COALESCING BINARY SYSTEMS: AN EFFECTIVE METRIC APPROACH [a]

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In order to have a more reliable description of the final stage of evolution of a binary system we have "reduced" the two-body problem in general relativity to an auxiliary one-body one. The effective description defines a particular re-summation of the original post-Newtonian expanded dynamics and should be able to specify some non-perturbative characteristics of the binary system as the existence of the innermost stable circular orbit.

1 Introduction

Coalescing compact binaries are among the most promising sources for the detection of gravita tional-waves by laser-interferometric observatories such as GEO600, LIGO and VIRGO. Because the gravitational signal depends sensitively on the transition from the adiabatic inspiral phase to the plunge one, it is very desirable to have a good understanding of the late-time dynamical evolution of the system.

Up till now there are not reliable and convincing results which can either assert the existence of the innermost stable circular orbit (ISCO), for a binary system, or point out its location. Clark and Eardley [1] and Blackburn and Detweiler [2] have found that the ISCO should be significantly more tightly bound than in the Schwarzschild case. On the contrary, Kidder, Will and Wiseman [3] (KWW) have predicted an ISCO that is less tightly bound then the one for a test particle in the Schwarzschild metric. This means that the inspiral evolution might enter a plunge phase before tidal disruption takes place. Using the Damour-Deruelle equations of motion [4], Kidder et al. have applied an hybrid approach in which the terms that are not linked to the symmetric mass ratio $\nu = m_1 m_2/(m_1 + m_2)^2$ (m_1 and m_2 are the masses of the two compact bodies) are treated exactly while the ν-dependent terms are considered as additional corrections. This method have been questioned later on by Schäfer and Wex [5]. They pointed out that the hybrid

[a]This talk is based on A. Buonanno and T. Damour, *Phys. Rev. D* 59 (1999) 084006.

approach gives different predictions if applied to the Hamiltonian rather than to the equations of motion and that it is also not robust under change of coordinates. Furthermore, it has been stressed by Damour, Iyer and Sathyaprakash [6] (DIS) that the hybrid approach suffers for an inconsistency because the ν-dependent contributions cannot be considered as formal corrections to the ν-independent ones, in some cases they are large modifications of them. Differently from Kidder et al., Damour et al. have predicted that the radial position of the ISCO for a binary system is smaller than the corresponding quantity in the Schwarzschild metric. All this controversy is mainly related with the fact that the Post-Newtonian (PN) expansion of the two-body dynamics is a badly convergent one, therefore it is very difficult to extract from it non-perturbative information.

In order to tackle these issues we have recently introduced a novel approach for studying the two-body problem in general relativity [7]. The key idea is to map the dynamics of the relative motion of a two-body system, made of neutron stars and/or black holes of comparable masses, onto the dynamics of one particle that is moving in some external effective metric. Turning off radiation damping, the effective metric will be a static spherically symmetric deformation of the Schwarzschild metric, with deformation parameter ν. In this metric we can solve exactly the dynamical problem of a test particle and we can define uniquely the ISCO. Our construction should be viewed as a non-perturbative way of re-summing the post-Newtonian expansion in the relativistic regime where $GM/c^2r \sim 1$, here M is the total mass of the system and r is the relative distance between the two bodies.

2 Two-body problem versus one-body problem

Let us concentrate for the moment on the time-symmetric dynamics of a binary system made of two compact bodies of mass m_1 and m_2 ($M = m_1 + m_2$ and $\mu = m_1 m_2/M$).

The Hamiltonian in the center of mass frame and in ADM coordinates is known up to 2PN level [8]. The relative-motion being invariant under time-translations and space rotations, it is possible to associate to the binary system two conserved quantities: the center of mass non-relativistic energy $\mathcal{E}^{\rm NR}_{\rm real} \equiv \mathcal{E}^{\rm NR}_{\rm c.m.}$ and the angular momentum $\mathcal{J}_{\rm real} \equiv \mathcal{J}_{\rm c.m.}$. In the Hamilton-Jacobi framework the dynamics of the two-body problem has been summarized in a coordinate-invariant way, up to 2PN level, by considering the relativistic "energy-levels" of the bound states [8]: $\mathcal{E}^{R}_{\rm real}(\mathcal{N}_{\rm real}, \mathcal{J}_{\rm real}) = \mathcal{E}^{\rm NR}_{\rm real} + M\,c^2$. They are function of the adiabatic invariants $\mathcal{N}_{\rm real} = I_{\rm real} + \mathcal{J}_{\rm real}$ (here $I_{\rm real}$ denotes the radial action variable) and $\mathcal{J}_{\rm real}$.

Let us introduce an effective one-body dynamics in a spacetime described by the action $S_{\rm eff}[z_0] = -\int m_0\,c\,ds_{\rm eff}$, where $ds_{\rm eff} = \sqrt{-g^{\rm eff}_{\mu\nu}\,dz_0^\mu\,dz_0^\nu}$ and $g^{\rm eff}_{\mu\nu}$ is some spherically symmetric static metric that, in the Schwarzschild gauge and at 2PN order, can be written as

$$ds^2_{\rm eff} = -A(R)\,c^2\,dt^2 + \frac{D(R)}{A(R)}\,dR^2 + R^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,,$$

$$A(R) = 1 + \frac{a_1}{c^2 R} + \frac{a_2}{c^4 R^2} + \frac{a_3}{c^6 R^3}\,, \qquad D(R) = 1 + \frac{d_1}{c^2 R} + \frac{d_2}{c^4 R^2}\,. \tag{1}$$

The coefficients a_i and d_i are unknowns and will be determined by the matching with the real two-body problem. Note that there are two mass parameters in the effective problem: the mass m_0 of the effective particle and some mass M_0 used to scale the coefficients a_i and d_i. Using the Hamilton-Jacobi equation we have derived up to 2PN level the "energy-levels", $\mathcal{E}^R_0(\mathcal{N}_0, \mathcal{J}_0) = \mathcal{E}^{\rm NR}_0 + m_0\,c^2$, of the bound states of the particle m_0 in the metric $g^{\rm eff}_{\mu\nu}$ [7].

We come now to define the rules to map the real two-body description to the effective one-body one. For the adiabatic invariants we have found quite natural, especially if we think in quantum terms, to use the following identification

$$\mathcal{J}_0 = \mathcal{J}_{\rm real}\,, \qquad \mathcal{N}_0 = \mathcal{N}_{\rm real}\,, \tag{2}$$

Method	$\mathcal{E}^{\rm NR}_{\rm real}/Mc^2$	z	$\widehat{\omega}_{\rm real}$	$f_\odot$ *(kHz)*
"Schwarzschild"	−0.01430	6	0.06804	2.199
Eff. action 1PN	−0.01440	5.942	0.06904	2.231
Eff. action 2PN	−0.01501	5.704	0.07340	2.372
KWW [3]	−0.00943	6.49	0.0605	1.96
DIS [6]	−0.01633	5.036	0.08850	2.860

Table 1: *ISCO values shown in Fig. 1. Here* $\widehat{\omega}_{\rm real} = (\omega_{\rm real}\, GM/c^3)$ *and* $f_{\rm real} = \omega_{\rm real}/(2\pi) \equiv f_\odot\,(M_\odot/M)$

while the matching prescription for the "energy-levels" is less trivial. Indeed, if we impose that they coincide modulo an overall shift then we are obliged to introduce either an energy dependence in the coefficients of the effective metric [10] or an effective mass m_0 which differs from the reduced mass μ even in the non-relativistic limit $c \to \infty$. On the contrary, we have found quite satisfactory to allow a transformation of the energy axis. At the level of 2PN this means:

$$\frac{\mathcal{E}^{\rm NR}_0}{m_0\,c^2} = \frac{\mathcal{E}^{\rm NR}_{\rm real}}{\mu\,c^2}\left[1+\alpha_1\,\frac{\mathcal{E}^{\rm NR}_{\rm real}}{\mu\,c^2}+\alpha_2\left(\frac{\mathcal{E}^{\rm NR}_{\rm real}}{\mu\,c^2}\right)^2\right]\,. \tag{3}$$

The coefficients α_i, as a_i and d_i, are then uniquely selected at 2PN level demanding that: i) the mass of the effective test particle be equal to the reduced mass ($m_0 = \mu$); ii) the linearized effective metric, which describes the one-graviton exchanges between the two bodies, coincide with the linearized Schwarzschild metric with mass equal to the total mass of the binary system ($M_0 = M$). The result of the matching is:

$$\alpha_1 = \frac{\nu}{2}\,,\quad a_1 = -2GM\,,\quad a_3 = 2\nu\,(GM)^3\,,\quad d_2 = -6\nu\,(GM)^2\,,\quad \alpha_2 = 0 = a_2 = 0 = d_1\,, \tag{4}$$

here $\nu = \mu/M$. The simplicity of the final result is quite striking. Moreover, using the values of α_1 and α_2 given by Eq. (4), we get that the rèlation between the real relativistic energy and the effective one, Eq. (3), coincides with the one that Brézin, Itzykson and Zinn-Justin [9] have derived for the analogous problem in electrodynamics. In that context they were interested in evaluating the relativistic Balmer formula for a two-body system including recoil effects. Note also that the same map (3) has already been used by Damour et al. [6].

3 Dynamics in the effective metric

In this section we propose to trust the physical consequences of the ν-deformed Schwarzschild-type metric, given by Eq. (1), even in the region where R is a few times GM/c^2. We think that this is meaningful because even in the extreme case of two bodies of equal mass, that is when $\nu = 1/4$, the ν-dependent terms that appear in the effective metric are relatively small.

Because gravitational radiation damping is known to circularize the orbits of a binary system, we are interested in analysing the stable circular orbits. From the effective Hamiltonian we can easily derive the effective radial potential and the angular frequency along circular orbits. We get [7]

$$W_{\mathcal{J}_0}(R) = A(R)\left[1+\frac{\mathcal{J}_0^2}{(\mu\,c\,R)^2}\right]\,,\qquad \omega_{\rm C} = \frac{\mathcal{J}_0}{\mu\,R^2}\,\frac{\sqrt{A(R)}}{\sqrt{1+\mathcal{J}_0^2/(\mu\,c\,R)^2}}\,. \tag{5}$$

The innermost stable circular orbit corresponds to the critical value of the angular momentum $\mathcal{J}_0^{\rm ISCO}$ where the maximum and the minimum of the effective potential fuse together to form an horizontal inflection point, that is $(\partial W_{\mathcal{J}_0}/\partial R)_{\rm ISCO} = 0 = (\partial^2 W_{\mathcal{J}_0}/\partial R^2)_{\rm ISCO}$. In Tab. 1 and Fig. 1 we summarize our predictions for the ISCO values in the case $\nu = 1/4$ and compare them with the results present in the literature. Note that we show the physical quantities defined in

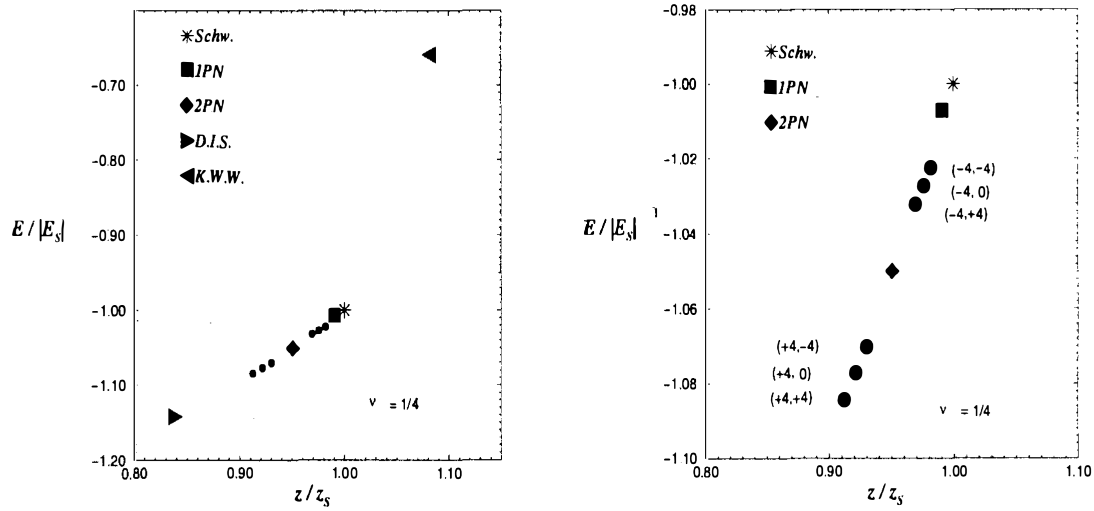

Figure 1: *ISCO values at $\nu = 1/4$ of the real non-relativistic energy $E \equiv \mathcal{E}^{\rm NR}_{\rm real}/\mu$, divided by the corresponding Schwarzschild value $E_S \equiv \mathcal{E}^{\rm NR}_S/\mu$, versus z/z_S ($E_S = -0.05719$, $z_S = 6$). The robustness of our approach is analysed in the right panel. We have introduced possible 3PN and 4PN contributions that are parametrized by the coefficients (a_4, a_5) (see the notations used in the text).*

the real two-body problem. Indeed, using the map given by Eq. (3) it is straightforward to get from the effective energy and frequency, Eq. (5), the corresponding real quantities. In Fig. 1 the non-relativistic real energy is represented as a function of $z = (GM\omega_{\rm real}/c^3)^{-2/3}$, which is an invariant measure of the radial position of the orbit.

We have found an ISCO which is more tightly bound than the one for a usual test-mass in the Schwarzschild metric. This result is consistent with what derived by Damour, Iyer and Sathyaprakash [6], but it goes in the opposite direction with respect to what Kidder, Will and Wiseman [3] obtained. Indeed, they have predicted that when ν increases the ISCO becomes less tightly bound.

In the right panel of Fig. 1 we analyse the robustness of our effective-action approach by exhibiting the points obtained introducing reasonable 3PN and 4PN contributions. More precisely, we include in the metric coefficient $A(R)$ the terms $\nu a_4\,(GM/c^2R)^4$ and $\nu a_5\,(GM/c^2R)^5$ and varied a_4, a_5 in the range -4 and $+4$. We obtain that the location of the ISCO is sensitive to the coefficient a_4 only at the 3% level and to the 4PN-coefficient a_5 at the 0.6% level.

4 Coming back to the real problem and taking into account radiation damping effect

To complete the analysis done in the previous sections we have to construct explicitly the transformation which maps the variables entering the effective problem onto those of the real one. This means that we have to relate the ADM coordinates (q,p) of the relative motion to the coordinates (q',p') of the effective description: $q'^i = \mathcal{Q}^i(q^j,p_j)$ and $p'^i = \mathcal{P}^i(q^j,p_j)$. Indeed, denoting by G the generating function of the canonical transformation we have

$$q'^i = q^i + \frac{\partial}{\partial p'_i}G(q,p')\,, \qquad p'_i = p_i + \frac{\partial}{\partial q'^i}G(q,p')\,, \tag{6}$$

where at our order of approximation $G = G_{\rm 1PN}/c^2 + G_{\rm 2PN}/c^4$. To determine G we impose that under the canonical transformation the effective reduced [b] Hamiltonian $\widehat{H}_{\rm eff}(q',p')$ is mapped

[b] In this section we deal with reduced quantities defined as $\widehat{H}_{\rm eff} \equiv H_{\rm eff}/\mu$ and $\widehat{H}_{\rm real} \equiv H_{\rm real}/M$.

to the real reduced Hamiltonian $\widehat{H}_{\rm real}(q,p)$ by the rule defined by Eq. (3). A quite involved calculation gives[7]

$$\frac{\partial \widehat{H}_{\rm Newt.}}{\partial q^i}\,\frac{\partial G_n}{\partial p_i} - \frac{\partial \widehat{H}_{\rm Newt.}}{\partial p_i}\,\frac{\partial G_n}{\partial q^i} = K_n(q,p) \qquad n = 1{\rm PN}, 2{\rm PN}\,, \tag{7}$$

where $H_{\rm Newt.}$ is the reduced Newtonian Hamiltonian while K_n is a source term which is known at every step of the iteration because it depends on quantities evaluated at the previous order. We have written down explicitly the solution of Eq. (7) and we have found that it is unique modulo the addition of terms generating a constant shift or a spatial rotation[7].

Up till now we have restricted ourselves only to the conservative dynamical evolution. To study the transition between the inspiral and the plunge phases we must take into account the radiation reaction effects. In the ADM canonical formalism they have been studied by Schäfer [11]. The total reduced Hamiltonian can be written as

$$\widehat{H}_{\rm tot}(q,p;t) = \widehat{H}^{\rm impr.}_{\rm real}(q,p) + \widehat{H}^{\rm reac}(q,p;t)\,, \tag{8}$$

where

$$\widehat{H}^{\rm reac}(q,p;t) = -h_{ij}^{\rm TTreac}(t)\left[\frac{1}{2}p^i p^j - \frac{1}{2}\frac{q^i q^j}{q^2}\right]\,, \qquad h_{ij}^{TTreac}(t) = -\frac{4}{5}\,\frac{G}{c^5}\,\frac{d^3\,Q_{ij}(t)}{dt^3}\,, \tag{9}$$

here Q_{ij} is the quadrupole moment of the two-body system and $h_{ij}^{\rm TTreac}$ has to be considered as an external given function of time. As improved-Hamiltonian we propose to use[7]

$$\widehat{H}^{\rm impr.}_{\rm real}(q,p) = \frac{c^2}{\nu}\,\left[\sqrt{1+2\,\nu\,\left[\frac{1}{c^2}\,\widehat{H}_{\rm eff}(q'(q,p),p'(q,p)) - 1\right]} - 1\right]\,. \tag{10}$$

Here $\widehat{H}^{\rm impr.}_{\rm real}$ should be considered as a particular re-summation of the initial 2PN-expanded dynamics. In $\widehat{H}_{\rm eff}$ we must apply the canonical transformation between the real and the effective description and use as metric coefficients our best estimate given by Eq. (4).

The transition from the adiabatic inspiral phase, driven by radiation damping, to the plunge phase, induced by strong curvature effects, can be studied integrating the Hamilton equations: $\dot{q}^i = \partial\widehat{H}_{\rm tot}/\partial p_i$ and $\dot{p}_i = -\partial\widehat{H}_{\rm tot}/\partial q^i$. The results with respect to the coordinates q', p' and for the values $\nu = 1/4$ and 0.1 are illustrated in Fig. 2.

5 Conclusions

We have mapped the complicated and badly convergent PN-expansion of the two-body dynamics onto a simpler auxiliary one-body problem. We have seen that the one-body dynamics defines a re-summation of the original 2PN-expanded one and captures some crucial non-perturbative aspects, such as the existence and the location of the innermost stable circular orbit. We predict an ISCO that is more tightly bound than the one for a test particle in the Schwarzschild metric. For $\nu = 1/4$ we get

$$\mathcal{E}^{\rm ISCO}_{\rm real} \simeq -0.015\,Mc^2\,, \qquad f^{\rm ISCO} \simeq 2372\,{\rm Hz}\left(\frac{M_\odot}{M}\right)\,, \qquad R^{\rm ISCO} \simeq 4.79\,\frac{GM}{c^2}\,. \tag{11}$$

This means that 1.5% of the mass-energy initially available in the binary system should be radiated in gravitational-waves before the plunge phase. This value should be compared to the ones obtained since now in the literature: 2.5% by Clark and Eardley[1], 17.5% by Blackburn and S. Detweiler[2], 1.6% and 0.9% predicted by Damour, Iyer and Sathyaprakash[6] and Kidder, Will and Wiseman[3], respectively.

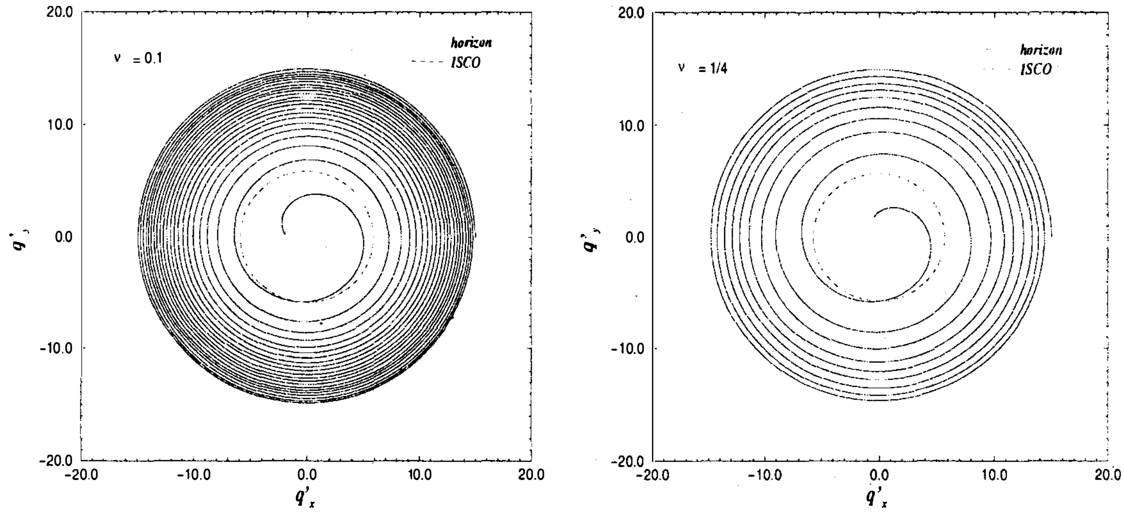

Figure 2: *Transition from the inspiral phase to the plunge one in (q', p') coordinates including radiation damping effects for $\nu = 1/4$ and $\nu = 0.1$. The position of the ISCO and of the horizon are also shown in the figures.*

Even if we believe that our approach points out the ISCO in a reliable way, moreover we think that the knowledge of the 3PN dynamics would be important to reduce the present uncertainty in the coefficients of the effective metric. Our results suggest that the inspiral phase of coalescence of binary neutron stars might terminate into tidal disruption without going through a well-defined plunge phase. If we were true the end of the inspiral phase might be very sensitive to the nuclear equations of state and the interferometric gravitational-waves detectors might provide for us useful and unique information about the dense nuclear matter of neutron stars.

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