Coupled quintessence and curvature-assisted acceleration

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Abstract

Spatially homogeneous models with a scalar field non-minimally coupled to the space-time curvature or to the ordinary matter content are analysed with respect to late-time asymptotic behaviour, in particular to accelerated expansion and isotropization. It is found that a direct coupling to the curvature leads to asymptotic de Sitter expansion in arbitrary exponential potentials, thus yielding a positive cosmological constant although none is apparent in the potential. This holds true regardless of the steepness of the potential or the smallness of the coupling constant. For matter-coupled scalar fields, the asymptotics are obtained for a large class of positive potentials, generalizing the well-known cosmic no-hair theorems for minimal coupling. In this case it is observed that the direct coupling to matter does not impact the late-time dynamics essentially.

1   Introduction

The term *coupled quintessence* was introduced by Amendola [3] and refers to a non-linear real scalar field within a cosmological model that is in general non-minimally coupled to the other forms of matter. Such a field is commonly used as a generalization of a cosmological constant or ordinary (minimally coupled) quintessence [10] to account for the observed late-time acceleration of the universe. Phenomenologically, the main motivation for considering an explicit coupling to matter is the existence of scaling attractors that exhibit accelerated expansion [2, 3, 4, 7, 16, 32], a feature which is closely related to the *cosmic coincidence* problem [28]. From the mathematical point of view, such couplings are interesting because they arise from the conformal transformation of scalar-tensor theories from the Jordan frame, where the scalar field is minimally coupled to matter, to the Einstein frame, in which the scalar field is minimally coupled to gravity, though at the expense of an explicit coupling to matter. Thus, knowledge on the dynamics in the Einstein frame can lead to an understanding of the

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behaviour of a large class of conformally related scalar-tensor or higher order gravity theories, prominent examples thereof may include Brans–Dicke theory [8] and $f(R)$-gravity [5].

A specific scalar-tensor theory belonging to the above-mentioned class that gained a lot of attention is obtained by directly coupling the quintessence field to the scalar curvature of the space-time. Such a coupling was originally introduced for the energy-momentum tensor of a self-interacting scalar particle to become renormalizable [9]. Models of this kind were studied with respect to different inflationary scenarios [1, 12, 13, 14, 15, 26] and were engaged to address the *cosmological constant* problem [21, 22] or to violate the null energy condition in order to obtain a barotropic index $w < -1$ for the corresponding perfect fluid [26]. Noteworthy too is the existence of stable scaling solutions in potentials other than exponential as found by Uzan [30] for the inverse power-law case. Recently, Tsujikawa [29] pointed out a particularly interesting consequence of a non-minimal coupling between scalar field and curvature, namely the possibility to obtain inflationary solutions in an exponential potential that is too steep to sustain inflation in the minimally coupled case. Following terminology for a similar effect in multi-field inflation models [20], this behaviour was called *curvature-assisted inflation*.

Quintessence models, minimally coupled or not, require the specification of a potential, in which the scalar field evolves. In work pioneered by Starobinsky [27] it was shown that, in principle, it is possible to reconstruct the quintessence potential from a given expansion history of the universe. While, phenomenologically, this certainly is a comfortable situation, it also means that basically any compatible dynamics can be obtained by just choosing an appropriate potential. As a consequence of this "scalar Synge trick", instead of regarding the potential as a free function, it seems physically reasonable to require some motivation for it. Exponential potentials can provide this easily, as they naturally arise from Kaluza-Klein type reductions of higher dimensional theories or from conformal rescalings, *e.g.* of a string frame cosmological constant. The existence of scaling attractors [6, 11] and power-law inflationary solutions [17] also rendered them attractive for cosmology.

The aim of the present paper is twofold. First, the *cosmic no-hair* theorems of Wald [31] for a positive cosmological constant, of Kitada and Maeda [18, 19] for quintessence in an exponential potential and of Rendall [23, 24, 25] for non-linear scalar fields in more generic potentials shall be generalized to the case of coupled quintessence. Those theorems concern initially expanding global solutions of Bianchi type I to VIII in the presence of ordinary matter and establish late-time acceleration and isotropization. Second, these results shall be applied to a scalar field in an exponential potential which is non-minimally coupled to the space-time curvature to demonstrate that an arbitrarily small positive coupling constant allows for an asymptotic de Sitter expansion, no matter how steep the potential actually is.

In section 2 the notation is set up and the equations of motion for spatially homogeneous models of coupled quintessence are obtained. Section 3 states the main assumptions, gives some basic estimates and derives the asymptotics in the

case where the potential energy density of the field is bounded below by a positive constant, leading to exponential acceleration and isotropization. Section 4 in contrast deals with evolutions where the potential energy density is allowed to approach zero which leads to a more subtle dynamics and, in general, intermediate acceleration only. Curvature-assisted acceleration is presented in section 5 for scalar fields non-minimally coupled to the scalar curvature of space-time by conformally relating them to the coupled quintessence models considered beforehand. Finally, section 6 summarizes the results and concludes with some comments.

2   Kinetics of coupled quintessence

The purpose of this section is to fix notations, to state the coupled quintessence model explicitly for the spatially homogeneous case and to obtain a $n + 1$-decomposition of the equations of motion with respect to the hypersurfaces of homogeneity. For this, let G be a Lie group of dimension $n \geq 3$ with Lie algebra $\mathfrak{g}$, $I \subset \mathbb{R}$ an interval in the reals $\mathbb{R}$ with non-empty interior and $\gamma \in C^2(I, T_2^0 \mathfrak{g})$ a family of Riemannian metrics on $\mathfrak{g}$. Then, if $M := G \times I$ with the canonical projections $\pi : M \to G$ and $t : M \to I$,

$$g := \pi^* \gamma - dt \otimes dt$$

is a Lorentzian metric on M, where the pullback of an arbitrary family $F \in C(I, T_s^0 \mathfrak{g})$ is defined by $(\pi^* F)(p, t) := (\pi^* F(t))(p, t) \in T_s^0 M$ for all $p \in G$, $t \in I$. Here, $F(t) \in T_s^0 \mathfrak{g}$ is identified with its $C^\infty(G)$-multilinear left-invariant extension to $T_s^0 G$ and $\mathcal{T}_s^r M$ stands for the set of continuous sections of the (r, s)-tensor bundle $T_s^r M$ over M. Having

$$k := -\frac{1}{2}\dot{g} \in C^1(I, T_2^0 \mathfrak{g})$$

the second fundamental form $I\!I$ of a t-hypersurface in M is given by

$$I\!I(X, Y) = -k(t)(T\pi X, T\pi Y)\partial_t , \qquad X, Y \in \mathcal{T}(G \times \{t\}).$$

A common convention in cosmology is to denote the *negative* of the mean curvature by H, so $H := -\mathrm{tr}_\gamma k/n$.

Similar to the metric on M, the energy-momentum tensor T of the ordinary matter content is constructed by choosing an energy density $\rho \in C^1(I)$, a flow covector $j \in C^1(I, T_1^0 \mathfrak{g})$ and a symmetric pressure tensor $S \in C^1(I, T_2^0 \mathfrak{g})$ and setting

$$T := \pi^* S + \pi^* j \otimes dt + dt \otimes \pi^* j + \rho\, dt \otimes dt.$$

For this indeed to describe ordinary matter, energy conditions will be imposed later on, but for the moment it is enough to know the decomposition given above.

Finally, for an interval $J \subset \mathbb{R}$ with non-empty interior fix coupling functions $C \in C^1(J)$ and $c \in C(J)$ as well as a potential $V \in C^1(J)$ for the quintessence

field $\phi \in C^2(I, J)$. Then, the coupled Einstein-scalar field-matter (ESM) equations on M read

$$Rc_g - \frac{1}{2}R_g g = \nabla_g\phi \otimes \nabla_g\phi - \frac{1}{2}|\nabla_g\phi|^2_g g - V(\phi)g + C(\phi)T \qquad (1)$$

$$\Box_g\phi - V'(\phi) = -c(\phi)\mathrm{tr}_g T \qquad (2)$$

$$\mathrm{div}_g C(\phi)T = c(\phi)(\mathrm{tr}_g T)\nabla_g\phi \qquad (3)$$

with Rc and R are Ricci tensor and scalar, ∇ is the Levi-Civita connection and $\Box$ the covariant wave operator on (M, g). Compositions with ϕ and t are understood implicitly where appropriate. Note the particular kind of direct coupling assumed between the scalar field ϕ and the matter energy-momentum tensor T in the right hand sides of these equations. Minimal coupling would correspond to $C = 1$ and $c = 0$ identically. While this is, of course, not the most general form of coupling conceivable, it encompasses the cases studied *e.g.* in [3] or [16] and is sufficient for the application to curvature-coupled models in section 5.

The $n + 1$-decomposition of the ESM equations results in the Hamiltonian and momentum constraints

$$n(n - 1)H^2 = \dot{\phi}^2 + 2V(\phi) + |\sigma|^2_\gamma - R_\gamma + 2C(\phi)\rho \qquad (4)$$

$$\mathrm{div}_\gamma \sigma = C(\phi)j, \qquad (5)$$

the evolution equations

$$n(n - 1)\dot{H} = -n\dot{\phi}^2 - n|\sigma|^2_\gamma + R_\gamma - C(\phi)(n\rho + \mathrm{tr}_\gamma S) \qquad (6)$$

$$\dot{\sigma} = -2\sigma \cdot \sigma - (n - 2)H\sigma + \hat{R}c_\gamma - C(\phi)\hat{S}, \qquad (7)$$

the scalar field equation

$$\ddot{\phi} + nH\dot{\phi} + V'(\phi) = c(\phi)\mathrm{tr}_g T \qquad (8)$$

and the equations of motion for the ordinary matter

$$[C(\phi)\rho]^{\cdot} + \mathrm{div}_\gamma C(\phi)j - C(\phi)\langle\sigma, \hat{S}\rangle + HC(\phi)(n\rho + \mathrm{tr}_\gamma S) = -c(\phi)(\mathrm{tr}_g T)\dot{\phi} \quad (9)$$

$$[C(\phi)j]^{\cdot} - \mathrm{div}_\gamma C(\phi)\hat{S} + nHC(\phi)j = 0 \qquad (10)$$

A hat shall denote the trace-free part of a tensor, $\sigma := k + H\gamma$ is the shear tensor and $\langle\cdot, \cdot\rangle_\gamma$ is the family of fibre metrics induced by γ on the tensor bundle $T^0_2\mathfrak{g}$ with the abbreviation $|\sigma|^2_\gamma := \langle\sigma, \sigma\rangle_\gamma$, while $(\sigma\cdot\sigma)(t)(X, Y) := \mathrm{tr}_{\gamma(t)}\left[\sigma(t)(X, \cdot)\otimes \sigma(t)(\cdot, Y)\right]$ for all $t \in I$ and $X, Y \in \mathfrak{g}$.

3   Exponential acceleration

To analyse the asymptotics of global solutions of the coupled quintessence model stated in section 2 some assumptions on the Lie group G, the potential V,

the coupling functions c and C as well as on the energy momentum tensor T describing the ordinary matter content will now be made. Two cases are then considered separately, namely one where the potential includes a positive cosmological constant that leads asymptotically to exponential inflation, and a second in the next section where the potential energy density of the scalar field approaches zero, which allows, in general, for intermediate late-time acceleration only. The argument closely follows that of [23].

The Lie group G is assumed such that every left invariant Riemannian metric has non-positive scalar curvature. In the three-dimensional simply connected case, this corresponds to groups of Bianchi type other than IX [31]. It follows now from the outline of section 2 that $R_\gamma \leq 0$. The energy momentum tensor T is supposed to satisfy the dominant and strong energy condition, which imply

(DEC) $$|j|_\gamma \leq \rho, \ |\mathrm{tr}_\gamma S| \leq n\rho$$
(SEC) $$\rho + \mathrm{tr}_\gamma S \geq 0.$$

Furthermore, it is assumed that the coupling function C is non-negative, so $C(\phi)T$ fulfills every energy condition T does, and that there is a constant C_0 such that

(C) $$|c| \leq C_0 C$$

holds. The potential V shall be positive and satisfy the following, more technical conditions:

(P1) There exists a positive lower semi-continuous minorant $\bar{V}$ on the closure $\bar{J}$ of J in $\mathbb{R}$, *i.e.* $\bar{V} : \bar{J} \to \mathbb{R}$ is lower semi-continuous, $\bar{V} > 0$ and $V(x) \geq \bar{V}(x)$ for all $x \in J$

(P2) V' is bounded on any set on which V is

(P3) V' extends to a continuous function on the closure of J in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ with values in $\bar{\mathbb{R}}$.

(P1) ensures that V is bounded away form zero on bounded subsets of J whereas (P3) yields the existence of limits $\lim_{x \to J_\pm} V'(x) = V'_\pm \in \bar{\mathbb{R}}$ when x approaches the endpoints $J_\pm$ of J in $\bar{\mathbb{R}}$. Finally, suppose that the model is initially expanding, so $t_0 = \min I \in \mathbb{R}$ and $H(t_0) > 0$. Lemma 1 below collects some immediate consequences of these assumptions.

Lemma 1. *The following properties hold:*

(i) *H is positive, bounded and monotonically decreasing*

(ii) *$\dot{\phi}$ is bounded and square-integrable with $\|\dot{\phi}\|_{L^\infty} \leq \sqrt{n(n-1)}\|H\|_{L^\infty}$ and $\|\dot{\phi}\|^2_{L^2} \leq (n-1)\|H\|_{L^\infty}$*

(iii) *$\ddot{\phi}$, R_γ, $|\sigma|^2_\gamma$, $V(\phi)$, $V'(\phi)$, $C(\phi)\rho$ and $c(\phi)\mathrm{tr}_g T$ are bounded with*

$$\|R_\gamma\|_{L^\infty}, \|\,|\sigma|^2_\gamma\|_{L^\infty}, 2\|V(\phi)\|_{L^\infty}, 2\|C(\phi)\rho\|_{L^\infty} \leq n(n-1)\|H\|^2_{L^\infty}$$

$$\|c(\phi)\mathrm{tr}_g T\|_{L^\infty} \leq \frac{1}{2}n(n-1)(n+1)C_0\|H\|_{L^\infty}$$

Proof. Since V is positive so is H^2 due the Hamiltonian constraint (4) and so is H itself because it is positive initially. The evolution equation (6) and (DEC) render $\dot{H}$ non-positive in the same way, so that H is monotonically decreasing and bounded by $0 < H \le H(t_0)$. In fact, the inequality $\dot{\phi}^2 \le -(n-1)\dot{H}$ holds, so integration over I yields $\|\dot{\phi}\|_{L^2}^2 \le (n-1)\|H\|_{L^\infty}$. The L^∞ bounds on $\dot{\phi}$, R_γ, $|\sigma|_\gamma^2$, $V(\phi)$ and $C(\phi)\rho$ follow form (4) directly, $V'(\phi) \in L^\infty(I)$ is then a consequence of (P2) and the estimate for $c(\phi)\mathrm{tr}_g T$ is obtained from the inequality $|c(\phi)\mathrm{tr}_g T| = |c(\phi)||\mathrm{tr}_\gamma S - \rho| \le C_0 C(\phi)(n+1)\rho$ using (C). The scalar field equation (8) finally gives a L^∞ bound for $\ddot{\phi}$. $\qquad\square$

Since V is a positive function, the infimum of the potential energy density of the field $V_0 := \inf (V \circ \phi)(I)$ is either positive or zero. The rest of this section is concerned with the late-time asymptotics in this first case. It is shown that expansion, isotropization and decay of matter terms take place exponentially in time. Note that the solution is assumed to exist globally, so I is not bounded from above.

Proposition 2. *For any $\delta > 0$ the spatial curvature R_γ, the shear $|\sigma|_\gamma^2$, the matter terms $C(\phi)\rho$, $C(\phi)|j|_\gamma$ and $C(\phi)\mathrm{tr}_\gamma S$ as well as the coupling term $c(\phi)\mathrm{tr}_g T$ decay at least as $e^{-(2-\delta)H_0 t}$, i.e.*

$$R_\gamma, \ |\sigma|_\gamma^2, \ C(\phi)\rho, \ C(\phi)|j|_\gamma, \ C(\phi)\mathrm{tr}_\gamma S, \ c(\phi)\mathrm{tr}_g T = \mathcal{O}\left(e^{-(2-\delta)H_0 t}\right) \quad (t \to \infty),$$

where $H_0 := \sqrt{2V_0/n(n-1)}$.

Proof. (i) From the square-integrability of $\dot{\phi}$ and the boundedness of $\ddot{\phi}$ (see Lemma 1) it follows that $\dot{\phi}(t) \to 0$ as $t \to \infty$. (ii) With $E := 1/2\,\dot{\phi}^2 + V(\phi) \in C^1(I)$ the field energy density, define a quantity [23]

$$Z := n(n-1)H^2 - 2E \in C^1(I).$$

Because of (8) $\dot{E} = c(\phi)(\mathrm{tr}_g T)\dot{\phi} - nH\dot{\phi}^2$, which implies with (SEC)

$$\dot{Z} = -2H\left(Z + (n-1)|\sigma|_\gamma^2 + C(\phi)\left[(n-2)\rho + \mathrm{tr}_\gamma S\right]\right) - 2c(\phi)(\mathrm{tr}_g T)\dot{\phi}$$
$$\le -2HZ - 2c(\phi)(\mathrm{tr}_g T)\dot{\phi}.$$

(iii) Fix a $0 < \delta < 1$, then

$$\delta H C(\phi)\rho + 2c(\phi)(\mathrm{tr}_g T)\dot{\phi} \ge \left[\delta H - 2(n+1)C_0|\dot{\phi}|\right]C(\phi)\rho$$

holds and with $-2HZ \le -(2-\delta)HZ - \delta H C(\phi)\rho$ and $H \ge H_0$ by (4) one obtains

$$Z \le -(2-\delta)H_0 Z \quad \text{eventually}$$

from (i). Integrating this yields

$$Z(t) = \mathcal{O}\left(e^{-(2-\delta)H_0 t}\right) \qquad (t \to \infty).$$

Noting again that $Z = |\sigma|_\gamma^2 - R_\gamma + C(\phi)\rho$ and using (DEC) and (C) now establishes the claims. $\qquad\square$

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With this information at hand, it is possible to obtain late-time limits for the mean curvature $-H$, the potential energy density of the field $V(\phi)$, its derivative, and for the derivatives of the field ϕ itself.

Proposition 3. *As t goes to infinity, the following limits are attained:*

   (i) $H(t) \to H_\infty$, *with* $H_\infty \geq H_0 > 0$

   (ii) $(V \circ \phi)(t) \to V_\infty := \frac{1}{2}n(n-1)H_\infty^2 > 0$

   (iii) $(V' \circ \phi)(t) \to 0$

   (iv) $\dot{\phi}(t)$, $\ddot{\phi}(t) \to 0$

Proof. (i) By Lemma 1 H is monotonically decreasing and bounded from below by $H_0 > 0$, so convergence follows immediately. (ii) Using the decay of the matter terms and of $\dot{\phi}$ from Proposition 2 as well as the convergence of H just obtained, the Hamiltonian constraint (4) yields $2(V \circ \phi)(t) \to n(n-1)H_\infty^2$ ($t \to \infty$). (iii) Let $\phi_- := \liminf_{t\to\infty} \phi(t)$ and $\phi_+ := \limsup_{t\to\infty} \phi(t)$ be the limes inferior and limes superior of ϕ in $\bar{\mathbb{R}}$ respectively. If $\phi_- = \phi_+$, then $\phi(t)$ converges for $t \to \infty$ and the limit lies in the closure of J in $\bar{\mathbb{R}}$, so $(V' \circ \phi)(t)$ converges in $\bar{\mathbb{R}}$ by assumption (P3). Thus, suppose $\phi_- < \phi_+$ and choose ϕ_0 in the open interval $]\phi_-, \phi_+[$. Then, there exists a sequence (s_n) in I with $s_n \to \infty$ and $\phi(s_n) \to \phi_0$ for $n \to \infty$ because ϕ is continuous. But then (ii) implies

$$V_\infty = \lim_{t\to\infty}(V \circ \phi)(t) = \lim_{n\to\infty} V(\phi(s_n)) = V(\phi_0),$$

so V is constant on $]\phi_-, \phi_+[$ and thus V' vanishes on $]\phi_-, \phi_+[$. This means that for every neighborhood N of zero, $U := (V')^{-1}(N)$ is a neighborhood of the closure of $]\phi_-, \phi_+[$ in $\mathbb{R}$. By construction, it follows that $\phi(t) \in U$ for t sufficiently large and therefore $(V' \circ \phi)(t) \in N$ eventually. This shows the convergence of $(V' \circ \phi)(t)$ in $\bar{\mathbb{R}}$ for $t \to \infty$ in any case. Employing the scalar field equation (8) and using Proposition 2 it follows that $-\ddot{\phi}$ converges to the same limit, so boundedness of $\dot{\phi}$ requires that limit to vanish. $\square$

The results of this section show in particular that if the potential contains a cosmological constant, *i.e.* is bounded below away from zero, the deceleration parameter $q := -1 - \dot{H}/H^2$ approaches -1 at late times, so the expansion of the universe is accelerated exponentially. Moreover, the density of ordinary matter $C(\phi)\rho$ as well as any anisotropy $|\sigma|_\gamma/H$ vanish exponentially fast.

4   Intermediate acceleration

In the case in which the potential energy density of the scalar field $V(\phi)$ can become arbitrarily small, $\inf(V \circ \phi)(I) = V_0 = 0$, the dynamics is more subtle. Nevertheless, for potentials falling off not too steeply, it is possible to prove late-time acceleration as well as decay estimates for the curvature and matter terms. In contrast to the findings in the case of positive $V_0 > 0$, the acceleration

is in general no longer exponential but may be asymptotically power-law. This will be referred to as *intermediate acceleration*. Of course, the solution again is supposed to exist globally in time.

In general, *i.e.* without having a positive lower bound V_0 on the field's potential energy density, proposition 2 is weakened to the following proposition 4.

Proposition 4. *The spatial curvature R_γ, the shear $|\sigma|_\gamma^2$, the matter terms $C(\phi)\rho$, $C(\phi)|j|_\gamma$ and $C(\phi)\mathrm{tr}_\gamma S$ as well as the coupling term $c(\phi)\mathrm{tr}_g T$ decay at least as t^{-2}*, i.e.

$$R_\gamma, \ |\sigma|_\gamma^2, \ C(\phi)\rho, \ C(\phi)|j|_\gamma, \ C(\phi)\mathrm{tr}_\gamma S, \ c(\phi)\mathrm{tr}_g T = \mathcal{O}\left(t^{-2}\right) \quad (t \to \infty).$$

Proof. Using the same quantity $Z = n(n-1)H^2 - \dot\phi^2 - 2V(\phi)$ as in the proof of proposition 2 it follows by Lemma 1 that $\dot\phi(t) \to 0$ as $t \to \infty$ and so both $Z \geq 0$ and $\dot Z \leq -HZ$ eventually are valid. The Hamiltonian constraint (4) in turn gives $Z \leq n(n-1)H^2$, so

$$\dot Z \leq -\frac{1}{\sqrt{n(n-1)}}Z^{\frac{3}{2}} \quad \text{eventually}$$

holds. Integrating this yields

$$Z(t) = \mathcal{O}\left(t^{-2}\right) \qquad (t \to \infty),$$

which establishes the claimed decay by (DEC) and (C). $\qquad\square$

Since $V(\phi)$ is positive and continuous on I, the condition $V_0 = 0$ is equivalent to $\liminf_{t\to\infty}(V \circ \phi)(t) = 0$. Monotonicity of H and the constraint equation (4) then give $\lim_{t\to\infty} H(t) = 0$ and, together with the proposition 4 just proven, that actually $\lim_{t\to\infty}(V \circ \phi)(t) = 0$. By presupposition (P1) it follows that J is unbounded and either $\lim_{t\to\infty} \phi(t) = -\infty$ or $\lim_{t\to\infty} \phi(t) = \infty$. Without loss of generality the latter will be assumed for the rest of this section.

Restricting the asymptotic steepness of the potential

$$\alpha := \limsup_{x\to\infty} \frac{|V'|}{V}(x)$$

will allow more detailed asymptotics to be obtained and late-time accelerated expansion to be shown. For this, consider the quantity

$$W := \frac{n(n-1)H^2}{2V(\phi)}.$$

According to the equations (4) and (6) it fulfills $W \geq 1$,

$$\frac{\dot\phi^2}{H^2} = n(n-1)\left(1 - \frac{1}{W}\right) - \frac{|\sigma|_\gamma^2}{H^2} + \frac{R_\gamma}{H^2} - \frac{2C(\phi)\rho}{H^2} \tag{11}$$

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and its time derivative is given by

$$\dot{W} = -\frac{2}{n-1}\left[\frac{\dot{\phi}^2}{H} + \frac{1}{2}(n-1)\frac{V'}{V}(\phi)\dot{\phi}\right.$$

$$\left.+ \frac{|\sigma|_\gamma^2}{H} - \frac{1}{n}\frac{R_\gamma}{H} + \frac{1}{n}\frac{C(\phi)}{H}(n\rho + \mathrm{tr}_\gamma S)\right]W \tag{12}$$

respectively. Also note that with the elementary inequality $\sqrt{a-b} \geq \sqrt{a}-b/\sqrt{a}$ for all $a > b \geq 0$ the relation

$$-\frac{\dot{\phi}^2}{H} - \frac{|\sigma|_\gamma^2}{H} + \frac{1}{n}\frac{R_\gamma}{H} - \frac{1}{n}\frac{C(\phi)}{H}(n\rho + \mathrm{tr}_\gamma S) \leq -\sqrt{n(n-1)}\left(1 - \frac{1}{W}\right)|\dot{\phi}|$$

$$-\left(1 - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\right)\frac{|\sigma|_\gamma^2}{H} + \left(\frac{1}{n} - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\right)\frac{R_\gamma}{H}$$

$$-\left(\frac{1}{2}\frac{n-1}{n} - \frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H}\right)\frac{2C(\phi)}{H} \tag{13}$$

follows from (11).

Proposition 5. *If $\alpha < 2/\sqrt{n(n-1)}$ then*

$$\limsup_{t\to\infty} W(t) \leq \left(1 - \frac{1}{2}\sqrt{\frac{n-1}{n}}\alpha\right)^{-1}.$$

Proof. Fix $\epsilon > 0$ with $\alpha + 2\epsilon < 2/\sqrt{n(n-1)}$, then there is a $T \in I$ with $(|V'|/V)(\phi(t)) \leq \alpha + \epsilon$ for all $t \geq T$. Define

$$1 < \delta := \left(1 - \sqrt{\frac{n-1}{n}}\frac{\alpha + 2\epsilon}{2}\right)^{-1} < \frac{n}{n-1}$$

and let $T \leq L \subset I$ be any subinterval of I bounded below by T on which W is not less than δ, $W|L \geq \delta$. For an arbitrary $t \in L$ distinguish two cases. First, assume that $(|\dot{\phi}|/H)(t) \geq \sqrt{(n-1)/n}$, then (12) directly gives

$$\dot{W}(t) \leq \left(-\frac{2}{\sqrt{n(n-1)}} + \alpha + \epsilon\right)|\dot{\phi}(t)|W(t) \leq -\epsilon|\dot{\phi}(t)|W(t).$$

Second, assume $(|\dot{\phi}|/H)(t) \leq \sqrt{(n-1)/n}$ instead, so

$$\frac{1}{\sqrt{n(n-1)}}\frac{|\dot{\phi}|}{H} \leq \frac{1}{n} \leq \frac{1}{2}\frac{n-1}{n} \leq 1,$$

then (13) yields

$$\dot{W}(t) \leq \left[-2\sqrt{\frac{n}{n-1}}\left(1 - \frac{1}{\delta}\right) + \alpha + \epsilon\right]|\dot{\phi}(t)|W(t) \leq -\epsilon|\dot{\phi}(t)|W(t).$$

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This, together, shows that on L the inequality $\dot{W} \leq -\epsilon W|\dot{\phi}|$ holds, so L is bounded because $\dot{\phi}$ is non-integrable. Since W is in particular monotonically decreasing on any such L the set where W is bigger than δ is itself bounded, but this means that $\limsup_{t\to\infty} W(t) \leq \delta$. $\qquad\square$

Combining the evolution and constraint equations (4),(6) with the relation (11) results in an upper bound on the deceleration parameter q, namely

$$q \leq n\left(1 - \frac{1}{W}\right) - 1, \tag{14}$$

leading to the following sufficient condition for accelerated expansion.

Proposition 6. *If $\alpha < 2/\sqrt{n(n-1)}$ then accelerated expansion occurs eventually.*

Proof. From proposition 5 it is known that

$$\limsup_{t\to\infty} W(t) \leq \left(1 - \frac{1}{2}\sqrt{\frac{n-1}{n}}\alpha\right)^{-1} < \frac{n}{n-1},$$

so $W < n/(n-1)$ eventually. By (14) this means that $q < 0$ eventually. $\qquad\square$

Isotropization can be proven by requiring the potential to be "flat" at infinity, which says that $\alpha = 0$.

Proposition 7. *If the potential is flat at infinity, i.e. $\lim_{x\to\infty}(V'/V)(x) = 0$, then the curvature and matter terms vanish faster than H^2, more precisely, as $t \to \infty$,*

$$\frac{\dot{\phi}}{H}(t), \frac{|\sigma|_\gamma^2}{H^2}(t), \frac{R_\gamma}{H^2}(t), \frac{C(\phi)\rho}{H^2}(t), \frac{C(\phi)|j|_\gamma}{H^2}(t), \frac{C(\phi)\mathrm{tr}_\gamma S}{H^2}(t), \frac{c(\phi)\mathrm{tr}_g T}{H^2}(t) \to 0.$$

Proof. When $\alpha = 0$, proposition 5 implies $W(t) \to 1$ as $t \to \infty$ and equation (11) together with (DEC) and (C) makes the claims evident immediately. $\qquad\square$

A question that arises in the treatment outlined so far is whether it is possible to decide *a priori*, *i.e.* without referring to the actual solution, to which of the two cases ($V_0 > 0$ or $V_0 = 0$) the evolution belongs. As an answer to this question in general not only depends on the form of the potential V but also on the initial data, no simple necessary and sufficient criterion can be expected. Instead, two rather rough conditions leading to each of the cases will be presented, that might be useful in some situations though. They can be employed to make contact with the findings of [24].

Take a barrier $B > 0$ and consider a component U of the set $\{x \in J : V(x) \leq B\}$. By monotonicity of H it is clear that if $H^2(t_0) \leq 2B/n(n-1)$ and ϕ belongs to U initially, $\phi(t_0) \in U$, then ϕ stays in U, $\phi(I) \subset U$. Now the conditions can be stated as

Lemma 8. *If U is bounded then $V_0 > 0$.*

Proof. With U its closure $\bar{U}$ in $\mathbb{R}$ is bounded too and thus compact. Because of $V(\phi) \geq \bar{V}(\phi)$ and $(\bar{V} \circ \phi)(I) \subset \bar{V}(\bar{U})$ it follows that $V_0 = \inf(V \circ \phi)(I) \geq \inf(\bar{V} \circ \phi)(I) \geq \inf \bar{V}(\bar{U}) > 0$ from (P1). $\qquad\square$

Lemma 9. *If $V' < 0$ on U and U is bounded below within J, then $\lim_{t \to \infty} \phi(t) = J_+$. In particular, if the potential has negative derivative everywhere and $V(x)$ decays as $x \to \infty$ then $V_0 = 0$.*

Proof. Let $U_\pm$ denote the endpoints of U in $\bar{\bar{\mathbb{R}}}$. By presupposition, U_- is an element of J and thus of U itself. It will now be shown that actually $U_+ = J_+$. Assume that this is not the case, so $U_+ < J_+$, then U_+ lies in the interior of J and $V' < 0$ on U implies that U can be extended in J beyond U_+ while remaining a connected subset of $\{x \in J : V(x) \leq B\}$. But this is a contradiction to U being a component of that set. So indeed $U_+ = J_+$. If $V_0 > 0$ proposition 3 gives $(V' \circ \phi)(t) \to 0$ for $t \to \infty$ and therefore $\phi(t) \to J_+$ $(t \to \infty)$. If $V_0 = 0$ nothing is left to be shown. The additional statement is immediate. $\qquad\square$

5 Curvature-assisted acceleration

The results of the previous sections 3 and 4 shall now be applied exemplarily to the case of a scalar field with an explicit coupling to the scalar curvature of space-time to demonstrate the mechanism of curvature-assisted acceleration. For simplicity an exponential potential is assumed although the method is not restricted to that case by any means. By a conformal transformation the direct coupling of the field to the Ricci curvature is first moved to the energy-momentum tensor where results are obtained by virtue of the propositions proved so far. Transforming back to the Jordan frame will then yield the desired statements.

On an interval $\tilde{I} := [\tilde{t}_0, \infty[$ suppose to have functions $\tilde{\phi} \in C^2(\tilde{I})$, $\tilde{\rho} \in C^1(\tilde{I})$, $\tilde{j} \in C^1(I, T_1^0\mathfrak{g})$ and families $\tilde{\gamma} \in C^2(\tilde{I}, T_2^0\mathfrak{g})$ and $\tilde{S} \in C^1(\tilde{I}, T_2^0\mathfrak{g})$ of Riemannian metrics and symmetric tensors on the Lie algebra $\mathfrak{g}$ respectively such that $\tilde{T} := \tilde{\pi}^*\tilde{S} + \tilde{\pi}^*\tilde{j} \otimes d\tilde{t} + d\tilde{t} \otimes \tilde{\pi}^*\tilde{j} + \tilde{\rho}\,d\tilde{t} \otimes d\tilde{t}$ satisfies the dominant and strong energy conditions (DEC) and (SEC) while $\tilde{g} := \tilde{\pi}^*\tilde{\gamma} - d\tilde{t} \otimes d\tilde{t}$ is a solution to the curvature-coupled Einstein-scalar field-matter (ESM) equations

$$(1 - \xi\tilde{\phi}^2)\left(Rc_{\tilde{g}} - \frac{1}{2}R_{\tilde{g}}\tilde{g}\right) = (1 - 2\xi)\nabla_{\tilde{g}}\tilde{\phi} \otimes \nabla_{\tilde{g}}\tilde{\phi} + \left(2\xi - \frac{1}{2}\right)|\nabla_{\tilde{g}}\tilde{\phi}|^2_{\tilde{g}}\tilde{g} \tag{15}$$
$$- 2\xi\tilde{\phi}\nabla^2_{\tilde{g}}\tilde{\phi} + 2\xi(\tilde{\phi}\Box_{\tilde{g}}\tilde{\phi})\tilde{g} - \tilde{V}(\tilde{\phi})\tilde{g} + \tilde{T}$$

$$\Box_{\tilde{g}}\tilde{\phi} - \xi R_{\tilde{g}}\tilde{\phi} - \tilde{V}'(\tilde{\phi}) = 0 \tag{16}$$

$$\mathrm{div}_{\tilde{g}}\tilde{T} = 0 \tag{17}$$

on $\tilde{M} := G \times \tilde{I}$ with potential $\tilde{V}$ of class C^1. Like before, $\tilde{\pi} : \tilde{M} \to G$ and $\tilde{t} : \tilde{M} \to \tilde{I}$ are the canonical projections, compositions with which are understood implicitly when appropriate. The coupling constant ξ is taken to be any non-zero real number.

Assume $1 - \xi\tilde{\phi}^2 > 0$ and define a conformal factor

$$\Omega := \sqrt[n-1]{1 - \xi\tilde{\phi}^2} \in C^2(\tilde{I}).$$

The transformation functions

$$p : \tilde{I} \to I, \quad \tilde{t} \mapsto \int_{\tilde{t}_0}^{\tilde{t}} \Omega \tag{18}$$

$$P : \tilde{J} \to \mathbb{R}, \quad \tilde{x} \mapsto \int_0^{\tilde{x}} \frac{\sqrt{1 - \eta\xi x^2}}{1 - \xi x^2} dx \tag{19}$$

are smooth diffeomorphisms from $\tilde{I}$ onto $I := p(\tilde{I})$ and from

$$\tilde{J} := \begin{cases}]-1/\sqrt{\xi}, 1/\sqrt{\xi}[& \text{if } \xi \text{ is positive} \\ \mathbb{R} & \text{if } \xi \text{ is negative} \end{cases}$$

onto $\mathbb{R}$ whose inverses will be denoted by q and Q respectively. For convenience the constant $\eta := 1 - 4\xi n/(n-1)$ is introduced. Note that the special case of conformal coupling is characterized by the vanishing of η. Having

$$\gamma := (\Omega^2\tilde{\gamma}) \circ q, \qquad \phi := P(\tilde{\phi}) \circ q$$
$$\rho := (\Omega^{-2}\tilde{\rho}) \circ q, \qquad j := (\Omega^{-1}\tilde{j}) \circ q, \qquad S := \tilde{S} \circ q$$

the metric g, the field ϕ and the energy-momentum tensor T built up as described in section 2 satisfy the matter-coupled ESM equations (1)–(3) on $M = G \times I$ with the smooth coupling functions

$$C := \frac{1}{1 - \xi Q^2} \qquad\qquad c := \frac{1}{n-1}C' = \frac{1}{n-1}\frac{2\xi Q}{\sqrt{1 - \eta\xi Q^2}}C$$

and the transformed potential $V := \tilde{V}(Q)C^{\frac{n+1}{n-1}}$. It is easily seen that (SEC) and (DEC) hold for T as well and the condition (C) is fulfilled due to the boundedness of $2\xi Q/\sqrt{1 - \eta\xi Q^2}$ on $\tilde{J}$.

Attention shall now be restricted to positive coupling constants $\xi > 0$ and fields evolving in exponential potentials $\tilde{V}(x) = \lambda e^{-\kappa x}$ for positive $\kappa, \lambda > 0$ and all $x \in \mathbb{R}$. Suppose $\tilde{H}(\tilde{t}_0) > -(\ln\Omega)\dot{}(\tilde{t}_0)$ and define

$$\tilde{\phi}_\infty := \sqrt{\frac{1}{\xi} + \frac{1}{\kappa^2}\left(\frac{n+1}{n-1}\right)^2} - \frac{1}{\kappa}\frac{n+1}{n-1}$$

so that $\phi_\infty := P(\tilde{\phi}_\infty)$ is the critical point of the transformed potential $V = \tilde{V}(Q)C^{\frac{n+1}{n-1}}$. Then the following proposition 10 holds true.

Proposition 10. *In the limit $\tilde{t} \to \infty$, the field $\tilde{\phi}$ and its derivatives converge,*

$$\tilde{\phi}(\tilde{t}) \to \tilde{\phi}_\infty, \quad \dot{\tilde{\phi}}(\tilde{t}) \to 0, \quad \ddot{\tilde{\phi}}(\tilde{t}) \to 0,$$

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so does the mean curvature $-\tilde{H}$,

$$\tilde{H}(\tilde{t}) \to \tilde{H}_\infty := \sqrt{\frac{2}{n(n-1)} \frac{\tilde{V}(\tilde{\phi}_\infty)}{1 - \xi \tilde{\phi}_\infty^2}}$$

and the curvature and matter terms vanish exponentially,

$$R_{\tilde{\gamma}}, \; |\tilde{\sigma}|_{\tilde{\gamma}}^2, \; \tilde{\rho}, \; |\tilde{j}|_{\tilde{\gamma}}, \; \mathrm{tr}_{\tilde{\gamma}}\tilde{S} = \mathcal{O}\left(e^{-(2-\delta)\tilde{H}_\infty \tilde{t}}\right)$$

for any $\delta > 0$. Moreover, asymptotically, the expansion is accelerated exponentially

$$\tilde{q} = -1 - \frac{\dot{\tilde{H}}}{\tilde{H}^2} \to -1.$$

Proof. (i) The presupposition $\tilde{H}(\tilde{t}_0) > -(\ln \Omega)\dot{}(\tilde{t}_0)$ is equivalent to $H(0) > 0$ while $0 = \min I$. Furthermore, the transformed potential V clearly fulfills the requirements (P1), (P2) and (P3) and $V(x)$ goes to infinity when $|x|$ does. Hence, it possesses a positive lower bound and Lemma 8 ensures the image $\phi(I)$ being relatively compact which implies that Ω is bounded away form zero. This renders $\Omega \notin L^1(\tilde{I})$ non-integrable and so the transformed evolution exists globally too, $I = [0, \infty[$. The results of section 3 thus apply. (ii) From proposition 3 convergence of $\phi(t)$ to ϕ_∞ and therefore of $\tilde{\phi}(\tilde{t})$ to $\tilde{\phi}_\infty$ follows. The decay of $\dot{\phi}$ and $\ddot{\phi}$ give those for the derivatives of $\tilde{\phi}$ as well as convergence of $\tilde{H}$ and $\tilde{q}$. The exponential decay of the curvature and matter terms $|\tilde{\sigma}|_{\tilde{\gamma}}^2 = \Omega^6(|\sigma|_\gamma^2 \circ p)$, $R_{\tilde{\gamma}} = \Omega^2(R_\gamma \circ p)$, $\tilde{\rho} = \Omega^{n+1}[C(\phi)\rho \circ p]$, $|\tilde{j}|_{\tilde{\gamma}} = \Omega^{n+1}[C(\phi)|j|_\gamma \circ p]$ and $\mathrm{tr}_{\tilde{\gamma}}\tilde{S} = \Omega^{n+1}[C(\phi)(\mathrm{tr}_\gamma S) \circ p]$ is then directly obtained from proposition 2. $\qquad\square$

This result shows that an arbitrarily small positive coupling constant ξ establishes a dynamics similar to the presence of a cosmological constant with value

$$\Lambda_{\mathrm{dyn}} = \frac{\tilde{V}(\tilde{\phi}_\infty)}{1 - \xi \tilde{\phi}_\infty^2}$$

although the potential lacks a positive lower bound. It causes exponential acceleration to occur asymptotically as well as exponentially fast isotropization and decay of matter independent of the steepness λ of the potential.

In the proof given above the particular form of the potential $\tilde{V}$ is used mainly to ensure the existence of exactly one critical point of V and thus to obtain convergence of the field $\tilde{\phi}$ and the conformal factor Ω at late times. For assumptions (P1), (P2) and (P3) to hold it is sufficient for instance to require the potential only to allow for a C^1-extension to the closure of $\tilde{J}$ in $\mathbb{R}$. Invoking Lemma 8 it can still be concluded that the field $\tilde{\phi}$ cannot approach the boundary of $\tilde{J}$ and hence the presumption $1 - \xi\tilde{\phi}^2 > 0$ does not restrict the dynamics even in this more general situation once it is fulfilled initially.

6 Conclusions

In this paper spatially homogeneous solutions to the Einstein equations in the presence of a non-minimally coupled scalar field and ordinary matter were investigated. Two different kinds of direct couplings, both to the matter and to the scalar curvature of space-time, were considered with respect to late-time acceleration and isotropization. The results are summarized in the following theorems that are immediate consequences of the propositions 2, 3 and 6, 7 respectively.

Theorem 11. *Consider a solution of Bianchi type I–VIII of the Einstein equations together with a non-linear scalar field evolving in a positive potential and coupled directly to ordinary matter satisfying the dominant and strong energy condition. Suppose the potential to possess a positive lower bound and the assumptions of section 3 to be fulfilled. If the solution is expanding initially and exists globally in the future then exponential acceleration and isotropization occur asymptotically.*

Theorem 12. *Consider the same situation as in theorem 11 above but suppose instead of a positive lower bound that the the asymptotic steepness α of the potential is less than $\sqrt{2/3}$. If the solution is expanding initially and exists globally in the future then accelerated expansion takes place eventually. Moreover if the potential is actually flat at infinity, $\alpha = 0$, the model isotropizes as well.*

The two statements just given are *cosmic no-hair* theorems for coupled quintessence. Note that the particular value of $\sqrt{2/3}$ in the condition on the asymptotic steepness of the potential appears too in [18] and [25].

In the case in which the scalar field is non-minimally coupled to the space-time curvature a conformal transformation was performed to cast the model into a coupled quintessence scenario where theorem 11 applied. As a consequence a late-time version of Tsujikawa's *curvature-assisted* mechanism [29] was obtained.

Theorem 13. *Consider a solution of Bianchi type I–VIII of the Einstein equations together with a non-linear scalar field coupled directly to the scalar curvature of space-time and evolving in an arbitrary exponential potential in the presence of ordinary matter satisfying the dominant and strong energy condition. Suppose that for a positive coupling constant ξ the conditions $1 - \xi\phi^2 > 0$ and $H > -[\ln(1 - \xi\phi^2)]\dot{}/(n-1)$ on the field ϕ and the expansion factor H are fulfilled initially. If the solution exists globally in the future then $1 - \xi\phi^2 > 0$ holds at any time and exponential acceleration and isotropization take place asymptotically.*

This result might be physically interesting since it provides a positive cosmological constant although the potential does not include one. Even more, this is true independent of the steepness of the potential, *i.e.* the expansion will be accelerated exponentially in potentials that would not even allow for power-law inflation in the minimally coupled case.

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