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Introduction to Functional Analysis for Engineers

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Abstract

These Lecture Notes Introduction to Functional Analysis for Engineers arose from a (2+1)-hour course *Functional Analysis* at Munich University of Technology for an international group of engineering students. Its focus is on developing the mathematical tools and skills needed for investigating boundary value problems and the finite element method as well as on developing mathematical understanding and insight.
In Part I of these lecture notes the basic concepts of Functional Analysis are developed. In Part II the treatment of boundary value problems is discussed. Appendix A.1 lists mathematical definitions which are assumed to be known in the main text, but were not known to *some* of the students. Appendix A.2 lists fifty homework problems. Appendix A.3 lists sample solutions to selected problems of Appendix A.2 and hints to some of the other ones.

Introduction to Functional Analysis for Engineers

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Preface

Functional Analysis emerged at the beginning of the 20th century. Dieudonné [Dieu81, p.110] wrote that Hilbert's paper of 1906 could be called the first paper in Functional Analysis. According to Lax, the first text book on Functional Analysis was Banach's book [Ba32] in 1932. In the same year, a first text book on Hilbert spaces appeared [Sto32], [Lax02, p.xvii]. For a detailed history of Functional Analysis see [Dieu81, Dev94].
Functional Analysis provides a unified view on historically different fields like linear algebra, ordinary differential equations, partial differential equations and integral equations, to name just a few. Thus its roots are much older, and many people have contributed to its development. Nowadays Functional Analysis is considered basic to mathematics and to the computational sciences because it gives a better understanding of the underlying mathematical structures. Moreover it provides methods and tools of investigation.
Because it provides a unified view on finite dimensional and infinite dimensional spaces, it provides very valuable tools for investigating the quality of numerical approximations to solutions of differential equations [Co68]. When discretising a given boundary value problem by finite differences, we approximate a differential operator by a sequence of matrices. These special matrices have very much in common with differential operators (see [Va62]). This is why these numerical methods work. On the other hand, these matrices operate on finite dimensional spaces, while differential operators operate on infinite dimensional spaces.
Also when we use other numerical methods such as finite element methods or finite volume methods, we approximate elements of infinite-dimensional function spaces by elements of finite-dimensional spaces. Thus there are some intrinsic differences between differential equations and their discretisations. To understand these differences, to analyse convergence of grid refinements and to estimate numerical errors, functional analysis is needed.
In this course we will focus on developing the mathematical tools and skills needed for investigating boundary value problems and the finite element method as well as on developing mathematical understanding and insight.
This text arose from a one semester course (2 hours per week of lecture, 1 hour per week of discussion of corrected homework problems; 4.5 ECTS credits) for **COME** students at Munich University of Technology, **TUM**. Come.tum is a Master of Science program in Computational Mechanics for students who obtained an above-average Bachelor's degree in Civil Engineering, Mechanical Engineering or comparable programs from a German or internationally acknowledged foreign university or a university of applied sciences (Fachhochschule). The foreign students come from such diverse countries as China, Nepal, India, Iran, Turkey, Greece, Kroatia, Spain, Mexico, Bolivia, USA, etc.
European Mathematical Tradition. Several students complained: *'Don't ask us to learn theorems and proofs. It will be much easier for us if you give us nothing but examples, many, many examples. This is how it is done in our home country. By examples we learn much faster.'* If you are *used to* learning by examples, this is indeed the faster method, *as long as you are dealing with simple problems.* As soon as you'll have to deal with *rather complex scientific problems* you'll encounter quite different phenomena in the different examples, and you'll start to wonder *what exactly makes the difference,* and how to know in advance which phenomenon to expect for a given problem. In such

cases it is *much more efficient* to take advantage of the insights, analyses and inventions of intelligent people of earlier centuries, instead of re-inventing the wheel yourself. - In Europe, mathematics is taught with theorems, proofs *and* examples for more than 2500 years, since Thales of Milet and Pythagoras. Nevertheless, scientists doing research in Cognitive Science and Mathematics Education are still discussing today what is the appropriate level of abstractness and the best method of teaching. J. Kaminski et al [KSH08] found in experiments with US students: *'Undergraduate students may benefit more from learning mathematics through a single abstract, symbolic representation than from learning multiple concrete examples'.* Not only undergraduate students! Anyway, you participants of come.tum came to Germany to get a German university degree, so you have to do it the German way.

When US citizens move within their own country, say from New York to Los Angeles (or from the West Coast to the East Coast) they say that they are experiencing a *culture shock*. Some of you are just experiencing much deeper changes than people do who move within the USA. *'Culture shock has many different effects, time spans, and degrees of severity. ... Many people are handicapped by its presence and do not recognize what is bothering them.'* [Wiki, 'Culture Shock', Oct. 2012]. It might be helpful to read about it though you do not think that you are having such problems.

In Part I of these lecture notes the basic concepts of Functional Analysis are developed. In Part II the treatment of boundary value problems is discussed. Appendix A.1 lists mathematical definitions which are assumed to be known in the main text, but were not known to *some* of the students. Appendix A.2 lists fifty homework problems. Appendix A.3 lists sample solutions to selected problems of Appendix A.2 and hints to some of the other ones. These sample solutions seemed to be necessary because some of the students never were given corrected homework problems during their previous studies. Thus they did not know how to provide what we expected from them.

These lecture notes are provided as searchable pdf with hypertext. An index thus seemed to be needless. - Some footnotes explain terms that were not familiar to some of the students. Other footnotes give some biographical information about important scientists. Note that the length of a footnote is *not* related to the importance of that person for mathematics/engineering. Note also how many of the mentioned scientists moved from engineering to mathematics – because they wanted to see theorems and proofs, I guess.

Acknowledgement I am indebted and very grateful to all persons who helped me when preparing these lecture notes: Prof. S. Hayes and Dr. G. Grammel taught this course in previous years. Both gave me a copy of their lecture notes and much additional advice. Dr. C. Bauer and Dr. A. Dall'Acqua designed and corrected the homework problems and discussed them in class. Questions and remarks by students also influenced these notes. And I got very interesting answers to Problem 40, especially by J. Lizarazu. Dr. H.-P. Kruse followed me in teaching this course and is using these lecture notes. His list of errata and his suggestions for improvements are also gratefully acknowledged.

Garching, 30. November 2012 R. Meyer-Spasche

Contents

Chapter 1

Basic Concepts

In this chapter basic concepts from functional analysis are introduced. Examples illustrate the concepts and prepare their application to differential equations.

1.1 Linear Spaces

In this section concepts are introduced which originally stem from linear algebra, but turned out to be very useful in the theoretical and numerical treatment of differential equations.

1.1.1 Spaces and Subspaces

Definition 1.1. *Let $I\!K$ be a field, $I\!K = \mathbb{R}$ or $I\!K = \mathbb{C}$. A* linear space over *$I\!K$ is a set V together with two operations*
addition $+ : V \times V \to V$, $(u, v) \mapsto u + v$, and
scalar multiplication $\cdot : I\!K \times V \to V$, $(\alpha, v) \mapsto \alpha \cdot v$,
with the following properties

$$\begin{aligned} u + v &= v + u, \\ u + (v + w) &= (u + v) + w, \\ (\alpha + \beta) \cdot u &= \alpha \cdot u + \beta \cdot u, \\ \alpha \cdot (u + v) &= \alpha \cdot u + \alpha \cdot v, \\ (\alpha\beta) \cdot u &= \alpha \cdot (\beta \cdot u), \\ 1 \cdot u &= u, \end{aligned}$$

for all $u, v, w \in V$ and all $\alpha, \beta \in I\!K$. And further
• *there is an element $0 \in V$ with $u + 0 = u$ for all $u \in V$, and*
• *for every $u \in V$ there is an element $\tilde{u} \in V$ with $u + \tilde{u} = 0$.*
$\tilde{u} =: -u$ and $u + (-u) =: u - u$.

Remark 1.2. *1. Let $0 \in I\!K$, $u \in V$. Then $0 \cdot u = 0 \in V$.*
This follows from $u = 1 \cdot u = (1 + 0) \cdot u = 1 \cdot u + 0 \cdot u = u + 0 \cdot u$. Now subtract u on both sides to get the result.

2. Let $-1 \in I\!K$, $u \in V$. Then $(-1) \cdot u = -u$.
This follows from $(-1) \cdot u + u = (-1) \cdot u + 1 \cdot u = (-1+1) \cdot u = 0 \cdot u = 0$.

Remark 1.3. *The '$\cdot$' is often omitted, i.e.*[1] $\alpha \cdot u = \alpha u$ *for* $\alpha \in \mathbb{R}$, $u \in V$. *A linear space is also called 'vector space' or 'function space', depending on the elements of the space. A linear space over* $\mathbb{R}$ *is also called a 'real linear space'. In this text we will mostly consider real linear spaces. We thus will often say 'linear space' when we mean 'real linear space'.*

Remark 1.4. *The axioms of addition and scalar multiplication required in Definition 1.1 are not satisfied in all collections of numbers. Let* ε_{mach} *be the* machine epsilon *of a given computer, i.e. the largest number of this computer such that* $1 + \varepsilon_{mach} = 1$. *Then* $(1 + \varepsilon_{mach}) + \varepsilon_{mach} = 1 \neq 1 + (\varepsilon_{mach} + \varepsilon_{mach})$. *The influence of rounding errors is an important topic of Numerical Analysis, see for instance [GvL96] or the book by Wilkinson [Wilk69], which revolutionized their treatment on its appearance.*

Example 1.5 ($\mathbb{R}^n$)**.** *Let* $n > 0$ *be an integer, i.e.* $n \in \mathbb{N}$. *Let* V *be the set of all real* n*-tuples* $(u_1, \ldots, u_n)^t$, $u_i \in \mathbb{R}$, $i = 1, \ldots n$, *with addition*

$$u + v = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}, \qquad u, v \in V,$$

and multiplication

$$\alpha \cdot u = \alpha \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha \cdot u_1 \\ \vdots \\ \alpha \cdot u_n \end{pmatrix} \qquad u \in V, \ \alpha \in \mathbb{R}.$$

V *together with these operations forms a linear space over* $\mathbb{R}$. *This space is called* $\mathbb{R}^n$.

Example 1.6 ($A(\Omega; \mathbb{R})$)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a subset. Let* V *be the set of all real functions* $f : \Omega \to \mathbb{R}, \quad x \mapsto f(x)$, *with addition*

$$(f + g)(x) = f(x) + g(x), \qquad \text{for all } f, g \in V, \ x \in \Omega, \tag{1.1}$$

and multiplication

$$(\alpha \cdot f)(x) = \alpha f(x), \qquad \text{for all } f \in V, \ \alpha \in \mathbb{R}, \ x \in \Omega. \tag{1.2}$$

V *together with these operations forms a linear space over* $\mathbb{R}$. *This space is called* $A(\Omega; \mathbb{R})$.

Lemma 1.7. *Let* V *be a linear space over* $I\!K$ *and* $W \subset V$ *a subset. Then* W *is a linear space w.r.t.*[2] *the addition and scalar multiplication of* V *iff*[3]
(i) $w_1 + w_2 \in W$ *for all* $w_1, w_2 \in W$;
(ii) $\alpha \cdot w \in W$ *for all* $\alpha \in I\!K$, $w \in W$.

[1] i.e. = id est (latin) = that is
[2] w.r.t. = with respect to
[3] iff = if and only if

Proof: left to the reader.

Definition 1.8. *Let V be a linear space and $W \subset V$ a subset. W is called*
* subspace of V *if W is a linear space w.r.t. the addition and scalar multiplication of V.*
* closed w.r.t. the addition and scalar multiplication of V *if it satisfies conditions (i) and (ii) of the Lemma.*

Example 1.9. *The smallest non-empty linear space is $V = \{0\}$. It is a subspace of all other non-empty linear spaces by Definition 1.1.*

Remark 1.10. *Note that Definition 1.1 alone does not guarantee that a linear space is not empty: by definition of the empty set $\emptyset$, every statement about its elements is both true and false. Thus Definition 1.1 also applies to $V = \emptyset$.*
But considering the empty set does not give new mathematical insights. Thus from now on 'linear space' *always means* 'non-empty linear space'.

Examples 1.11 (Subspaces of $A(\Omega;\mathbb{R})$)**.** *The following subsets of $A(\Omega;\mathbb{R})$ are also subspaces:*
$B(\Omega;\mathbb{R})$, the set of all bounded functions $f : \Omega \to \mathbb{R}$, i.e. of all functions $f : \Omega \to \mathbb{R}$ such that for f there is some $b \in \mathbb{R},\ b > 0$ with $|f(x)| < b$ for all $x \in \Omega$;
$C^k(\Omega;\mathbb{R})$, $k \in \mathbb{N}$, the set of all k-times continuously[4] *differentiable functions $f : \Omega \to \mathbb{R}$;*
$C^0(\Omega;\mathbb{R}) := C(\Omega;\mathbb{R})$, the set of all continuous functions $f : \Omega \to \mathbb{R}$;
$C^\infty(\Omega;\mathbb{R})$, the space of all functions $f : \Omega \to \mathbb{R}$ which are infinitely often continuously differentiable.

Example 1.12 ($P(\mathbb{R},\mathbb{R})$)**.** *The set of polynomials $p : \mathbb{R} \to \mathbb{R}$ is a real linear space, $P(\mathbb{R},\mathbb{R})$. For $0 \le m \le n,\ \alpha, a_i, b_i \in \mathbb{R},\ x \in \mathbb{R}$ let*

$$p(x) := \sum_{i=0}^{m} a_i x^i, \qquad q(x) := \sum_{i=0}^{n} b_i x^i$$

define two polynomials. According to (1.1) and (1.2) we define the addition by

$$(p+q)(x) = \sum_{i=0}^{m} (a_i + b_i) x^i + \sum_{i=m+1}^{n} b_i x^i$$

and the scalar multiplication by

$$(\alpha \cdot p)(x) = \sum_{i=0}^{m} (\alpha a_i) x^i.$$

The degree $\deg(p)$ *of a polynomial $p \in P(\mathbb{R},\mathbb{R})$ is the highest power i with non-zero coefficient. Thus* $\deg(a + x^n) = n$.

Remark 1.13. *Some of the spaces we considered until now have non-zero intersections. For example $P(\mathbb{R},\mathbb{R}) \subset C^\infty(\mathbb{R},\mathbb{R})$; $C^{k+m}(\Omega,\mathbb{R}) \subset C^k(\Omega,\mathbb{R})$ for $k, m \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$. Note, however, that some inclusions depend on the domain Ω: $C([0,1],\mathbb{R}) \subset B([0,1],\mathbb{R})$, but $C(]0,1[,\mathbb{R}) \not\subset B(]0,1[,\mathbb{R})$.*

[4] see Definitions A.14 and A.15 in the Appendix

Here we used the following definitions:

$$]a, b[:= \{x \in \mathbb{R} : a < x < b\}, \qquad [a, b] := \{x \in \mathbb{R} : a \le x \le b\} = \overline{]a, b[}. \tag{1.3}$$

$]a, b[$ is *open*, $[a, b]$ is *closed*, $\overline{]a, b[}$ is the *closure of* $]a, b[$; see the definitions in Section A.1.1 in the Appendix.

Differentiable functions are very important in many applications, especially for formulating differential equations. In other applications, however, it is more adequate to work with integrable functions and to formulate integral equations. Later on we will consider variational equations which contain both derivatives and integrals. The integrals considered will always be **Lebesgue integrals**.
The Riemann[5] integral is adequate for many applications. But it has several drawbacks. For example, the limits of sequences of certain Riemann integrable functions are not Riemann integrable. Using the more general Lebesgue[6] integral, these problems are avoided. Whenever a function is Riemann integrable, it is also Lebesgue integrable, and both integrals agree. Certain functions are not Riemann integrable but Lebesgue integrable. The limits of sequences of Lebesgue integrable functions are always Lebesgue integrable. An introduction to Lebesgue integrals and to L^p spaces may be found in [Red86, p.33 - 41] and [VG81, Chap. 11].

Example 1.14 ($L^p(\Omega; \mathbb{R})$ spaces, $p = 1, 2$)**.** *Assume $\Omega \subset \mathbb{R}^n$ is a subset. With addition (1.1) and scalar multiplication (1.2) $L^p(\Omega; \mathbb{R})$ is the real linear space of all Lebesgue-integrable functions $f : \Omega \to \mathbb{R}$ satisfying*

$$\int_\Omega |f(x)|^p dx < \infty. \tag{1.4}$$

In L^p-spaces we say that $f_1 = f_2$ if $f_1(x) = f_2(x)$ a.e.[7] in Ω.

Proof that $L^2(\Omega; \mathbb{R})$ is a real linear space with addition (1.1) and scalar multiplication (1.2). The proof for $L^1(\Omega; \mathbb{R})$ is similar, but simpler.
For $\alpha \in \mathbb{R}$, $f \in L^2(\Omega; \mathbb{R})$ we get that $(\alpha \cdot f)(x) = \alpha f(x)$ and

$$\int_\Omega |(\alpha \cdot f)(x)|^2 dx = \alpha^2 \int_\Omega |f(x)|^2 dx < \infty. \tag{1.5}$$

Let $f, g \in L^2(\Omega; \mathbb{R})$. We have to show that $f + g \in L^2(\Omega; \mathbb{R})$, i.e. that the sum is square-integrable.

$$\int_\Omega |f(x) + g(x)|^2 dx \;=\; \int_\Omega \left| f(x)^2 + g(x)^2 + 2f(x)g(x) \right| dx$$

[5]Bernhard Riemann (* 1826 near Lüneburg, + 1866 near Lago Maggiore, Italy), studied math in Göttingen and Berlin, Prof. of math in Göttingen 1859 - 1866, suffering from health problems after 1862

[6]Henri Léon Lebesgue (* 1885 in Beauvais, + 1941 in Paris), father worker, mother school teacher, higher education financed by stipends and own work; Ph.D. in 1902 at Ecole Normale, military research during World War I; 1919 Prof. at Sorbonne, 1921 - 1941 Prof. at Collège de France. Revised and generalized several basic concepts of math, e.g. in differential geometry, measure theory and integration

[7]a.e. in Ω = 'almost everywhere in Ω'; i.e. the set of those points $x \in \Omega$ with $f_1(x) \neq f_2(x)$ is negligible, i.e. it has Lebesgue measure zero. Definition of *equivalence classes: [Lax02, p.4].*

$$\begin{aligned} &\le \int_\Omega |f(x)|^2 dx + \int_\Omega |g(x)|^2 dx + 2\int_\Omega |f(x)||g(x)| dx \\ &\le 2\int_\Omega |f(x)|^2 + 2\int_\Omega |g(x)|^2 < \infty. \end{aligned} \tag{1.6}$$

To see that inequality (1.6) is correct, put $a = |f(x)|, \quad b = |g(x)|$ for fixed $x \in \Omega$ and remember that $a, b \in \mathbb{R}$ satisfy

$$2ab \le a^2 + b^2 \tag{1.7}$$

because

$$0 \le (a-b)^2 = a^2 - 2ab + b^2.$$

Since $f, \ g \in L^2(\Omega; \mathbb{R})$, estimate (1.7) applies for almost all $x \in \Omega$. Now we apply the rules of Lebesgue measure and integration to get the result. $\diamond$

The reader is supposed to know the rules of Lebesgue measure and integration (see [Red86, p.33 - 41] and [VG81, Chap. 11]).

Definition 1.15 (Basis)**.** *Let V be a linear space over $I\!K$. A subset $B \subset V$ is called* basis *of V iff*
(i) B is a generating system *of V, i.e. for every $v \in V$ there are $k \in \mathbb{N}$, $\{b_1, \ldots, b_k\} \subset B$ and $\{\alpha_1, \ldots, \alpha_k\} \subset I\!K$ such that*

$$v = \sum_{i=1}^k \alpha_i \cdot b_i, \tag{1.8}$$

(ii) this representation is unique, i.e. the elements of B are linearly independent, *i.e. if for some $k \in \mathbb{N}$, $\{\alpha_1, \ldots, \alpha_k\} \subset I\!K$,*

$$\sum_{i=1}^k \alpha_i \cdot b_i = 0 \qquad \Longrightarrow \qquad \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0. \tag{1.9}$$

The cardinality of B is called the dimension of V.

Examples 1.16.
* $V = \{0\} : B = \emptyset$, *the empty set,* $\dim V = 0$;
* $\mathbb{R}^3 : \dim \mathbb{R}^3 = 3, \quad B = \{e_1, e_2, e_3\}$ *with*

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.10}$$

Another basis of $\mathbb{R}^3$ is $\tilde{B} = \{e_1 + e_2, 2e_2, e_2 + e_3\}$.
No basis of $\mathbb{R}^3$ are $\{e_1, e_2\}$ (not a generating system) or $\{e_1, e_2, e_3, e_1 + e_2 + e_3\}$ (not linearly independent).
* $\mathbb{R}^n$: *analogous to* $\mathbb{R}^3$, $B = \{e_1, \ldots, e_n\}$, $\dim \mathbb{R}^n = n$.
* $P(\mathbb{R}, \mathbb{R})$, *the linear space of real polynomials of arbitrary degree,* $B = \{1, x, x^2, \ldots\}$, $\dim P(\mathbb{R}, \mathbb{R}) = \infty$.

Note that the basis of a linear space V is not unique, but the representation of the elements of V w.r.t. a given basis is unique, and the dimension of a linear space is uniquely determined (proof?).

1.1.2 Maps, Functions, Operators, Functionals and Forms

Definition 1.17 (Map). *Let X, Y be sets. $f : X \to Y,\ x \mapsto f(x)$ is called a* map *or a* mapping, *if f assigns exactly one $f(x) = y \in Y$ to every $x \in X$. The set X is called the* domain of definition of f, *the set Y is called the* range of f, *$f(X) \subset Y$ is called* the image of X under f in Y.

The term 'mapping' (='map' = 'Abbildung') originates from the theory of complex functions and/or differential geometry. Today it is of very general use. Functions are also maps. Typically the term 'function' is used if $X, Y \subset \mathbb{R}^n$ or $X, Y \subset \mathbb{C}^n$ and properties like 'differentiable' are important. Operators are also maps. Typically the term 'operator' is used if X and Y are function spaces. Operators mapping a function space into the underlying field $I\!K$ are called 'functionals'. Note that the 'forms' defined in this section are linear or bilinear; functionals may be linear or nonlinear (quadratic, for example).
If X and Y are not just sets but have additional properties, then important questions are: how does f transport these properties between X and Y? Is the image of an open set open? If X has a basis B, is $f(B)$ a basis of Y?, and so on.

Definition 1.18. *Let V, W be linear spaces over $I\!K$. A map $L : V \to W$ is called* linear *if for all $u, v \in V$, $\alpha \in I\!K$*

$$\begin{aligned} L(u+v) &= L(u) + L(v), \\ L(\alpha \cdot v) &= \alpha \cdot L(v). \end{aligned}$$

Note that L maps the operations '+' and '$\cdot$' in V onto those in W. Also, it maps $0 \in V$ onto $0 \in W$: $L(0) = L(0 \cdot 0) = 0 \cdot L(0) = 0$.

Examples 1.19.
* $L : V \to W, \quad v \mapsto 0$; *the range of L is W, the image of L is $\{0\}$;*
* $L : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto Ax$, *A a real $n \times n$-matrix,*
this is also written as $A : \mathbb{R}^n \to \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}$.

When dealing with both linear and nonlinear maps at the same time, linearity is often made visible by omitting brackets: $Lu + N(u) = 0$ means that L is linear and N nonlinear.

Remark 1.20. *Let V, W be linear spaces over $I\!K$ and let $B = \{b_1, b_2, \ldots\}$ be a basis of V, i.e. $V = span\{b_1, b_2, \ldots\}$. Then every linear map $L : V \to W$ is completely determined by its values on B, i.e. by $L(B)$.*

Proof: Assume $v \in V$ arbitrary, then $v = \sum_{i=1}^{k} \alpha_i b_i$ for some $k \in \mathbb{N},\ \alpha_i \in I\!K,\ b_i \in B$, and

$$Lv = L(\sum_{i=1}^{k} \alpha_i b_i) = \sum_{i=1}^{k} \alpha_i L b_i.$$

Definition 1.21. *Let V be a linear space over $I\!K$. A linear map $L : V \to I\!K$ is called a* linear form.

Examples 1.22.
* $L : C([0,1],\mathbb{R}) \to \mathbb{R}, \quad f \mapsto f(0)$;
* $L : C^1([0,1],\mathbb{R}) \to \mathbb{R}, \quad f \mapsto f'(\frac{1}{2})$;
* *Let* $y := (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$ *be fixed.* $L_y : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto y^t x := \sum_{i=1}^n y_i x_i$.
* $L : C([a,b],\mathbb{R}) \to \mathbb{R}, \quad f \mapsto \int_a^b f(x)dx$;
* *The Kepler-Simpson[8] quadrature operator A of Problem 18 is a linear form. It numerically approximates the linear form L of the preceding example.*

Note that it is desirable almost always that numerical approximations have the same properties as the mathematical objects that are approximated.

Definition 1.23 (bilinear forms). [9] *Let V, W be linear spaces over $\mathbb{R}$. A map*

$$\begin{aligned} a : V \times W &\to \mathbb{R}, \\ (v,w) &\mapsto a(v,w) \end{aligned}$$

is a bilinear form *if it is a linear form in both of its arguments, i.e. if the maps*

$$\begin{aligned} a_w : V &\to \mathbb{R}, \\ u &\mapsto a_w(u) := a(u,w), \end{aligned}$$

and

$$\begin{aligned} a_v : W &\to \mathbb{R}, \\ u &\mapsto a_v(u) := a(v,u) \end{aligned}$$

are linear forms for all $v \in V$ and all $w \in W$.

Examples 1.24.
* $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (y,x) \mapsto y^t x := \sum_{i=1}^n y_i x_i$;
* $a : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}, \quad (y,x) \mapsto y_1 x_1 + y_2 x_2$;

1.1.3 Inner Products

Definition 1.25. *Let V be a real linear space.*
A bilinear form $a : V \times V \to \mathbb{R}$ is called
* symmetric *iff* $\quad a(v,w) = a(w,v) \quad$ *for all* $v, w \in V$;
* positive semi-definite [10] *iff* $\quad a(v,v) \geq 0 \quad$ *for all* $v \in V$;
* positive definite *if it is positive semi-definite and* $\quad (a(v,v) = 0 \quad$ *iff* $v = 0)$;
A bilinear form $a : V \times V \to \mathbb{R}$ is called inner product on V *if it is symmetric and positive definite.*
A real linear space with inner product is called inner product space.

[8] Johannes Kepler (*1571 in Weil der Stadt, +1630 in Regensburg); Thomas Simpson (*1710 in Market-Bosworth, GB, +1761 in Market-Bosworth); The 'Keplersche Fassregel' is a special case of Simpson's formular for computing a volume.
[9] 'bi-linear' = 'two times linear'; 'bi' (latin) = 'double', 'two times';
[10] 'semi' (latin) = 'hemi' (greek) = 'demi' (french) = 'half';

Examples 1.26.

$$V = \mathbb{R}^n, \qquad a(v,w) =< v, w >:= v^t w = \sum_{i=1}^{n} v_i w_i; \tag{1.11}$$

this inner product is also called Euclidean product*;*

$$V = C([0,1],\mathbb{R}), \qquad a(f,g) := \int_0^1 f(x)g(x)dx. \tag{1.12}$$

Remark 1.27. *Let V be an inner product space with inner product $< \cdot, \cdot >$. Then $< v, 0 >=< 0, v >= 0$ for all $v \in V$, because $< v, 0 >=< v, 0 \cdot v >= 0 \cdot < v, v >= 0$.*

In an inner product space, $< u, v >= 0$ is possible for $u, v \neq 0,\ u \neq v$. Take for instance $n = 2$ in (1.11), $u = (1,0)^t,\ v = (0,1)^t$.

Definition 1.28. *Let V be an inner product space with inner product $< \cdot, \cdot >$. $u, v \in V$ are called* orthogonal w.r.t. *$< \cdot, \cdot >$ iff $u, v \neq 0$ and $< u, v >= 0$.*

Theorem 1.29 (Bunjakowski-Schwarz inequality). [11] *Let V be an inner product space. Then the inequality*

$$< u, v >^2 \leq < u, u >< v, v > \tag{1.13}$$

is valid for all $u, v \in V$.

Proof: The statement is true for $v = 0$, because of Remark 1.27. Let $v \neq 0$. Then

$$\begin{aligned}
0 &\leq \left\langle u - \frac{< u,v >}{< v,v >}v, u - \frac{< u,v >}{< v,v >}v \right\rangle, \\
&= < u,u > - \left\langle u, \frac{< u,v >}{< v,v >}v \right\rangle - \left\langle \frac{< u,v >}{< v,v >}v, u \right\rangle + \left\langle \frac{< u,v >}{< v,v >}v, \frac{< u,v >}{< v,v >}v \right\rangle \\
&= < u,u > -2\frac{< u,v >}{< v,v >} < u,v > + \frac{< u,v >^2}{< v,v >^2} < v,v > \\
&= < u,u > -2\frac{< u,v >^2}{< v,v >} + \frac{< u,v >^2}{< v,v >}, \\
&= < u,u > - \frac{< u,v >^2}{< v,v >}.
\end{aligned}$$

Now multiply by $< v, v >$ and add $< u, v >^2$ to complete the proof. ◇

[11] Viktor Jakowlewitsch Bunjakowski (*1804 in Bar, +1889 in St. Petersburg) studied in Coburg, Lausanne and Paris (with Cauchy). Got his PhD in Paris, then taught 'new' mathematics in St. Petersburg, 1846 - 1880 Prof. at U St. Petersburg. Proved the inequality for the special case of definite integrals in one real variable.
Hermann Amandus Schwarz (*1843 in Hermsdorf, now Sobiecin in Poland; +1921 in Berlin)

1.1.4 Norms and Metrics

Definition 1.30. *Let V be a real linear space. A map*

$$\begin{aligned} ||\cdot|| : V &\to \mathbb{R}, \\ v &\mapsto ||v||, \end{aligned}$$

is called a norm on V *iff*

$$||v|| \geq 0 \text{ for all } v \in V, \tag{1.14}$$
$$||v|| = 0 \text{ iff } v = 0, \tag{1.15}$$
$$||\alpha v|| = |\alpha|\ ||v|| \text{ for all } \alpha \in \mathbb{R}, v \in V, \tag{1.16}$$
$$||u+v|| \leq ||u|| + ||v|| \text{ for all } u, v \in V. \tag{1.17}$$

Inequality (1.17) is called triangle inequality*. A linear space with norm is called* normed space*.*

Examples 1.31 (Norms on $\mathbb{R}^n$)**.**
* *abs* $: \mathbb{R} \to \mathbb{R}, \quad x \mapsto |x|,$ *the absolute value defines a norm on $\mathbb{R} = \mathbb{R}^1$;*
Now let $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n,\ n \geq 1$,
* $\mathbb{R}^n \to \mathbb{R}, \quad x \mapsto ||x||_1 := \sum_{i=1}^n |x_i|,$ *is called the* 1-norm*;*
* $\mathbb{R}^n \to \mathbb{R}, \quad x \mapsto ||x||_2 := \left(\sum_{i=1}^n |x_i|^2\right)^{1/2},$ *is called the* 2-norm *or the* Euclidean norm*; as we will see, it generates the 'Euclidean space';*
* $\mathbb{R}^n \to \mathbb{R}, \quad x \mapsto ||x||_\infty := \max_{1 \leq i \leq n} |x_i|,$ *is called the* ∞-norm *or* maximum norm*;*
Note that all three norms reduce to the absolute value for $n = 1$.

Examples 1.32 (Norms on C-Spaces)**.**
* *For $f \in C([0,1], \mathbb{R})$ we define the* ∞-norm *or* sup-norm*:*

$$||f||_\infty := \sup_{x \in [0,1]} |f(x)| \ ; \tag{1.18}$$

* *For $f \in C^1([0,1], \mathbb{R})$ we define the* C^1-norm

$$||f||_{C^1} := \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)| \ . \tag{1.19}$$

* *For $f \in C([0,1], \mathbb{R}) \cap C^1([0,1], \mathbb{R})$ there are thus two norms:*

$$||f||_\infty \leq ||f||_{C^1} \ .$$

Accordingly, norms are defined on $C^m(\Omega, \mathbb{R}),\ \Omega \subset \mathbb{R}^n,\ n, m \in \mathbb{N}$.

Theorem 1.33 (Natural norms)**.** *Let V be an inner product space, i.e. a real linear space with inner product, $< \cdot, \cdot >$. Then*

$$||v|| := < v, v >^{1/2} \qquad \text{for all } v \in V \tag{1.20}$$

is a norm on V. This norm is called the natural norm of the inner product space*.*

Proof:
$||\cdot||$ defined by (1.20) is positive definite because inner products are positive definite.
$||\alpha v|| =< \alpha v, \alpha v >^{1/2} = (\alpha^2 < v, v >)^{1/2} = (\alpha^2)^{1/2} < v, v >^{1/2} = |\alpha|\ ||v||$ for all $\alpha \in \mathbb{R}$, $v \in V$.
The triangle inequality is valid because

$$\begin{aligned} ||u+v||^2 &= < u+v, u+v > \\ &= < u,u > + < u,v > + < v,u > + < v,v >, \\ &= ||u||^2 + ||v||^2 + 2 < u,v >, \\ &\leq ||u||^2 + ||v||^2 + 2\ | < u,v > |, \\ &\leq ||u||^2 + ||v||^2 + 2\ ||u||\ ||v||, \\ &= (||u|| + ||v||)^2. \end{aligned}$$

The last '$\leq$' follows from the Schwarz inequality (1.13): $< u,v >^2 \leq < u,u >< v,v >= ||u||^2||v||^2$, and because 'square root' is a monotonic function. Thus

$$| < u,v > | \leq ||u||\ ||v||. \tag{1.21}$$

Applying the square root one more time completes the proof. ◇

Theorem 1.34 (Parallelogram identity)**.** *Let V be an inner product space. Then the* parallelogram identity *is valid:*

$$||u+v||^2 + ||u-v||^2 = 2\ ||u||^2 + 2\ ||v||^2 \qquad \textit{for all } u, v \in V. \tag{1.22}$$

Conversely, if V is a normed space such that the parallelogram identity is valid for all $u, v \in V$, then V is an inner product space with the inner product

$$< u,v >:= \frac{1}{4}(||u+v||^2 - ||u-v||^2), \tag{1.23}$$

such that (1.20) is valid for $< \cdot, \cdot >$ and $||\cdot||$.

The parallelogram identity provides a criterion if a given norm is a natural norm.
Proof: left to the reader (Homework Problem 30). The '$\Leftarrow$' part of the statement was discovered and first proved by John von Neumann[12] [Lax02, p. 53]. ◇

Example 1.35 (parallelogram identity does not hold)**.**
Consider $C([a,b], \mathbb{R})$, $0 < a < b$, with $||f||_\infty = \sup_{x\in[a,b]} |f(x)|$. Take $u(x) := 1$ and $v(x) := x$ and choose the values of a and b such that

$$\begin{aligned} ||u+v||_\infty &= \sup_{x\in[a,b]} |u(x)+v(x)| = \sup_{x\in[a,b]} |1+x| = 1+b, \\ ||u-v||_\infty &= \sup_{x\in[a,b]} |u(x)-v(x)| = \sup_{x\in[a,b]} |1-x| = 1-a. \end{aligned}$$

[12]John (Janos, Johann) Neumann von Margitta (*1903 in Budapest, +1957 in Washington, D.C.) studied mathematics in Berlin and Budapest and chemistry in Zürich; worked 1926 with Hilbert in Göttingen, 1927 - 1933 at U Berlin and U Hamburg and concurrently from 1930 on in Princeton, USA. Moved fully to Princeton in 1933. *Ehrendoktor (Dr.rer.nat. h.c.)* of Fakultät für Mathematik, TUM in 1953

Then we get with $a := 0.5$ *and* $b = 1$

$$\begin{aligned} ||u+v||^2 + ||u-v||^2 &= (1+b)^2 + (1-a)^2 = 4 + 0.25 && (1.24) \\ 2||u||^2 + 2||v||^2 &= 2 + 2b^2 = 4. && (1.25) \end{aligned}$$

Note that eq. (1.25) depends only on b*, but eq. (1.24) depends both on* a *and* b*.*

Examples 1.36 (norms induced by inner products)**.**
* *Let* $V := \mathbb{R}^n, n \geq 1$*, and*

$$||x||_2 :=< x, x >^{\frac{1}{2}}, \qquad \textit{with } < x, y >:= x^t y = \sum_{i=1}^{n} x_i y_i. \tag{1.26}$$

This is the Euclidean norm introduced in Examples 1.31.

* *Let* $V := \{u \in C^1([0,1], \mathbb{R}) : u(0) = u(1) = 0\}$*.* $V \subset C^1([0,1], \mathbb{R})$ *is a linear subspace and thus a real linear space. The norm*

$$||u|| := a(u,u)^{1/2} \qquad \textit{with } a(u,v) := \int_0^1 u'(x)v'(x)dx \tag{1.27}$$

is called the energy norm*.*

* *Let* $V := L^2([0,1], \mathbb{R})$ *be the real linear space of square-integrable functions introduced in Example 1.14.*

$$||u||_{L^2} := a(u,u)^{1/2} \qquad \textit{with } a(u,v) := \int_0^1 u(x)v(x)dx. \tag{1.28}$$

Note that $|a(u,v)| < \infty$ *for square-integrable functions* u, v*, because*

$$|a(u,v)| \leq \int_0^1 |u(x)||v(x)|dx < \infty,$$

see (1.6) and (1.7). Those estimates were necessary for concluding that L^2 *is a linear space. Thus* $a(u,v)$ *is well defined.*
It is easy to show that $a(u,v)$ *is a bilinear form, symmetric and positive semi-definite. It is also definite:* $a(u,u) = 0$ *iff* $u(x) = 0$ *a.e.. Since we decided in Example 1.14 that* $u = 0$ *iff* $u(x) = 0$ *a.e., we get that* $a(u,u) = 0$ *iff* $u = 0$*. Thus* $a(u,v)$ *is positive definite. Now that we know that* $L^2([0,1], \mathbb{R})$ *is an inner product space, we could apply the Schwarz inequality and get* $a(u,v)^2 \leq a(u,u)a(v,v) < \infty$ *for all* $u, v \in L^2([0,1], \mathbb{R})$*.*

Definition 1.37. *Let* U *be a set. A map* $d : U \times U \to \mathbb{R}$*,* $(u,v) \mapsto d(u,v)$ *is called a* metric on U *iff*

$$\begin{aligned} d(u,v) &\geq 0 \quad \textit{and} \quad (d(u,v) = 0 \textit{ iff } u = v); \\ d(u,v) &= d(v,u); \\ d(u,v) &\leq d(u,w) + d(w,v); \end{aligned}$$

for all $u, v, w \in U$*. A set* U *with a metric is called* metric space*.*

Lemma 1.38. *Let V be a linear space with norm $||\cdot||$. Then*

$$d(u,v) := ||u-v|| \tag{1.29}$$

defines a metric on V.

Proof: left to the reader (Homework Problem 24). $\diamond$

Remark 1.39. *Norms thus allow to measure lengths and distances.*

Remark 1.40. $\mathbb{R}^n$ *with the metric which is induced by the Euclidean norm (1.26) is called* Euclidean Space. *Because the usual Euclidean Geometry is valid in this space (compare Homework Problem 21).*

1.2 Convergence, Continuity and Completeness

In this section concepts are introduced which mostly stem from Analysis (Calculus). The basic versions of these concepts are expected to be known to the reader. They may be looked up in [VG81, AnOb, MVa], for instance.

1.2.1 Convergence

Given some set with a metric, convergence may be defined. For the basic definitions related to convergence in subsets of $\mathbb{R}^n$, $n \geq 1$ with Euclidean metric, see [VG81, chap.3] or [AnOb, MVa]. Here we will mostly consider convergence in normed function spaces.

Definition 1.41. *Let V be a normed space and $(v_n)_{n\in\mathbb{N}} \subset V$ a sequence. We say that* $(v_n)_{n\in\mathbb{N}}$ converges in the norm $||\cdot||$ to the limit[13] $v \in V$ *iff*

$$\lim_{n\to\infty} ||v_n - v|| = 0. \tag{1.30}$$

We then also write: $v_n \to v$ for $n \to \infty$, or $\lim_{n\to\infty} v_n = v$ in the norm $||\cdot||$.

Remark 1.42. *Note that the norm maps convergence properties in a function space onto convergence properties in the real numbers.*

As is illustrated in Homework Problem 21, different norms induce different metrics. Does this lead to different types of convergence? Not in $\mathbb{R}^n$, but often in function spaces.

Examples 1.43. *Consider $(f_n)_{n\in\mathbb{N}} \subset C([0,2],\mathbb{R})$ with*

$$f_n(x) := \begin{cases} x^n & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x \leq 2 \end{cases} . \tag{1.31}$$

** Choose $x \in [0,2]$ arbitrary, but fixed. Then we get a sequence of real numbers with $\lim_{n\to\infty} f_n(x) = f(x)$, with*

$$f(x) := \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x \leq 2 \end{cases} . \tag{1.32}$$

[13] 'lim' short for 'limes' (latin) = 'limit';

This type of convergence of a sequence of functions with fixed argument is called pointwise convergence.
* f_n *and* f *are square-integrable. We thus now consider convergence w.r.t.*[14] *the* L^2*-norm. We get*

$$||f_n - f||_{L^2}^2 = \int_0^2 |f_n(x) - f(x)|^2 dx = \int_0^1 x^{2n} dx = \frac{1}{2n+1} \to 0 \textit{ for } n \to \infty. \tag{1.33}$$

* *Since* $(f_n)_{n\in\mathbb{N}} \subset C([0,2],\mathbb{R})$ *we may consider convergence in the sup-norm* $||\cdot||_\infty$. *In this case the sequence does not converge because* f *is not continuous at* $x = 1$. *This follows from the next theorem.*

Theorem 1.44. *Let* $(f_n)_{n\in\mathbb{N}} \subset C([a,b],\mathbb{R})$ *be a sequence and* $f : [a,b] \to \mathbb{R}$ *be bounded, i.e.* $|f(x)| < c$ *for all* $x \in [a,b] \subset \mathbb{R}$ *and some* $c > 0$. *If* $||f_n - f||_\infty \to 0$ *for* $n \to \infty$, *then* f *is continuous, i.e.* $f \in C([a,b],\mathbb{R})$.

Proof: Choose a fixed $x \in [a,b]$ and $\varepsilon > 0$. Also choose some sequence $(x_k)_{k\in\mathbb{N}} \subset [a,b]$ such that $\lim_{k\to\infty} x_k = x$.
By assumption $||f_n - f||_\infty \to 0$ for $n \to \infty$. Thus we may find some $n_o \in \mathbb{N}$ such that $||f_n - f||_\infty < \frac{\varepsilon}{3}$ for all $n \geq n_o$.
By assumption all f_n are continuous. Thus $\lim_{k\to\infty} f_n(x_k) = f_n(x)$ for n fixed. We thus may choose $k_0 \in \mathbb{N}$ s.th. $|f_n(x_k) - f_n(x)| < \frac{\varepsilon}{3}$ for all $k \geq k_0$.
Using $n \geq n_o$ and $k \geq k_o$ and applying the triangle inequality we get

$$\begin{aligned} |f(x_k) - f(x)| &\leq |f(x_k) - f_n(x_k)| + |f_n(x_k) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{1.34}$$

It follows that for all $\varepsilon > 0$ there is a $k_o \in \mathbb{N}$ such that $|f(x_k) - f(x)| \leq \varepsilon$ for all $k > k_o$, i.e. $\lim_{k\to\infty} f(x_k) = f(x)$.
Thus $\lim_{k\to\infty} x_k = x$ implies $\lim_{k\to\infty} f(x_k) = f(x)$. This means that f is continuous in $x \in [a,b]$. Since x was arbitrary, this means that f is continuous in $[a,b]$. ◇

Definition 1.45. *Let* V *be a linear space and* $||\cdot||_1$, $||\cdot||_2$ *two norms on* V. *We say that*
* $||\cdot||_1$ is stronger than $||\cdot||_2$ *iff there is a real constant* $C > 0$ *s.th.*

$$||v||_2 \leq C||v||_1, \qquad \textit{for all } v \in V; \tag{1.35}$$

* $||\cdot||_1$ and $||\cdot||_2$ are equivalent *iff there are real constants* C_1, $C_2 > 0$ *s.th.*

$$||v||_2 \leq C_1||v||_1 \textit{ and } ||v||_1 \leq C_2||v||_2, \qquad \textit{for all } v \in V. \tag{1.36}$$

Examples 1.46.
* *Let* $f \in C^1([0,1],\mathbb{R}) \subset C([0,1],\mathbb{R})$. *Then* $||f||_\infty \leq ||f||_{C^1}$, *according to Examples 1.32. Norm* $||\cdot||_{C^1}$ *is stronger than* $||\cdot||_\infty$.
* *Consider* $\mathbb{R}^n$ *with* $||x||_1 = \sum_{i=1}^n |x_i|$ *and* $||x||_\infty = \max_i |x_i|$, *Examples 1.31. Then*

$$||x||_1 \leq n\ ||x||_\infty \qquad \textit{and} \qquad ||x||_\infty \leq ||x||_1. \tag{1.37}$$

[14]w.r.t. = with respect to

Thus both norms are equivalent.
* *Consider* $\mathbb{R}^2$ *with* $||\cdot||_1$ *and* $||\cdot||_2 = (x_1^2 + x_2^2)^{\frac{1}{2}}$. *Then* $||x||_1^2 = x_1^2 + x_2^2 + 2\ |x_1|\ |x_2|$ *and* $||x||_2^2 = x_1^2 + x_2^2$. *We thus get using inequality (1.7)*

$$||x||_1^2 \leq 2\ ||x||_2^2 \qquad \text{and} \qquad ||x||_2^2 \leq ||x||_1^2. \tag{1.38}$$

Since 'sqrt' is a monotonic function we thus see that on $\mathbb{R}^2$ *the 1-norm, 2-norm and* ∞*-norm are equivalent.*

Actually, a much more general result has been proved:

Lemma 1.47. *On* $\mathbb{R}^n$, $n \geq 1$, *all norms are equivalent.*

1.2.2 Continuity

Definition 1.48 (Continuity)**.**
Let $V,\ W$ *be normed spaces,* $\Omega \subset V$. *A function* $f : \Omega \to W$ *is*
* continuous at $v \in \Omega$ *iff*

$$\lim_{n\to\infty} ||f(v_n) - f(v)|| = 0 \text{ for all sequences } (v_n)_{n\in\mathbb{N}} \subset \Omega \text{ with } \lim_{n\to\infty} ||v_n - v|| = 0\ ; \tag{1.39}$$

* continuous in Ω *iff it is continuous in all* $v \in \Omega$.

Note that this is a straight-forward generalization of 'Continuity of real functions', Definition A.14.

Examples 1.49. *Consider again* $f : [0, 2] \to \mathbb{R}$ *from Examples 1.43,*

$$f(x) := \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } 1 \leq x \leq 2 \end{cases} . \tag{1.40}$$

If we apply Definition 1.48 with the sequence $(x_n)_{n\in\mathbb{N}},\ x_n := 1 + \frac{1}{n}$, *we get* $f(x_n) = 1$ *for all* $n \in \mathbb{N}$ *and thus* $\lim_{n\to\infty} f(x_n) = 1 = f(1)$.
If we apply Definition 1.48 with the sequence $(x_n)_{n\in\mathbb{N}},\ x_n := 1 - \frac{1}{n}$, *we get* $f(x_n) = 0$ *for all* $n \in \mathbb{N}$ *and thus* $\lim_{n\to\infty} f(x_n) = 0 \neq 1 = f(1)$.
Condition (1.39) thus is not satisfied at $x_o = 1$ *for* all *sequences.* $f : [0, 2] \to \mathbb{R}$ *thus is not continuous at* $x_o = 1$.
Note, however, that $\tilde{f} : [1, 2] \to \mathbb{R}$ *is continuous at the point* $x_o = 1$ *and in the interval* $[1, 2]$. *Also,* $\hat{f} : (0, 1) \to \mathbb{R}$ *is continuous in the interval* $(0, 1)$. *Since* $x_o = 1$ *is the only point in the domain of definition of* f *where* f *is not continuous, we call* $f : [0, 2] \to \mathbb{R}$ piecewise continuous *or* continuous a.e..

Lemma 1.50. *Let* $V,\ W$ *be normed linear spaces. A linear map* $L : V \to W$ *is continuous on* V *iff it is continuous in* $0 \in V$.

Proof: '$\Rightarrow$': Since L is continuous on V, it is continuous in $0 \in V$.
'$\Leftarrow$': Assume that L is continuous in $0 \in V$ but not on V. Then there is some $v \in V$ s.th. L is not continuous at that v. Then there is some sequence $(v_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} v_n = v$

and $\lim_{n\to\infty} Lv_n \neq Lv$. But $||Lv_n - Lv|| = ||L(v_n - v)|| = ||Lw_n|| \to L0 = 0$ for $n \to \infty$, because $w_n := v_n - v \to 0$ for $n \to \infty$, and L is continuous at 0. $\diamond$

Note that the linearity of L is substantial for such a result: given some nonlinear function, continuity at one point does not have consequences for continuity at other points.

Example 1.51. *Let* $V := C^1([0,\pi],\mathbb{R}),\ W := C([0,\pi],\mathbb{R}),\ f \mapsto Df = f'$.
D is a linear map since $(f_1+f_2)' = f_1' + f_2'$ *and* $(\alpha f)' = \alpha f'$ *for all* $f, f_1, f_2 \in V$ *and* $\alpha \in \mathbb{R}$. *Is D continuous? As we will see, this depends on the norms defined in V and W.*
** Equip both V and W with* $||\cdot||_\infty$. *Define*

$$f_n(x) := \frac{\sin nx}{n} \qquad \text{for all } x \in [0,\pi]. \tag{1.41}$$

Then

$$\begin{aligned} ||f_n||_\infty &= ||\frac{\sin nx}{n}||_\infty = \frac{1}{n} \to 0 \qquad \text{for } n \to \infty, \\ ||Df_n||_\infty &= ||\frac{n\cos nx}{n}||_\infty = \|\cos nx\|_\infty = 1 \neq 0 = D0 \qquad \text{for } n \to \infty. \end{aligned}$$

Thus D is not continuous at $f = 0$ and thus not on $C^1([0,\pi],\mathbb{R})$.
** Now equip W with* $||\cdot||_\infty$ *and V with the stronger norm* $||\cdot||_{C^1}$ *introduced in (1.19). For the f_n defined in (1.41) we now get*

$$||f_n||_{C^1} = ||f_n||_\infty + ||f_n'||_\infty \to (0+1) = 1 \qquad \text{for } n \to \infty.$$

Thus the previous example does not apply under the C^1-norm. – Now let $(g_n)_{n\in\mathbb{N}} \subset V$ *be an arbitrary sequence with* $||g_n||_{C^1} = \sup_{x\in[0,\pi]} |g_n(x)| + \sup_{x\in[0,\pi]} |g_n'(x)| \to 0$ *for* $n \to \infty$. *Then*

$$||Dg_n||_\infty = ||g_n'||_\infty \leq ||g_n||_\infty + ||g_n'||_\infty = ||g_n||_{C^1} \to 0 \qquad \text{for } n \to \infty.$$

Thus $D : (C^1([0,\pi],\mathbb{R}), ||\cdot||_{C^1}) \to (C([0,\pi],\mathbb{R}), ||\cdot||_\infty)$ *is continuous,*
but $D : (C^1([0,\pi],\mathbb{R}), ||\cdot||_\infty) \to (C([0,\pi],\mathbb{R}), ||\cdot||_\infty)$ *is not continuous.*

Remark 1.52. *On* $C^n([a,b],\mathbb{R}),\ n \in \mathbb{N}$ *the following norm may be defined:*

$$||f||_{C^n} := \sum_{k=0}^{n} ||D^k f||_\infty, \qquad f \in C^n,$$

with $D^0 f = f$.[15]

[15] Note that this definition cannot be extended to C^∞: Let $[a,b] = [0,\pi]$, then $||\sin x||_{C^n} = n+1$, and this is not bounded for $n \to \infty$. Indeed, C^∞ is a *not normable topological vector space* in the following sense: every norm induces a metric (see Lemma 1.38), every metric induces a topology (i.e. it induces which sets are open or closed: see the definitions in Section A.1.1 in the Appendix, where we used the metric induced by the Euclidean norm for defining open sets in $\mathbb{R}$ and in $\mathbb{R}^2$). C^∞ has a 'natural' topology which makes it a topological vector space (i.e. vector addition and scalar multiplication are continuous maps in this topology), but there is no norm which induces this topology; see [Rud91, Chap.1, Examples, p.33 - 36] where this is explained in detail for $C^\infty(\Omega,\mathbb{C}),\ \Omega \subset \mathbb{R}^m$ open. This is of theoretical interest. In numerical practice, $||\cdot||_{C^n}$ is used typically for $n < 10$.

Definition 1.53. *Let V, W be normed linear spaces with norms $||\cdot||_V$, $||\cdot||_W$. A linear map $L : V \to W$ is called* bounded *if there exists a constant $C \geq 0$ s.th.*

$$||Lv||_W \leq C||v||_V \qquad \text{for all } v \in V.$$

Remark 1.54. *If L is bounded then*

$$\frac{||Lv||_W}{||v||_V} \leq C \qquad \text{for all } v \neq 0,$$

and also

$$||L|| := \sup_{||v||_V=1} ||Lv||_W \leq C. \tag{1.42}$$

$||L||$ is called operator norm of L, compatible to the norms $||\cdot||_V, ||\cdot||_W$.

Example 1.55. *Let $V = W = \mathbb{R}^n$, $A : \mathbb{R}^n \to \mathbb{R}^n$ a real $n \times n$-Matrix. Then we may define*

$$||A||_\infty := \max_{||x||_\infty=1} ||Ax||_\infty.$$

It can be shown that this definition leads to the same norm as the definition given in Homework Problem 31.

Lemma 1.56. *Let V, W be normed spaces, $L : V \to W$ a linear map. Then L is continuous iff L is bounded.*

Proof: '$\Leftarrow$' Assume L is bounded; $(v_n)_{n\in\mathbb{N}} \subset V$ a sequence with $v_n \to 0$ for $n \to \infty$. Then there is a constant $C \geq 0$ s.th. $||Lv_n||_W \leq C\ ||v_n||_V$. But $||v_n||_V \to 0$ for $v_n \to 0$. Thus $||Lv_n||_W \to 0$ for $v_n \to 0$. L is continuous in V according to Lemma 1.50.
'$\Rightarrow$' Assume that L is continuous, but not bounded. Then there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset V \setminus \{0\}$ with

$$\frac{||Lv_n||_W}{||v_n||_V} > n \qquad \text{for all } n.$$

Note that

$$||\frac{v_n}{||v_n||_V}||_V = \frac{1}{||v_n||_V}||v_n||_V = 1 \qquad \text{and} \qquad ||\frac{v_n}{n||v_n||_V}||_V \to 0 \text{ for } n \to \infty.$$

Define

$$w_n := \frac{v_n}{n||v_n||_V}. \quad \text{Then} \quad ||Lw_n||_W = \frac{1}{n}\frac{||Lv_n||_W}{||v_n||_V} > \frac{1}{n}n = 1.$$

We thus got a contradiction: $||w_n||_V \to 0$, but $||Lw_n||_W > 1$ for all n. Thus L is not continuous at 0. $\diamond$

1.2.3 Completeness; Banach Spaces, Hilbert Spaces

Definition 1.57 (Cauchy sequence). [16]
Let V be a normed space. A sequence $(v_n)_{n\in\mathbb{N}} \subset V$ is called Cauchy sequence *if for any real $\varepsilon > 0$ there is a number $n_o \in \mathbb{N}$ s.th.*

$$||v_n - v_m|| < \varepsilon \qquad \text{for all } n, m \geq n_o. \tag{1.43}$$

Lemma 1.58. *Let V be a normed space and $(v_n)_{n\in\mathbb{N}} \subset V$ a convergent sequence with $\lim_{n\to\infty} v_n = v \in V$. Then $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof: Let $\varepsilon > 0$ be given. Because (v_n) is convergent, there is an $n_o \in \mathbb{N}$ s.th.

$$||v_n - v|| < \frac{\varepsilon}{2} \qquad \text{for all } n \geq n_o.$$

Thus we get

$$||v_n - v_m|| \leq ||v_n - v|| + ||v - v_m|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } n, m \geq n_o. \qquad \diamond$$

Examples 1.59. *Consider again the sequence $(f_n)_{n\in\mathbb{N}}$ introduced in Examples 1.43, f as defined in (1.32).*
** Let $V := L^2([0,2],\mathbb{R})$, with the L^2-norm. As we have seen in Examples 1.43, $\lim_{n\to\infty} f_n = f$ in the L^2-norm, and $f, f_n \in (V, ||\cdot||_{L^2})$. According to Lemma 1.58, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(V, ||\cdot||_{L^2})$.*
** Let $V := C([0,2],\mathbb{R})$, with the L^2-norm. Again we get $\lim_{n\to\infty} f_n = f$ in the L^2-norm, and $f_n \in (V, ||\cdot||_{L^2})$ for all $n \in \mathbb{N}$. Thus $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence also in $(V, ||\cdot||_{L^2})$. It converges to f, but $f \notin (V, ||\cdot||_{L^2})$. Thus we have a Cauchy sequence without limit in $(C([0,2],\mathbb{R}), ||\cdot||_{L^2})$.*

Thus we see: every convergent sequence is a Cauchy sequence, but there are Cauchy sequences that do not converge in a given normed space. This is a deficit of that space and motivates the following definition:

Definition 1.60 (Completeness). *Let V be a normed space. V is called* complete, *iff every Cauchy sequence in V has a limit in V.*

Examples 1.61.
** $L^2([0,2],\mathbb{R}), ||\cdot||_{L^2}$ is complete;*
** $C([0,2],\mathbb{R}), ||\cdot||_{L^2}$ is not complete;*
** $C([0,2],\mathbb{R}), ||\cdot||_\infty$ is complete (Theorem 1.44).*

Remark 1.62. *In many cases it is much easier to show that a given sequence $(v_n)_{n\in\mathbb{N}} \subset V$ is a Cauchy sequence than finding explicitly the element $v \in V$ such that eq. (1.30) is satisfied. Indeed, if a normed space V is known to be complete, some existence theorems may be proved through constructing Cauchy sequences.*

---

[16] Augustin-Louis Cauchy (* 1789 in Paris, + 1857 near Paris), studied engineering at Ecole Polytechnique (E.P.) and worked as an engineer for several years; became a mathematician around 1811; later Prof. of math at E.P., member of 'Academie Francaise'. Main fields of interest: theory of complex functions, fluid mechanics, theory of elasticity; wrote several text books [GIS90].

Definition 1.63 (Banach space)**.** *A complete normed space is called* Banach space.[17]

Examples 1.64.
* $L^1([0,2],\mathbb{R}), ||\cdot||_{L^1}$ *is a Banach space,* $||f||_{L^1} := \int_0^2 |f(x)|dx$;
* $C([0,2],\mathbb{R}), ||\cdot||_\infty$ *is a Banach space;*
* $C^1([0,2],\mathbb{R}), ||\cdot||_{C^1}$ *is a Banach space;*
* *many important infinite-dimensional function spaces are Banach spaces: they are defined such that they are complete;*
* *all finite-dimensional normed spaces are complete.*

Inner product spaces with their natural norm have important additional properties. This motivates the following definition:

Definition 1.65 (Hilbert space)**.** *An inner product space which is complete w.r.t. the natural norm induced by the inner product is called* Hilbert space.[18]

Examples 1.66.
* $\mathbb{R}^n$ *with inner product* $< x, y >:= \sum_{i=1}^n x_i y_i$ *and norm* $||x||_2 =< x, x >^{\frac{1}{2}}$ *is a Hilbert space (and a Banach space).*
* $L^2([a,b],\mathbb{R}), ||\cdot||_{L^2}$ *is a Hilbert space.*

Later on we will use the following two lemmas for normed spaces.

Lemma 1.67. *Let V be an inner product space with natural norm $||\cdot||$. Consider the product space $V \times V$ with norm*

$$||(v_1, v_2)||_{V\times V} := \sqrt{||v_1||^2 + ||v_2||^2}. \tag{1.44}$$

Then the map

$$P : V \times V \to \mathbb{R}, \qquad (v_1, v_2) \mapsto P(v_1, v_2) :=< v_1, v_2 > \tag{1.45}$$

is continuous.

Proof: Let $v_1, v_2, w_1, w_2 \in V$. Then $|P(v_1, v_2) - P(w_1, w_2)| \le \tilde{C}||(v_1, v_2) - (w_1, w_2)||_{V\times V}$ for some $\tilde{C} \ge 0$, and it follows that P is continuous. We get

$$\begin{aligned} |P(v_1, v_2) &- P(w_1, w_2)| = |< v_1, v_2 > - < w_1, w_2 >| \\ &\le |< v_1, v_2 > - < w_1, v_2 >| + |< w_1, v_2 > - < w_1, w_2 >| \end{aligned}$$

[17] Stefan Banach (* 1892 in Kraków, + 1945 in Lvov=Lemberg, now Ukraine) 1910 - 1914: studied engineering in Lemberg (Poland), no exam; 1916 'discovered' by H. Steinhaus, who was a Ph.D. student of D. Hilbert in Göttingen and then prof. of math in Lemberg. Banach became a Ph.D. student of Steinhaus, got his Ph.D. in math in 1920. Later Banach was leader of a very important and successful polish mathematical school at Lvov. After the Nazis invaded Poland, Banach was tortured in medical experiments in a concentration camp. He survived the camp, but only for a few months. [GIS90], [Lax02, p.172].

[18] David Hilbert (* 1862 in Königsberg=Kaliningrad, + 1943 in Göttingen) studied math in Königsberg 1880 - 1885; Ph.D. 1885; full Prof. 1893; Prof. in Göttingen 1895 - 1930; see [GIS90] and also the book *C. Reid: Hilbert. Springer Verlag 1970*

$$
\begin{aligned}
&= |<v_1-w_1,v_2>|+|<w_1,v_2-w_2>| \\
&\leq ||v_1-w_1||\ ||v_2||+||w_1||\ ||v_2-w_2|| \qquad \text{Schwarz inequality} \\
&\leq C\ (||v_1-w_1||+||v_2-w_2||), \quad C:=\max\{||v_2||,||w_1||\}\geq 0, \\
&\leq \sqrt{2}\ C\ \sqrt{||v_1-w_1||^2+||v_2-w_2||^2}, \quad \text{since } (a+b)^2\leq 2(a^2+b^2) \text{ for } a,b\in\mathbb{R}, \\
&\leq \tilde{C}\ ||(v_1-w_1,v_2-w_2)||_{V\times V}, \quad \text{definition (1.44)},\ \tilde{C}=C\sqrt{2}, \\
&= \tilde{C}\ ||(v_1,v_2)-(w_1,w_2)||_{V\times V} \quad \text{addition in } V\times V. \qquad \diamond
\end{aligned}
$$

Lemma 1.68. *Let V and W be normed linear spaces and $a: V\times W\to\mathbb{R}$ a bilinear map. Define*

$$||(v,w)||_{V\times W} := ||v||_V+||w||_W. \tag{1.46}$$

Then a is continuous iff there is a constant $C\geq 0$ such that

$$|a(v,w)|\leq C||v||_V||w||_W \quad \textit{for all } v\in V,\ w\in W. \tag{1.47}$$

Proof: "$\Rightarrow$": Assume that a is continuous. According to Lemma 1.56 it is bounded w.r.t. each of its arguments, i.e. there are constants $C_w, C_v\geq 0$ such that

$$
\begin{aligned}
|a(v,w)| &\leq C_w||v||_V \qquad \text{for all } v\in V, \text{ and any fixed } w\in W, \\
|a(v,w)| &\leq C_v||w||_W \qquad \text{for all } w\in W, \text{ and any fixed } v\in V,
\end{aligned}
$$

but that does not ensure a common bound for all (v,w). To get ineq. (1.47) for all $(v,w)\in V\times W$, we adapt the proof of Lemma 1.56 to the present case:
Assume that a is continuous, but that the bound C does not exist. Then there is a sequence $(v_n,w_n)_{n\in\mathbb{N}}\subset V\times W\setminus(0,0)$ such that

$$|a(v_n,w_n)|>n\ \|v_n\|\ \|w_n\|.$$

Define

$$\hat{v}_n:=\frac{v_n}{\sqrt{n}\|v_n\|}, \qquad \hat{w}_n:=\frac{w_n}{\sqrt{n}\|w_n\|}.$$

Then $\lim_{n\to\infty}(\hat{v}_n,\hat{w}_n)=(0,0)$, but $|a(\hat{v}_n,\hat{w}_n)|>1$ for all $n\in\mathbb{N}$. Thus a is not continuous at $(0,0)$. This is a contradiction. Thus a constant $C\geq 0$ does exist s.th. inequality (1.47) is satisfied.
"$\Leftarrow$": Assume that (1.47) is valid and let $v_1,v_2\in V,\ w_1,w_2\in W$. Then we get very similarly to the proof of Lemma 1.67:

$$
\begin{aligned}
|a(v_1,w_1) &- a(v_2,w_2)|\leq|a(v_1,w_1)-a(v_2,w_1)|+|a(v_2,w_1)-a(v_2,w_2)| \\
&= |a(v_1-v_2,w_1)|+|a(v_2,w_1-w_2)| \\
&\leq C(||v_1-v_2||_V\ ||w_1||_W+||v_2||_V\ ||w_1-w_2||_W) \qquad \text{from (1.47)} \\
&\leq \tilde{C}\ (||v_1-v_2||_V+||w_1-w_2||_W), \quad \tilde{C}:=C\max\{||v_2||_V,||w_1||_W\}\geq 0, \\
&= \tilde{C}\ ||(v_1-v_2,w_1-w_2)||_{V\times W}, \quad \text{definition (1.46)}, \\
&= \tilde{C}\ ||(v_1,w_1)-(v_2,w_2)||_{V\times W} \quad \text{addition in } V\times W.
\end{aligned}
$$

Thus a is continuous. $\diamond$

Remark 1.69. *Consider the special case* $W = V$, *a an inner product in* V, $||\cdot||_V$ *its natural norm. Then (1.47) follows from the Schwarz inequality (1.13). Inner products thus are continuous maps from* $(V \times V, ||\cdot||_{V\times V})$ *to* $\mathbb{R}$ *if* $||\cdot||_{V\times V}$ *is equivalent to (1.46) or (1.44).*

Chapter 2

Boundary Value Problems (BVPs)

2.1 Classical and Weak Solutions

The first two parts of this section are mostly heuristic. Their emphasis is on giving motivations for the precise definitions to be introduced in the third part.

2.1.1 Classical and weak derivatives

We have seen earlier: there was a need to generalize the classical Riemann integral to the Lebesgue integral. Also, functions which are elements of C^k-spaces, $k \geq 0$, are Riemann integrable. $|\int f^2| < \infty$ is a desirable property, but not an adequate requirement for C^k-spaces (see for instance Examples 1.59). It is an adequate requirement for L^p-spaces, $p \geq 1$. L^p-functions are not required to be continuous in every point of the domain of definition, continuity a.e. is sufficient. Thus we cannot expect them to be differentiable everywhere, but only a.e. in the best case.

Example 2.1. *Consider $f : [-1,1] \to \mathbb{R},\ x \mapsto f(x) = |x|$. Then $f \in C^0$. However $\tilde{f} : [-1,0] \to \mathbb{R},\ x \mapsto \tilde{f}(x) = |x|$ and $\hat{f} : [0,1] \to \mathbb{R},\ x \mapsto \hat{f}(x) = |x|$ satisfy $\tilde{f}, \hat{f} \in C^\infty$. f itself is piecewise C^∞, but $f \notin C^k,\ k > 0$. This is a very abrupt change of properties. A milder change of properties is desirable. It is desirable to define*

$$f'(x) := \begin{cases} \tilde{f}'(x) & \text{for } -1 \leq x < 0 \\ \hat{f}'(x) & \text{for } \ \ 0 \leq x \leq 1 \end{cases} \tag{2.1}$$

and then call f' a weak derivative of f in $[-1,1]$.

As we will see later in this section, weak derivatives are defined such that they coincide with a classical derivative whenever a classical derivative exists. A function $g \in C^m([a,b],\mathbb{R})$ has m classical derivatives, and $g^{(m)} \in C^0([a,b],\mathbb{R})$. A function q in the Sobolev[1] space $H^m([a,b],\mathbb{R})$ has m weak derivatives, and the mth weak derivative is an element of $L^2([a,b],\mathbb{R})$. The relationship between classical and weak derivatives is further

[1] Sergej Sobolev (* 1908 in St. Petersburg, + 1989 in Leningrad = St. Petersburg) worked mostly in Moscow and Novosibirsk

characterized by so-called *'embedding theorems'* [Red86, p.168f].These give conditions under which C^k-spaces can be embedded into L^p-spaces, or Sobolev spaces can be embedded into C^k-spaces.
See the definition of 'embedding' given in the Appendix, Definition A.13. Accordingly we define here:

Definition 2.2. *1. Let V, W be linear spaces. A linear map $j : V \to W,\ v \mapsto j(v) = w$ is called an* injection *or an* embedding *iff $j(v_1) = j(v_2)$ only for $v_1 = v_2$.*
2. Let B_1, B_2 be Banach spaces. A bounded linear map $l : B_1 \to B_2,\ v \mapsto l(v) = w$ is called a continuous embedding *iff $l(v_1) = l(v_2)$ only for $v_1 = v_2$.*

Remember Lemma 1.56: linear maps between normed spaces are continuous iff they are bounded.

Examples 2.3.
** the identity map $i : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto x$ is an embedding;*

$$* \ j : \mathbb{R}^3 \to \mathbb{R}^n,\ n > 3,\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ is an embedding, but}$$

$$* \ p : \mathbb{R}^n \to \mathbb{R}^3,\ n > 3,\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ is not an embedding. Actually, it is a projection.}$$

Part of the structure of $\mathbb{R}^n$ is conserved under the map p, but part of its structure gets lost.

2.1.2 Why do we need weak solutions of differential equations?

Definition 2.4 (classical solution)**.** *Given a differential equation (DE) on $\Omega \subset \mathbb{R}^n,\ \ \Omega$ an open set. A function u defined on $\bar{\Omega}$ is called* classical solution *of this DE iff u satisfies the DE and all derivatives of u occurring in the DE are continuous in every $x \in \Omega$.*

Let us look at some examples from mathematics, physics and civil engineering:

- Given

$$\begin{aligned} u'' &= f \text{ in }]a, b[\subset \mathbb{R}, \\ u(a) &= u(b) = 0. \end{aligned} \tag{2.2}$$

  If $u \in C([a, b], \mathbb{R}) \cap C^2(]a, b[, \mathbb{R})$ and u satisfies (2.2) then u is a classical solution of (2.2).

- Let $R \in \mathbb{R},\ R > 0$.

$$\begin{aligned} \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= f \text{ in } \Omega_R := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}, \qquad (2.3) \\ u &= g \text{ on } \partial\Omega_R := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\} \end{aligned}$$

  Suppose that $u \in C(\bar{\Omega}_R, \mathbb{R}) \cap C^2(\Omega_R, \mathbb{R}),\ \bar{\Omega}_R := \Omega_R \cup \partial\Omega_R$, solves the DE. Then u is a classical solution.

- Given $H \in \mathbb{R}$, $H > 0$, let $q \in C^1([0,H], \mathbb{R})$ and $c_v \in C(\bar{D}, \mathbb{R})$, with
  $D := \{(z,t) \in \mathbb{R}^2 : 0 < z < H,\ \ 0 < t < \infty\}$ and
  $\bar{D} := \{(z,t) \in \mathbb{R}^2 : 0 \le z \le H,\ \ 0 \le t < \infty\}$ and

$$\begin{aligned} \frac{\partial u}{\partial t} &= c_v \frac{\partial^2 u}{\partial z^2} \qquad \text{in } D, \qquad (2.4) \\ u(z,0) &= q \qquad \text{on } 0 \le z \le H, \quad \textit{with} \quad \frac{dq}{dz}(0) = 0 \quad \textit{and} \quad q(H) = 0, \end{aligned}$$

$$\frac{\partial u}{\partial z}(0,t) = 0, \qquad u(H,t) = 0 \quad \text{for} \quad 0 \le t < \infty.$$

  Suppose that u solves eqs. (2.4) and also that u is continuously differentiable w.r.t. t for $t \in (0,\infty)$ and two times continuously differentiable w.r.t. x in $(0,H)$, and that $u \in C(\bar{D}, \mathbb{R})$. Then u is a classical solution.

  Let us have a closer look at eq. (2.4). With smooth data (as assumed above) it is the spatially-one-dimensional version of a mathematical model for many smooth processes in nature and/or engineering. In mathematics and physics it is called **heat equation** or **diffusion equation** [MRT05].
  As *heat equation*, it models the evolution of temperature u in a thin rod of length H under the influence of heat conduction, when the initial distribution of temperature along the rod and the temperature or heat flux at the end points of the rod are known/prescribed. Under many circumstances, heat conduction is a smooth process, and thus it is adequate to describe it by classical solutions.
  As *diffusion equation*, eq.(2.4) models the diffusion of some chemical with concentration u in time. Also diffusion is a process of usually smooth nature.
  In soil mechanics, eq. (2.4) describes the vertical consolidation of a soil layer between an impermeable base and a freely draining upper surface [BR87, Chapter 4: Consolidation Theory and Settlement Analysis], [Li07]. The soil layer consists of soil grains and porewater, both incompressible. It has thickness H in the vertical z-direction. The upper surface is loaded by some heavy structure (a building, a bridge, a dam, ...), and this drives excess porewater out of the soil layer and makes the heavy structure sink and hopefully finally settle. In the horizontal x,y-directions, the configuration is assumed to be infinitely extended, homogeneous and in steady state. With u the excess porewater pressure, eq. (2.4) is the governing one-dimensional **consolidation equation**, derived by Terzaghi[2] in 1923 [BR87, p.104]. Consolidation is a smooth process, and thus adequately described by classical solutions.

[2]Karl von Terzaghi(*1883 in Prague, 1963 in Winchester, Mass., USA), studied mechanical engineer-

Now we look at several problems which cannot be solved in a satisfactory way within the framework of C^k-solutions:

- Let $\Omega \subset \mathbb{R}^2$ be connected and open. Consider a BVP

$$\begin{aligned} \Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{2.5}$$

  What are necessary and sufficient conditions on the data of the DE (2.5) such that it has a classical solution?
  necessary: suppose $u \in C^2(\bar{\Omega}, \mathbb{R})$ solves (2.5); then $f \in C(\bar{\Omega}, \mathbb{R})$ and $g \in C^2(\partial\Omega, \mathbb{R})$, if the boundary $\partial\Omega$ is smooth (i.e. without singularities like corners or cusps: see Definition A.17 (Smoothness of boundaries) in the Appendix).
  Let us assume that the boundary $\partial\Omega$ is smooth. Then the necessary conditions are *not sufficient* for the existence of a classical solution. A counterexample is given in [Zeid93, Problem 6.3, p.249f].
  There are two ways of finding both necessary and sufficient conditions on the data $\Omega,\ f,\ g$ of the DE (2.5): require more smoothness of the data/solutions (i.e. Hölder continuity, see [Zeid93, pp. 230 - 236]), or require less smoothness (i.e. consider weak solutions which solve the DE almost everywhere and are elements of Sobolev spaces and L^p spaces). There are good reasons for each of both ways, and both ways have been persued successfully in the past. Today, considering weak solutions is more popular, because it is the more general concept. If a DE has a classical solution, then its weak solution is equal to the classical solution.

- It is desirable to consider also equations (2.5) in domains with only piecewise smooth or even non-smooth boundaries $\partial\Omega$, for instance domains with corners. In such cases the existence theory using Hölder continuity is not applicable. The following example shows that less smoothness of the data of the DE *might* lead to less smoothness of the solutions:

  Examples 2.5 (Bratu problem)**.** [3]
  ** Consider*

$$-\Delta u = 2e^u \qquad \textit{in} \;\; \Omega_1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \tag{2.6}$$

ing, civil engineering and geology in Graz, k.u.k-Austria; worked with a Vienna engineering company on the construction of some of the first hydroelectric power plants in several European countries; Ph.D. in 1911. In 1912-13 he took off to study major dam construction sites in the West of USA. After leading engineering battalions of the Austrian Army in WW1, he moved from university to university and developed there the new discipline of soil mechanics (Prof. 1916 in Istanbul, 1925 at MIT in USA, 1929 - 38 in Vienna, from 1938 on at Harvard University, USA), and also he was a consultant for many large-scale projects, for instance a military airport in Vienna-Aspern (together with Theodor von Karman and Richard von Mises), the Chicago subway and the Aswan High Dam in Egypt. Questions about how to construct the bed for the gigantic Nazi buildings in Nuremberg led to discussions with Adolf Hitler in 1935. Emigration to USA in 1938; US-citizenship in 1943 [Wiki, Terzaghi; German, English, 2012].

[3] Emilian A. Bratu (*1904 in Bucharest, +1991 in Bucharest), studied industrial chemistry at polytechnical schools/institutes of technology in Bucharest, Vienna, Karlsruhe and Berlin-Charlottenburg (1923 - 1930), dr. ing. in Vienna 1936; Professor of Industrial Chemistry/Chemical Engineering (1948 - 1974) at the polytechnic institute of Bucharest; member of the Romanian Academy, see *Revue Roumaine de Chimie* **39** (1994), 1239 - 1239 ; *Revista de Chimie* **45** (1994), 643 - 651 and `www.library.pub.ro/doc/bratu.doc`

$$u = 0 \qquad on\ \partial\Omega_1.$$

The solution of eq. (2.6) is known explicitly [Ban80]. It is in $C^\infty(\bar{\Omega}_1, \mathbb{R})$, radially symmetric and given by

$$U(x,y) := \ln \frac{4}{(1+x^2+y^2)^2}. \tag{2.7}$$

** Now consider the same DE in the square $\Omega_2 \subset \Omega_1$ touching $\partial\Omega_1$, and impose as boundary values the values of (2.7) on $\partial\Omega_2$:*

$$\begin{aligned} -\Delta u &= 2e^u \qquad in\ \ \Omega_2 := (-s,s)\times(-s,s),\ \ s := \frac{1}{\sqrt{2}}, \\ u &= U \qquad on\ \ \partial\Omega_2. \end{aligned} \tag{2.8}$$

This solution is also explicitly known: $u(x,y) = U(x,y)$ on $\bar{\Omega}_2$ and $u \in C^\infty(\bar{\Omega}_2, \mathbb{R})$.
** Now consider the same DE in the same square $\Omega_2 \subset \Omega_1$ touching $\partial\Omega_1$, but impose vanishing boundary values on $\partial\Omega_2$:*

$$\begin{aligned} -\Delta u &= 2e^u \qquad in\ \ \Omega_2 = (-s,s)\times(-s,s),\ \ s = \frac{1}{\sqrt{2}}, \\ u &= 0 \qquad on\ \ \partial\Omega_2. \end{aligned} \tag{2.9}$$

Problem (2.9) still has a classical solution, but with minimal smoothness. We get $u \in C^2(\Omega_2,\mathbb{R}) \cap C(\bar{\Omega}_2,\mathbb{R})$. Suppose $u \in C^2(\bar{\Omega}_2,\mathbb{R})$, smooth up to the boundary. Then u satisfies the DE also on the boundary, i.e. $\Delta u = 2 = u_{,xx} + u_{,yy}$ on $\partial\Omega_2$. On $Q_{x,-s} := \{(x,y) \in \mathbb{R}^2 : x \in (-s,s),\ y = -s\}$ we get $u_{,yy} = 0$ and thus $u_{,xx} = 2$. On $Q_{-s,y} := \{(x,y) \in \mathbb{R}^2 : y \in (-s,s),\ x = -s\}$ we get $u_{,xx} = 0$ and thus $u_{,yy} = 2$. Thus both second derivatives of u jump at the corner where $Q_{-s,y}$ and $Q_{x,-s}$ meet. Thus $u \notin C^2(\bar{\Omega}_2,\mathbb{R})$.

- When we compute classical solutions by finite element methods (FEMs), we often use basis functions which do not have classical derivatives. In such cases it is adequate to first show convergence of the numerical approximation to a weak solution of the original equation, and then maybe use an embedding theorem to conclude that the weak solution is actually a classical solution.

- There is an old saying ‘natura non saltat’, nature does not jump. Nevertheless, in many cases it is adequate and convenient to model certain physics phenomena by functions which are only piecewise continuous. Even if a more detailed and more advanced mathematical model with continuous functions is known: it may involve several time scales and/or several spatial scales. An example are **shock waves** evolving, for instance, if a body moves in air (i.e. in a compressible fluid) with a speed larger than the speed of sound. In this case the density of the air changes so fast in a very small domain (‘the shock’) that it does make sense to model the density by a function that jumps at the locus of the shock. If only the evolution of the shock is of interest, but not the details inside the shock, then both the numerical and theoretical treatment of shocks are simplified by considering weak solutions of the wave equation. Actually, Sobolev generalized ‘solutions of the wave equation’ by introducing weak solutions in 1934.

- An example from Civil Engineering is the so-called **water hammer**: in a badly designed pipe system or pipeline a pressure surge may not only cause a hammering noise, but dammage a pipe or even make it explode/implode when a valve is abruptly opened or closed [Stre71, Chapter 12: unsteady flows in closed conduits], [Li07]. Consider a simplified mathematical model for the water-hammer phenomenon:
  let x be the distance along the pipe measured from the upstream end, t the time, H the elevation of hydraulic grade line above a fixed datum, and V the average velocity at a cross section. Then $H(x,t)$ and $V(x,t)$ have to satisfy the following hyperbolic system [Stre71, p.651 - 655]:

$$\begin{aligned} g\frac{\partial H}{\partial x} + V\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} + \frac{fV|V|}{2D} &= 0, \\ \frac{a^2}{g}\frac{\partial V}{\partial x} + V\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t} + V\sin\theta &= 0. \end{aligned} \tag{2.10}$$

  This system develops shock waves of pressure, no matter how smooth its coefficients are. It thus has weak solutions, and it is important to compute these weak solutions in order to avoid them in practice.

These are some examples from mathematics, physics and engineering why we need weak solutions. It would be easy to give many more examples.

2.1.3 Weak Derivatives and Sobolev Spaces

Definition 2.6 (weak derivative). *Let $f \in L^2([a,b],\mathbb{R})$. We say that $g \in L^2([a,b],\mathbb{R})$ is a* weak derivative *of f, iff*

$$\int_a^b g(x)v(x)dx + \int_a^b f(x)v'(x)dx = 0 \qquad \text{for all } v \in C_0^1([a,b],\mathbb{R}), \tag{2.11}$$

with

$$C_0^1([a,b],\mathbb{R}) := \{v \in C^1([a,b],\mathbb{R}) : v(a) = v(b) = 0\}. \tag{2.12}$$

Remark 2.7. *$C_0^1([a,b],\mathbb{R})$ is a linear space and thus a subspace of $C^1([a,b],\mathbb{R})$. Under the C^1-norm it is complete and thus a Banach space. The proof is left to the reader.*

Remark 2.8. *We use the terms* classical – weak *here, other authors prefer* strong – weak *or* derivative – generalized derivative*. This is a matter of taste.*

Lemma 2.9. *Suppose $f \in L^2([a,b],\mathbb{R})$ has a continuous classical derivative f'. Then f' is a weak derivative of f.*

Proof: Let $v \in C_0^1([a,b],\mathbb{R})$ be arbitrary. Then partial integration gives

$$\int_a^b f'(x)v(x) = f(x)v(x)|_a^b - \int_a^b f(x)v'(x)dx. \tag{2.13}$$

Since f is continuously differentiable in the closed interval $[a,b]$, we get $|f(a)| < \infty$ and $|f(b)| < \infty$, and we conclude $f(x)v(x)|_a^b = 0$ for all $v \in C_0^1([a,b],\mathbb{R})$. Thus

$$\int_a^b f'(x)v(x)dx + \int_a^b f(x)v'(x)dx = 0 \qquad \text{for all } v \in C_0^1([a,b],\mathbb{R}), \tag{2.14}$$

and this is eq. (2.11) with $g = f'$. $\diamond$

Remark 2.10. *Since there will be no confusion, we will denote by f' both weak and classical first derivatives of f.*

Example 2.11. *Let $f : [-1,1] \to \mathbb{R},\ x \mapsto |x|$. Then $g : [-1,1] \to \mathbb{R}$,*

$$g(x) := \begin{cases} -1 & \textit{for } \ -1 \le x < 0 \\ 1 & \textit{for } \ \ 0 \le x \le 1 \end{cases} \tag{2.15}$$

is a weak derivative of f, because

$$\begin{aligned}
\int_{-1}^1 g(x)v(x)dx &= \int_{-1}^0 g(x)v(x)dx + \int_0^1 g(x)v(x)dx \\
&= \int_{-1}^0 -v(x)dx + \int_0^1 v(x)dx \\
&= \int_{-1}^0 f'(x)v(x)dx + \int_0^1 f'(x)v(x)dx \\
&= f(x)v(x)|_{-1}^0 - \int_{-1}^0 f(x)v'(x)dx + f(x)v(x)|_0^1 - \int_0^1 f(x)v'(x)dx \\
&= f(0)v(0) - 0 - \int_{-1}^0 f(x)v'(x)dx + 0 - f(0)v(0) - \int_0^1 f(x)v'(x)dx \\
&= -\int_{-1}^1 f(x)v'(x)dx \qquad \textit{for all } v \in C_0^1([a,b],\mathbb{R}).
\end{aligned}$$

We thus verified what was mentioned as desirable in Example 2.1.

Remark 2.12. *By repeated application of Def. 2.6 we obtain the* second weak derivative f'', third weak derivative f''', $\cdots$, *m-th weak derivative $f^{(m)}$.*

Definition 2.13. *Let $[a,b] \subset \mathbb{R}$ and let $f^{(\mu)}$ denote a weak derivative of f of order μ.*

$$H^m([a,b],\mathbb{R}) := \{f \in L^2([a,b],\mathbb{R}) : f^{(\mu)} \in L^2([a,b],\mathbb{R}),\ \mu = 1,\cdots,m\} \tag{2.16}$$

is called Sobolev space of order m .

Remark 2.14. *We have seen that classical derivatives are also weak derivatives, Lemma 2.9. On the other hand we have seen that not all weak derivatives are classical derivatives, Example 2.11. Thus $C^1([a,b],\mathbb{R}) \subset H^1([a,b],\mathbb{R})$ and $C^1([a,b],\mathbb{R}) \neq H^1([a,b],\mathbb{R})$.*

Theorem 2.15. *The Sobolev spaces $H^m([a,b],\mathbb{R})$, $m \in \mathbb{N}$, are Hilbert spaces with the inner product*

$$< u, v >_{H^m} := \int_a^b \sum_{j=0}^m u^{(j)}(x) v^{(j)}(x) dx \qquad (2.17)$$

and corresponding norm $||u||_{H^m} := \sqrt{< u, u >_{H^m}}$.

Proof: see [Red86, p.166]. $\diamond$

Remark 2.16. *In Remark 1.52 the general formula for C^m-norms was given. Note that the H^m-norms add up differently. When we exchange finite sum and integration in eq. (2.17) and rewrite it for the norms, we get*

$$||u||_{H^m}^2 = \sum_{j=0}^m ||u^{(j)}(x)||_{L^2}^2, \qquad u \in H^m.$$

For $m = 1$ we thus get

$$||f||_{C^1} = ||f||_\infty + ||f'||_\infty, \qquad ||u||_{H^1}^2 = ||u||_{L^2}^2 + ||u'||_{L^2}^2.$$

In the following we exploit the relationships between C^m-spaces and H^m-spaces a bit further. As we have seen earlier (Problem 15 of the Exercises),

$$(u, v)_m := \int_a^b \sum_{j=0}^m u^{(j)}(x) v^{(j)}(x) dx \qquad (2.18)$$

is an inner product on $C^m([a,b],\mathbb{R})$. This is also still true for more general domains $\Omega \subset \mathbb{R}^n$. For such more general domains, however, $||f||_{H^m} < \infty$ is not always satisfied. We thus define

$$\hat{C}^m(\Omega,\mathbb{R}) := \{f \in C^m(\Omega,\mathbb{R}) : ||f||_{H^m} < \infty\}. \qquad (2.19)$$

$\hat{C}^m(\Omega,\mathbb{R})$ is an inner product space with $\hat{C}^m(\Omega,\mathbb{R}) \subset C^m(\Omega,\mathbb{R})$, $m \in \mathbb{N}$. Is it complete? The answer is *no*, its completion is a Sobolev space (compare Example 1.59):

Theorem 2.17.
* *$L^2(\Omega,\mathbb{R}) =: H^0(\Omega,\mathbb{R})$ is the smallest Banach space V s.th. $\hat{C}(\Omega,\mathbb{R}) \subset V$ and all Cauchy sequences w.r.t. $||\cdot||_{L^2}$ contained in $\hat{C}(\Omega,\mathbb{R}) = \hat{C}^0(\Omega,\mathbb{R})$ converge in V.*
In other words: $L^2(\Omega,\mathbb{R})$ is the completion of $\hat{C}(\Omega,\mathbb{R})$ w.r.t. $||\cdot||_{L^2}$.

* *$H^m(\Omega,\mathbb{R})$, $m \in \mathbb{N}$ is the smallest Banach space V s.th. $\hat{C}^m(\Omega,\mathbb{R}) \subset V$ and all Cauchy sequences w.r.t. $||\cdot||_{H^m}$ contained in $\hat{C}^m(\Omega,\mathbb{R})$ converge in V.*
In other words: $H^m(\Omega,\mathbb{R})$ is the completion of $\hat{C}^m(\Omega,\mathbb{R})$ w.r.t. $||\cdot||_{H^m}$.
* *$H^m(\Omega,\mathbb{R})$ is the completion of $\hat{C}^\infty(\Omega,\mathbb{R}) := \{f \in C^\infty(\Omega,\mathbb{R}) : ||f||_{H^m} < \infty\}$ w.r.t. $||\cdot||_{H^m}$ as well.*

Proof: see [Red86, p.166]. ◇

When we solve BVPs of mth order on $[a, b]$, we usually search for solutions in one of the spaces $C_0^m([a, b], \mathbb{R}) \subset H_0^m([a, b], \mathbb{R}) \subset H^m([a, b], \mathbb{R})$, with $H_0^m([a, b], \mathbb{R})$ the completion of $C_0^m([a, b], \mathbb{R})$ under the norm $||\cdot||_{H^m}$. Here the spaces C_0^m are defined analoguously to (2.12):

$$C_0^m([a, b], \mathbb{R}) := \{v \in C^m([a, b], \mathbb{R}) : v(a) = v(b) = 0\}, \quad m \geq 1. \tag{2.20}$$

Examples 2.18.

$$-u'' = f \quad \text{in } (0, 1), \qquad u(0) = \alpha,\ u(1) = \beta. \tag{2.21}$$

** Suppose $\alpha = 0$; $\beta = 0$. Then solutions will be in $C_0^2([0, 1], \mathbb{R})$ if $f \in C([0, 1], \mathbb{R})$; solutions will be in $H_0^2([0, 1], \mathbb{R})$ if $f \in L^2([0, 1], \mathbb{R})$.*
** Suppose $\alpha^2 + \beta^2 \neq 0$. Then a different approach is necessary because the set*

$$\{f \in C^2([0, 1], \mathbb{R}) : f(0) = \alpha,\ f(1) = \beta\}$$

is not a linear space. One possibility is to define $v(x) := u(x) - \alpha(1 - x) - \beta x$. Then v is a solution of

$$-v'' = f \quad \text{in } (0, 1), \qquad v(0) = 0,\ v(1) = 0.$$

When v is known, it is easy to get u from v.
** For quasilinear DEs like $-u'' = f(u)$ or nonlinear DEs like $g(u, u')u'' = f(u)$ or $g(u, u', u'') = 0$ the transformation changes the differential equation, but in principle the transformation can be done same way.*

Thus it is no loss of generality to impose vanishing boundary conditions in theoretical considerations, though it is inconvenient in practice.

2.2 Variational Problems

We now study the relation between variational problems and quadratic minimization problems.

Theorem 2.19. *Let V be a linear space, $a : V \times V \to \mathbb{R}$ a bilinear form, symmetric, positive definite (i.e. a is an inner product on V) and continuously differentiable, and assume $L : V \to \mathbb{R},\ v \mapsto Lv$ is a continuously differentiable linear form and $F : V \to \mathbb{R}$ is given by*

$$F(v) := \frac{1}{2}a(v, v) - Lv.$$

Then $u \in V$ minimizes $F(\cdot)$, $F(u) = \min_{v \in V} F(v)$, iff $u \in V$ solves the variational problem

$$a(u, v) = Lv \qquad \text{for all } v \in V. \tag{2.22}$$

Proof: '⇒' Let $t \in \mathbb{R},\ v \in V$ and consider

$$F(u+tv) = F(u) + t(a(u,v) - Lv) + \frac{1}{2}t^2 a(v,v).$$

Thus

$$\frac{F(u+tv) - F(u)}{t} = (a(u,v) - Lv) + \frac{1}{2}ta(v,v).$$

Since u minimizes $F(\cdot)$ and F is continuously differentiable, we get

$$\lim_{t\to 0} \frac{F(u+tv) - F(u)}{t} = 0 = a(u,v) - Lv \qquad \text{for all } v \in V.$$

Note: F is continuously differentiable because a and L have this property. Because F is continuously differentiable, all partial derivatives of F are continuous at u, and the directional derivative of F at u in the direction of v vanishes at u for all directions v, see [VG81, Chap. 7, Section A] or [Rud00, AnOb, MVa].
'⇐' Assume

$$a(u,v) - Lv = 0 \qquad \text{for all } v \in V.$$

For $t = 1$ we get

$$F(u+v) = F(u) + \frac{1}{2}a(v,v) > F(u) \text{ for } v \neq 0$$

since a is positive definite. Thus u minimizes F. ◇
Does such a u exist? Yes, under the assumptions of the following theorem.

Theorem 2.20 (Riesz Representation Theorem)**.** [4]
Let V be a Hilbert space with inner product $<\cdot,\cdot>$ and $L : V \to \mathbb{R}$ a continuous linear form. Then there exists a unique $u \in V$ s.th.

$$Lv = <u,v> \qquad \text{for all } v \in V. \tag{2.23}$$

Proof: [Red86, p.112].

Example 2.21. *$V := L^2([0,1],\mathbb{R}),\ L : V \to \mathbb{R},\ Lv := \int_0^{\frac{1}{2}} v(x)dx$.*
$Lv = <u,v>$ for all $v \in V \Rightarrow \int_0^{\frac{1}{2}} v(x)dx = \int_0^1 u(x)v(x)dx \ \Rightarrow$

$$u(x) := \begin{cases} 1 & \text{for } \ 0 \le x \le \frac{1}{2} \\ 0 & \text{for } \ \frac{1}{2} < x \le 1 \end{cases}.$$

The Riesz representation theorem can be generalized to bilinear forms with apropriate properties:

Theorem 2.22 (Lax-Milgram Lemma)**.** [5] *Let V be a Hilbert space with natural norm $||\cdot||$, $L : V \to \mathbb{R}$ a continuous linear form, $a : V \times V \to \mathbb{R}$ a bilinear form satisfying*

$$c\ ||v||^2 \le a(v,v) \quad \text{for all } v \in V \text{ and some } c > 0, \tag{2.24}$$

[4] Frigyes (=Fritz) Riesz (* 1880 in Györ, + 1956 in Budapest) studied in Budapest, Göttingen and Zürich, taught in Hungary; one of the founders of functional analysis.

[5] Peter Lax (* 1926 in Budapest), Ph.D. 1949 at New York U, Prof. of math at NYU, Abel prize in 2005; - Arthur Norton Milgram (* 1912 in Philadelphia, + 1961), Ph.D. 1937 at U Pennsylvania, [Wiki, Oct. 2012]

and

$$a(u,v) \leq K||u||\ ||v|| \quad \text{for all } u, v \in V \text{ and some } K > 0. \tag{2.25}$$

Then there exists a unique $u \in V$ s.th.

$$a(u,v) = Lv \quad \text{for all } v \in V.$$

Remark 2.23. *Consider the case $a(u,v) =< u, v >$, where $< \cdot, \cdot >$ is the inner product of V. Then (2.24) is trivally satisfied with $c = 1$, and (2.25) follows from the Schwarz inequality with $K = 1$. Thus the Lax-Milgram Lemma reduces to the Riesz Representation Theorem.*

Definition 2.24. *Let V be an inner product space with inner product $< \cdot, \cdot >$ and related norm $|| \cdot ||$. A bilinear form $a : V \times V \to \mathbb{R}$ is called*
* coercive *iff there is a constant $c > 0$ such that*

$$|a(v,v)| \geq c < v, v > \qquad \text{for all } v \in V; \tag{2.26}$$

* continuous[6] w.r.t. $|| \cdot ||$ *iff there is a constant $K > 0$ such that*

$$|a(u,v)| \leq K||u||\ ||v|| \qquad \text{for all } u, v \in V. \tag{2.27}$$

Remark 2.25. *Let V be an inner product space with inner product $< \cdot, \cdot >$ and related norm $|| \cdot ||$, and let $a(\cdot, \cdot)$ be another inner product on V with related norm $|| \cdot ||_a$. If a is coercive and continuous w.r.t. $|| \cdot ||$, then*

$$\alpha||v||^2 \leq ||v||_a^2 \leq K||v||^2 \qquad \text{for some } \alpha > 0,\ K > 0, \quad \text{and for all } v \in V,$$

i.e. the two norms are equivalent.

A closely related theorem is the following

Theorem 2.26. *Let V be a Hilbert space and $L : V \to \mathbb{R}$ be a bounded linear form, i.e. $|Lv| \leq C||v||$ for all $v \in V$ and some $C \geq 0$; assume $a : V \times V \to \mathbb{R}$ is a symmetric positive definite bilinear form satisfying*

$$a(v,v) \geq \alpha < v, v > \qquad \text{for all } v \in V \text{ and some } \alpha > 0,$$

and

$$|a(u,v)| \leq K||u||\ ||v|| \qquad \text{for all } u, v \in V \text{ and some } K > 0.$$

Then there exists a unique $u \in V$ s.th.

$$a(u,v) = Lv \qquad \text{for all } v \in V.$$

Moreover, the following estimate is valid:

$$||u|| \leq \frac{2C}{\alpha}. \tag{2.28}$$

[6]remember Lemma 1.68 and Remark 1.69

Proof: By assumption, a defines an inner product on V. From Remark 2.25 it follows that $||\cdot||$ and $||\cdot||_a$ are equivalent. Thus sequences converge in $||\cdot||$ iff they converge in $||\cdot||_a$. Thus V is a Hilbert space w.r.t. $||\cdot||_a$ and L is continuous w.r.t. $||\cdot||_a$ (Lemma 1.56). Now apply the Riesz representation theorem for the inner product a. It follows that there is exactly one $u \in V$ with $a(u,v) = Lv$ for all $v \in V$.
To apply Theorem 2.19 for estimating u we note that continuous linear maps are continuously differentiable [Co68, Chap.2, 16.1, 16.2]. From Theorem 2.19 we now conclude that u minimizes $F(v) = \frac{1}{2}a(v,v) - Lv$. Thus $F(u) \leq F(0) = 0$ and

$$F(u) = \frac{1}{2}a(u,u) - Lu \geq \frac{1}{2}\alpha||u||^2 - C||u||.$$

If $u = 0$ then estimate (2.28) is trivially satisfied. So now assume $u \neq 0$. Then we obtain

$$\begin{aligned}
\frac{1}{2}\alpha||u||^2 - C||u|| &\leq 0 \\
\frac{1}{2}\alpha||u||^2 &\leq C||u||, \\
||u|| &\leq \frac{2C}{\alpha}.
\end{aligned}$$

Thus Theorem 2.26 is proved. ◇

2.3 Existence of Solutions

In this section we will apply Theorem 2.26 to show existence of solutions of boundary value problems. We will prove the following theorem:

Theorem 2.27. *Assume $f \in L^2([0,1],\mathbb{R})$ and $\lambda \in \mathbb{R},\ \lambda > -8$. Then*

$$\begin{aligned}
-u''(x) + \lambda u(x) &= f(x) \qquad \text{in } (0,1), \qquad (2.29)\\
u(0) = u(1) &= 0,
\end{aligned}$$

has a unique solution $u \in H_0^1([0,1],\mathbb{R})$.

Remark 2.28. *Assume $f = 0$. Then there is the unique solution $u(x) \equiv 0$ for all $\lambda \neq \lambda_k = -k^2\pi^2,\ k \in \mathbb{N}$, but for $\lambda = \lambda_k = -k^2\pi^2,\ k \in \mathbb{N}$, solutions are not unique. Note that the bound -8 in the theorem satisfies $-8 > \lambda_1 = -\pi^2$ and is related to the interval $[0,1]$.*

Proof of Remark: $u(x) \equiv 0$ solves eq. (2.29) with $f = 0$ for all λ. Are there other solutions? Equation (2.29) with $f = 0$ is a linear second order equation with constant coefficients. Thus it has two linearly independent fundamental solutions.Thus the general solution can be written as $w(x) = \alpha \sin \mu x + \beta \cos \nu x$. But $\beta \cos \nu x$ satisfies the boundary conditions only for $\beta = 0$. So let $v(x) := \alpha \sin \mu x,\ \alpha, \mu \in \mathbb{R}$. Then $v''(x) = -\alpha\mu^2 \sin \mu x = -\mu^2 v(x)$, and ($v(0) = 0 = v(1)$ iff $\mu^2 = k^2\pi^2,\ k \in \mathbb{N}$). Thus $v_k(x) := \alpha \sin k\pi x$ solves eq. (2.29) with $f = 0$ for arbitrary $\alpha \in \mathbb{R}$ and $\lambda_k := -k^2\pi^2,\ k \in \mathbb{N}$. This proves that solutions are unique for $\lambda \neq \lambda_k$ and not unique for $\lambda = \lambda_k$. ◇

$\lambda > -8$ in the theorem is an approximation to $\lambda > -\pi^2$ that has been chosen to simplify the proof of the theorem.
For proving Theorem 2.27 we need

Lemma 2.29 (Poincaré - Friedrichs). [7]
1. General case: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth enough boundary, i.e. with a Lipschitzian boundary[8], and let $v \in H_0^1(\Omega, \mathbb{R})$. Then there is a constant $K = K(\Omega) > 0$ s.th.

$$\int_\Omega v^2 dx \leq K \int_\Omega \nabla v \cdot \nabla v \quad dx. \tag{2.30}$$

2. special case: Let $\Omega = [0,1] \subset \mathbb{R}$, and let $v \in H_0^1([0,1], \mathbb{R})$. Then

$$\int_0^1 v^2 dx \leq \frac{1}{8} \int_0^1 (v'(x))^2 \quad dx. \tag{2.31}$$

Proof: see [Red86, p.224f and p.239, exercise 36.1] for the general case. ◇

Proof of Theorem 2.27: Multiply the differential equation

$$-u''(x) + \lambda u(x) = f(x)$$

by arbitrary $v \in C_0^1([0,1], \mathbb{R})$ and integrate to get

$$\int_0^1 u'(x)v'(x)dx + \lambda \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{2.32}$$

Now the proof is done in the following steps:
1. Define

$$a(w,v) := \int_0^1 w'(x)v'(x)dx + \lambda \int_0^1 w(x)v(x)dx \tag{2.33}$$

and show that $a(w,v)$ is a symmetric, positive definite bilinear form, i.e. an inner product, which is continuous and coercive for $\lambda > -8$, i.e. which satisfies for $\lambda > -8$

$$a(v,v) \geq \alpha < v,v > \qquad \text{for some } \alpha > 0 \text{ and all } v \in H_0^1([0,1], \mathbb{R}), \tag{2.34}$$

$$a(u,v) \leq K||u||_{H^1}||v||_{H^1} \qquad \text{for some } K > 0 \text{ and all } u,v \in H_0^1([0,1], \mathbb{R}). \tag{2.35}$$

2. Define the linear form $L : H_0^1([0,1], \mathbb{R}) \to \mathbb{R}$,

$$Lv := \int_0^1 f(x)v(x)dx \tag{2.36}$$

and show that it is continuous.
3. Use that $H_0^1([0,1], \mathbb{R})$ is a Hilbert space (subsection 2.1.3) and apply Theorem 2.26 to

[7] Jules-Henri Poincaré (* 1854 in Nancy, + 1912 in Paris), engineer, mathematician, physicist, astronomer, philosopher. – Kurt-Otto Friedrichs (* 1901 in Kiel, + 1982 in New Rochelle, USA) mathematician; worked mostly at U Göttingen and New York U, emigrated 1937 with his jewish wife and to continue cooperation with his Göttingen colleagues

[8] see Definition A.17

conclude that equation (2.32) has a unique solution $u \in H_0^1([0,1],\mathbb{R})$. This then completes the proof.
Step 1: Let $v, w \in H_0^1([0,1],\mathbb{R})$. Symmetry and bilinearity of $a(w,v)$ are shown in the same way as in Problem 15. 'Positive definite' will follow from coercivity.
Assume $\lambda \geq 1$. Then $\alpha = 1$ because

$$a(v,v) = \int_0^1 v'(x)^2 dx + \lambda \int_0^1 v(x)^2 dx \geq \int_0^1 v'(x)^2 dx + \int_0^1 v(x)^2 dx =< v, v >_{H^1} .$$

Assume $-8 < \lambda \leq 1$, i.e. $0 < \lambda + 8 \leq 9$. Then using the Poincaré - Friedrichs Lemma we get

$$\begin{aligned}
a(v,v) &= \int_0^1 v'(x)^2 dx + \lambda \int_0^1 v(x)^2 dx \\
&= \left(\frac{8+\lambda}{9}\right) \int_0^1 v'(x)^2 dx + \left(1 - \frac{8+\lambda}{9}\right) \int_0^1 v'(x)^2 dx + \lambda \int_0^1 v(x)^2 dx \\
&\geq \left(\frac{8+\lambda}{9}\right) \int_0^1 v'(x)^2 dx + 8\left(1 - \frac{8+\lambda}{9}\right) \int_0^1 v(x)^2 dx + \lambda \int_0^1 v(x)^2 dx \\
&= \left(\frac{8+\lambda}{9}\right) \int_0^1 v'(x)^2 dx + \left(\frac{8+\lambda}{9}\right) \int_0^1 v(x)^2 dx \\
&= \left(\frac{8+\lambda}{9}\right) < v, v >_{H^1} .
\end{aligned}$$

Thus (2.34) is satisfied for $\lambda > -8$ with $\alpha = (8+\lambda)/9,\ 0 < \alpha \leq 1$. To prove (2.35), consider

$$|a(w,v)| \leq (1+|\lambda|) \left| \int_0^1 w'(x)v'(x)dx + \int_0^1 w(x)v(x)dx \right| \leq (1+|\lambda|)\ ||w||_{H^1}\ ||v||_{H^1}.$$

Step 2: For L in eq. (2.36) we get

$$|Lv| \leq ||f||_{L^2}\ ||v||_{L^2} \leq ||f||_{L^2}\ ||v||_{H^1} \qquad \text{for all } v \in H_0^1([0,1],\mathbb{R}).$$

Step 3: We apply Theorem 2.26 and conclude that there exists a unique solution $u \in H_0^1([0,1],\mathbb{R})$. This solution satisfies

$$||u||_{H^1} \leq 2||f||_{L^2} \qquad \text{for } \lambda \geq 1 \tag{2.37}$$

and

$$||u||_{H^1} \leq \frac{18||f||_{L^2}}{8+\lambda} \qquad \text{for} \quad 1 \geq \lambda > -8. \tag{2.38}$$

We notice that the estimate (2.38) approaches $2||f||_{L^2}$ for $\lambda \to 1$, and $+\infty$ for $\lambda \to -8$ and $f \neq 0$. This singularity is caused by the fact that eq. (2.29) has infinitely many solutions for $\lambda = -\pi^2,\ f = 0$, but no solution at all for $\lambda = -\pi^2,\ f \neq 0$.
The proof is completed. $\diamond$

2.4 Approximation of Solutions

Definition 2.30 (dense subsets). *Let V be a normed space with norm $||\cdot||$.* A subset $X \subset V$ is dense in V *iff for every $v \in V$ and every $\varepsilon > 0$ there is an $x \in X$ such that $||x - v|| < \varepsilon$.*

Remark 2.31. *If X is dense in V, then every $v \in V$ can be approximated by a sequence $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = v$. To find such a sequence, choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and apply Definition 2.30 to find x_n with $||x_n - v|| < \varepsilon_n$.*

Example 2.32. *The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ w.r.t. $|\cdot|$. This is basic for using computers.*

Note that the concepts *dense* and *completion* are closely related:

Theorem 2.33. *$C([a,b],\mathbb{R})$ is dense in $L^2([a,b],\mathbb{R})$ under $||\cdot||_{L^2}$. For $m \in \mathbb{N}$, $C^m([a,b],\mathbb{R})$ is dense in $H^m([a,b],\mathbb{R})$ under $||\cdot||_{H^m}$.*

Proof: Since $L^2([a,b],\mathbb{R})$ is the completion of $C([a,b],\mathbb{R})$ under $||\cdot||_{L^2}$, every $f \in L^2([a,b],\mathbb{R})$ is either in $C([a,b],\mathbb{R})$, or there is a Cauchy sequence of elements of $C([a,b],\mathbb{R})$ which converges to f. Thus $C([a,b],\mathbb{R})$ is dense in $L^2([a,b],\mathbb{R})$. In the same way, the second part of Theorem 2.33 follows from (2.19) and Theorem 2.17. $\diamond$

2.4.1 Approximation of Functions

Taylor expansions.[9] Remember the linear space of polynomials

$$P([a,b],\mathbb{R}) = \{p : [a,b] \to \mathbb{R} : p(x) := \sum_{i=0}^{m} a_i x^i,\ m \in \mathbb{N}\cup\{0\},\ a_i \in \mathbb{R},\ i = 1,\dots,m\}$$

introduced earlier, $P([a,b],\mathbb{R}) \subset C([a,b],\mathbb{R})$.

Theorem 2.34 (Weierstrass Approximation Theorem). [10] *Every continuous function $f : [a,b] \to \mathbb{R}$ can be approximated arbitrarily well by polynomials, i.e. $P([a,b],\mathbb{R})$ is dense in $C([a,b],\mathbb{R})$ under $||\cdot||_\infty$.*

Proof: see for instance [Krey, p.280] or [Yosh80, p.8f]. $\diamond$

How do we find the approximating polynomials, how do we estimate the error of the approximation? One way is using Taylor expansion:

Theorem 2.35. *Let $f \in C^{N+1}([a,b],\mathbb{R})$, $f^{(j)}$ the jth derivative of f, $f^{(0)} = f$. Then*

$$\begin{aligned} T_f^N(x) &:= f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots \\ &= \sum_{j=0}^{N} f^{(j)}(a)\frac{(x-a)^j}{j!} \end{aligned} \tag{2.39}$$

[9]First discovered in 1671 by James Gregory (*1638 near Aberdeen - + 1675 in Edinburgh), published in 1715 by Brook Taylor (*1685 in Middlesex - + 1731 in London).

[10]Karl Theodor Wilhelm Weierstraß (*1815 Ostenfelde, + 1897 Berlin)

is the N-th Taylor polynomial of f at a. *It satisfies on the interval* $[a,b]$

$$||f - T_f^N||_\infty \leq ||f^{(N+1)}||_\infty \, \frac{(b-a)^{N+1}}{(N+1)!}. \tag{2.40}$$

Proof: see books on Analysis. ◇

If $f \in C^\infty([a,b],\mathbb{R})$, a Taylor series $\sum_{j=0}^{\infty} f^{(j)}(a)\frac{(x-a)^j}{j!}$ does exist, at least formally. But for $N \to \infty$ the expression (2.40) does not always converge to zero. Some Taylor series do converge to f, some do not converge, some do converge, but not to f. How comes? It turned out that it is not adequate to investigate Taylor series for real functions directly. Instead, complex functions $f : \mathbb{C} \to \mathbb{C}$ and their Taylor series with differentiation w.r.t. $z \in \mathbb{C}$ have to be considered. In the *Theory of Complex Functions* holomorphic functions play a prominent role, i.e. functions that are infinitely often differentiable and then are represented by their Taylor series. When investigating real Taylor series, first complex Taylor series have to be considered, and then their real parts, restricted to the real axis. Numerically, Taylor expansion methods are useful for solving certain initial value problems. Taylor expansions are fundamental for the error analysis of many numerical methods, especially for numerical methods for the solution of initial value problems.

Fourier Analysis.[11] Consider periodic functions in $L^2([0,2\pi],\mathbb{R})$. They form a linear subspace $V_{2\pi}$. Define

$$\begin{aligned} s_k(x) &:= \frac{1}{\sqrt{\pi}}\sin kx, \qquad k \in \mathbb{N}\cup\{0\}, \\ c_k(x) &:= \frac{1}{\sqrt{\pi}}\cos kx, \qquad k \in \mathbb{N}, \qquad c_0(x) := \frac{1}{\sqrt{2\pi}} \end{aligned}$$

Then

$$\int_0^{2\pi} c_k(x)c_j(x)dx = \delta_{kj} = \left\{ \begin{array}{ll} 1 & \text{if} \quad k = j, \\ 0 & \text{if} \quad k \neq j, \end{array} \right. , \qquad \text{for } k,j \in \mathbb{N}\cup\{0\}, \tag{2.41}$$

$$\int_0^{2\pi} s_k(x)s_j(x)dx = \delta_{kj}, \qquad \text{for } k,j \in \mathbb{N},$$

$$\int_0^{2\pi} c_k(x)s_j(x)dx = 0 \qquad \text{for } k,j \in \mathbb{N}\cup\{0\}.$$

Thus $\{c_0, c_1, s_1, c_2, s_2, \cdots\}$ are orthonormal w.r.t. $< \cdot,\cdot >_{L^2}$. They are linearly independent over $\mathbb{R}$, and they form a complete system, a basis, in the subspace $V_{2\pi}$ of 2π-periodic functions. [12] Thus every function $f \in V_{2\pi}$ has a unique representation

$$f(x) = \sum_{k=0}^{\infty} a_k c_k(x) + \sum_{k=1}^{\infty} b_k s_k(x) \quad \text{ with } a_k := < c_k, f >_{L^2},\ b_k := < s_k, f >_{L^2}\ . \tag{2.42}$$

[11] Jean-Baptiste-Joseph de Fourier (*1768 in Auxerre - + 1830 in Paris)

[12] δ_{kj} is called 'the Kronecker delta', after Leopold Kronecker, mathematician (* 1823 in Liegnitz, now Legnica, Poland; + 1891 in Berlin)

Let $F_f^N(x) := \sum_{k=0}^N a_k c_k(x) + \sum_{k=1}^N b_k s_k(x)$ with $a_k,\ b_k$ as defined in (2.42) be the trigonometric polynomial of order N approximating f. Then it can be proved that

$$||f - F_f^N||_{L^2}^2 = \sum_{k=N+1}^{\infty} (a_k^2 + b_k^2) \tag{2.43}$$

and that $||f||_{L^2}^2 = a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) < \infty$. Thus the Fourier series does converge to f in $V_{2\pi}$.

Linear Splines. Consider an equidistant partition of the interval $[0,1] \subset \mathbb{R}$: choose $N \in \mathbb{N}$, define $x_k := \frac{k}{N},\ k = 0, \cdots, N$, such that

$$[0,1] = \cup_{k=1}^N [x_{k-1}, x_k]. \tag{2.44}$$

Let $b_k:\ [0,1] \to \mathbb{R},\ x \mapsto b_k(x)$ with

$$b_k(x) := \begin{cases} 1 + (x - x_k)N & \text{for } \ x_k - \frac{1}{N} \le x \le x_k \\ 1 - (x - x_k)N & \text{for } \ x_k \le x \le x_k + \frac{1}{N} \\ 0 & \text{for } \ |x - x_k| \ge \frac{1}{N} \end{cases}.$$

Note that the b_k are piecewise affin linear continuous functions on $[0,1]$. The linear space spanned by the basis $B_N := \{b_0, \cdots, b_N\} \subset C([0,1],\mathbb{R})$ is called *the space of linear splines*,

$$S_N^1([0,1],\mathbb{R}) := span\{b_0, \cdots, b_N\} \subset C([0,1],\mathbb{R}). \tag{2.45}$$

We have $\dim S_N^1([0,1],\mathbb{R}) = N+1$.

Theorem 2.36 (Approximation with linear splines)**.** *Let $f \in C^2([0,1],\mathbb{R}), N \in \mathbb{N},\ s \in S_N^1([0,1],\mathbb{R})$,*

$$s(x) := \sum_{k=0}^N f(x_k) b_k(x). \tag{2.46}$$

Then

$$||f - s||_\infty \le \frac{1}{N^2} \frac{||f''||_\infty}{2}. \tag{2.47}$$

This theorem relies on the Taylor expansion and can be used for proving the following

Theorem 2.37 (Trapezoidal quadrature rule)**.** *Let $f \in C^2([0,1],\mathbb{R}), N \in \mathbb{N}$,*

$$I_f^N := \frac{f(0) + 2f(x_1) + \cdots 2f(x_{N-1}) + f(1)}{N}. \tag{2.48}$$

Then

$$\left| \int_0^1 f(x)dx - I_f^N \right| \le \frac{1}{N^2} \frac{||f''||_\infty}{2}. \tag{2.49}$$

Proof: Let $s \in S_N^1$ be defined according to (2.46). Then we get

$$I_f^N = \int_0^1 s(x)dx \tag{2.50}$$

and

$$\left| \int_0^1 f(x)dx - \int_0^1 s(x)dx \right| \leq \int_0^1 ||f - s||_\infty dx \leq \frac{1}{N^2} \frac{||f''||_\infty}{2}. \tag{2.51}$$

This proves the theorem. ◇

If we compare the three error estimates (2.40), (2.43) and (2.47) we see: only for (2.47) we can be sure that it will decay *monotonically with* N for large enough N. It is thus more convenient in practical computations than the other two.

Closely related to these three N-dependent error estimates is the basic idea of the Galerkin[13] method. It will be discussed next.

2.4.2 The Galerkin Method

Let V be a Hilbert space with inner product $< \cdot, \cdot >$ and norm $|| \cdot ||$, and with an orthonormal basis $\{v_i\}_{i=1}^\infty$. Thus $< v_i, v_j >= \delta_{ij}$ for $i, j \in \mathbb{N}$, and for every $v \in V$ there exist $\{\alpha_i\}_{i=1}^\infty$ such that $v = \sum_{i=1}^\infty \alpha_i v_i$.

Now suppose an equation

$$Lu = f \tag{2.52}$$

is to be solved in V, with given linear operator $L : V \to V$ and $f \in V$. The essential idea of the Galerkin method is: define a sequence of subspaces $(V_N)_{N \in \mathbb{N}} \subset V$ with $V_N \subset V_{N+1}$,

$$V_N := span\{v_i\}_{i=1}^N, \tag{2.53}$$

and solve eq. (2.52) in V_N instead of in V, by requiring

$$< v_j, Lu - f >= 0 \qquad \text{for } j = 1, \cdots, N. \tag{2.54}$$

Find γ_i such that $f = \sum_{i=1}^n \gamma_i v_i$ and make the ansatz $u_N := \sum_{k=1}^N \alpha_k v_k$ for the solution, then

$$< v_j, \sum_{k=1}^N (\alpha_k L v_k - \gamma_k v_k) >= 0, \qquad j = 1, \cdots, N, \tag{2.55}$$

[13] Boris Grigorewitsch Galerkin (*1871 in Polosk, White Russia; +1945 in Moscow) engineer, mathematically oriented scientist, architect, military engineer of high rank. Important work in several fields, especially in theory of elasticity (mostly for plates and shells) and in the numerical treatment of boundary value problems. Studied until 1899 at the institute of technology in St. Petersburg = Petrograd = Leningrad. Took part in the revolution of 1906 and was put in prison for about 18 months. In 1909 he started teaching at the Petrograd/Leningrad polytechnic institute; Professor for constructive mechanics after 1920; member of the Academy after 1928. He introduced then modern methods of mathematical analysis and computation for planning the construction of buildings; planned and advised planners of power plants (both heat and water power plants). His design of the Petrograd power plant (1913 - 1915) was praised as 'bold and inventive'. [Co68, p.131], [GIS90]

gives N equations for the N unknown coefficients $\{\alpha_k\}$ of u_N.
Convergence of the method for $N \to \infty$ is ensured, because $(v_i)_{i=1}^{\infty}$ is a basis of V. To get a good approximation of u for moderate N, the spaces V_N (i.e. the space V, the basis of V, the numbering of its elements and $N \geq n$) have to be chosen appropriately: if $V_N \cap W_N$ is 'too small', $W_N := span\{Lv_i\}_{i=1}^{N} \cup span\{f\}$, then u_N is a poor approximation to u. If $V_N \cap W_N = \{0\}$, then $u \in V \setminus V_N$ and $u_N = 0$. Application of the method is most efficient if $W_N = V_N$, N small.
The method is explained here for linear equations $Lu = f$. It is also in use for nonlinear systems of equations, for instance for the spatial discretization of the Navier-Stokes equations [Me99, p. 36-39; p. 66-76]. For BVPs, popular choices for $\{v_i\}_{i=1}^{N}$ are splines, trigonometric functions, and Čebyšev polynomials; but also Bessel functions or Legendre polynomials are used where appropriate.

2.4.3 Approximation of Boundary Values

Consider again a BVP (2.5) on some open and connected domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} \Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega, \end{aligned}$$

with a *classical* solution u. For specifying the solution and getting uniqueness, boundary values g have to be prescribed on $\partial\Omega$. *How to generalize the role of the boundary values to the case of weak solutions?*
For $u \in C(\bar{\Omega}, \mathbb{R})$, $u(x)|_{x \in \partial\Omega} =: u|_{\partial\Omega}$ is uniquely defined.
Example: let $u(x) = x + 2$ on $[0, 1]$, then $u|_{\partial[0,1]}$ is given by $u(0) = 2$; $u(1) = 3$. $\diamond$
For $u \in H^m(\bar{\Omega}, \mathbb{R})$, u is uniquely defined only a.e., *almost everywhere*, i.e. everywhere except on a set of Lebesgue measure zero. If $\Omega \subset \mathbb{R}$ is an interval, $\partial\Omega \subset \mathbb{R}$ consists of two points; if $\Omega \subset \mathbb{R}^2$ is a surface, then $\partial\Omega \subset \mathbb{R}^2$ is a curve, etc. In the Lebesgue measure of Ω, $\partial\Omega$ is always a set of measure zero. But also weak solutions have to be specified on the boundary, to ensure uniqueness in an appropriate way. *How?*
We know: $L^2(\bar{\Omega}, \mathbb{R})$ is the closure of $C(\bar{\Omega}, \mathbb{R})$ under $|| \cdot ||_{L^2}$. Given $u \in L^2(\bar{\Omega}, \mathbb{R})$, there is a Cauchy sequence $\{v_n\} \subset C(\bar{\Omega}, \mathbb{R})$ with $\lim_{n\to\infty} v_n = u$. For each v_n, $\gamma(v_n) := v_n|_{\partial\Omega}$ is uniquely defined. Does this define $\gamma(u)$ uniquely? The answer is *no*: an example with $\lim_{n\to\infty} v_n = u = \lim_{n\to\infty} u_n$ in $L^2(\bar{\Omega}, \mathbb{R})$ and $\lim_{n\to\infty} v_n|_{\partial\Omega} \neq \lim_{n\to\infty} u_n|_{\partial\Omega}$ is given in [Red86, p.172]. It turns out that more smoothness is necessary: both more smoothness of u and enough smoothness of $\partial\Omega \subset \mathbb{R}^n$ in case of $n > 1$.

Recall Definition A.17 and Remark A.18.

Theorem 2.38 (Trace Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded with Lipschitz boundary $\partial\Omega$.*
(i) There exists a unique bounded linear operator $\gamma : H^1(\Omega, \mathbb{R}) \to L^2(\partial\Omega, \mathbb{R})$ with

$$||\gamma(u)||_{L^2(\partial\Omega,\mathbb{R})} \leq C \; ||u||_{H^1(\partial\Omega,\mathbb{R})}; \tag{2.56}$$

For $u \in C^1(\bar{\Omega}, \mathbb{R})$, $\gamma(u) = u|_{\partial\Omega}$ is valid in the usual way.
(ii) The range of γ is dense in $L^2(\partial\Omega, \mathbb{R})$.

Proof: see [Red86, p.173f]. ◇

Remark 2.39. *Apply the theorem to $u \in H^m(\Omega, \mathbb{R})$, m appropriate, to obtain a trace theorem involving derivatives of the function u on $\partial\Omega$.*

Appendix A

A.1 Preliminary Definitions

A.1.1 Open and closed

Definition A.1 (Open ball)**.** *Given* $x_0 \in \mathbb{R}$ *and* $r > 0$*, then* $B(x_0, r) := \{x \in \mathbb{R} : |x - x_0| < r\}$ *is the* open ball in $\mathbb{R}$ with radius r and center x_0*.*
Given $\mathbf{x}_0 \in \mathbb{R}^2$ *and* $r > 0$*, then* $B(\mathbf{x}_0, r) := \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{x}_0| < r\}$ *is the open ball in* $\mathbb{R}^2$ *with radius* r *and center* $\mathbf{x}_0$*.*

Here, $|x|$ denotes the modulus of x: $|x| := \sqrt{x^2}$ for $x \in \mathbb{R}$ and $|\mathbf{x}| := \sqrt{x^2 + y^2}$ for $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Similarly open balls in $\mathbb{R}^n$ are defined for all $n \in \mathbb{N}$.

Definition A.2 (Open set)**.** *A subset* E *of* $\mathbb{R}$ *(*$\mathbb{R}^2$*) is said to be* ***open*** *if it contains an open ball in* $\mathbb{R}$ *(*$\mathbb{R}^2$*) about each of its points.*
A subset E *of* $\mathbb{R}$ *is said to be* ***closed*** *if its complement (* $\mathbb{R} \setminus E$*) is open.*

From the definition it follows: an open ball is open; the set $\mathbb{R}$ itself is both open and closed in $\mathbb{R}$; its complement, the empty set $\emptyset$, is also both open and closed in $\mathbb{R}$. Remember that a set is **not** like a door.

Definition A.3. *The* ***closure*** $\bar{E}$ ***of a set*** E *is the smallest closed set containing* E*.*

The closure of an open ball $B(x_0, r)$ is $\overline{B(x_0, r)} := \{x \in \mathbb{R} : |x - x_0| \leq r\}$, a 'closed ball'.

Definition A.4 (Open and closed intervals)**.** *Given a real interval with endpoints* $a, b \in \mathbb{R}$*. Depending on (the bracket-environment of) the context, we write open intervals as*

$$(a, b) =]a, b[:= \{x \in \mathbb{R} : a < x < b\},$$

and closed intervals as

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} = \overline{]a, b[} = \overline{(a, b)}.$$

A.1.2 Bounds, max, min, sup, inf

Let E be an arbitrary non-empty set of real numbers. $F := \{\frac{n}{n+1};\ n \in \mathbb{N}\}$.

Definition A.5 (Bounds)**.** *$k \in \mathbb{R}$ is called an **upper (lower) bound for** E if $x \leq k$ ($x \geq k$) for all $x \in E$.*
*E is called **bounded above (bounded below)** if it has an upper bound (lower bound).*
*E is called **bounded** if it is bounded above and below.*

Example A.6. *The set F has upper bounds 1 and 100 and lower bounds 0 and -5. Hence F is bounded. Note that bounds are not unique.*

Definition A.7 (max, min)**.** *An element $m \in \mathbb{R}$ is called **maximum of** E, $m = \max(E)$ (**minimum of** E, $m = \min(E)$) if $m \in E$ and $m \geq x$ ($m \leq x$) for all $x \in E$.*

Remark A.8. *An important point to remember is that the maximum and the minimum of a set E are elements of E. They are uniquely determined.*
*Every **finite** non-empty set of real numbers has a maximum. This is not true in general for subsets of $\mathbb{R}$ with infinitely many elements. Example: F has no maximum. Indeed, the numbers $\frac{n}{n+1}$ are arbitrarily close to 1 but 1 is not an element of the set, and every real number strictly smaller than 1 is not an upper bound for F. We need an extension of the definition of maximum. That is the notion of supremum.*

Definition A.9 (sup, inf)**.** *Let E be bounded above. The **supremum** or **least upper bound of** E, denoted $\sup(E)$, is the real number λ such that:*

- *λ is an upper bound for E;*
- *for all $\lambda' < \lambda$ there exists $x \in E$ such that $x > \lambda'$.*

$\inf(E)$, the infimum of E, is defined similarly for real sets E which are bounded below.
If E is unbounded above we set $\sup(E) := +\infty$. Similarly, if E is unbounded below $\inf(E) := -\infty$.

Note that the supremum of a bounded set is the minimum of its upper bounds. The importance of the definition of supremum can be seen in the following theorem

Theorem A.10. *Every non-empty subset of $\mathbb{R}$ bounded from above has a supremum.*

Remark A.11. *If $\sup(E) \in E$ then $\sup(E) = \max(E)$. If $\inf(E) \in E$ then $\inf(E) = \min(E)$.*

For more details see [VG81, Chap.1].

A.1.3 Embeddings

Definition A.12 (Isomorphism)**.** *Let V, U be linear spaces, $v_1, v_2 \in V$. A linear map $L : V \to U$ is called*

- injective *iff $Lv_1 = Lv_2$ only for $v_1 = v_2$;*

- surjective *or* onto *iff* $L(V) = U$;
- bijective *or* an isomorphism *iff* L *is both injective and surjective.*

Isomorphic linear spaces have the same dimension and the same structure and may be viewed as 'the same space'.

Definition A.13 (Embedding). *Let* V, U *be linear spaces and* W *a subspace of* V. *Let* $L : V \to U$ *be an injective linear map. Then* $W \subset V$ *and* $L(W) \subset U$ *are isomorphic. We say that* L *embeds* W *into* U. L *is an embedding.*

A.1.4 Continuity and Smoothness

Definition A.14 (Continuity of Real Functions). *A function* $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$, *is*
* continuous at $x_o \in \Omega$
iff for every $\varepsilon > 0$ *there is a* $\delta > 0$ *s.th.* $|f(x) - f(x_o)| < \varepsilon$ *for all* x *with* $|x - x_o| < \delta$,
or, equivalently,
iff $\lim_{x \to x_o} |f(x) - f(x_o)| = 0$.
* continuous in Ω *iff it is continuous in every* $x_o \in \Omega$.

Definition A.15 (Continuously differentiable). *A function* $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$, *is*
* continuously differentiable at $x_o \in \Omega$
iff a derivative $f' : \Omega \to \mathbb{R}$ *exists in some ball around* x_o *and is continuous at* x_o;
* continuously differentiable in Ω *iff it is continuously differentiable in every* $x_o \in \Omega$.
If a function f *is* m*-times continuously differentiable, we define*

$$u^{(j)}(x) := \frac{d^j u}{dx^j} \qquad j = 1, \ldots, m. \tag{A.1}$$

Definition A.16 (Lipschitz Continuity). [1] *A function* $f : [a, b] \mapsto \mathbb{R}$ *is called* Lipschitz-continuous *if there is a constant* $L \geq 0$ *such that for all* x_1, $x_2 \in [a, b]$

$$|f(x_1) - f(x_2)| \leq L\, |x_1 - x_2|$$

The space of all such Lipschitz-continuous functions is called $Lip([a, b], \mathbb{R})$.

Note that Lipschitz continuity is a special case of continuity. Every Lipschitz-continuous function is continuous. Moreover, *Lip* satisfies (A.3).

Definition A.17 (Smoothness of boundaries). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded, connected domain,* $n \geq 2$. *Let* $x_0 \in \partial\Omega$, $\varepsilon > 0$, $B(x_0, \varepsilon) := \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$. *Suppose there are a map* $g : B(x_0, \varepsilon) \to \mathbb{R}$ *such that* $g(\zeta_1, \cdots, \zeta_n) = 0$ *on* $B(x_0, \varepsilon) \cap \partial\Omega$ *and a map* f *such that* $g(\zeta_1, \cdots, \zeta_n) = f(\zeta_2, \cdots, \zeta_n) - \zeta_1$, *i.e. such that* $\zeta_1 = f(\zeta_2, \cdots, \zeta_n)$ *parameterizes the segment of* $\partial\Omega$ *contained in* $B(x_0, \varepsilon)$.

- *If* f *is Lipschitz continuous at* x_0, *i.e. if* f *satisfies*

$$|f(\zeta_2, \cdots, \zeta_n) - f(\eta_2, \cdots, \eta_n)| \leq L\; |(\zeta_2, \cdots, \zeta_n) - (\eta_2, \cdots, \eta_n)| \tag{A.2}$$

  in $B(x_0, \varepsilon)$ *for some constant* $L > 0$, *we say that* $\partial\Omega$ is Lipschitz continuous at x_0.

[1] Rudolf Otto Sigismund Lipschitz (*1832 near Königsberg=Kaliningrad - +1903 in Bonn)

- *If f is m times continuously differentiable at x_0, we say that $\partial\Omega$* is m times continuously differentiable at x_0.
- *If $\partial\Omega$ is Lipschitz continuous at every $x_0 \in \partial\Omega$, we say that $\partial\Omega$* is Lipschitz continuous.
- *If $\partial\Omega$ is m times continuously differentiable at every $x_0 \in \partial\Omega$, we say that $\partial\Omega$* is m times continuously differentiable.

Remark A.18. *A C^m-boundary is a C^m-curve or -surface and thus smooth. A Lipschitz boundary may have corners, but no cusps or horns and no slots.*

A.2 Homework Problems

1. **(3 points)** Answer the following questions.

   - What is an open ball in $\mathbb{R}$? and in $\mathbb{R}^2$?
   - Is an open ball in $\mathbb{R}$ an open subset of $\mathbb{R}^2$?
   - Give an example of a closed set in $\mathbb{R}$. Is it closed in $\mathbb{R}^2$?
   - Is the empty set $\emptyset$ closed? Is it open?

   Give an example of a subset of $\mathbb{R}$ that is neither open nor closed.

2. **(3 points)** Prove Lemma 1.7.

3. **(1 point)** Fix $n \in \mathbb{N}$. Let $P^n(\mathbb{R}, \mathbb{R}) \subset P(\mathbb{R}, \mathbb{R})$ be the set of polynomials p of degree $\deg(p) \leq n$ with real coefficients. Show that $P^n(\mathbb{R}, \mathbb{R})$ is a linear space over $\mathbb{R}$.

4. **(2 points)** Let $V_a := \{(u_1, \ldots, u_n)^t \in \mathbb{R}^n \mid u_1 = a\}$. Is V_a a subspace of $\mathbb{R}^n$

   (a) for $a = 0$?

   (b) for $a = 1$?

   Why?

5. **(3 points)** Let V be a linear space and W_1 and W_2 linear subspaces of V. Is $W_1 \cap W_2$ a linear subspace of V? And $W_1 + W_2 := \{v = u + w \in V \mid u \in W_1, w \in W_2\}$? And $W_1 \cup W_2$? Prove your answers.

6. **(4 points)** Consider a set $V \subset \mathbb{C}$. Define addition and scalar multiplication in V through addition and multiplication in the field $\mathbb{C}$.
   • Is $V := \mathbb{R}$ a linear space over the field $\mathbb{C}$?
   • Is $V := \mathbb{C}$ a linear space over the field $\mathbb{R}$?
   Prove your statements!

7. ***(3 points)**[2] A function $f : [a, b] \mapsto \mathbb{R}$ is called *Lipschitz-continuous* if there is a constant $L \geq 0$ such that for all $x_1, x_2 \in [a, b]$

   $$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$$

   The space of all such Lipschitz-continuous functions is called $Lip([a, b], \mathbb{R})$. Show that

   $$C^1([0, 1], \mathbb{R}) \subset Lip([0, 1], \mathbb{R}) \subset C([0, 1], \mathbb{R}). \tag{A.3}$$

8. **(2 points)** Let $E := \{y \in \mathbb{R} : y = x^2 \text{ for } x \in (-1, 1)\}$. Is E bounded? If yes, give a lower and/or upper bound of E. Give also $\inf(E)$ and $\sup(E)$. Is the infimum a minimum and/or the supremum a maximum?

[2] Problems marked with a '*' are voluntary. The additional points gathered by solving them can make up for missing points from other problems.

9. (2 points) Consider $B := \{1, 1+x, 1+x^2, x^3\} \subset C(\mathbb{R}, \mathbb{R})$. Are the elements of B linearly independent? Name the space they generate, i.e. the subspace $W = span\{B\}$ of $C(\mathbb{R}, \mathbb{R})$ which is spanned by B. Give a basis of $span\{B\}$.

10. (2 points) Let $A: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map

$$x \mapsto Ax := \begin{pmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} x$$

Determine the dimensions of

$$\begin{aligned} \mathrm{Ker}(A) &:= \{x \in \mathbb{R}^3 \,|\, Ax = 0\}, \\ \mathrm{Im}(A) &:= \{y \in \mathbb{R}^3 \,|\, y = Ax \text{ for some } x \in \mathbb{R}^3\}. \end{aligned}$$

11. (2 points) To which spaces $C^m(\Omega, \mathbb{R})$ do the following functions belong?

$$\begin{aligned} u(x) &= |x| \text{ for } x \in (-1, 1), \\ u(x, y) &= \sin(x)(1-y) \text{ for } (x, y) \in [0, \pi] \times [0, 1] \end{aligned}$$

12. (2 points) Let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ and define the map

$$\begin{aligned} L: C^2(\Omega, \mathbb{R}) &\to C(\Omega, \mathbb{R}) \\ u &\mapsto \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \end{aligned}$$

Is L a linear map? Is there some $u \in C^2$, $u(x, y) \not\equiv 0$, such that $Lu = 0$? Give an example!

13. (8 points) Let $V := \{g \in C^2([-1, 1], \mathbb{R}) : g(-1) = g(1) = 0\}$ and define

$$\begin{aligned} < \cdot, \cdot >: & \qquad V \times V \to \mathbb{R} \\ < u, v > &:= 2\int_{-1}^{1} u''(x)v''(x)dx \end{aligned}$$

(a) Show that V is a linear space.

(b) Show that $< \cdot, \cdot >$ is an inner product.

(c) Define the corresponding natural norm of the inner product space V.

(d) Are $u = x^2 - 1$, $v = \sin(\pi x)$ orthogonal with respect to $< \cdot, \cdot >$?

14. (2 points) Do the following maps

$$\begin{aligned} a_i &: \quad C^2([0, 1], \mathbb{R}) \times C^2([0, 1], \mathbb{R}) \to \mathbb{R}, \quad i = 1, 2 \\ a_1(u, v) &:= \int_0^1 u'(x)v'(x)\, dx \\ a_2(u, v) &:= \int_0^1 \left(u(x)v(x) + u''(x)v''(x)\right)\, dx. \end{aligned}$$

define inner products on the linear space $C^2([0, 1], \mathbb{R})$?

15. (3 points) Consider the space $C^m([0,1],\mathbb{R})$ and define a mapping by

$$\begin{aligned} <\cdot,\cdot>_m: &\quad C^m([0,1],\mathbb{R}) \times C^m([0,1],\mathbb{R}) \to \mathbb{R} \\ <u,v>_m \;:= &\quad \int_0^1 \sum_{j=0}^m u^{(j)}(x)v^{(j)}(x)dx = \int_0^1 (uv + u'v' + ... + u^{(m)}v^{(m)})dx \\ &\quad \text{with} \quad u^{(j)}(x) := \frac{d^j u}{dx^j} \end{aligned}$$

(a) Show that $<\cdot,\cdot>_m$ is a bilinear form and an inner product.
(b) Show that $u(x) := x^{\frac{3}{2}}$, $v(x) := 1 - 2x^{\frac{5}{2}}$ are orthogonal with respect to the inner product $<\cdot,\cdot>_0$. Are u and v orthogonal with respect to $<\cdot,\cdot>_1$?
(c) Give an example of functions u and v in $C^2([0,1],\mathbb{R})$ that are orthogonal with respect to $<\cdot,\cdot>_2$.

16. (1 point) The Euclidean scalar product in $\mathbb{R}^n$ is defined in eq. (1.11). Give a direct proof of the Schwarz inequality in the case $n = 2$.

17. (1 point) Let V be a real linear space with a scalar product $<\cdot,\cdot>$. Show Pythagoras' theorem: If x is orthogonal to y then

$$<x+y, x+y> = <x,x> + <y,y> . \tag{A.4}$$

18. (1 point) Show that the quadrature operator of Kepler-Simpson

$$A: \; C([a,b],\mathbb{R}) \to \mathbb{R}, \quad Af := \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

is linear. (Remark: This operator approximates $\int_a^b f(x)dx$.)

19. (6 points) Let $\|\cdot\|$ be a *natural* norm on V. Show that for any $x, y, z \in V$

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z - \frac{x+y}{2}\|^2.$$

20. (3 points) (a) Show that

$$\|f\| := \left(\int_0^1 (f''(x))^2 dx\right)^{\frac{1}{2}} \tag{A.5}$$

defines a norm in $V := \{g \in \mathcal{C}^2([0,1],\mathbb{R}) : g(0) = g(1) = 0\}$.
(b) Is this a norm induced by an inner product?
(c) Is it a norm in $\mathcal{C}^2([0,1],\mathbb{R})$?

21. (1 point) We define the unit sphere in a-norm in $\mathbb{R}^2$ as

$$S_a(0,1) := \{x \in \mathbb{R}^2 : \|x\|_a = 1\} \text{ for } a \in [1,+\infty) \cup \{\infty\}. \tag{A.6}$$

Draw in the same picture $S_a(0,1)$ for $a = 1$, 2 and ∞.

22. (3 points) We define on $[0, 2\pi]$ the family of functions

$$u_0(t) := 1, \quad u_n(t) = \cos(nt) \quad \text{and} \quad v_n(t) = \sin(nt) \qquad \text{for } n \in \mathbb{N}.$$

(a) Do u_n, v_n belong to $L^2([0, 2\pi], \mathbb{R})$?
(b) We define a scalar product in $L^2([0, 2\pi], \mathbb{R})$ as

$$< f, g >:= \int_0^{2\pi} f(x)g(x)dx. \tag{A.7}$$

Show that $< u_n, u_m >= 0$ for all $n \neq m$ (Hint: Assume $n < m$ and use repeated integration by parts). Show that $< u_n, v_m >= 0$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$.
(c) Compute the norm of u_n for $n \in \mathbb{N}_0$ in the natural norm of $L^2([0, 2\pi], \mathbb{R})$.

23. *(1 point) Show that $\mathbb{R}^2$ is embeddable into $\mathbb{R}^3$ (see Definition A.13).

24. (1 point) Prove Lemma 1.38 in the lecture notes.

25. (1 point) Let V be a set with a metric $d : V \times V \to \mathbb{R}$ and B a subset of V. Given a point $x \in V$ we define the distance of x from B as $D(x, B) := \inf_{b \in B} d(x, b)$. Show that

$$|D(x, B) - D(y, B)| \leq d(x, y) \text{ for all } x, y \in V. \tag{A.8}$$

26. (3 points) (Gram-Schmidt-Orthonormalisation)
Let $B = (x_n)_{n \in \mathbb{N}} = \{x_1, x_2, ...\}$ be a basis of an infinite dimensional space V with scalar product $< \cdot, \cdot >_V$ and norm $\|v\|_V := \sqrt{< \cdot, \cdot >_V}$. We define the sequence $(z_n)_{n \in \mathbb{N}}$ as

$$\begin{aligned} z_1 &:= x_1/\|x_1\|_V \\ \tilde{z}_n := x_n - \sum_{j=1}^{n-1} < x_n, z_j >_V z_j, \quad z_n &:= \tilde{z}_n/\|\tilde{z}_n\|_V, \text{ for } n > 1. \end{aligned}$$

(a) Show that $\|z_n\|_V = 1$ and that z_n is orthogonal to $z_1, ..., z_{n-1}$.
(b) Show that $z_1, ..., z_n$ are linearly independent. Is $z_1, \ldots, z_n$ a basis of $span\{x_1, \ldots, x_n\}$?
(c) Calculate $z_1, ..., z_4$ for $V := P^\infty([-1, 1], \mathbb{R})$ (the space of polynomials of arbitrary degree), $B := (t^n)_{n \in \mathbb{N}_0} = \{1, t, t^2, ...\}$ and $< p, q >_V := \int_{-1}^{1} p(t)q(t)dt$.

27. (12 points) For $x \in [-1, 1] \subset \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 2$, define

$$f_n(x) := \begin{cases} 1 + x & \text{for } -1 \leq x < -\frac{1}{n} \\ 1 - \frac{2}{n} - x & \text{for } -\frac{1}{n} \leq x \leq 0 \\ 1 - \frac{2}{n} + x & \text{for } 0 < x \leq \frac{1}{n} \\ 1 - x & \text{for } \frac{1}{n} < x \leq 1 \end{cases}$$

(a) Draw a sketch of f_n!

(b) Calculate the point-wise limit $f(x) := \lim_{n \to \infty} f_n(x)$ for fixed $x \in [-1, 1]$!

(c) To which spaces $C^m([-1,1],\mathbb{R})$ do f_n and f belong?

(d) Does $(f_n)_{n\in\mathbb{N}}$ converge to f in the supremum norm on $C([-1,1],\mathbb{R})$?

(e) Does $(f_n)_{n\in\mathbb{N}}$ converge to f in the L^2-norm on $C([-1,1],\mathbb{R})$?

28. (1 point) Continuous functions $f:[0,1]\to\mathbb{R}$ are square-integrable, $C([0,1],\mathbb{R})\subset L^2([0,1],\mathbb{R})$. Hence we may consider $(C([0,1],\mathbb{R}),\|\cdot\|_\infty)$ and $(C([0,1],\mathbb{R}),\|\cdot\|_{L^2([0,1],\mathbb{R})})$. Show that $\|\cdot\|_\infty$ is stronger than $\|\cdot\|_{L^2([0,1],\mathbb{R})}$ in $C([0,1],\mathbb{R})$.

29. (4 points) Show that the following mappings are continuous:

$$\begin{aligned} D:(C^1([0,1],\mathbb{R}),\|\cdot\|_{C^1([0,1],\mathbb{R})}) &\to (C([0,1],\mathbb{R}),\|\cdot\|_\infty):\ f\mapsto f',\\ I:(C([0,1],\mathbb{R}),\|\cdot\|_\infty) &\to (C([0,1],\mathbb{R}),\|\cdot\|_{L^2([0,1],\mathbb{R})}):\ f\mapsto f. \end{aligned}$$

Fix a vector $a\in\mathbb{R}^n$, $a=(a_1,\ldots,a_n)^T$. Define

$$F_a:(\mathbb{R}^n,\|\cdot\|_1)\to(\mathbb{R}^n,\|\cdot\|_2):\ x\mapsto 2x+a.$$

Is the map F_a linear? Is it continuous?

Is the following mapping continuous?

$$I:(C([0,1],\mathbb{R}),\|\cdot\|_{L^2([0,1],\mathbb{R})})\to(C([0,1],\mathbb{R}),\|\cdot\|_\infty):\ f\mapsto f.$$

30. (6 points) Prove Theorem 1.34 of the lecture notes.

31. (5 points) Consider the set $\mathbb{R}^{n\times n}$ of real $n\times n$ matrices:

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & & \ddots & \vdots\\ a_{n1} & a_{n2} & \ldots & a_{nn}\end{pmatrix}\in\mathbb{R}^{n\times n}.$$

Is this a linear space over $\mathbb{R}$?

Define

$$\|A\|_\infty:=\max_{1\le j\le n}\sum_{k=1}^n|a_{jk}|\ ,\qquad \|A\|_1:=\max_{1\le k\le n}\sum_{j=1}^n|a_{jk}|.$$

Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are norms on $\mathbb{R}^{n\times n}$. Are these norms equivalent? Show moreover that

$$\|Ax\|_\infty\le\|A\|_\infty\|x\|_\infty\quad\text{and}\quad\|Ax\|_1\le\|A\|_1\|x\|_1\quad\text{for } x\in\mathbb{R}^n.$$

32. *(1 point) In the set $\mathbb{R}^{n\times n}$ of real $n\times n$ matrices another important norm is the Frobenius norm:

$$\|A\|_F:=(\sum_{k=1}^n\sum_{j=1}^n|a_{jk}|^2)^{\frac{1}{2}}.$$

Are $\|\cdot\|_F$ and $\|\cdot\|_\infty$ equivalent norms in $\mathbb{R}^{n\times n}$?

33. (4+*1 points) Let V, W be real linear spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$.
(a) Define addition and scalar multiplication on $V \times W$ through those on V and W and prove that $V \times W$ is a real linear space.
(b) Show that

$$\|(v,w)\|_{V\times W,1} := \|v\|_V + \|w\|_W \text{ for all } v \in V,\ w \in W,$$

defines a norm in $V \times W$. Moreover show that $(V \times W, \|\cdot\|_{V\times W,1})$ is complete if $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are complete.
(c) Additionally, suppose now that V and W are inner product spaces and that $\|\cdot\|_V$, $\|\cdot\|_W$ are the corresponding natural norms. Let

$$\|(v,w)\|_{V\times W,2} := \sqrt{\|v\|_V^2 + \|w\|_W^2} \text{ for all } v \in V,\ w \in W.$$

$\|\cdot\|_{V\times W,2}$ is a norm in $V \times W$ (you do not need to prove this). Is this a natural norm (i.e induced by an inner product)? If so, which is the inner product associated?
*(d) Is $\|\cdot\|_{V\times W,1}$ a natural norm in $V \times W$? If so, which is the inner product associated? If not, give an example.

34. (3 points) Let A be the Kepler-Simpson quadrature operator of Problem 18:

$$\begin{aligned} A : (C([a,b],\mathbb{R}), \|\cdot\|_\infty) &\to (\mathbb{R}, |\cdot|), \\ f &\mapsto Af := \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] \end{aligned}$$

(a) Show that $|Af| \le |b-a|\ \|f\|_\infty$.

(b) Find an $\tilde{f} \in C([0,1],\mathbb{R})$ such that $\|\tilde{f}\|_\infty = 1$ and $|A\tilde{f}| = |b-a|$.

(c) Conclude that the operator norm of A is $|b-a|$.

35. (3 points) Consider the sequence $\{v_n\}_{n\in\mathbb{N}} \subset C([0,1],\mathbb{R})$ defined for every $n \in \mathbb{N}$ by

$$v_n(x) := \cos\left(\frac{x}{n}\right) \text{ for } x \in [0,1].$$

(a) For fixed $x \in [0,1]$ calculate $\lim_{n\to\infty} v_n(x)$ and denote the limit by $v(x)$.

(b) Does $(v_n)_{n\in\mathbb{N}}$ converge to its pointwise limit v in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$?

(c) Is (v_n) a Cauchy sequence in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$?

Hint: use that cos is a continuous and decreasing function in $[0,1]$.

36. *(3 points) Consider for $a = const.$ the following linear mappings

$$\begin{aligned} A_0 : \ (C^1([0,1],\mathbb{R}), \|\cdot\|_{L^2([0,1],\mathbb{R})}) &\to (C([0,1],\mathbb{R}), \|\cdot\|_{L^2([0,1],\mathbb{R})}), \\ f(x) &\mapsto A_0 f(x) := af'(x); \\ A_1 : \ (C^1([0,1],\mathbb{R}), \|\cdot\|_{H^1}) &\to (C([0,1],\mathbb{R}), \|\cdot\|_{L^2([0,1],\mathbb{R})}), \\ f(x) &\mapsto A_1 f(x) := af'(x), \end{aligned}$$

where $\|\cdot\|_{H^1}$ is defined by

$$\|f\|_{H^1} := \left(\int_0^1 f^2(x) + f'^2(x)\ dx\right)^{\frac{1}{2}}.$$

(a) Find a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1([0,1],\mathbb{R})$ such that $\|f_n\|_{L^2([0,1],\mathbb{R})} \to 0$ and $\|A_0 f_n\|_{L^2([0,1],\mathbb{R})} \nrightarrow 0$.

(b) Is A_0 continuous? Is it bounded?

(c) Is A_1 continuous? Is it bounded?

37. (5 points) (Normalised Chebyshev Polynomials) For $u, v \in C([-1,1],\mathbb{R})$ we define

$$< u, v >_\rho := \int_{-1}^1 \rho(t)u(t)v(t)dt, \quad \rho(t) = (1-t^2)^{-1/2}.$$

(a) Show that $< \cdot,\cdot >_\rho$: $C([-1,1],\mathbb{R}) \times C([-1,1],\mathbb{R}) \to \mathbb{R}$ is an inner product.

(b) Repeat Exercise 26 c) with $V := P^\infty([-1,1],\mathbb{R})$, $B := \{1, t, t^2, ...\}$ and $< \cdot,\cdot >_V := < \cdot,\cdot >_\rho$. Calculate z_1, z_2, z_3 !

(c) Why is z_4 an odd function: $z_4(-t) = -z_4(t)$?

38. (4 points) Calculate the operator norms of the following operators, using steps similar to those for solving Problem 34:
(a)

$$\begin{aligned} I : (C([0,1],\mathbb{R}), \|\cdot\|_\infty) &\to (\mathbb{R}, |\cdot|), \\ f &\mapsto If := \int_0^1 f(x)dx \end{aligned}$$

(b) For $v \in \mathbb{R}^n$ ($\|v\|_2 = 1$) the projection operator

$$\begin{aligned} P : (\mathbb{R}^n, \|\cdot\|_2) &\to (\mathbb{R}^n, \|\cdot\|_2) \\ u &\mapsto Pu := \langle u, v\rangle v. \end{aligned}$$

projecting u onto the subspace $span(v) \subset \mathbb{R}^n$.

39. (1+*2 points) The **Legendre polynomials** are defined as

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \quad t \in [-1,1], \ \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

(a) Why is $\frac{d^k}{dt^k}(t^2-1)^l|_{t=\pm 1} = 0$ for $l > k$, $l, k \in \mathbb{N}_0$?

(b) * $< P_n, P_m >_{L^2(]-1,1[,\mathbb{R})} := \int_{-1}^1 P_n(t)P_m(t)dt = 0$ for all $n \neq m$
Hint: Assume $n < m$ and use repeated integration by parts.

(c) * Does P_n solve the Legendre differential equation
$(1-t^2)y'' - 2ty' + n(n+1)y = 0$?

40. *(10 points) Give *simple* examples from your field of interest for

(a) a differential equation with classical solutions;

(b) a differential equation with weak solutions.

41. (2 points) Consider the sequence $(f_n)_{n\in\mathbb{N}\setminus\{1\}}$ given in $C([0,1],\mathbb{R})$ by

$$f_n(x) := \begin{cases} xn & \text{for } 0 \le x < \frac{1}{n} \\ 2 - nx & \text{for } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{for } \frac{2}{n} \le x \le 1 \end{cases} \qquad n \ge 2.$$

(a) Draw a sketch of f_n.

(b) Compute the pointwise limit.

42. (5 points) Let $f, g\ : (0,2) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}, \qquad g(x) = \begin{cases} 2x & \text{if } 0 < x \le 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}.$$

(a) Draw a sketch of f and g.

(b) Show that g is the weak derivative of f by using the definition of weak derivatives.

(c) Is g also the weak derivative of

$$h(x) = \begin{cases} x^2 & \text{if } 0 < x \le 1 \\ 2 & \text{if } 1 < x < 2 \end{cases} ?$$

43. Remark (Sketches) In many situations, sketches are very helpful for finding mathematical proofs. Note however: A sketch is not equivalent to a mathematical proof and cannot replace a proof. It might be misleading.

Example [mathe+07]: Given two identical triangles and two identical trapezia. Combine them in two different ways to form two different rectangles[3].
triangles: 1 right angle, side lengths are 25 cm, 40 cm and $\sqrt{25^2 + 40^2}$ cm.
trapezia: 2 right angles, side lengths 40 cm, 25 cm, 25 cm and $\sqrt{25^2 + (40-25)^2}$ cm.
(a) Draw a sketch of a triangle and of a trapezium.
(b) Combine the two triangles to get a rectangle of 25×40 cm^2. Combine the two trapezia to form a rectangle of $25 \times (40+25)$ cm^2. In total you get a rectangle of $25 \times (40+40+25)$ cm^2. *Right?* Draw a sketch.
(c) Now form a rectangle of $(40+25) \times 40$ cm^2. Place the triangles inside so that the 40-cm side of each of them lies on one of the (40+25)-sides of the rectangle and the adjacent accute angle lies in one corner of the rectangle (symmetric figure). Draw a sketch.
The triangles touch each other at the diagonal of the rectangle. *Right?*

[3] A trapezium has two parallel sides; a rectangle has two pairs of parallel sides, and all four angles are right angles.

And the two trapezia fit exactly into the spaces left. *Right?*
(d) The area of the rectangle of (b) is $25\times\ 105 = 2625$ cm^2, and the area of the rectangle of (c) is $65 \times 40 = 2600$ cm^2. *Right?*
What's wrong?

44. (3 points) Verify that

$$U(x,y) := \ln \frac{4}{(1+x^2+y^2)^2}$$

solves the Bratu problem:

$$\left\{\begin{array}{rcll} -\Delta u & = & 2e^u & \text{in } \Omega_1 := \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}, \\ u & = & 0 & \text{on } \partial\Omega_1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\} \end{array}\right.$$

(See Examples 2.5 of the lecture notes.)

45. *(2 points) Let A be a matrix, $A\in\mathbb{R}^{n\times n}$. In Problem 31 we defined the norm

$$\|A\|_1 \quad := \quad \max_{1\le k\le n} \sum_{j=1}^n |a_{jk}|,$$

and it was shown that $\|Ax\|_1 \le \|A\|_1\|x\|_1$ for $x\in\mathbb{R}^n$, $\|x\|_1 = \sum_{j=1}^n |x_j|$).

(a) Find $x\in\mathbb{R}^n$ such that $\|x\|_1 = 1$ and $\|Ax\|_1 = \|A\|_1\|x\|_1$.

(b) Show that $\|A\|_1$ is the operator norm of the operator

$$\begin{array}{rcl} A : (\mathbb{R}^n, \|\cdot\|_1) & \to & (\mathbb{R}^n, \|\cdot\|_1) \\ x & \mapsto & Ax. \end{array}$$

46. (1 point) Let $m\in\mathbb{N}$. Show that $H^m((a,b),\mathbb{R}) \subset L^2((a,b),\mathbb{R})$ is a linear space.

47. (2 points) Find the weak derivative of $u(x) = |\cos(x)|$ on $[0,2\pi]$ and prove that it is indeed the weak derivative!

48. (7 points) Let

$$u(x) := \left\{\begin{array}{ll} 1+x & -1<x\le 0 \\ 1-x & 0<x<1 \end{array}\right. , \quad v(x) := \sin(\pi x), \quad w(x) := \cos(\frac{\pi x}{2})$$

(u is a linear approximation of w.)

(a) Show that u and v are in $H^1((-1,1),\mathbb{R})$!

(b) Show that u and v are orthogonal in $L^2((-1,1),\mathbb{R})$ as well as in $H^1((-1,1),\mathbb{R})$!

(c) Calculate the distance of u and w in $L^2((-1,1),\mathbb{R})$ as well as in $H^1((-1,1),\mathbb{R})$!

(d) Verify the Schwarz inequality with u and $u-v$ in $L^2((-1,1),\mathbb{R})$ as well as in $H^1((-1,1),\mathbb{R})$!

49. (8 + *3 points) Consider the Euclidean Hilbert space $(\mathbb{R}^n, < \cdot, \cdot >_2)$. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix having n real eigenvalues $\lambda_j > 0,\ j = 1, \ldots, n$ corresponding to eigenvectors $\{v_j\}$. Then $\{v_1, ..v_n\}$ is a basis of $\mathbb{R}^n$.

(a) Consider the map $A : (\mathbb{R}^n, \| \cdot \|_2) \to (\mathbb{R}^n, \| \cdot \|_2)$. Show that the operator norm is

$$\|A\| = \max_{j \in \{1,..,n\}} \lambda_j.$$

(b) Define

$$\begin{array}{lll} a: & \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, & a(u,v) := u^t A v, \\ L: & \mathbb{R}^n \to \mathbb{R}, & Lv := b^t v, \qquad b \in \mathbb{R}^n \text{ fixed.} \end{array}$$

(c) Show that a is a bilinear, symmetric, continuous, coercive and positive definite form!

(d) * Show that the maps a and L are continuously differentiable.

(e) Show that there is a unique minimizer of

$$F(v) := \frac{1}{2} a(v,v) - L(v),$$

and for the minimizer v_0 give an estimate of $\|v_0\|_2$!

(f) Find the minimum of the functional $F : \mathbb{R}^2 \to \mathbb{R}$ for

$$A := \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad b := (1, -1)^t$$

and check the estimate.

50. (4 points) Consider the boundary value problem (BVP)

$$-u''(x) = f(x) \quad \text{in } (0,2), \quad u(0) = u(2) = 0,$$
$$f(x) = \begin{cases} 0 & \text{for } x \in (0,1) \\ 6x - 4 & \text{for } x \in (1,2) \end{cases}.$$

Show that $f \in L^2((0,2), \mathbb{R})$ and that

$$u(x) = \begin{cases} x & \text{for } x \in [0,1) \\ -x^3 + 2x^2 & \text{for } x \in [1,2] \end{cases}$$

is in $H^2([0,2], \mathbb{R})$ and that it is weak solution of the BVP.

A.3 Answers and Hints for Selected Problems

1. (3 points) *Answer the following questions.*

- *What is an open ball in $\mathbb{R}$? and in $\mathbb{R}^2$?*
  Repeat Definition A.1.
- *Is an open ball in $\mathbb{R}$ an open subset of $\mathbb{R}^2$?*
  No. An open interval (open ball in $\mathbb{R}$) does not contain open circles (open balls in $\mathbb{R}^2$).
- *Give an example of a closed set in $\mathbb{R}$. Is it closed in $\mathbb{R}^2$?*
  The interval $\overline{B(0,1)} := \{x \in \mathbb{R} : |x| \leq 1\} =: [-1, 1]$ is closed both in $\mathbb{R}$ and in $\mathbb{R}^2$ because both complements are open.
- *Is the empty set $\emptyset$ closed? Is it open?*
  Yes; Yes; because its complement, the full set, is both open and closed.

Give an example of a subset of $\mathbb{R}$ that is neither open nor closed.
An interval $[a, b[:= \{x \in \mathbb{R} : a \leq x < b\}$ is closed at a but open at b. Thus it is neither open nor closed.

2. (3 points) *Prove Lemma 1.7.*
Let V be a linear space over $I\!K$ and $W \subset V$ a subset. We have to show that Definition 1.1 is satisfied for W with the addition and scalar multiplication defined in V iff W satisfies (i) and (ii).
'$\Rightarrow$': Suppose that $W \subset V$ is a linear space w.r.t. the addition and scalar multiplication of V. Then '$+$' and '$\cdot$' satisfy for all $u, v \in W,\ \alpha \in I\!K$ according to Definition 1.1: $+ : W \times W \to W,\ (u, v) \mapsto u + v \in W$ and $\cdot : I\!K \times W \to W,\ (\alpha, v) \mapsto \alpha \cdot v \in W$. Thus (i) and (ii) are satisfied.
'$\Leftarrow$': suppose $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$ and $\alpha \cdot w \in W$ for all $\alpha \in I\!K,\ w \in W$. In this case $+ : V \times V \to V,\ (u, v) \mapsto u + v$ defined in V and restricted to $W \subset V$ satisfies also $+ : W \times W \to W,\ (u, v) \mapsto u + v \in W$. Same way it follows that the scalar multiplication defined on $I\!K \times V$ satisfies $I\!K \times W \to W,\ (\alpha, w) \mapsto \alpha \cdot w \in W$ when restricted to $W \subset V$. All other conditions of Definition 1.1 are satisfied in $W \subset V$ because they are satisfied in V. In particular $v = 0 \in W \cap V$ and $\bar{w} = -w \in W \cap V$ because $0,\ -1 \in I\!K$ and $\alpha \cdot w \in W$, see Remark 1.2. Thus W is a linear space w.r.t. the addition and multiplication of V.

3. (1 point) *Fix $n \in \mathbb{N}$. Let $P^n(\mathbb{R}, \mathbb{R}) \subset P(\mathbb{R}, \mathbb{R})$ be the set of polynomials p of degree $\deg(p) \leq n$ with real coefficients. Show that $P^n(\mathbb{R}, \mathbb{R})$ is a linear space over $\mathbb{R}$.*

It has to be shown that $P^n(\mathbb{R}, \mathbb{R})$ is closed w.r.t. the addition and scalar multiplication of $P(\mathbb{R}, \mathbb{R})$ (see Definition 1.8). The statement then follows from Lemma 1.7 and Example 1.12.

4. (2 points) *Let $V_a := \{(u_1, \ldots, u_n)^t \in \mathbb{R}^n \mid u_1 = a\}$. Is V_a a subspace of $\mathbb{R}^n$*
• *for $a = 0$?* • *for $a = 1$?* *Why?*
Let $v, w \in V_a,\ \alpha \in \mathbb{R}$.
Assume $a = 0$. Then $v_1 + w_1 = 0$ and $v + w \in V_a$; also $\alpha \cdot v_1 = 0$ and $\alpha \cdot v = 0 \in V_a$.

Thus V_a is closed w.r.t. addition and scalar multiplication in $\mathbb{R}^n$. According to Lemma 1.7, V_a is a subspace of $\mathbb{R}^n$.
For $a = 1$, $v_1 + w_1 = 2a \neq a$, thus $v + w \notin V_a$: not a subspace.

5. **(3 points)** *Let V be a linear space and W_1 and W_2 linear subspaces of V. Is $W_1 \cap W_2$ a linear subspace of V?*
Let $u, v \in W_1 \cap W_2$ and $\alpha \in \mathbb{R}$. Then $u, v, \alpha \cdot u$ and $u + v \in W_i$, $i = 1, 2$. Thus $\alpha \cdot u$ and $u + v \in W_1 \cap W_2$ and $W_1 \cap W_2$ is a linear subspace of V.
And $W_1 \cup W_2$?
Let $u, w \in W_1 \cup W_2$. Then, in general, we do not know if u is in W_1 or in W_2 and what about $u+v \ldots$. Thus we have to find a counterexample to prove that $W_1 \cup W_2$ is *not* a subspace of V. Let $V = \mathbb{R}^2$, $W_1 := \{u \in V | u_2 = 0\}$, $W_2 := \{u \in V | u_1 = 0\}$. Then $v := (1, 0)^t \in W_1$ and $w := (0, 1)^t \in W_2$, but $v + w = (1, 1)^t \notin W_1 \cup W_2$.
And $W_1 + W_2 := \{v = u + w \in V |\ u \in W_1, w \in W_2\}$?
Let $v_1, v_2 \in W_1 + W_2$, $\alpha \in I\!\!K$. Then v_i may be written as $v_i = u_i + w_i$, $i = 1, 2$. Then $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$ in $V \Rightarrow v_1 + v_2 \in W_1 + W_2$. Also we get $\alpha v_1 = (\alpha u_1) + (\alpha w_1) \in V \cap (W_1 + W_2)$. Now we apply Lemma 1.7.

6. **(4 points)** *Consider a set $V \subset \mathbb{C}$. Define addition and scalar multiplication in V through addition and multiplication in the field $\mathbb{C}$.*
• *Is $V := \mathbb{R}$ a linear space over the field $\mathbb{C}$?*
Let $\alpha = i \in \mathbb{C}$ and $v = 1 \in \mathbb{R}$. Then $\alpha \cdot v = i \notin V = \mathbb{R}$. Thus $\mathbb{R}$ is not a linear space over the field $\mathbb{C}$.
• *Is $V := \mathbb{C}$ a linear space over the field $\mathbb{R}$?*
Yes, because $\mathbb{C}$ is a linear space over $\mathbb{C}$ and $\mathbb{C}$ is closed w.r.t. addition in $\mathbb{C}$ and w.r.t. scalar multiplication by real numbers.

7. ***(3 points)** *A function $f : [a, b] \mapsto \mathbb{R}$ is called* Lipschitz-continuous *if there is a constant $L \geq 0$ such that for all x_1, $x_2 \in [a, b]$*
$$|f(x_1) - f(x_2)| \leq L\, |x_1 - x_2|. \tag{A.9}$$
The space of all such Lipschitz-continuous functions is called $Lip\,([a, b], \mathbb{R})$. Show that
$$C^1([0, 1], \mathbb{R}) \subset Lip\,([0, 1], \mathbb{R}) \subset C([0, 1], \mathbb{R}). \tag{A.10}$$
Let $f \in C^1([0, 1], \mathbb{R})$. Then a continuous derivative f' exists on $[0, 1]$ and is bounded by $\max_{x\in[0,1]} |f'(x)| =: L$. Furthermore, the Mean Value Theorem is applicable. For every $x_1, x_2 \in [0, 1]$, $x_1 < x_2$, there exists some $\xi \in [x_1, x_2]$ such that
$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2).$$
Thus $|f(x_1) - f(x_2)| \leq |f'(\xi)|\ |(x_1 - x_2)| \leq L|(x_1 - x_2)|$. Thus (A.9) is satisfied and $C^1([0, 1], \mathbb{R}) \subset Lip\,([0, 1], \mathbb{R})$.
Now let $f \in Lip\,([0, 1], \mathbb{R})$. Given $\varepsilon > 0$, choose $\delta = \varepsilon/L$ to obtain
$$|f(x_1) - f(x_2)| < \varepsilon \qquad \text{for } |x_1 - x_2| < \delta.$$
Thus f is continuous and $Lip\,([0, 1], \mathbb{R}) \subset C([0, 1], \mathbb{R})$.

8. (2 points) *Let* $E := \{y \in \mathbb{R} : y = x^2 \text{ for } x \in (-1,1)\}$. *Is* E *bounded? If yes, give a lower and/or upper bound of* E. *Give also* $\inf(E)$ *and* $\sup(E)$. *Is the infimum a minimum and/or the supremum a maximum?*
For $-1 < x < 1$ we get $0 \leq x^2 < 1$. Thus $E = [0,1)$. Lower bounds of E are for instance -1 and 0, upper bounds are for instance 1 and 2. $\inf(E) = \min(E) = 0 \in E$; $\sup(E) = 1 \notin E$. $\max(E)$ does not exist.

9. (2 points) *Consider* $B := \{1, 1+x, 1+x^2, x^3\} \subset C(\mathbb{R},\mathbb{R})$.
Are the elements of B *linearly independent over* $\mathbb{R}$*?*
Yes. Suppose there are real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that
$\alpha \cdot 1 + \beta \cdot 1 + \beta \cdot x + \gamma \cdot 1 + \gamma \cdot x^2 + \delta \cdot x^3 = 0$.
Then $\delta = \gamma = \beta = 0$ and thus also $\alpha = 0$.
Name the space they generate, i.e. the subspace $W = span\{B\}$ *of* $C(\mathbb{R},\mathbb{R})$ *which is spanned by* B.
$W = P^3(\mathbb{R},\mathbb{R})$ defined in Problem 3.
Give a basis of $span\{B\}$.
Since the elements of B are linearly independent over $\mathbb{R}$ and $span\{B\}$ is the linear space generated by B, B is a basis of $span\{B\}$.

10. (2 points) *Let* $A: \mathbb{R}^3 \to \mathbb{R}^3$ *be the linear map*

$$x \mapsto Ax := \begin{pmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{pmatrix} x$$

Determine the dimensions of

$$\begin{aligned} \mathrm{Ker}(A) &:= \{x \in \mathbb{R}^3 \,|\, Ax = 0\}, \\ \mathrm{Im}(A) &:= \{y \in \mathbb{R}^3 \,|\, y = Ax \text{ for some } x \in \mathbb{R}^3\}. \end{aligned}$$

Ker $(A) = span\{(9,2,-4)^t\}$, dim Ker $(A) = 1$. Im $(A) = span\{(2,0,0)^t, (1,2,2)^t\}$, dim Im $(A) = 2$. The proofs are left to the reader.

11. (2 points) *To which spaces* $C^m(\Omega,\mathbb{R})$ *do the following functions belong?*

$$\begin{aligned} u(x) &= |x| \text{ for } x \in (-1,1), \\ u(x,y) &= \sin(x)(1-y) \text{ for } (x,y) \in [0,\pi] \times [0,1] \end{aligned}$$

Results: $u(x) = |x| \in C(\Omega,\mathbb{R})$ for $\Omega = (-1,1)$ (not differentiable in $x = 0$);
$u(x,y) \in C^\infty(\Omega,\mathbb{R})$ for $\Omega = [0,\pi] \times [0,1]$.

12. (2 points) *Let* $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ *and define the map*

$$\begin{aligned} L:\ C^2(\Omega,\mathbb{R}) &\to C(\Omega,\mathbb{R}) \\ u &\mapsto \Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \end{aligned}$$

Is L *a linear map?* Yes: verify that the conditions of Definition 1.18 are satisfied.
Is there some $u \in C^2$, $u(x,y) \not\equiv 0$, *such that* $Lu = 0$*? Give an example!*
$u(x,y) := x^2 - y^2$. Then $\Delta u = 2 - 2 = 0$ for all $(x,y) \in \Omega$. Thus $\mathrm{Ker}(L) \neq \{0\}$.

13. (8 points) *Let $V := \{g \in C^2([-1,1],\mathbb{R}) : g(-1) = g(1) = 0\}$ and define*

$$\begin{aligned} <\cdot,\cdot>: &\qquad V \times V \to \mathbb{R} \\ <u,v> &:= 2\int_{-1}^{1} u''(x)v''(x)dx \end{aligned}$$

(a) *Show that V is a linear space.*
$V \subset C^2([-1,1],\mathbb{R})$ is a subset of a linear space, and $f,g \in V$, $\alpha,\beta \in \mathbb{R} \Rightarrow \alpha f(-1) + \beta g(-1) = 0 = \alpha f(1) + \beta g(1); \Rightarrow (\alpha f + \beta g)(-1) = 0 = (\alpha f + \beta g)(1)$. $\Rightarrow V$ is closed in $C^2([-1,1],\mathbb{R})$ w.r.t. the addition and scalar multiplication of $C^2([-1,1],\mathbb{R})$. From Lemma 1.7 it follows that V is a subspace and thus a linear space.

(b) *Show that $<\cdot,\cdot>$ is an inner product.*
According to Definition 1.25 we have to show that $<\cdot,\cdot>$ is bilinear, symmetric and positive definite.
positive semi-definite: $u \in C^2 \Rightarrow u'' \in C$, u'' is continuous and integrable, and $(u''(x))^2 \geq 0 \Rightarrow <u,u>= 2\int_{-1}^{1}(u''(x))^2\,dx \geq 0$.
definite: $u(x) = 0 \Rightarrow u''(x) = 0 \Rightarrow 2\int_{-1}^{1}(u''(x))^2\,dx = 0$; Now assume $2\int_{-1}^{1}(u''(x))^2\,dx = 0$; $\Rightarrow u''(x) = 0 \Rightarrow u'(x) = c \Rightarrow u(x) = cx + d$ for some $c,d \in \mathbb{R}$; and from the boundary conditions $u(-1) = -c+d = 0 = c+d = u(1)$ $\Rightarrow 2d = 0 \Rightarrow c = 0 \Rightarrow u(x) = 0$. Thus $<\cdot,\cdot>$ is positive definite.
symmetric: $<u,v>= 2\int_{-1}^{1} u''(x)\,v''(x)\,dx = 2\int_{-1}^{1} v''(x)\,u''(x)\,dx =<v,u>$.
linear in the first variable: $<\alpha u_1 + \beta u_2, v>= 2\int_{-1}^{1}(\alpha u_1''(x) + \beta u_2''(x))\,v''(x)\,dx$ $= 2\alpha\int_{-1}^{1} u_1''(x)v''(x)\,dx + 2\beta\int_{-1}^{1} u_2''(x)v''(x)\,dx = \alpha <u_1,v> +\beta <u_2,v>$.
From this and the symmetry it follows that it is bilinear.

(c) *Define the corresponding natural norm of the inner product space V.*
$||u|| := (<u,u>)^{1/2}$ is the natural norm of the inner product $<\cdot,\cdot>$.

(d) *Are $u = x^2 - 1$ and $v = \sin(\pi x)$ orthogonal with respect to $<\cdot,\cdot>$?*
$u(x) = x^2 - 1,\ u'(x) = 2x,\ u''(x) = 2;$
$v(x) = \sin(\pi x),\ v'(x) = \pi\cos(\pi x),\ v''(x) = -\pi^2\sin(\pi x);$

$$\begin{aligned} <u,v> &= 2\int_{-1}^{1} 2\left(-\pi^2\sin(\pi x)\right)\,dx = 4\pi\int_{-1}^{1}\left(-\pi\sin(\pi x)\right)\,dx \\ &= 4\pi\,\cos(\pi x)|_{-1}^{1} = 4\pi\left(\cos(\pi) - \cos(-\pi)\right) = 0. \end{aligned}$$

Yes, $u = x^2 - 1$ and $v = \sin(\pi x)$ are orthogonal with respect to $<\cdot,\cdot>$.

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By now you are quite experienced in solving homework problems,
so we will skip from now on some of the sample solutions.
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19. (6 points) *Let $\|\cdot\|$ be a* natural *norm on V. Show that for any $x,y,z \in V$*

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\|z - \frac{x+y}{2}\|^2.$$

A natural norm satisfies $||x||^2 =< x, x >$. We give two different proofs.
First proof:

$$\begin{aligned}
||z-x||^2 &= < z-x, z-x > \\
&= < z,z > - < z,x > - < x,z > + < x,x > \\
&= < z,z > -2 < z,x > + < x,x >, \qquad \text{symmetry of inner products} \\
||z-y||^2 &= < z,z > -2 < z,y > + < y,y > \\
||x-y||^2 &= < x,x > -2 < x,y > + < y,y > \\
||z-\frac{x+y}{2}||^2 &= < z,z > -2 < z, \frac{x+y}{2} > + < \frac{x+y}{2}, \frac{x+y}{2} >
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\,||x-y||^2 + 2\,||z-\frac{x+y}{2}||^2 = &\frac{1}{2} < x,x > - < x,y > + \frac{1}{2} < y,y > + \\
&+ 2 < z,z > -2 < z, x+y > + < x+y, \frac{x+y}{2} > \\
= &\frac{1}{2} < x,x > - < x,y > + \frac{1}{2} < y,y > +2 < z,z > -2 < z,x > -2 < z,y > + \\
&+ \frac{1}{2} < x,x > + \frac{1}{2} < y,y > + < x,y > \\
= &< x,x > + < y,y > +2 < z,z > -2 < z,x > -2 < z,y > \\
= &2 < z,z > -2 < x,z > -2 < y,z > + < x,x > + < y,y > \\
= &\|z-x\|^2 + \|z-y\|^2
\end{aligned}$$

Second, more elegant proof:
Apply the Parallelogram Identity

$$2\,||u||^2 + 2\,||v||^2 = ||u+v||^2 + ||u-v||^2$$

with $u := z-x$ and $v := z-y \Rightarrow$

$$\begin{aligned}
2\,||z-x||^2 + 2\,||z-y||^2 &= ||z-x+z-y||^2 + ||z-x-z+y||^2 \\
&= ||2z-(x+y)||^2 + ||-(x-y)||^2 \\
&= 4\,||z-\frac{x+y}{2}||^2 + ||x-y||^2 \qquad \text{use } \ ||\alpha x||^2 = \alpha^2 ||x||^2
\end{aligned}$$

27. (12 points) *For $x \in [-1,1] \subset \mathbb{R},\ n \in \mathbb{N},\ n \geq 2,$ define*

$$f_n(x) := \begin{cases} 1+x & \text{for } -1 \leq x < -\frac{1}{n} \\ 1-\frac{2}{n}-x & \text{for } -\frac{1}{n} \leq x \leq 0 \\ 1-\frac{2}{n}+x & \text{for } 0 < x \leq \frac{1}{n} \\ 1-x & \text{for } \frac{1}{n} < x \leq 1 \end{cases}$$

(a) *Draw a sketch of f_n!*
$f_n,\ n \geq 2$ fixed, is piecewise linear. Its graph connects in $\mathbb{R}^2$ the points $P_1 = (-1,0);\ P_2 = (-\frac{1}{n}, 1-\frac{1}{n});\ P_3 = (0, 1-\frac{2}{n});\ P_4 = (1/n, 1-\frac{1}{n});\ P_5 = (1,0).$
It thus has the shape of a letter 'M' and is symmetric w.r.t. the y-axis.

(b) *Calculate the point-wise limit $f(x) := \lim_{n\to\infty} f_n(x)$ for fixed $x \in [-1,1]$!*
We note that $P_2(n)$ and $P_4(n)$ converge for $n \to \infty$ to $P_3 = (0,1)$. We thus have to consider negative and positive x separately.
For every fixed $x \in [-1,0)$ exists some $N \in \mathbb{N}$ such that $-1 \le x < -\frac{1}{n}$ for all $n > N$, $\Rightarrow f_n(x) = 1 + x$ for all $n > N$, $\Rightarrow$

$$\lim_{n\to\infty} f_n(x) = 1 + x \qquad \text{for} \qquad x \in [-1,0).$$

For every fixed $x \in (0,1]$ exists some $\tilde{N} \in \mathbb{N}$ such that $\frac{1}{n} < x \le 1$ for all $n > \tilde{N}$; $\Rightarrow f_n(x) = 1 - x$ for all $n > \tilde{N}$, $\Rightarrow$

$$\lim_{n\to\infty} f_n(x) = 1 - x \qquad \text{for} \qquad x \in (0,1].$$

Moreover $f_n(0) = 1 - 2/n$, $\Rightarrow \lim_{n\to\infty} f_n(0) = 1$. Thus

$$f(x) = \begin{cases} 1 + x & \text{for } \; -1 \le x \le 0 \\ 1 - x & \text{for } \quad 0 < x \le 1 \end{cases}.$$

(c) *To which spaces $C^m([-1,1],\mathbb{R})$ do f_n and f belong?*
$f_n(-1/n) = 1 - 2/n + 1/n = 1 - 1/n = \lim_{x\to(-1/n)^-} f_n(x)$. Thus f_n is continuous in $P_2(n)$. In the same way it has to be shown that f_n is also continuous in $P_3(n)$ and in $P_4(n)$.
f is continuous in $x = 0$ because

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 + x = 1 = \lim_{x\to 0^+} 1 - x = \lim_{x\to 0^+} f(x).$$

f is not continuously differentiable in $x = 0$ because

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} 1 = 1 \ne -1 = \lim_{x\to 0^+} f'(x).$$

f is a polynomial in $[-1,0)$ and in $(0,1]$ $\Rightarrow$ it is infinitely often differentiable there. Collecting these results follows that $f \in C^0([-1,1],\mathbb{R})$ and $f \notin C^1([-1,1],\mathbb{R}) \Rightarrow f \in C^m([-1,1],\mathbb{R})$ only for $m = 0$. Same way it has to be shown that $f_n \in C^m([-1,1],\mathbb{R})$ only for $m = 0$.

(d) *Does $(f_n)_{n\in\mathbb{N}}$ converge to f in the supremum norm on $C([-1,1],\mathbb{R})$?*
$||f||_\infty = \sup_{x\in[-1,1]} |f(x)|$, $\quad (f_n)$ converges to f iff $\lim_{n\to\infty} ||f_n - f|| = 0$;

$$\begin{aligned} f_n(x) - f(x) &= 0 \quad \text{for} \quad x \in [-1,-1/n), \\ f_n(x) - f(x) &= 1 - 2/n - x - 1 - x = -2/n - 2x \;\text{ in }\; [-1/n,0], \end{aligned}$$

and $-2/n \le -2/n - 2x \le 0$ in $[-1/n,0]$, and both bounds are attained. Thus $0 \le |f_n(x) - f(x)| \le 2/n$ in $[-1,0]$. As can be shown in the same way, also $0 \le |f_n(x) - f(x)| \le 2/n$ in $[0,1]$. Thus

$$||f_n - f||_\infty = \sup_{x\in[-1,1]} |f_n(x) - f(x)| = \frac{2}{n} \qquad \text{and} \qquad \lim_{n\to\infty} ||f_n - f||_\infty = 0.$$

Using sketches as arguments is usually faster, but often misleading and thus no mathematical proof, see Remark 43.

(e) *Does $(f_n)_{n\in\mathbb{N}}$ converge to f in the L^2-norm on $C([-1,1],\mathbb{R})$?*
$||f||_{L^2} = \left(\int_{-1}^1 f(x)^2dx\right)^{1/2}$
Yes, $||f_n-f||_{L^2} = \left(\int_{-1}^1 (f_n(x)-f(x))^2dx\right)^{1/2}$ does converge to zero for $n\to\infty$.
This will be proved in two different ways.
First proof:

$$\begin{aligned}
||f_n-f||_{L^2}^2 &= \int_{-1}^1 (f_n(x)-f(x))^2dx \\
&= 0+\int_{-1/n}^0 (-\frac{2}{n}-2x)^2dx+\int_0^{1/n}(-\frac{2}{n}+2x)^2dx+0, \quad \text{see } (d) \\
&= \int_{-1/n}^0 (4x^2+\frac{8x}{n}+\frac{4}{n^2})dx+\int_0^{1/n}(4x^2-\frac{8x}{n}+\frac{4}{n^2})dx, \\
&= \left(\frac{4}{3}x^3+\frac{4x^2}{n}+\frac{4}{n^2}x\right)\Big|_{-1/n}^0+\left(\frac{4}{3}x^3-\frac{4x^2}{n}+\frac{4}{n^2}x\right)\Big|_0^{1/n} \\
&= \left(\frac{4}{3n^3}-\frac{4}{n^3}+\frac{4}{n^3}\right)+\left(\frac{4}{3n^3}-\frac{4}{n^3}+\frac{4}{n^3}\right) \\
&= \frac{8}{3n^3} \qquad \to 0 \quad \text{for} \quad n\to\infty.
\end{aligned}$$

Second, more elegant proof:
In $C([-1,1],\mathbb{R})$, $||\cdot||_\infty$ is stronger than $||\cdot||_{L^2}$. Thus a constant $C>0$ exists s.th.

$$||f_n-f||_{L^2}\le C\,||f_n-f||_\infty\to 0 \quad \text{for} \quad n\to\infty.$$

We have

$$||f||_{L^2}^2=\int_{-1}^1 f(x)^2dx\le\int_{-1}^1(\sup|f(x)|)^2dx=||f||_\infty^2\int_{-1}^1 1dx$$

$\sup_{x\in[-1,1]}|f(x)|$ exists because $[-1,1]$ is bounded and closed and f is continuous; $C:=\left(\int_{-1}^1 1dx\right)^{1/2}=\sqrt{2}$.

30. (3 points) *Prove Theorem 1.34 of the lecture notes.*
'$\Rightarrow$': obmitted here because it is straightforward and easy.
'$\Leftarrow$': Let V be a normed space with norm $||\cdot||$. Assume that the parallelogram identity (1.22) is valid and define

$$<u,v>:=\frac{1}{4}(||u+v||^2-||u-v||^2). \tag{A.11}$$

V is a linear space because it is a normed space, see Definition 1.30. Thus it remains to show that $<\cdot,\cdot>$ defined in (A.11) is an inner product such that (1.20) is valid for $<\cdot,\cdot>$ and $||\cdot||$.
Equality (1.20) follows immediately: $<u,u>=\frac{1}{4}||u+u||^2-0=\frac{1}{4}4\,||u||^2=||u||^2$.

$< \cdot, \cdot >$ is an inner product:
symmetric: $4 < u, v >= ||u+v||^2 - ||u-v||^2 = ||v+u||^2 - (-1)^2||v-u||^2 = 4 < v, u >$.
positive definite: from the validity of (1.20) follows that $< \cdot, \cdot >$ is positive definite because norms are positive definite.
Now we show the linearity of $< \cdot, \cdot >$ w.r.t. one of its variables. Bilinearity then follows from the symmetry.

$$\begin{aligned} < u, v > + < w, v > &= \frac{1}{4}\left(||u+v||^2 - ||u-v||^2 + ||w+v||^2 - ||w-v||^2\right) \\ &= \frac{1}{2}\left(||\frac{u+w}{2} + v||^2 - ||\frac{u+w}{2} - v||^2\right) \qquad \text{by eq. (1.22)} \\ &= 2 < \frac{u+w}{2}, v > \end{aligned} \tag{A.12}$$

From this follows $< u, v > + < w, v >=< u+w, v >$ because

$$\begin{aligned} 2 < \frac{u}{2}, v > &= < u, v > + < 0, v > \qquad w = 0 \text{ in (A.12)}, \\ &= < u, v > \qquad u = 0 \text{ in (A.11)}. \end{aligned}$$

Iterating this we get $\alpha < u, v >=< \alpha\, u, v >$ for $\alpha = m\ 2^{-n},\ m, n \in \mathbb{Z}$.
Now let an arbitrary $\alpha \in \mathbb{R}$ be given. Then we can find a sequence $(\alpha_\nu) = (m_\nu 2^{-n_\nu})$, $\nu \in \mathbb{N}$, $m_\nu,\ n_\nu \in \mathbb{Z}$ such that $\lim_{\nu\to\infty} \alpha_\nu = \alpha$. V is a normed space, thus $\lim_{\nu\to\infty} ||\alpha_\nu u + v|| = ||\alpha u + v||$ and similarly $\lim_{\nu\to\infty} ||\alpha_\nu u - v|| = ||\alpha u - v||$. Thus (A.11) leads to $\alpha < u, v >=< \alpha\, u, v >$ for arbitrary $\alpha \in \mathbb{R},\ u, v \in V$. $\diamond$

40. *(10 points) *Give* simple *examples from your field of interest for*

(a) a differential equation with classical solutions;

(b) a differential equation with weak solutions.

see Section 2.1.2 for examples by other people.

41. (2 points) *Consider the sequence $(f_n)_{n\in\mathbb{N}\setminus\{1\}}$ given in $C([0,1],\mathbb{R})$ by*

$$f_n(x) := \begin{cases} xn & \textit{for } 0 \le x < \frac{1}{n} \\ 2 - nx & \textit{for } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \textit{for } \frac{2}{n} \le x \le 1 \end{cases} \qquad n \ge 2.$$

(a) Draw a sketch of f_n.

(b) Compute the pointwise limit.

Results: (a) the graph of f_n connects the points $P_1(n) = (0,0)$; $P_2(n) = (1/n, 1)$; $P_3(n) = (2/n, 0)$ and $P_4(n) = (1, 0)$.
(b) The sequence (f_n) has the pointwise limit $f \equiv 0$.
For methods of proof see Problem 27.

42. (5 points) *Let $f, g : (0,2) \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 & \textit{if } 0 < x \le 1 \\ 1 & \textit{if } 1 < x < 2 \end{cases}, \quad g(x) = \begin{cases} 2x & \textit{if } 0 < x \le 1 \\ 0 & \textit{if } 1 < x < 2 \end{cases}.$$

(a) Draw a sketch of f and g.

(b) Show that g is the weak derivative of f by using the definition of weak derivatives.

(c) Is g also the weak derivative of

$$h(x) = \begin{cases} x^2 & \textit{if } 0 < x \le 1 \\ 2 & \textit{if } 1 < x < 2 \end{cases} ?$$

(b) To satisfy Definition 2.6 , $f, g \in L^2([0,2],\mathbb{R})$ is necessary. Note that both f and g are not defined in the end points of the interval $(0,2)$. But they are defined a.e. in $[0,2]$ as is required for Lebesgue integrals and L^2-functions.

$$\begin{aligned} \int_0^2 |f(x)|^2\,dx &= \int_0^1 x^4\,dx + \int_1^2 1\,dx = \frac{1}{5} + 1 < \infty, \\ \int_0^2 |g(x)|^2\,dx &= \int_0^1 4\,x^2\,dx + \int_1^2 0\,dx = \frac{4}{3} + 0 < \infty. \end{aligned}$$

Thus $f, g \in L^2([0,2],\mathbb{R})$. Now let $v \in C_0^1([0,2],\mathbb{R})$ be arbitrary. Then

$$\begin{aligned} \int_0^2 g(x)v(x)\,dx &= \int_0^1 2x\,v(x)\,dx + \int_1^2 0\,v(x)\,dx \\ &= \int_0^1 (x^2)'\,v(x)\,dx + \int_1^2 (1)'\,v(x)\,dx \qquad \text{now integrate by parts} \\ &= x^2\,v(x)\big|_0^1 - \int_0^1 x^2\,v'(x)\,dx + 1\,v(x)|_1^2 - \int_1^2 1\,v'(x)\,dx \\ &= v(1) - \int_0^1 f(x)\,v'(x)\,dx - v(1) - \int_1^2 f(x)\,v'(x)\,dx \\ &= -\int_0^2 f(x)\,v'(x)\,dx. \end{aligned}$$

Thus g is the weak derivative of f.
(c) No, g is not the weak derivative of h:

$$\begin{aligned} \int_0^2 g(x)v(x)\,dx &= \int_0^1 2x\,v(x)\,dx + \int_1^2 0\,v(x)\,dx \\ &= \int_0^1 (x^2)'\,v(x)\,dx + \int_1^2 (2)'\,v(x)\,dx \qquad \text{now integrate by parts} \\ &= x^2\,v(x)\big|_0^1 - \int_0^1 x^2\,v'(x)\,dx + 2\,v(x)|_1^2 - \int_1^2 2\,v'(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= v(1) - \int_0^1 h(x)\, v'(x)\, dx - 2\, v(1) - \int_1^2 h(x)\, v'(x)\, dx \\
&= -v(1) - \int_0^2 h(x)\, v'(x)\, dx, \\
&\neq - \int_0^2 h(x)\, v'(x)\, dx \qquad \text{if } \; v(1) \neq 0.
\end{aligned}
$$

Such v exists: Let $v(x) := x\,(x-2)$. Then $v \in C_0^1([0,2],\mathbb{R})$ and $v(1) = -1$. Thus g and h do not satisfy Definition 2.6 with arbitrary $v \in C_0^1([0,2],\mathbb{R})$.

43. Remark (Sketches) *In many situations, sketches are very helpful for finding mathematical proofs. Note however: A sketch is not equivalent to a mathematical proof and cannot replace a proof. It might be misleading.*

Example: *Given two identical triangles and two identical trapezia. Combine them in two different ways to form two different rectangles*[4].
triangles: *1 right angle, side lengths are 25 cm, 40 cm and* $\sqrt{25^2+40^2}$ *cm.*
trapezia: *2 right angles, side lengths 40 cm, 25 cm, 25 cm and* $\sqrt{25^2+(40-25)^2}$ *cm.*
(a) Draw a sketch of a triangle and of a trapezium.
(b) Combine the two triangles to get a rectangle of 25 × 40 cm^2*. Combine the two trapezia to form a rectangle of 25 × (40+25) cm*^{2}*. In total you get a rectangle of 25 × (40+40+25) cm*^{2}*. Right? Draw a sketch.*
Yes, right.
(c) Now form a rectangle of (40+25) × 40 cm^2*. Place the triangles inside so that the 40-cm side of each of them lies on one of the (40+25)-cm sides of the rectangle and the adjacent accute angle lies in one corner of the rectangle (symmetric figure). Draw a sketch. The triangles touch each other at the diagonal of the rectangle. Right?*
No, wrong. The inclination of the diagonal is $40/65 \approx 0.615$, the inclination of the side of the triangle is $25/40 = 0.625$.
And the two trapezia fit exactly into the spaces left. Right?
No, wrong again. The inclination of the side of the trapezium is $15/25 = 0.6$.
In a sketch, it looks like the three different lines have the same inclination, but actually they differ, by less than 5%. Such small deviations can hardly be detected by the naked eye.
(d) The area of the rectangle of (b) is 25× 105 = 2625 cm^2*, and the area of the rectangle of (c) is 65 × 40 = 2600 cm*^{2}*. Right?*
Yes, right. The area of the rectangle of (c) is slightly smaller (about 1%) since the figures inside slightly overlap.

45. *(2 points) *Let A be a matrix, $A \in \mathbb{R}^{n\times n}$. In Problem 31 we defined the norm*

$$\|A\|_1 \;:=\; \max_{1\le k\le n} \sum_{j=1}^{n} |a_{jk}|,$$

[4] A trapezium has two parallel sides; a rectangle has two pairs of parallel sides, and all four angles are right angles.

and it was shown that $\|Ax\|_1 \le \|A\|_1 \|x\|_1$ *for* $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_{j=1}^n |x_j|$*).*

(a) *Find* $x \in \mathbb{R}^n$ *such that* $\|x\|_1 = 1$ *and* $\|Ax\|_1 = \|A\|_1 \|x\|_1$.
Assume $\|Ax\|_1 = \sum_{j=1}^n |a_{jm}|$, i.e. that the maximum is attained for $k = m$, in the m-th column. Then choose $\tilde{x} \in \mathbb{R}^n$ such that $\tilde{x}_j := \delta_j^m,\ j = 1, \ldots, n$,

$$\delta_j^m := \begin{cases} 1 & \text{if} \quad j = m, \\ 0 & \text{if} \quad j \neq m, \end{cases} . \tag{A.13}$$

Then

$$\tilde{x}||_1 = \sum_{j=1}^n |\delta_j^m| = 1, \qquad ||A\tilde{x}||_1 = \sum_{j=1}^n | \sum_{j=1}^n a_{jk} \delta_k^m | = \sum_{j=1}^n |a_{jm}| = ||A||_1.$$

(b) *Show that* $\|A\|_1$ *is the operator norm of the operator*

$$\begin{aligned} A : (\mathbb{R}^n, \|\cdot\|_1) &\rightarrow (\mathbb{R}^n, \|\cdot\|_1) \\ x &\mapsto Ax. \end{aligned}$$

According to (1.42) $||A|| = \sup_{||x||_1 = 1} ||Ax||_1$. The statement follows from $\|Ax\|_1 \le \|A\|_1 \|x\|_1$ for all x and $\|A\tilde{x}\|_1 = \|A\|_1 \|\tilde{x}\|_1 = \|A\|_1$.

47. (2 points) *Find the weak derivative of* $u(x) = |\cos(x)|$ *on* $[0, 2\pi]$ *and prove that it is indeed the weak derivative!*
Result:

$$u(x) = \begin{cases} \cos x & \text{for} \quad x \in [0, \pi/2], \\ -\cos x & \text{for} \quad x \in (\pi/2, 3\pi/2], \\ \cos x & \text{for} \quad x \in (3\pi/2, 2\pi], \end{cases} \qquad u'(x) = \begin{cases} -\sin x & \text{for} \quad x \in [0, \pi/2], \\ \sin x & \text{for} \quad x \in (\pi/2, 3\pi/2], \\ -\sin x & \text{for} \quad x \in (3\pi/2, 2\pi]. \end{cases}$$

It has to be shown that $u, u' \in L^2([0, 2\pi], \mathbb{R})$ and that they satisfy Definition 2.6.

49. (8 + *3 points) *Consider the Euclidean Hilbert space* $(\mathbb{R}^n, < \cdot, \cdot >_2)$*. Let* $A \in \mathbb{R}^{n \times n}$ *be a symmetric matrix having n real eigenvalues* $\lambda_j > 0,\ j = 1, \ldots, n$ *corresponding to eigenvectors* $\{v_j\},\ j = 1, \ldots, n$*. Then* $\{v_1, ..v_n\}$ *is a basis of* $\mathbb{R}^n$*.*
We may choose the corresponding basis vectors $\{v_j\}$ as orthonormal, i.e. such that $< v_j, v_k >_2 = \delta_j^k$, with δ_j^k defined as in (A.13):

$$\begin{aligned} < v_j, v_k >_2 &= \frac{\lambda_j}{\lambda_j} < v_j, v_k >_2 = \frac{1}{\lambda_j} < A\, v_j, v_k >_2 = \frac{1}{\lambda_j} (Av_j)^t v_k \\ &= \frac{1}{\lambda_j} v_j^t A^t v_k = \frac{1}{\lambda_j} v_j^t A v_k \\ &= \frac{\lambda_k}{\lambda_j} < v_j, v_k >_2 . \end{aligned}$$

For $j \neq k,\ \lambda_j \neq \lambda_k$ it follows that $< v_j, v_k >_2 = 0$. If A has some multiple eigenvalue λ_i and a corresponding eigenspace spanned by several linearly independent vectors, we may also assume them to be orthogonal to each other. Then we normalise to get $||v_j||_2 = 1,\ j = 1, \ldots, n$.

(a) *Consider the map* $A : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|_2)$. *Show that the operator norm is*

$$\|A\| = \max_{j\in\{1,..,n\}} \lambda_j.$$

With the $\{v_j\}$ chosen we get $||Av_j||_2 = ||\lambda_j v_j||_2 = |\lambda_j| = \lambda_j,\ j = 1, \ldots, n$, and thus from (1.42)

$$||A|| \geq \max_{j=1,\ldots,n} \lambda_j. \tag{A.14}$$

Given an arbitrary $u \in \mathbb{R}^n$, there are $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$ with

$$u = \sum_{j=1}^{n} \xi_j\, v_j, \qquad A\,u = \sum_{j=1}^{n} \xi_j\, \lambda_j\, v_j, \qquad ||u||_2^2 = \sum_{j=1}^{n} \xi_j^2,$$

since $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Thus

$$\begin{aligned}
||Au||_2^2 &= \ < Au, Au >_2 = \sum_{j=1}^{n} \xi_j\lambda_j < v_j, Au > && \text{(A.15)}\\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} \xi_j\xi_k\lambda_j\lambda_k < v_j, v_k > \\
&= \sum_{k=1}^{n} \xi_k^2\lambda_k^2 \qquad \text{since } < v_j, v_k >= \delta_j^k \\
&\leq \left(\max_k \lambda_k\right)^2 ||u||_2^2 \\
\text{Thus} \quad ||Au||_2 &\leq \max_{j=1,\ldots,n} \lambda_j\, ||u||_2 \qquad \text{for all } u \in \mathbb{R}^n \\
\Rightarrow \quad ||A||_2 &\leq \max_{j=1,\ldots,n} \lambda_j. && \text{(A.16)}
\end{aligned}$$

Estimates (A.14) and (A.16) together now give $||A||_2 = \max_{j=1,\ldots,n} \lambda_j$.

(b) *Define*

$$\begin{aligned}
a :&\quad \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad a(u, v) := u^t Av,\\
L :&\quad \mathbb{R}^n \to \mathbb{R}, \qquad Lv := b^t v, \qquad b \in \mathbb{R}^n \textit{ fixed.}
\end{aligned}$$

Show that a is a bilinear, symmetric, continuous, coercive and positive definite form!
- symmetric: for arbitrary $u, v \in \mathbb{R}^n$ we get

$$a(u, v) = u^t Av = (v^t A^t u)^t = (v^t Au)^t = (a(v, u))^t = a(v, u),$$

since $A^t = A$ by assumption and $\alpha^t = \alpha$ for all $\alpha \in \mathbb{R}$.
- bilinear: For all $u_1, u_2, v \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ we get from the linearity of

matrix and scalar multiplications

$$\begin{aligned} a(\alpha u_1 + \beta u_2, v) &= (\alpha u_1 + \beta u_2)^t A v \\ &= (\alpha {u_1}^t + \beta {u_2}^t) A v \\ &= \alpha {u_1}^t A v + \beta {u_2}^t A v \\ &= \alpha a(u_1, v) + \beta a(u_2, v). \end{aligned}$$

linear in the first variable and symmetric $\Rightarrow$ bilinear.
- continuous: With the Bunjakowski-Schwarz inequality (1.13) and (A.16) we get

$$|a(u,v)| = |u^t A v| = \ | < u, Av >_2 | \ \leq \ ||u||_2 ||Av||_2 \ \leq \max_{j=1,\dots,n} \lambda_j \ ||u||_2 \ ||v||_2.$$

Now we apply the definition (2.27) to conclude that a is continuous.
- coercive: again, let $u = \sum_{j=1}^n \xi_j \, v_j$, then

$$\begin{aligned} a(u,u) &= \ < u, Au >_2 = \sum_{j=1}^n \sum_{k=1}^n \xi_k \ \xi_j < v_k, A v_j >_2 \\ &= \sum_{j=1}^n {\xi_j}^2 \lambda_j \geq \min_{k=1,\dots,n} \lambda_k \ \sum_{j=1}^n {\xi_j}^2 = \min_{k=1,\dots,n} \lambda_k \ < u, u >_2 . \end{aligned}$$

We define $c := \min_{k=1,\dots,n} \lambda_k$. Then $c > 0$ by the assumption on the eigenvalues of A. Now we apply the definition (2.26) to conclude that a is coercive.
- positive definite: for arbitrary $u \in \mathbb{R}^n$ we get from the proof of 'coercive' and from the positive definiteness of $< \cdot, \cdot >_2$

$$a(u,u) \geq \min_{k=1,\dots,n} \lambda_k \ < u, u >_2 \ \begin{array}{l} \geq \ 0, \\ = \ 0 \quad \text{iff} \quad u = 0. \end{array}$$

(c) ** Show that the maps a and L are continuously differentiable.*
To show that a is continuously differentiable, we have to show that all partial derivatives of a exist and are continuous.

$$a(u,v) = u^t A v = \sum_{j=1}^n \sum_{k=1}^n u_j a_{jk} v_k$$

is a polynomial in the components $u_1, \dots, u_n, v_1 \dots, v_n$ of u and v and thus continuously differentiable w.r.t. these u_j and v_k. We get

$$\frac{\partial a(u,v)}{\partial u_j} = \sum_{k=1}^n a_{jk} v_k, \quad j = 1, \dots, n, \qquad \frac{\partial a(u,v)}{\partial v_k} = \sum_{j=1}^n u_j a_{jk}, \quad k = 1, \dots, n.$$

The map L is given by

$$Lv = b^t v = \sum_{k=1}^n b_k v_k, \qquad b^t = (b_1, \dots, b_n).$$

Lv is a polynomial in the components of v. It follows that L is also continuously differentiable.

(d) *Show that there is a unique minimizer of*

$$F(v) := \frac{1}{2}a(v,v) - L(v), \tag{A.17}$$

and for the minimizer v_0 give an estimate of $\|v_0\|_2$!
L is a bounded linear operator:
let $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$, than

$$\begin{aligned} L(\alpha u + \beta v) &= b^t(\alpha u + \beta v) = \alpha b^t u + \beta b^t v = \alpha L u + \beta L v \\ |Lu| &= |< b, u >| \leq ||b||_2 ||u||_2 \end{aligned} \tag{A.18}$$

by the Schwarz inequality.
Since a, F are continuously differentiable, a is positive definite, symmetric and bilinear and L is linear, we know by Theorem 2.19 that $u \in \mathbb{R}^n$ minimizes eq (A.17) iff u is a solution of

$$a(u,v) = Lv \qquad \text{for all } v \in \mathbb{R}^n. \tag{A.19}$$

Moreover, since L is bounded and a is continuous and coercive, we know by Theorem 2.26 that there exists a unique $u \in \mathbb{R}^n$ solving eq (A.19). This solution satisfies the estimate

$$||u||_2 \leq 2\frac{||b||_2}{\min_{j=1,\dots,n}\lambda_j}. \tag{A.20}$$

(e) *Find the minimum of the functional $F : \mathbb{R}^2 \to \mathbb{R}$ for*

$$A := \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad b := (1, -1)^t$$

and check estimate (A.20).
The minimum u has to satisfy (A.19) in $\mathbb{R}^2$. Since

$$\begin{aligned} a(u,v) &= u^t A v = (u_1, u_2)\begin{pmatrix} 3v_1 + v_2 \\ v_1 + 3v_2 \end{pmatrix} = \ldots = v_1(3u_1 + u_2) + v_2(u_1 + 3u_2), \\ Lv &= v_1 - v_2, \end{aligned} \tag{A.21}$$

it follows that $3u_1 + u_2 = 1, \quad u_1 + 3u_2 = -1$ and thus $u_1 = -u_2 = 1/2$. The eigenvalues of the matrix A are $\lambda_1 = 2, \ \lambda_2 = 4$, and

$$||b||_2 = ||\begin{pmatrix} 1 \\ -1 \end{pmatrix}||_2 = \sqrt{2}, \qquad ||u||_2 = \frac{1}{\sqrt{2}}.$$

With these values estimate (A.20) reads

$$\frac{1}{\sqrt{2}} = ||u||_2 \leq 2\frac{||b||_2}{\min_{j=1,\dots,n}\lambda_j} = \sqrt{2}. \tag{A.22}$$

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