The Relativistic Theory of Electromagnetic Susceptibility and its Application to Plasmas

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R. A. O'Sullivan
H. Derfler

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Abstract

Covariant constitutive relations are developed for an arbitrary, anisotropic, dispersive, dissipative medium, thus allowing for relaxation phenomena, and the covariant wave dispersion relation is derived. The susceptibility tensor of order four is given explicitly for a Vlasov plasma with arbitrary velocity distribution and non-zero d.c. electric field. It is then shown how to determine the constitutive relation explicitly from the moment of the single particle current, without using the Vlasov equation. The familiar "dielectric tensor" is obtained as a special case.

I. INTRODUCTION

Since the pioneering work of Clemmow and Wilson [1, 2], Buneman [3] and others, the kinetic theory of relativistic plasmas has acquired an extensive literature. See for example the articles listed by the authors of references [4 - 9]. Basic to many treatments is the dielectric tensor derived by Trubnikov [10] from the three-dimensional relativistic Vlasov equation for a stationary, Maxwellian plasma with zero d.c. electric field. Since the dielectric tensor is not covariant, however, one of the main advantages of a relativistic treatment, namely the direct application to drifting plasmas, is lost by this method. Furthermore, the extension of Trubnikov's dielectric tensor to cases where the d.c. electric field is not zero is far from obvious. In order to write covariant constitutive equations, one needs the four-dimensional magnetization-polarization tensor, which was introduced by Dällenbach [11] and Pauli [12]. They showed, on the basis of intuitive physical arguments, that it can be expressed formally in terms of the averaged moment of the single particle current density. They did not, however, obtain the relationship between this tensor and the electromagnetic field tensor.

In this paper, we develop the theory of the covariant constitutive relations for an arbitrary, anisotropic, dispersive medium, allowing for relaxation phenomena. To do this we replace the familiar dielectric tensor of order two by a susceptibility tensor of order four, which relates the electromagnetic field tensor to the magnetization-polarization tensor already mentioned. Taking the example of a Vlasov plasma, we find an explicit expression for the susceptibility tensor, and hence for the unspecified terms occurring in Dällenbach's expression for the magnetization-polarization tensor. The susceptibility tensor would be of particular importance in applications to a multiple drift plasma where it would be difficult to obtain the same result by Lorentz transformations. In deriving the susceptibility tensor, we regard all particles in the plasma as bound particles, so that the free current density is zero. This model is particularly useful for bounded plasmas, since it removes the need to work with surface charges and currents. The model has been used by Derfler and Omura [13] who worked non relativistically. They divided the total plasma current into magnetization and polarization currents, from which they derived separately the magnetic permeability and dielectric permittivity tensors, whereas all previous treatments lump both terms into one equivalent expression, commonly called "the dielectric tensor" [14, 15].

In section II, we define the fourth order susceptibility and permittivity tensors, and derive the general wave dispersion relation in covariant form. In section III, we solve the manifestly covariant Vlasov equation for a drifting plasma with non-zero d.c. electric field in terms of a four-dimensional gyro-tensor. In section IV, the explicit form of the magnetization-polarization tensor is first derived from the first-order solution of the Vlasov equation. Then, after solving the first-order equation of motion in terms of Lagrangian variables, we show how the magnetization-polarization tensor can be obtained, independently of the Vlasov equation, from the moment of the single particle current. In section V we integrate the magnetization-polarization tensor for a drifting equilibrium plasma and finally obtain Trubnikov's dielectric tensor as a special case.

II. COVARIANT CONSTITUTIVE EQUATIONS FOR AN ANISOTROPIC MEDIUM

Throughout the paper, we adopt the following conventions: Unless otherwise indicated, Roman subscripts take the values 1 to 4 and Greek subscripts go from 1 to 3. We use Minkowski coordinates, in which the four vector f has components f_i, where f_α are the components of a spacelike vector $\underset{\sim}{f}$ and $f_4 = if_o$ is the timelike component. For example, x denotes the world-point, with $x_o = ct$. We denote the volume element $df_o\, df_1\, df_2\, df_3$ by df.

For an arbitrary medium, Maxwell's equations are written covariantly in terms of the electromagnetic field tensor $\underset{\sim}{B}$, the excitation tensor $\underset{\sim}{H}$ and the four-vector free current density J^{free}, as follows:

$$\partial_j B^*_{ij} = 0 \;, \qquad \partial_j H_{ij} = J_i^{free} \;, \tag{2.1}$$

where $\underset{\sim}{B}^*$ is the dual of $\underset{\sim}{B}$, defined by $B^*_{ij} = \frac{1}{2}\eta_{ijkl} B_{kl}$, η_{ijkl} being the alternating tensor of order four. In terms of the familiar three-vectors, we write, following Sommerfeld [16],

$$\underset{\sim}{B} = ((\underset{\sim}{B}, \underset{\sim}{E}/ic)) \equiv \begin{pmatrix} 0 & B_3 & -B_2 & E_1/ic \\ -B_3 & 0 & B_1 & E_2/ic \\ B_2 & -B_1 & 0 & E_3/ic \\ -E_1/ic & -E_2/ic & -E_3/ic & 0 \end{pmatrix}$$

and

$$\underset{\sim}{H} = ((\underset{\sim}{H}, -ic\underset{\sim}{D}))$$

where $\underset{\sim}{E}$ and $\underset{\sim}{\mathfrak{H}}$ are the electric and magnetic field strength respectively, $\underset{\sim}{\mathfrak{B}}$ is the magnetic induction and $\underset{\sim}{\mathfrak{D}}$ the electric displacement. $\underset{\sim}{B}$ and $\underset{\sim}{H}$ are related by the equation

$$\underset{\sim}{H} = \underset{\sim}{B}/\mu_o - \underset{\sim}{M} \tag{2.2}$$

where $\underset{\sim}{M} = ((\underset{\sim}{\mathfrak{M}}, ic\underset{\sim}{P}))$ is the magnetization-polarization tensor, involving the magnetization and polarization vectors, $\underset{\sim}{\mathfrak{M}}$ and $\underset{\sim}{P}$ respectively, which describes the effect of the current density due to bound charges,

$$J_i = \partial_j M_{ij}$$

In the kinetic treatment of a plasma, $\underset{\sim}{B}$ is the average of the microfield existing between the particles. Regarding all particles as bound particles, we set $J^{free} = o$. The electromagnetic properties of the plasma are then completely described by the tensor $\underset{\sim}{M}$, which in general is a non-linear functional of the field tensor $\underset{\sim}{B}$. In the following, we shall consider small perturbations, $\underset{\sim}{B}^1$, from the stationary state $\underset{\sim}{B}^o$, with

$$\underset{\sim}{B} = \underset{\sim}{B}^o + \underset{\sim}{B}^1 \qquad \text{etc.}$$

and

$$J = J^o + J^1 + J^{ex}$$

where the higher order contributions of the bound charges are lumped into an equivalent excitation current J^{ex}. For a homogeneous medium, we can then represent the functional relating the first order perturbations as a linear convolution of the form

$$M^1_{ij}(x) = -X_{ijk\ell} * B^1_{k\ell} \equiv -\int_{-\infty}^{\infty} d\xi\, X_{ijk\ell}(x-\xi)\, B^1_{k\ell}(\xi), \qquad (2.3)$$

where $\underset{\sim}{X}$ represents the first order susceptibility kernel and the arguments are invariant functionals of x. Hence the convolutions are manifestly covariant with respect to Lorentz transformations,

$$x'_i = A_{ij} x_j \quad , \quad \underset{\sim}{A} = \begin{pmatrix} \delta_{\alpha\beta} + \lambda_\alpha \lambda_\beta/(\gamma+1) & i\lambda_\beta \\ -i\lambda_\alpha & \gamma \end{pmatrix} \qquad (2.4)$$

where $\lambda_\alpha = v_\alpha \gamma/c$, $\gamma = (1 - v_\alpha v_\alpha/c^2)^{-\frac{1}{2}}$, $\underset{\sim}{v}$ being the three-velocity. The convolutions exist in the mathematical sense, even in the presence of instabilities, due to the finite speed of propagation, provided the perturbations are applied at some finite time. For the same reason, the two-sided Laplace transform,

$$B^1_{ij}[k] \equiv \iiiint_{-\infty}^{\infty} dx\, e^{ik_j x_j}\, B^1_{ij}(x) \quad , \qquad (2.5)$$

exists and is covariant with respect to Lorentz transformations. Also, due to the finite speed of propagation, we can always find upper bounds in each inertial frame, such that

$$\begin{aligned} |\underset{\sim}{B}^1(x)| &< C e^{-\varkappa |x_\alpha|} && \text{for } x_0 \text{ const}, |x_\alpha| \to \infty , \\ |\underset{\sim}{B}^1(x)| &< C e^{\varkappa_0 x_0} && \text{for } x_\alpha \text{ const}, x_0 > ct^* , \qquad (2.6) \\ |\underset{\sim}{B}^1(x)| &= 0 && \text{for } x_\alpha \text{ const}, x_0 < ct^* . \end{aligned}$$

It follows from a theorem of Doetsch [17] that $\underset{\sim}{B}^1[k]$ is a regular, analytic function of the complex variables k_i in the strips

$$|\mathrm{Im}\, k_\alpha| \leq \kappa \,, \tag{2.7}$$

which include the real wave number axis, and a lower frequency plane

$$\mathrm{Im}\, k_0 = \mathrm{Im}\, \omega/c \leq -\kappa_0 \,, \tag{2.8}$$

where κ_0 is positive in the presence of instabilities. Analytic functions of k_i satisfying the conditions of Eqs.(2.7) and (2.8) will henceforth be referred to as $L(\kappa_0)$ functions. These can be inverted in the form

$$B^1_{ij}(x) = (2\pi)^{-4} \iiiint_{i\sigma_j - \infty}^{i\sigma_j + \infty} dk\, e^{-ik_j x_j}\, B^1_{ij}[k] \,, \tag{2.9}$$

where we restrict the Bromwich contours of integration to the regions where $\underset{\sim}{B}^1[k]$ is regular and analytic, i.e. where $\sigma_0 \leq -\kappa_0$, $|\sigma_\alpha| \leq \kappa$. Since $\underset{\sim}{X}(\xi)$ represents the response of the medium to an impulse localized in space-time, i.e. $\underset{\sim}{B}^1(x)$ $\propto \delta(x)$, the susceptibility kernel satisfies conditions similar to Eqs.(2.6) and is therefore an $L(\kappa_0)$ function. Under Laplace analysis Eq.(2.3) takes the form

$$M^1_{ij}[k] = -X_{ijk\ell}[k]\, B^1_{k\ell}[k] \tag{2.1o}$$

and $\underset{\sim}{M}^1[k]$ is thus an $L(\kappa_0)$ function. Since, in the following, we shall be concerned only with the Laplace transforms of the tensors, we shall delete the square brackets identifying the transforms wherever no confusion can arise. The calculations in Appendices B and C will be performed for k_α real and $\mathrm{Im}\, k_o < 0$, it being understood that the results apply to the region specified in Eqs.(2.7) and (2.8) and can be extended by analytical continuation.

By the skew-symmetry of M^1_{ij} and $B^1_{k\ell}$, $X_{ijk\ell}$ must be skew-symmetric in the index pairs (i,j) and (k,ℓ). Taking the transform of Eq.(2.2) and substituting Eq.(2.1o) we obtain

$$H^1_{ij} = T_{ijk\ell} B^1_{k\ell} \tag{2.11}$$

where

$$T_{ijk\ell} = X_{ijk\ell} + \eta_{ijmn}\eta_{k\ell mn}/4\mu_o .$$

In terms of this tensor, we can write the Laplace transforms of Maxwell's Eqs.(2.1) in the form

$$\eta_{mpqr} k_p B^1_{qr} = 0 , \quad -iH^1_{mp} k_p = T_{mpqr} k_p B^1_{qr} = J^{ex}_m . \tag{2.12}$$

These equations can be solved most conveniently by means of the vector potential A, defined in the usual manner, such that

$$B^1_{qr}(x) = \partial_q A_r(x) - \partial_r A_q(x) \Rightarrow B^1_{qr} = k_q A_r - k_r A_q \equiv k_{[q} A_{r]}$$

satisfies the first of Eqs.(2.12) identically. The problem thus reduces to the solution of the second set of Maxwell's equations, which, in terms of the vector potential, takes the form

$$K_{mr} A_r = \tfrac{1}{2} i J^{ex}_m , \quad K_{mr} \equiv T_{mpqr} k_p k_q . \tag{2.13}$$

From the skew-symmetry of $\underset{\sim}{T}$, it follows that the rows and columns respectively of the matrix $\underset{\sim}{K}$ are linearly dependent, i.e.

$$K_{mr} k_r = 0 \quad , \quad k_m K_{mr} = 0 \; . \tag{2.14}$$

Thus the determinant $\| K_{mr} \|$ vanishes identically for general k. It is therefore obvious that the equation $\| K_{mr} \| = 0$ cannot be the wave dispersion relation, as has been claimed [18]. By substituting from Eqs.(2.14) for rows and columns in the 3 x 3 minors of K_{mr}, it is easy to show that the cofactor $\bar{K}_{mr}$ of K_{mr} equals $K\, k_m k_r$, where K is an invariant. We can express K in terms of the trace, $\bar{K}_{ii}$:

$$K = \bar{K}_{ii} / k_j k_j \quad . \tag{2.15}$$

Thus the cotensor can be written in the form

$$\bar{K}_{mr} = k_m k_r K = k_m k_r \bar{K}_{ii} / k_j k_j \quad . \tag{2.16}$$

Provided $K \neq 0$, it follows that the system of Eqs.(2.13) has rank three and hence can be solved for general k when the continuity equation, $k_m J_m = o$, is satisfied. If K were zero, then by Eq.(2.16), all the factors $\bar{K}_{mr}$ would vanish, so that the system of Eqs.(2.13) would have rank < 3. In this case, we would have yet another condition on J for general k, which would be unphysical.

By deleting in Eqs.(2.13) the Mth row and Rth column, we get the system of equations

$$K_{\mu\rho} A_\rho + K_{\mu R} A_R = \tfrac{1}{2} i J_\mu^{ex} , \quad \mu \neq M , \tag{2.17}$$

$$K_{M\rho} A_\rho + K_{MR} A_R = \tfrac{1}{2} i J_M^{ex} , \quad \rho \neq R , \tag{2.18}$$

where the summation convention is suspended for capital subscripts. We first solve Eq.(2.17) in the form

$$A_\rho = \tfrac{1}{2} i K^{-1}_{\rho\mu} J_\mu^{ex} - K^{-1}_{\rho\mu} K_{\mu R} A_R \tag{2.19}$$

and substitute from Eqs.(2.14) the relations

$$K_{\mu R} = -K_{\mu\rho} k_\rho / k_R , \quad K_{M\rho} = -K_{\mu\rho} k_\mu / k_M$$

into Eqs.(2.18) and (2.19). We thus confirm that the continuity equation $k_m J_m = o$ must be satisfied, and obtain the solution of Eqs.(2.13) in the form

$$B^1_{R\rho} = k_{[R} A_{\rho]} = \tfrac{1}{2} i K^{-1}_{\rho\mu} k_R J_\mu^{ex} . \tag{2.2o}$$

To write this result covariantly, we first recall that the 3 x 3 matrix $K_{\mu\rho}$ can be inverted by Cramer's rule in the form

$$K^{-1}_{\mu\rho} = \bar{K}_{M\mu R\rho} / \bar{K}_{MR} , \tag{2.21}$$

where

$$\bar{K}_{mr} \equiv \eta_{mnpq}\,\eta_{rstu}\,K_{ns}\,K_{pt}\,K_{qu}/3!$$

and

$$\bar{K}_{mnrs} \equiv \eta_{mnpq}\,\eta_{rstu}\,K_{pt}\,K_{qu}/2!$$

are the cotensors to K_{mr} of second and fourth order respectively. Substituting Eq.(2.21) into Eq.(2.2o) and using Eq.(2.16), we obtain, on multiplication with k_M^2 ,

$$k_M^2\, \overset{1}{B}_{R\rho} = \tfrac{1}{2}\, i\, \bar{K}_{M\mu R\rho}\, k_M\, J_\mu^{ex}\, k_j\, k_j / \bar{K}_{ii} \,. \tag{2.22}$$

By the skew symmetry of $\bar{K}_{M\mu R\rho}$, setting $\mu = M$ and $\rho = R$ would give zero contribution, so we can lift the restrictions $\mu \neq M$ and $\rho \neq R$, replacing μ by n and ρ by s. We can then sum over M from 1 to 4, which produces a factor $k_j k_j$ on the left-hand side of Eq.(2.22) and cancels the one on the right-hand side. Then M and R lose their uniqueness and can be replaced by m and r respectively. We can now write the solution of the inhomogeneous Maxwell's equations for general k in the manifestly covariant form

$$4\overset{1}{B}_{rs} = i\,\bar{K}_{mnrs}\, k_{[m}\, J^{ex}_{n]} / \bar{K}_{ii} = i\,\bar{K}_{mnrs}\, k_{[m} J^{ex}_{n]} / k_i k_i K \,. \tag{2.23}$$

It can be verified immediately that the excitation tensor

$$\overset{1}{H}_{mn} = -ik_{[m} J^{ex}_{n]} / k_i k_i$$

satisfies the second set of Maxwell's Eq.(2.12) for general k. Therefore, by substitution into Eq.(2.23), we obtain the inverse of the permittivity tensor, defined in Eq.(2.11), in the form

$$B^1_{rs} = T^{-1}_{rsmn} H^1_{mn} \tag{2.24}$$

where

$$T^{-1}_{rsmn} = -\tfrac{1}{4}\bar{K}_{mnrs} / K . \tag{2.25}$$

Using the tensors $\underset{\sim}{T}$ and $\underset{\sim}{T}^{-1}$, we can now rewrite the constitutive Eq.(2.2) in either of the alternative forms

$$\begin{aligned} M^1_{ij} &= -\left(T_{ijk\ell} - \delta_{i[k}\delta_{\ell]j}/2\mu_o\right) B^1_{k\ell} , \\ M^{1*}_{ij} &= \left(T^{-1*}_{ijk\ell} - \delta_{i[k}\delta_{\ell]j}/2\mu_o\right) H^{1*}_{k\ell} , \end{aligned} \tag{2.26}$$

where

$$T^{-1*}_{ijk\ell} \equiv \tfrac{1}{4}\,\eta_{ijmn} T^{-1}_{mnpq}\,\eta_{pqk\ell} .$$

We have already argued that K does not vanish for general k. On the other hand, K must vanish in regions where the current density in Eq.(2.23) vanishes in order to allow for finite fields $\underset{\sim}{B}$, e.g. to explain the phenomenon of propagating waves. The covariant wave dispersion function is therefore given by the invariant K defined in Eq.(2.15). One may be tempted to conclude from Eq.(2.25) that the inverse tensors $\underset{\sim}{T}^{-1}$ and $\underset{\sim}{X}^{-1}$ would not exist in the presence of waves. This apparent contradiction can be resolved immediately in the construction of the tensor kernels $\underset{\sim}{T}^{-1}(x)$ and $\underset{\sim}{X}^{-1}(x)$ in the space-time domain by means of the inverse Laplace transform. In this process, one encounters the problem of residue calculus at discrete and/or continuous sets of poles $k^{(n)}$ of $\underset{\sim}{T}^{-1}[k]$ and $\underset{\sim}{X}^{-1}[k]$ which are in fact solutions of the wave dispersion relation K = o.

Provided Min Im $k_0^{(n)} > -\kappa_0$, one can adjust the Bromwich contour so that $\sigma_0 < -\kappa_0$, implying that $\underset{\sim}{T}^{-1}[k]$ is an $L(\kappa_0)$ function. The proof that this condition is always satisfied can be given only on the basis of the kinetic theory of $\underset{\sim}{T}[k]$, as developed in sections IV and V. Any finding to the contrary would mean that the problem was ill-posed to start with. Since Maxwell's equations are satisfied for $k^{(n)}$, the residue calculus leads to an expression for the propagators or Green's functions $\underset{\sim}{T}^{-1}(x)$ and $\underset{\sim}{X}^{-1}(x)$ in terms of waves as discussed by Derfler [19-22] in the context of his wave stability criteria.

For purposes of comparison with the familiar non-covariant dielectric permittivity and diamagnetic permeability tensors of order two, we will now determine the components of $\underset{\sim}{T}$ and $\underset{\sim}{T}^{-1}$ in the rest frame. In the present context, this is unambiguously defined as that frame in which the three-vector fields are related in the form

$$D_\alpha = \epsilon_0 \epsilon_{\alpha\beta} E_\beta \quad , \quad H_\alpha = \mu^{-1}_{\alpha\beta} B_\beta / \mu_0 \quad ,$$

which, following Sommerfeld [16], is the natural way of writing the constitutive relations. Substituting these relationships into Eqs.(2.11) and (2.24) and using the fact that, in the rest frame, $\underset{\sim}{\mathfrak{H}}$ is independent of $\underset{\sim}{E}$ and $\underset{\sim}{\mathfrak{D}}$ of $\underset{\sim}{\mathfrak{B}}$, we obtain

$$\begin{aligned} &T_{\alpha\beta\gamma 4} = T_{\alpha 4\beta\gamma} = 0 = T^{-1*}_{\alpha\beta\gamma 4} = T^{-1*}_{\alpha 4\beta\gamma} \\ &T_{\alpha\beta\gamma\delta} = \eta_{\alpha\beta\eta}\mu^{-1}_{\eta\zeta}\eta_{\zeta\gamma\delta}/2\mu_0 \, , \quad T_{\alpha 4\gamma 4} = \epsilon_{\alpha\gamma}/2\mu_0 \, , \\ &T^{-1*}_{\alpha\beta\gamma\delta} = \tfrac{1}{2}\mu_0 \eta_{\alpha\beta\eta}\epsilon^{-1}_{\eta\zeta}\eta_{\zeta\gamma\delta} \, , \quad T_{\alpha 4\gamma 4} = \tfrac{1}{2}\mu_0\mu_{\alpha\gamma} \, . \end{aligned} \qquad (2.27)$$

It should be noted for completeness that the so-called "plasma dielectric tensor", $\tilde{\epsilon}_{\alpha\beta}$, defined by

$$J^1_\alpha = -i\omega\,(\tilde{\epsilon}_{\alpha\beta} - \delta_{\alpha\beta})\,E^1_\beta/4\pi \;,$$

is actually a combination of the dielectric permittivity and diamagnetic permeability in the form

$$\tilde{\epsilon}_{\alpha\beta} = \epsilon_{\alpha\beta} + (k_4)^{-2}\eta_{\alpha\gamma\delta}\,k_\gamma\,(\delta_{\delta\epsilon} - \mu^{-1}_{\delta\epsilon})\,k_\zeta\,\eta_{\epsilon\zeta\beta} \;. \quad (2.28)$$

Substitution of Eqs.(2.27) and (2.28) into the covariant dispersion function of Eq.(2.15) gives the familiar three dimensional form of the dispersion relation in the rest frame, viz.

$$\| k_0^2\,\tilde{\epsilon}_{\alpha\beta} + k_\alpha k_\beta - k_\gamma k_\gamma \delta_{\alpha\beta} \| = 0 \;.$$

In the case of an isotropic, non-dispersive medium, where ϵ and μ are scalars, the tensors $\underset{\sim}{T}$ and $\underset{\sim}{T}^{-1}$ simplify, such that in the rest frame

$$T_{\alpha\beta\gamma\delta} = \eta_{\alpha\beta\lambda}\,\eta_{\lambda\gamma\delta}/2\mu_0\mu \;, \quad T_{\alpha 4\gamma 4} = \epsilon\,\delta_{\alpha\gamma}/2\mu_0 \;,$$

$$T^{-1*}_{\alpha\beta\gamma\delta} = \mu_0\,\eta_{\alpha\beta\lambda}\,\eta_{\lambda\gamma\delta}/2\epsilon \;, \quad T^{-1*}_{\alpha 4\gamma 4} = \tfrac{1}{2}\,\mu_0\mu\,\delta_{\alpha\gamma} \;.$$

By Lorentz transformation using Eq.(2.4), we can obtain expressions for the tensors $\underset{\sim}{T}$ and $\underset{\sim}{T}^{-1}$ in terms of ϵ and μ, in any inertial frame. Eqs.(2.26) then reduce to the constitutive equations for an isotropic, non-dispersive medium given in three-dimensional form by Sommerfeld and in four-dimensional form by Pauli and Synge [24], viz.:

$$M^{1}_{ij}\lambda_j = -(\epsilon-1)B^{1}_{ij}\lambda_j/\mu_0 = -\chi_e B^{1}_{ij}\lambda_j/\mu_0 \qquad (2.29)$$

$$M^{1*}_{ij}\lambda_j = (\mu-1)H^{1*}_{ij}\lambda_j = \chi_m H^{1*}_{ij}\lambda_j$$

where χ_e and χ_m are the electric and magnetic susceptibilities. The analogy between Eqs. (2.26) and (2.29) was anticipated in designating the tensor $\underset{\sim}{X}$ as the covariant electromagnetic susceptibility tensor. Whereas, in the previous treatments, the two equations (2.29) were needed to describe the properties of the medium, either one of our Eqs. (2.26) contains all the required information, since the tensor $\underset{\sim}{X}$ combines the electric and magnetic susceptibilities.

III. SOLUTION OF THE COVARIANT VLASOV EQUATION

In covariant form, the Vlasov equation is

$$u_i \frac{\partial f}{\partial x_i} - \frac{e}{mc} B_{ij} \frac{\partial f}{\partial u_i} u_j = 0 , \qquad (3.1)$$

where e is the electronic charge, m is the proper mass of an electron and $u_i = dx_i/ds$ is the four-velocity of an electron, s being defined by $ds^2 = -dx_i dx_i$. The analysis can be generalized to a multi-species plasma, but, to avoid extra subscripts, we shall confine our attention to electrons within a background of very heavy ions. For comparison with the three-dimensional formulation, we note that $u_\alpha = v_\alpha \gamma / c$, $u_4 = i\gamma$, where v_α is the classical three-velocity and $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$.

The distribution function is restricted by the relativistic energy-momentum relationship, $u_i u_i = -1$, to a hypersphere in the eight dimensional phase space whose volume element is
$$dx\,du \equiv dx_0\, dx_1\, dx_2\, dx_3\, du_0\, du_1\, du_2\, du_3 \;.$$
Hence we can use the Dirac delta function to write

$$\begin{aligned} f(x,u) &= F(x,u)\, \delta(u_i u_i + 1) \\ &= F(x\;,u\quad)\left[\delta\left(u_0 - \sqrt{u_\alpha u_\alpha + 1}\right) + \delta\left(u_0 + \sqrt{u_\alpha u_\alpha + 1}\right)\right] / 2u_0 \end{aligned} \qquad (3.2)$$

where $\frac{1}{2} F(x_\alpha, u_\alpha, t)$ gives the usual distribution function over six-dimensional phase space. The four-vector current density is given by

$$J_i(x) = -ec \int du\, u_i\, f(x,u) \qquad (3.3)$$

where we define

$$\int du \equiv \iiiint_{u_i=\rho_i}^{\infty} du_0\, du_1\, du_2\, du_3$$

with $\rho_\alpha = -\infty$, $\rho_0 = 1$. This integral is covariant under Lorentz transformation since $u_0 = \gamma \geq 1$ in all inertial frames. The definition of $J_i(x)$ is compatible with the normalization condition obtained by setting the number of particles in the system,

$$\begin{aligned} N &= \int dx_1 dx_2\, dx_3\, n(x) = \int dx_1 dx_2 dx_3\, du\, u_0 f(x,u) \\ &= \int dx_1 dx_2 dx_3\, du_1\, du_2 du_3\, \tfrac{1}{2} F(x_\alpha, u_\alpha, t) \end{aligned} \qquad (3.4)$$

The Lorentz invariance of these expressions is demonstrated by Akama [9]. We note that the first integral in Eq. (3.4) involves the product of relative volume by relative number density in the case of a drifting plasma and proper volume by proper number density in the case of a stationary plasma.

We now linearize Eq. (3.1) with the ansatz $f(x,u) = f^0(u) + f^1(x,u)$ which gives to zeroth order

$$B^0_{ij}\, u_j\, \partial f^0/\partial u_i = 0 \qquad (3.5)$$

and to first order

$$u_i\, \partial f^1/\partial x_i - e(mc)^{-1}(B^0_{ij}\, \partial f^1/\partial u_i + B^1_{ij}\, \partial f^0/\partial u_i) u_j = 0\,. \qquad (3.6)$$

Taking the two-sided Laplace transform, we have

$$-ik_i u_i f^1 - e(mc)^{-1}(B^0_{ij}\, \partial f^1/\partial u_i + B^1_{ij}\, \partial f^0/\partial u_i) u_j = 0 \qquad (3.7)$$

where Eq. (3.7) involves functions of k and u.

We solve Eqs. (3.5) and (3.7) by the method of characteristics. The zeroth order equation of motion

$$du_i(s)/ds = -e(mc)^{-1} B^{\circ}_{ij}\, u_j(s) \tag{3.8}$$

is formally equivalent to a characteristic equation of both (3.5) and (3.7). Its solution, in matrix notation, is $u(s) = u(s')\exp \underset{\sim}{\Phi}(s)$ where

$$\exp \underset{\sim}{\Phi} \equiv \sum_{n=0}^{\infty} \frac{\underset{\sim}{\Phi}^n}{n!} \quad , \quad \Phi_{ij}(s) = -\frac{e}{mc}\int_{s'}^{s} B^{\circ}_{ij}(s'')\,ds'' .$$

A necessary and sufficient condition for the existence of this solution is that $\underset{\sim}{B}^{\mathrm{o}}(s)$ has a set of eigenvectors independent of s. In particular, the solution holds if $\underset{\sim}{B}^{\mathrm{o}}$ is independent of s. By the Cayley-Hamilton theorem, $\underset{\sim}{\Phi}$ satisfies its own characteristic equation, viz.

$$0 = \| \underset{\sim}{\Phi} - \lambda \underset{\sim}{I} \| = \| \underset{\sim}{\Phi} \| - \alpha\lambda + \beta\lambda^2 - \gamma\lambda^3 + \lambda^4 , \tag{3.9}$$

where α, β, γ are independent of λ. Since $\underset{\sim}{\Phi}$ is skew-symmetric, it follows that $\| \underset{\sim}{\Phi} \| = \alpha = \gamma = 0$ and $\beta = \frac{1}{2}\Phi_{ij}\Phi_{ij}$. If $\underset{\sim}{B}^{\mathrm{o}}$ is independent of s (i.e. for constant external fields), then

$$\beta = (s-s')^2 \left[(\underset{\sim}{B}^{\circ})^2 - (E^{\circ})^2/c^2 \right] e^2/m^2c^2$$

hence we write $\beta = (s-s')^2\,\Omega^2/c^2$ covariantly, where Ω may be regarded as the relativistically covariant cyclotron frequency. Introducing the variable $\phi = \sqrt{\beta}$, for $s > s'$, and using Eq. (3.9) to generate a recurrence relation for $\underset{\sim}{\Phi}^n$ we obtain the covariant gyrotensor $\underset{\sim}{D}$ in the form

$$D_{ij}(-\phi) \equiv \left(e^{\Phi}\right)_{ij} = \delta_{ij} - \frac{e}{m\Omega} B^{\circ}_{ij}\,\phi - \left(\frac{e}{m\Omega}\right)^2 B^{\circ}_{ik} B^{\circ}_{kj} (\cos\phi - 1) + \left(\frac{e}{m\Omega}\right)^3 B^{\circ}_{ik} B^{\circ}_{k\ell} B^{\circ}_{\ell j} (\sin\phi - \phi) .$$

Under the usual physical condition that $\underset{\sim}{E}^{\circ} \perp \underset{\sim}{B}^{\circ}$ in the laboratory frame, it is easily verified that $\underset{\sim}{\Phi}^3 = -\beta \underset{\sim}{\Phi}$. Since this equation is covariant, we have in general

$$D_{ij}(-\phi) = \delta_{ij} - \left(\frac{e}{m\Omega}\right)^2 B^{\circ}_{ik} B^{\circ}_{kj} (\cos\phi - 1) - \frac{e}{m\Omega} B^{\circ}_{ij} \sin\phi . \tag{3.10}$$

Further properties of $\underset{\sim}{D}(-\phi)$ are established in Appendix A. In general, the solution of Eq. (3.8) is

$$u_i(s) = D_{ij}(-\phi)\, u_j(s') . \tag{3.11}$$

The most general solution of Eq. (3.5) is thus a function $f^{\circ}(u)$ such that

$$f^{\circ}[u(s)] = f^{\circ}[u(s')] . \tag{3.12}$$

For example, $f^{\circ}(u) = F(u_i u_i, u_i \lambda_i)$ satisfies Eq. (3.5) provided

$$\lambda_i B^{\circ}_{ij} = 0 . \tag{3.13}$$

In particular, λ_i can represent the four-vector plasma drift velocity. In this case, the drift velocity observed in the laboratory frame is restricted to a component parallel to $\underset{\sim}{B}^{\circ}$ and a component of magnitude E°/B° in the direction $\underset{\sim}{E}^{\circ} \times \underset{\sim}{B}^{\circ}$. Under these conditions, the Synge-Jüttner distribution,[25]

$$f^{\circ}(u) = \alpha e^{\mu\lambda_i u_i}\,\delta(u_i u_i + 1)\,, \qquad (3.14)$$

α and μ constants, is a solution of Eq.(3.5).

We now solve the remaining characteristic equation of Eq.(3.7), viz.

$$df^1(s)/ds = e(mc)^{-1}B^1_{ij}\,u_j(s)\,\partial f^{\circ}(s)/\partial u_i + ik_i u_i(s)\,f^1(s) \qquad (3.15)$$
$$\equiv g(u) + ik_i u_i f^1 .$$

The homogeneous part has the solution

$$f^H(s) = \text{const.}\ \exp\left[i\int^s k_i u_i(s')\,ds'\right]$$

where u(s′) is given by inversion of Eq.(3.11), using Eq.(A.1). The inhomogeneous equation can now be solved by variation of constant. Substituting $f^1(s) = C(s)\,f^H(s)$, we obtain

$$C'(s)f^H(s) = g[u(s)]$$
$$\therefore\ C(s) = \int^s \frac{g[u(s')]}{f^H(s')}\,ds' .$$
$$\therefore f^1[k,u(s)] = f^1[k,u(s_0)]\exp\left[i\int_{s_0}^{s}k_i u_i(s')ds'\right] + \int_{s_0}^{s}\exp\left[i\int_{s'}^{s}k_i u_i(s'')ds''\right]g[u(s')]\,ds' .$$

Since $\mathrm{Im}\,k_0 < 0$, we can let $s_0 \to -\infty$. Then we have

$$f^1[k,u(s)] = \int_{-\infty}^{s}\exp\left[i\int_{s'}^{s}k_i u_i(s'')ds''\right] g[u(s')]\,ds' . \qquad (3.16)$$

Changing the variables of integration to $\phi = (s-s')\Omega/c$ and $\phi' = (s-s'')\Omega/c$, we have

$$u_i(s') = D_{ij}(\phi)\,u_j(s), \quad u_i(s'') = D_{ij}(\phi')\,u_j(s) \quad . \qquad (3.17)$$

The variable s is now eliminated from Eq. (3.16), since it appears only in u(s). Introducing the tensor $\underset{\sim}{L}(\phi)=\int_0^{\phi}\underset{\sim}{D}(\phi')\,d\phi'$ we obtain

$$f^1[k,u] = \frac{c}{\Omega}\int_0^{\infty} \exp\left[ick_i L_{ij}(\phi)u_j/\Omega\right] g\left[D_{k\ell}(\phi)u_\ell\right] d\phi \,.$$

We now evaluate $g\left[D_{k\ell}(\phi)u_\ell\right]$ by substitution into Eq. (3.15). By Eq. (3.12), it follows that

$$\partial f^\circ(u)/\partial u_k \Big|_{u_k \to D_{k\ell}u_\ell} = D_{k\ell}\,\partial f^\circ(u)/\partial u_\ell \,.$$

The Laplace-transformed solution of the first order Vlasov equation is therefore:

$$f^1[k,u] = \frac{e}{m\Omega}\int_0^{\infty} d\phi \exp\left[ick_i L_{ij}(\phi)u_j/\Omega\right] B'_{k\ell}[k] D_{km}(\phi)\frac{\partial f^\circ(u)}{\partial u_m} D_{\ell n}(\phi)u_n \,. \quad (3.18)$$

IV. COVARIANT MAGNETIZATION-POLARIZATION TENSOR

From Eq. (3.3), the Laplace-transformed plasma current density due to the first order perturbation is given by

$$J_i^1[k] = -ec\int du\, u_i\, f^1[k,u] \ . \tag{4.1}$$

It has been shown[9] that Eq. (4.1) implies that J^1 satisfies the continuity equation, which in k-space takes the form

$$k_i J_i^1[k] = 0 \ . \tag{4.2}$$

We now seek to express $J^1[k]$ in terms of the magnetization-polarization tensor $\underset{\sim}{M}^1[k]$ such that

$$J_i^1[k] = -ik_j M_{ij}^1 [k] \ .$$

The continuity Eq. (4.2) confirms that $\underset{\sim}{M}^1[k]$ is skew-symmetric. From Eqs. (3.18) and (4.1) we have, for the first order current density,

$$J_i^1[k] = -\frac{e^2c}{m\Omega}\int_0^\infty d\phi\, B_{kl}^1 D_{km} D_{ln} \int du\, e^{ir_ju_j} u_i u_n \frac{\partial f^0}{\partial u_m} \tag{4.3}$$

where

$$r_j \equiv ck_p L_{pj}/\Omega \ .$$

In Appendix B, we show by partial integration that

$$J^1_i[k] = -k_j \frac{ie^2c^2}{m\Omega^2} B^1_{k\ell} \int du \int_0^\infty d\phi\, f^0 e^{ir_p u_p} L_{km} D_{\ell n} u_n u_q\, \delta_{i[m}\delta_{q]j} \, . \tag{4.4}$$

We can therefore write

$$M^1_{ij}[k] = -ec \int du\, f^0(u)\, \Delta x_{[i} u_{j]} \quad , \tag{4.5}$$

where

$$\Delta x_i \equiv -\frac{ec}{m\Omega^2} B^1_{k\ell}[k] \int_0^\infty d\phi\, e^{ir_p u_p} L_{ki} D_{\ell n} u_n \, .$$

Comparing Eq.(4.5) with Eq.(2.10), one sees that we have now obtained an explicit expression for the covariant susceptibility tensor $X_{ijk\ell}$, in the case of a drifting Vlasov plasma with an arbitrary zero-order velocity distribution. In Appendix C, we show that Δx is the Laplace-transformed perturbation to the zero order orbit of an electron, expressed in terms of the Lagrangian variables u_i. It is thus clear that we can obtain the expression (4.5) for $\underset{\sim}{M}^1$ directly from the equation of motion, without using the Vlasov equation. The same result could be obtained by means of the Hamilton-Jacobi theory given by Pfirsch [23].

If we now recall the positive ions needed to maintain charge neutrality in the plasma, it is clear that the zero-order contribution to the plasma current density is zero and we can write

$$M_{ij} = M^0_{ij} + M^1_{ij} = \langle \tfrac{1}{2}\rho_0 c\, x_{[i} u_{j]} \rangle_{av.} \quad , \tag{4.6}$$

where ρ_0 is the microscopic charge distribution in the plasma and x is the position of a particle under the influence of the perturbing electromagnetic field. Writing

$$M_{ij} = \frac{1}{2}\eta_{ijkl}\left\langle \frac{1}{2}\rho_0 c\,\eta_{klmn}x_m u_n \right\rangle_{av.}\,,$$

we see that M_{ij} is the dual of the averaged four-dimensional moment of current density. We have now derived explicitly the magnetization-polarization tensor for a drifting Vlasov plasma and expressed it in the form arrived at by Dällenbach on the basis of physical arguments and symmetry requirements. Whereas, however, Dällenbach wrote the Eq.(4.6) in real space, interpreting the average as a volume integral and restricting the application to nondispersive, lossless media, we have extended its application to dispersive, dissipative media, where the equation now holds in Laplace transform space and involves an ensemble average.

V. MAGNETIZATION-POLARIZATION TENSOR FOR AN EQUILIBRIUM PLASMA

In Eqs. (4.5) we have given the magnetization-polarization tensor $\underset{\sim}{M}^1$ for an arbitrary velocity distribution $f^0(u)$. For a given zero-order distribution in the rest frame, the co-variant form of $\underset{\sim}{M}^1$ could be obtained by Lorentz transformation. For an equilibrium plasma, however, we can substitute for $f^0(u)$ the covariant Synge-Jüttner distribution of Eq. (3.14), which, in the rest frame, reduces to the relativistic Maxwell-Boltzmann distribution. The constant α occurring in Eq. (3.14) is the normalizing factor and $\mu = mc^2/kT$, kT being the kinetic temperature in the rest frame of the plasma. Performing the velocity integration for a Synge-Jüttner distribution in Appendix D, we obtain

$$M^1_{ij}[k] = -\frac{e^2c^2n_0\mu}{m\Omega^2K_2(\mu)} B^1_{k\ell}[k]\delta_{i[m}\delta_{n]j}\int_0^\infty d\phi\, D_{\ell p}(\phi)L_{km}(\phi)\left(s_p s_n \mathcal{K}_3 - \delta_{pn}\mathcal{K}_2\right) \tag{5.1}$$

where n_0 is the proper number density of the plasma,

$$\mathcal{K}_n(\phi) = (s_i s_i)^{-\frac{1}{2}n} K_n\left(\sqrt{s_i s_i}\right),$$

K_n being the modified Hankel function of order n and

$$s_i(\phi) = r_i(\phi) - i\mu\lambda_i \ .$$

We recall that the drift velocity λ is restricted by Eq. (3.13) in the laboratory frame to components parallel to $\underset{\sim}{B}^0$ and in the $\underset{\sim}{E}^0 \times \underset{\sim}{B}^0$ direction.

For a stationary plasma with no d.c. electric field, we have $\lambda_\alpha = o = D_{\alpha 4} = L_{\alpha 4}$ and the current density, derived in Appendix D, is

$$J^1_\alpha[k] = -ik_j M^1_{\alpha j} = -\frac{e^2 n_o \mu^2}{m\Omega K_2(\mu)} E^1_\beta \int_0^\infty d\phi \left(\mathcal{K}_2 D_{\beta\alpha} - \mathcal{K}_3 r_\alpha r_\gamma D_{\beta\gamma}\right).$$

In this case, we thus obtain the well-known "dielectric tensor" of Trubnikov,

$$\tilde{\epsilon}_{\alpha\beta} = \delta_{\alpha\beta} + \frac{i\omega_p^2 \mu^2}{\omega\Omega K_2(\mu)} \int_0^\infty d\phi \left(\mathcal{K}_2 D_{\beta\alpha} - \mathcal{K}_3 r_\alpha r_\gamma D_{\beta\gamma}\right)$$

where $\omega_p = (4\pi e^2 n_o/m)^{\frac{1}{2}}$ is the plasma frequency.

VI. DISCUSSION

We have developed a covariant description of an arbitrary, anisotropic, dispersive medium, allowing for relaxation phenomena, such as Landau damping. The covariant formulation is especially useful in cases where each of several species of particle has a different drift velocity, for example in a two-stream plasma. In such cases, it would be difficult to obtain the correct constitutive relations by means of Lorentz transformations from the respective rest frames. In Eq. (2.23) we obtained the complete solution of inhomogeneous Maxwell's equations in covariant form. Using the permittivity tensor of order four, $\underset{\sim}{T}$, we wrote in Eq. (2.15) the general wave dispersion function. The inverse permittivity tensor, $\underset{\sim}{T}^{-1}$ of Eq. (2.25) represents the Green's function which is a prerequisite for the covariant solution of the nonlinear problem of wave-wave interaction. Of particular practical interest here is the phenomenon of stimulated synchrotron emission which may well be responsible for the sharp resonance observed by Fidone et al.[26,27] in experiments intended to determine the self-magnetic field of the toroidal current in Tokamak devices.

In Eq. (4.5) the magnetization-polarization tensor $\underset{\sim}{M}$, and hence the susceptibility tensor $\underset{\sim}{X}$ and the permittivity tensor $\underset{\sim}{T}$, are given explicitly for a drifting Vlasov plasma with arbitrary velocity distribution and non-zero d.c. electric field. We allow for drift velocities parallel to the magnetic field as well as in the $\underset{\sim}{E} \times \underset{\sim}{B}$ direction. The tensor $\underset{\sim}{M}$ is expressed in terms of the averaged moment of the single particle current density in Laplace transform space. This result applies even in the case of a dissipative plasma and allows for the occurrence of resonance phenomena. It is thus of more general validity than the formally similar result obtained by Dällenbach. Finally, we showed that in the special case of a stationary Maxwellian plasma with zero d.c. electric field, our results reduces to the relativistic dielectric tensor of Trubnikov.

The theory developed here now makes it possible to formulate Derfler's stability criteria [19 - 22] in a covariant fashion. Furthermore, the M tensor, derived in sections IV and V, represents the relativistic extension of Derfler's and Omura's separation of the plasma current into magnetization and polarization currents in the non-relativistic case. Using this non-relativistic treatment, Puri and Tutter have shown numerically [28] that the magnetic contribution to classical wave dispersion is negligible away from the immediate neighbourhood of the cyclotron harmonic frequencies. It can now be established whether this classical magnetic contribution is meaningful at the cyclotron harmonics or whether it is dominated by relativistic effects. These, according to Shkarovsky [29] are important at the harmonics for perpendicular wave propagation, even at moderate plasma temperatures.

In sections III and IV, we excluded gradients in plasma density and in the unperturbed electromagnetic field. This enabled the tensor $\underset{\sim}{M}$ to be written as a simple product in transform space. In the future, we will extend the present treatment to include gradients. The expressions obtained will then be convolutions in the transform space, like those obtained by Derfler and Leuterer [3o] in the non-relativistic treatment of Bernstein waves. Also, the possibility of a covariant formulation involving anisotropic temperature must be investigated because of its importance in plasma physics.

Finally, our treatment of the Vlasov plasma provides a concrete example in which the mechanical response of the medium to an electromagnetic perturbation is given explicitly in covariant form. One is therefore in a position to scrutinize the discrepancies arising out of the differing definitions of the energy-momentum tensor given by Minkowski and Abraham [31]. Thus it is hoped that the results obtained in this paper may contribute to a solution of what may be considered the last open question in classical electrodynamics.

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APPENDIX A: PROPERTIES OF THE GYROTENSOR AND ITS INTEGRAL

The solution of the first order equation of motion for electrons in a Vlasov plasma given in Eqs. (3.10) and (3.11) holds for a varying field $\underset{\sim}{B}^{o}$ which commutes with $\underset{\sim}{\Phi}$, provided Ω is some scalar. In section III, we showed that Ω has special significance when $\underset{\sim}{B}^{o}$ is independent of s. Since $\underset{\sim}{\Phi}$ is skew-symmetric, $\underset{\sim}{D}$ is orthogonal and therefore

$$D_{ik}(\phi)\, D_{jk}(\phi) = \delta_{ij} \quad . \tag{A.1}$$

Denoting differentiation with respect to ϕ by a dot, and using the commutativity of $\underset{\sim}{\Phi}$ and $\underset{\sim}{B}^{o}$ we obtain

$$\dot{\underset{\sim}{D}} = \dot{\underset{\sim}{\Phi}} \cdot \underset{\sim}{D} = \underset{\sim}{D} \cdot \dot{\underset{\sim}{\Phi}} \quad , \tag{A.2}$$

which becomes

$$\dot{D}_{ij}(\phi) = -\frac{e}{m\Omega} B^{o}_{ik} D_{kj} = -\frac{e}{m\Omega} D_{ik} B^{o}_{kj} \quad , \tag{A.3}$$

when $\underset{\sim}{B}^{o}$ is independent of s. In this case, we also have

$$D_{ik}(\phi) D_{kj}(\psi) = D_{ij}(\phi+\psi), \tag{A.4}$$

$$D_{ij}(\phi) = D_{ji}(-\phi) \Rightarrow L_{ij}(\phi) = -L_{ji}(-\phi), \tag{A.5}$$

where

$$\underset{\sim}{L}(\phi) = \int_0^{\phi} \underset{\sim}{D}(\phi')\, d\phi' \; .$$

We also obtain two useful relationships involving $\underset{\sim}{D}$ and $\underset{\sim}{L}$.
By Eq.(A.2) $\underset{\sim}{D}-\underset{\sim}{I} = \int_0^{\phi} \dot{\underset{\sim}{D}}\, d\phi' = \int_0^{\phi} \underset{\sim}{D}\cdot\dot{\underset{\sim}{\Phi}}\, d\phi'$.
Integrating by parts then gives, since $\ddot{\underset{\sim}{\Phi}} = 0$,

$$\underset{\sim}{D} - \underset{\sim}{I} = \underset{\sim}{L}\cdot\dot{\underset{\sim}{\Phi}} = \dot{\underset{\sim}{\Phi}}\cdot\underset{\sim}{L} \quad . \tag{A.6}$$

By Eqs.(A.4,5),

$$D_{ik}(\phi)L_{jk}(\phi) = \int_0^{\phi} D_{ij}(\phi-\phi')d\phi' = \int_0^{\phi} D_{ij}(\psi)d\psi = L_{ij}(\phi) . \tag{A.7}$$

APPENDIX B: INTEGRATION BY PARTS OF $J_i(k)$

In order to derive Eq. (4.4) from Eq. (4.3) we first integrate by parts, with respect to u_m, each term in the sum represented by the repeated index m. Then

$$J_i^1[k] = \frac{e^2c}{m\Omega}\int_0^\infty d\phi\, B^1_{k\ell} D_{km} D_{\ell n} \int du\, f^\circ e^{ir_ju_j}(ir_m u_i u_n + \delta_{im} u_n + \delta_{nm} u_i) . \quad \text{(B.1)}$$

The integrated parts depend on $\left[\int du^3 e^{ir_ju_j} u_i u_n f^\circ\right]^\infty_{u_m=p_m}$, where du^3 represents the volume element of the subspace obtained by holding u_m constant. Now we have

$$\left| e^{ir_ju_j} u_i u_n f^\circ \right| = \left| \exp\{-ik_4[x_4^\circ(s') - x_4^\circ(s)]\} u_i u_n f^\circ \right| ,$$

where $x_4^\circ(s') = -c\Omega^{-1} L_{4i}(\phi) u_i(s) + x_4^\circ(s)$ is the timelike component of the zero-order particle displacement. On writing $\nu = Im(ck_o) < 0$, the modulus of the exponential factor becomes $e^{-\nu[t(s') - t(s)]}$ which is bounded by 0 and 1 over the range of ϕ integration, since $-\infty < s' \le s$ and therefore $-\infty < t(s') \le t(s)$. The integrated parts thus vanish under the usual physical requirement that $[u_i u_n f^\circ]^\infty_{u_m=p_m} = 0$.

The third term in Eq. (B.1) vanishes since $B^1_{k\ell} D_{km} D_{\ell n} \delta_{nm} = B^1_{k\ell}\delta_{k\ell} = 0$ by Eq. (A.1) and the skew-symmetry of B^1_{kl}. We integrate the second term by parts with respect to ϕ, obtaining

$$J_i^{(2)}[k] = -\frac{e^2c}{m\Omega} B^1_{k\ell} \int du\, f^\circ u_n \int_0^\infty d\phi\, L_{ki} (\dot{D}_{\ell n} + i\dot{r}_p u_p D_{\ell n}) e^{ir_ju_j} . \quad \text{(B.2)}$$

The integrated part depends on

$$\left[D_{\ell n} L_{ki} e^{ir_ju_j}\right]^\infty_{\phi=0} = \left[D_{\ell n} L_{ki} \exp\left(-\frac{ic}{\Omega} k_\alpha L_{\alpha 4} u_4 - \nu t' + \nu t\right)\right]^{-\infty}_{t'=t} ,$$

where $t' = t(s')$ and $t = t(s)$. The lower limit vanishes since $L_{ki} = o$ for $t' = t$ and the upper limit vanishes due to the presence of $e^{-\nu t'}$. By Eq. (A.3), the first term of $J_i^{(2)}$ becomes

$$J_i^{(2a)}[k] = e^3 c (m\Omega)^{-2} B^1_{k\ell} B^\circ_{mn} \int du\, f^\circ u_n \int_0^\infty d\phi\, L_{ki} D_{\ell p} e^{ir_j u_j} \delta_{mp}$$

$$= e^3 c (m\Omega)^{-2} B^1_{k\ell} B^\circ_{mn} \int_0^\infty d\phi\, L_{\ell i} D_{kp} \int du\, e^{ir_j u_j} \partial (f^\circ u_n u_p)/\partial u_m \,,$$

since $B^\circ_{mn} u_n \partial f^\circ / \partial u_m = 0$ by Eq. (3.5) and $B^\circ_{mn} \delta_{mn} = 0$ by the skew-symmetry of B^o_{mn}. We now integrate by parts with respect to u_m, the integrated parts vanishing by the argument already used, to obtain

$$J_i^{(2a)}[k] = -e^3 c (m\Omega)^{-2} B^1_{k\ell} B^\circ_{mn} \int_0^\infty d\phi\, L_{\ell i} D_{kp} \int du\, e^{ir_j u_j} f^\circ u_n u_p\, ick_q L_{qm}/\Omega$$

Eliminating B^o_{mn} by the use of Eq. (A.6), we find

$$J_i^{(2)}[k] = -\frac{ie^2 c}{m\Omega^2} B^1_{k\ell} \int du\, f^\circ \int_0^\infty d\phi\, L_{ki}\, e^{ir_j u_j} k_n u_n D_{\ell p} u_p \,.$$

Substituting into Eq. (B.1) and simplifying the product $D_{km} r_m$ by use of Eq. (A.7), we obtain Eq. (4.4).

APPENDIX C: THE PERTURBED PARTICLE ORBIT

To zero order, the equation of motion is given by Eq. (3.8). The velocity perturbation $u^1(s)$, due to the perturbing field $\mathbf{B}^1$, must be determined from the first order equation

$$du_k^1(s)/ds = -e(mc)^{-1}\left[B^0_{k\ell}u^1_\ell(s) + (2\pi)^{-4}\int dk\, B^1_{k\ell}[k]e^{-ik_qx_q^0(s)}u^0_\ell(s)\right] \quad \text{(C.1)}$$

where we have used the inverse Fourier-Laplace transform for $\mathbf{B}^1(x)$ and denoted the unperturbed velocity and displacement of the particle by $u^0(s)$ and $x^0(s)$. We seek a solution for $u^1(s)$ by variation of constant, using the ansatz

$$u^1_i(s) = D_{ij}\left[-\Omega c^{-1}(s-s')\right]w_j(s,s') .$$

We then have from Eq. (C.1), using Eq. (A.1):

$$dw_j(s,s')/ds = -e(mc)^{-1}D_{kj}\left[-\Omega c^{-1}(s-s')\right](2\pi)^{-4}\int dk\, B^1_{k\ell}\, e^{-ik_qx_q^0(s)}u^0_\ell(s) .$$

$$\therefore \quad u^1_i(s) = -e(mc)^{-1}(2\pi)^{-4}\int dk\, B^1_{k\ell}[k]\int_{s_0}^{s} ds'\, D_{ki}\left[-\Omega c^{-1}(s'-s)\right]e^{-ik_qx_q^0(s')}u^0_\ell(s'),$$

where s_0 is the initial proper time and we have used Eqs. (A.4) and (A.5). Integrating and applying Eq. (A.4) again, we obtain

$$\Delta x_i(s) \equiv \int_{s_0}^{s} ds'\, u^1_i(s') = -e(mc)^{-1}(2\pi)^{-4}\int dk\, B^1_{k\ell}\int_{s_0}^{s} ds'\, D_{im}\left[-\Omega c^{-1}(s'-s)\right]\int_{s_0}^{s'} ds''\, D_{km}\left[-\Omega c^{-1}(s''-s)\right]e^{-ik_qx_q^0(s'')}u^0_\ell(s'')$$

Using Eq. (3.16) and its integral to represent $u^0(s'')$ and $x^0(s'')$ in terms of the Lagrangian variables $u^0(s)$ and $x^0(s)$ and changing the variables of integration to $\phi = \Omega c^{-1}(s-s')$ and $\phi' = \Omega c^{-1}(s-s'')$, we can eliminate s, since it occurs only in the arguments of the displacements and velocity. Then we have

$$\Delta x_i = -\frac{ec}{(2\pi)^4 m\Omega^2}\int dk\, B^1_{k\ell}[k]e^{-ik_q x_q^0}\int_0^\infty d\phi\, D_{im}(\phi)\int_\phi^\infty d\phi'\, D_{km}(\phi')D_{\ell n}(\phi')u_n^0 \exp\left[ick_q L_{qp}(\phi')u_p^0/\Omega\right].$$

Integrating by parts with respect to ϕ, and using Eq. (A.7), we obtain

$$\Delta x_i = -\frac{ec}{(2\pi)^4 m\Omega^2}\int dk\, B^1_{k\ell}[k]e^{-ik_q x_q}\int_0^\infty d\phi\; e^{ir_p u_p} L_{ki} D_{\ell n} u_n \quad . \qquad \text{(C.2)}$$

We have deleted the superscript o since all velocities and displacements occurring on the right hand side are those of the unperturbed motion. The integrated part vanishes by the argument used already in Appendix B. The Laplace transform of Δx is obtained immediately by comparing Eq. (C.2) with Eq. (2.9).

APPENDIX D: VELOCITY INTEGRATION OF M_{ij} FOR A SYNGE-JÜTTNER DISTRIBUTION

Substituting the Synge-Jüttner distribution of Eq.(3.14) for $f^o(u)$ in Eq.(4.5) we obtain

$$M^1_{ij}[k] = \frac{e^2c^2\alpha}{m\Omega^2} B^1_{k\ell} \int_0^\infty d\phi\, D_{\ell n}(\phi) L_{km}(\phi)\, \delta_{i[m}\, \delta_{q]j} \int du\, e^{is_p u_p} \delta(u_r u_r + 1) u_n u_q \qquad \text{(D.1)}$$

where $s_p = r_p(\phi) - i\mu\lambda_p$. We will now evaluate

$$I_{nq} \equiv \int du\, e^{is_p u_p} \delta(u_r u_r + 1) u_n u_q \ .$$

Integrating first over the variable u_o, and introducing cylindrical polar coordinates u_z, $u_\perp$, θ for u_α, we obtain

$$I_{nq} = \frac{1}{2}\int_{-\pi}^{\pi} d\theta \int_0^\infty du_\perp \int_{-\infty}^\infty du_z\, u_\perp\, F_{nq}(u) \exp\left[is_\perp u_\perp \cos(\theta - \zeta) + is_z u_z - s_4 \left(u_\perp^2 + u_z^2 + 1\right)^{\frac{1}{2}}\right],$$

where

$$F_{\alpha\alpha} = u_z^2 / w \qquad \text{(no summation implied)}$$

$$F_{\alpha\beta} = u_z u_\perp \cos\theta / w\ , \qquad \alpha \neq \beta$$

$$F_{\alpha 4} = i u_z\ , \qquad F_{44} = -w\ ,$$

$$w = \left(u_\perp^2 + u_z^2 + 1\right)^{\frac{1}{2}}$$

and we have expressed s in terms of its cylindrical polar components $s_z, s_\perp, \zeta$. After changing the variable to ψ, where $\sinh\psi = u_z(1 + u_\perp^2)^{-\frac{1}{2}}$, we obtain the integral over u_z in terms of modified Hankel functions, K_n, by Eq.(9.6.24) of Abramowitz and Stegun [32], such that

$$I_{nq} = \int_{-\pi}^{\pi} d\theta \int_0^\infty du_\perp u_\perp q\, G_{nq}(u_\perp, \theta) \exp\left[is_\perp u_\perp \cos(\theta - \zeta)\right],$$

where

$$G_{\alpha\alpha} = \tfrac{1}{2} q \left[\cos 2\xi\, K_2(y) - K_0(y)\right],$$
$$G_{\alpha\beta} = -iu_\perp \cos\theta \sin\xi\, K_1(y),$$
$$G_{\alpha 4} = \tfrac{1}{2} q \sin 2\xi\, K_2(y),$$
$$G_{44} = -\tfrac{1}{2} q \left[\cos 2\xi\, K_2(y) + K_0(y)\right], \quad q = (1+u_\perp^2)^{\frac{1}{2}}, \quad y = q(s_4^2 + s_z^2)^{\frac{1}{2}},$$
$$\sin\xi = -s_z(s_4^2 + s_z^2)^{-\frac{1}{2}}, \qquad \cos\xi = s_4(s_4^2 + s_z^2)^{-\frac{1}{2}}.$$

To perform the θ integration, we use the fact that, by the periodicity of the cosine function

$$\int_{-\pi}^{\pi} d\theta \exp\left[is_\perp u_\perp \cos(\theta - \zeta)\right] = 2\int_0^\pi d\phi \exp(is_\perp u_\perp \cos\phi).$$

Then, by Eq. (9.1.21) of Abramowitz and Stegun, we obtain the θ integral in terms of Bessel functions J_n, such that

$$I_{nq} = 2\pi \int_0^\infty du_\perp u_\perp q\, H_{nq}(u_\perp), \tag{D.2}$$

where

$$H_{\alpha\alpha} = \tfrac{1}{2} q\, J_0(s_\perp u_\perp)\left[\cos 2\xi\, K_2(y) - K_0(y)\right],$$
$$H_{\alpha\beta} = u_\perp \cos\zeta\, J_1(s_\perp u_\perp) \sin\xi\, K_1(y),$$
$$H_{\alpha 4} = \tfrac{1}{2} q\, J_0(s_\perp u_\perp) \sin 2\xi\, K_2(y),$$
$$H_{44} = -\tfrac{1}{2} q\, J_0(s_\perp u_\perp)\left[\cos 2\xi\, K_2(y) + K_0(y)\right].$$

The integrals I_{nq} of Eq. (D.2) are now in the form given by Watson[33] in Eq. (13.47(2)) and, combining the results, we obtain

$$I_{nq} = -2\pi\left(s_n s_q \mathcal{K}_3 - \delta_{nq} \mathcal{K}_2\right), \tag{D.3}$$

where $\mathcal{K}_n$ is defined in section V.

The normalizing factor α which occurs in Eq.(D.1) is determined by the condition, $n = \int du\, u_0\, f^0(u)$ where n is the relative number density of the plasma. Integration along similar lines to that performed above gives $\alpha = n\mu/2\pi\lambda_0 K_2(\mu) = n_0\mu/2\pi K_2(\mu)$ where n_0 is the proper number density. Substitution of this result together with Eq.(D.3) into Eq.(D.1) gives Eq.(5.1).

The expression given for $M_{ij}[k]$ in Eq.(5.1) is arbitrary in terms which make zero contribution to $J_i = -ik_j M_{ij}$. We proceed to eliminate such terms to obtain a slightly simpler expression for $M_{ij}[k]$. From Eq.(5.1), using Eq.(A.7) we have

$$M^1_{ij}[k] = A\, B^1_{k\ell} \int_0^\infty d\phi \left[(-L_{ki} s_j + L_{kj} s_i)(-ck_p L_{\ell p}/\Omega + i D_{\ell p} \mu\lambda_p) \mathcal{K}_3 - (D_{\ell j} L_{ki} - D_{\ell i} L_{kj}) \mathcal{K}_2 \right],$$

where

$$A = -e^2 c^2 n_0 \mu / m\Omega^2 K_2(\mu).$$

The product of the terms $L_{kj} s_i$ and $k_p L_{\ell p}$ contributes zero to J_i and so is dropped. By partial integration of the term $D_{\ell j} L_{ki} \mathcal{K}_2$, we obtain

$$M^1_{ij}[k] = A\, B^1_{k\ell} \int_0^\infty d\phi \Big\{ \left[(-L_{ki} s_j + L_{kj} s_i) i D_{\ell n} \mu\lambda_n + L_{ki} s_j c k_p L_{\ell p}/\Omega \right] \mathcal{K}_3 + L_{\ell j} L_{ki}\, d\mathcal{K}_2/d\phi$$
$$+ (L_{\ell j} \dot{D}_{ki} + D_{\ell i} L_{kj}) \mathcal{K}_2 \Big\},$$

since $\mathcal{K}_2 \to 0$ as $\phi \to \infty$. The last term of the integrand is symmetric in (k, ℓ) and so can be dropped. Since $d\mathcal{K}_2/d\phi = -\dot{s}_r s_r \mathcal{K}_3$, we have

$$M^1_{ij}[k] = A\, B^1_{k\ell} \int_0^\infty d\phi \left[(-L_{ki} s_j + L_{kj} s_i) i D_{\ell n} \mu\lambda_n + c k_p L_{ki} (L_{\ell p} s_j - D_{pr} s_r L_{\ell j})/\Omega \right] \mathcal{K}_3 .$$

The second term can be dropped since its contribution to J_i is $\Lambda = AB^1_{k\ell}\int_0^\infty d\phi c\Omega^{-1}k_j k_p L_{ki} L_{\ell p} s_r(\delta_{jr} - D_{jr})\mathcal{K}_3$ where we have interchanged the indices j and p in the second term. By Eq.(A.6)

$$\Lambda = AB^1_{k\ell}\int_0^\infty d\phi\, ec(m\Omega^2)^{-1}k_j k_p L_{ki} L_{\ell p}(i\mu\lambda_r + r_r)L_{jq}B^\circ_{qr}\mathcal{K}_3 = 0,$$

since $B^o_{qr}\lambda_r = o$ by Eq.(3.13) and $k_j L_{jq} B^o_{qr} r_r = o$ by the skew-symmetry of B^o_{qr}. Hence we have

$$M^1_{ij}[k] = \frac{ie^2c^2n_o\mu}{m\Omega^2 K_2(\mu)}B^1_{k\ell}[k]\delta_{i[m}\delta_{q]j}\int_0^\infty d\phi D_{\ell n}(\phi)\mu\lambda_n L_{km}(\phi)s_q(\phi)\mathcal{K}_3 . \quad \text{(D.4)}$$

Trubnikov's dielectric tensor applies to the case of zero drift velocity (λ_α = o) and zero d.c. electric field ($D_{\alpha 4} = L_{\alpha 4}$ = o). Under these conditions, Eq.(D.4) gives

$$M^1_{\alpha j}[k] = -iAB^1_{\beta 4}[k]\int_0^\infty d\phi\mu\lambda_4(L_{\beta j}s_\alpha - L_{\beta\alpha}s_j)\mathcal{K}_3 ,$$

$$\therefore J^1_\alpha[k] = -ik_j M^1_{\alpha j} = -iAB^1_{\beta 4}\mu\int_0^\infty d\phi(k_\gamma L_{\beta\gamma}r_\alpha - \Omega c^{-1}L_{\beta\alpha}s'_j s_j)\mathcal{K}_3 , \quad \text{(D.5)}$$

because

$$k_j s_j = k_j c\Omega^{-1}k_m L_{mj} - ik_m\delta_{mj}\mu\lambda_j = k_m\{c\Omega^{-1}k_j L_{jn}D_{mn} - i[D_{mj} + ec(m\Omega)^{-1}L_{mn}B^\circ_{nj}]\mu\lambda_j\}$$

by Eqs.(A.7) and (A.6), and therefore $k_j s_j = \dot{s}_j s_j \Omega/c$ by Eq.(3.13). Integrating the second term of Eq.(D.5) by parts and using Eq.(A.7) again, we have

$$J^1_\alpha[k] = -iA\mu\Omega c^{-1}B^1_{\beta 4}\int_0^\infty d\phi\left(D_{\beta\gamma}r_\gamma r_\alpha\mathcal{K}_3 - D_{\beta\alpha}\mathcal{K}_2\right) .$$

We thus obtain Trubnikov's dielectric tensor $\tilde{\epsilon}_{\alpha\beta}$, defined by the relationship

$$J_{\alpha}^{1} = -i\omega\left(\tilde{\epsilon}_{\alpha\beta} - \delta_{\alpha\beta}\right)E_{\beta}^{1}/4\pi \ .$$

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