 4.5) is wrong for $|\Sigma| = 2$, since our new algorithm has a running time $\tilde{\mathcal{O}}(n + \delta M/n)$ which is not monotone. Hence, it is impossible to use general strings from $LCS_{\leq}(\alpha, \Sigma)$ as a hard core for $LCS(\alpha, \Sigma)$. Instead, we use strings from a different, appropriately chosen parameter setting $LCS_{\leq}(\alpha', \Sigma')$ as a hard core, see, e.g., Observation 10.18. Moreover, instead of padding with new strings x_p, y_p for each parameter, we need an integrated construction where we control all parameters at once. This is a technically demanding task to which we devote a large part of this paper (Section 10). Since the cases $|\Sigma| = 2$, $|\Sigma| = 3$, and $|\Sigma| \geq 4$ adhere to different relations of Table 2, these three cases have to be treated separately. Furthermore, as for large alphabet we consider cases $\alpha_{\delta} = \alpha_m$ and $\alpha_L = \alpha_m$. Hence, our reductions are necessarily rather involved and we need to very carefully fine-tune our constructions.

5 Organization

The remainder of the paper contains the proofs of Theorems 3.2, 3.3, 3.4, and 3.5, following the outline given in Section 4. Specifically, Section 6 lists and proves our complete set of parameter relations (proving Lemma 4.1). In Section 7, we prove basic facts and technical tools easing the constructions and proofs in later sections – this includes a simple greedy prefix matching property as well as a surprising technique to reduce the number of dominant pairs of two given strings. In Section 8, we show how to pad each parameter individually (proving Lemma 4.2). Section 9 then constructs hard instances for large alphabet (for the downward closure of any parameter setting, proving Theorem 4.6 and thus Theorem 3.3). Finally, the much more intricate case of small constant alphabet sizes such as $|\Sigma| = 2$ is handled in Section 10, which takes up a large fraction of this paper (proving Theorem 3.4). We present our new algorithm in Section 11 (proving Theorem 3.5). Finally, Section 12 describes the necessary modifications to our hardness proofs to show the same conditional lower bounds also under the weaker variant of SETH used in [2].

We remark that for some intermediate strings x, y constructed in the proofs, the assumption $|x| \ge |y|$ may be violated; in this case we use the definitions given in Figure 1b (and thus we may have n(x,y) < m(x,y) and $\Delta(x,y) < \delta(x,y)$). Since L, M, d, and Σ are symmetric in the sense

L(x,y) = L(y,x), these parameters are independent of the assumption $|x| \ge |y|$. For simplicity, we will always work with the following equivalent variant of OVH.

Unbalanced Orthogonal Vectors Hypothesis (UOVH): For any $\alpha, \beta \in (0, 1]$, and computable functions $f(n) = n^{\alpha - o(1)}$, $g(n) = n^{\beta - o(1)}$, the following problem requires time $n^{\alpha + \beta - o(1)}$: Given a number n, solve a given OV instance with $D = n^{o(1)}$ and $|\mathcal{A}| = f(n)$ and $|\mathcal{B}| = g(n)$.

Lemma 5.1 (Essentially folklore). UOVH is equivalent to OVH.

Proof. Clearly, UOVH implies OVH (using $\alpha = \beta = 1, f(n) = g(n) = n$). For the other direction, assume that UOVH fails and let $\alpha, \beta \in (0,1], f(n) = n^{\alpha - o(1)}, \text{ and } g(n) = n^{\beta - o(1)}$ be such that OV with $D = n^{o(1)}$ and $|\mathcal{A}| = f(n)$ and $|\mathcal{B}| = g(n)$ can be solved in time $\mathcal{O}(n^{\alpha + \beta - \varepsilon})$ for some constant $\varepsilon > 0$. Consider an arbitrary OV instance $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^D$ with $D = n^{o(1)}$. We partition \mathcal{A} into $s := \lceil \frac{n}{f(n)} \rceil$ sets $\mathcal{A}_1, \ldots, \mathcal{A}_s$ of size f(n) and \mathcal{B} into $t := \lceil \frac{n}{g(n)} \rceil$ sets $\mathcal{B}_1, \ldots, \mathcal{B}_t$ of size g(n) (note that the last set of such a partition might have strictly less elements, but can safely be filled up using all-ones vectors). By assumption, we can solve each OV instance $\mathcal{A}_i, \mathcal{B}_j$ in time $\mathcal{O}(n^{\alpha + \beta - \varepsilon})$. Since there exist $a \in \mathcal{A}, b \in \mathcal{B}$ with $\langle a, b \rangle = 0$ if and only if there exist $a \in \mathcal{A}_i, b \in \mathcal{B}_j$ with $\langle a_i, b_j \rangle = 0$ for some $i \in [s], j \in [t]$, we can decide the instance \mathcal{A}, \mathcal{B} by sequentially deciding the $s \cdot t = \mathcal{O}(n^{2-(\alpha+\beta)+o(1)})$ OV instances $\mathcal{A}_i, \mathcal{B}_j$. This takes total time $\mathcal{O}(s \cdot t \cdot n^{\alpha+\beta-\varepsilon}) = \mathcal{O}(n^{2-\varepsilon'})$ for any $\varepsilon' < \varepsilon$, which contradicts OVH and thus proves the claim.

6 Parameter Relations

In this section we prove relations among the studied parameters, summarized in Table 3. Some of these parameter relations can be found at various places in the literature, however, our set of relations is complete in the sense that any parameter setting α is non-trivial if and only if it satisfies all our relations, see Theorem 3.2.

Consider a relation like $d(x,y) \leq L(x,y) \cdot m(x,y)$, given in Lemma 6.3(i) below. Fix exponents α_d, α_L , and α_m , and consider all instances x, y with $d(x,y) = \Theta(n^{\alpha_d}), L(x,y) = \Theta(n^{\alpha_L})$, and $m(x,y) = \Theta(n^{\alpha_m})$. Note that the relation may be satisfied for infinitely many instances if $\alpha_d \leq \alpha_L + \alpha_m$. On the other hand, if $\alpha_d > \alpha_L + \alpha_m$ then the relation is satisfied for only finitely many instances. This argument translates Table 3 into Table 2 (using $\alpha_n = 1$), thus generating a complete list of restrictions for non-trivial parameter settings.

Let x, y be any strings. In the remainder of this section, for convenience, we write p = p(x, y) for any parameter $p \in \mathcal{P}^*$. Recall that by possibly swapping x and y, we may assume $m = |y| \le |x| = n$. This assumption is explicit in our definition of parameter settings. For some other strings x, y considered in this paper, this assumption may be violated. In this case, the parameter relations of Table 3 still hold after replacing n by $\max\{n, m\}$ and m by $\min\{n, m\}$, as well as Δ by $\max\{\Delta, \delta\}$ and δ by $\min\{\Delta, \delta\}$ (as the other parameters are symmetric).

Note that Assumption 2.1 (i.e., every symbol in Σ appears at least once in x and y) implies $|\Sigma| \leq m$ and ensures that any symbol of x has at least one matching symbol in y, and thus $M \geq n$.

We next list trivial relations. The length of the LCS L satisfies $L \leq m$. The numbers of deleted positions satisfy $\Delta = n - L \leq n$, $\delta = m - L \leq m$, and $\delta \leq \Delta$. Since any dominant pair is also a matching pair, we have $d \leq M$. Moreover, $d \geq L$ since for any $1 \leq k \leq L$ there is at least one k-dominant pair.

To prepare the proofs of the remaining relations, recall that we defined L[i,j] = L(x[1..i],y[1..j]). Moreover, observe that $L(x,y) \leq \sum_{\sigma \in \Sigma} \min\{\#_{\sigma}(x),\#_{\sigma}(y)\}$, which we typically exploit without explicit notice. Furthermore, we will need the following two simple facts.

Observation 6.1. For any $\sigma \in \Sigma$, we have $\#_{\sigma}(x) \leq L$ or $\#_{\sigma}(y) \leq L$.

Proof. If some $\sigma \in \Sigma$ occurs at least L+1 times in both x and y, then σ^{L+1} is an LCS of x and y of length L+1>L, which is a contradiction.

Observation 6.2. Fix $1 \le k \le L$ and $1 \le \overline{i} \le n$. Then there is at most one k-dominant pair (\overline{i}, j) with $1 \le j \le m$, namely the pair (\overline{i}, j^*) with $j^* = \min\{j \mid L[\overline{i}, j] = k\}$ if it exists. Symmetrically for every $1 \le k \le n$ and $1 \le \overline{j} \le m$, there is at most one k-dominant pair (i, \overline{j}) with $1 \le i \le n$.

Proof. All pairs (\bar{i},j) with $j \neq j^*$ and $L[\bar{i},j] = k$ satisfy $j \geq j^*$, so they are dominated by (\bar{i},j^*) . \square

We are set up to prove the more involved relations of Table 3. We remark that while the inequality $d \leq Lm$ is well known since the first formal treatment of dominant pairs, the bound $d \leq L^2|\Sigma|$ seems to go unnoticed in the literature.

Lemma 6.3. It holds that (i) $d \leq Lm$ and (ii) $d \leq L^2|\Sigma|$.

Proof. (i) Let $1 \le k \le L$. For any $1 \le \bar{j} \le m$ there is at most one k-dominant pair (i, \bar{j}) by Observation 6.2. This proves $|D_k| \le m$ and thus $d = \sum_{i=1}^L |D_k| \le Lm$.

(ii) Let $\sigma \in \Sigma$. By Observation 6.1, we may assume that $\#_{\sigma}(x) \leq L$ (the case $\#_{\sigma}(y) \leq L$ is symmetric). For any occurrence i_{σ} of σ in x and any $1 \leq k \leq L$, there can be at most one k-dominant pair (i_{σ}, j) by Observation 6.2. Hence, σ contributes at most L k-dominant pairs. Summing over all $\sigma \in \Sigma$ and $k = 1, \ldots, L$ yields the claim.

Lemma 6.4. We have $d \leq 2L(\Delta + 1)$.

Proof. Fix an LCS z of x and y. Since z can be obtained by deleting at most $\Delta = n - L$ positions from x or by deleting at most $\delta = m - L$ positions from y, x[1..i] and y[1..j] contain $z[1..i - \Delta]$ and $z[1..j - \delta]$, respectively, as a subsequence. Hence, we have $\min\{i - \Delta, j - \delta\} \leq L[i, j] \leq \min\{i, j\}$. Let $1 \leq k \leq L$. By the previous property, if L[i, j] = k then (i) $k \leq i \leq k + \Delta$ or (ii)

 $k \leq j \leq k + \delta$. Note that for each $\bar{i} \in \{k, \dots, k + \Delta\}$ we have (by Observation 6.2) at most one k-dominant pair (\bar{i}, j) , and similarly, for each $\bar{j} \in \{k, \dots, k + \delta\}$ we have at most one k-dominant pair (i, \bar{j}) . This proves $|D_k| \leq \Delta + \delta + 2 \leq 2(\Delta + 1)$, from which the claim follows.

Lemma 6.5. We have $d \geq |\Sigma|$.

Proof. By Assumption 2.1, every symbol $\sigma \in \Sigma$ appears in x and y. Let i be minimal with $x[i] = \sigma$ and j be minimal with $y[j] = \sigma$. We show that (i,j) is a dominant pair of x,y, and thus $d \geq |\Sigma|$. Let k = L[i,j] = L(x[1..i],y[1..j]). Since x[i] = y[j], we have L[i-1,j-1] = k-1. Moreover, since the last symbol in x[1..i] does not appear in y[1..j-1], it cannot be matched, and we obtain L[i,j-1] = L[i-1,j-1] = k-1. Similarly, L[i-1,j] = k-1. This proves that (i,j) is a k-dominant pair of x,y, as desired.

Lemma 6.6. We have (i) $M \ge L^2/|\Sigma|$ and (ii) $M \le 2Ln$.

Proof. (i) Let z be an LCS of x and y. We have $M = \sum_{\sigma \in \Sigma} \#_{\sigma}(x) \cdot \#_{\sigma}(y) \ge \sum_{\sigma \in \Sigma} \#_{\sigma}(z)^2$. By $\sum_{\sigma \in \Sigma} \#_{\sigma}(z) = L$ and the arithmetic-quadratic mean inequality, the result follows.

(ii) Let $\Sigma_w := \{ \sigma \in \Sigma \mid \#_{\sigma}(w) \leq L \}$ for $w \in \{x, y\}$. By Observation 6.1, we have $\Sigma_x \cup \Sigma_y = \Sigma$. We can thus bound

$$M = \sum_{\sigma \in \Sigma} \#_{\sigma}(x) \cdot \#_{\sigma}(y) \le \sum_{\sigma \in \Sigma_{y}} L \cdot \#_{\sigma}(x) + \sum_{\sigma \in \Sigma_{x}} L \cdot \#_{\sigma}(y) \le L(n+m) \le 2Ln. \quad \Box$$

Small alphabets. Not surprisingly, in the case of very small alphabets there are more relations among the parameters. The following relations are specific to $|\Sigma| = 2$ and $|\Sigma| = 3$.

Lemma 6.7. Let $\Sigma = \{0, 1\}$. Then $M \ge Lm/4$.

Proof. Let z be an LCS of x and y. Without loss of generality we may assume $\#_0(x) \ge n/2$ (by possibly exchanging 0 and 1). If $\#_0(y) \ge L/2$, then $M \ge \#_0(x) \cdot \#_0(y) \ge Ln/4 \ge Lm/4$. Otherwise we have $\#_0(y) < L/2 \le m/2$, which implies $\#_1(y) \ge m/2$. By $\#_0(y) < L/2$, we must have that $\#_1(x) \ge L/2$, since otherwise $L \le \min\{\#_0(x), \#_0(y)\} + \min\{\#_1(x), \#_1(y)\} \le \#_0(y) + \#_1(x) < L$, which is a contradiction. Hence $M \ge \#_1(x) \cdot \#_1(y) \ge Lm/4$, proving the claim. \square

The following lemma (which is also applicable for large alphabets) asserts that if most positions in x and y are the same symbol, say 0, then the number of dominant pairs is small.

Lemma 6.8. Let 0 be a symbol in Σ and set $\lambda := \sum_{\sigma \in \Sigma \setminus \{0\}} \min\{\#_{\sigma}(x), \#_{\sigma}(y)\}$. Then $d \leq 5\lambda L$. In particular, for $\Sigma = \{0, 1\}$, we have $d \leq 5L \cdot \#_1(y)$.

Proof. Let $1 \le k \le L$ and $\sigma \in \Sigma \setminus \{0\}$. By Observation 6.2, there are at most $\min\{\#_{\sigma}(x), \#_{\sigma}(y)\}$ k-dominant pairs (i, j) with $x[i] = y[j] = \sigma$. Hence in total, there are at most $\lambda \cdot L$ dominant pairs (i, j) with $x[i] = y[j] \ne 0$.

To count the remaining dominant pairs, which are contributed by 0, we use a similar argument to Lemma 6.4. Let $1 \le k \le L$ and consider any pair (i,j) with x[i] = y[j] = 0, say x[i] is the ℓ_x -th occurrence of 0 in x and y[j] is the ℓ_y -th occurrence of 0 in y. If (i,j) is a k-dominant pair then $k - \lambda \le \min\{\ell_x, \ell_y\} \le k$. Indeed, if $\min\{\ell_x, \ell_y\} < k - \lambda$, then

$$L[i,j] \le \sum_{\sigma \in \Sigma} \min\{\#_{\sigma}(x[1..i]), \#_{\sigma}(y[1..j])\} \le \min\{\ell_x, \ell_y\} + \lambda < k,$$

contradicting the definition of a k-dominant pair. Moreover, if $\min\{\ell_x, \ell_y\} > k$, then $0^{\min\{\ell_x, \ell_y\}}$ is a common subsequence of x[1..i], y[1..j] of length strictly larger than k, which is again a contradiction to (i, j) being a k-dominant pair.

Hence, we have $k - \lambda \le \ell_x \le k$ or $k - \lambda \le \ell_y \le k$. Since any choice of ℓ_x uniquely determines i by Observation 6.2 (and symmetrically ℓ_y determines j), there are at most $2\lambda + 2$ k-dominant pairs with x[i] = y[j] = 0. In total, we have at most $(3\lambda + 2)L \le 5\lambda L$ dominant pairs (note that $\lambda \ge 1$ by Assumption 2.1 and $|\Sigma| \ge 2$).

Lemma 6.9. If $\Sigma = \{0, 1\}$ then $M \ge nd/(5L)$.

Proof. Without loss of generality assume that $\min\{\#_0(x), \#_0(y)\} \ge \min\{\#_1(x), \#_1(y)\}$ (by possibly exchanging 0 and 1). Then $\lambda = \min\{\#_1(x), \#_1(y)\}$ satisfies $\#_{\sigma}(y) \ge \lambda$ for all $\sigma \in \Sigma$. Thus, $M = \sum_{\sigma \in \Sigma} \#_{\sigma}(x) \cdot \#_{\sigma}(y) \ge (\#_0(x) + \#_1(x)) \cdot \lambda = \lambda n$. By Lemma 6.8, we have $\lambda \ge d/(5L)$ and the claim follows.

For ternary alphabets, the following weaker relation holds.

Lemma 6.10. If $\Sigma = \{0, 1, 2\}$ then $M \ge md/(80L)$.

Proof. Let z be an LCS of x and y. We permute the symbols such that $\sigma = 0$ maximizes $\#_{\sigma}(z)$. Thus, we have $\#_{0}(x) \geq \#_{0}(z) \geq L/|\Sigma| = L/3$ and symmetrically $\#_{0}(y) \geq L/3$.

If $\#_0(y) \ge m/2$ then we have $M \ge \#_0(x) \cdot \#_0(y) \ge Lm/6 \ge dm/(18L)$ by Lemma 6.3(ii). Similarly, if $\#_0(x) \ge n/2$ then we obtain $M \ge Ln/6 \ge dn/(18L) \ge dm/(18L)$. Hence, it remains to consider the case $\#_0(x) \le n/2$ and $\#_0(y) \le m/2$. Let x', y' be the subsequences obtained by

deleting all 0s from x and y, respectively, and note that $|x'|, |y'| \ge m/2$. Since x', y' have alphabet size 2, Lemma 6.7 is applicable and yields $M(x', y') \ge L(x', y') \cdot m(x', y')/4$. Observe that there is a common subsequence of x', y' of length at least $\lambda/2$, where $\lambda = \sum_{\sigma \in \Sigma \setminus \{0\}} \min\{\#_{\sigma}(x), \#_{\sigma}(y)\}$ (consider the longer subsequence of $1^{\min\{\#_1(x), \#_1(y)\}}$ and $2^{\min\{\#_2(x), \#_2(y)\}}$). Hence,

$$M(x,y) \ge M(x',y') \ge \frac{1}{4} \cdot L(x',y') \cdot m(x',y') \ge \frac{1}{4} \cdot \frac{\lambda}{2} \cdot \frac{m}{2} = \frac{\lambda m}{16}.$$

The claim now follows from $\lambda \geq d/(5L)$ as proven in Lemma 6.8.

7 Technical Tools and Constructions

To prepare later constructions and ease their analysis, this section collects several technical results. We start off with the simple fact that equal prefixes can be greedily matched.

Lemma 7.1 (Greedy Prefix Matching). For any strings w, x, y, we have L(wx, wy) = |w| + L(x, y) and d(wx, wy) = |w| + d(x, y).

Proof. For the first statement, it suffices to prove the claim when w=0 is a single symbol (by induction and renaming of symbols). Consider any common subsequence z of 0x, 0y. If z is of the form 0z', then z' is a common subsequence of x, y, so $|z| \le 1 + L(x, y)$. If the first symbol of z is not 0, then the first symbols of 0x, 0y are not matched, so z is a common subsequence of x, y and we obtain $|z| \le L(x, y)$. In total, $L(0x, 0y) \le 1 + L(x, y)$. The converse holds by prepending 0 to any LCS of x, y.

For the second statement, let x' = wx and y' = wy. For $i \in [|w|]$, we have L(x'[1..i], y'[1..i]) = i. Hence (i, i) is the unique i-dominant pair of x', y' and no other dominant pairs (i, j) with $i \leq |w|$ or $j \leq |w|$ exist. This yields |w| dominant pairs. Consider now any (i, j) with $i = |w| + \bar{i}$ and $j = |w| + \bar{j}$ where $\bar{i} \in [|x|], \bar{j} \in [|y|]$. By the first statement, $L(x'[1..i], y'[1..j]) = |w| + L(x[1..\bar{i}], y[1..\bar{j}])$. Thus (i, j) is a (|w| + k)-dominant pair of x' and y' if and only if (\bar{i}, \bar{j}) is a k-dominant pair of x and y. This yields d(x, y) additional dominant pairs, proving the claim.

For bounding the number of dominant pairs from below we often use the following observation.

Observation 7.2. For any strings a, x, b, y, we have $d(ax, by) \ge d(a, b)$.

Proof. This trivially follows from the fact that any prefixes a', b' of a, b are also prefixes of ax, by. \square

7.1 Generating dominant pairs

The dependency of d on the other parameters is quite complicated, and hence it is a rather complex task to generate a desired number of dominant pairs while respecting given bounds on all other parameters. We present different constructions for this purpose in the following.

The following lemma establishes the first such construction, illustrated in Figure 2. We remark that statements (iii) and (iv) are technical tools that we will only use for $|\Sigma| = \mathcal{O}(1)$ in Section 10.

Lemma 7.3 (Generating dominant pairs). Let $R, S \ge 0$ and define $a := (01)^{R+S}$ and $b := 0^R (01)^S$. The following statements hold.

- (i) We have L(a,b) = |b| = R + 2S.
- (ii) We have $R \cdot S \leq d(a,b) \leq \min\{2(R+1),5S\} \cdot (R+2S) = \mathcal{O}(R \cdot S)$.

```
0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1
0
   1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
   0
   1 1 2 2 3 3 3 3 3 3 3 3 3 3 3 3 3
0
   0
   1 1 2 2 3 3 4 4 5 5 5 5 5 5 5 5 5 5
0
   1 2 2 3 3 4 4 5 5 6 6 6 6 6 6
   1 2 3 3 4 4 5 5 6 6
                       7 7 7 7 7
   1 2 3 4 4 5 5 6 6 7 7 8 8 8 8 8 8 8 8
1
   1 2 3 4 5 5 6 6 7 7 8 8 9 9 9 9 9 9
   1 2 3 4 5 6 6 7 7 8 8 9 9 10 10 10 10 10
   1 2 3 4 5 6 7 7 8 8 9 9 10 10 11 11 11 11
   1 2 3 4 5 6 7 8 8 9 9 10 10 11 11 12 12 12
1
   1 2 3 4 5 6 7 8 9 9 10 10 11 11 12 12 13 13
   1 2 3 4 5 6 7 8 9 10 10 11 11 12 12 13 13 14
```

Figure 2: The *L*-table for the strings $a = (01)^{R+S}$ and $b = 0^R (01)^S$ with R = 4, S = 5 (where the entry in row j and column i denotes L(a[1..i], b[1..j])). The indicated dominant pairs visualize the results of Lemma 7.3.

- (iii) For any $\alpha, \beta, \beta' \geq 0$, we have $L(a1^{\alpha}, 0^{\beta}b0^{\beta'}) = |b| = R + 2S$.
- (iv) For any $\alpha, \beta, \beta' \geq 0$, we have $R \cdot S \leq d(a1^{\alpha}, 0^{\beta}b0^{\beta'}) \leq 2(\max\{R + \alpha, \beta + \beta'\} + 1)(R + 2S)$. Proof. All statements follow from the following fact.
 - (*) For any $0 \le s \le S, s \le r \le R + s$ and $\beta \ge 0$, we have $L((01)^r, 0^\beta 0^R (01)^s) = r + s$.

To prove (*), note that by Lemma 7.1 (reversing the strings) we have

$$L((01)^r, 0^{\beta+R}(01)^s) = 2s + L((01)^{r-s}, 0^{\beta+R}) = 2s + \min\{\#_0((01)^{r-s}), \beta+R\} = 2s + (r-s) = r+s.$$

Statement (i) now follows from setting s = S, r = R + S, and $\beta = 0$.

To see (iii), note that $L(a1^{\alpha}, 0^{\beta}b0^{\beta'}) \geq L(a, b) = |b|$ by (i). For the upper bound, we compute

$$\begin{array}{lcl} L(a1^{\alpha},0^{\beta}b0^{\beta'}) & \leq & \min\{\#_0(a1^{\alpha}),\#_0(0^{\beta}b0^{\beta'})\} + \min\{\#_1(a1^{\alpha}),\#_1(0^{\beta}b0^{\beta'})\} \\ & = & \min\{R+S,R+S+\beta+\beta'\} + \min\{R+S+\alpha,S\} = R+2S = L(a,b). \end{array}$$

To prove (ii), note that $d(a,b) \leq 5 \cdot \#_1(b) \cdot L(a,b) = 5S(R+2S)$ by Lemma 6.8. The bound $d(a,b) \leq 2(R+1) \cdot (R+2S)$ follows from (iv) by setting $\alpha = \beta = \beta' = 0$, hence it remains to prove (iv).

For the lower bound, we use $d(a1^{\alpha}, 0^{\beta}b0^{\beta'}) \geq d(a, 0^{\beta}b)$ (by Observation 7.2) and consider $L'[r, s] := L((01)^r, 0^{\beta+R}(01)^s)$. We prove that for any $1 \leq s \leq S$, $s < r \leq R+s$, we have at least one (r+s)-dominant pair (i,j) with $2(r-1) < i \leq 2r$ and $\beta+R+2(s-1) < j \leq \beta+R+2s$. Indeed, L'[r,s] = r+s (by (i)) implies that an (r+s)-dominant pair (i,j) with $i \leq 2r$ and $j \leq \beta+R+2s$ exists. If we had $i \leq 2(r-1)$, then by monotonicity of L(x[1..i], y[1..j]) in i and j we would have $L'[r-1,s] \geq r+s$, contradicting L'[r-1,s] = r+s-1 (by (i)). Thus, we obtain i > 2(r-1), and symmetrically we have $j > \beta+R+2(s-1)$. Hence, for every $1 \leq s \leq S$, $s < r \leq R+s$, we have at least one dominant pair which is not counted for any other choice of r and s. Since for any $1 \leq s \leq S$ there are R choices for $s < r \leq R+s$, we conclude that $d(a,b) \geq S \cdot R$. For the upper bound, note that $\Delta(a1^{\alpha}, 0^{\beta}b0^{\beta'}) = \max\{R+\alpha, \beta+\beta'\}$ by (iii), and hence Lemma 6.4 yields $d(a1^{\alpha}, 0^{\beta}b0^{\beta'}) \leq 2 \cdot L(a1^{\alpha}, 0^{\beta}b0^{\beta'}) \cdot (\Delta(a1^{\alpha}, 0^{\beta}b0^{\beta'}) + 1) = 2(R+2S)(\max\{R+\alpha, \beta+\beta'\}+1)$. \square

The previous construction uses alphabet size $|\Sigma| = 2$, enforcing $M(a,b) \ge L(a,b)^2/2$. If the desired number of matching pairs is much smaller than L^2 , which can only be the case if the desired alphabet size is large, we have to use a more complicated construction exploiting the larger alphabet. To make the analysis more convenient, we first observe a simple way to bound the number of dominant pairs from below: if we can find k pairwise non-dominating index pairs (i,j) that have the same LCS value (of the corresponding prefixes) and whose predecessors (i-1,j-1) have a strictly smaller LCS value, then at least k/2 dominant pairs exist.

Lemma 7.4. For any strings x, y set L[i, j] := L(x[1..i], y[1..j]). Suppose that there are index pairs $(i_1, j_1), \ldots, (i_k, j_k)$ with $i_1 < i_2 < \cdots < i_k$ and $j_1 > j_2 > \cdots > j_k$ such that for some γ and all $1 \le \ell \le k$ we have $L[i_\ell, j_\ell] = \gamma$ and $L[i_\ell - 1, j_\ell - 1] = \gamma - 1$. Then the number of γ -dominant pairs of x and y is at least k/2.

Proof. For each k, fix any γ -dominant pair $p_{\ell} = (i_{\ell}^*, j_{\ell}^*)$ that dominates (i_{ℓ}, j_{ℓ}) . Note we may have $p_{\ell} = (i_{\ell}, j_{\ell})$ if (i_{ℓ}, j_{ℓ}) is itself a dominant pair, and thus p_{ℓ} always exists. Set $i_0 := 1$. We argue that for every ℓ , we have $i_{\ell-1} \leq i_{\ell}^* \leq i_{\ell}$.

Note that for all ℓ , we have $L[i_{\ell}-1,j_{\ell}-1]<\gamma$ and hence $L[i,j]<\gamma$ for all $i< i_{\ell}, j< j_{\ell}$. Thus, we have either (1) $i_{\ell}^*=i_{\ell}$ (and $j_{\ell}^*\leq j_{\ell}$) or (2) $j_{\ell}^*=j_{\ell}$ (and $i_{\ell}^*\leq i_{\ell}$). Case (1) trivially satisfies $i_{\ell-1}\leq i_{\ell}^*\leq i_{\ell}$. Thus, it remains to argue that in case (2) we have $i_{\ell-1}\leq i_{\ell}^*$. Indeed, otherwise we have $i_{\ell}^*< i_{\ell-1}$ and $j_{\ell}^*=j_{\ell}< j_{\ell-1}$, which implies $L[i_{\ell}^*,j_{\ell}^*]<\gamma$ and (i_{ℓ}^*,j_{ℓ}^*) is no γ -dominant pair.

Note that the above property implies $p_{\ell} \neq p_{\ell+2}$ for all $1 \leq \ell \leq k-2$, since $i_{\ell}^* \leq i_{\ell} < i_{\ell+1} \leq i_{\ell+2}^*$. Thus, the number of γ -dominant pairs is bounded from below by $|\{p_{\ell} \mid 1 \leq \ell \leq k\}| \geq k/2$.

Note that the previous lemma would not hold without the condition $L[i_{\ell}-1, j_{\ell}-1] = \gamma - 1$. We are set to analyze our next construction, which is illustrated in Figure 3.

Lemma 7.5 (Generating dominant pairs, large alphabet). Let $t \geq 2$, $1 \leq t' \leq t$, and $S \geq R \geq 1$. Over alphabet $\Sigma = \{1, \ldots, t\}$ we define the strings

$$a := ((1 \dots t) \circ (t' \dots 1))^R \circ (1 \dots t)^{S-R},$$

 $b := (1 \dots t)^S.$

It holds that

- (i) L(a,b) = |b| = St,
- (ii) Assume that $S \geq R(t'+1)$. Then $(St)(Rt')/8 \leq d(a,b) \leq 4(St)(Rt')$,
- (iii) $tS^2 \le M(a,b) \le t(S+R)S$.

Proof. Note that (i) trivially follows from the fact that b is a subsequence of a. For (iii), observe that for all $\sigma \in \Sigma$, we have $S \leq \#_{\sigma}(a) \leq R+S$ and $\#_{\sigma}(b) = S$, from which the claim follows immediately. The upper bound of (ii) follows from $d(a,b) \leq 2 \cdot L(a,b) \cdot (\Delta(a,b)+1) = 2(St)(Rt'+1) \leq 4(St)(Rt')$ (by Lemma 6.4). To prove the remaining lower bound, we establish the following fact.

(*) Let
$$w := 1 \dots t$$
 and $w' := t' \dots 1$. Then $L((ww')^R, w^{R+k}) = Rt + k$ for all $0 \le k \le t'R$.

Let us postpone the proof and show that (ii) follows from (*). Define $v := w^{S-R}$. For $0 \le k \le K := \min\{S-R,Rt'\}$ and $0 \le \ell \le (S-R-k)t$, we let $a(k,\ell) := (ww')^R v[1..\ell]$ and $b(k,\ell) := w^{R+k} v[1..\ell]$.

```
1 2 3 3 2 1 1 2 3 3 2 1 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3
  1
   2
3
   1
    2
    3
    1
    2
3
    2 3 4 5 6 6 6 7 7 7 8 8 8 9 10 10 10 10 10 10 10 10 10 10 10 10
1
   1 2 3 4 5 6 6 7 7 7 8 8 8 9 9 10 11 11 11 11 11 11 11 11 11 11 11
2
3
   1 2 3 4 5 6 6 7 8 8 8 8 8 9 10 10 11 12 12 12 12 12 12 12 12 12 12
   1 2 3 4 5 6 7 7 8 8 8 9 9 9 1011 11 1213 13 13 13 13 13 13 13
1
2
    2 3 4 5 6 7 8 8 8 9 9 9 10 10 11 12 12 13 14 14 14 14 14 14 14 14
    2 3 4 5 6 7 8 9 9 9 9 10 11 11 12 13 13 14 15 15 15 15 15 15 15 15
    2 3 4 5 6 7 8 9 9 10 10 10 11 12 12 13 14 14 15 16 16 16 16 16 16
1
   1 2 3 4 5 6 7 8 9 9 10 10 10 11 11 12 13 13 14 15 15 16 17 17 17 17 17
   1 2 3 4 5 6 7 8 9 10 10 10 10 11 12 12 13 14 14 15 16 16 17 18 18 18 18
   1 2 3 4 5 6 7 8 9 10 10 11 11 11 12 13 13 14 15 15 16 17 17 18 19 19 19
   1 2 3 4 5 6 7 8 9 10 11 11 11 12 12 13 14 14 15 16 16 17 18 18 19 20 20
   1 2 3 4 5 6 7 8 9 10 11 11 11 12 13 13 14 15 15 16 17 17 18 19 19 20 21
```

Figure 3: The L-table for strings a, b of Lemma 7.5 with t = t' = 3, R = 2, S = 7.

Note that $a(k, \ell)$ and $b(k, \ell)$ are prefixes of a and b, respectively. By greedy suffix matching (i.e., Lemma 7.1 applied to the reversed strings) and (*), we obtain

$$L(a(k,\ell),b(k,\ell)) = \ell + L((ww')^R,w^{R+k}) = Rt + k + \ell.$$

Hence, any $0 \le k \le K$, $1 \le \ell \le (S-R-k)t$ give rise to an index pair (i,j) with $L(a[1..i],b[1..j]) = L(a(k,\ell),b(k,\ell)) > L(a(k,\ell-1),b(k,\ell-1)) = L(a[1..i-1],b[1..j-1])$. Let I_{γ} denote the set of all such index pairs (i,j) that additionally satisfy $L(a[1..i],b[1..j]) = \gamma$. Then for any γ , no $(i,j) \in I_{\gamma}$ dominates another $(i',j') \in I_{\gamma}$. Thus by Lemma 7.4, $d(a,b) \ge \sum_{\gamma} |I_{\gamma}|/2$. By counting all possible choices for k and ℓ , we obtain $\sum_{\gamma} |I_{\gamma}| = \sum_{0 \le k \le K} t(S-R-k) \ge tK(S-R)/2$. This yields $d(a,b) \ge t \cdot \min\{S-R,Rt'\} \cdot (S-R)/4$. For $S \ge R(t'+1)$, we have $S-R \ge S/2$ as well as $S-R \ge Rt'$ and the lower bound of (ii) follows.

To prove (*), let $a' := (ww')^R$ and $b' := w^{R+k}$. For the lower bound, it is easy to see that we can completely match R of the copies of w in b' to copies of w in a', and at the same time match a single symbol in each of the remaining k copies of w in b to a single symbol in some copy of w' in a' (since $k \le R|w'| = Rt'$). This yields $L(a',b') \ge R|w| + k = Rt + k$.

For the upper bound, we write $b' = \bigcirc_{j=1}^{R+k} b_j$ with $b_j := w$ and consider a partitioning $a' = \bigcirc_{j=1}^{R+k} a_j$ such that $L(a',b') = \sum_{j=1}^{R+k} L(a_j,b_j)$. For any a_j , let $w(a_j)$ denote the number of symbols that a_j shares with any occurrence of w (if, e.g., $a_j = xw'y$ for some prefix x of w and some suffix y of w, then $w(a_j) = |x| + |y|$). We first show that

$$L(a_j, b_j) \le \begin{cases} 1 & \text{if } w(a_j) = 0, \\ \min\{w(a_j), |w|\} & \text{otherwise.} \end{cases}$$
 (1)

Note that trivially $L(a_j,b_j) \leq |b_j| = |w|$. Hence for an upper bound, we may assume that $w(a_j) < |w|$, and in particular that a_j is a subsequence of $a'_j = xw'y$ for some prefix $x = \sigma_x \dots t$ of w and

some suffix $y = 1 \dots \sigma_y$ of w with $\sigma_y \leq \sigma_x$, where $|x| + |y| = w(a_j)$. Note that any longest common subsequence z of a'_j and $b_j = w = 1 \dots t$ is an increasing subsequence of a'_j . Hence, if z starts with a symbol $\sigma' \geq \sigma_y$, then z is an increasing subsequence in x $t' \dots \sigma'$; easy inspection shows that in this case $|z| \leq \max\{|x|, 1\}$. If z starts with a symbol $\sigma' \leq \sigma_x$, then z is an increasing subsequence in $\sigma' \dots 1$ y; again, one can see that $|z| \leq \max\{|y|, 1\}$ holds in this case. Thus, $L(a_j, b_j) \leq L(a'_j, b_j) = |z| \leq \max\{|x|, |y|, 1\} \leq \max\{|x| + |y|, 1\} = \max\{w(a_j), 1\}$, concluding the proof of (1).

Let $J = \{j \mid w(a_j) \ge 1\}$. We compute

$$L(a',b') = \sum_{j=1}^{R+k} L(a_j,b_j) \le \left(\sum_{j\in J} \min\{w(a_j),|w|\}\right) + (R+k-|J|)$$

$$\le \min\left\{\sum_{j=1}^{R+k} w(a_j),|J|\cdot|w|\right\} + (R+k-|J|)$$

$$\le \min\{R\cdot|w|,|J|\cdot|w|\} + R+k-|J| \le R|w| + k = Rt + k,$$

where the last inequality follows from the observation that |J| = R maximizes the expression $\min\{R \cdot |w|, |J| \cdot |w|\} - |J|$. This finishes the proof of (*) and thus the lemma.

7.2 Block elimination and dominant pair reduction

We collect some convenient tools for the analysis of later constructions. The first allows us to "eliminate" 0^{ℓ} -blocks when computing the LCS of strings of the form $x0^{\ell}y$, $0^{\ell}z$, provided that ℓ is sufficiently large.

Lemma 7.6. For any strings x, y, z and $\ell \ge \#_0(x) + |z|$ we have $L(x0^{\ell}y, 0^{\ell}z) = \ell + L(0^{\#_0(x)}y, z)$.

Proof. Let $u:=x0^\ell y$ and $v:=0^\ell z$. In case we match no symbol in the 0^ℓ -block of v with a symbol in $0^\ell y$ in u, then at most $\min\{\#_0(x),\ell\}$ symbols of the 0^ℓ -block of v are matched. The remainder z yields at most |z| matched symbols. Otherwise, in case we match any symbol in the 0^ℓ -block of v with a symbol in $0^\ell y$ in u, then no symbol $\sigma \neq 0$ of x can be matched. Thus, in this case we may replace x by $0^{\#_0(x)}$. Together this case distinction yields $L(u,v) = \max\{\min\{\#_0(x),\ell\} + |z|, L(0^{\#_0(x)+\ell}y,0^\ell z)\}$. Using Lemma 7.1, we obtain $L(u,v) = \max\{\#_0(x) + |z|, \ell + L(0^{\#_0(x)}y,z)\}$. The assumption $\ell \geq \#_0(x) + |z|$ now yields the claim.

The following lemma bounds the number of dominant pairs of strings of the form x' = yx, y' = zy by $d(x', y') = \mathcal{O}(|z| \cdot |y'|)$. If $|x'| \geq |y'|$, this provides a bound of $\mathcal{O}(\delta(x', y') \cdot m(x', y'))$ instead of the general, weaker bound $\mathcal{O}(\Delta(x', y') \cdot m(x', y'))$ of Lemma 6.4.

Lemma 7.7. For any strings x, y, z, let x' = yx, y' = zy. Then

$$d(x',y') \le |y| \cdot (|z|+1) + d(x',z) \le |y| \cdot (|z|+1) + |z|^2.$$

Proof. For every prefix $\tilde{y} = y'[1..j]$, we bound the number of dominant pairs (i, j) of x', y'. Clearly, all prefixes \tilde{y} of z (i.e., $j \leq |z|$) contribute $d(x', z) \leq L(x', z) \cdot m(x', z) \leq |z|^2$ dominant pairs.

It remains to consider $\tilde{y} = z y[1..\ell]$ (i.e., $j = |z| + \ell$) for $\ell \in [|y|]$. For $i < \ell$, the string $\tilde{x} = x'[1..i] = y[1..i]$ is a subsequence of \tilde{y} , i.e., $L(\tilde{x}, \tilde{y}) = i$, but the prefix z y[1..i] of \tilde{y} already satisfies $L(\tilde{x}, z y[1..i]) = i$. Hence, there are no dominant pairs with $i < \ell$. Thus, consider $i \ge \ell$ and let $\tilde{x} = x'[1..i]$. Clearly, y[1..j] is a common subsequence of \tilde{x}, \tilde{y} . This yields $L(\tilde{x}, \tilde{y}) \ge j = |\tilde{y}| - |z|$

```
\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 
       0
      0
       0
                                                    4 4 4 4 4 4 4 5 5 6 6 6 6 6 6 6
       1
       1
```

Figure 4: Illustration of Lemma 7.8. The strings $x' = y2^{\ell}x, y' = 2^{\ell}y$ are defined using $x = (01)^{R+S}$, $y = 0^R(01)^S$, R = 4, S = 5, and $\ell = 2$. The number of dominant pairs is significantly reduced compared to Figure 2.

and hence any such dominant pair (i, j) satisfies $j-|z| \leq L(x'[1..i], y'[1..i]) \leq j$. By Observation 6.2, there are at most |z|+1 such dominant pairs for fixed j. This yields at most $|y| \cdot (|z|+1)$ dominant pairs (i, j) with $|z| < j \leq |y'|$, concluding the proof.

The above lemma gives rise to a surprising technique: Given strings x, y, we can build strings x', y' such that L(x', y') lets us recover L(x, y), but the number of dominant pairs may be reduced significantly, namely to a value $d(x', y') = \mathcal{O}(\delta(x, y) \cdot n(x, y))$, independently of d(x, y). The effect of this technique is illustrated in Figure 4.

Lemma 7.8 (Dominant Pair Reduction). Consider strings x, y and a number $\ell > |y| - L(x, y)$.

- (i) If 2 is a symbol not appearing in x, y, then $x' := y2^{\ell}x$ and $y' := 2^{\ell}y$ satisfy $L(x', y') = L(x, y) + \ell$ and $d(x, y) \leq 3\ell \cdot |y|$.
- (ii) For any symbols 0,1 (that may appear in x, y) set $x'' := 0^k 1^k y 1^\ell 0^k 1^k x$ and $y'' := 1^\ell 0^k 1^k y$ with k := 2|y| + |x| + 1. Then $L(x'', y'') = L(x, y) + \ell + 2k$ and $d(x, y) \le \mathcal{O}(\ell(|x| + |y| + \ell))$.

Proof. (i) We clearly have $L(x',y') \ge L(2^{\ell},2^{\ell}) + L(x,y) \ge \ell + L(x,y)$. For the other direction, let z be a common subsequence of x',y'. If z contains no 2, then by inspecting y' we obtain $|z| \le |y| = L(x,y) + (|y| - L(x,y)) < L(x,y) + \ell$, so z is no LCS. Otherwise, if z contains a 2, then no symbol from the copy of y in x' can be matched by z, implying $|z| \le L(2^{\ell}x,y') = \ell + L(x,y)$.

For the dominant pairs, we apply Lemma 7.7 to obtain $d(x', y') \leq |y|(\ell+1) + d(x', 2^{\ell})$. Note that $d(x', 2^{\ell}) = d(2^{\ell}, 2^{\ell}) = \ell$, since we can delete all characters different from 2 in x' without affecting the dominant pairs of $x', 2^{\ell}$ and then apply Lemma 7.1. Thus, $d(x', y') \leq |y|(\ell+1) + \ell \leq 3\ell \cdot |y|$.

(ii) The argument is slightly more complicated when the padding symbols may appear in x, y. Clearly, we have $L(x'', y'') \ge L(1^{\ell}0^k 1^k x, 1^{\ell}0^k 1^k y) \ge \ell + 2k + L(x, y)$. For the other direction, let z be a common subsequence of x'', y''. If z does not match any 0 in the 0^k -block of y'' with a symbol in a 0^k -block in x'', then from the 0^k -block of y'' we match at most |y| + |x| symbols, and we obtain $|z| \le (|y| + |x|) + |1^{\ell}1^k y| = |x| + 2|y| + \ell + k < \ell + 2k$, since k > |x| + 2|y|,

so z is no longest common subsequence. If z matches a 0 in the 0^k -block of y'' with a symbol in the left 0^k -block of x'', then no symbol in the 1^ℓ -block in y'' is matched by z, so we obtain $|z| \leq |0^k 1^k y| = 2k + L(x,y) + (|y| - L(x,y)) < 2k + L(x,y) + \ell$, so z is no longest common subsequence. It remains the case where z matches some 0 in the 0^k -block of y'' with a symbol in the right 0^k -block of x''. Then the part $1^k y$ of y'' has to be matched to a subsequence of $0^k 1^k x$ in x''. This yields $|z| \leq \ell + k + L(0^k 1^k x, 1^k y)$. Since k > |y| we can apply Lemma 7.6 (swapping the roles of 0 and 1) to obtain $L(0^k 1^k x, 1^k y) = k + L(x, y)$, so as desired we have $|z| \leq \ell + 2k + L(x, y)$. For the dominant pairs, we apply Lemma 7.7 to obtain $d(x'', y'') \leq |0^k 1^k y| \cdot (\ell + 1) + \ell^2 = \mathcal{O}(\ell(|x| + |y| + \ell))$.

8 Paddings

In this section we construct paddings that allow to augment any strings from $LCS_{\leq}(\alpha)$ to become strings in $LCS(\alpha)$. Specifically, we prove Lemma 4.2. So let α be a parameter setting satisfying Table 2, let $p \in \mathcal{P}^* = \{n, m, L, \delta, \Delta, |\Sigma|, M, d\}$ be a parameter, and let $n \geq 1$. We say that strings x, y prove Lemma 4.2 for parameter p if (n, x, y) is an instance of $LCS_{\leq}(\alpha)$ with $p(x, y) = \Theta(n^{\alpha_p})$, and given n we can compute x = x(n), y = y(n), and L(x, y) in time $\mathcal{O}(n)$. Note that for the first requirement of being an instance of $LCS_{\leq}(\alpha)$, we have to show that $p'(x, y) \leq \mathcal{O}(n^{\alpha_{p'}})$ for any parameter $p' \in \mathcal{P}^*$. Recall that we write $p = n^{\alpha_p}$ for the target value of parameter p.

Lemma 8.1. Let Σ' be an alphabet of size $\min\{|\Sigma|, L\}$. Then the strings $x := y := \bigcap_{\sigma \in \Sigma'} \sigma^{\lfloor L/|\Sigma'| \rfloor}$ prove Lemma 4.2 for parameter L.

Proof. Note that $\lfloor L/|\Sigma'|\rfloor = \Theta(L/|\Sigma'|)$, since $|\Sigma'| \leq L$. Thus, indeed $L(x,y) = |x| = \Theta(L/|\Sigma'|) \cdot |\Sigma'| = \Theta(L)$. Moreover, L(x,y) can be computed in time $\mathcal{O}(n)$, as well as x and y. For the number of matching pairs we note that $M(x,y) \leq |\Sigma'| \cdot (L/|\Sigma'|)^2$, which is $\max\{L,L^2/|\Sigma|\}$ by choice of $|\Sigma'|$. This is $\mathcal{O}(M)$, using the parameter relations $M \geq n \geq m \geq L$ and $M \geq L^2/|\Sigma|$ (see Table 3).

The remaining parameters are straight-forward. Using $m(x,y) = n(x,y) = L(x,y) = \Theta(L)$ and the parameter relations $L \leq m \leq n$ we obtain that $m(x,y) \leq \mathcal{O}(m)$ and $n(x,y) \leq \mathcal{O}(n)$. Moreover, $\delta(x,y) = \Delta(x,y) = 0 \leq \delta \leq \Delta$. The alphabet size $|\Sigma(x,y)| = |\Sigma'|$ is at most $|\Sigma|$ by choice of $|\Sigma'|$. By Lemma 7.1 we obtain $d(x,y) = |x| = \Theta(L) \leq \mathcal{O}(d)$ using the parameter relation $L \leq d$.

Lemma 8.2. The strings $x := 1^{\Delta+1}$ and y := 1 prove Lemma 4.2 for parameter Δ . The strings x := 1 and $y := 1^{\delta+1}$ prove Lemma 4.2 for parameter δ .

Proof. The analysis is straight-forward. Note that indeed $\Delta(x,y) = \Delta$, and that $L(x,y) = 1 \le \mathcal{O}(L)$. We have $n(x,y) = \Delta + 1 \le \mathcal{O}(n)$ and $m(x,y) = 1 \le \mathcal{O}(m)$. Clearly, L(x,y), x, and y can be computed in time $\mathcal{O}(n)$. Moreover, $\delta(x,y) = 0 \le \delta$ and the alphabet size is $1 \le \mathcal{O}(\Sigma)$. Finally, we have $M(x,y) = \Theta(\Delta) \le \mathcal{O}(n) \le \mathcal{O}(M)$ using the parameter relations $L \le n \le M$, and using the relation $d \le Lm$ we obtain $d(x,y) \le 1 \le \mathcal{O}(d)$.

The analysis for δ is symmetric; the same proof holds almost verbatim.

Lemma 8.3. The strings constructed in Lemma 8.1 or the strings constructed in Lemma 8.2 prove Lemma 4.2 for parameters n and m.

Proof. Since $n = L + \Delta$ we have $L = \Theta(n)$ or $\Delta = \Theta(n)$, i.e., $\alpha_L = 1$ or $\alpha_\Delta = 1$. In the first case, in Lemma 8.1 we construct strings of length $\Theta(L) = \Theta(n)$, and thus these strings prove Lemma 4.2 not only for parameter L but also for parameter n. In the second case, the same argument holds for the first pair of strings constructed in Lemma 8.2. The parameter m is symmetric.

Lemma 8.4. Let $w := 12 \dots |\Sigma|$ be the concatenation of $|\Sigma|$ unique symbols. The strings w, w or the strings w, rev(w) prove Lemma 4.2 for parameter $|\Sigma|$.

Proof. Clearly, both pairs of strings realize an alphabet of size exactly $|\Sigma|$. Since $m = L + \delta$ we have $L = \Theta(m)$ or $\delta = \Theta(m)$. In the first case, we use $L(w, w) = |\Sigma| \le m = \Theta(L)$ and $\delta(w, w) = \Delta(w, w) = 0 \le \delta \le \Delta$. In the second case, we have $L(w, \text{rev}(w)) = 1 \le \mathcal{O}(L)$ and $\delta(w, \text{rev}(w)) = \Delta(w, \text{rev}(w)) = |\Sigma| - 1 \le m = \Theta(\delta) \le \mathcal{O}(\Delta)$.

The remaining parameters are straight-forward. Let $(x,y) \in \{(w,w),(w,\operatorname{rev}(w))\}$. We have $n(x,y) = m(x,y) = |\Sigma| \le m \le n$. Moreover, $d(x,y) \le M(x,y) = |\Sigma| \le d \le M$ using the relations $|\Sigma| \le d \le M$. Clearly, the strings and their LCS length can be computed in time $\mathcal{O}(n)$.

8.1 Matching Pairs

Lemma 8.5. If $\alpha_{\Delta} = 1$ then $x := 1^{\lfloor M/n \rfloor + \Delta}$ and $y := 1^{\lfloor M/n \rfloor}$ prove Lemma 4.2 for parameter M.

Proof. Note that $\lfloor M/n \rfloor = \Theta(M/n)$ by the parameter relation $M \geq n$. We have $M(x,y) = \Theta((M/n)^2 + \Delta M/n)$. By the parameter relations $M \leq 2Ln \leq 2n^2$ the first summand is $\mathcal{O}(n \cdot M/n) = \mathcal{O}(M)$. Since $\alpha_{\Delta} = 1$, the second summand is $\Theta(M)$. Thus, we indeed have $M(x,y) = \Theta(M)$.

The remainder is straight-forward. Clearly, x, y, and $L(x, y) = \lfloor M/n \rfloor$ can be computed in time $\mathcal{O}(n)$. Since $M \leq 2Ln$ we also obtain $L(x,y) = m(x,y) = \lfloor M/n \rfloor \leq \mathcal{O}(L) \leq \mathcal{O}(m)$. Moreover, $n(x,y) = \lfloor M/n \rfloor + \Delta \leq \mathcal{O}(n)$ by the relations $M/n \leq 2L \leq 2n$ and $\Delta \leq n$. Note that $\Delta(x,y) = \Delta$ and $\delta(x,y) = 0 \leq \delta$. The alphabet size is 1. By Lemma 7.1 we have $d(x,y) = \lfloor M/n \rfloor \leq 2L \leq 2d$. \square

Lemma 8.6. Assume $\alpha_{\Delta} < 1$ and let Σ' be an alphabet of size $\min\{\lceil m^2/M \rceil, |\Sigma|\}$. Then $x := y := \bigcap_{\sigma \in \Sigma'} \sigma^{\lfloor m/|\Sigma'| \rfloor}$ prove Lemma 4.2 for parameter M.

Proof. Observe that $\alpha_{\Delta} < 1$ implies $\alpha_L = \alpha_m = 1$, so that $n = \Theta(L) = \Theta(m)$ (see Table 2). The number of matching pairs is $M(x,y) = |\Sigma'| \cdot \lfloor m/|\Sigma'| \rfloor^2$. By the parameter relation $m \geq |\Sigma|$ and $|\Sigma| \geq |\Sigma'|$ we have $\lfloor m/|\Sigma'| \rfloor = \Theta(m/|\Sigma'|)$, and by $M \leq 2Ln = \Theta(m^2)$ we obtain $\lceil m^2/M \rceil = \Theta(m^2/M)$. Thus, $M(x,y) = \Theta(m^2/|\Sigma'|) = \Theta(\max\{M, m^2/|\Sigma|\})$ by choice of $|\Sigma'|$. Using $m = \Theta(L)$ and the parameter relation $M \geq L^2/|\Sigma|$ we indeed obtain $M(x,y) = \Theta(M)$.

The remainder is straight-forward. Since x = y we have $L(x,y) = m(x,y) = n(x,y) = |\Sigma'| \cdot \lfloor m/|\Sigma'| \rfloor \le m = \Theta(L) = \Theta(n)$. Moreover, $\delta(x,y) = \Delta(x,y) = 0 \le \delta \le \Delta$. The alphabet size is $|\Sigma(x,y)| = |\Sigma'| \le |\Sigma|$ by choice of $|\Sigma'|$. By Lemma 7.1 we have $d(x,y) = L(x,y) \le \mathcal{O}(L) \le \mathcal{O}(d)$. Clearly, x, y, and L(x,y) can be computed in time $\mathcal{O}(n)$.

8.2 Dominant Pairs

We start with a simple construction that always works on constant-sized alphabets ($\alpha_{\Sigma} = 0$).

Lemma 8.7. Assume $\alpha_d \leq 2\alpha_L \leq \alpha_M$ and set $x := (01)^{R+S}$ and $y := 0^R (01)^{R+S}$ (as analyzed in Lemma 7.3), instantiated with $R = \lfloor \min\{\Delta, \sqrt{d}\} \rfloor$, $S = \lceil d/R \rceil$. Then x, y prove Lemma 4.2 for parameter d.

Proof. Note that by definition $R \leq \sqrt{d}$, and hence $S \geq d/R \geq R$. By Lemma 7.3(ii), we obtain $d(x,y) = \Theta(R \cdot S) = \Theta(R \cdot d/R) = \Theta(d)$. For the other parameters, note that $n(x,y) = 2(R+S) = \mathcal{O}(d/R) = \mathcal{O}(d/\Delta + \sqrt{d})$. By the relation $d \leq 2L(\Delta + 1)$, we have $d/\Delta \leq \mathcal{O}(L)$, and by the assumption $\alpha_d \leq 2\alpha_L$, we have $d \leq \mathcal{O}(L^2)$ and hence $\sqrt{d} \leq \mathcal{O}(L)$. Thus, $n(x,y) \leq \mathcal{O}(L)$.

The remainder is straight-forward. By $L(x,y) \leq m(x,y) \leq n(x,y) \leq \mathcal{O}(L) \leq \mathcal{O}(m) \leq \mathcal{O}(n)$ we have verified n, m, L. Consequently, also $M(x,y) \leq n(x,y)^2 \leq \mathcal{O}(L^2) \leq \mathcal{O}(M)$ by the assumption $2\alpha_L \leq \alpha_M$. Trivially, $|\Sigma(x,y)| = 2 = \mathcal{O}(|\Sigma|)$. Observing that $\delta(x,y) = 0 \leq \mathcal{O}(\delta)$ and $\Delta(x,y) = 0 \leq \mathcal{O}(\delta)$

 $R \leq \mathcal{O}(\Delta)$ by Lemma 7.3(i) concludes the parameter verification. Since x, y and L(x, y) = R + 2S (by Lemma 7.3(i)) can be computed in time $\mathcal{O}(n)$, the claim follows.

The construction above creates a long LCS of length $L(x,y) = \Theta(m(x,y))$ which forces $d(x,y) = \mathcal{O}(L(x,y)^2)$. With super-constant alphabet sizes, one can construct larger numbers of dominant pairs (compared to L(x,y)) by exploiting the crossing gadgets defined in Definition 9.4.

Lemma 8.8. Assume $\alpha_d > 2\alpha_L$ and set $v := (01)^{R+S}$ and $w := 0^R (01)^S$ with R = S = L. Construct $x := \operatorname{cr}^{\mathbf{x}}(v, \ldots, v)$ and $y := \operatorname{cr}^{\mathbf{y}}(w, \ldots, w)$ on $\lfloor d/L^2 \rfloor$ copies of v and w. Then x, y prove Lemma 4.2 for parameter d.

Proof. Note that $\lfloor d/L^2 \rfloor = \Theta(d/L^2)$ since we assume $\alpha_d > 2\alpha_L$. By the Crossing Alphabets Lemma (Lemma 9.3), we obtain L(x,y) = L(v,w) = 3L, and in particular L(x,y), x, and y can be computed in time $\mathcal{O}(n)$.

Furthermore, the Crossing Alphabets Lemma also yields $d(x,y) = \Theta(d/L^2) \cdot d(v,w) = \Theta(d)$, where the bound $d(v,w) = \Theta(L^2)$ follows from Lemma 7.3(i). Similarly, we observe that $\Delta(x,y) \leq n(x,y) = \Theta(d/L^2) \cdot n(v,w) = \Theta(d/L)$, which is at most $\mathcal{O}(\Delta) \leq \mathcal{O}(n)$ by the parameter relation $d \leq 2L(\Delta+1)$. Likewise, $m(x,y) \leq n(x,y) = \mathcal{O}(d/L) \leq \mathcal{O}(m)$ by the parameter relation $d \leq Lm$. Moreover, $M(x,y) = \mathcal{O}(d/L^2) \cdot M(v,w) = \mathcal{O}(d) \leq \mathcal{O}(M)$ by $d \leq M$. Finally, the assumption $2\alpha_L < \alpha_d$ together with the parameter relation $d \leq Lm$, i.e., $\alpha_d \leq \alpha_L + \alpha_m$, forces $\alpha_L < \alpha_m$. Hence, $\alpha_\delta = \alpha_m$, i.e., $\delta = \Theta(m)$ (see Table 2), and thus $\delta(x,y) \leq m(x,y) \leq \mathcal{O}(m) = \mathcal{O}(\delta)$. Since v and w have alphabet size 2 and we use $\lfloor d/L^2 \rfloor$ copies over disjoint alphabets, we have $\lfloor \Sigma(x,y) \rfloor = 2\lfloor d/L^2 \rfloor \leq \mathcal{O}(|\Sigma|)$ by the parameter relation $d \leq L^2|\Sigma|$, which concludes the proof. \square

For super-constant alphabet sizes, the number of matching pairs M(x, y) can attain values much smaller than $L(x, y)^2$, which is an orthogonal situation to the lemma above. In this case, we use a different generalization of the first construction (that we already prepared in Section 7).

Lemma 8.9. Assume $\alpha_M < 2\alpha_L$ and set $x := (1 \dots t \ t' \dots 1)^R (1 \dots t)^{S-R}$ and $y := (1 \dots t)^S$ (as analyzed in Lemma 7.5) instantiated with

$$t := \left\lfloor \frac{L^2}{M} \right\rfloor, \hspace{1cm} t' := \min\{r, t\} \hspace{1cm} R := \left\lceil \frac{r}{t} \right\rceil \hspace{1cm} S := 4 \left\lceil \frac{d}{rt} \right\rceil,$$

where $r := \min\{\Delta, \lfloor \sqrt{d/t} \rfloor\}$. Then x, y prove Lemma 4.2 for parameter d.

Proof. We first verify the conditions of Lemma 7.5. Observe that by the assumption $\alpha_M < 2\alpha_L$ we indeed have $t = \Theta(L^2/M)$ and $t \geq 2$ (for sufficiently large n). From the parameter relation $M \geq L^2/|\Sigma|$ we obtain $t \leq |\Sigma|$, and from the parameter relation $d \geq |\Sigma|$ (and $\alpha_{\Delta} > 0$) this yields $r \geq 1$. Thus, $1 \leq t' \leq t$. Moreover, $r = \Theta(\min\{\Delta, \sqrt{d/t}\})$. Observe that $r \leq Rt' \leq 2r$. Indeed, if $r \leq t$ then R = 1 and t' = r, and if r > t then $r/t \leq R \leq 2r/t$ and t' = t. In particular, we have

$$Rt' = \Theta\left(\min\{\Delta, \sqrt{d/t}\}\right) \text{ and } S = \Theta\left(\frac{d}{Rt' \cdot t}\right).$$

Note that $R(t'+1) \leq 2Rt' \leq 4r \leq S$, since $r \leq \sqrt{d/t}$. In particular, this yields $1 \leq R \leq S$, so that all conditions of Lemma 7.5 are satisfied.

In the remainder we show that x, y satisfy the parameter constraints. We have $n(x, y) \le (R+S)t = \mathcal{O}(St) = \mathcal{O}(d/(Rt')) = \mathcal{O}(d/\Delta + \sqrt{dt})$. Note that $d/\Delta = \mathcal{O}(L)$ by the parameter relation $d \le 2L(\Delta+1)$, and that $\sqrt{dt} = \mathcal{O}(\sqrt{dL^2/M}) = \mathcal{O}(L)$ by the parameter relation $d \le M$. Thus, $L(x,y) \le m(x,y) \le n(x,y) \le \mathcal{O}(L) \le \mathcal{O}(m) \le \mathcal{O}(n)$, which satisfies the parameters L, m, n.

For d, note that by Lemma 7.5(ii), we have $d = \Theta((Rt') \cdot (St)) = \Theta((Rt') \cdot d/(Rt')) = \Theta(d)$. For M, Lemma 7.5(iii) shows that $M(x,y) = \mathcal{O}(S^2t) = \mathcal{O}((d/(Rt'))^2 \cdot (1/t)) = \mathcal{O}(L^2 \cdot (M/L^2)) = \mathcal{O}(M)$, where we used $d/(Rt') = \mathcal{O}(L)$ as shown above. Since L(x,y) = |b| = St, we obtain $\delta(x,y) = 0 \le \mathcal{O}(\delta)$ and $\Delta(x,y) = Rt' \le \mathcal{O}(\Delta)$. Finally, $|\Sigma(x,y)| = t = \Theta(L^2/M) \le \mathcal{O}(|\Sigma|)$ follows from the parameter relation $M \ge L^2/|\Sigma|$. Observing that x,y, and L(x,y) = St can be computed in time $\mathcal{O}(n)$ concludes the proof.

9 Hardness for Large Alphabet

In this section, we consider a parameter setting α satisfying the relations of Table 2, and we prove a lower bound for $LCS_{\leq}(\alpha)$ assuming OVH, thus proving Theorem 4.6. We split our proof into the two cases $\alpha_{\delta} = \alpha_m$ (where L may be small) and $\alpha_L = \alpha_m$ (where L is large). For readability, but abusing notation, for the target value $\lceil n^{\alpha_p} \rceil$ of parameter p we typically simply write p.

In this section we can assume that

$$\alpha_L, \alpha_m, \alpha_\delta, \alpha_\Delta > 0 \quad \text{and} \quad \alpha_d > 1,$$
 (LB)

since otherwise the known $\tilde{\mathcal{O}}(n + \min\{d, \delta m, \delta \Delta\})$ algorithm runs in (near-)optimal time $\tilde{\mathcal{O}}(n)$ and there is nothing to show (here we used the parameter relations $d \leq Lm$ and $L, m, \delta, \Delta \leq n$).

9.1 Small LCS

Assume $\alpha_{\delta} = \alpha_m$, i.e., $\delta = \Theta(m)$. In this case, the longest common subsequence might be arbitrarily small, i.e., any value $0 < \alpha_L \le \alpha_m$ is admissible.

9.1.1 Hard Core

At the heart of our constructions lies the previous reduction from OV to LCS of [28], which we restate here for our purposes.

Lemma 9.1. Let two sets $A = \{a_1, \ldots, a_A\}$ and $B = \{b_1, \ldots, b_B\}$ of vectors in $\{0, 1\}^D$ with $A \ge B$ be given. In time O(AD), we can construct strings x_1, \ldots, x_{2A} and y_1, \ldots, y_B over $\{0, 1\}$ and $\gamma, \gamma' = \Theta(D)$ such that the strings x and y defined by

$$x := x_1 \ 0^{\gamma} \ x_2 \ 0^{\gamma} \ \dots \ x_{2A-1} \ 0^{\gamma} \ x_{2A},$$
$$y := 0^{A\gamma'} \ y_1 \ 0^{\gamma} \ y_2 \ 0^{\gamma} \ \dots \ y_{B-1} \ 0^{\gamma} \ y_B \ 0^{A\gamma'},$$

satisfy the following properties:

- (i) We can compute some ρ in time $\mathcal{O}(AD)$ such that $L(x,y) \geq \rho$ if and only if there is a pair i, j with $\langle a_i, b_j \rangle = 0$.
- (ii) $|x|, |y| = \mathcal{O}(AD)$.
- (iii) $\#_1(y) = \mathcal{O}(BD)$.
- (iv) For all $\beta \geq 0$, we have $L(x, 0^{\beta}y) = L(x, y)$.

Proof. Claims (i) and (ii) are a restatement of the main result in [28], instantiated for LCS. Claim (iii) follows directly from the construction. It is easy to check that in [28] the value of γ' is chosen large enough to satisfy $A\gamma' \geq \#_0(x)$. This yields claim (iv), since any common subsequence z of $x, 0^{\beta}y$ starts with at most $\#_0(x)$ symbols 0, which can all be matched to the $0^{A\gamma'}$ -block of y, so z is also a common subsequence of x, y, and we obtain $L(x, y) = L(x, 0^{\beta}y)$.

9.1.2 Constant Alphabet

First assume $\alpha_{\Sigma} = 0$ and thus $|\Sigma| = \mathcal{O}(1)$. Consider any $n \geq 1$ and target values $p = n^{\alpha_p}$ for $p \in \mathcal{P}$. Let $\mathcal{A} = \{a_1, \ldots, a_A\}$, $\mathcal{B} = \{b_1, \ldots, b_B\} \subseteq \{0, 1\}^D$ be a given OV instance with $D = n^{o(1)}$ and where we set $A := \lfloor L/D \rfloor$ and $B := \lfloor d/(LD) \rfloor$. Note that $A = n^{\alpha_L - o(1)} = n^{\Omega(1)}$ and $B = n^{\alpha_d - \alpha_L - o(1)} = n^{\Omega(1)}$ by (LB). Also note that UOVH implies that solving such OV instances takes time $(AB)^{1-o(1)} = n^{\alpha_d - o(1)} = d^{1-o(1)}$.

Construct strings x, y as in Lemma 9.1. Then from the LCS length L(x, y) we can infer whether \mathcal{A}, \mathcal{B} has an orthogonal pair of vectors by Lemma 9.1(i). Moreover, this reduction runs in time $\mathcal{O}(AD) = \mathcal{O}(L) \leq \mathcal{O}(d^{1-\varepsilon})$ for sufficiently small $\varepsilon > 0$ (since $\alpha_L \leq \alpha_m \leq 1 < \alpha_d$ by Table 2 and (LB)). We claim that (n, x, y) is an instance of $LCS_{\leq}(\alpha)$. This shows that any algorithm solving $LCS_{\leq}(\alpha)$ in time $\mathcal{O}(d^{1-\varepsilon})$ implies an algorithm for our OV instances with running time $\mathcal{O}(d^{1-\varepsilon})$, contradicting UOVH. Hence, in the current case $\alpha_{\delta} = \alpha_m$ and $\alpha_{\Sigma} = 0$, any algorithm for $LCS_{\leq}(\alpha)$ takes time $d^{1-o(1)}$, proving part of Theorem 4.6.

It remains to show the claim that (n, x, y) is an instance of $LCS_{\leq}(\alpha)$. Using the parameter relations $L \leq m \leq n$, Lemma 9.1(ii), and the definition of A, we have $L(x,y) \leq m(x,y) \leq n(x,y) = |x| \leq \mathcal{O}(AD) = \mathcal{O}(L) \leq \mathcal{O}(m) \leq \mathcal{O}(n)$, so indeed $p(x,y) \leq \mathcal{O}(p) = \mathcal{O}(n^{\alpha_p})$ for $p \in \{L, m, n\}$. Similarly, we obtain $\delta(x,y) \leq \Delta(x,y) \leq n(x,y) \leq \mathcal{O}(m) = \mathcal{O}(\delta) \leq \mathcal{O}(\Delta)$, where the equality holds in the current case $\alpha_{\delta} = \alpha_m$. Since x,y use the binary alphabet $\{0,1\}$, we have $|\Sigma(x,y)| = 2 \leq \mathcal{O}(n^{\alpha_{\Sigma}})$. For the number of matching pairs we have $M(x,y) \leq n(x,y)^2 = \mathcal{O}((AD)^2) = \mathcal{O}(L^2)$. Since we are in the case $\alpha_{\Sigma} = 0$, from the parameter relation $M \geq L^2/|\Sigma|$ (Lemma 6.6(i)) we obtain $L^2 \leq \mathcal{O}(M)$ and thus also M(x,y) is sufficiently small. Finally, we use Lemmas 6.8 and 9.1(iii) to bound $d(x,y) \leq \mathcal{O}(L(x,y) \cdot \#_1(y)) \leq \mathcal{O}(AD \cdot BD)$, which by definition of A, B is $\mathcal{O}(d)$. This proves that (n,x,y) belongs to $LCS_{\leq}(\alpha)$.

We remark that our proof also yields the following lemma, which we will use for small alphabets.

Lemma 9.2. Let α be a parameter setting satisfying Table 2 with $\alpha_{\Sigma} = 0$ and $\alpha_{\delta} = \alpha_{m}$. There is a constant $\gamma \geq 1$ such that any algorithm for $LCS_{\leq}^{\gamma}(\alpha, \{0, 1\})$ takes time $d^{1-o(1)}$, unless OVH fails. This holds even restricted to instances (n, x, y) of $LCS_{\leq}^{\gamma}(\alpha, \{0, 1\})$ with $|x|, |y| \leq \gamma \cdot n^{\alpha_{L}}$ and $\#_{1}(y) \leq \gamma \cdot n^{\alpha_{d} - \alpha_{L}}$ satisfying $L(x, 0^{\beta}y) = L(x, y)$ for all $\beta \geq 0$.

9.1.3 Superconstant Alphabet

To tackle the general case $\alpha_{\Sigma} \geq 0$ (while still assuming $\alpha_{\delta} = \alpha_m$), we use the following fact which is similar to the Disjoint Alphabets Lemma.

Lemma 9.3 (Crossing Alphabets). Let $\Sigma_1, \ldots, \Sigma_k$ be disjoint alphabets and let x_i, y_i be strings over alphabet Σ_i . Consider $x := x_1 \ldots x_k$ and $y := y_k \ldots y_1$, i.e., the order in y is reversed. For any parameter $p \in \{n, m, |\Sigma|, M, d\}$ we have $p(x, y) = \sum_{i=1}^k p(x_i, y_i)$. Moreover, $L(x, y) = \max_i L(x_i, y_i)$.

Proof. The statement is trivial for the string lengths n, m, alphabet size $|\Sigma|$, and number of matching pairs M. For the LCS length L we observe that any common subsequence z that matches a symbol in Σ_i cannot match any symbols in other alphabets, which yields $L(x, y) \leq \max_i L(x_i, y_i)$. Since any common subsequence of x_i, y_i is also a common subsequence of x, y, we obtain equality.

Since every dominant pair is also a matching pair, every dominant pair of x, y stems from prefixes $x_1
ldots x_{j-1}x'$ and $y_k
ldots y_{j+1}y'$, with x' a prefix of x_j and y' a prefix of y_j for some j. Since $L(x_1
ldots x_{j-1}x', y_k
ldots y_{j+1}y') = L(x', y')$, we obtain that the dominant pairs of x, y of the form $x_1
ldots x_{j-1}x', y_k
ldots y_{j+1}y'$ are in one-to-one correspondence with the dominant pairs of x_j, y_j . Since these dominant pairs of x, y are incomparable, this yields the claim for parameter d.

We make use of the above lemma by invoking the following construction.

Definition 9.4. Let $\Sigma_1, \ldots, \Sigma_t$ be a collection of disjoint two-element alphabets. For any string z over $\{0,1\}$ and Σ_i , let $z \uparrow \Sigma_i$ denote the string z lifted to Σ_i , i.e., we replace the symbols $\{0,1\}$ in z bijectively by Σ_i . Then for given $x_1, \ldots, x_t, y_1, \ldots, y_t \in \{0,1\}^*$ we construct

$$\operatorname{cr}^{X}(x_{1}, \dots, x_{t}) := x_{1} \uparrow \Sigma_{1} \qquad x_{2} \uparrow \Sigma_{2} \qquad \dots \qquad x_{t} \uparrow \Sigma_{t},$$

$$\operatorname{cr}^{Y}(y_{1}, \dots, y_{t}) := y_{t} \uparrow \Sigma_{t} \qquad y_{t-1} \uparrow \Sigma_{t-2} \qquad \dots \qquad y_{1} \uparrow \Sigma_{1}.$$

We adapt the construction from Lemma 9.1 using the following trick that realizes an "OR" of $t \leq \mathcal{O}(\Sigma)$ instances, without significantly increasing the parameters d and M.

Consider any $n \geq 1$ and target values $p = n^{\alpha_p}$ for $p \in \mathcal{P}$. Let $\mathcal{A} = \{a_1, \dots, a_A\}$, $\mathcal{B} = \{b_1, \dots, b_B\} \subseteq \{0, 1\}^D$ be a given OV instance with $D = n^{o(1)}$ and where we set $A = \lfloor \frac{d}{\min\{L, \sqrt{d}\} \cdot D} \rfloor$ and $B = \lfloor \frac{\min\{L, \sqrt{d}\}}{D} \rfloor$. Note that UOVH implies that solving such OV instances takes time $(AB)^{1-o(1)} = n^{\alpha_d - o(1)} = d^{1-o(1)}$. Since clearly $A \geq B$, we can partition \mathcal{A} into $t := \lceil A/B \rceil$ groups $\mathcal{A}_1, \dots, \mathcal{A}_t$ of size B (filling up the last group with all-ones vectors). Using the relation $d \leq L^2|\Sigma|$ (Lemma 6.3(ii)), we obtain $t \leq \mathcal{O}(d/L^2 + 1) \leq \mathcal{O}(\Sigma)$.

For each $i=1,\ldots,t$ we construct strings x_i and y_i for the sets \mathcal{A}_i and \mathcal{B} using Lemma 9.1. Finally, we set $x:=\operatorname{cr}^{\mathsf{x}}(x_1,\ldots,x_t)$ and $y:=\operatorname{cr}^{\mathsf{y}}(y_1,\ldots,y_t)$. By the Crossing Alphabets Lemma and Lemma 9.1.(i) from L(x,y) we can infer whether \mathcal{A},\mathcal{B} has an orthogonal pair of vectors. We claim that (n,x,y) is an instance of $\operatorname{LCS}_{\leq}(\boldsymbol{\alpha})$. This shows that any algorithm solving $\operatorname{LCS}_{\leq}(\boldsymbol{\alpha})$ in time $\mathcal{O}(d^{1-\varepsilon})$ implies an algorithm for our OV instances with running time $\mathcal{O}(d^{1-\varepsilon})$, contradicting UOVH. Hence, in the current case $\alpha_{\delta}=\alpha_m$, any algorithm for $\operatorname{LCS}_{\leq}(\boldsymbol{\alpha})$ takes time $d^{1-o(1)}$, proving part of Theorem 4.6.

It remains to show the claim that (n,x,y) is an instance of $LCS_{\leq}(\alpha)$. This is similar to the proof for the case $\alpha_{\Sigma}=0$, additionally using the Crossing Aphabets Lemma. Specifically, we obtain $m(x,y) \leq n(x,y) = |x| = \sum_{i=1}^t |x_i| \leq \mathcal{O}(t \cdot BD) = \mathcal{O}(AD) = \mathcal{O}(\max\{d/L,\sqrt{d}\})$, which is at most $\mathcal{O}(m) \leq \mathcal{O}(n)$ using the parameter relations $d \leq Lm \leq m^2$ (Lemma 6.3.(i)). Similarly, we obtain $\delta(x,y) \leq \Delta(x,y) \leq n(x,y) \leq \mathcal{O}(m) = \mathcal{O}(\delta) \leq \mathcal{O}(\Delta)$, where the equality holds in the current case $\alpha_m = \alpha_\delta$. For L we obtain $L(x,y) = \max_i L(x_i,y_i) \leq |y_i| = \mathcal{O}(BD) \leq \mathcal{O}(L)$. Since $t \leq \mathcal{O}(\Sigma)$ we have $|\Sigma(x,y)| \leq \mathcal{O}(\Sigma)$. Using the parameter relation $d \leq M$ we have $d(x,y) \leq M(x,y) = \sum_{i=1}^t M(x_i,y_i) \leq t \cdot |x_i| \cdot |y_i| = t \cdot \mathcal{O}((BD)^2) = \mathcal{O}(AD \cdot BD) = \mathcal{O}(d) \leq \mathcal{O}(M)$. This proves that (n,x,y) belongs to $LCS_{\leq}(\alpha)$.

9.2 Large LCS

Now assume $\alpha_L = \alpha_m$, i.e., $L = \Theta(m)$. Then the number of deletions in the shorter string might be arbitrary small, i.e., any value $0 < \alpha_{\delta} \le \alpha_m$ is admissible. In this case, the construction of [28] is no longer applicable. The new 1vs1/2vs1 gadgets that we design for constructing hard strings for small δ can be seen as one of our main contributions.

9.2.1 Hard Core

The following lemma (which effectively represents an intermediate step in the proof of [28]) yields the basic method to embed sets of vectors into strings x and y.

Lemma 9.5. Let two sets $A = \{a_1, \ldots, a_A\}$ and $B = \{b_1, \ldots, b_B\}$ of vectors in $\{0, 1\}^D$ be given. In time $\mathcal{O}((A+B)D)$ we can construct strings x_1, \ldots, x_A of length ℓ_X and y_1, \ldots, y_B of length ℓ_Y over alphabet $\{0, 1\}$, as well as integers $\rho_1 < \rho_0$, such that for all $i \in [A]$, $j \in [B]$ we have

- (i) $\ell_{\rm Y} \leq \ell_{\rm X} \leq \mathcal{O}(D)$,
- (ii) $L(x_i, y_j) = \rho_0$ if $\langle a_i, b_j \rangle = 0$,
- (iii) $L(x_i, y_j) = \rho_1$ if $\langle a_i, b_j \rangle \neq 0$, and
- (iv) $L(x_i, y_i) > \ell_Y/2$.

Proof. We can construct strings x'_1, \ldots, x'_A of length $\ell'_{\mathsf{X}} = \mathcal{O}(D)$ and y'_1, \ldots, y'_B of length $\ell'_{\mathsf{Y}} = \mathcal{O}(D)$ and integers $\rho'_1 < \rho'_0$ as in [28, Claim III.6] (using the so called normalized vector gadget) that satisfy $L(x'_i, y'_j) = \rho'_0$ if $\langle a_i, b_j \rangle = 0$ and $L(x'_i, y'_j) = \rho'_1$ otherwise. To additionally enforce conditions (i) and (iv), we define $x_i := 1^{\ell'_{\mathsf{Y}}} 0^{\ell'_{\mathsf{Y}}+1} x'_i$ and $y_j := 0^{\ell'_{\mathsf{Y}}+1} y'_j$. Since $L(x_i, y_j) = L(x'_i, y'_j) + \ell'_{\mathsf{Y}} + 1$ by Lemmas 7.6 and 7.1, we thus obtain conditions (ii) and (iii) for $\rho_0 := \rho'_0 + \ell'_{\mathsf{Y}} + 1$ and $\rho_1 := \rho'_1 + \ell'_{\mathsf{Y}} + 1$. Since by definition $\ell_{\mathsf{Y}} = 2\ell'_{\mathsf{Y}} + 1$ holds, the first condition follows directly and the trivial bound $L(x_i, y_j) \geq \ell'_{\mathsf{Y}} + 1 > \ell_{\mathsf{Y}}/2$ shows that the last condition is fulfilled.

1vs1/2vs1 gadget. The aim of the following construction is to embed given strings y_1, \ldots, y_Q into a string y and strings x_1, \ldots, x_P into x, where $P = \Theta(Q)$, such that in an LCS each y_j is either aligned with a single string x_i or with several strings $x_i, x_{i+1}, \ldots, x_{i'}$. In the first case, $|y_j| - L(x_i, y_j)$ characters of y_j are not contained in an LCS of x and y, while in the second case y_j can be completely aligned. By choosing P = 2Q - N for an arbitrary $1 \le N \le Q$, it will turn out that the LCS aligns N strings y_j with a single partner x_i , and the remaining Q - N strings y_j with two strings x_i, x_{i+1} each. Thus, only N strings y_j are not completely aligned.

To formalize this intuition, let $P \geq Q$. We call a set $\Lambda = \{(i_1, j_1), \dots, (i_k, j_k)\}$ with $0 \leq k \leq Q$ and $1 \leq i_1 < i_2 < \dots < i_k \leq P$ and $1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq Q$ a (partial) multi-alignment. Let $\Lambda(j) = \{i \mid (i, j) \in \Lambda\}$. We say that every $j \in [Q]$ with $|\Lambda(j)| = k$ is k-aligned. We will also refer to a 1-aligned $j \in [Q]$ as being uniquely aligned to i, where $\Lambda(j) = \{i\}$. Every $j \in [Q]$ with $\Lambda(j) = \emptyset$ is called unaligned. Note that each $i \in [P]$ occurs in at most one $(i, j) \in \Lambda$. We denote the set of multi-alignments as $\Lambda_{P,Q}^{\text{multi}}$.

We will also need the following specialization of multi-alignments. We call a multi-alignment $\Lambda \in \Lambda_{P,Q}^{\text{multi}}$ a (1,2)-alignment, if each j is either 1-aligned or 2-aligned. Let $\Lambda_{P,Q}^{1,2}$ denote the set of all (1,2)-alignments.

Given strings x_1, \ldots, x_P of length ℓ_X and y_1, \ldots, y_Q of length ℓ_Y , we define the value $v(\Lambda)$ of a multi-alignment $\Lambda \in \mathbf{\Lambda}_{P,Q}^{\text{multi}}$ as $v(\Lambda) = \sum_{j=1}^{Q} v_j$ where

$$v_j := \begin{cases} 0 & \text{if } j \text{ is unaligned,} \\ L(x_i, y_j) & \text{if } j \text{ is uniquely aligned to } i, \\ \ell_Y & \text{if } j \text{ is } k\text{-aligned for } k \geq 2. \end{cases}$$

Lemma 9.6. Given strings x_1, \ldots, x_P of length ℓ_X and y_1, \ldots, y_Q of length ℓ_Y , construct

$$x := G(x_1) G(x_2) \dots G(x_P),$$

 $y := G(y_1) G(y_2) \dots G(y_Q),$

where $G(w) := 0^{\gamma_1} 1^{\gamma_2} (01)^{\gamma_3} w 1^{\gamma_3}$ with $\gamma_3 := \ell_x + \ell_y$, $\gamma_2 := 8\gamma_3$ and $\gamma_1 := 6\gamma_2$. Then we have

$$\max_{\Lambda \in \mathbf{\Lambda}_{P,Q}^{1,2}} v(\Lambda) \le L(x,y) - Q(\gamma_1 + \gamma_2 + 3\gamma_3) \le \max_{\Lambda \in \mathbf{\Lambda}_{P,Q}^{\text{multi}}} v(\Lambda). \tag{2}$$

Proof. For the first inequality of (2), let $\Lambda \in \Lambda_{P,Q}^{1,2}$. For every y_j , we define $z_j = \bigcap_{i \in \Lambda(j)} G(x_i)$. Consider a 1-aligned j and let $i \in [P]$ be the index j is uniquely aligned to. We have that $z_j = G(x_i) = 0^{\gamma_1} 1^{\gamma_2} (01)^{\gamma_3} x_i 1^{\gamma_3}$ and hence by Lemma 7.1, we obtain $L(z_j, G(y_j)) = \gamma_1 + \gamma_2 + 3\gamma_3 + L(x_i, y_j) = \gamma_1 + \gamma_2 + 3\gamma_3 + v_j$. Likewise, consider a 2-aligned j and let $i, i' \in [P]$ be such that $\Lambda(j) = \{i, i'\}$. Then $z_j = G(x_i)G(x_{i'})$. We compute

$$L(z_j, G(y_j)) = \gamma_1 + \gamma_2 + 3\gamma_3 + L(x_i 1^{\gamma_3} 0^{\gamma_1} 1^{\gamma_2} (01)^{\gamma_3} x_{i'}, y_j)$$

$$\geq \gamma_1 + \gamma_2 + 3\gamma_3 + L((01)^{\gamma_3}, y_j)$$

$$= \gamma_1 + \gamma_2 + 3\gamma_3 + \ell_Y = \gamma_1 + \gamma_2 + 3\gamma_3 + v_j,$$

where the first line follows from Lemma 7.1, the second line from monotonicity and the third line from $\gamma_3 \geq \ell_Y = |y_j|$. Observe that $z_1 z_2 \dots z_Q$ is a subsequence of x. We conclude that

$$L(x,y) \ge \sum_{j=1}^{Q} L(z_j, G(y_j)) = Q(\gamma_1 + \gamma_2 + 3\gamma_3) + \sum_{j=1}^{Q} v_j.$$

It remains to prove the second inequality of (2). Write $x = z_1 z_2 \dots z_Q$ such that $L(x,y) = \sum_{j=1}^Q L(z_j, G(y_j))$. We define a multi-alignment Λ by letting $(i,j) \in \Lambda$ if and only if z_j contains strictly more than half of the 0^{γ_1} -block of $G(x_i)$. Note that the thus defined set satisfies the definition of a multi-alignment, since no two z_j 's can contain more than half of $G(x_i)$'s 0^{γ_1} -block and if $(i,j), (i',j') \in \Lambda$, then j < j' implies i < i'. It remains to show that $L(z_j, G(y_j)) \leq \gamma_1 + \gamma_2 + 3\gamma_3 + v_j$ for all j to prove the claim.

In what follows, we use the shorthand $H(w) := 1^{\gamma_2} (01)^{\gamma_3} w 1^{\gamma_3}$. Note that $G(w) = 0^{\gamma_1} H(w)$. Consider an unaligned $j \in [Q]$. By definition, z_j is a subsequence of $0^{\gamma_1/2} H(x_i) 0^{\gamma_1/2}$ for some $i \in [P]$. We can thus bound (using Lemma 7.1)

$$L(z_j, \mathcal{G}(y_j)) \leq L(0^{\gamma_1/2}H(x_i)0^{\gamma_1/2}, 0^{\gamma_1}H(y_j)) = \frac{\gamma_1}{2} + L(H(x_i)0^{\gamma_1/2}, 0^{\gamma_1/2}H(y_j)).$$

By Lemma 7.6 with $\ell := \gamma_1/2 \ge 2\gamma_2 + 6\gamma_3 + \ell_X + \ell_Y = |H(x_i)| + |H(y_i)| \ge \#_0(H(x_i)) + |H(y_i)|$,

$$L(H(x_i)0^{\gamma_1/2},0^{\gamma_1/2}H(y_j)) = \gamma_1/2 + L(0^{\#_0(H(x_i))},H(y_j)) \le \gamma_1/2 + \#_0(H(y_j)) \le \gamma_1/2 + \gamma_3 + \ell_Y.$$

Hence, in total we have $L(z_j, G(y_j)) \leq \gamma_1 + \gamma_3 + \ell_Y \leq \gamma_1 + \gamma_2 + 3\gamma_3 = \gamma_1 + \gamma_2 + 3\gamma_3 + v_j$, as desired. Consider a $j \in [Q]$ that is uniquely aligned (under Λ) to some i. Then z_j is a subsequence of $0^{\gamma_1/2}H(x_{i-1})0^{\gamma_1}H(x_i)0^{\gamma_1/2}$. Analogously to above we compute

$$\begin{split} L(z_{j}, \mathbf{G}(y_{j})) & \leq \frac{\gamma_{1}}{2} + L(H(x_{i-1})0^{\gamma_{1}}H(x_{i})0^{\gamma_{1}/2}, 0^{\gamma_{1}/2}H(y_{j})) \\ & = \gamma_{1} + L(0^{\#_{0}(H(x_{i-1})) + \gamma_{1}}H(x_{i})0^{\gamma_{1}/2}, H(y_{j})) \\ & = \gamma_{1} + L(0^{\#_{0}(H(x_{i-1})) + \gamma_{1}}1^{\gamma_{2}}(01)^{\gamma_{3}}x_{i}1^{\gamma_{3}}0^{\gamma_{1}/2}, 1^{\gamma_{2}}(01)^{\gamma_{3}}y_{j}1^{\gamma_{3}}). \end{split}$$

Using Lemma 7.6 with symbol 0 replaced by 1 yields, since $\ell := \gamma_2 \ge 3\gamma_3 + \ell_Y = |(01)^{\gamma_3} y_j 1^{\gamma_3}|$ and $\#_1(0^{\#_0(H(x_{i-1}))+\gamma_1}) = 0$,

$$L(z_j, G(y_j)) \le \gamma_1 + \gamma_2 + L((01)^{\gamma_3} x_i 1^{\gamma_3} 0^{\gamma_1/2}, (01)^{\gamma_3} y_j 1^{\gamma_3}) = \gamma_1 + \gamma_2 + 2\gamma_3 + L(x_i 1^{\gamma_3} 0^{\gamma_1/2}, y_j 1^{\gamma_3}).$$

Similarly, using Lemma 7.6 with symbol 0 replaced by 1 on the reversed strings yields, since $\ell := \gamma_3 \ge \ell_Y = |y_j|$ and $\#_1(0^{\gamma_1/2}) = 0$,

$$L(x_i 1^{\gamma_3} 0^{\gamma_1/2}, y_i 1^{\gamma_3}) = \gamma_3 + L(x_i, y_i).$$

Hence, we obtain the desired $L(z_i, G(y_i)) \leq \gamma_1 + \gamma_2 + 3\gamma_3 + L(x_i, y_i) = \gamma_1 + \gamma_2 + 3\gamma_3 + v_i$.

It remains to consider $j \in [Q]$ that is k-aligned for $k \geq 2$. In this case, the claim follows from the trivial bound $L(z_j, G(y_j)) \leq |G(y_j)| = \gamma_1 + \gamma_2 + 3\gamma_3 + v_j$.

Thus z_1, \ldots, z_Q defines a multi-alignment $\Lambda \in \Lambda_{P,Q}^{\text{multi}}$ with

$$L(x,y) = \sum_{j=1}^{Q} L(z_j, G(y_j)) \le Q(\gamma_1 + \gamma_2 + 3\gamma_3) + v(\Lambda),$$

proving the second inequality of (2).

We can now show how to embed an OV instance $\mathcal{A} = \{a_1, \ldots, a_A\}, \mathcal{B} = \{b_1, \ldots, b_B\} \subseteq \{0, 1\}^D$ with $A \leq B$ into strings x and y of length $\mathcal{O}(B \cdot D)$ whose LCS can be obtained by deleting at most $\mathcal{O}(A \cdot D)$ symbols from y. For this we will without loss of generality assume that A divides B by possibly duplicating some arbitrary element of \mathcal{B} up to A-1 times without affecting the solution of the instance.

The key idea is that for any P and Q = 2P - N with $N \in \{0, \dots, P\}$, $\Lambda_{P,Q}^{1,2}$ is non-empty and each $\Lambda \in \Lambda_{P,Q}^{1,2}$ has exactly N uniquely aligned $j \in [Q]$ and exactly P - N 2-aligned $j \in [Q]$. At the same time each $\Lambda \in \Lambda_{P,Q}^{\text{multi}}$ leaves at least N indices $j \in [Q]$ either unaligned or uniquely aligned.

Lemma 9.7. Let $a_1, \ldots, a_A, b_1, \ldots b_B \subseteq \{0,1\}^D$ be given with $A \mid B$. Construct the corresponding strings x_1, \ldots, x_A of length ℓ_X , y_1, \ldots, y_B of length $\ell_Y \leq \ell_X \leq \mathcal{O}(D)$, and integers ρ_0, ρ_1 as in Lemma 9.5 and define

$$\tilde{x} := (\tilde{x}_1, \dots, \tilde{x}_P) = (\overbrace{x_1, \dots, x_A, x_1, \dots, x_A, \dots, x_1, \dots, x_A}^{2 \cdot (B/A) + 3 \text{ groups of size } A}),$$

$$\tilde{y} := (\tilde{y}_1, \dots, \tilde{y}_Q) = (\underbrace{y_1, \dots, y_1}_{A \text{ copies of } y_1}, y_1, \dots, y_B, \underbrace{y_1, \dots, y_1}_{A \text{ copies of } y_1}),$$

where P := 2B + 3A and Q := B + 2A. Then the instance $x := \bigcap_i G(\tilde{x}_i)$, $y := \bigcap_j G(\tilde{y}_j)$ of Lemma 9.6 (with the corresponding choice of γ_1, γ_2 and γ_3) satisfies the following properties:

- (i) For every $i \in [A], j \in [B]$, there is a (1,2)-alignment $\Lambda \in \Lambda_{P,Q}^{1,2}$ such that some $\ell \in [Q]$ is uniquely aligned to some $k \in [P]$ with $\tilde{x}_k = x_i$ and $\tilde{y}_\ell = y_j$.
- (ii) We have $L(x,y) \geq Q(\gamma_1 + \gamma_2 + 3\gamma_3) + (A-1)\rho_1 + \rho_0 + (Q-A)\ell_Y$ if and only if there are $i \in [A], j \in [B]$ with $\langle a_i, b_i \rangle = 0$.
- (iii) We have $|y| \le |x| \le \mathcal{O}(B \cdot D)$ and $\delta(x, y) = \mathcal{O}(A \cdot D)$.

Proof. For (i), we let $j \in [B]$ and note that $y_j = \tilde{y}_\ell$ for $\ell := A + j$. We will show that for every $\lambda \in \{0, \ldots, A-1\}$, there is a (1,2)-alignment Λ with $(k,\ell) \in \mathbf{\Lambda}_{P,Q}^{1,2}$ where $k := 2(A+j)-1-\lambda$. By the cyclic structure of \tilde{x} , $(\tilde{x}_k)_{0 \leq \lambda < A}$ cycles through all values x_1, \ldots, x_A . Hence, for some choice of λ the desired $\tilde{x}_k = x_i$ follows, yielding the claim.

To see that for any $\lambda \in \{0, \dots, A-1\}$, some $\Lambda \in \Lambda_{P,Q}^{1,2}$ with $(k,\ell) \in \Lambda$ exists, observe that there are $\ell-1=A+j-1$ predecessors of \tilde{y}_ℓ and $k-1=2(A+j-1)-\lambda=2(\ell-1)-\lambda$ predecessors of \tilde{x}_k . Hence there is a (1,2)-alignment $\Lambda_1 \in \Lambda_{k-1,\ell-1}^{1,2}$ (leaving λ indices $j \in [Q]$ uniquely aligned). Similarly, observe that there are $Q-\ell=B+A-j$ successors of \tilde{y}_ℓ and $P-k=2B+A-2j+\lambda+1=2(Q-\ell)-(A-\lambda-1)$ successors of \tilde{x}_k , hence there is a (1,2)-alignment $\Lambda_2 \in \Lambda_{P-k,Q-\ell}^{1,2}$ (which leaves $A-(\lambda+1)$ indices j uniquely aligned). By canonically composing Λ_1 , (k,ℓ) and Λ_2 we can thus obtain $\Lambda \in \Lambda_{P,Q}^{1,2}$ with $(k,\ell) \in \Lambda$.

For (ii), assume that there are $i \in [A], j \in [B]$ satisfying $\langle a_i, b_j \rangle = 0$. By (i), there is some $\Lambda \in \Lambda_{P,Q}^{1,2}$ where some $\ell \in [Q]$ is uniquely aligned to some $k \in [P]$ such that $\tilde{x}_k = x_i$ and $\tilde{y}_\ell = y_j$. To apply Lemma 9.6, observe that Λ has Q - A 2-aligned $j \in [Q]$, which contribute value ℓ_Y to $v(\Lambda)$, and A uniquely aligned $j \in [Q]$, in particular, ℓ is uniquely aligned to k. Since any \tilde{x}_i corresponds to some $x_{i'}$, every \tilde{y}_j corresponds to some $y_{j'}$ and $L(x_{i'}, y_{j'}) \in \{\rho_0, \rho_1\}$, we conclude that ℓ contributes ρ_0 to $v(\Lambda)$ and the other A-1 uniquely aligned j contribute at least ρ_1 . Hence by the lower bound in Lemma 9.6, we obtain $L(x, y) \geq Q(\gamma_1 + \gamma_2 + 3\gamma_3) + v(\Lambda)$, where $v(\Lambda) \geq (A-1)\rho_1 + \rho_0 + (Q-A)\ell_Y$.

Assume now that no $i \in [A], j \in [B]$ satisfy $\langle a_i, b_j \rangle = 0$, and let $\Lambda \in \Lambda_{P,Q}^{\text{multi}}$. Then any $j \in [Q]$ uniquely aligned to some $i \in [P]$ contributes $L(\tilde{x}_i, \tilde{y}_j) = \rho_1$ to $v(\Lambda)$. Let λ be the number of $j \in [Q]$ that are k-aligned for any $k \geq 2$, each contributing ℓ_Y to $v(\Lambda)$. Then there are at most $\min\{P-2\lambda,Q-\lambda\}$ uniquely aligned $j \in [Q]$ (since every k-aligned j blocks at least two $i \in [P]$ for other alignments), and the remaining $j \in [Q]$ are unaligned, with no contribution to $v(\Lambda)$. Hence $v(\Lambda) \leq \lambda \ell_Y + \min\{P-2\lambda,Q-\lambda\} \cdot \rho_1 = \min\{P\rho_1 + (\ell_Y-2\rho_1)\lambda,Q\rho_1 + (\ell_Y-\rho_1)\lambda\}$. Note that $\ell_Y/2 < \rho_1 \leq \ell_Y$ (by Lemma 9.5(iv)), hence this minimum of linear functions with leading coefficients $\ell_Y - 2\rho_1 < 0$ and $\ell_Y - \rho_1 \geq 0$ is maximized when both have the same value, i.e., when $\lambda = P - Q = Q - A$. Thus, $v(\Lambda) \leq (Q - A)\ell_Y + A\rho_1 < (Q - A)\ell_Y + (A - 1)\rho_1 + \rho_0$. Thus by the upper bound of Lemma 9.6 we conclude that $L(x,y) < Q(\gamma_1 + \gamma_2 + 3\gamma_3) + (Q - A)\ell_Y + (A - 1)\rho_1 + \rho_0$.

For (iii), since $P \geq Q$ and $\ell_{X} \geq \ell_{Y}$ we have $|x| \geq |y|$, and by $P \leq \mathcal{O}(A)$ and $|G(\tilde{x}_{i})| \leq \mathcal{O}(\ell_{X}) \leq \mathcal{O}(D)$ we obtain $|x| \leq \mathcal{O}(AD)$. Note that for any (1,2)-alignment $\Lambda \in \Lambda_{P,Q}^{1,2}$, we have

$$v(\Lambda) = Q \cdot \ell_{\mathrm{Y}} - \sum_{j \text{ uniquely aligned to } i} (\ell_{\mathrm{Y}} - L(x_i, y_j)) = Q \cdot \ell_{\mathrm{Y}} - \mathcal{O}(A \cdot D),$$

since by P = 2Q - A the number of uniquely aligned indices j in Λ equals A, and $\ell_Y = \mathcal{O}(D)$. Hence by Lemma 9.6, $L(x,y) \geq Q(\gamma_1 + \gamma_2 + 3\gamma_3) + Q\ell_Y - \mathcal{O}(A \cdot D) = |y| - \mathcal{O}(A \cdot D)$, implying $\delta(x,y) = |y| - L(x,y) \leq \mathcal{O}(A \cdot D)$.

9.2.2 Constant Alphabet

First assume $\alpha_{\Sigma} = 0$ and thus $|\Sigma| = \mathcal{O}(1)$. Consider any $n \geq 1$ and target values $p = n^{\alpha_p}$ for $p \in \mathcal{P}$. We write $\lfloor x \rfloor_2$ for the largest power of 2 less than or equal to x. Let $\mathcal{A} = \{a_1, \ldots, a_A\}$, $\mathcal{B} = \{b_1, \ldots, b_B\} \subseteq \{0, 1\}^D$ be a given OV instance with $D = n^{o(1)}$ and where we set

$$A := \left\lfloor \frac{1}{D} \min \left\{ \delta, \frac{d}{\min\{m, \Delta\}} \right\} \right\rfloor_2 \quad \text{and} \quad B := \left\lfloor \frac{1}{D} \min\{m, \Delta\} \right\rfloor_2.$$

By $\alpha_m, \alpha_\Delta \leq 1$ and (LB) we obtain $A \geq n^{\min\{\alpha_L, \alpha_d - 1\} - o(1)} = n^{\Omega(1)}$ and $B = n^{\min\{\alpha_m, \alpha_\Delta\} - o(1)} = n^{\Omega(1)}$. Also note that UOVH implies that solving such OV instances takes time $(AB)^{1-o(1)} = \min\{d, \delta m, \delta \Delta\}^{1-o(1)}$, which is the desired bound. We claim that $A \leq B$, implying $A \mid B$. Indeed, if $\delta \leq d/\min\{m, \Delta\}$ this follows from the simple parameter relations $\delta \leq m$ and $\delta \leq \Delta$. Otherwise, if $\delta > d/\min\{m, \Delta\}$, then in particular $\delta \Delta > d$, implying $d < \Delta^2$. Together with the parameter relations $d \leq Lm \leq m^2$ we indeed obtain $d/\min\{m, \Delta\} \leq \min\{m, \Delta\}$.

Thus, we may construct strings x,y as in Lemma 9.7. We finish the construction by invoking the Dominant Pair Reduction (Lemma 7.8) to obtain strings $x' := 0^k 1^k y 1^\ell 0^k 1^k x$ and $y' := 1^\ell 0^k 1^k y$ with k := 2|y| + |x| + 1 and $\ell := \Theta(A \cdot D)$ with sufficiently large hidden constant, so that $\ell > \delta(x,y)$. Then from the LCS length L(x',y') we can infer whether \mathcal{A},\mathcal{B} has an orthogonal pair of vectors by $L(x',y') = L(x,y) + \ell + 2k$ and Lemma 9.7(ii). Moreover, this reduction runs in time $\mathcal{O}(|x'| + |y'|) = \mathcal{O}(|x| + |y|) = \mathcal{O}(BD) \leq \mathcal{O}(\min\{d,\delta m,\delta \Delta\}^{1-\varepsilon})$ for sufficiently small $\varepsilon > 0$ (since $\alpha_{\delta} > 0$ and $\alpha_{d} > 1 \geq \alpha_{m}, \alpha_{\Delta}$ by (LB) and Table 2). We claim that (n,x',y') is an instance of LCS $\leq (\alpha)$. This

shows that any algorithm solving $LCS_{\leq}(\boldsymbol{\alpha})$ in time $\mathcal{O}(\min\{d, \delta m, \delta \Delta\}^{1-\varepsilon})$ implies an algorithm for our OV instances with running time $\mathcal{O}(\min\{d, \delta m, \delta \Delta\}^{1-\varepsilon})$, contradicting UOVH. Hence, in the current case $\alpha_L = \alpha_m$ and $\alpha_{\Sigma} = 0$, any algorithm for $LCS_{\leq}(\boldsymbol{\alpha})$ takes time $\min\{d, \delta m, \delta \Delta\}^{1-o(1)}$, proving part of Theorem 4.6.

It remains to show the claim that (n, x', y') is an instance of $LCS_{\leq}(\alpha)$. From Lemmas 7.8 and 9.7(iii) we obtain $L(x', y') = \ell + 2k + L(x, y) = \ell + 2k + |y| - \delta(x, y) \ge |y'| - \mathcal{O}(AD)$, and thus $\delta(x', y') \le \mathcal{O}(AD) \le \mathcal{O}(\delta)$. Using the parameter relations $L \le m \le n$, Lemma 9.7(iii), and the definition of B, we have $L(x', y') \le m(x', y') \le n(x', y') = |x'| \le \mathcal{O}(BD) = \mathcal{O}(\min\{m, \Delta\})$, which together with the relation $m \le n$ and the assumption $\alpha_L = \alpha_m$ shows that $p(x', y') \le \mathcal{O}(p) = \mathcal{O}(n^{\alpha_p})$ for $p \in \{L, m, n\}$. Similarly, we obtain $\Delta(x', y') \le n(x', y') \le \mathcal{O}(\min\{m, \Delta\}) \le \mathcal{O}(\Delta)$. Since x', y' use the binary alphabet $\{0, 1\}$, we have $|\Sigma(x', y')| = 2 \le \mathcal{O}(n^{\alpha_{\Sigma}})$. For the number of matching pairs we have $M(x', y') \le n(x', y')^2 = \mathcal{O}((BD)^2) = \mathcal{O}(L^2)$. Since we are in the case $\alpha_{\Sigma} = 0$, from the parameter relation $M \ge L^2/|\Sigma|$ (Lemma 6.6(i)) we obtain $L^2 \le \mathcal{O}(M)$ and thus also M(x', y') is sufficiently small. Finally, we use Lemma 7.8 to bound $d(x', y') \le \mathcal{O}(\ell \cdot |y|) \le \mathcal{O}(AD \cdot BD)$, which by definition of A, B is $\mathcal{O}(d)$. This proves that (n, x', y') belongs to $LCS_{\le}(\alpha)$.

We remark that our proof also yields the following, which we will use for small alphabets.

Lemma 9.8. Let α be a parameter setting satisfying Table 2 with $\alpha_{\Sigma} = 0$ and $\alpha_{L} = \alpha_{m}$. There is a constant $\gamma \geq 1$ such that any algorithm for $LCS_{\leq}^{\gamma}(\alpha, \{0, 1\})$ takes time $\min\{d, \delta m, \delta \Delta\}^{1-o(1)}$, unless OVH fails. This holds even restricted to instances (n, x, y) of $LCS_{\leq}^{\gamma}(\alpha, \{0, 1\})$ with $|y| \leq |x| \leq \gamma \cdot \min\{n^{\alpha_{m}}, n^{\alpha_{\Delta}}\}$.

9.2.3 Superconstant Alphabet

The crucial step in extending our construction to larger alphabets is to adapt the 1vs1/2vs1 gadget such that the strings use each symbol in the alphabet Σ roughly evenly, thus reducing the number of matching pairs by a factor $|\Sigma|$.

Recall that given a 2-element alphabet Σ' and a string z over $\{0,1\}$, we let $z \uparrow \Sigma'$ denote the string z lifted to alphabet Σ' by bijectively replacing $\{0,1\}$ with Σ' .

Lemma 9.9. Let P = 2B + 3A and Q = B + 2A for some $A \mid B$. Given strings x_1, \ldots, x_P of length ℓ_X and y_1, \ldots, y_Q of length ℓ_Y , we define, as in Lemma 9.6, $G(w) := 0^{\gamma_1} 1^{\gamma_2} (01)^{\gamma_3} w 1^{\gamma_3}$ with $\gamma_3 := \ell_X + \ell_Y$, $\gamma_2 := 8\gamma_3$ and $\gamma_1 := 6\gamma_2$. Let $\Sigma_1, \ldots, \Sigma_t$ be disjoint alphabets of size 2 with $Q/t \geq A/2 + 1$. We define

$$x := H(x_1) H(x_2) \dots H(x_P),$$

$$y := G(y_1) \uparrow \Sigma_{f(1)} \quad G(y_2) \uparrow \Sigma_{f(2)} \quad \dots \quad G(y_Q) \uparrow \Sigma_{f(Q)},$$

where $f(j) = \lceil \frac{j}{Q} \cdot t \rceil$ and

$$H(x_i) := \begin{cases} G(x_i) \uparrow \Sigma_{k+1} G(x_i) \uparrow \Sigma_k & \text{if } \bigcup_{j=\lceil i/2 \rceil}^{\lfloor (i+A)/2 \rfloor} \{f(j)\} = \{k, k+1\} \\ G(x_i) \uparrow \Sigma_k & \text{if } \bigcup_{j=\lceil i/2 \rceil}^{\lfloor (i+A)/2 \rfloor} \{f(j)\} = \{k\} \end{cases}$$

Then we have

$$\max_{\Lambda \in \mathbf{\Lambda}_{P,Q}^{1,2}} v(\Lambda) \le L(x,y) - Q(\gamma_1 + \gamma_2 + 3\gamma_3) \le \max_{\Lambda \in \mathbf{\Lambda}_{P,Q}^{\text{multi}}} v(\Lambda). \tag{3}$$

Proof. Note that $H(\cdot)$ is well-defined, since $f(\cdot)$ maps $\{1,\ldots,Q\}$ to constant-valued intervals of length at least $Q/t-1\geq A/2$, as f(j)=k if and only if $j\in\left(\frac{Qk}{t}-\frac{Q}{t},\frac{Qk}{t}\right]$, containing at least

Q/t-1 integers. Hence for every i, the $\leq A/2$ values $f(\lceil i/2 \rceil), \ldots, f(\lfloor (i+A)/2 \rfloor)$ can touch at most 2 different constant-valued intervals.

The proof of (3) is based on the proof of Lemma 9.6 (the analogous lemma for alphabet $\Sigma = \{0,1\}$). For the first inequality of (3), let $\Lambda \in \Lambda_{P,Q}^{1,2}$ and define for every j the substring $z'_j = \bigcup_{i \in \Lambda(j)} H(x_i)$. Note that under Λ , each $\Lambda(j)$ consists of one or two elements from $\{2j - A, \dots, 2j\}$, since there are at most 2Q - P = A uniquely aligned j. In other words, for any $i \in \Lambda(j)$ we have $j \in \{\lceil i/2 \rceil, \dots, \lfloor (i+A)/2 \rfloor\}$. Thus, by definition each $H(x_i)$ for $i \in \Lambda(j)$ contains $G(x_i) \uparrow \Sigma_{f(j)}$ as a substring and hence z'_j contains $\bigcap_{i \in \Lambda(j)} G(x_i) \uparrow \Sigma_{f(j)}$ as a subsequence. This proves

$$L(z_j', G(y_j) \uparrow \Sigma_{f(j)}) \ge L\Big(\underset{i \in \Lambda(j)}{\bigcirc} G(x_i) \uparrow \Sigma_{f(j)}, G(y_j) \uparrow \Sigma_{f(j)} \Big) = L\Big(\underset{i \in \Lambda(j)}{\bigcirc} G(x_i), G(y_j) \Big),$$

which reduces the proof to the case of $\Sigma = \{0, 1\}$ – note that the last term is equal to $L(z_j, G(y_j))$ in the proof of the same inequality of Lemma 9.6 and thus the remainder follows verbatim.

It remains to show the second inequality of (3). Essentially as in the proof of Lemma 9.6, we write $x=z_1'z_2'\ldots z_Q'$ with $L(x,y)=\sum_{j=1}^Q L(z_j',\mathsf{G}(y_j)\uparrow\Sigma_{f(j)})$. For every z_j' , we obtain a string z_j by deleting all symbols not contained in $\Sigma_{f(j)}$ and then lifting it to the alphabet $\{0,1\}$. We conclude that $L(z_j',\mathsf{G}(y_j)\uparrow\Sigma_{f(j)})=L(z_j,\mathsf{G}(y_j))$. We claim that $z:=z_1z_2\ldots z_Q$ is a subsequence of $x_{\{0,1\}}:=\mathsf{G}(x_1)\ldots\mathsf{G}(x_P)$ (which is equal to the string x that we constructed in the case of $\Sigma=\{0,1\}$). Indeed, if $H(x_i)$ is of the form $w_{k+1}w_k$ for some k with $w_\ell=\mathsf{G}(x_i)\uparrow\Sigma_\ell$, then symbols of at most one of w_k and w_{k+1} are contained in z. To see this, note that if w_k is not deleted then at least one of its symbols is contained in some z_j' with f(j)=k, but then no symbol in w_{k+1} can be contained in z_j' with f(j')=k+1, since this would mean j'>j, so w_{k+1} is deleted. Thus,

$$L(x,y) = \sum_{j=1}^{Q} L(z'_{j}, G(y_{j}) \uparrow \Sigma_{f(j)}) = \sum_{j=1}^{Q} L(z_{j}, G(y_{j})) \le L(x_{\{0,1\}}, y_{\{0,1\}}),$$

where $y_{\{0,1\}} := G(y_1) \dots G(y_Q)$ is the string y that we constructed in the case of $\Sigma = \{0,1\}$. Hence, the second inequality of (3) follows from the proof of Lemma 9.6.

By the same choice of vectors as in Lemma 9.7, we can embed orthogonal vectors instances.

Lemma 9.10. Let $a_1, \ldots, a_A, b_1, \ldots b_B \subseteq \{0, 1\}^D$ be given with $A \mid B$. Construct the corresponding strings x_1, \ldots, x_A of length ℓ_X , y_1, \ldots, y_B of length $\ell_Y \leq \ell_X \leq \mathcal{O}(D)$ and integers ρ_0, ρ_1 as in Lemma 9.5 and define

$$\tilde{x} := (\tilde{x}_1, \dots, \tilde{x}_P) = (x_1, \dots, x_A, x_1, \dots, x_A, \dots, x_1, \dots, x_A),$$

$$\tilde{y} := (\tilde{y}_1, \dots, \tilde{y}_Q) = (\underbrace{y_1, \dots, y_1}_{A \text{ copies of } y_1}, y_1, \dots, y_B, \underbrace{y_1, \dots, y_1}_{A \text{ copies of } y_1}),$$

where P := 2B+3A and Q := B+2A. For disjoint alphabets $\Sigma_1, \ldots, \Sigma_t$ of size 2 with $Q/t \ge A/2+1$, we construct the instance $x := \bigcirc_i H(\tilde{x}_i)$, $y := \bigcirc_j G(\tilde{y}_j)$ of Lemma 9.9 (with the corresponding choice of γ_1, γ_2 and γ_3). This satisfies the following properties:

- (i) We have that $L(x,y) \ge Q(\gamma_1 + \gamma_2 + 3\gamma_3) + (A-1)\rho_1 + \rho_0 + (Q-A)\ell_Y$ if and only if there are $i \in [A], j \in [B]$ with $\langle a_i, b_j \rangle = 0$.
- (ii) We have $|y| \le |x| \le \mathcal{O}(B \cdot D)$ and $\delta(x, y) = \mathcal{O}(A \cdot D)$.

Proof. The lemma and its proof are a slight adaptation of Lemma 9.7: For (i), since Lemma 9.9 proves (3) which is identical to (2), we can follow the proof of Lemma 9.7(i) and (ii) verbatim (since we have chosen \tilde{x} and \tilde{y} as in this lemma). For (ii), the bounds $|y| \leq |x| \leq \mathcal{O}(B \cdot D)$ and $\delta(x,y) = \mathcal{O}(A \cdot D)$ follow exactly as in Lemma 9.6 (note that only |x| has increased by at most a factor of 2, so that $|x| \leq \mathcal{O}(B \cdot D)$ still holds by the trivial bound).

We can now finish the proof of Theorem 4.6 for the case of $\alpha_L = \alpha_m$ and $\alpha_{\Sigma} > 0$. Consider any $n \ge 1$ and target values $p = n^{\alpha_p}$ for $p \in \mathcal{P}$. Let $\mathcal{A} = \{a_1, \dots, a_A\}$, $\mathcal{B} = \{b_1, \dots, b_B\} \subseteq \{0, 1\}^D$ be a given OV instance with $D = n^{o(1)}$ and where we set, as in the case $\alpha_{\Sigma} = 0$,

$$A := \left\lfloor \frac{1}{D} \min \left\{ \delta, \frac{d}{\min\{m, \Delta\}} \right\} \right\rfloor_2 \quad \text{and} \quad B := \left\lfloor \frac{1}{D} \min\{m, \Delta\} \right\rfloor_2.$$

As before, we have $A \mid B$, so we may construct strings x,y as in Lemma 9.10, where we set $t := \min\{\lfloor Q/(A/2+1)\rfloor, |\Sigma|\} = \Theta(\min\{B/A, |\Sigma|\})$. We finish the construction by invoking the Dominant Pair Reduction (Lemma 7.8) to obtain strings $x' := y2^{\ell}x$ and $y' := 2^{\ell}y$, where 2 is a symbol not appearing in x,y and we set $\ell := \Theta(A \cdot D)$ with sufficiently large hidden constant, so that $\ell > \delta(x,y)$.

For the remainder of the proof we can follow the case $\alpha_{\Sigma} = 0$ almost verbatim. The only exception is the bound on the number of matching pairs. Note that symbol 2 appears $\mathcal{O}(AD)$ times in x' and y'. As in x and y every symbol appears roughly equally often and the total alphabet size is $\Theta(t)$, for any symbol $\sigma \neq 2$ we have $\#_{\sigma}(x) \leq \mathcal{O}(|x|/t)$ and $\#_{\sigma}(y) \leq \mathcal{O}(|y|/t)$, implying $\#_{\sigma}(x'), \#_{\sigma}(y') \leq \mathcal{O}(BD/t)$. Hence, $M(x',y') \leq \mathcal{O}((AD)^2 + t \cdot (BD/t)^2)$. Using $t = \Theta(\min\{B/A,|\Sigma|\})$ and $A \leq B$, we obtain $M(x',y') \leq \mathcal{O}(\max\{AD \cdot BD,(BD)^2/|\Sigma|\}) \leq \mathcal{O}(\max\{d,m^2/|\Sigma|\})$. The assumption $\alpha_L = \alpha_m$ and the parameter relations $M \geq L^2/|\Sigma|$ and $M \geq d$ now imply $M(x',y') \leq \mathcal{O}(M)$. This concludes the proof of Theorem 4.6.

10 Hardness for Small Constant Alphabet

In this section, we show hardness of the parameter settings $LCS(\alpha, \Sigma)$ for alphabets of constant size $|\Sigma| \geq 2$, i.e., we prove Theorem 3.4. The general approach, as outlined in Section 4, is to take the hard instances x, y of $LCS_{\leq}(\alpha, \{0, 1\})$ constructed in Section 9 and pad them to instances x', y' of $LCS(\alpha, \Sigma)$. Notably, unlike the black-box method of Lemma 4.5 that effectively considered each parameter separately, we now cannot make extensive use of the Disjoint Alphabets Lemma, as this would introduce more symbols than admissible. Instead, for small alphabet size such as $|\Sigma| = 2$ we need to pad all parameters simultaneously in a combined construction, taking care of the interplay of the parameters manually. Additionally, for $|\Sigma| \in \{2,3\}$, more complex parameter relations hold.

Unfortunately, this general approach fails for $\Sigma = \{0, 1\}$, i.e., we cannot always pad hard strings x, y of $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ to $LCS(\boldsymbol{\alpha}, \{0, 1\})$. Surprisingly, the reason is that by an $\mathcal{O}(n + \delta M/n)$ -time algorithm (given in Section 11), some parameter settings $LCS(\boldsymbol{\alpha}, \{0, 1\})$ are indeed simpler to solve than $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ (conditional on SETH). In these cases, we take hard instances (n, x, y) from $LCS_{\leq}(\boldsymbol{\alpha}', \{0, 1\})$ for a suitably defined "simpler" parameter setting $\boldsymbol{\alpha}'$ and pad x, y to instances of $LCS(\boldsymbol{\alpha}, \{0, 1\})$.

As in Section 9, we distinguish between the two cases $\alpha_{\delta} = \alpha_m$ (i.e., $\delta = \Theta(m)$ and any $0 < \alpha_L \le \alpha_m$ is admissible) and $\alpha_L = \alpha_m$ (i.e., $L = \Theta(m)$ and any $0 < \alpha_{\delta} < \alpha_m$ is admissible).

10.1 Small LCS

In this section, we assume $\alpha_{\delta} = \alpha_m$. It can be checked that this assumption implies $\alpha_{\Delta} = 1$, i.e., $\Delta = \Theta(n)$. Moreover, if $|\Sigma| = 2$ then the assumption and the parameter relation $M \geq 1$

 $nd/(80L) \ge \Omega(nd/m)$ imply $\delta M/n = \Omega(d)$. Thus, the desired running time bound simplifies to $d^{1-o(1)}$. Theorem 3.4 in this regime follows from the following statement (and Lemma 4.1).

Lemma 10.1. Let (α, Σ) be a parameter setting satisfying Table 2 with $\alpha_{\delta} = \alpha_m$. There is a constant $\gamma \geq 1$ such that any algorithm for $LCS^{\gamma}(\alpha, \Sigma)$ takes time $d^{1-o(1)}$ unless OVH fails.

We prove the above lemma in the remainder of this section. Note that any parameter setting $(\boldsymbol{\alpha}, \Sigma)$ satisfying Table 2 gives rise to a parameter setting $\boldsymbol{\alpha}$ satisfying Table 2 with $\alpha_{\Sigma} = 0$ (where the converse does not hold in general). Recall that for any such $\boldsymbol{\alpha}$, in Lemma 9.2 we constructed hard instances (n, x, y) of $\mathrm{LCS}_{\leq}^{\gamma}(\boldsymbol{\alpha}, \{0, 1\})$ with an additional threshold ρ such that deciding $L(x, y) \geq \rho$ decides the corresponding OV instance, yielding hardness of $\mathrm{LCS}_{\leq}^{\gamma}(\boldsymbol{\alpha}, \{0, 1\})$. Furthermore, the constructed instances have the additional guarantees that $|x|, |y| \leq \gamma \cdot n^{\alpha_L}$ and $\#_1(y) \leq \gamma \cdot n^{\alpha_d - \alpha_L}$ and for any $\beta \geq 0$ we have $L(x, 0^{\beta}y) = L(x, y)$.

Hence, to prove the above lemma it suffices to show how to compute, given any such instance (n, x, y) and threshold ρ , an instance x', y' of $LCS^{\gamma'}(\boldsymbol{\alpha}, \Sigma)$ (for some $\gamma' \geq 1$) and an integer ρ' in time $\mathcal{O}(n)$ such that $L(x', y') \geq \rho'$ if and only if $L(x, y) \geq \rho$. More precisely, we show how to compute an integer τ in time $\mathcal{O}(n)$ such that $L(x', y') \geq \tau + \rho$ if and only if $L(x, y) \geq \rho$.

We will do this successively for alphabet sizes $|\Sigma| = 2, |\Sigma| = 3, |\Sigma| = 4$, and $|\Sigma| \geq 5$. To this end, the following basic building block will be instantiated with different parameters. Recall that in Lemma 7.3, we defined strings $a = (01)^{R+S}$ and $b = 0^R (01)^S$ with the properties L(a, b) = |b| = R + 2S and $d(a, b) = \Theta(R \cdot S)$.

Lemma 10.2 (Basic building block). Let x, y be given strings. Given $\alpha, \beta, R, S \geq 0$, we set $\ell := |x| + |y|$, and define

Then we have $L(x', y') = L(a, b) + \ell + L(x, 0^{\beta}y) = R + 2S + \ell + L(x, 0^{\beta}y)$. If $L(x, 0^{\beta}y) = L(x, y)$ then we even have $L(x', y') = R + 2S + \ell + L(x, y)$.

Proof. Clearly, $L(x', y') \ge L(a, b) + L(0^{\ell}, 0^{\ell}) + L(x, 0^{\beta}y) = (R + 2S) + \ell + L(x, 0^{\beta}y)$ since L(a, b) = |b| = R + 2S by Lemma 7.3. To prove a corresponding upper bound, note that we can partition y' = wz such that $L(x', y') = L(a1^{\alpha}, w) + L(0^{\ell}x, z)$. Consider first the case that z is a subsequence of y. Then

$$L(x', y') = L(a1^n, w) + L(0^{\ell}x, z) \le L(a1^n, y') + L(0^{\ell}x, y),$$

since w, z are subsequences of y', y, respectively. Using $L(u, v) \leq \sum_{\sigma \in \Sigma} \min\{\#_{\sigma}(u), \#_{\sigma}(v)\}$ for any strings u, v, we obtain

$$L(x', y') \le (\#_0(a1^n) + \#_1(y')) + (\#_0(y) + \#_1(0^{\ell}x))$$

= $(R+S) + (S+\#_1(y)) + \#_0(y) + \#_1(x)$
 $\le (R+2S) + \ell + L(x, 0^{\beta}y),$

since $\ell \geq |x| + |y| \geq \#_0(x) + \#_0(y) + \#_1(y)$. It remains to consider the case that z is not a subsequence of y and hence w is a subsequence of $b0^{\beta+\ell}$. By Lemma 7.3(iii), we can without loss of generality assume that w is a subsequence of b, since $L(a1^{\alpha}, b0^{\beta+\ell}) = L(a, b)$. We write z = z'z'' such that z'' is a subsequence of $0^{\ell+\beta}y$ and maximal with this property. Hence, wz' is a subsequence of b. Using the facts $L(u, v) \leq |v|$ and $L(u, v'v'') \leq |v'| + L(u, v'')$, we bound

$$L(x',y') = L(a1^n, w) + L(0^{\ell}x, z'z'') \le |w| + (|z'| + L(0^{\ell}x, z'')),$$

Since wz' is a subsequence of b and z'' is a subsequence of $0^{\ell+\beta}y$, this yields

$$L(x', y') \le |b| + L(0^{\ell}x, 0^{\ell+\beta}y) = (R+2S) + \ell + L(x, 0^{\beta}y)$$

where we used greedy prefix matching. This finishes the proof.

We now instantiate the basic building block to prove Lemma 10.1 for $\Sigma = \{0, 1\}$. Note that in the remainder we again simply write p for the target value $\lceil n^{\alpha_p} \rceil$ of parameter $p \in \mathcal{P}$, while the parameter value attained by any strings x, y is denoted by p(x, y), as in Section 9. Note that the additional guarantees for (n, x, y) are satisfied by Lemma 9.2.

Lemma 10.3. Consider a parameter setting $(\alpha, \{0, 1\})$ satisfying Table 2 with $\alpha_{\delta} = \alpha_{m}$. Let (n, x, y) be an instance of $LCS_{\leq}^{\gamma}(\alpha, \{0, 1\})$ with $|x|, |y| \leq \gamma \cdot L$ and $\#_{1}(y) \leq \gamma \cdot d/L$ satisfying $L(x, 0^{\beta'}y) = L(x, y)$ for any $\beta' \geq 0$. We obtain strings x', y' from Lemma 10.2 (recall that in this lemma we set $\ell := |x| + |y|$), where we choose

$$R := L,$$
 $S := |d/L|,$ $\beta := \tilde{m} := \max\{m, 2|x|\},$ $\alpha := \tilde{n} := \max\{n, \tilde{m} + |y|\}.$

Then, setting $\kappa := |M/n|$, the strings defined by

$$x'' := 1^{\kappa} x' = 1^{\kappa} a 1^{\tilde{n}} 0^{\ell} x = 1^{\kappa} (01)^{R+S} 1^{\tilde{n}} 0^{\ell} x,$$

$$y'' := 1^{\kappa} y' = 1^{\kappa} b 0^{\ell+\tilde{m}} y = 1^{\kappa} 0^{R} (01)^{S} 0^{\ell+\tilde{m}} y.$$

are an instance of LCS $^{\gamma'}(\boldsymbol{\alpha}, \{0,1\})$ (for some constant $\gamma' \geq \gamma$) and can be computed in time $\mathcal{O}(n)$, together with an integer τ such that $L(x'', y'') \geq \tau + \rho$ if and only if $L(x, y) \geq \rho$.

Proof. Note that

$$L(x'', y'') = \kappa + L(x', y') = \kappa + (R + 2S) + \ell + L(x, y), \tag{4}$$

where the first equality follows from greedy prefix matching and the second follows from Lemma 10.2. Thus by setting $\tau = \kappa + (R+2S) + \ell$, we have that $L(x'', y'') \ge \tau + \rho$ if and only if $L(x, y) \ge \rho$. Clearly, x'', y'', and τ can be computed in time $\mathcal{O}(n)$, and $\Sigma(x'', y'') = \{0, 1\}$.

We first verify that $|x|, |y|, \ell, R, S, |a|, |b|, \kappa = \mathcal{O}(L)$. By assumption, $|x|, |y| = \mathcal{O}(L)$ and thus $\ell = |x| + |y| = \mathcal{O}(L)$. By the parameter relation $d \leq |\Sigma| \cdot L^2 = 2L^2$, we note that $d/L = \mathcal{O}(L)$ and hence by choice of R, S, we have $|a|, |b| = \Theta(R+S) = \Theta(L+d/L) = \Theta(L)$. Furthermore, the parameter relation $M \leq 2Ln$ implies $\kappa \leq M/n \leq 2L$. Since $L(x,y) \leq |x| = \mathcal{O}(L)$, the bound $L(x'',y'') = \kappa + R + 2S + \ell + L(x,y) = R + \mathcal{O}(L) = \Theta(L)$ follows directly from (4).

Observe that \tilde{n} is chosen such that $|x''| \geq |y''|$. Also, $\tilde{m} = \Theta(m)$ and $\tilde{n} = \Theta(n)$. Since $L \leq m \leq n$, we thus have $|x''| = \kappa + |a| + \tilde{n} + \ell + |x| = \tilde{n} + \mathcal{O}(L) = \Theta(n)$ and $|y| = \kappa + |b| + \tilde{m} + \ell + |y| = \tilde{m} + \mathcal{O}(L) = \Theta(m)$.

Note that by (4), $\delta(x'', y'') = (\tilde{m} + |y|) - L(x, y) \ge \tilde{m} - |x| \ge m/2$. Hence, $\delta(x'', y'') = \Theta(m) = \Theta(\delta)$ (by the assumption $\alpha_{\delta} = \alpha_m$). Moreover, since $\delta = \Theta(m)$, for some constant $\varepsilon > 0$ we have $\Delta = \delta + (n - m) \ge \varepsilon m + n - m = n - (1 - \varepsilon)m \ge n - (1 - \varepsilon)n = \Omega(n)$ (where we used the parameter relation $m \le n$). Since also $\Delta \le n$ we have $\Delta = \Theta(n)$. By the same argument, using $\delta(x'', y'') = \Theta(m) = \Theta(m(x'', y''))$ and $n(x'', y'') = \Theta(n)$ as shown above, we obtain $\Delta(x'', y'') = \Theta(n(x'', y'')) = \Theta(n)$, and thus $\Delta(x'', y'') = \Theta(\Delta)$.

For M, observe that $\#_1(x'') = \kappa + \#_1(a) + \tilde{n} + \#_1(x) = \tilde{n} + \mathcal{O}(L) = \Theta(n)$. Moreover, $\#_1(y) = \mathcal{O}(d/L)$ (by assumption) and $\#_1(b) = S = \mathcal{O}(d/L)$ yield $\#_1(y'') = \kappa + \#_1(b) + \#_1(y) = \Theta(M/n) + \mathcal{O}(d/L)$ (here $\kappa = \Theta(M/n)$ follows from the parameter relation $M \geq n$). This yields $\#_1(x'') \cdot \#_1(y'') = \Theta(M) + \mathcal{O}(dn/L) = \Theta(M)$ (using the parameter relation $M \geq nd/(5L)$). Also

note that $\#_0(x'') = \#_0(a) + \ell + \#_0(x) = \mathcal{O}(L)$ and $\#_0(y'') = \#_0(b) + \ell + \tilde{m} + \#_0(y) = \tilde{m} + \mathcal{O}(L) = \mathcal{O}(m)$. This yields $\#_0(x'') \cdot \#_0(y'') = \mathcal{O}(Lm) = \mathcal{O}(M)$ (using the parameter relation $M \geq Lm/4$). Combining these bounds, we obtain $M(x'', y'') = \#_0(x'') \cdot \#_0(y'') + \#_1(x'') \cdot \#_1(y'') = \Theta(M)$. Note that the last two parameter relations used here exploited that we have $\Sigma = \{0, 1\}$.

It remains to determine the number of dominant pairs. Since $L(x',y') = \Theta(L)$ (as argued above) and $\#_1(y') = \mathcal{O}(d/L)$, Lemma 6.8 yields $d(x',y') \leq 5L(x',y') \cdot \#_1(y') = \mathcal{O}(L \cdot d/L) = \mathcal{O}(d)$. For a corresponding lower bound, from Observation 7.2 and Lemma 7.3 we obtain $d(x',y') \geq d(a,b) \geq R \cdot S = \Omega(d)$. By Lemma 7.1, the claim now follows from $d(x'',y'') = \kappa + d(x',y') = \mathcal{O}(L) + \Theta(d) = \Theta(d)$, where we use $\kappa = \mathcal{O}(L)$ and the parameter relation $d \geq L$.

The case $\Sigma = \{0, 1, 2\}$ is similar to $\{0, 1\}$, except that we use the new symbol 2 to pad the parameter n, we use symbol 1 to pad m, and we have to swap the constructions for x'' and y''.

Lemma 10.4. Consider a parameter setting $(\boldsymbol{\alpha}, \{0, 1, 2\})$ satisfying Table 2 with $\alpha_{\delta} = \alpha_{m}$. Let (n, x, y) be an instance of $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}, \{0, 1\})$ with $|x|, |y| \leq \gamma \cdot L$ and $\#_{1}(y) \leq \gamma \cdot d/L$ satisfying $L(x, 0^{\beta'}y) = L(x, y)$ for any $\beta' \geq 0$. We obtain strings x', y' from Lemma 10.2 (recall that in this lemma we set $\ell := |x| + |y|$), where we choose

$$R := L,$$
 $S := |d/L|,$ $\beta := 0,$ $\alpha := m.$

Then, setting $\kappa := \lfloor M/n \rfloor$ and $\tilde{n} := \max\{n, \kappa + |a| + m + |x|\}$, the strings defined by

$$x'' := 2^{\tilde{n}} y' = 2^{\tilde{n}} b \quad 0^{\ell} y = 2^{\tilde{n}} 0^{R} (01)^{S} \quad 0^{\ell} y,$$

$$y'' := 2^{\kappa} x' = 2^{\kappa} a 1^{m} 0^{\ell} x = 2^{\kappa} (01)^{R+S} 1^{m} 0^{\ell} x.$$

are an instance of $LCS^{\gamma'}(\alpha, \{0, 1, 2\})$ (for some constant $\gamma' \geq \gamma$) and can be computed in time $\mathcal{O}(n)$, together with an integer τ such that $L(x'', y'') \geq \tau + \rho$ if and only if $L(x, y) \geq \rho$.

Proof. Note that unlike the case $\{0,1\}$ the string x now appears in y'' and y appears in x'', so the constructions are swapped. This is necessary to realize m and M, using the parameter relation $M \ge md/(80L)$ that holds for $\Sigma = \{0,1,2\}$. Observe that as usual $|x''| \ge |y''|$.

We first compute

$$L(x'', y'') = L(2^{\tilde{n}}, 2^{\kappa}) + L(y', x') = \kappa + (R + 2S) + \ell + L(x, y), \tag{5}$$

where the first equality follows from the Disjoint Alphabets Lemma and the second equality from greedy prefix matching and Lemma 10.2. Thus, by setting $\tau = \kappa + (R+2S) + \ell$, we have $L(x'', y'') \ge \tau + \rho$ if and only if $L(x,y) \ge \rho$. Clearly, x'', y'', and τ can be computed in time $\mathcal{O}(n)$, and $\Sigma(x'', y'') = \{0, 1, 2\}$.

As in the case $\{0,1\}$, we have $|x|, |y|, \ell, R, S, |a|, |b|, \kappa = \mathcal{O}(L)$. Thus, by (5), we have $L(x'', y'') = R + \mathcal{O}(L) = \Theta(L)$. Furthermore, note that $\tilde{n} = \Theta(n)$. Thus, $|y''| = \kappa + |a| + m + \ell + |x| = m + \mathcal{O}(L) = \Theta(m)$ and $|x''| = \tilde{n} + |b| + \ell + |y| = \tilde{n} + \mathcal{O}(L) = \Theta(n)$. Since $L(x, y) \leq |x| = \mathcal{O}(L)$, the bound $L(x'', y'') = R + \mathcal{O}(L) = \Theta(L)$ follows directly from (5).

By (5), we see that $\delta(x'', y'') = |y''| - L(x'', y'') = R + m + (|x| - L(x, y)) \ge m$. Hence $\delta(x'', y'') = \Theta(m)$. Thus, $\Delta(x'', y'') = \delta(x'', y'') + (|x''| - |y''|) = \Theta(n)$ follows as in the case $\{0, 1\}$.

For M, observe that $\#_1(y'') = \#_1(a) + m + \#_1(x) = m + \mathcal{O}(L) = \Theta(m)$. Moreover, $\#_1(y) = \mathcal{O}(d/L)$ (by assumption) yields $\#_1(x'') = \#_1(b) + \#_1(y) = S + \mathcal{O}(d/L) = \mathcal{O}(d/L)$. Also note that $\#_0(y'') = \#_0(a) + \ell + \#_0(x) = \mathcal{O}(L)$ and $\#_0(x'') = \#_0(b) + \ell + \#_0(y) = \mathcal{O}(L)$. Since furthermore $\#_2(y'') = \Theta(M/n)$ (by the parameter relation $M \geq n$) and $\#_2(x'') = \Theta(n)$, we conclude that

 $M(x'', y'') = \sum_{\sigma \in \{0,1,2\}} \#_{\sigma}(x'') \cdot \#_{\sigma}(y'') = \mathcal{O}(dm/L + L^2) + \Theta(M)$. By the parameter relations $M \ge md/(80L)$ (using that $\Sigma = \{0,1,2\}$) and $M \ge L^2/|\Sigma| = \Omega(L^2)$, this yields $M(x'', y'') = \Theta(M)$.

For the remaining parameter d, by the disjoint alphabets lemma and Lemma 7.1 we have $d(x'', y'') = d(2^{\tilde{n}}, 2^{\kappa}) + d(y', x') = \kappa + d(x', y')$ (using symmetry d(x, y) = d(y, x)). The remaining arguments are the same as in the case $\{0, 1\}$.

In the case $\Sigma = \{0, 1, 2, 3\}$ we can use the new symbol 3 to pad m (instead of using symbol 1, as in the previous case). Note that now x appears in x'' and y in y'', as in the case $\{0, 1\}$.

Lemma 10.5. Consider a parameter setting $(\boldsymbol{\alpha}, \{0, 1, 2, 3\})$ satisfying Table 2 with $\alpha_{\delta} = \alpha_{m}$. Let (n, x, y) be an instance of $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}, \{0, 1\})$ with $|x|, |y| \leq \gamma \cdot L$ and $\#_{1}(y) \leq \gamma \cdot d/L$ satisfying $L(x, 0^{\beta'}y) = L(x, y)$ for any $\beta' \geq 0$. We obtain strings x', y' from Lemma 10.2 (recall that in this lemma we set $\ell := |x| + |y|$), where we choose

$$R := L,$$
 $S := \lfloor d/L \rfloor,$ $\beta := 0,$ $\alpha := 0.$

Then, setting $\kappa := \lfloor M/n \rfloor$ and $\tilde{n} := \max\{n, m + \kappa + |y|\}$, the strings defined by

$$x'' := 3 \quad 2^{\tilde{n}} x' = 3 \quad 2^{\tilde{n}} a \quad 0^{\ell} x = 3 \quad 2^{\tilde{n}} (01)^{R+S} \quad 0^{\ell} x,$$

 $y'' := 3^{m} \quad 2^{\kappa} \quad y' = 3^{m} \quad 2^{\kappa} \quad b \quad 0^{\ell} \quad y = 3^{m} \quad 2^{\kappa} \quad 0^{R} (01)^{S} \quad 0^{\ell} \quad y,$

are an instance of $LCS^{\gamma'}(\boldsymbol{\alpha}, \{0, 1, 2, 3\})$ (for some constant $\gamma' \geq \gamma$) and can be computed in time $\mathcal{O}(n)$, together with an integer τ such that $L(x'', y'') \geq \tau + \rho$ if and only if $L(x, y) \geq \rho$.

Proof. We compute

$$L(x'', y'') = L(3, 3^m) + L(2^{\tilde{n}}, 2^{\kappa}) + L(x', y') = 1 + \kappa + (R + 2S) + \ell + L(x, y), \tag{6}$$

where the first equality follows from the Disjoint Alphabets Lemma and the second follows from greedy prefix matching and Lemma 10.2. Hence, by setting $\tau = 1 + \kappa + R + 2S + \ell$, we have $L(x,y) \ge \rho$ if and only if $L(x'',y'') \ge \tau + \rho$. Clearly, x'',y'', and τ can be computed in time $\mathcal{O}(n)$, and $\Sigma(x'',y'') = \{0,1,2,3\}$.

As for the cases $\{0,1\}$ and $\{0,1,2\}$, we have $|x|,|y|,\ell,R,S,|a|,|b|,\ell,\kappa=\Theta(L)$. Note that by choice of \tilde{n} , we have again $|x''|\geq |y''|$ and $\tilde{n}=\Theta(n)$. Hence, $|x''|=1+\tilde{n}+|a|+\ell+|x|=\tilde{n}+\mathcal{O}(L)=\Theta(n)$ and $|y''|=m+\kappa+|b|+\ell+|y|=m+\mathcal{O}(L)=\Theta(m)$. Since $L(x,y)\leq |x|=\mathcal{O}(L)$, the bound $L(x'',y'')=R+\mathcal{O}(L)=\Theta(L)$ follows directly from (6). Note that (6) also implies that $\delta(x'',y'')=|y''|-L(x'',y'')=m-1+|x|-L(x,y)\geq m-1$ and hence $\delta(x'',y'')=\Theta(m)$. Thus, $\Delta(x'',y'')=\delta(x'',y'')+(|x''|-|y''|)=\Theta(n)$ follows as for the case $\{0,1\}$.

For M, observe that $|a0^{\ell}x|, |b0^{\ell}y| = \mathcal{O}(L)$ implies that $M(a0^{\ell}x, b0^{\ell}y) = \mathcal{O}(L^2)$. By the Disjoint Alphabets Lemma, we obtain

$$M(x'', y'') = M(2^{\tilde{n}}, 2^{\kappa}) + M(3, 3^m) + M(a0^{\ell}x, b0^{\ell}y) = \kappa \tilde{n} + \mathcal{O}(m + L^2) = \Theta(M),$$

where we used $\kappa \tilde{n} = \Theta(M/n \cdot n) = \Theta(n)$ (note that $M \ge n$ implies $\kappa = \Theta(M/n)$) and the parameter relations $M \ge n \ge m$ and $M \ge L^2/|\Sigma| = \Omega(L^2)$.

For the remaining parameter d, as in the case $\{0,1\}$ we show that $d(x',y') = \Theta(d)$. Now the Disjoint Alphabets Lemma and Lemma 7.1 prove that $d(x'',y'') = d(3,3^m) + d(2^{\tilde{n}},2^{\kappa}) + d(x',y') = 1 + \kappa + d(x',y') = d(x',y') + \mathcal{O}(L) = \Theta(d)$ using $\kappa = \mathcal{O}(L)$ and the parameter relation $d \geq L$. \square

Finally, observe that for any parameter setting $(\boldsymbol{\alpha}, \Sigma)$ with $|\Sigma| \geq 5$ satisfying Table 2, also the parameter setting $(\boldsymbol{\alpha}, \{0, 1, 2, 3\})$ satisfies Table 2. Hence, the following lemma transfers the hardness of $LCS(\boldsymbol{\alpha}, \{0, 1, 2, 3\})$ to $LCS(\boldsymbol{\alpha}, \Sigma)$.

Lemma 10.6. Let α be a parameter setting satisfying Table 2 with $\alpha_{\delta} = \alpha_m$. Let Σ be an alphabet of size $|\Sigma| \geq 5$. If there is an $\mathcal{O}(n^{\beta})$ -time algorithm for $LCS(\alpha, \Sigma)$, then also $LCS(\alpha, \{0, 1, 2, 3\})$ admits an $\mathcal{O}(n^{\beta})$ -time algorithm.

Proof. Given an instance (x, y) of $LCS(\boldsymbol{\alpha}, \{0, 1, 2, 3\})$ with n := |x|, we show how to compute in time $\mathcal{O}(n)$ an instance (x', y') of $LCS(\boldsymbol{\alpha}, \Sigma)$ such that L(x', y') = 1 + L(x, y). The claim then follows from applying the $\mathcal{O}(n^{\beta})$ -time algorithm on x', y' (and subtracting 1 from the result).

Without loss of generality, let $\Sigma = \{0, \ldots, \sigma\}$ with $\sigma \geq 4$. Define x' := wx and $y' = w^R y$, where $w = 4 \ldots \sigma$ and $w^R = \sigma \ldots 4$. Then by the Disjoint Alphabets and Crossing Alphabets Lemmas (Lemmas 4.3 and 9.3), we obtain $L(x',y') = L(w,w^R) + L(x,y) = 1 + L(x,y)$. It remains to show that (x',y') is an instance of $LCS(\alpha,\Sigma)$. By the Crossing Alphabets Lemma, for all parameters $p \in \{d,M,n,m\}$ we have $p(w,w^R) = \sum_{\sigma'=4}^{\sigma} p(\sigma',\sigma') = |\Sigma| - 4$, and hence the Disjoint Alphabets Lemma yields $p(x',y') = p(w,w^R) + p(x,y) = |\Sigma| - 4 + p(x,y) = \Theta(p(x,y))$, by the parameter relations $n \geq m \geq |\Sigma|$ and $M \geq d \geq |\Sigma|$. For L we obtain $L(x',y') = L(w,w^R) + L(x,y) = 1 + L(x,y) = \Theta(L(x,y))$. For $p \in \{\delta,\Delta\}$ this yields $p(x',y') = (|w|-1) + p(x,y) = \Theta(p(x,y))$, since $\alpha_\Delta \geq \alpha_\delta = \alpha_m \geq \alpha_\Sigma$ (by the assumption $\alpha_\delta = \alpha_m$ and the parameter relations $\Delta \geq \delta$ and $m \geq |\Sigma|$) and thus $\Delta(x,y) \geq \delta(x,y) \geq \Omega(|w|-1)$. Hence, (x',y') has the same parameters as (x,y) up to constant factors, so all parameter relations satisfied by (x,y) are also satisfied by (x',y'). Since clearly x',y' use alphabet Σ , indeed (x',y') is an instance of $LCS(\alpha,\Sigma)$.

Lemmas 10.3, 10.4, 10.5, and 10.6 of this section, together with the construction of hard strings in $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ in Lemma 9.2, prove hardness of $LCS(\boldsymbol{\alpha}, \Sigma)$ for any constant alphabet size in the case $\alpha_{\delta} = \alpha_{m}$, i.e., Lemma 10.1.

10.2 Large LCS, Alphabet Size at least 3

In this section, we study the case that $\alpha_L = \alpha_m$ (and $\alpha_\delta, \alpha_\Delta$ may be small). Additionally, we assume that $|\Sigma| \geq 3$. In this regime, Theorem 3.4 follows from the following statement (and Lemma 4.1).

Lemma 10.7. Let $(\boldsymbol{\alpha}, \Sigma)$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$ and $|\Sigma| \geq 3$. There is a constant $\gamma \geq 1$ such that any algorithm for LCS $^{\gamma}(\boldsymbol{\alpha}, \Sigma)$ takes time min $\{d, \delta m, \delta \Delta\}^{1-o(1)}$ unless OVH fails.

By the following lemma, it suffices to prove the result for $\Sigma = \{0, 1, 2\}$ (note that for any $(\boldsymbol{\alpha}, \Sigma)$ satisfying Table 2 with $\alpha_L = \alpha_m$ and $|\Sigma| \geq 4$, also $(\boldsymbol{\alpha}, \{0, 1, 2\})$ satisfies Table 2, since the only additional constraint $\alpha_M \geq \alpha_m + \alpha_d - \alpha_L$ for ternary alphabets simplifies, by $\alpha_L = \alpha_m$, to the constraint $\alpha_M \geq \alpha_d$, which is satisfied by $\boldsymbol{\alpha}$).

Lemma 10.8. Let α be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$. Let Σ be an alphabet of size $|\Sigma| \geq 4$. If there is an $\mathcal{O}(n^{\beta})$ -time algorithm for $LCS(\alpha, \Sigma)$, then also $LCS(\alpha, \{0, 1, 2\})$ admits an $\mathcal{O}(n^{\beta})$ -time algorithm.

Proof. Given an instance (x, y) of $LCS(\alpha, \{0, 1, 2\})$ with n := |x|, we show how to compute in time $\mathcal{O}(n)$ an instance (x', y') of $LCS(\alpha, \Sigma)$ such that $L(x', y') = |\Sigma| - 3 + L(x, y)$. The claim then follows from applying the $\mathcal{O}(n^{\beta})$ -time algorithm on x', y' (and subtracting $|\Sigma| - 3$ from the result).

Without loss of generality, let $\Sigma = \{0, \ldots, \sigma\}$ with $\sigma \geq 3$. Define x' := wx and y' = wy, where $w = 3 \ldots \sigma$. Then by the Disjoint Alphabets Lemma (Lemma 4.3), we obtain $L(x', y') = L(w, w) + L(x, y) = |\Sigma| - 3 + L(x, y)$. It remains to show that (x', y') is an instance of LCS (α, Σ) . By the Disjoint Alphabets Lemma, for all parameters $p \in \mathcal{P}^* = \{n, m, L, \delta, \Delta, |\Sigma|, M, d\}$ we have $p(x', y') = p(w, w) + p(x, y) = \sum_{\sigma'=3}^{\sigma} p(\sigma', \sigma') + p(x, y)$. For $p \in \{n, m, L, M, d\}$ we have $p(\sigma', \sigma') = 1$

and thus $p(x', y') = |\Sigma| - 3 + p(x, y) = \Theta(p(x, y))$ by the assumption $\alpha_L = \alpha_m$ and the parameter relations $n \ge m \ge |\Sigma|$ and $M \ge d \ge |\Sigma|$. For $p \in \{\delta, \Delta\}$ this yields p(x', y') = 0 + p(x, y) = p(x, y). Hence, (x', y') has the same parameters as (x, y) up to constant factors, so all parameter relations satisfied by (x, y) are also satisfied by (x', y'). Since clearly x', y' use alphabet Σ , indeed (x', y') is an instance of $LCS(\alpha, \Sigma)$.

To prepare the construction, we adapt the construction of Lemma 10.2 to obtain a desired value for d (and, in later sections, δ). Recall that Lemma 7.3 defines $a = (01)^{R+S}$, $b = 0^R (01)^S$ with L(a,b) = |b| = R + 2S and $d(a,b) = \Theta(R \cdot S)$.

Lemma 10.9 (Basic building block). Given strings x, y and $R, S, \ell, \beta \ge 0$ with $\ell \ge R + |x| + |y|$ we define

$$x' := a 0^{\ell} x = (01)^{R+S} 0^{\ell} x,$$

 $y' := 0^{\beta} b 0^{\ell} y = 0^{\beta} 0^{R} (01)^{S} 0^{\ell} y.$

Assume that $S \geq |x|$ or $L(x, 0^{\beta}y) = L(x, y)$. Then we have $L(x', y') = R + 2S + \ell + L(x, y)$.

Proof. Clearly, $L(x', y') \ge L(a, b) + L(0^{\ell}, 0^{\ell}) + L(x, y) = R + 2S + \ell + L(x, y)$, since L(a, b) = |b| = R + 2S. To prove the corresponding upper bound, we partition $y' = y_1y_2$ such that $L(x', y') = L(a, y_1) + L(0^{\ell}x, y_2)$. Consider first the case that y_2 is a subsequence of y. Then

$$L(x', y') \le L(a, y_1) + L(0^{\ell}x, y) \le |a| + |y| = 2(R+S) + |y| \le (R+2S) + \ell + L(x, y),$$

since $\ell \geq R + |y|$.

It remains to consider the case that y_2 is not a subsequence of y and hence y_1 is a subsequence of $0^{\beta}b0^{\ell}$. By Lemma 7.3(iii), we can without loss of generality assume that y_1 is a subsequence of $0^{\beta}b$, since $L(a,0^{\beta}b0^{\ell}) = |b| = L(a,0^{\beta}b)$. Hence, we can partition $0^{\beta}b = y_1z$ with $L(x',y') \leq L(a,y_1) + L(0^{\ell}x,z0^{\ell}y)$. We bound

$$L(a, y_1) \le \min\{\#_0(a), \#_0(y_1)\} + \min\{\#_1(a), \#_1(y_1)\} \le (R + S) + \#_1(y_1).$$

Observe that $L(0^{\ell}x, z0^{\ell}y) \leq \#_1(z) + L(0^{\ell}x, 0^{\#_0(z)+\ell}y)$, since each "1" in z can increase the LCS by at most 1. By greedy prefix matching, we obtain $L(0^{\ell}x, z0^{\ell}y) \leq \#_1(z) + \ell + L(x, 0^{\#_0(z)}y)$. With the assumption $L(x, 0^{\beta}y) = L(x, y)$ for any $\beta \geq 0$, this yields

$$L(x',y') \leq L(a,y_1) + L(0^{\ell}x,z0^{\ell}y) \leq (R+S+\#_1(y_1)) + (\#_1(z) + \ell + L(x,y)) = R + 2S + \ell + L(x,y),$$

since $0^{\beta}b = y_1z$ and hence $\#_1(y_1) + \#_1(z) = \#_1(0^{\beta}b) = S$.

With the alternative assumption $S \ge |x|$, we bound $L(0^{\ell}x, z0^{\ell}y) \le |0^{\ell}x| \le \ell + S$. If $\#_1(y_1) = 0$ this yields

$$L(x',y') \le L(a,y_1) + L(0^{\ell}x,z0^{\ell}y) \le (R+S) + (\ell+S) \le R + 2S + \ell + L(x,y).$$

Otherwise, if $\#_1(y_1) \ge 1$, by inspecting the structure of $0^{\beta}b = 0^{\beta+R}(01)^S$ we see that z is a subsequence of $(01)^{S-\#_1(y_1)}$. Consider an LCS of $0^{\ell}x, z0^{\ell}y$. If no "1" in z is matched by the LCS, then we obtain

$$L(0^{\ell}x, z0^{\ell}y) \le L(0^{\ell}x, 0^{\#_0(z) + \ell}y) \le \#_0(z) + L(0^{\ell}x, 0^{\ell}y) = \#_0(z) + \ell + L(x, y),$$

where we used the fact $L(u, vw) \leq |v| + L(uw)$ and greedy prefix matching. Otherwise, if a "1" in z is matched by the LCS to a "1" in $0^{\ell}x$, i.e., to a "1" in x, then the 0^{ℓ} -block of $0^{\ell}x$ is matched to a subsequence of z and hence

$$L(0^{\ell}x, z0^{\ell}y) \le L(0^{\ell}, z) + L(x, z0^{\ell}y) \le \#_0(z) + |x| \le \#_0(z) + \ell + L(x, y),$$

where we used $\ell \ge |x|$. Since z is a subsequence of $(01)^{S-\#_1(y_1)}$ and thus $\#_0(z) \le S - \#_1(y_1)$, in both cases we obtain

$$L(x',y') \le L(a,y_1) + L(0^{\ell}x,z0^{\ell}y) \le (R+S+\#_1(y_1)) + (\#_0(z) + \ell + L(x,y)) \le R + 2S + \ell + L(x,y),$$

which finally proves the claim.

Lemma 10.10. Consider x', y' as in Lemma 10.9 with $\beta = 0$, and assume $S \ge |x|$. Then

$$R \cdot S \le d(x', y') \le (R+1)(4R+6S+\ell) + d(x, y).$$

Proof. Note that the simple fact $L(u'u'',v) \leq |u''| + L(u',v)$ implies that for any strings w,z and any i we have $L(w,z) \leq |w[(i+1)..|w|]| + L(w[1..i],z) = |w| - i + L(w[1..i],z)$, and hence $L(w[1..i],z) \geq i - (|w| - L(w,z))$. Recall that L(a,b) = |b| = R + 2S by Lemma 7.3(iii). This yields $L(a[1..i],b) \geq i - (|a| - (R+2S)) = i - R$ and $L(a,b[1..j]) \geq j - (|b| - |b|) = j$.

The lower bound follows from $d(x',y') \ge d(a,b) \ge R \cdot S$ by Observation 7.2 and Lemma 7.3(iv). For the upper bound, we consider all possible prefixes $\tilde{x} := x'[1..i], \tilde{y} := y'[1..j]$ of x', y' and count how many of them correspond to dominant pairs. Clearly, for $i \le |a|, j \le |b|$, there are d(a,b) dominant pairs.

By the above observation, for any $i \leq |a|$, we have $L(a[1..i], b) \geq i - R$. Hence, any dominant pair of the form (i,j) satisfies $i - R \leq L(a[1..i], b) \leq |a[1..i]| = i$. By Observation 6.2, there are at most R+1 such dominant pairs for fixed i. Thus, there are at most $|a| \cdot (R+1)$ dominant pairs with $i \leq |a|$ and j > |b|. Similarly, for $j \leq |b|$, we have $L(a, b[1..j]) \geq j$. Hence, there are no dominant pairs with i > |a| and $j \leq |b|$, since already the prefix a of x'[1..i] includes b[1..j] as a subsequence. In total, there are at most $d(a, b) + |a| \cdot (R+1)$ dominant pairs with $i \leq |a|$ or $j \leq |b|$.

Let i = |a| + k, j = |b| + k with $k \in [\ell]$. Then $L(\tilde{x}, \tilde{y}) = L(a0^k, b0^k) = L(a, b) + k = |b| + k$ by greedy suffix matching and Lemma 7.3(iii). As any such choice could correspond to a dominant pair, we count at most ℓ dominant pairs. Analogously to above, for i = |a| + k, there can be at most $i - L(a0^k, b0^k) \le R$ dominant pairs with j > |b| + k. Symmetrically, for j = |b| + k, there are at most $j - L(a0^k, b0^k) = 0$ dominant pairs with i > |a| + k. This yields at most $(R + 1) \cdot \ell$ dominant pairs with $|a| < i \le |a| + \ell$ or $|b| < j \le |b| + \ell$.

It remains to count dominant pairs with $i=|a|+\ell+\tilde{i}$ and $j=|b|+\ell+\tilde{j}$, with $\tilde{i}\in[|x|],\tilde{j}\in[|y|]$. Here, Lemma 10.9 bounds $L(\tilde{x},\tilde{y})=L(a0^{\ell}x[1..\tilde{i}],b0^{\ell}y[1..\tilde{j}])=L(a,b)+\ell+L(x[1..\tilde{i}],y[1..\tilde{j}])$. Hence, the dominant pairs of this form are in one-to-one correspondence to the dominant pairs of x,y.

Summing up all dominant pairs, we obtain

$$d(x', y') \le d(a, b) + (R+1)(|a|+\ell) + d(x, y)$$

$$\le (R+1)(4R+6S+\ell) + d(x, y),$$

since |a| = 2R + 2S and Lemma 7.3(iv) yields $d(a, b) \le 2(R+1)(R+2S)$.

Finally, to pad δ (and later, in Section 10.3.1, Δ), we need the following technical lemma.

Lemma 10.11. Let x, y be arbitrary and μ, ν such that $\nu \ge \mu + |y|$. We define

$$x' := 0^{\mu} 1^{\nu} 0^{\mu} x,$$

$$y' := 1^{\nu} 0^{\mu} y.$$

Then $L(x', y') = \mu + \nu + L(x, y)$ and $d(x', y') = 2\mu + \nu + \#_1(y) + d(x, y)$.

Proof. Note that for any prefix w, z of $0^{\mu}x, 0^{\mu}y$, we have

$$L(0^{\mu}1^{\nu}w, 1^{\nu}z) = \nu + L(w, z), \tag{7}$$

by Lemma 7.6 (swapping the role of "0"s and "1"s). In particular, we obtain $L(x', y') = \nu + L(0^{\mu}x, 0^{\mu}y) = \mu + \nu + L(x, y)$ by greedy prefix matching.

For the second statement, we consider all possible prefixes $\tilde{x} := x'[1..i], \tilde{y} := y'[1..j]$ of x', y' and count how many of them correspond to dominant pairs. Note that these prefixes have to end in the same symbol, since any dominant pair is a matching pair. Recall that $\tilde{x} := x'[1..i], \tilde{y} := y'[1..j]$ gives rise to a dominant pair if and only if $L(\tilde{x}, \tilde{y}) > L(x'[1..i-1], \tilde{y})$ and $L(\tilde{x}, \tilde{y}) > L(\tilde{x}, y'[1..j-1])$.

- $\tilde{x} = 0^{\mu}1^{\nu}w$, $\tilde{y} = 1^{\ell}$ (with w non-empty prefix of $0^{\mu}x$, $\ell \in [\nu]$): These prefixes do not correspond to a dominant pair, since $L(0^{\mu}1^{\nu}w, 1^{\ell}) = L(0^{\mu}1^{\ell}, 1^{\ell}) = \ell$ is obtained already by a shorter prefix of \tilde{x} .
- $\tilde{x} = 0^{\mu}1^{\nu}w, \tilde{y} = 1^{\nu}z$ (with w non-empty prefix of $0^{\mu}x, z$ non-empty prefix of $0^{\mu}y$): These prefixes correspond to a dominant pair if and only if w, z correspond to a dominant pair of $0^{\mu}x, 0^{\mu}y$, since by (7) we have $L(\tilde{x}, \tilde{y}) = \nu + L(w, z)$. This yields $d(0^{\mu}x, 0^{\mu}y)$ dominant pairs, which by Lemma 7.1 evaluates to $\mu + d(x, y)$.
- $\tilde{x} = 0^k, \tilde{y} = 1^{\mu}0^{\ell}$ (with $k, \ell \in [\mu]$): Clearly, $L(\tilde{x}, \tilde{y}) = \min\{k, \ell\}$. It follows that \tilde{x}, \tilde{y} corresponds to a dominant pair if and only if $k = \ell$. This yields exactly μ dominant pairs.
- $\tilde{x} = 0^{\mu}1^k$, $\tilde{y} = 1^{\ell}$ (with $k, \ell \in [\nu]$): Analogously to above, $L(\tilde{x}, \tilde{y}) = \min\{k, \ell\}$, hence this corresponds to a dominant pair if and only if $k = \ell$. This yields exactly ν dominant pairs.
- $\tilde{x} = 0^k$, $\tilde{y} = 1^{\nu}0^{\mu}z$ (with $k \in [\mu]$, z non-empty prefix of y): We have $L(\tilde{x}, \tilde{y}) = L(0^k, 1^{\nu}0^k) = k$, hence these prefixes do not correspond to a dominant pair, since the LCS is already obtained for a shorter prefix of \tilde{y} .
- $\tilde{x} = 0^{\mu}1^{k}$, $\tilde{y} = 1^{\nu}0^{\mu}z$ (with $k \in [\nu]$, z non-empty prefix of y): Since we can either match some "1" of the 1^{ν} -block in \tilde{y} to a "1" in \tilde{x} (necessarily discarding the initial 0^{μ} -block of \tilde{x}) or delete the complete 1^{ν} -block, we obtain

$$L(0^{\mu}1^{k}, 1^{\nu}0^{\mu}z) = \max\{L(1^{k}, 1^{\nu}0^{\mu}z), L(0^{\mu}1^{k}, 0^{\mu}z)\}$$
$$= \max\{k, \mu + L(1^{k}, z)\} = \max\{k, \mu + \min\{k, \#_{1}(z)\}\}.$$

Consider the case $L(\tilde{x}, \tilde{y}) = L(x'[1..i], y'[1..j]) = k$. Then also L(x'[1..i], y'[1..(j-1)]) = k, and hence (i, j) is no dominant pair. If, however, $L(\tilde{x}, \tilde{y}) = \mu + \min\{k, \#_1(z)\}$, then this corresponds to a dominant pair if and only if $k = \#_1(z)$ (and z ends in "1"): if $k > \#_1(z)$, then also $L(x'[1..(i-1)], y'[1..j]) = \mu + \#_1(z)$, if $k < \#_1(z)$, then also $L(x'[1..i], y'[1..(j-1)]) = \mu + k$. Thus, there are exactly $\min\{\nu, \#_1(y)\} = \#_1(y)$ such dominant pairs.

In total, we have counted $\nu + 2\mu + \#_1(y) + d(x,y)$ dominant pairs.

We start with the basic building block from Lemma 10.9 and then further pad the strings to obtain the desired n, m, Δ, δ, M as follows. Note that the guarantee $|y| \leq |x| \leq \mathcal{O}(\min\{\Delta, m\})$ is satisfied by Lemma 9.8.

Lemma 10.12. Let $(\alpha, \{0, 1, 2\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$. Let (n, x, y) be an instance of $LCS_{<}^{\gamma}(\alpha, \{0, 1\})$ with $|y| \leq |x| \leq \mathcal{O}(\min\{\Delta, m\})$. We set

$$S = \max\{m, |x|\}, \qquad R = |d/m|,$$

to instantiate the basic building block $x' = a0^{\ell}x = (01)^{R+S}0^{\ell}x$ and $y' = b0^{\ell}y = 0^{R}(01)^{S}0^{\ell}y$ of Lemma 10.9 with $\ell := R+S+|x|+|y|$. Moreover, we define $\kappa := \lfloor M/n \rfloor$ and $\tilde{m} := \max\{m, \delta + 2R + 2S + \ell + |x|\}$ to further pad the instance to

$$x'' = 2^{\kappa} 2^{\Delta} 1^{\tilde{m}} 0^{\delta} a 0^{\ell} x,$$

$$y'' = 2^{\kappa} 0^{\delta} 1^{\tilde{m}} 0^{\delta} b 0^{\ell} y.$$

Then x'', y'' is an instance of LCS $^{\gamma'}(\alpha, \{0, 1, 2\})$ for some constant $\gamma' \geq 1$ and can be computed in time $\mathcal{O}(n)$, together with an integer τ such that $L(x'', y'') = \tau + L(x, y)$.

Proof. We first use the Disjoint Alphabets Lemma and greedy prefix matching to obtain

$$L(x'', y'') = \kappa + L(1^{\tilde{m}}0^{\delta}x', 0^{\delta}1^{\tilde{m}}0^{\delta}y') = \kappa + \tilde{m} + \delta + L(x', y') = \kappa + \tilde{m} + \delta + R + 2S + \ell + L(x, y), (8)$$

where we used Lemma 10.11 for the second equality (with the roles of x, y swapped) and Lemma 10.9 for the last equality. Observe that x'', y'' and $\tau = \kappa + \tilde{m} + \delta + R + 2S + \ell$ can be computed in time $\mathcal{O}(n)$.

It remains to verify that x, y is an instance of $LCS^{\gamma'}(\alpha, \{0, 1, 2\})$ for some $\gamma' \geq 1$. We first observe that $S = \Theta(m)$ by $|x| = \mathcal{O}(m)$ and $R = \mathcal{O}(S)$ by the parameter relation $d \leq Lm \leq m^2$. Since also $|y| = \mathcal{O}(m)$, we conclude $R, S, |x|, |y|, \ell = \mathcal{O}(m)$. Observe that $\kappa = \mathcal{O}(M/n) = \mathcal{O}(m)$ by the relation $M \leq mn$ and $\tilde{m} = \Theta(m)$ (using that $R, S, \ell, |x| = \mathcal{O}(m)$ and the parameter relation $\delta \leq m$). Thus, $|x''| = \kappa + \Delta + \tilde{m} + \delta + 2(R + S) + \ell + |x| = \Theta(m + \Delta) + \mathcal{O}(m) = \Theta(n)$. Similarly, $|y''| = \kappa + \tilde{m} + \delta + R + 2S + \ell + |y| = \Theta(m + \delta) + \mathcal{O}(m) = \Theta(m)$. By (8), we also have $L(x'', y'') = \tilde{m} + \mathcal{O}(m) = \Theta(m)$, as desired.

Note that by $\Delta \geq \delta$, $|a| \geq |b|$ and $|x| \geq |y|$, we indeed have $|x''| \geq |y''|$. Furthermore, using the relation $d \leq 2L(\Delta+1) = \mathcal{O}(m\Delta)$, we obtain $R = \mathcal{O}(d/m) = \mathcal{O}(\Delta)$. By (8), we obtain $\Delta(x'',y'') = \Delta + R + (|x| - L(x,y)) = \Delta + \mathcal{O}(\Delta) = \Theta(\Delta)$ since $|x| = \mathcal{O}(\Delta)$. Similarly, (8) yields $\delta(x'',y'') = \delta + (|y| - L(x,y)) = \delta + \mathcal{O}(\delta) = \Theta(\delta)$, since $|y| - L(x,y) = \delta(x,y) = \mathcal{O}(\delta)$.

For the dominant pairs, we apply the disjoint alphabets lemma, Lemma 7.1 and Lemma 10.11 to compute

$$d(x'', y'') = \kappa + d(1^{\tilde{m}}0^{\delta}x', 0^{\delta}1^{\tilde{m}}0^{\delta}y') = \kappa + \tilde{m} + 2\delta + d(x', y').$$
(9)

Lemma 10.10 yields the lower bound $d(x', y') \ge R \cdot S = \Omega(d)$ and the corresponding upper bound

$$d(x', y') \le (R+1)(4R+6S+\ell) + d(x, y) = \mathcal{O}(R \cdot S + d(x, y)) = \mathcal{O}(d).$$

Thus, (9) yields $d(x'', y'') = \Theta(d) + \mathcal{O}(m) = \Theta(d)$, where we used that $d \geq L = \Omega(m)$ since $\alpha_L = \alpha_m$.

For M, we count $\#_2(x'') = \Delta + \kappa$ and $\#_2(y'') = \kappa$, as well as $\#_0(y''), \#_1(y'') \leq |y''| = \mathcal{O}(m)$, $\#_0(x'') \leq \delta + |x'| = \mathcal{O}(m)$ and $\#_1(x'') \leq \tilde{m} + |x'| = \mathcal{O}(m)$. Thus, $M(x'', y'') = (\Delta + \kappa)\kappa + \mathcal{O}(m^2)$ and by $M \geq L^2/|\Sigma| = \Omega(m^2)$ (since $\alpha_L = \alpha_M$), it suffices to prove that $(\Delta + \kappa)\kappa = \Theta(M)$ to

verify $M(x'',y'') = \Theta(M)$. Indeed, we have $\kappa = \Theta(M/n)$, since $M \geq n$. If $\alpha_{\Delta} < 1$, we have $\alpha_m = \alpha_n = 1$ and $M = \Omega(m^2)$ together with the relation $M \leq mn = \mathcal{O}(m^2)$ implies $M = \Theta(m^2)$. Thus, $\kappa = \Theta(m)$ and hence $(\kappa + \Delta)\kappa = \Theta(m^2) = \Theta(M)$. If $\alpha_{\Delta} \geq \alpha_m$, then $\alpha_{\Delta} = 1$ and hence $\Delta + \kappa = \Delta + \mathcal{O}(m) = \Theta(n)$, which implies $(\kappa + \Delta)\kappa = \Theta(n \cdot M/n) = \Theta(M)$. Finally, note that indeed $\Sigma(x'', y'') = \{0, 1, 2\}$.

Combining Lemma 10.12 with Lemma 9.8 finally proves Lemma 10.7.

10.3 Large LCS, Alphabet Size 2

In this section, we study the case that $\alpha_L = \alpha_m$ (and $\alpha_\delta, \alpha_\Delta$ may be small) for the case of binary alphabets, i.e., $\Sigma = \{0, 1\}$. In this regime, Theorem 3.4 follows from the following statement (and Lemma 4.1).

Lemma 10.13. Let $(\alpha, \{0, 1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$. There is a constant $\gamma \geq 1$ such that any algorithm for LCS $^{\gamma}(\alpha, \{0, 1\})$ takes time min $\{d, \delta\Delta, \delta M/n\}^{1-o(1)}$ unless OVH fails.

We present different constructions for three subcases, that we discuss shortly in the following paragraphs and in detail in the remainder of this section. We hope that this short discussion conveys enough intuition about the "complexity" of the task to make it believable that our lengthy and technical case distinction is indeed necessary.

Case 1: $\alpha_{\Delta} \leq \alpha_m = \alpha_L$. Then $n = \Theta(m)$ and it follows that any binary strings x, y satisfy $M(x,y) = \Theta(m^2)$, so M poses no constraints, in particular there are no constraints on the numbers of "0"s and "1"s in the constructed strings. On the other hand, the potentially small value of Δ renders some of our gadgets useless (e.g., Lemma 10.25). Since δ may be small, we use the hardness construction from Section 9.2 (for large LCS).

Otherwise, we have $\alpha_{\Delta} > \alpha_m$ and thus $\Delta = \Theta(n) \gg m$. Note that any string x of length n contains at least n/2 "0"s or "1"s, say x contains many "1"s. Then to obtain $\Theta(M)$ matching pairs, y must contain at most $\mathcal{O}(M/n)$ "1"s. Thus, we need to pay close attention to the number of "1"s in the constructed string y. We split the case $\alpha_{\Delta} > \alpha_m$ into two subcases. Case 2: $\alpha_{\Delta} > \alpha_m = \alpha_L$ and $\alpha_{\delta} \geq \alpha_M - 1$. Here, the constraint on $\#_1(y)$ is stronger than the constraint on δ , and we use the hardness construction from Section 9.1 (for small LCS), since it introduces few "1"s. Case 3: $\alpha_{\Delta} > \alpha_m = \alpha_L$ and $\alpha_{\delta} < \alpha_M - 1$. Here, the constraint on δ is stronger than the constraint on $\#_1(y)$, and we use the hardness construction from Section 9.2 (for large LCS), since it keeps δ small.

10.3.1 Case $\alpha_{\Delta} \leq \alpha_m = \alpha_L$

Since $n = \Delta + L$, the assumptions $\alpha_L = \alpha_m$ and $\alpha_\Delta \leq \alpha_m$ imply $n = \Theta(m)$. Together with the parameter relations $L^2/|\Sigma| \leq M \leq 2Ln$ and $\Sigma = \{0,1\}$ we obtain $M = \Theta(m^2)$. In particular, in this regime the $\tilde{\mathcal{O}}(\delta\Delta)$ time bound beats $\tilde{\mathcal{O}}(\delta M/n)$, and Lemma 10.13 simplifies to the following result.

Lemma 10.14. Let $(\boldsymbol{\alpha}, \Sigma)$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$ and $\alpha_\Delta \leq \alpha_m$. There is a constant $\gamma \geq 1$ such that any algorithm for $LCS^{\gamma}(\boldsymbol{\alpha}, \Sigma)$ takes time $\min\{d, \delta\Delta\}^{1-o(1)}$ unless OVH fails.

We can now instantiate the parameters of Lemma 10.9 to create a string with the desired number of dominant pairs. The remaining parameters will be padded in an additional construction. Note that the preconditions, specifically the additional guarantee, are satisfied by Lemma 9.8.

Lemma 10.15. Let $(\boldsymbol{\alpha}, \Sigma)$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m$ and $\alpha_\Delta \leq \alpha_m$. Given any instance (n, x, y) of $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}, \{0, 1\})$ with the additional guarantee $|y| \leq |x| \leq \gamma \cdot m$, we can construct an instance (n, x', y') of $LCS_{\leq}^{\gamma'}(\boldsymbol{\alpha}, \{0, 1\})$ (for some constant $\gamma' \geq \gamma$) and τ in time $\mathcal{O}(n)$ such that

- (i) $L(x', y') = \tau + L(x, y)$,
- (ii) $d(x', y') = \Theta(d)$.
- (iii) $|x'| \ge |y'|$.

Proof. We construct x', y' as in Lemma 10.9 with $S = \max\{|x|, m\}$, $R = \lceil d/S \rceil$, $\beta = 0$ and $\ell = (R+S+|x|+|y|)$. Note that indeed $|x'| \ge |y'|$ by $|x| \ge |y|$, $|a| \ge |b|$, and $\beta = 0$. The assumption $|x|, |y| = \mathcal{O}(m)$ yields $S = \Theta(m)$. By the parameter relation $d \le 2(\Delta + 1) \cdot L = \mathcal{O}(\Delta \cdot m)$, we also have $R = \mathcal{O}(d/S+1) = \mathcal{O}(d/m+1) = \mathcal{O}(\Delta)$. We conclude that $|x'|, |y'| = \mathcal{O}(R+S+|x|+|y|) = \mathcal{O}(m+\Delta) = \mathcal{O}(m)$, since by assumption $\alpha_{\Delta} \le \alpha_{m}$. This yields $L(x', y') = \mathcal{O}(m) = \mathcal{O}(L)$ (by the assumption $\alpha_{L} = \alpha_{m}$) and $M(x', y') = \mathcal{O}(m^{2}) = \mathcal{O}(M)$ (note that $M \ge L^{2}/|\Sigma| = \Omega(m^{2})$ by the assumption $\alpha_{L} = \alpha_{m}$).

By Lemma 10.9, we have $L(x',y')=R+2S+\ell+L(x,y)$, satisfying (i). This yields $\delta(x',y')=\delta(x,y)=\mathcal{O}(\delta)$ and $\Delta(x',y')=R+\Delta(x,y)=\mathcal{O}(d/m+\Delta)=\mathcal{O}(\Delta)$, by the parameter relation $d\leq 2L(\Delta+1)=\mathcal{O}(m\Delta)$. For d, we first observe that $\lceil d/S \rceil=\Theta(d/m)$ (by the parameter relation $d\geq L=\Theta(m)$) and $\ell=\mathcal{O}(R+S+|x|+|y|)=\mathcal{O}(m)$. Lemma 10.10 yields the lower bound $d(x',y')\geq R\cdot S=\Omega(d/m\cdot m)=\Omega(d)$ as well as the corresponding upper bound $d(x',y')=\mathcal{O}(R\cdot \ell+d(x,y))=\mathcal{O}(d/m\cdot m+d)=\mathcal{O}(d)$.

These bounds prove that (n, x', y') is an instance of $LCS_{<}^{\gamma'}(\alpha, \{0, 1\})$ for some $\gamma' \ge 1$.

We use Lemma 10.11 to finally pad δ , Δ , and m.

Lemma 10.16. Let x, y, x', y', τ be as given in Lemma 10.15. Then, in time $\mathcal{O}(n)$ we can construct an instance x''', y''' of $LCS^{\gamma''}(\boldsymbol{\alpha}, \{0,1\})$ (for some constant $\gamma'' \geq 1$) and an integer τ' such that $L(x''', y''') = \tau' + L(x', y') = (\tau + \tau') + L(x, y)$.

Proof. As an intermediate step, let $\tilde{m} := \max\{m, |x'|, |y'|, \Delta\}$ and construct

$$x'' := 0^{\Delta} 1^{\tilde{m}+\Delta} 0^{\Delta} x',$$

$$y'' := 1^{\tilde{m}+\Delta} 0^{\Delta} y'.$$

We obtain the final instance as

$$x''' := 1^{5\tilde{m}+\delta} 0^{\delta} x'',$$

 $y''' := 0^{\delta} 1^{5\tilde{m}+\delta} 0^{\delta} y''.$

Note that by definition of \tilde{m} , we have $\tilde{m} + \Delta \ge \Delta + |y'|$ and $\delta + 5\tilde{m} \ge \delta + (3\Delta + \tilde{m} + |x'|) = \delta + |x''|$, satisfying the conditions of Lemma 10.11. Hence, this lemma yields

$$L(x''', y''') = 5\tilde{m} + 2\delta + L(x'', y'') = 6\tilde{m} + 2\delta + 2\Delta + L(x', y'). \tag{10}$$

Clearly, x', y', τ and hence also x'', y'' and $\tau' := 6\tilde{m} + 2\delta + 2\Delta$ can be computed in time $\mathcal{O}(n)$.

We now verify all parameters. Clearly, $\Sigma(x''', y''') = \{0, 1\}$. Since by assumption $\alpha_{\Delta} \leq \alpha_m = 1$, we have $|x'| = \mathcal{O}(n) = \mathcal{O}(m)$, $|y'| = \mathcal{O}(m)$, and $\Delta = \mathcal{O}(m)$. This implies $\tilde{m} = \Theta(m)$ and consequently $|x'''| = 6\tilde{m} + 2\delta + 3\Delta + |x'| = 6\tilde{m} + \mathcal{O}(m) = \Theta(m) = \Theta(n)$ (using $\delta \leq \Delta$ and

 $\alpha_{\Delta} \leq \alpha_m = 1$). Similarly, $|y'''| = 6\tilde{m} + 3\delta + 2\Delta + |y'| = 6\tilde{m} + \mathcal{O}(m) = \Theta(m)$. By (10), we have $L(x''', y''') = 6\tilde{m} + 2\delta + 2\Delta + L(x', y') = 6\tilde{m} + \mathcal{O}(L) = \Theta(L)$ (using the assumption $\alpha_L = \alpha_m$).

Note that $|x'''| \ge |y'''|$ follows from $\Delta \ge \delta$ and $|x'| \ge |y'|$ (by Lemma 10.15(iii)). Hence, (10) provides $\Delta(x''', y''') = \Delta + \Delta(x', y') = \Theta(\Delta)$ and $\delta(x''', y''') = \delta + \delta(x', y') = \Theta(\delta)$.

The number of matching pairs satisfies $M(x''', y''') = \#_0(x''') \#_0(y''') + \#_1(x''') \#_1(y''') = \Theta(m^2) = \Theta(M)$, where the last bound follows from $M \ge L^2/|\Sigma| = \Omega(m^2)$ and $M \le 2Ln = \mathcal{O}(m^2)$ by $\alpha_L = \alpha_m = 1$. For the number of dominant pairs, we apply Lemma 10.11 to bound

$$d(x''', y''') = 3\delta + 5\tilde{m} + \#_1(x'') + d(x'', y'')$$

$$= 3\delta + 5\tilde{m} + (\tilde{m} + \Delta + \#_1(x')) + (3\Delta + \tilde{m} + \#_1(y') + d(x', y'))$$

$$= 7\tilde{m} + 3(\delta + \Delta) + \#_1(x') + \#_1(y') + d(x', y') = \Theta(d) + \mathcal{O}(m) = \Theta(d),$$

where the last two bounds follow from \tilde{m} , |x'|, |y'|, δ , $\Delta = \mathcal{O}(m)$, $d(x', y') = \Theta(d)$ by Lemma 10.15 and the parameter relation $d \geq L = \Omega(m)$.

Combining Lemmas 10.15 and 10.16 with Lemma 9.8 finally proves Lemma 10.14.

10.3.2 Case $\alpha_{\Delta} > \alpha_m = \alpha_L$ and $\alpha_{\delta} \geq \alpha_M - 1$

In this section, we consider the case where $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_\delta \ge \alpha_M - 1$. In this case, we have $\Delta = \Theta(n) \gg m$ and $M/n \le \mathcal{O}(\delta)$. Since $M/n \le \mathcal{O}(m) \le \mathcal{O}(\Delta)$, the fastest known algorithm runs in time $\tilde{\mathcal{O}}(\min\{d,\delta M/n\})$. Consequently, in this regime Lemma 10.13 simplifies to the following statement.

Lemma 10.17. Let $(\alpha, \{0, 1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_M - 1 \le \alpha_\delta$. There is a constant $\gamma \ge 1$ such that any algorithm for $LCS^{\gamma}(\alpha, \Sigma)$ takes time $\min\{d, \delta M/n\}^{1-o(1)}$ unless OVH fails.

To obtain this result, we cannot simply pad our hard instances (n, x, y) of $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ to $LCS(\boldsymbol{\alpha}, \{0, 1\})$, since the desired running time bound $\min\{d, \delta M/n\}$ is not monotone. In other words, for $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ we have a lower bound of $\min\{\delta \Delta, \delta m, d\}^{1-o(1)}$ (see Lemma 9.8) which can be higher than the running time $\mathcal{O}(n+\delta M/n)$ of our new algorithm (Theorem 3.5) and thus we would violate SETH. In fact, we even cannot start from an instance as constructed in Lemma 9.8, since this would generate too many "1"s. Instead, we use instances of a different parameter setting $LCS_{\leq}(\boldsymbol{\alpha}', \{0, 1\})$ with $\alpha'_{\delta} = \alpha'_{m}$, i.e., we invoke Lemma 9.2.

Observation 10.18. Let $(\alpha, \{0, 1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_M - 1 \le \alpha_\delta$. Then $\alpha' := \alpha'(\alpha)$ defined by

$$\begin{aligned} \alpha_d' &= \min\{\alpha_d, \alpha_\delta + \alpha_M - 1\}, & \alpha_M' &= 2\alpha_L', \\ \alpha_d' - \alpha_L' &= \min\{\alpha_M - 1, \alpha_d/2\}, & \alpha_\Delta' &= 1, \\ \alpha_m' &= \alpha_\delta' = \alpha_L', & \alpha_\Sigma' &= 0, \end{aligned}$$

yields a parameter setting $(\alpha', \{0,1\})$ satisfying Table 2. The definition of α' implies

$$\alpha_L' = \min\{\alpha_\delta, \max\{\alpha_d - \alpha_M + 1, \alpha_d/2\}\}. \tag{11}$$

Moreover, there is some constant $\gamma \geq 1$ such that no algorithm solves $LCS_{\leq}^{\gamma}(\alpha', \{0, 1\})$ in time $\min\{d, \delta M/n\}^{1-o(1)}$ unless OVH fails. This holds even restricted to instances (n, x, y) with $|x|, |y| \leq \gamma n^{\alpha'_L} = \mathcal{O}(\min\{\delta, \max\{dn/M, \sqrt{d}\}\})$ and $\#_1(y) \leq \gamma \cdot n^{\alpha'_d - \alpha'_L} = \mathcal{O}(\min\{M/n, \sqrt{d}\})$ satisfying $L(x, 0^{\beta}y) = L(x, y)$ for any $\beta \geq 0$.

Proof. We first prove (11). Consider the case that $\alpha_d/2 \leq \alpha_M - 1$. Then $\alpha_d \leq 2(\alpha_M - 1) \leq \alpha_\delta + (\alpha_M - 1)$, where we used the assumption $\alpha_M - 1 \leq \alpha_\delta$. Thus, $\alpha'_d = \alpha_d$ and by definition

$$\alpha_L' = \alpha_d' - \min\{\alpha_M - 1, \alpha_d/2\} = \alpha_d - \alpha_d/2 = \alpha_d/2.$$

From $\alpha_d/2 \leq \alpha_M - 1$, it follows that $\alpha_d - \alpha_M + 1 \leq \alpha_d/2$ and $\alpha_d/2 \leq \alpha_M - 1 \leq \alpha_\delta$, hence $\alpha'_L = \alpha_d/2 = \min\{\alpha_\delta, \max\{\alpha_d - \alpha_M + 1, \alpha_d/2\}\}\$, as desired.

Consider the remaining case that $\alpha_d/2 > \alpha_M - 1$. Then by definition

$$\alpha_L' = \alpha_d' - (\alpha_M - 1) = \min\{\alpha_d - \alpha_M + 1, \alpha_\delta\}.$$

Since $\alpha_d/2 > \alpha_M - 1$ implies $\alpha_d - \alpha_M + 1 \ge \alpha_d/2$, this indeed yields

$$\alpha_L' = \min\{\max\{\alpha_d - \alpha_M + 1, \alpha_d/2\}, \alpha_\delta\},\$$

as desired, concluding the proof of (11).

Checking all constraints from Table 2 is straight-forward, except for the inequalities $0 \le \alpha_L' \le 1$ and $\alpha_d' \le 2\alpha_L'$. From (11), $0 \le \alpha_L' \le 1$ follows immediately by the parameter relations $\alpha_\delta, \alpha_d \ge 0$ and $\alpha_\delta \le \alpha_m \le 1$. For the other inequality, note that $\alpha_d' \le 2\alpha_L'$ is equivalent to $\min\{\alpha_M - 1, \alpha_d/2\} = \alpha_d' - \alpha_L' \le \alpha_L' = \min\{\alpha_\delta, \max\{\alpha_d - \alpha_M + 1, \alpha_d/2\}\}$, which directly follows from the assumption $\alpha_M - 1 \le \alpha_\delta$ and the trivial fact that $\alpha_d/2 \le \max\{\alpha_d - \alpha_M + 1, \alpha_d/2\}$.

The last statement directly follows from Lemma 9.2.

It remains to pad strings x, y of $LCS_{\leq}(\alpha', \{0, 1\})$ to $LCS(\alpha, \{0, 1\})$. The first step is the following construction which pads δ and d.

Lemma 10.19. Let $(\alpha, \{0, 1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_M - 1 \le \alpha_\delta$, and construct α' as in Observation 10.18. Let (n, x, y) be an instance of $LCS_{\le}(\alpha', \{0, 1\})$ with $|x|, |y| = \mathcal{O}(\min\{\delta, \max\{dn/M, \sqrt{d}\}\})$ and $\#_1(y) = \mathcal{O}(\min\{M/n, \sqrt{d}\})$ satisfying $L(x, 0^\beta y) = L(x, y)$ for any $\beta \ge 0$. We set

$$S = |\min\{M/n, \sqrt{d}\}|, \qquad R = |d/S|, \qquad \ell = |x| + |y| + R + S, \qquad \beta = \delta,$$

and define, as in Lemma 10.9,

Then x', y' is an instance of LCS $_{<}^{\gamma'}(\alpha, \{0, 1\})$ (for some $\gamma' \geq 1$) with the additional guarantees that

- (i) $|x'|, |y'| = \mathcal{O}(m),$
- (ii) $\#_1(y') = \mathcal{O}(M/n)$ and $\#_1(y') \cdot |b0^{\ell}y| = \mathcal{O}(d)$.
- (iii) $d(x', y') = \Theta(d)$,
- (iv) $|y'| L(x', y') = \Theta(\delta)$,
- (v) In time $\mathcal{O}(n)$ we can compute a number τ such that $L(x',y') = \tau + L(x,y)$.

Proof. Note that $S \leq \sqrt{d}$ implies that $S \leq R$. Note also that by assumption, $|x|, |y| = \mathcal{O}(\min\{\delta, R\})$ and hence $R, S, \ell, |x|, |y| = \mathcal{O}(R)$. Furthermore, $R = \Theta(d/S) = \Theta(d/Mn + \sqrt{d}) = \mathcal{O}(m)$, where the first bound follows from $\alpha_d \geq 0$ and $\alpha_M \geq 1$ and the second follows from the parameter relations $d/Mn \leq 5L = \mathcal{O}(m)$ and $d \leq Lm \leq m^2$. Hence $|x'| = \mathcal{O}(R) = \mathcal{O}(m)$. Additionally, $\tilde{\delta} = \Theta(\delta)$ and thus $|y'| = \mathcal{O}(\delta + m) = \mathcal{O}(m)$ by $\delta \leq m$; thus we have proven (i). This also implies $L(x', y') \leq \mathcal{O}(m) = \mathcal{O}(L)$ since $\alpha_L = \alpha_m$.

Note that $\#_1(y) = S + \#_1(y) = \mathcal{O}(S)$ by definition of S and assumption on $\#_1(y)$. We compute $d(x',y') \leq 5L(x',y') \cdot \#_1(y') \leq 5|x'| \cdot \#_1(y') = \mathcal{O}(R \cdot S) = \mathcal{O}(d)$. Furthermore, we obtain by Observation 7.2 and Lemma 7.3 that $d(x',y') \geq d(a,0^\beta b) \geq R \cdot S = \Omega(d)$. Note that in particular $\#_1(y') = \mathcal{O}(S) = \mathcal{O}(M/n)$ and $\#_1(y') \cdot |b0^\ell y| = \mathcal{O}(S \cdot R) = \mathcal{O}(d)$, proving (ii). The bound $M(x',y') \leq \mathcal{O}(m^2) = \mathcal{O}(M)$ trivially follows from $|x'|,|y'| = \mathcal{O}(m)$ and $M \geq L^2/|\Sigma| = \Omega(m^2)$ since $\alpha_L = \alpha_m$.

By Lemma 10.9 we have $L(x',y') = R + 2S + \ell + L(x,y)$, which immediately yields (v). Moreover, we obtain $\delta(x',y') = \delta + \delta(x,y) = \Theta(\delta)$, since $\delta(x,y) \le |x| \le \mathcal{O}(\delta)$, proving (iv). Finally, $\Delta(x',y') \le |x'| \le \mathcal{O}(m) \le \mathcal{O}(\Delta)$ by the assumption $\alpha_{\Delta} > \alpha_{m}$.

To finally pad all remaining parameters, we first prepare the following technical tool.

Lemma 10.20. Let $x = 1^{\kappa}0^{\mu}w$ and $y = 0^{\mu}0^{\nu}z$ with $\mu > |z|$. Then it holds that $L(x,y) = \mu + L(w, 0^{\nu}z)$, as well as $d(x,y) \ge d(w, 0^{\nu}z)$ and $d(x,y) \le \min\{\kappa, |z|\} + \mu + d(1^{\kappa}0^{\mu}, z) + d(w, 0^{\nu}z)$.

Proof. By Lemma 7.6, we have that $L(x,y) = \mu + L(w,0^{\nu}z)$.

This immediately shows that $d(x,y) \ge d(w,0^{\nu}z)$, since the above statement implies, for any prefixes \tilde{w}, \tilde{z} of $w, 0^{\nu}z$, that $L(1^{\kappa}0^{\mu}\tilde{w}, 0^{\mu}\tilde{z}) = \mu + L(\tilde{w}, \tilde{z})$ and hence any k-dominant pair (i,j) of w and $0^{\nu}z$ gives rise to a $(\mu + k)$ -dominant pair $(\kappa + \mu + i, \mu + j)$ of x and y.

For the upper bound, we count the number of prefixes \tilde{x}, \tilde{y} of x, y corresponding to dominant pairs. Note that \tilde{x}, \tilde{y} have to end in the same symbol to be a dominant pair. Consider first the case that $\tilde{x} = 1^k$. Hence we must have $\tilde{y} = 0^\mu 0^\nu \tilde{z}$ for some prefix $\tilde{z} = z[1..\ell]$ of \tilde{z} . Clearly, $L(\tilde{x}, \tilde{y}) = \min\{k, \#_1(\tilde{z})\}$. Hence, \tilde{x}, \tilde{y} corresponds to a dominant pair if and only if $\#_1(\tilde{z}) = \#_1(z[1..\ell]) = k$ and $\#_1(z[1..\ell-1]) < k$, i.e., \tilde{z} is determined by the k-th occurrence of "1" in z. Thus, there can be at most $\min\{\kappa, \#_1(z)\} \leq \min\{\kappa, |z|\}$ such dominant pairs.

Consider the case that $\tilde{x} = 1^{\kappa}0^k$ with $k \in [\mu]$. We separately regard the following types of prefixes of y.

- $\tilde{y} = 0^{\ell}$ with $\ell \in [\mu + \nu]$: By greedy suffix matching, $L(\tilde{x}, \tilde{y}) = L(1^{\kappa}0^{k}, 0^{\ell}) = \min\{k, \ell\}$, hence as above there can be at most μ dominant pairs, since there are only μ choices for k.
- $\tilde{y} = 0^{\mu}0^{\nu}\tilde{z}$: We have $L(\tilde{x},\tilde{y}) = \max\{k, L(1^{\kappa}0^k,\tilde{z})\}$. To see that this holds, note that the longest common subsequence either includes none of the ones of 1^{κ} of \tilde{y} , in which case 0^k is the LCS of \tilde{y} and \tilde{x} , or otherwise it matches at least one 1 in \tilde{y} , which means that the LCS deletes all "0"s preceding the first "1" in \tilde{y} , i.e., the whole $0^{\mu+\nu}$ block in y'.

If $L(\tilde{x}, \tilde{y}) = k$, then \tilde{x}, \tilde{y} cannot correspond to a dominant pair since already the prefix 0^k of \tilde{y} satisfies $L(\tilde{x}, 0^k) = k = L(\tilde{x}, \tilde{y})$. Hence \tilde{x}, \tilde{y} can only correspond to a dominant pair if $L(\tilde{x}, \tilde{y}) = L(1^{\kappa}0^k, \tilde{z})$ and hence $1^{\kappa}0^k, \tilde{z}$ correspond to a dominant pair of $1^{\kappa}0^{\mu}, z$. This yields at most $d(1^{\kappa}0^{\mu}, z)$ dominant pairs.

Finally, consider the case that $\tilde{x} = 1^{\kappa}0^{\mu}\tilde{w}$ with \tilde{w} a prefix of w. There are no dominant pairs for $\tilde{y} = 0^{\ell}$ with $\ell \in [\mu]$: Already for the prefix $1^{\kappa}0^{\ell}$ of \tilde{x} , we have $L(1^{\kappa}0^{\ell}, 0^{\ell}) = \ell = L(\tilde{x}, \tilde{y})$, hence these prefixes cannot correspond to dominant pairs. It remains to consider $\tilde{y} = 0^{\mu}\tilde{z}$ for a prefix \tilde{z} of $0^{\nu}z$. Again by Lemma 7.6, we have $L(\tilde{x}, \tilde{y}) = \mu + L(\tilde{w}, \tilde{z})$ and hence such dominant pairs are

in one-to-one correspondence with the dominant pairs of w and $0^{\nu}z$. This yields at most $d(w, 0^{\nu}z)$ further dominant pairs.

By summing up over the three cases, we conclude that there are at most $\min\{\kappa, |z|\} + \mu + d(1^{\kappa}0^{\mu}, z) + d(w, 0^{\nu}z)$ dominant pairs.

We can finally pad to LCS(α , {0, 1}).

Lemma 10.21. Let x, y, x', y', τ be as in Lemma 10.19. We set $\kappa := \lfloor M/n \rfloor$, $\tilde{\Delta} := \max\{\Delta, |y'|\}$, and $\tilde{m} := \max\{m, |y'|\}$ and define

$$x'' = 1^{\kappa + \tilde{\Delta}} 0^{\tilde{m}} x',$$

$$y'' = 1^{\kappa} 0^{\tilde{m}} y'.$$

Then x'', y'' is an instance of LCS(α , {0,1}). Moreover, we can compute a number τ' in time $\mathcal{O}(n)$ such that $L(x'', y'') = \tau' + L(x, y)$.

Proof. Note that $\kappa = \mathcal{O}(M/n) = \mathcal{O}(m)$ by the parameter relation $M \leq mn$. Lemma 10.19(i) yields $|x'|, |y'| = \mathcal{O}(m)$ and hence $\tilde{m} = \Theta(m)$ and $\tilde{\Delta} = \Theta(\Delta)$ (since $\alpha_m \leq \alpha_\Delta = 1$). Thus, $|x''| = \kappa + \tilde{\Delta} + \tilde{m} + |x'| = \Theta(\Delta) + \mathcal{O}(m) = \Theta(n)$ since $\alpha_\Delta = 1$. Furthermore, $|y''| = \kappa + \tilde{m} + |y'| = \tilde{m} + \mathcal{O}(m) = \Theta(m)$.

Observe that $\tilde{\Delta}$ has been defined such that $|x''| \geq |y''|$. By greedy prefix matching and Lemma 10.20, we obtain

$$L(x'', y'') = \kappa + L(1^{\tilde{\Delta}} 0^{\tilde{m}} x', 0^{\tilde{m}} y') = \kappa + \tilde{m} + L(x', y').$$
(12)

Since $L(x',y') = \tau + L(x,y)$, we satisfy the last claim by setting $\tau' := \kappa + \tilde{m} + \tau$. Moreover, $L(x'',y'') = \tilde{m} + \mathcal{O}(m) = \Theta(m) = \Theta(L)$ since $|x'|,|y'| \leq \mathcal{O}(m)$ and $\alpha_L = \alpha_m$. Furthermore, (12) yields $\Delta(x'',y'') = \tilde{\Delta} + (|x'| - L(x',y')) = \Theta(\Delta) + \mathcal{O}(m) = \Theta(\Delta)$ and $\delta(x'',y'') = |y'| - L(x',y') = \Theta(\delta)$ by Lemma 10.19(iv).

For the dominant pairs, we apply Lemma 7.1 to bound $d(x'', y'') = \kappa + d(1^{\tilde{\Delta}}0^{\tilde{m}}x', 0^{\tilde{m}}y')$. To bound the latter term, note that Lemma 10.20 yields $d(1^{\tilde{\Delta}}0^{\tilde{m}}x', 0^{\tilde{m}}y') \geq d(x', y') = \Omega(d)$ by Lemma 10.19(iii). For the upper bound, we first recall that $y' = 0^{\tilde{\delta}}b0^{\ell}y$ and that $\#_1(y') \cdot |b0^{\ell}y| = \mathcal{O}(d)$ by Lemma 10.19(ii). Hence we have $d(1^{\tilde{\Delta}}0^{\tilde{m}}, b0^{\ell}y) \leq 5 \cdot L(1^{\tilde{\Delta}}0^{\tilde{m}}, b0^{\ell}y) \cdot \#_1(b0^{\ell}y) = \mathcal{O}(|b0^{\ell}y| \cdot \#_1(y')) = \mathcal{O}(d)$. We can finally compute, using Lemma 10.20,

$$d(1^{\tilde{\Delta}}0^{\tilde{m}}x', 0^{\tilde{m}}y') \le \min{\{\tilde{\Delta}, |y'|\}} + \tilde{m} + d(1^{\tilde{\Delta}}0^{\tilde{m}}, b0^{\ell}y) + d(x', y')$$

$$\le |y'| + \tilde{m} + \mathcal{O}(d) + d(x', y') = \mathcal{O}(d),$$

where the last bound follows from |y'|, $\tilde{m} = \mathcal{O}(m) = \mathcal{O}(d)$ by the relation $d \geq L = \Omega(m)$ (since $\alpha_L = \alpha_m$) and $d(x', y') = \mathcal{O}(d)$.

It remains to count the number of matching pairs. We have $\#_0(x''), \#_0(y'') \leq |x'| + |y'| + \tilde{m} = \mathcal{O}(m)$, as well as $\#_1(x'') = \kappa + \tilde{\Delta} + \#_1(x') = \Theta(\Delta) + \mathcal{O}(m) = \Theta(n)$ (since $\alpha_{\Delta} = 1$) and $\#_1(y'') = \kappa + \#_1(y') = \kappa + \mathcal{O}(M/n) = \Theta(M/n)$ by Lemma 10.19(ii). Thus $M(x'', y'') = \#_1(x'')\#_1(y'') + \#_0(x'')\#_0(y'') = \Theta(M) + \mathcal{O}(m^2) = \Theta(M)$, where the last bound follows from $M \geq L^2/|\Sigma| = \Omega(m^2)$ since $\alpha_L = \alpha_m$.

Combining Lemmas 10.19 and 10.21 with Observation 10.18 finally proves Lemma 10.17.

10.3.3 Case $\alpha_{\Delta} > \alpha_m = \alpha_L$ and $\alpha_{\delta} \leq \alpha_M - 1$

In this section, we consider the case where $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_\delta \le \alpha_M - 1$. In this case, we have $\Delta = \Theta(n) \gg m$ and $\delta \le \mathcal{O}(M/n)$. Since $M/n \le \mathcal{O}(m) \le \mathcal{O}(\Delta)$, the fastest known algorithm runs in time $\tilde{\mathcal{O}}(\min\{d,\delta M/n\})$. Consequently, in this regime Lemma 10.13 simplifies to the following statement.

Lemma 10.22. Let $(\alpha, \{0,1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_\delta \leq \alpha_M - 1$. There is a constant $\gamma \geq 1$ such that any algorithm for $LCS^{\gamma}(\alpha, \Sigma)$ takes time $\min\{d, \delta M/n\}^{1-o(1)}$ unless OVH fails.

As in the previous section, to prove this result we cannot simply pad instances of $LCS_{\leq}(\boldsymbol{\alpha}, \{0, 1\})$ to $LCS(\boldsymbol{\alpha}, \{0, 1\})$, since the desired running time bound $\min\{d, \delta M/n\}$ is not monotone. Instead, we start from instances of a suitably chosen different parameter setting $LCS_{\leq}(\boldsymbol{\alpha}', \{0, 1\})$ with $\alpha'_L = \alpha'_m$, i.e., we invoke Lemma 9.8.

Observation 10.23. Let $(\alpha, \{0, 1\})$ be a non-trivial parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$ and $\alpha_\delta \leq \alpha_M - 1$. Define $\alpha' := \alpha'(\alpha)$ by

$$\begin{split} \alpha_{\delta}' &= \min\{\alpha_{\delta}, \alpha_d/2\}, & \alpha_M' &= 1 + \alpha_m', \\ \alpha_L' &= \alpha_m' &= \min\{\alpha_M - 1, \alpha_d - \alpha_\delta'\}, & \alpha_{\Delta}' &= 1, \\ \alpha_d' &= \min\{\alpha_{\delta} + \alpha_M - 1, \alpha_d\}, & \alpha_{\Sigma}' &= 0, \end{split}$$

Then the parameter setting $(\alpha', \{0, 1\})$ satisfies Table 2. Furthermore, there is some constant $\gamma \geq 1$ such that no algorithm solves $LCS_{\leq}^{\gamma}(\alpha', \{0, 1\})$ in time $n^{\alpha'_d(1-o(1))} = \min\{d, \delta M/n\}^{1-o(1)}$ unless OVH fails. This holds even restricted to instances (n, x, y) with $|x|, |y| \leq \gamma \cdot n^{\alpha'_m} = \mathcal{O}(\min\{M/n, \max\{d/\delta, \sqrt{d}\}\})$.

Proof. We only discuss the inequalities from Table 2 that are not straight-forward to verify. To see $\alpha'_{\delta} \leq \alpha'_{m}$, note that $\alpha_{\delta} \leq \alpha_{M} - 1$ by assumption and $\alpha_{d}/2 = \alpha_{d} - \alpha_{d}/2 \leq \alpha_{d} - \alpha'_{\delta}$. The inequality $\alpha'_{L} \leq \alpha'_{d}$ follows from $\alpha_{M} - 1 \leq \alpha_{M} - 1 + \alpha_{\delta}$ and $\alpha_{d} - \alpha'_{\delta} \leq \alpha_{d}$. Furthermore, $\alpha'_{d} \leq 2\alpha'_{L}$ follows from $\alpha_{\delta} + \alpha_{M} - 1 \leq 2(\alpha_{M} - 1)$ (by assumption) and $\alpha_{d} = 2(\alpha_{d} - \alpha_{d}/2) \leq 2(\alpha_{d} - \alpha'_{\delta})$. From $\alpha'_{d} \leq 2\alpha'_{L}$ and $\alpha'_{L} = \alpha'_{m}$ we also obtain $\alpha'_{M} = 1 + \alpha'_{m} = 1 + \alpha'_{L} \geq 1 + \alpha'_{d} - \alpha'_{L}$, which corresponds to the parameter relation $M \geq nd/(5L)$ that only holds for $\Sigma = \{0,1\}$. Finally, $\alpha'_{M} \geq \alpha'_{d}$ follows from $\alpha'_{M} = 1 + \alpha'_{m} = \min\{\alpha_{M}, 1 + \alpha_{d} - \alpha'_{\delta}\} \geq \min\{\alpha_{M}, \alpha_{d}\}$ by $\alpha'_{\delta} \leq \alpha_{\delta} \leq 1$ and similarly $\alpha'_{d} = \min\{\alpha_{\delta} + \alpha_{M} - 1, \alpha_{d}\} \leq \min\{\alpha_{M}, \alpha_{d}\}$.

Lemma 9.8 shows that some $\gamma \geq 1$ exists such that $\mathrm{LCS}_{\leq}^{\gamma}(\alpha',\{0,1\})$ cannot be solved in time $\min\{n^{\alpha'_d},n^{\alpha'_\delta+\alpha'_m},n^{\alpha'_\delta+\alpha'_\Delta}\}^{(1-o(1))}$, even restricted to instances (n,x,y) with $|x|,|y| \leq \gamma \cdot n^{\alpha'_m} = \mathcal{O}(\min\{M/n,\max\{d/\delta,\sqrt{d}\}\})$. We simplify the running time bound by noting that $\alpha'_\Delta=1\geq\alpha'_m$, so that $n^{\alpha'_\delta+\alpha'_m}\leq n^{\alpha'_\delta+\alpha'_\Delta}$. Moreover, we have $\alpha'_\delta+\alpha'_m=\alpha'_d=\min\{\alpha_\delta+\alpha_M-1,\alpha_d\}$. Indeed, if $\alpha_\delta\leq\alpha_d/2$, we have $\alpha'_\delta=\alpha_\delta$ and hence $\alpha'_\delta+\alpha'_m=\min\{\alpha_\delta+\alpha_M-1,\alpha_d\}=\alpha'_d$. Otherwise, $\alpha_d/2<\alpha_\delta\leq\alpha_M-1$, forcing $\alpha'_\delta=\alpha_d/2$ and $\alpha'_m=\alpha_d/2$, which yields $\alpha'_\delta+\alpha'_m=\alpha_d=\min\{\alpha_\delta+\alpha_M-1,\alpha_d\}=\alpha'_d$. Thus, $\min\{\alpha'_d,\alpha'_\delta+\alpha'_m,\alpha'_\delta+\alpha'_\Delta\}=\alpha'_d$ and the lower bound simplifies to $n^{\alpha'_d(1-o(1))}=\min\{\delta M/n,d\}^{1-o(1)}$.

In this section, to pad the number of dominant pairs, we will construct instances $x' = a0^{\ell}x = (01)^{R+S}0^{\ell}x$, $y' = b0^{\ell}y = 0^{R}(01)^{S}0^{\ell}y$, where we choose R, S proportional to $n^{\alpha_R}, n^{\alpha_S}$ with

$$\alpha_S := \min\{\alpha_M - 1, \max\{\alpha_d - \alpha_\delta, \alpha_d/2\}\},$$

$$\alpha_R := \alpha_d - \alpha_S.$$

Note that the use of α_S , α_R is a slight abuse of notation, since R, S are not actual input parameters, but α_S , α_R depend only on α . We will later set $R = c \cdot n^{\alpha_R}$, $S = c' \cdot n^{\alpha_S}$ with suitably large constants c, c'. However, depending on whether $\alpha_S \leq \alpha_R$ or $\alpha_S > \alpha_R$, we will extend and analyze the basic construction differently. We start with the simpler case of $\alpha_S \leq \alpha_R$.

Lemma 10.24 (Construction for $\alpha_S \leq \alpha_R$). Let $(\boldsymbol{\alpha}, \{0, 1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$, $\alpha_\delta \leq \alpha_M - 1$, and $\alpha_S \leq \alpha_R$. Given an instance (n, x, y) of $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}', \{0, 1\})$ with $|y| \leq |x| = \mathcal{O}(\min\{M/n, \max\{d/\delta, \sqrt{d}\}\})$, we use the parameters

$$S = \max\{|x|, \lfloor n^{\alpha_S} \rfloor\}, \qquad R = \lfloor d/S \rfloor,$$

$$\ell = R + S + |x| + |y|, \qquad \beta = \delta,$$

to instantiate $x' = a0^{\ell}x = (01)^{R+S}0^{\ell}x$ and $y' = 0^{\beta}b0^{\ell}y = 0^{\beta}0^{R}(01)^{S}0^{\ell}y$ as in Lemma 10.9. We furthermore set $\tilde{m} := \max\{m, \delta + R + 2S + \ell + |y|\}$ and $\kappa := |M/n|$ and define

$$x'' := 1^{\Delta + \kappa} 0^{\tilde{m}} x' = 1^{\Delta + \kappa} 0^{\tilde{m}} \quad a \ 0^{\ell} x = 1^{\Delta + \kappa} 0^{\tilde{m}} \quad (01)^{R+S} 0^{\ell} x,$$

$$y'' := 1^{\kappa} 0^{\tilde{m}} y' = 1^{\kappa} 0^{\tilde{m}} 0^{\delta} b \ 0^{\ell} y = 1^{\kappa} 0^{\tilde{m}} 0^{\delta} 0^{R} (01)^{S} 0^{\ell} y.$$

Then x'', y'' is an instance of $LCS^{\gamma'}(\alpha, \{0, 1\})$ for some constant $\gamma' \geq 1$ and can be constructed in time $\mathcal{O}(n)$ together with some integer τ such that $L(x'', y'') = \tau + L(x, y)$.

Proof. By greedy prefix matching and Lemma 7.6, we obtain

$$L(x'', y'') = \kappa + L(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}y') = \kappa + \tilde{m} + L(x', y') = \kappa + \tilde{m} + R + 2S + \ell + L(x, y),$$
(13)

where the last equality follows from Lemma 10.9, which applies since $S \ge |x|$. Clearly, x'', y'' and $\tau = \kappa + \tilde{m} + R + 2S + \ell$ can be computed in time $\mathcal{O}(n)$.

It remains to verify that x, y is an instance of $LCS^{\gamma'}(\alpha, \{0, 1\})$ for some $\gamma' \geq 1$. We observe that $|x|, |y| = \mathcal{O}(n^{\alpha_S})$ by assumption, and hence $S = \Theta(n^{\alpha_S}) = \Theta(\min\{M/n, \max\{d/\delta, \sqrt{d}\}\})$. Thus, $R = \Theta(d/S) = \Theta(n^{\alpha_R}) = \mathcal{O}(dn/M + \min\{\delta, \sqrt{d}\}) = \mathcal{O}(m)$, where the last bound follows from the parameter relations $M \geq nd/(5L) = \Omega(nd/m)$ (since $\alpha_L = \alpha_m$) and $\delta \leq m$. By assumption $\alpha_S \leq \alpha_R$, we have $S = \mathcal{O}(R) = \mathcal{O}(m)$. Furthermore, we have $\kappa = \mathcal{O}(M/n) = \mathcal{O}(m)$ by the relation $M \leq mn$ and $\tilde{m} = \Theta(m)$ by $R, S, |x|, |y|, \ell = \mathcal{O}(m)$ and $\delta \leq m$. Thus, $|x''| = \kappa + \Delta + \tilde{m} + 2(R + S) + \ell + |x| = \Theta(\Delta + m) = \Theta(n)$ and $|y''| = \kappa + \tilde{m} + \delta + R + 2S + \ell + |y| = \Theta(m)$. By (13), we also conclude that $L(x'', y'') = \tilde{m} + \mathcal{O}(m) = \Theta(m) = \Theta(L)$ by the assumption $\alpha_L = \alpha_m$.

Note that by $\Delta \geq \delta$, $|a| \geq |b|$ and $|x| \geq |y|$, we have $|x''| \geq |y''|$. Hence by (13), $\Delta(x'', y'') = \Delta + R + (|x| - L(x, y)) = \Theta(\Delta)$, since $R, |x| = \mathcal{O}(m) = \mathcal{O}(\Delta)$ by $\alpha_{\Delta} > \alpha_{m}$. Likewise, (13) yields $\delta(x'', y'') = \delta + (|y| - L(x, y)) = \delta + \delta(x, y) = \Theta(\delta)$ since $\delta(x, y) = \mathcal{O}(\delta)$ by $\delta(x, y) = \mathcal{O}(n^{\alpha'_{\delta}})$ and $\alpha'_{\delta} \leq \alpha_{\delta}$.

For the dominant pairs, we first compute

$$d(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}y') \ge d(x', y') \ge d(a, 0^{\delta}b) \ge R \cdot S = \Omega(d),$$

using Lemma 10.20, Observation 7.2, and Lemma 7.3. For a corresponding upper bound, we use Lemma 10.20 to obtain

$$d(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}y') = d(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}0^{\delta}b0^{\ell}y) \le |y'| + \tilde{m} + d(1^{\Delta}0^{\tilde{m}}, b0^{\ell}y) + d(x', y').$$

By Lemma 6.8 we have $d(1^{\Delta}0^{\tilde{m}}, b0^{\ell}y) \leq 5 \cdot L(1^{\Delta}0^{\tilde{m}}, b0^{\ell}y) \cdot \#_1(b0^{\ell}y) = \mathcal{O}(|b0^{\ell}y| \cdot (S + |y|)) = \mathcal{O}(R \cdot S) = \mathcal{O}(d)$. Since $|y'| + \tilde{m} = \mathcal{O}(m) = \mathcal{O}(d)$ (using $d \geq L = \Omega(m)$ since $\alpha_L = \alpha_m$) and $d(x', y') \leq 1$

 $5 \cdot L(x', y') \cdot \#_1(y') = \mathcal{O}(|x'| \cdot \#_1(y')) = \mathcal{O}(R \cdot S) = \mathcal{O}(d)$, we conclude that $d(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}y') = \Theta(d)$. Finally, Lemma 7.1 yields $d(x'', y'') = \kappa + d(1^{\Delta}0^{\tilde{m}}x', 0^{\tilde{m}}y') = \Theta(d) + \mathcal{O}(m) = \Theta(d)$, as desired.

It remains to count the matching pairs. Note that $\#_0(x''), \#_0(y'') = \mathcal{O}(\tilde{m} + |x'| + |y'|) = \mathcal{O}(m)$. Furthermore $\#_1(x'') = \Delta + \kappa + \#_1(x') = \Theta(\Delta) + \mathcal{O}(m) = \Theta(n)$ (since $\alpha_{\Delta} > \alpha_m$ implies $\alpha_{\Delta} = 1$) and $\#_1(y'') = \kappa + S + \#_1(y) = \Theta(\kappa) = \Theta(M/n)$, where we used $S, |y| = \mathcal{O}(M/n)$ and $\kappa = \Theta(M/n)$ (since $M \ge n$). Thus, $M(x'', y'') = \#_1(x'') \#_1(y'') + \#_0(x'') \#_1(y'') = \Theta(n \cdot M/n) + \mathcal{O}(m^2) = \Theta(M)$ using that $M \ge L^2/|\Sigma| = \Omega(m^2)$ by $\alpha_L = \alpha_m$. Note that indeed $\Sigma(x'', y'') = \{0, 1\}$.

Before giving the construction for the case $\alpha_S > \alpha_R$, we present a technical lemma that is similar to the dominant pair reduction technique of Lemma 7.8.

Lemma 10.25. Let $x' = a0^{\ell}x = (01)^{R+S}0^{\ell}x$, $y' = b0^{\ell}y = 0^{R}(01)^{S}0^{\ell}y$ be an instance of Lemma 10.2 and $R \ge |y| - L(x,y)$. We set t := R + |y'| + 1 and define

$$\begin{split} \bar{x} \; := & \quad 1^t \; 0^t \; y' \; 0^R \; 1^{t+\Delta} \; 0^t \; x', \\ \bar{y} \; := & \quad 0^R \; 1^t \; \quad 0^t \; y'. \end{split}$$

Then

(i)
$$L(\bar{x}, \bar{y}) = R + 2t + L(x', y'),$$

(ii)
$$d(\bar{x}, \bar{y}) \le (2t + |y'|)(R+1) + R^2$$
,

(iii)
$$d(\bar{x}, \bar{y}) \geq R \cdot S$$
.

Proof. We first prove the following property.

(*) For any prefixes \tilde{x}, \tilde{y} of x', y', we have

$$L(1^t 0^t y' 0^R 1^{t+\Delta} 0^t \tilde{x}, 0^R 1^t 0^t \tilde{y}) = \max\{2t + |\tilde{y}|, R + 2t + L(\tilde{x}, \tilde{y})\}.$$

Note that the lower bound immediately follows from either matching $1^t0^t\tilde{y}$ with $1^t0^ty'$, or by matching $0^R1^t0^t\tilde{y}$ with $0^R1^{t+\Delta}0^t\tilde{x}$. For the upper bound, fix an LCS and observe that it cannot match symbols in the 1^t -prefix of \bar{x} and symbols in the 0^R -prefix of \bar{y} , so at least one of the two prefixes stays unmatched. Thus,

$$L(1^t0^ty'0^R1^{t+\Delta}0^t\tilde{x},0^R1^t0^t\tilde{y}) \leq \max\{L(0^ty'0^R1^{t+\Delta}0^t\tilde{x},0^R1^t0^t\tilde{y}), L(1^t0^ty'0^R1^{t+\Delta}0^t\tilde{x},1^t0^t\tilde{y})\}$$

$$= \max\{R + L(0^{t-R}y'0^R1^{t+\Delta}0^t\tilde{x},1^t0^t\tilde{y}), 2t + |\tilde{y}|\},$$

where the second line follows from greedy prefix matching. Setting $\hat{x} := 0^{t-R}y' 0^R 1^{t+\Delta} 0^t \tilde{x}$ and $\hat{y} := 1^t 0^t \tilde{y}$, it remains to provide the upper bound $R + L(\hat{x}, \hat{y}) \leq \max\{2t + |\tilde{y}|, R + 2t + L(\tilde{x}, \tilde{y})\}$ to prove (*). Assume that an LCS z of \hat{x}, \hat{y} matches less than t - R symbols of the 1^t -prefix of \hat{y} . Then $|z| \leq t - R + L(\hat{x}, 0^t \tilde{y}) \leq 2t - R + |\tilde{y}|$, yielding $R + L(\hat{x}, \hat{y}) \leq 2t + |\tilde{y}|$. Hence, assume instead that at least t - R symbols of the 1^t -prefix of \hat{y} are matched. Since the number of "1"s in the prefix $0^{t-R}y'0^R$ of \hat{x} is only $\#_1(y') \leq |y'| < t - R$, all zeroes of this prefix have to be deleted, resulting in

$$\begin{split} L(\hat{x},\hat{y}) &= |z| \leq L(1^{\#_1(y') + t + \Delta} 0^t \tilde{x}, 1^t 0^t \tilde{y}) \\ &= t + L(1^{\#_1(y') + R + \Delta} 0^t \tilde{x}, 0^t \tilde{y}) \\ &= 2t + L(\tilde{x}, \tilde{y}), \end{split}$$

where the second line follows from greedy prefix matching and the third follows from Lemma 7.6. Thus, we have verified $R + L(\hat{x}, \hat{y}) \leq \max\{2t + |\tilde{y}|, R + 2t + L(\tilde{x}, \tilde{y})\}$, which implies (*).

As an immediate consequence, (*) yields $L(\bar{x}, \bar{y}) = \max\{2t + |y'|, R + 2t + L(x', y')\} = R + 2t + L(x', y')$ since $R \ge |y| - L(x, y) = |y'| - L(x', y')$ (where the equality follows from Lemma 10.2). This proves (i).

For (ii), an application of Lemma 7.7 yields $d(\bar{x}, \bar{y}) \leq (2t + |y'|)(R+1) + R^2$.

Finally, for (iii) we consider, analogous to Lemma 7.3, for any $0 \le s \le S$ and $s \le r \le R + s$, the prefixes $\tilde{x} = (01)^r$ of x' and $\tilde{y} = 0^R (01)^s$ of y'. Then by (*),

$$L(1^t 0^t y' 0^R 1^{t+\Delta} 0^t \tilde{x}, 0^R 1^t 0^t \tilde{y}) = \max\{2t + |\tilde{y}|, R + 2t + L(\tilde{x}, \tilde{y})\}$$
$$= \max\{2t + R + 2s, R + 2t + r + s\} = (R + 2t) + r + s,$$

where we used Lemma 7.3(*) for the second equality and $r \geq s$ for the last equality. Analogously to the proof of Lemma 7.3(ii), this yields $d(\bar{x}, \bar{y}) \geq R \cdot S$, since any feasible choice of r, s gives rise to a unique dominant pair.

We can now give the construction for $\alpha_S > \alpha_R$. Recall that

$$\alpha_S := \min\{\alpha_M - 1, \max\{\alpha_d - \alpha_\delta, \alpha_d/2\}\},$$

$$\alpha_R := \alpha_d - \alpha_S.$$

Lemma 10.26 (Construction for $\alpha_S > \alpha_R$). Let $(\boldsymbol{\alpha}, \{0,1\})$ be a parameter setting satisfying Table 2 with $\alpha_L = \alpha_m < \alpha_\Delta$, $\alpha_\delta \leq \alpha_M - 1$, and $\alpha_S > \alpha_R$. Given an instance (n, x, y) of $LCS_{\leq}^{\gamma}(\boldsymbol{\alpha}', \{0,1\})$ with $|y| \leq |x| = \mathcal{O}(\min\{M/n, \max\{d/\delta, \sqrt{d}\}\})$, we can construct an instance $x^{(4)}, y^{(4)}$ of $LCS_{\leq}^{\gamma'}(\boldsymbol{\alpha}, \{0,1\})$ (for some constant $\gamma' \geq 1$) in time $\mathcal{O}(n)$ together with some integer τ such that $L(x^{(4)}, y^{(4)}) = \tau + L(x, y)$.

Proof. We first set $\ell_1 := |x|$ to define

$$x^{(1)} := 0^{\ell_1} x,$$

$$y^{(1)} := 1^{\delta} 0^{\ell_1} y.$$

which pads the parameter δ . For convenience, define $\delta_1 := |y^{(1)}| - L(x^{(1)}, y^{(1)})$. We use the parameters

$$S := \lfloor n^{\alpha_S} \rfloor, \qquad \qquad R := \max\{\lfloor n^{\alpha_R} \rfloor, \delta_1\}, \qquad \qquad \ell_2 := |x^{(1)}| + |y^{(1)}|,$$

to define, as in Lemma 10.2,

$$x^{(2)} := a \ 0^{\ell_2} \ x^{(1)},$$

 $y^{(2)} := b \ 0^{\ell_2} \ y^{(1)}.$

We then use the dominant pair reduction trick of Lemma 10.25, that additionally pads Δ , and define

$$\begin{split} x^{(3)} &:= 1^{\ell_3} \ 0^{\ell_3} \ y^{(2)} \ 0^R \ 1^{\ell_3 + \Delta} \ 0^{\ell_3} \ x^{(2)}, \\ y^{(3)} &:= 0^R \ 1^{\ell_3} \quad 0^{\ell_3} \ y^{(2)}, \end{split}$$

where $\ell_3 := R + |y^{(2)}| + 1$. The final instance is then constructed as

$$x^{(4)} = 1^{\kappa} 0^{m} x^{(3)},$$

$$y^{(4)} = 1^{\kappa} 0^{m} y^{(3)},$$

where $\kappa := \lfloor M/n \rfloor$. We first compute

$$L(x^{(4)}, y^{(4)}) = \kappa + m + L(x^{(3)}, y^{(3)})$$

$$= \kappa + m + R + 2\ell_3 + L(x^{(2)}, y^{(2)})$$

$$= \kappa + m + 2R + 2S + \ell_2 + 2\ell_3 + L(x^{(1)}, x^{(1)})$$

$$= \kappa + m + 2R + 2S + \ell_1 + \ell_2 + 2\ell_3 + L(x, y),$$
(14)

where we used greedy prefix matching in the first line, Lemma 10.25(i) in the second, Lemma 10.2 in the third, and Lemma 7.6 in the last line. Note that $x^{(4)}, y^{(4)}$, and $\tau := \kappa + m + 2R + 2S + \ell_1 + \ell_2 + 2\ell_3$ can be computed in time $\mathcal{O}(n)$.

It remains to verify that $x^{(4)}, y^{(4)}$ is an instance of $LCS^{\gamma'}(\alpha, \{0, 1\})$ for some $\gamma' \geq 1$. We first observe that $|x|, |y| = \mathcal{O}(n^{\alpha_S})$ and hence $|x^{(1)}|, |y^{(1)}| = \mathcal{O}(n^{\alpha_S} + \delta)$. Note that by definition $S = \Theta(n^{\alpha_S})$.

Assume for contradiction that $\alpha_R < \alpha_\delta$. Note that by definition $\alpha_R = \alpha_d - \alpha_S = \max\{\alpha_d - \alpha_M + 1, \min\{\alpha_\delta, \alpha_d/2\}\}$ and hence $\alpha_R < \alpha_\delta$ only holds if $\alpha_d - \alpha_M + 1 \le \alpha_R = \alpha_d/2 < \alpha_\delta$. But then $\alpha_d - \alpha_\delta \le \alpha_d/2 < \alpha_\delta \le \alpha_M - 1$. This forces $\alpha_S = \alpha_d/2$ by definition, which contradicts the assumption $\alpha_S > \alpha_R$. We therefore obtain $\alpha_R \ge \alpha_\delta$.

Note that $\delta_1 = |y^{(1)}| - L(x^{(1)}, y^{(1)}) = \delta + \delta(x, y)$ by Lemma 7.6. Since $\delta(x, y) \leq \mathcal{O}(n^{\alpha'_{\delta}})$ and $\alpha'_{\delta} \leq \alpha_{\delta}$, this yields $\delta_1 = \Theta(\delta)$, and hence $R = \Theta(n^{\alpha_R})$. Thus, $R = \mathcal{O}(S)$, since $\alpha_R < \alpha_S$. It is immediate that $|x^{(2)}|, |y^{(2)}| = \mathcal{O}(R + S + |x^{(1)}| + |y^{(1)}|) = \mathcal{O}(S)$. Furthermore, it follows that $|x^{(3)}| = \Delta + \mathcal{O}(S)$ and $|y^{(3)}| = \mathcal{O}(S)$. Finally $|y^{(4)}| = m + \mathcal{O}(M/n + S) = \Theta(m)$, where the last bound follows from $S = \mathcal{O}(M/n) = \mathcal{O}(m)$ by the parameter relation $M \leq mn$. Likewise, $|x^{(4)}| = m + \Delta + \mathcal{O}(M/n + S) = \Theta(m + \Delta) = \Theta(n)$. Finally, $L(x^{(4)}, y^{(4)}) = m + \mathcal{O}(M/n + S) = \Theta(m) = \Theta(L)$ by (14) and $\alpha_L = \alpha_m$.

Since $|x| \ge |y|$, $|a| \ge |b|$ and $\Delta \ge \delta$ it is easy to see that $|x^{(4)}| \ge |y^{(4)}|$. Hence (14) yields

$$\Delta(x^{(4)}, y^{(4)}) = 2\ell_3 + |y^{(2)}| + \Delta + R + \Delta(x, y) = \Delta + \mathcal{O}(m) = \Theta(\Delta),$$

since in particular $\Delta(x,y) \leq |x| = \mathcal{O}(m)$. Similarly, $\delta(x^{(4)},y^{(4)}) = \delta + \delta(x,y) = \Theta(\delta)$ as above.

For the number of dominant pairs, we observe that Lemma 10.25(iii) yields $d(x^{(3)}, y^{(3)}) \ge R \cdot S = \Omega(d)$. From Lemma 10.25(ii), the corresponding upper bound $d(x^{(3)}, y^{(3)}) \le (2\ell_3 + |y^{(2)}|) \cdot (R+1) + R^2 = \mathcal{O}(S \cdot R + R^2) = \mathcal{O}(d)$ follows, since $R = \mathcal{O}(S)$ by $\alpha_R < \alpha_S$. Thus, by Lemma 7.1 we obtain $d(x^{(4)}, y^{(4)}) = \kappa + m + d(x^{(3)}, y^{(3)}) = \Theta(d) + \mathcal{O}(m) = \Theta(d)$ by $d \ge L = \Omega(m)$ since $\alpha_L = \alpha_m$.

It remains to count the number of matching pairs. We have $\#_1(y^{(4)}) = \kappa + \#_1(y^{(3)}) = \Theta(M/n)$, since $\kappa = \Theta(M/n)$ by the parameter relation $M \geq n$, and $\#_1(y^{(3)}) \leq |y^{(3)}| = \mathcal{O}(S) = \mathcal{O}(M/n)$. Since $|y^{(4)}| = \mathcal{O}(m)$, we have $\#_0(y^{(4)}) = \mathcal{O}(m)$. Note that $\#_1(x^{(4)}) = \kappa + 2\ell_3 + \Delta + |x^{(2)}| + |y^{(2)}| = \Delta + \mathcal{O}(S + \kappa) = \Theta(n)$, since $\alpha_{\Delta} > \alpha_m$ implies $\alpha_{\Delta} = 1$. Finally, $\#_0(x^{(4)}) = m + 2\ell_3 + R + \#_0(y^{(2)}) + \#_0(x^{(2)}) = \mathcal{O}(m)$. Thus, we obtain $M(x^{(4)}, y^{(4)}) = \#_1(x^{(4)}) \cdot \#_1(y^{(4)}) + \#_0(x^{(4)}) \cdot \#_0(y^{(4)}) = \Theta(n \cdot M/n) + \mathcal{O}(m^2) = \Theta(M)$ by the relation $M \geq L^2/|\Sigma| = \Omega(m^2)$, since $\alpha_L = \alpha_m$.

Note that combining Lemmas 10.24 and 10.26 with Observation 10.23 yields Lemma 10.22.

11 New Algorithm for Binary Alphabet

In this section we prove Theorem 3.5, i.e., we assume that $\Sigma = \{0,1\}$ and provide an algorithm running in time $\mathcal{O}(n + \delta M/n)$. More precisely, for any input x, y, by $\#_0(x) + \#_1(x) = n$ we have $\max\{\#_0(x), \#_1(x)\} \geq n/2$, so without loss of generality assume $\#_1(x) \geq n/2$ (otherwise exchange

0 and 1). Since $M = \#_0(x) \cdot \#_0(y) + \#_1(x) \cdot \#_1(y)$, it follows that $\#_1(y) \leq 2M/n$. Hence, it suffices to design an algorithm running in time $\mathcal{O}(n + \delta \cdot \#_1(y))$.

Theorem 11.1. For $\Sigma = \{0,1\}$, LCS has an $\mathcal{O}(n + \delta \cdot \#_1(y))$ algorithm.

We preprocess x in time $\mathcal{O}(n)$ to support the following queries. For $\sigma \in \{0,1\}$, $0 \le i \le n$, and $t \ge 1$, $\operatorname{Next}_{\sigma}^t(i)$ returns the position of the t-th occurrence of symbol σ after position i in x, i.e., $\operatorname{Next}_{\sigma}^t(i) = i'$ if and only if $x[i'] = \sigma$ and $\#_{\sigma}(x[i+1..i']) = t$ (if such a number i' does not exist then $\operatorname{Next}_{\sigma}^t(i) := \infty$). For convenience, we let $\operatorname{Next}_{\sigma}^0(i) := i$. For t = 1 we also write $\operatorname{Next}_{\sigma}^1(i) = \operatorname{Next}_{\sigma}^1(i)$. It is easy to implement $\operatorname{Next}_{\sigma}^t$ in time $\mathcal{O}(1)$ using rank/select data structures on the 0's and 1's in x, which can be built in time $\mathcal{O}(n)$ [52, 73]. The symbol succeeding i is $\operatorname{Next}_{\Sigma}(i) := i + 1$ if $i + 1 \le n$, or ∞ otherwise, which can be computed in time $\mathcal{O}(1)$.

Let $\lambda = \#_1(y)$ and $1 \leq j_1 < \ldots < j_{\lambda} \leq m$ be the positions of all 1's in y, and for convenience set $j_0 := 0$. We can assume that the last symbol in each of x and y is a 1, in particular $j_{\lambda} = m$, because appending symbol 1 to both x and y increases the LCS by exactly 1 (by Lemma 7.1). We write z_{ℓ} for the number of 0's between $j_{\ell-1}$ and j_{ℓ} in y, i.e., $y = 0^{z_1} 10^{z_2} 1 \dots 10^{z_{\lambda}} 1$ with $z_{\ell} \geq 0$.

Consider the dynamic programming table T that contains for all $0 \le \ell \le \lambda$ and $k \ge 0$ (it remains to fix an upper bound on k) the value

$$T[\ell, k] = \min\{0 \le i \le n \mid L(x[1..i], y[1..j_{\ell}]) = j_{\ell} - k\},\tag{15}$$

where we set $\min \emptyset = \infty$. Observe that from T we can read off the LCS length as $L(x,y) = m - \min\{k \mid T[\lambda,k] < \infty\}$. In particular, we may initialize $\tilde{\delta} := 1$, and compute the table T for $0 \le \ell \le \lambda$ and $0 \le k \le \tilde{\delta}$. If there is no $0 \le k \le \tilde{\delta}$ with $T[\lambda,k] < \infty$ then we double $\tilde{\delta}$ and repeat. This exponential search ends once we find a value $\tilde{\delta} \in [\delta,2\delta)$.

Next we show how to recursively compute $T[\ell, k]$. For $\ell = 0$, we clearly have T[0, 0] = 0 and $T[0, k] = \infty$ for any k > 0. For $\ell > 0$, the following dynamic programming recurrence computes $T[\ell, k]$, as shown in Lemma 11.2 below.

$$T[\ell, k] = \min \left\{ \min \{ \operatorname{Next}_{\Sigma}(\operatorname{Next}_{0}^{z_{\ell} - k + k'}(T[\ell - 1, k'])) \mid \max\{0, k - z_{\ell}\} \leq k' < k \}, \right.$$

$$\operatorname{Next}_{1}(\operatorname{Next}_{0}^{z_{\ell}}(T[\ell - 1, k])),$$

$$T[\ell - 1, k - z_{\ell} - 1] \right\}.$$

$$(16)$$

Note that the third line only applies if $k - z_{\ell} - 1 \ge 0$, as $T[\ell', k'] = \infty$ for k' < 0.

Let us discuss how to efficiently implement the above algorithm, assuming we already have computed the values $T[\ell-1,k]$, $0 \le k \le \tilde{\delta}$. Clearly, we can evaluate the second and third line in constant time, using the Next data structures that we built in the preprocessing. For the first line, observe that $\operatorname{Next}_0^t(i)$ is the position of the $(t+\#_0(x[1..i]))$ -th 0 in x. Hence, $\operatorname{Next}_0^{z_\ell-k+k'}(T[\ell-1,k'])$ is the position of the $(z_\ell-k+k'+\#_0(x[1..T[\ell-1,k']]))$ -th 0 in x, so it is minimized if $k'+\#_0(x[1..T[\ell-1,k']])$ is minimized⁴. Thus, it suffices to compute a range minimum query over the interval $[\max\{0,k-z_\ell\},k)$ on the array $A_\ell[0..\tilde{\delta}]$ with $A_\ell[k']:=k'+\#_0(x[1..T[\ell-1,k']])$. From the answer $A_\ell[r]$ to this range minimum query we can infer $T[\ell,k]$ in time $\mathcal{O}(1)$. Specifically, the first line evaluates to the next symbol after the position of the $(z_\ell-k+A[r])$ -th 0 in x, i.e., $\operatorname{Next}_{\Sigma}(\operatorname{Next}_0^{z_\ell-k+A_\ell[r]}(0))$.

Note that range minimum queries can be performed in time $\mathcal{O}(1)$, after a preprocessing of $\mathcal{O}(|A_{\ell}|) = \mathcal{O}(\tilde{\delta})$ [30, 76], where $|A_{\ell}|$ is the size of array A_{ℓ} . Since we can reuse the array A_{ℓ} for all $0 \leq k \leq \tilde{\delta}$, we spend (amortized) preprocessing time $\mathcal{O}(1)$ per entry of $T[\ell, \cdot]$. In total, this

⁴Here we interpret $\#_0(x[1..\infty])$ as ∞ .

yields time $\mathcal{O}(\tilde{\delta} \cdot \lambda) = \mathcal{O}(\tilde{\delta} \cdot \#_1(y))$ to build the table T. The exponential search for δ yields time $\mathcal{O}(\delta \cdot \#_1(y))$. Adding the preprocessing time $\mathcal{O}(n)$, we obtain an $\mathcal{O}(n + \delta \cdot \#_1(y))$ algorithm. It remains to prove correctness of the recursive formula (16).

Lemma 11.2. Table (15) follows the recursive formula (16).

Proof. For any $1 \le \ell \le \lambda$ and $0 \le k \le \tilde{\delta}$ we show that the value $T[\ell, k]$ of (15) follows the recursive formula (16). Let $i = T[\ell, k]$ and let i' be minimal with

$$L(x[1..i], y[1..j_{\ell}]) = L(x[1..i'], y[1..j_{\ell-1}]) + L(x[i'+1..i], y[j_{\ell-1}+1..j_{\ell}]).$$
(17)

Let $k' = j_{\ell-1} - L(x[1..i'], y[1..j_{\ell-1}])$. Then we claim $i' = T[\ell - 1, k']$. Indeed, since i' satisfies the condition $L(x[1..i'], y[1..j_{\ell-1}]) = j_{\ell-1} - k'$ of (15) we have $i' \geq T[\ell - 1, k']$. Moreover, if we had $i' > T[\ell - 1, k']$ then we could replace i' by $T[\ell - 1, k']$, as both values satisfy the condition $L(x[1..i'], y[1..j_{\ell-1}]) = j_{\ell-1} - k'$, contradicting minimality of i'.

Let $r = L(x[i'+1..i], y[j_{\ell-1}+1..j_{\ell}])$. By (17) we have $j_{\ell} - k = j_{\ell-1} - k' + r$, and we obtain $r = 1 + z_{\ell} - k + k'$ using $z_{\ell} = j_{\ell} - j_{\ell-1} - 1$. Note that $i \geq i'$ is the smallest value attaining $L(x[i'+1..i], y[j_{\ell-1}+1..j_{\ell}]) = r$. Indeed, if there was a smaller value $i' \leq i^* < i$ with $L(x[i'+1..i^*], y[j_{\ell-1}+1..j_{\ell}]) = r$, then $L(x[1..i^*], y[1..j_{\ell}]) \geq L(x[1..i'], y[1..j_{\ell-1}]) + L(x[i'+1..i^*], y[j_{\ell-1}+1..j_{\ell}]) = L(x[1..i], y[1..j_{\ell}]) = j_{\ell} - k$. Then there also exists $0 \leq i'' \leq i^* < i$ with equality, i.e., $L(x[1..i''], y[1..j_{\ell}]) = j_{\ell} - k$. Indeed, if $L(x[1..i^*], y[1..j_{\ell}]) > j_{\ell} - k$ then we can repeatedly reduce i^* by 1, this reduces $L(x[1..i^*], y[1..j_{\ell}])$ by at most 1, and we eventually reach $j_{\ell} - k$ since $L(x[1..i], y[1..j_{\ell}]) = 0$ for t = 0. However, existence of i'' < i contradicts minimality of $i = T[\ell, k]$.

Now we show that i is one of the terms on the right hand side of (16), considering three cases. Case 1: If $1 \le r < z_\ell + 1$, then the LCS of x[i'+1..i] and $y[j_{\ell-1}+1..j_\ell] = 0^{z_\ell}1$ consists of r-1 0's and one additional symbol which is 0 or 1. Thus, the smallest i attaining r is $\mathrm{Next}_{\Sigma}(\mathrm{Next}_0^{r-1}(i'))$, accounting for r-1 0's and one additional symbol. Since $r-1 = z_\ell - k + k'$ and $i' = T[\ell-1, k']$, we have shown that $i = T[\ell, k]$ is of the form $\mathrm{Next}_{\Sigma}(\mathrm{Next}_0^{z_\ell - k + k'}(T[\ell-1, k']))$ for some k'. Observe that $1 \le r < z_\ell + 1$ implies $k - z_\ell \le k' < k$. We clearly also have $k' \ge 0$. This corresponds to the first line of (16).

Case 2: If $r = z_{\ell} + 1$ then x[i' + 1..i] contains $y[j_{\ell-1} + 1..j_{\ell}] = 0^{z_{\ell}} 1$. Thus, $i = \text{Next}_1(\text{Next}_0^{z_{\ell}}(i'))$, accounting for z_{ℓ} 0's followed by a 1. In this case, we have $k' = k + r - z_{\ell} - 1 = k$ so that $i = T[\ell, k]$ is of the form $\text{Next}_1(\text{Next}_0^{z_{\ell}}(T[\ell-1, k]))$. This corresponds to the second line of (16).

Case 3: If r = 0 then i = i', since the smallest value $i \ge i'$ attaining $L(x[i'+1..i], y[j_{\ell-1}+1..j_{\ell}]) = 0$ is i'. In this case, we have $k' = k - z_{\ell} - 1$, and we obtain $T[\ell, k] = i = i' = T[\ell - 1, k'] = T[\ell - 1, k - z_{\ell} - 1]$. This only applies if $k - z_{\ell} - 1 \ge 0$. This corresponds to the third line of (16).

This case distinction shows that i is one of the terms on the right hand side of (16). Also observe that we have $i \leq \operatorname{Next}_1(\operatorname{Next}_0^{z_\ell}(T[\ell-1,k]))$, since the number $\operatorname{Next}_1(\operatorname{Next}_0^{z_\ell}(T[\ell-1,k]))$ is part of the set of which $i = T[\ell,k]$ is the minimum. Similarly, we have $i \leq \operatorname{Next}_{\Sigma}(\operatorname{Next}_0^{z_\ell-k+k'}(T[\ell-1,k']))$ for any $\max\{0,k-z_\ell\} \leq k' < k$, and $i \leq T[\ell-1,k-z_\ell-1]$ if $k-z_\ell-1 \geq 0$. This proves that i is the minimum over all expressions on the right hand side of (16), proving that $i = T[\ell,k]$ follows the recursive formula (16).

12 Strengthening Hardness via BP-SETH

A recent and surprising result by Abboud, Hansen, Virginia Williams and Williams [2] proves conditional lower bounds for LCS and related problems under a natural weaker variant of SETH, called

BP-SETH. In this variant, the role of CNF formulas in SETH is replaced by branching programs, providing a much more expressive class of Boolean functions – intuitively, the corresponding satisfiability problem becomes much harder. As a consequence, refuting a conditional lower bound based on BP-SETH would yield stronger algorithmic consequences, strengthening the conditional lower bound significantly. Furthermore, Abboud et al. show that even a sufficiently strong *polylogarithmic* improvement for LCS would imply faster (formula) satisfiability algorithms than currently known.

In this section, we show how to adapt the proofs of our conditional lower bounds to also hold under BP-SETH. To this end, we first introduce this weaker assumption, briefly state the main construction of Abboud et al. (specialized to LCS) and then describe the necessary modifications to our proofs.

Branching Programs and BP-SETH. Branching programs provide a popular model for non-uniform computation. Formally, a branching program P on N variables, length T and width W consists of a directed graph G, whose vertex set V is divided into T disjoint layers V_1, \ldots, V_T . Each layer contains at most W vertices. Every edge in G starts in some layer V_j and ends at the following layer V_{j+1} . Each edge is annotated with a constraint $X_i = b$, where X_1, \ldots, X_N are the Boolean input variables and $b \in \{0,1\}$. The constraints of all edges starting in layer V_j must use the same variable X_i , but may have potentially different values of b. There are two distinguished nodes, namely some start node $v_0 \in V_1$, and an accept node $v^* \in V_T$. Given an assignment $X \in \{0,1\}^N$ to the variables, we say that P accepts x if and only if there is a path from v_0 to v^* such that the constraints on all edges on the path are satisfied by X.

The corresponding satisfiability problem BP-SAT asks, given a branching program P on N variables, length T and width W, to determine whether there exists an assignment $X \in \{0,1\}^N$ that is accepted by P. The results in [2] rely on the following hypothesis.

Hypothesis 12.1. For any $\varepsilon > 0$, BP-SAT with $(\log W)(\log T) = o(N)$ cannot be solved in time $O((2-\varepsilon)^N)$.

Note that this hypothesis is much weaker than SETH: By Barrington's theorem, already branching programs of constant width and T = polylog(N) can simulate any NC circuit, and thus any CNF formula.

Central Construction of Abboud et al. We present an intermediate step of the construction of Abboud et al. in a convenient form for our purposes.

Lemma 12.2 (Normalized BP-Vector Gadgets [2, Implicit lemma in the proof of Theorem 6, specialized to LCS]). Let P be a branching program on $N_1 + N_2$ variables, width W and length T. Let $\mathcal{A} = \{a_1, \ldots, a_A\} \subseteq \{0, 1\}^{N_1}, \mathcal{B} = \{b_1, \ldots, b_B\} \subseteq \{0, 1\}^{N_2}$ be sets of partial assignments. In linear time in the output size, we can construct binary strings x_1, x_2, \ldots, x_A of length ℓ_X and y_1, y_2, \ldots, y_B of length ℓ_Y (called normalized vector gadgets) with the following properties.

- 1. There is some function $f(W,T) = T^{\mathcal{O}(\log W)}$ with $\ell_x, \ell_Y \leq f(W,T)$,
- 2. We can compute ρ_0, ρ_1 with $\rho_0 > \rho_1$ such that $L(x_i, y_j) \ge \rho_0$ if the assignment given by assigning a_i to the variables X_1, \ldots, X_{N_1} and assigning b_j to $X_{N_1+1}, \ldots, X_{N_1+N_2}$ is accepted by P. Otherwise, we have $L(x_i, y_j) = \rho_1$.

Proof modifications. Equipped with the above lemma, we sketch how to modify our lower bounds to also hold under BP-SETH. Consider the following problem LCS-Pair with Guarantees: Given binary strings x_1, x_2, \ldots, x_A of length ℓ_X and y_1, y_2, \ldots, y_B of length ℓ_Y , where $\ell_X, \ell_Y = (A + B)^{o(1)}$, and given integers $\rho_0 > \rho_1$, decide whether either (1) $L(x_i, y_j) \ge \rho_0$ for some i, j, or (2) $L(x_i, y_j) = \rho_1$ for all i, j. The naive solution for this problem runs in time $(AB)^{1+o(1)}$.

Lemma 12.2 shows that any $\mathcal{O}((AB)^{1-\varepsilon})$ -time algorithm for LCS-Pair with Guarantees would break BP-SETH. Indeed, given a branching program P on N variables, length T and width W, we first split the variables into two sets of size N/2, and let \mathcal{A}, \mathcal{B} be the sets of all assignments to these two sets of variables. Then testing whether there are partial assignments $a_i \in \mathcal{A}$ and $b_j \in \mathcal{B}$ that together are accepted by P is equivalent to deciding satisfiability of P. Since $A = |\mathcal{A}| = B = |\mathcal{B}| = 2^{N/2}$, any $\mathcal{O}((AB)^{1-\varepsilon})$ -time algorithm would break BP-SETH. Moreover, in BP-SETH we can assume $\log T \cdot \log W = o(N)$ and thus $\ell_X, \ell_Y = T^{\mathcal{O}(\log W)} = 2^{o(N)} = (A+B)^{o(1)}$, as required in the definition of LCS-Pair with Guarantees. Furthermore, the same $(AB)^{1-o(1)}$ lower bound under BP-SETH also holds restricted to instances with $B = A^{\beta \pm o(1)}$ for any $0 < \beta \le 1$. To see this, in the above reduction instead of splitting the N variables into equal halves $N_1 = N_2 = N/2$, we choose N_1, N_2 such that $N_2 \approx \beta N_1$, so that the sets of partial assignments $\mathcal{A} = \{0,1\}^{N_1}$ and $\mathcal{B} = \{0,1\}^{N_2}$ satisfy $B = |\mathcal{B}| = |\mathcal{A}|^{\beta \pm o(1)} = A^{\beta \pm o(1)}$.

Observe that the normalized vector gadgets constructed in Lemma 9.5 show the similar claim that any $\mathcal{O}((AB)^{1-\varepsilon})$ -time algorithm for LCS-Pair with Guarantees would break the OV hypothesis, and thus SETH. Since all reductions presented in this paper use normalized vector gadgets either directly via Lemma 9.5 or indirectly via Lemma 9.1, they all implicitly go via LCS-Pair with Guarantees. Hence, we can easily replace the first part of our reductions, i.e., the reduction from SAT/OV to LCS-Pair with Guarantees, by the reduction from BP-SAT to LCS-Pair with Guarantees given in Lemma 12.2. This yields a conditional lower bound based on BP-SETH.

There are two steps where we have to be careful: First, LCS-Pair with Guarantees does not immediately give properties (i) and (iv) of Lemma 9.5, however, they can be ensured easily as in the proof of Lemma 9.5. Second, Lemma 9.1 is the construction from [28], and thus to check that it goes via LCS-Pair with Guarantees one needs to check that the proof in [28] indeed only uses Lemma 9.5. Along these lines we obtain the following strengthening of our main result.

Theorem 12.3. Theorems 3.3 and 3.4 also hold after replacing SETH by BP-SETH.

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