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Holographic Entanglement Entropy for the Most General Higher Derivative Gravity

Rong-Xin Miao*, Wu-zhong Guo††

*Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
Am Mühlenberg 1, 14476 Golm, Germany

†Kavli Institute for Theoretical Physics, Key Laboratory of Frontiers in Theoretical Physics,
Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190

†Yukawa Institute for Theoretical Physics (YITP),
Kyoto University, Kyoto 606-8502, Japan

Abstract

The holographic entanglement entropy for the most general higher derivative gravity is investigated. We find a new type of Wald entropy, which appears in the dynamic spacetime and reduces to usual Wald entropy for stationary black holes. Furthermore, we obtain a formal formula of HEE for the most general higher derivative gravity and work it out exactly for some squashed cones. As an important application, we derive HEE for gravitational action with one derivative of the curvature when the extrinsic curvature vanishes. We also study some toy models with non-zero extrinsic curvature. We prove that our formula yields the correct universal term of entanglement entropy for 4d CFTs. Furthermore, we solve the puzzle raised by Hung, Myers and Smolkin that the logarithmic term of entanglement entropy derived from Weyl anomaly of CFTs does not match the holographic result even if the extrinsic curvature vanishes. We find that such mismatch comes from the 'anomaly of entropy' of the derivative of curvature. After considering such contributions carefully, we resolve the puzzle successfully.

*Email: rong-xin.miao@aei.mpg.de
†Email: wuzhong@itp.ac.cn

Contents

1 Introduction

In [1, 2], Ryu and Takayanagi develop a holographic approach to calculate entanglement entropy (EE) of quantum (conformal) field theories in the context of AdS/CFT correspondence [3]. For a subsystem A on the boundary, they propose an elegant formula of EE

$$S_A = \frac{\text{Area of } \gamma_A}{4G}, \tag{1}$$

where γ_A is the minimal surface in the bulk whose boundary is given by ∂A and G is the bulk Newton constant. Their formula yields the correct EE for two-dimensional CFTs and satisfies the strong subadditivity of EE [27]

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}. \tag{2}$$

Recently, the conjecture eq.(1) was proved by Lewkowycz and Maldacena [5]. See also [6, 7] for the proof of Ryu-Takayanagi conjecture. Besides the gravity side there are also many interesting progress in the field theory side, please refer to [8, 9, 10, 11, 12, 13, 14] for more details.

The formula of Ryu and Takayanagi applies to quantum field theories dual to Einstein Gravity. Thus the corresponding CFTs have only one independent central charge. To cover more general field theories, one need to generalize their work to higher derivative gravity. A natural candidate of holographic entanglement entropy (HEE) for higher derivative gravity would be Wald entropy:

$$S_{\text{Wald}} = -2\pi \int d^d y \sqrt{g} \frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}. \tag{3}$$

However, as pointed out by Hung, Myers and Smolkin[15], Wald entropy does not give the correct universal logarithmic term of EE for CFTs when the extrinsic curvature is non-zero. For Lovelock gravity, we have another entropy formula: the Jacobson-Myers entropy [16] which differs from Wald entropy by some extrinsic-curvature terms. It turns out that the Jacobson-Myers entropy [16] yields the correct CFT results [15, 17]. However, there is no similar entropy formula for general higher derivative gravity. One do not know how to derive HEE from the first principle when the extrinsic curvature appears.

The first breakthrough was made by Fursaev, Patrushev and Solodukhin [18]. They develop a regularization procedure to deal with the squashed conical singularities. Using this regularization procedure, they successfully obtain HEE for the curvature-squared gravity. Soon after [18], another important breakthrough was made by Dong [19]. Dong find that, similar to holographic Weyl anomaly, the would-be logarithmic terms also contribute to HEE. Dong call such contribution as the 'anomaly of entropy'. For the so-called 'general higher derivative gravity' whose action including no derivatives of the curvature $S(g, R)$, Dong derive an elegant formula of HEE:

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \Big[\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}} + \sum_\alpha \Big(\frac{\partial^2 L}{\partial R_{zizl} \partial R_{\bar{z}k\bar{z}l}} \Big)_\alpha \frac{8 K_{zij} K_{\bar{z}kl}}{q_\alpha + 1} \Big], \tag{4}$$

where the first term is Wald entropy and the second term is the anomaly of entropy. Please refer to [19] for the definition of q_α. It should be mentioned that Camps [20] also made important contributions in this direction. For recent developments of HEE, please refer to [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

So far, HEE for gravitational actions which include derivatives of the curvature is not known. In this paper, we fill this gap by generalizing Dong's work to 'the most general higher derivative gravity' $S(g, R, \nabla R, ...)$. We find all the possible would-be logarithmic terms and derive a formal formula of HEE for 'the most general higher derivative gravity'. To get more exact formulas, we focus on gravity theories whose action $S(g, R, \nabla R)$ includes only one derivative of the curvature. A natural guess of HEE for $S(g, R, \nabla R)$ would be Dong's formula eq.(4) with all ∂ be replaced by δ. This is however not the case. Instead, we find that new terms should be added to both Wald entropy and anomaly of entropy even if we replace all ∂ by δ. The generalized Wald entropy for $S(g, R, \nabla R)$ is

$$\begin{aligned} S_{\text{G-Wald}} &= 2\pi \int d^d y \sqrt{g} \Big[\frac{\delta L}{\delta R_{z\bar{z}z\bar{z}}} + 2 \Big(\frac{\partial L}{\partial \nabla_z R_{\bar{z}i\bar{z}j}} K_{\bar{z}ij} + c.c \Big) \Big] \\ &= 2\pi \int d^d y \sqrt{g} \Big[-\frac{\delta L}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} + 2 \frac{\partial L}{\partial \nabla_\alpha R_{\mu\rho\nu\sigma}} K_{\beta\rho\sigma} (n^\beta{}_\mu n_{\alpha\nu} - \epsilon^\beta{}_\mu \epsilon_{\alpha\nu}) \Big] \end{aligned} \tag{5}$$

By 'generalized Wald entropy', we means the total entropy minus the anomaly of entropy. Interestingly, a new term proportional to the extrinsic curvature appears in the generalized Wald entropy. This new term only appears in dynamic space-time, thus it is consistent with Wald's results for stationary black holes. While for the anomaly of entropy, since the general case is very complicated, we set $K_{aij}=0$ for simplicity. If the anomaly of entropy is just Dong's formula with ∂ be replaced by δ, it should vanish after we set $K_{aij}=0$. However, we get

$$\begin{aligned} S_{\text{Anomaly}} &= 2\pi\int d^dy\sqrt{g}\big[64\big(\frac{\partial^2 L}{\partial\nabla_z R_{zizl}\partial\nabla_{\bar{z}} R_{\bar{z}k\bar{z}l}}\big)_{\alpha_1}\frac{Q_{zzij}Q_{\bar{z}\bar{z}kl}}{\beta_{\alpha_1}} \\ &+ 96i\big(\frac{\partial^2 L}{\partial\nabla_z R_{zizl}\partial\nabla_{\bar{z}} R_{\bar{z}z\bar{z}k}}\big)_{\alpha_1}\frac{Q_{zzij}V_{\bar{z}k}}{\beta_{\alpha_1}}+c.c \\ &+ 144\big(\frac{\partial^2 L}{\partial\nabla_z R_{z\bar{z}zl}\partial\nabla_{\bar{z}} R_{\bar{z}z\bar{z}k}}\big)_{\alpha_1}\frac{V_{zl}V_{\bar{z}k}}{\beta_{\alpha_1}}\big]. \end{aligned} \tag{6}$$

Applying the above formula, we resolve the puzzle raised by Huang, Myers and Smolkin that the logarithmic term of EE derived from Weyl anomaly of CFTs does not match the holographic result even if the extrinsic curvature vanishes [15]. We find that such mismatch comes from the contributions of the derivative of the curvature. After considering these contributions carefully by using the above formula, we resolve the HMS puzzle successfully.

For non-zero extrinsic curvature, we investigate a toy model with Lagrangian $L=\lambda_1\nabla_\alpha R\nabla^\alpha R+\lambda_2\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu}+\lambda_3\nabla_\alpha R_{\mu\nu\rho\sigma}\nabla^\alpha R^{\mu\nu\rho\sigma}$. We derive HEE and prove it yields the correct logarithmic terms of EE for 4d CFTs. We also compute HEE of the toy model by using FPS regularization [18] and find full agreement with the results by using Dong's method.

The paper is organized as follows. In Sect. 2, we briefly review Dong's derivation of HEE for 'general higher derivative gravity'. In Sect. 3, we generalize Dong's method to the most general cases. We obtain a formal formula of HEE for the most general higher derivative gravity. As an exercise, we work out the exact formula for some interesting cone metrics. In Sect. 4, we check our formula by using the FPS method. We also prove that our formula yields the correct logarithmic term of EE for 4d CFTs. In Sect. 5, we resolve the HMS puzzle. We derive the logarithmic term of entanglement entropy for 6d CFTs from Weyl anomaly and find it is consistent with the holographic result for entangling surfaces with zero extrinsic curvature but without rotational symmetry. We conclude in Sect 6.

2 Dong's proposal of HEE for higher derivative gravity

In this section, we briefly review Dong's derivation of HEE for higher derivative gravity. The key observation of Dong is that, similar to the holographic Weyl anomaly, the would-be logarithmic term also contributes to HEE. As a result, corrections of entropy from the extrinsic curvature emerge:

$$\delta S = 32\pi\int d^dy\sqrt{g}\big(\frac{\partial^2 L}{\partial R_{zizl}\partial R_{\bar{z}k\bar{z}l}}\big)_{\alpha_1}\frac{K_{zij}K_{\bar{z}kl}}{\beta_{\alpha_1}}. \tag{7}$$

Dong calls such corrections as the anomaly of entropy. For simplicity, he focuses on the gravity theories without derivatives of the curvature, $S=S(g,R)$. We review Dong's derivation of HEE in this section and generalize it to the most general case $S=S(g,R,\nabla R,...)$ in the next section.

2.1 The replica trick

A useful method to derive HEE is by applying the replica trick. Let us take Einstein Gravity as an example. Recall that the Renyi entropy is defined as

$$S_n = -\frac{1}{n-1}\log tr[\rho^n] = -\frac{1}{n-1}(\log Z_n - n\log Z_1) \tag{8}$$

$$Z_n = Tr[\hat{\rho}^n],\ \rho = \frac{\hat{\rho}}{Tr[\hat{\rho}]}, \tag{9}$$

where Z_n is the partition function of the field theory on a suitable manifold M_n known as the n-fold cover.

For theories with a holographic dual we can build a suitable bulk solution B_n whose boundary is M_n. Then the gauge-gravity duality identifies the field theory partition function on M_n with the on-shell bulk action on B_n

$$Z_n = Z[M_n] = e^{-S[B_n]}. \tag{10}$$

We can derive the HEE by taking the limit $n \to 1$ of Renyi entropy

$$\begin{aligned} S_{EE} &= \lim_{n\to 1} S_n = -\partial_n(\log Tr[\rho^n]|_{n\to 1} = -Tr[\rho\log\rho] \\ &= -\partial_n(\log Z_n - n\log Z_1)|_{n\to 1} = \partial_n(S[B_n] - nS[B_1])|_{n\to 1} \\ &= -\partial_\epsilon S_{reg}, \end{aligned} \tag{11}$$

where $S_{reg} = (nS[B_1] - S[B_n])$ is the regularized action and $\epsilon = 1 - \frac{1}{n}$. For Einstein gravity, we have

$$S_{reg} = \frac{1}{16\pi G}\int_{Reg\ Cone} dx^D \sqrt{G}\ R = \epsilon\ \frac{\text{Area}}{4G}. \tag{12}$$

Then we can derive HEE of Einstein gravity as $S = -\frac{\text{Area}}{4G}$. Note that we work in the Euclidean signature. So entropy formula differs from its usual Lorentzian form by a minus sign.

There is still one question need to be answered. On which surface shall we apply this formula? We know the answer is the minimal surface for Einstein gravity. In general, according to [5], we require that the analytically continued solution satisfies the linearized equations of motion near the cone $\rho = 0$. We call this method the "boundary condition method". The metric of regularized cone is

$$ds^2 = e^{2A}dzd\bar{z} + (g_{ij} + 2zK_{zij} + 2\bar{z}K_{\bar{z}ij})dy^i dy^j + o(\rho^2), \tag{13}$$

where $z = \rho e^{i\tau}$, $dzd\bar{z} = d\rho^2 + \rho^2 d\tau^2$, $A = -\epsilon\log(\rho)$ and K_z is the extrinsic curvature. Let us compute the linearized equations of motion $\delta G_{zz} = 8\pi G\delta T_{zz}$. We focus on the divergent terms, going like $1/\rho$ near the origin. Since the stress tensor is not expected to be singular, we have

$$\delta R_{zz} = -\frac{\epsilon}{z}K_z + \text{regular terms}. \tag{14}$$

Requiring the above equation to be regular near the cone, we get $K_z = K_{\bar{z}} = 0$. This is just the condition of the minimal surface.

There is another method to derive the minimal surface conditions. We call it the 'cosmic brane method'. Consider the action

$$S_{total} = S_{EH} + S_B = -\frac{1}{16\pi G_N}\int_{Reg} d^D x\sqrt{G}R + \frac{\epsilon}{4G_N}\int d^{D-2}y\sqrt{g}. \tag{15}$$

In the limit $\epsilon \to 0$, we can treat S_B as the action of a probe brane and find its location by minimizing S_B without back reaction on the bulk fields. This gives exactly the minimal surface.

We have shown how to derive HEE for Einstein Gravity and how to derive the location of the cone. Now let us try to generalize it to higher derivative gravity.

2.2 Would-be logarithmic terms

According to [19], the metric of regularized cone is

$$\begin{aligned} ds^2 = e^{2A}[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2] + \big(g_{ij} + 2K_{aij}x^a + Q_{abij}x^a x^b\big)dy^i dy^j \\ +2ie^{2A}(U_i + V_{ai}x^a)(\bar{z}dz - zd\bar{z})dy^i + ..., \end{aligned} \tag{16}$$

where $T, g_{ij}, K_{aij}, Q_{abij}, U_i, V_{ai}$ are independent of z and $\bar{z}$, with the exception that $Q_{z\bar{z}ij} = Q_{\bar{z}zij}$ contains a factor e^{2A}. The warp factor A is regularized by a thickness parameter a as $A = -\frac{\epsilon}{2}\lg(z\bar{z} + a^2)$. As we shall show below, the result is independent of the choice of regularization.

The key observation of [19] is that

$$\int \rho d\rho \partial_z A \partial_{\bar{z}} A e^{-\beta A} = -\frac{\epsilon}{4\beta}, \tag{17}$$

where $z = \rho e^{i\tau}$. Naively the left hand of eq.(17) is in order $o(\epsilon^2)$. Magically it becomes in order $o(\epsilon)$ after regularization. The magic happens because would-be logarithmic divergence gets a $\frac{1}{\epsilon}$ enhancement:

$$\int d\rho \frac{1}{\rho^{1-\beta\epsilon}} \sim \frac{1}{\beta\epsilon}. \tag{18}$$

As we know, the coefficient of a would-be logarithmic divergence is universal (like anomaly). So eq.(17) is independent of the regularization. In fact, we can give a very simple proof. It is known that the following formula is universal

$$\int dzd\bar{z}e^{-\beta A}\partial_z\partial_{\bar{z}}A = -\pi\epsilon. \tag{19}$$

This formula is usually used to derive Wald entropy. Performing integration by parts, we get

$$\int dzd\bar{z}e^{-\beta A}\partial_z A\partial_{\bar{z}}A = \frac{-\pi\epsilon}{\beta}. \tag{20}$$

which is exactly eq.(17). It should be mentioned that we can drop the boundary terms safely. One can check that the boundary term is zero after regularization.

2.3 Dong's formula: HEE for four-derivative gravity

Now let us focus on the four-derivative gravity whose action $S(g,R)$ contains no derivatives of the curvature. By four-derivative gravity, we means the equations of motion are four order differential equations. This is the case investigated in [19]. From the regularized metric eq.(16), we can derive the curvature with non-vanishing derivatives of A as

$$\begin{aligned}
R_{z\bar{z}z\bar{z}} &= e^{2A}\partial_z\partial_{\bar{z}}A + ...,\\
R_{zizj} &= 2K_{zij}\partial_z A + ...,\\
R_{z\bar{z}zi} &= ie^{2A}U_i\partial_z(z\partial_z A) + ...,
\end{aligned} \tag{21}$$

where "..." denotes terms without derivatives of A. One can get the other curvatures by exchanging $z, \bar{z}, i, j$ and complex conjugate. For the reason will be clear in sect. 3, $R_{z\bar{z}zi} \sim e^{2A}U_i\partial_z(z\partial_z A) \sim 0$ actually does not contribute to HEE. Thus, from eqs.(21,19,20), we can derive the HEE as

$$S_{EE} \quad = \quad 2\pi\int d^dy\sqrt{g}\Big[\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}} + 16\Big(\frac{\partial^2 L}{\partial R_{zizl}\partial R_{\bar{z}k\bar{z}l}}\Big)_{\alpha_1}\frac{K_{zij}K_{\bar{z}kl}}{\beta_{\alpha_1}}\Big] \tag{22}$$

The first term above is just the Wald entropy, and the second term denotes the anomly of entropy [19]. It should be stressed that, unlike K_{aij}, U_i could not appear in the formula of HEE eq.(22). Otherwise, it would yield wrong results of entropy for stationary black holes. As we shall show in next section, $R_{z\bar{z}zi} = ie^{2A}U_i\partial_z(z\partial_z A)$ indeed do not contribute to HEE.

3 HEE for the most general higher derivative gravity

In this section, we investigate HEE for the most general higher derivative gravity. Firstly, we find all the possible would-be logarithmic terms. Then we derive a formal formula of HEE for the most general higher derivative gravity. Finally, we work out the formal formula exactly for some special cone metrics.

3.1 General would-be logarithmic terms

To start, let us firstly briefly review the squashed cone. According to [19], the general squashed cone metric is

$$\begin{aligned}
ds^2 = e^{2A}[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2] + 2ie^{2A}V_i(\bar{z}dz - zd\bar{z})dy^i\\
+(g_{ij} + Q_{ij})dy^i dy^j,
\end{aligned} \tag{23}$$

where g_{ij} is the metric on the transverse space and is independent of $z, \bar{z}$. $A = -\frac{\epsilon}{2}\lg(z\bar{z}+a^2)$ is regularized warp factor. T, V_i, Q_{ij} are defined as

$$T = \sum_{n=0}^{\infty}\sum_{m=0}^{P_{a_1...a_n}} e^{2mA}T_{i\ a_1...a_n}x^{a_1}...x^{a_n},$$

$$V_i=\sum_{n=0}^{\infty}\sum_{m=0}^{P_{a_1...a_n}} e^{2mA}V_{i\ a_1...a_n i}x^{a_1}...x^{a_n},$$

$$Q_{ij}=\sum_{n=1}^{\infty}\sum_{m=0}^{P_{a_1...a_n}} e^{2mA}Q_{i\ a_1...a_n ij}x^{a_1}...x^{a_n}. \tag{24}$$

Here $z,\bar{z}$ are denoted by x^a and $P_{a_1...a_n}$ is the number of pairs of $z,\bar{z}$ appearing in $a_1...a_n$. For example, we have $P_{zz\bar{z}}=P_{z\bar{z}z}=P_{\bar{z}zz}=1$, $P_{z\bar{z}z\bar{z}}=2$ and $P_{zz...z}=0$. Expanding T,V,Q to the second order and using Dong's notations, we have

$$\begin{aligned}
&T=T_0+T_a x^a+T_{0\ ab}x^a x^b+2e^{2A}T_{1\ z\bar{z}}\ z\bar{z}...,\\
&V_i=U_i+V_a x^a+V_{0\ abi}x^a x^b+2e^{2A}V_{1\ z\bar{z}i}\ z\bar{z}...,\\
&Q_{ij}=2K_{aij}x^a+Q_{0\ abij}x^a x^b+2e^{2A}Q_{1\ z\bar{z}ij}\ z\bar{z}....
\end{aligned} \tag{25}$$

How to split W (W denote T,V,Q) into $\{W_0,W_1,...,W_P\}$ is an important problem[1]. Equations of motion may help to fix this splitting. We leave this problem to future research. In this paper, for simplicity we keep only the highest order term W_P to illustrate our approach. It does not affect our main results (eqs.(5,6,27,29) and the results in Sect. 4 and Sect. 5))[2].

Using the squashed cone metric (23), we can calculate the action of most general higher derivative gravity and then select the relevant terms to derive HEE. Now let us discuss all the possible terms relevant to HEE.

Let us denote the general derivatives by

$$\hat{\partial}=c^{mn}\partial_z^m\partial_{\bar{z}}^n, \tag{26}$$

where c^{mn} are arbitrary constants. Since only $o(\epsilon)$ terms contribute to HEE, we only need to consider terms with at most two A: $\hat{\partial}A,\hat{\partial}A\hat{\partial}A$. For the first case $\hat{\partial}A$, it is easy to find that only the following terms contribute to HEE

$$\begin{aligned}
\int dz d\bar{z} z^m \bar{z}^n \partial_z^{m+1}\partial_{\bar{z}}^{n+1}A &= \int dz d\bar{z}(-1)^{m+n}m!n!\partial_z\partial_{\bar{z}}A\\
&= (-1)^{m+n+1}m!n!\pi\epsilon.
\end{aligned} \tag{27}$$

Equivalently, we have

$$\partial_z^{m+1}\partial_{\bar{z}}^{n+1}A=-\pi\epsilon\partial_z^m\partial_{\bar{z}}^n\bar{\delta}(z,\bar{z}). \tag{28}$$

[1] We thank Dong and Camps for reminding us this problem.

[2] Let us focus on the splitting of $Q_{z\bar{z}ij}$. For simplicity, we set $Q_{0\ z\bar{z}ij}=0$ in this paper. Calculations show that $Q_{0\ z\bar{z}ij}$ does not appear in the generalized Wald entropy eq.(5). However, it does contribute to the anomaly of entropy. To be consistent with Wald entropy, $Q_{0\ z\bar{z}ij}$ must vanish in the stationary spacetime. This implies $Q_{0\ z\bar{z}ij}$ is either zero or a function of the extrinsic curvature K_{aij}. Recall that eq.(6) is derived under the condition $K_{aij}=0$. Thus $Q_{0\ z\bar{z}ij}$ does not appear in eq.(6) either. Because we only use eq.(6) to resolve the HMS puzzle, so $Q_{0\ z\bar{z}ij}$ does not affect our conclusion in Sect. 5. As we shall show in Sect. 4, only the leading term $Q_{1\ z\bar{z}ij}=\frac{1}{2}G_{ij}$ contributes to the logarithmic term of EE. And the subleading term $Q_{0\ z\bar{z}ij}\sim o(K^2)$ is irrelevant with our discussions of Sect. 4.

These terms contribute to the Wald entropy. Note that the delta function is defined as $\int dzd\bar{z}\bar{\delta}(z,\bar{z}) = 1$.

As for the second case $\hat{\partial}A\hat{\partial}A$, we should focus on the would-be logarithmic terms. That is because only such terms could gain a $\frac{1}{\epsilon}$ enhancement. The only possible terms are

$$\begin{aligned}\int dzd\bar{z}z^m\bar{z}^n\partial_z^{m+1}A\partial_{\bar{z}}^{n+1}Ae^{-\beta A} &= \int dzd\bar{z}(-1)^{m+n}m!n!\partial_z A\partial_{\bar{z}}Ae^{-\beta A} \\ &= (-1)^{m+n+1}m!n!\frac{\pi\epsilon}{\beta}. \end{aligned} \tag{29}$$

Equivalently, we have

$$\partial_z^{m+1}A\partial_{\bar{z}}^{n+1}Ae^{-\beta A} = -\frac{\pi\epsilon}{\beta}\partial_z^m\partial_{\bar{z}}^n\bar{\delta}(z,\bar{z}). \tag{30}$$

These terms contribute to the anomaly of entropy.

The simplest method to prove eq.(29) is by applying integration by part and dropping the irrelevant terms such as $\hat{\partial}\partial_z\partial_{\bar{z}}A\hat{\partial}A, \hat{\partial}A\hat{\partial}A\hat{\partial}A$ and so on. This is the method we used in eq.(29). We can also prove eq.(29) by using Dong's method. Recall that $A = -\frac{\epsilon}{2}\log(z\bar{z})$, we have $z^m\partial_z^{m+1}A = -\frac{\epsilon}{2}(-1)^m\frac{m!}{z}$. Thus we can derive

$$\begin{aligned}\int \rho d\rho z^m\bar{z}^n\partial_z^{m+1}A\partial_{\bar{z}}^{n+1}Ae^{-\beta A} &= \int d\rho(-1)^{m+n}m!n!\frac{\epsilon^2}{4}\rho^{-1+\beta\epsilon} \\ &= (-1)^{m+n}m!n!\frac{\epsilon}{4\beta}\rho^{\beta\epsilon}|_0^\infty \\ &\cong (-1)^{m+n+1}\frac{\epsilon}{4\beta}m!n!. \end{aligned} \tag{31}$$

Here $\cong$ denotes equivalence after regularization.

It should be stressed that terms contains $\partial_z\partial_{\bar{z}}A$, $z\partial_z A$ or $\bar{z}\partial_{\bar{z}}A$ in the second case would not contribute to HEE,

$$\begin{aligned}&\hat{\partial}\partial_z\partial_{\bar{z}}A\hat{\partial}A = 0, \\ &\hat{\partial}(z\partial_z A)\hat{\partial}A = \hat{\partial}(\bar{z}\partial_{\bar{z}}A)\hat{\partial}A = 0.\end{aligned}$$

That is because $\partial_z\partial_{\bar{z}}A = -\frac{\epsilon}{2}\frac{a^2}{(a^2+r^2)^2}$, so $\hat{\partial}\partial_z\partial_{\bar{z}}A\hat{\partial}A$ at least in order ϵ^2a^2. Note that (a^2+r^2) always appear as a whole in the denominator. To cancel a^2, we must have $\epsilon^2\frac{a^2}{a^2+r^2}$ after integration. However this is a r^{-2} term rather than a would be logarithmic term $\frac{1}{\epsilon}r^{-\epsilon}$. So we can not cancel ϵ and a^2 at the same time. Similar for the second case, $\hat{\partial}(z\partial_z A)\hat{\partial}A$ is also at least in order ϵ^2a^2. Thus, it does not contribute to HEE either. Maybe the most quick way to see that $\partial_z\partial_{\bar{z}}A$ and $z\partial_z A$ do not contribute to HEE is by identifying $A = -\frac{\epsilon}{2}\log(z\bar{z})$. So we have $\hat{\partial}\partial_z\partial_{\bar{z}}A = \hat{\partial}(z\partial_z A) = 0$, which can not contribute to HEE at all.

Using eqs.(27,29), we can derive HEE for most general higher derivative gravity as

$$\begin{aligned}S_{HEE} &= -\partial_\epsilon S_{reg}|_{\epsilon=0} \\ &= 2\pi\delta(z,\bar{z})\hat{g}^{ab}\Big(\frac{\delta S}{\delta\partial_a\partial_b A} + \frac{1}{\beta_\alpha}[\frac{\delta}{\delta\partial_b A}(\frac{\delta S}{\delta\partial_a A}|_{\partial_z\partial_{\bar{z}}A=0})]_\alpha\Big)|_{\epsilon=0}, \end{aligned} \tag{32}$$

where a sum over α is implied. Note that, we drop all the $\hat{\partial}\partial_z\partial_{\bar{z}}A$ terms after one variation of $\partial_a A$ in the second term of eq.(32). This formula applies to the most general higher derivative gravity. It is one of the main results of this paper. Let us comment on our formula (32).

Firstly, the first term of eq.(32) is the generalized Wald entropy. It should be stressed that not only $R_{z\bar{z}z\bar{z}}$ and its covariant derivative $\nabla^n R_{z\bar{z}z\bar{z}}$ but aslo many other terms may contribute to the generalized Wald entropy. For example, we have

$$\nabla_z R_{\bar{z}i\bar{z}j} = K_{\bar{z}ij}\partial_z\partial_{\bar{z}}A + ... \tag{33}$$

Clearly, the above term contributes to the generalized Wald entropy and is not included in the usual Wald entropy $\frac{\delta S}{\delta R_{\mu\nu\rho\sigma}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}$. Note that such new generalized Wald entropy appears only in the dynamic space-time. Thus nothing goes wrong with Wald's formula which is designed for the stationary black holes. We shall discuss the generalized Wald entropy in details in the next subsection.

Secondly, the second term of eq.(32) is the anomaly of entropy. In general, it is very difficult to calculate such terms for the most general higher derivative gravity. Let us play a trick. Setting $A = -\frac{\epsilon}{2}\log[z\bar{z}]$ and keeping only the would-be logarithmic term $\frac{1}{2}dzd\bar{z}e^{-\beta A}\frac{\epsilon^2}{z\bar{z}}$ in the action, then replacing it by $\frac{2\pi}{\beta}$, we obtain the final result.

$$\begin{aligned}
S_{\text{Action}} &= \int \frac{1}{2}dzd\bar{z}\sum_{\alpha} C_\alpha e^{-\beta_\alpha A}\frac{\epsilon^2}{z\bar{z}} + ...\\
S_{\text{Anomaly of entropy}} &= \sum_{\alpha} C_\alpha \frac{2\pi}{\beta_\alpha}.
\end{aligned} \tag{34}$$

Thirdly, we have found all the relevant terms with HEE in order $O(A)$ and $O(A^2)$. A natural question is whether terms in higher order $O(A^{n+2})$ contribute to HEE or not. In general, only would-be $(\log\rho)^{n+1}$ terms may get an enhancement after regularization. Let us discuss these terms briefly. Recall that we have

$$e^{-\beta A}\partial_z A\partial_{\bar{z}}A = \frac{-\pi\epsilon}{\beta}\delta(z,\bar{z}). \tag{35}$$

Taking the derivatives of the above equation by β, we can derive

$$A^n e^{-\beta A}\partial_z A\partial_{\bar{z}}A = \frac{-\pi n!\epsilon}{\beta^{n+1}}\delta(z,\bar{z}). \tag{36}$$

Naively, the left hand side of eq.(36) is in order $o(\epsilon^{n+2})$. However it becomes in order $o(\epsilon)$ after regularization. Actually, this is the would be $(\log\rho)^{n+1}$ terms. This kind of terms may contribute to HEE for some crazy regularized cone metrics. However, if we focus on higher derivative gravity with the regularized cone eq.(23) , only eq.(29) is already enough. That is because the factor $e^{\beta A}$ always appear as an entirety in the regularized metric and the action [19], and $A^n\hat{\partial}A\hat{\partial}A$ terms never appear separately. Thus only the would-be logarithmic term contribute to HEE of higher derivative gravity. Based on eqs.(27,29), in Sect.4 we shall prove that our formulas of HEE yield the correct universal logarithmic terms of EE for 4d CFTs and our results agree with those derived by applying the FPS method [18]. This can be regarded as a support of the fact that terms in higher order $O(A^{n+2})$ do not contribute to HEE.

To summary, we have found all the would-be logarithmic terms and obtained a formal formula of HEE for the most general higher derivative gravity. In the next section, we shall work out this formula exactly for some squashed cone metrics.

3.2 HEE for six-derivative gravity

In this subsection, we investigate HEE of six-derivative gravity. By six-derivative gravity, we mean the equations of motion are six order differential equations. Its action can always be rewritten in the form $S(g,R,\nabla R)$. We firstly derive the generalized Wald entropy for the general cone metric and then calculate the anomaly of entropy for some special cone metric.

Let us firstly investigate the generalized Wald entropy. It come from the first term of eq.(32). As we have mentioned in the above section, in addition to $R_{z\bar{z}z\bar{z}}$ and its covariant derivative $\nabla_\mu R_{z\bar{z}z\bar{z}}$, many other terms may contribute to the generalized Wald entropy. We list all the possible terms relevant to the generalized Wald entropy below.

$$\begin{aligned}
&R_{z\bar{z}z\bar{z}} = e^{2A}\partial_z\partial_{\bar{z}}A + ...,\\
&\nabla_z R_{z\bar{z}z\bar{z}} = e^{2A}\partial_z^2\partial_{\bar{z}}A + ...,\\
&\nabla_z R_{\bar{z}z\bar{z}i} = 2iU_i e^{2A}\partial_z\partial_{\bar{z}}A + ...,\\
&\nabla_i R_{z\bar{z}zj} = 2K_{\bar{z}ij}\partial_z\partial_{\bar{z}}A + ...,\\
&\nabla_z R_{\bar{z}i\bar{z}j} = 2K_{\bar{z}ij}\partial_z\partial_{\bar{z}}A + ...
\end{aligned}\tag{37}$$

Using the above formulae, we can derive

$$\begin{aligned}
S_{\text{G-Wald}} = 2\pi\int d^dy\sqrt{g}\big[\quad & \frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}}\\
- \quad & \frac{1}{\sqrt{g}}\partial_z(\sqrt{g}\frac{\partial L}{\partial \nabla_z R_{\bar{z}z\bar{z}z}}) + c.c\\
+ \quad & 4i\frac{\partial L}{\partial \nabla_z R_{\bar{z}z\bar{z}i}}U_i + c.c\\
+ \quad & 2\frac{\partial L}{\partial \nabla_z R_{\bar{z}i\bar{z}j}}K_{\bar{z}ij} + c.c\\
+ \quad & 4\frac{\partial L}{\partial \nabla_i R_{z\bar{z}zj}}K_{zij} + c.c \quad \big].
\end{aligned}\tag{38}$$

Take into account that $\Gamma^z_{zi} = -2iU_i, \Gamma^i_{jz} = K_{zj}^{\ \ i}, \Gamma^z_{ij} = -2K_{\bar{z}ij}$, we obtain the generalized Wald entropy as

$$\begin{aligned}
S_{\text{G-Wald}} &= 2\pi\int d^dy\sqrt{g}\big[\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}} - \nabla_\mu\frac{\partial L}{\partial \nabla_\mu R_{z\bar{z}z\bar{z}}} + 2(\frac{\partial L}{\partial \nabla_z R_{\bar{z}i\bar{z}j}}K_{\bar{z}ij} + c.c) \ \big]\\
&= 2\pi\int d^dy\sqrt{g}\big[\frac{\delta L}{\delta R_{z\bar{z}z\bar{z}}} + 2(\frac{\partial L}{\partial \nabla_z R_{\bar{z}i\bar{z}j}}K_{\bar{z}ij} + c.c) \ \big].
\end{aligned}\tag{39}$$

Remarkably, a new term proportional to the extrinsic curvature K_{aij} appears in the generalized Wald entropy. This new term vanishes for stationary black holes and thus is consistent with Wald's results. In general, self conjugate terms such as $T_0, U_i, Q_{z\bar{z}ij}$... could not contribute new terms to the generalized Wald entropy, otherwise it conflicts with Wald entropy for stationary black holes. That is because, in general, these self conjugate terms are non-zero in stationary spacetime. Indeed, $T_0, U_i, Q_{z\bar{z}ij}$ do not appear in our generalized Wald entropy eq.(39) for six-derivative gravity. The above generalized Wald entropy can be written in a covariant form as

$$S_{\text{G-Wald}} = 2\pi\int d^dy\sqrt{g}\big[\ -\frac{\delta L}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} + 2\frac{\partial L}{\partial \nabla_\alpha R_{\mu\rho\nu\sigma}}K_{\beta\rho\sigma}(n^\beta{}_\mu n_{\alpha\nu} - \epsilon^\beta{}_\mu\epsilon_{\alpha\nu}) \ \big].\tag{40}$$

Let us go on to study the anomaly of entropy. Because the general case is quite complicated we consider some special cone metrics below. Recall that the squashed cone metric is

$$ds^2 = e^{2A}[dzd\bar{z} + e^{2A}T(\bar{z}dz - zd\bar{z})^2] + 2ie^{2A}V_i(\bar{z}dz - zd\bar{z})dy^i + (g_{ij} + Q_{ij})dy^i dy^j. \tag{41}$$

For simplicity, we firstly consider the case with zero extrinsic curvature. Thus, we have

$$\begin{aligned} T &= T_0 + T_a x^a + T_{ab}x^a x^b + ..., \\ V_i &= U_i + V_a x^a + V_{abi}x^a x^b + ..., \\ Q_{ij} &= Q_{abij}x^a x^b + ... \end{aligned} \tag{42}$$

Note that there is a factor e^{2A} before $T_{z\bar{z}}, V_{z\bar{z}i}$ and $Q_{z\bar{z}ij}$. Let us calculate $R, \nabla R$, and select all the possible terms relevant to HEE. We have

$$\begin{aligned} R_{z\bar{z}z\bar{z}} &= e^{2A}\partial_z\partial_{\bar{z}}A + ..., \\ \nabla_z R_{z\bar{z}z\bar{z}} &= e^{2A}\partial_z^2\partial_{\bar{z}}A + ..., \\ \nabla_z R_{zizj} &= 4Q_{zzij}\partial_z A + ..., \\ \nabla_z R_{z\bar{z}zj} &= -3ie^{2A}V_{zj}\partial_z A + ... \end{aligned} \tag{43}$$

Note that to derive $\nabla_z R_{zizj}$ and $\nabla_z R_{z\bar{z}zj}$, we have identified $z\partial_z^2 A \cong -\partial_z A$ and $z^2\partial_z^3 A \cong 2\partial_z A$. In general, we have $z^m\partial_z^{m+1}A \cong (-1)^m m!\partial_z A$. We can read out these indentities from eq.(29). Using eqs.(27,29,43), we can derive HEE for six-derivative gravity as

$$\begin{aligned} S_{HEE} &= 2\pi\int d^d y\sqrt{g}\Big[\frac{\delta L}{\delta R_{z\bar{z}z\bar{z}}} + 64\Big(\frac{\partial^2 L}{\partial\nabla_z R_{zizl}\partial\nabla_{\bar{z}}R_{\bar{z}k\bar{z}l}}\Big)_{\alpha_1}\frac{Q_{zzij}Q_{\bar{z}\bar{z}kl}}{\beta_{\alpha_1}} \\ &+ 96i\Big(\frac{\partial^2 L}{\partial\nabla_z R_{zizl}\partial\nabla_{\bar{z}}R_{\bar{z}z\bar{z}k}}\Big)_{\alpha_1}\frac{Q_{zzij}V_{\bar{z}k}}{\beta_{\alpha_1}} + c.c \\ &+ 144\Big(\frac{\partial^2 L}{\partial\nabla_z R_{z\bar{z}zl}\partial\nabla_{\bar{z}}R_{\bar{z}z\bar{z}k}}\Big)_{\alpha_1}\frac{V_{zl}V_{\bar{z}k}}{\beta_{\alpha_1}}\Big]. \end{aligned} \tag{44}$$

Now let us consider a more complicated case. We set $V_i = 0$ but with general $T,\ Q_{ij}$. For simplicity, we only investigate a special action, $S = \int dx^D\sqrt{G}\nabla_\mu R_{\nu\alpha\beta\gamma}\nabla^\mu R^{\nu\alpha\beta\gamma}$. Applying the formulae in the appendix, we obtain the anomaly of entropy

$$\begin{aligned} S_{\text{Anomaly}} &= 32\pi\int d^d y\sqrt{g}\Big[4Q_{\bar{z}\bar{z}ij}Q_{zz}{}^{ij} + 8K_{\bar{z}}{}^{ij}K_{zj}{}^{k}Q_{z\bar{z}ki} - 2K_{\bar{z}p}{}^{p}K_{\bar{z}}{}^{ij}Q_{zzij} \\ &- 2K_{zp}{}^{p}K_{z}{}^{ij}Q_{\bar{z}\bar{z}ij} + 2K_{zij}K_{\bar{z}}{}^{ij}Q_{z\bar{z}p}{}^{p} + (K_{zij}K_{\bar{z}}{}^{ij})^2 + K_{zp}{}^{p}K_{\bar{z}q}{}^{q}K_{zij}K_{\bar{z}}{}^{ij} \\ &- 4K_{z}{}^{ij}K_{\bar{z}jk}K_{\bar{z}}{}^{kl}K_{zli} + \bigtriangledown_s^{(y)}K_{zij}\bigtriangledown^{(y)s}K_{z}{}^{ij} - 40TK_{zij}K_{\bar{z}}{}^{ij} \\ &+ K_{zij}K_{z}{}^{ij}K_{\bar{z}kl}K_{\bar{z}}{}^{kl} + R_{zijk}R_{\bar{z}}{}^{ijk} - 6Q_{zz\bar{z}ij}K_{\bar{z}}{}^{ij} - 6Q_{\bar{z}\bar{z}zij}K_{z}{}^{ij}\Big]. \end{aligned} \tag{45}$$

Using the above formula, we can calculate HEE of squashed cones [18] in Minkowski spacetime. The cylindrical and spherical cone metrics in Minkowski spacetime are given by

$$ds^2 = e^{2A}dzd\bar{z} + \big(a^2 + az + a\bar{z} + \frac{z^2 + \bar{z}^2}{4} + \frac{e^{2A}z\bar{z}}{2}\big)d\phi^2 + dz^2, \tag{46}$$

$$ds^2 = e^{2A}dzd\bar{z} + \big(a^2 + az + a\bar{z} + \frac{z^2 + \bar{z}^2}{4} + \frac{e^{2A}z\bar{z}}{2}\big)(d\theta^2 + \sin^2\theta d\phi^2). \tag{47}$$

Applying eq.(45), we can derive HEE as

$$S_{\text{HEE}} = \frac{36\pi^2 L}{a^3}, \text{ for the cylindrical cone,} \tag{48}$$

$$S_{\text{HEE}} = \frac{192\pi^2}{a^2}, \text{ for the spherical cone.} \tag{49}$$

Note that the generalized Wald entropy is zero for the above cone metrics in Minkowski spacetime.

To summary, we have found a new type of Wald entropy, the generalized Wald entropy, for six-derivative gravity. This generalized Wald entropy appears in dynamic spacetime and reduces to Wald entropy for stationary black holes. It would be interesting to study the physical meaning of this generalized Wald entropy. We leave it to the future work. We also derive the anomaly of entropy for cone metrics with zero extrinsic curvature. As for non-zero extrinsic curvature, we study a toy model of six-derivative gravity. In sect. 4, we shall check our results by using the method of Fursaev et al. We shall also prove that our results give the correct logarithmic term of EE for 4d CFTs.

3.3 HEE for 2n-derivative gravity

We calculate HEE of 2n-derivative gravity in this subsection. By 2n-derivative gravity, we mean the equations of motion are 2n-order differential equations. Its action can always be rewritten as $S(g, R, \nabla R, ..., \nabla^{n-2} R)$. In general, the formula of HEE becomes more and more complicated when higher and higher derivatives are involved. For simplicity, we consider only one special case here.

We choose the cone metric (23) with

$$\begin{aligned}
T &= z^{n-3} T_{\underbrace{z...z}_{n-3}} + \bar{z}^{n-3} T_{\underbrace{\bar{z}...\bar{z}}_{n-3}} + \sum_{m=n-2}^{\infty} e^{2AP_{a_1...a_m}} T_{a_1...a_m} x^{a_1} ... x^{a_m}, \\
V_i &= z^{n-2} V_{\underbrace{z...z}_{n-2} i} + \bar{z}^{n-2} V_{\underbrace{\bar{z}...\bar{z}}_{n-2} i} + \sum_{m=n-1}^{\infty} e^{2AP_{a_1...a_m}} V_{a_1...a_m i} x^{a_1} ... x^{a_m}, \\
Q_{ij} &= z^{n-1} Q_{\underbrace{z...z}_{n-1} ij} + \bar{z}^{n-1} Q_{\underbrace{\bar{z}...\bar{z}}_{n-1} ij} + \sum_{m=n}^{\infty} e^{2AP_{a_1...a_m}} Q_{a_1...a_m ij} x^{a_1} ... x^{a_m}.
\end{aligned} \tag{50}$$

We call this kind of cone as 'the highest-order cone'. That is because only the highest-order derivative of curvature $\nabla^{n-2} R$ contributes to the anomaly of entropy in this case. We have

$$\begin{aligned}
\nabla_z^{n-2} R_{zizj} &= (n-1)\Gamma[n] Q_{\underbrace{z...z}_{n-1} ij} \partial_z A + ..., \\
\nabla_z^{n-2} R_{z\bar{z}zj} &= -i \frac{n-2}{n-1} \Gamma[n+1] e^{2A} V_{\underbrace{z...z}_{n-2} j} \partial_z A + ..., \\
\nabla_z^{n-2} R_{z\bar{z}z\bar{z}} &= \frac{n-3}{n-2} \Gamma[n+1] e^{4A} T_{\underbrace{z...z}_{n-3}} \partial_z A + ...
\end{aligned} \tag{51}$$

In the derivation of the above fromulas, we have identified $z^m \partial_z^{m+1} A$ with $(-1)^m m! \partial_z A$, which can be read out from eq.(29).

Using eqs.(27,29,51), we can derive HEE for 2n-derivative gravity as

$$\begin{aligned}
S_{EE} &= 2\pi \int d^d y \sqrt{g} \Big[\frac{\delta L}{\delta R_{z\bar{z}z\bar{z}}} + 4(n-1)^2 \Gamma[n]^2 \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{zizj} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}k\bar{z}l}} \Big)_{\alpha_1} Q_{\underbrace{z...z}_{n-1} ij} Q_{\underbrace{\bar{z}...\bar{z}}_{n-1} kl} / \beta_{\alpha_1} \\
&+ i8(n-2)\Gamma[n]\Gamma[n+1] \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{zizj} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}z\bar{z}k}} \Big)_{\alpha_1} Q_{\underbrace{z...z}_{n-1} ij} V_{\underbrace{\bar{z}...\bar{z}}_{n-2} k} / \beta_{\alpha_1} + c.c \\
&+ 4\frac{(n-1)(n-3)}{n-2} \Gamma[n]\Gamma[n+1] \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{zizj} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}z\bar{z}z}} \Big)_{\alpha_1} Q_{\underbrace{z...z}_{n-1} ij} T_{\underbrace{\bar{z}...\bar{z}}_{n-3}} / \beta_{\alpha_1} + c.c \\
&+ 16\frac{(n-2)^2}{(n-1)^2} \Gamma[n+1]^2 \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{z\bar{z}zl} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}z\bar{z}k}} \Big)_{\alpha_1} V_{\underbrace{z...z}_{n-2} l} V_{\underbrace{\bar{z}...\bar{z}}_{n-2} k} / \beta_{\alpha_1} \\
&+ -i8\frac{(n-3)}{n-1} \Gamma[n+1]^2 \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{z\bar{z}zi} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}z\bar{z}z}} \Big)_{\alpha_1} V_{\underbrace{z...z}_{n-2} i} T_{\underbrace{\bar{z}...\bar{z}}_{n-3}} / \beta_{\alpha_1} + c.c \\
&+ 4\frac{(n-3)^2}{(n-2)^2} \Gamma[n+1]^2 \Big(\frac{\partial^2 L}{\partial \nabla_z^{n-2} R_{z\bar{z}z\bar{z}} \partial \nabla_{\bar{z}}^{n-2} R_{\bar{z}z\bar{z}z}} \Big)_{\alpha_1} T_{\underbrace{z...z}_{n-3}} T_{\underbrace{\bar{z}...\bar{z}}_{n-3}} / \beta_{\alpha_1} \Big]. \qquad (52)
\end{aligned}$$

As for the general case, the formula of HEE is quite complicated. Like the holographic Weyl anomaly, it seems very difficult (if not impossible) to derive an exact expression. Actually, there is no need to work it out exactly. Instead, for any given action and cone metric, we can directly use eqs.(27,29) to calculate HEE.

4 Checks of our formulas

In this section, we shall check our the formula of HEE. Firstly, we use the FPS method to derive HEE of six-derivative gravity and show it in full agreement with our formula derived in sect. 3.2. Secondly, we prove that our holographic formula yields the correct logarithmic term of entanglement entropy for 4d CFTs. For simplicity, we only focus on a toy model with the action $S = \int dx^D \sqrt{G} \nabla_\mu R_{\nu\alpha\beta\gamma} \nabla^\mu R^{\nu\alpha\beta\gamma}$ in this section.

4.1 The FPS method

In [18], Fursaev et al develop a regularization procedure to calculate the integrals of polynomial curvature invariants on manifolds with squashed conical singularities. By studying the cylindrical and spherical entangling surfaces in Minkowski spacetime, they obtain a formula of HEE for curvature-squared gravity. However, they do not show many details about the calculations. In this subsection, we firstly recover the key point in their derivations and then derive HEE for the six-derivative gravity $S = \int dx^D \sqrt{G} \nabla_\mu R_{\nu\alpha\beta\gamma} \nabla^\mu R^{\nu\alpha\beta\gamma}$ for some cone metrics. For simplicity, we focus on the anomaly of entropy in this section.

Let us start with regularized cylindrical and spherical cone metrics in Minkowski spacetime.

$$ds^2 = \frac{r^2}{n^2} d\tau^2 + dr^2 + (a^2 + 2ar^n \cos\tau + \frac{r^{2n}\cos 2\tau}{2} + \frac{r^2}{2}) d\phi^2 + dz^2, \qquad (53)$$

$$ds^2 = \frac{r^2}{n^2} d\tau^2 + dr^2 + (a^2 + 2ar^n \cos\tau + \frac{r^{2n}\cos 2\tau}{2} + \frac{r^2}{2})(d\theta^2 + \sin^2\theta d\phi^2). \qquad (54)$$

Here we use the regularized cone metric equivalent to Dong's [19] and Camps' [20] but different from [18]. That means there is a factor e^{2A} before $Q_{z\bar{z}ij}$ if we transform the above cone metrics (53,54) into the form of eq.(23). It should be mentioned that, if we take the FPS ansatz for the regularized cone, we also get the same results by applying Dong's method and the FPS method respectively. To avoid the singularity, we firstly do the integral $\int_0^{2n\pi} d\tau$ for an integer n and then analytically continue n to 1. The key point which does not be pointed out clearly in [18] is that one only need to select the would-be logarithmic terms in the integral:

$$S_{\text{Action}} = \int dr d\tau \sum_m C_m (n-1)^2 r^{mn-m-1} + ...,$$
$$S_{\text{Anomaly of HEE}} = \sum_m C_m \frac{2\pi}{m}. \tag{55}$$

For examples, let us derive HEE for the curvature-squared gravity. This is the case investigated in [18]. Note that the Wald entropy of curvature-squared gravity vanishes in Minkowski spacetime. Thus we only need to consider the anomaly of entropy. For the cylindrical metric (53), we obtain

$$\int dx^4 \sqrt{G} R_{\nu\alpha\beta\gamma} R^{\nu\alpha\beta\gamma} = \int dr \frac{32 L n^2 \pi^2 (n-1)^2 r^{2n-3}}{a} \to S_{HEE} = 16\pi^2 \frac{L}{a},$$
$$\int dx^4 \sqrt{G} R_{\mu\nu} R^{\mu\nu} = \int dr \frac{8 L n^2 \pi^2 (n-1)^2 r^{2n-3}}{a} \to S_{HEE} = 4\pi^2 \frac{L}{a},$$
$$\int dx^4 \sqrt{G} R^2 = o(n-1)^2 \to S_{HEE} = 0. \tag{56}$$

Similarly for the spherical metric (54), we have

$$\int dx^4 \sqrt{G} R_{\nu\alpha\beta\gamma} R^{\nu\alpha\beta\gamma} = \int dr \ 128 n^2 \pi^2 (n-1)^2 r^{2n-3} \to S_{HEE} = 64\pi^2,$$
$$\int dx^4 \sqrt{G} R_{\mu\nu} R^{\mu\nu} = \int dr \ 64 n^2 \pi^2 (n-1)^2 r^{2n-3} \to S_{HEE} = 32\pi^2,$$
$$\int dx^4 \sqrt{G} R^2 = o(n-1)^2 \to S_{HEE} = 0. \tag{57}$$

Eqs. (56,57) agree with those of [18].

Let us go on to calculate HEE for the six-derivative gravity $S = \int dx^D \sqrt{G} \nabla_\mu R_{\nu\alpha\beta\gamma} \nabla^\mu R^{\nu\alpha\beta\gamma}$. After some complicated calculations, we get

$$S = \int dr \frac{16L(-1+n)^2 n^2 \pi^2 \left((40-80n+39n^2) r^{4n-5} + (4+4n-3n^2) r^{2n-3}\right)}{a^3} + ...$$
$$\to S_{HEE} = \frac{36\pi^2 L}{a^3}, \tag{58}$$

for the cylindrical metric (53) and

$$S = \int dr \frac{128(-1+n)^2 n^2 \pi^2 \left(2(1-n)(9n-5) r^{4n-5} + (n^2-4) r^{2n-3}\right)}{a^2} + ...$$
$$\to S_{HEE} = \frac{192\pi^2}{a^2}, \tag{59}$$

for the spherical metric (54). Compare eqs.(58-59) with eq.(48) in sect. 3.2, we find full agreement. This can be regraded as a double check of our results in sect. 3.2.

4.2 Logarithmic term of EE for 4d CFTs

In this section, we prove that our formula of HEE yields the correct logarithmic term of EE for 4d CFTs. This is a nontrivial check of our results. For simplicity, we focus on an example of 6-derivative gravity in five-dimensional space-time as follows:

$$S = \frac{1}{16\pi}\int d^5x\sqrt{-\hat{G}}(\hat{R}+\frac{12}{l^2}+\lambda_1\nabla_\mu\hat{R}\nabla^\mu\hat{R}+\lambda_2\nabla_\alpha\hat{R}_{\mu\nu}\nabla^\alpha\hat{R}^{\mu\nu}+\lambda_3\nabla_\alpha\hat{R}_{\mu\nu\rho\sigma}\nabla^\alpha\hat{R}^{\mu\nu\rho\sigma}). \tag{60}$$

According to [33] , the expected logarithmic term of EE for the dual CFTs is

$$S_{EE} = \log(l/\delta)\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}[aR_\Sigma - c(C^{abcd}h_{ac}h_{bd} - k^{\iota ab}k_{\iota ab} + \frac{1}{2}k_a^{\iota a}k_{\iota b}^b)], \tag{61}$$

where the central charges a and c is given by [34]

$$a = \frac{\pi}{8}, \quad c = \frac{\pi}{8} + 8\pi\lambda_3. \tag{62}$$

Thus, it is expected that HEE of $\nabla_\mu\hat{R}\nabla^\mu\hat{R}$ and $\nabla_\alpha\hat{R}_{\mu\nu}\nabla^\alpha\hat{R}^{\mu\nu}$ do not contribute to the logarithmic term, while HEE of $\nabla_\alpha\hat{R}_{\mu\nu\rho\sigma}\nabla^\alpha\hat{R}^{\mu\nu\rho\sigma}$ yields a logarithmic term as

$$-4\lambda_3\log(l/\delta)\int_\Sigma d^2x\sqrt{h}[C^{abcd}h_{ac}h_{bd} - k^{\iota ab}k_{\iota ab} + \frac{1}{2}k_a^{\iota a}k_{\iota b}^b]. \tag{63}$$

As we shall prove below, this is indeed the case.

Let us firstly compute the generalized Wald entropy. Applying the formula (40), we get

$$\begin{aligned} S_{\text{G-Wald}} = \frac{1}{4}\int d\rho d^2y\sqrt{h}[\quad & -1+2\lambda_1\Box R+\lambda_2 n^{\mu\nu}\Box R_{\mu\nu}+\lambda_3\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\Box R_{\mu\nu\rho\sigma} \\ & +2\lambda_2\nabla^\alpha R^{\mu\nu}K_\beta(n^\beta{}_\mu n_{\alpha\nu}-\epsilon^\beta{}_\mu\epsilon_{\alpha\nu}) \\ & +2\lambda_3\nabla^\alpha R^{\mu\rho\nu\sigma}K_{\beta\rho\sigma}(n^\beta{}_\mu n_{\alpha\nu}-\epsilon^\beta{}_\mu\epsilon_{\alpha\nu})\ ]. \end{aligned} \tag{64}$$

Note that we work in the Euclidean signature. So HEE is different from the Lorentzian one by a minus sign. The first term of the above equation is just the Bekenstein-Hawking entropy. According to [33, 15], it gives a logarithmic term as

$$\log(l/\delta)\frac{1}{16}\int_\Sigma d^2x\sqrt{h}[R_\Sigma - (C^{abcd}h_{ac}h_{bd} - k^{\iota ab}k_{\iota ab} + \frac{1}{2}k_a^{\iota a}k_{\iota b}^b)]. \tag{65}$$

Thus we only need to consider the other terms of eq.(64) below.

For asymptotically AdS space-time, we can expand the bulk metric in the Fefferman-Graham gauge

$$ds^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu = \frac{1}{4\rho^2}d\rho^2 + \frac{1}{\rho}g_{ij}dx^i dx^j, \tag{66}$$

where $g_{ij} = \overset{(0)}{g}_{ij} + \rho\overset{(1)}{g}_{ij} + ... + \rho^{\frac{d}{2}}(\overset{(\frac{d}{2})}{g}_{ij} + \overset{(\frac{d}{2})}{h}_{ij}\log\rho) + ...$ Interestingly,

$$\overset{(1)}{g}_{ij} = -\frac{1}{d-2}(\overset{(0)}{R}_{ij} - \frac{\overset{(0)}{R}}{2(d-1)}\overset{(0)}{g}_{ij}), \tag{67}$$

can be determined completely by PBH transformation [35, 36] and thus is independent of equations of motion. However, the higher order terms $\overset{(2)}{g}_{ij}, \overset{(3)}{g}_{ij}...$ are indeed constrained by equations of motion.

Fortunately, for the logarithmic terms of HEE in 5-dimensional space-time, we only need to expand the metric to the subleading order $\overset{(1)}{g}_{ij}$. Let us define a useful quantity $\hat{R}$ as

$$
\begin{aligned}
\hat{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} + (G_{\mu\rho}G_{\nu\sigma} - G_{\mu\sigma}G_{\nu\rho}),\\
\hat{R}_{\mu\nu} &= R_{\mu\nu} + dG_{\mu\nu},\\
\hat{R} &= R + d(d+1).
\end{aligned}
\tag{68}
$$

According to [34], we have

$$
\begin{aligned}
&\hat{R} \sim o(\rho^2), \quad \hat{R}_{ij} \sim o(\rho), \quad \hat{R}_{i\rho} \sim o(\rho), \quad \hat{R}_{\rho\rho} \sim o(1)\\
&\hat{R}_{i\rho j\rho} \sim o(\frac{1}{\rho}), \quad \hat{R}_{\rho ijk} \sim o(\frac{1}{\rho})\\
&\hat{R}_{ijkl} = \frac{C_{ijkl}}{\rho}.
\end{aligned}
\tag{69}
$$

Note that eq.(67) is used in the derivation of above equations.

Denote the transverse space of the squashed cone by m. The embedding of the 3-dimensional submanifold m into 5-dimensional bulk is described by $X^\mu = X^\mu(\sigma^\alpha)$, where $X^\mu = \{x^i, \rho\}$ are bulk coordinates and $\sigma^\alpha = \{y^a, \tau\}$ are coordinates on m. We choose a gauge

$$\tau = \rho, \quad h_{a\tau} = 0, \tag{70}$$

where $h_{\alpha\beta}$ is the induced metric on m. Let us expand the embedding functions as

$$X^i(\tau, y^i) = \overset{(0)}{X}{}^i(y^a) + \overset{(1)}{X}{}^i(y^a)\tau + ... \tag{71}$$

Diffeomorphism preserving the FG gauge (66) and above gauge (70) uniquely fixes a transformation rule of the embedding functions $X^\mu(y^a, \tau)$ [37]. From this transformation rule, we can identity $\overset{(1)}{X}{}^i(y^a)$ with $\frac{1}{4}k^i(y^a)$

$$\overset{(1)}{X}{}^i(y^a) = \frac{1}{4}k^i(y^a), \tag{72}$$

where k^i is the trace of the extrinsic curvature of the entangling surface Σ in the boundary where CFTs live. From eq.(71), we can derive the induced metric on m as

$$h_{\tau\tau} = \frac{1}{4\tau^2}\Big(1 + \frac{1}{4}k^i k^j \overset{(0)}{g}_{ij}\,\tau + \cdots\Big), \qquad h_{ab} = \frac{1}{\tau}\Big(\overset{(0)}{h}_{ab} + \overset{(1)}{h}_{ab}\,\tau + ...\Big), \tag{73}$$

with

$$\overset{(0)}{h}_{ab} = \partial_a \overset{(0)}{X}{}^i \partial_b \overset{(0)}{X}{}^j \; \overset{(0)}{g}_{ij}, \qquad \overset{(1)}{h}_{ab} = \overset{(1)}{g}_{ab} - \frac{1}{2}k^i k^j_{ab} \overset{(0)}{g}_{ij}. \tag{74}$$

Thus, we have

$$\sqrt{h} = \sqrt{\overset{(0)}{h}}\frac{1}{2\rho^2} + ... \tag{75}$$

Using eq.(71), we can also derive the extrinsic curvature K of m as

$$K^i_{ab} = (k^i_{ab} - \frac{k^i}{2}\overset{(0)}{h}_{ab}) + ... \tag{76}$$

Note that all the other components of $K^{\mu}_{\alpha\beta}$ are higher order terms which do not contribute to the logarithmic terms.

Now let us begin to derive the logarithmic term from the generalized Wald entropy eq.(64). Note that $\Box \sim o(1)$ and $(\epsilon^{\mu\nu}, n^{\mu\nu})$ take the same order as $G^{\mu\nu}$. Applying eqs.(69,75,76), we find that, in addition to the Bekenstein-Hawking entropy, only $\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\Box R_{\mu\nu\rho\sigma} \sim o(\rho)$ in the generalized Wald entropy eq.(64) contribute to the logarithmic terms. After some calculations, we can derive

$$\begin{aligned}
S_{\text{G-Wald}} &= \frac{1}{4}\int d\rho d^2y\sqrt{h}\left[\lambda_3\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\Box R_{\mu\nu\rho\sigma}+...\right] \\
&= \frac{1}{4}\int d\rho d^2y\frac{\sqrt{\overset{(0)}{h}}}{2\rho^2}\left[\lambda_3\epsilon^{ij}\epsilon^{kl}(4\rho^2\nabla_\rho\nabla_\rho\hat{R}_{ijkl}+\rho\overset{(0)}{g}{}^{mn}\nabla_m\nabla_n\hat{R}_{ijkl})+...\right] \\
&= \frac{1}{4}\int d\rho d^2y\frac{\sqrt{\overset{(0)}{h}}}{2\rho}\left[\lambda_3\overset{(0)}{\epsilon}{}^{ij}\overset{(0)}{\epsilon}{}^{kl}(-8C_{ijkl})+...\right] \\
&= -4\int d\rho d^2y\frac{\sqrt{\overset{(0)}{h}}}{2\rho}\left[\lambda_3(h^{ik}h^{jl}C_{ijkl})+...\right] \\
&= -4\lambda_3\log(l/\delta)\int d^2y\sqrt{\overset{(0)}{h}}(h^{ac}h^{bd}C_{abcd})+...
\end{aligned} \tag{77}$$

It agrees with the expected logarithmic term of EE for 4d CFTs with zero extrinsic curvature eq.(63). In the above derivations, we have used the following useful formulae

$$\nabla_\rho\nabla_\rho\hat{R}_{ijkl}=\frac{C_{ijkl}}{\rho^3},\quad g^{mn}\nabla_m\nabla_n\hat{R}_{ijkl}=-12\frac{C_{ijkl}}{\rho^2},\quad \overset{(0)}{\epsilon}{}^{ij}\overset{(0)}{\epsilon}{}^{kl}C_{ijkl}=2h^{ac}h^{bd}C_{abcd},\quad \rho_0=\delta^2. \tag{78}$$

Now let us go on to compute the logarithmic term from the entropy eq.(45). Recall that the squashed cone metric is

$$ds^2=e^{2A}[dzd\bar{z}+e^{2A}T(\bar{z}dz-zd\bar{z})^2]+2ie^{2A}V_i(\bar{z}dz-zd\bar{z})dy^i+(g_{ij}+Q_{ij})dy^idy^j \tag{79}$$

with

$$\begin{aligned}
&T=T_0+T_ax^a+..., \\
&V_i=U_i+V_ax^a+V_{abi}x^ax^b+..., \\
&Q_{ij}=2K_{aij}x^a+Q_{abij}x^ax^b+Q_{abcij}x^ax^bx^c....
\end{aligned} \tag{80}$$

Note that there is a factor e^{2A} before $Q_{z\bar{z}}, Q_{zz\bar{z}}, Q_{z\bar{z}\bar{z}}$ and $V_{z\bar{z}}$. It should be stressed that, for asymptotically AdS space-time the submanifold m is very close to the boundary, thus we cannot choose T, V_i, Q_{ij} freely. Instead, they should approach to the value for AdS. On the leading order, we have

$$T_0=-\frac{1}{12},\ U_i=0,\ V_{ai}=0,\ Q_{z\bar{z}ij}=\frac{1}{2}G_{ij},\ Q_{zzij}=K_{zil}K_{zj}^{\ \ l},\ Q_{zz\bar{z}ij}=\frac{4}{9}K_{zij}. \tag{81}$$

Let us derive the above formulas. For simplicity, we focus on pure AdS below. It is expected that it gives the leading value of T, V, Q for asymptotically AdS.

According to [19], we have

$$\begin{aligned}
R_{abcd} &= 12T_0\varepsilon_{ab}\varepsilon_{cd}, \\
R_{abci} &= 3\varepsilon_{ab}V_{ci}, \\
R_{abij} &= 2\varepsilon_{ab}(\partial_i U_j - \partial_j U_i) + G^{kl}(K_{ajk}K_{bil} - K_{aik}K_{bjl}), \\
R_{aibj} &= [\varepsilon_{ab}(\partial_i U_j - \partial_j U_i) + 4G_{ab}U_iU_j] + G^{kl}K_{ajk}K_{bil} - Q_{abij}, \\
R_{ikjl} &= r_{ikjl} + G^{ab}(K_{ail}K_{bjk} - K_{aij}K_{bkl}).
\end{aligned} \tag{82}$$

Comparing the above formula with $R_{\mu\nu\rho\sigma} = -G_{\mu\rho}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\rho}$, we get

$$\begin{aligned}
&T_0 = -\frac{1}{12}, \\
&V_{ci} = 0, \\
&(\partial_i U_j - \partial_j U_i) = 0,\ G^{kl}(K_{ajk}K_{bil} - K_{aik}K_{bjl}) = 0, \\
&G_{ab}G_{ij} + 4G_{ab}U_iU_j + G^{kl}K_{ajk}K_{bil} - Q_{abij} = 0, \\
&r_{ikjl} + G_{ij}G_{kl} - G_{il}G_{kj} + G^{ab}(K_{ail}K_{bjk} - K_{aij}K_{bkl}) = 0.
\end{aligned} \tag{83}$$

Let us make a brief discussion. Since $F_{ij} = \partial_i U_j - \partial_j U_i = 0$, we can always set $U_i = 0$ locally. Since K is in higher order, from the last equation above, we find G_{ij} is the metric of AdS_3 on leading order. To derive the leading order of $Q_{zz\bar{z}ij}$, one need to compute $\nabla_{\bar{z}}R_{zizi}$. To leading order, we have

$$\nabla_{\bar{z}}R_{zizj} = -4T_0K_{zij} + 2K_{zl(i}Q_{z\bar{z}j)}^{\ \ l} - 3Q_{zz\bar{z}ij} + o(K^3) = 0. \tag{84}$$

Taking into account $T_0 = -\frac{1}{12}, Q_{z\bar{z}ij} = \frac{1}{2}G_{ij}$, we get $Q_{zz\bar{z}ij} = \frac{4}{9}K_{zij} + o(K^3)$. Now we can calculate the logarithmic term from the anomaly of entropy.

Without loss of generality, to the leading order, we can choose the cone metric as

$$\begin{aligned}
ds^2 &= e^{2A}[dzd\bar{z} - \frac{1}{12}e^{2A}(\bar{z}dz - zd\bar{z})^2] + \frac{1 + e^{2A}z\bar{z}}{4\rho^4}d\rho^2 \\
&+ \frac{\eta_{ab}(1 + e^{2A}z\bar{z}) + \sqrt{\rho}\big((2z + \frac{4}{3}zz\bar{z})\bar{k}_{zab} + (2\bar{z} + \frac{4}{3}z\bar{z}\bar{z})\bar{k}_{\bar{z}ab}\big)}{\rho}dy^a dy^b
\end{aligned} \tag{85}$$

where we have replaced K by k by using eq.(76) and $\bar{k}_{zab} = (k_{zab} - \frac{k_z}{2}h_{ab})$ is the traceless part of k_{zab}. Substituting the above squashed cone metric into eqs.(109,110,111), we get

$$\begin{aligned}
S_{Anomaly} &= 16\lambda_3 \int d\rho d^2y \frac{1}{2\rho}\sqrt{h}\big[\bar{k}_{zab}\bar{k}_{\bar{z}}^{ab} + o(\rho)\big] \\
&= 4\lambda_3 \log(l/\delta) \int_\Sigma d^2x\sqrt{h}(k^{\iota ab}k_{\iota ab} - \frac{1}{2}k_a^{\iota a}k_{\iota b}^b)
\end{aligned} \tag{86}$$

Combining eqs.(65,77,86), we finally obtain the logarithmic term of HEE as

$$S_{EE} = \log(l/\delta) \int_\Sigma d^2x\sqrt{h}[(\frac{1}{16})R_\Sigma - (\frac{1}{16} + 4\lambda_3)(C^{abcd}h_{ac}h_{bd} - k^{\iota ab}k_{\iota ab} + \frac{1}{2}k_a^{\iota a}k_{\iota b}^b)], \tag{87}$$

which exactly agrees with the CFT results eq.(63). Now we finish the proof.

5 Resolution of the HMS puzzle

Hung, Myers and Smolkin find that the logarithmic term of EE derived from the trace anomaly of 6d CFTs agrees with the holographic result for entangling surfaces with rotational symmetry. However, mismatch appears when the entangling surfaces have no rotational symmetry even if the extrinsic curvature vanishes [15]. We clarify this problem in this section. After considering the anomaly of entropy from the higher-derivative term $C^{ijkl}\nabla^2 C_{ijkl}$, we resolve this problem successfully.

Let us first review the approach of calculating the logarithmic term of EE from the trace anomaly for 6d CFTs [15, 38]. In six dimensions, the trace anomaly takes the following form

$$\langle T^i{}_i \rangle = \sum_{n=1}^{3} B_n\, I_n + 2A\, E_6, \tag{88}$$

where E_6 is the Euler density and I_i are conformal invariants defined by

$$\begin{aligned} I_1 &= C_{kijl}C^{imnj}C_m{}^{kl}{}_n\,, \qquad I_2 = C_{ij}{}^{kl}C_{kl}{}^{mn}C_{mn}{}^{ij}\,, \\ I_3 &= C_{iklm}(\nabla^2\,\delta^i_j + 4R^i{}_j - \frac{6}{5}\,R\,\delta^i_j)C^{jklm}\,. \end{aligned} \tag{89}$$

According to [15, 33, 38], the universal logarithmic term of EE can be identified with HEE of the trace anomaly. For entangling surfaces with the rotational symmetry, only Wald entropy contribute to HEE of the trace anomaly (88). Thus, we have

$$S_{\text{EE}} = \log(\ell/\delta) \int d^4x\sqrt{h}\left[2\pi \sum_{n=1}^{3} B_n\, \frac{\partial I_n}{\partial R^{ij}{}_{kl}}\,\tilde{\varepsilon}^{ij}\,\tilde{\varepsilon}_{kl} + 2\,A\,E_4\right]_\Sigma\,, \tag{90}$$

where

$$\begin{aligned} \frac{\partial I_1}{\partial R^{ij}{}_{kl}}\,\tilde{\varepsilon}^{ij}\,\tilde{\varepsilon}_{kl} &= 3\left(C^{jmnk}\,C_m{}^{il}{}_n\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} - \frac{1}{4}\,C^{iklm}\,C^j{}_{klm}\,\tilde{g}^\perp_{ij} + \frac{1}{20}\,C^{ijkl}\,C_{ijkl}\right)\,, \\ \frac{\partial I_2}{\partial R^{ij}{}_{kl}}\,\tilde{\varepsilon}^{ij}\,\tilde{\varepsilon}_{kl} &= 3\left(C^{klmn}\,C_{mn}{}^{ij}\,\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} - C^{iklm}\,C^j{}_{klm}\,\tilde{g}^\perp_{ij} + \frac{1}{5}\,C^{ijkl}\,C_{ijkl}\right)\,, \\ \frac{\partial I_3}{\partial R^{ij}{}_{kl}}\,\tilde{\varepsilon}^{ij}\,\tilde{\varepsilon}_{kl} &= 2\left(\Box\, C^{ijkl} + 4\,R^i{}_m C^{mjkl} - \frac{6}{5}\,R\,C^{ijkl}\right)\tilde{\varepsilon}_{ij}\,\tilde{\varepsilon}_{kl} - 4\,C^{ijkl}\,R_{ik}\,\tilde{g}^\perp_{jl} \\ &\quad + 4\,C^{iklm}\,C^j{}_{klm}\,\tilde{g}^\perp_{ij} - \frac{12}{5}\,C^{ijkl}\,C_{ijkl}\,. \end{aligned} \tag{91}$$

The above result can be reliably applied for entangling surfaces with rotational symmetry. However, Myers et al find that it is inconsistent with the holographic result for entangling surfaces with zero extrinsic curvature but without a rotational symmetry. Assuming the conditions

$$K_{aij} = 0, R_{abci} = 3\epsilon_{ab}V_{ci} = 0, \tag{92}$$

they derive the holographic result for Einstein gravity as

$$S_{HEE} = 2\pi \log\delta\, \frac{\tilde{L}^5}{\ell_{\text{P}}^5} \int_\Sigma d^4x\sqrt{h}\left[\frac{1}{2}h^{ij}\, \overset{(2)}{g}{}_{ij} + \frac{1}{8}(h^{ij}\, \overset{(1)}{g}{}_{ij})^2 - \frac{1}{4}\, \overset{(1)}{g}{}_{ij}\, h^{jk}\, \overset{(1)}{g}{}_{kl}\, h^{li}\right] \tag{93}$$

The mismatch between holographic result eq.(93) and CFT result eq.(90) becomes

$$\Delta S = 4\pi B_3 \log\delta \int_\Sigma d^4x\sqrt{h} \quad (\quad C_{mn}{}^{rs}C^{mnkl}\tilde{g}^\perp_{sl}\tilde{g}^\perp_{rk} - C_{mnr}{}^{s}C^{mnrl}\tilde{g}^\perp_{sl} \tag{94}$$

$$+2C_m{}^n{}_r{}^s C^{mkrl}\tilde{g}^{\perp}_{ns}\tilde{g}^{\perp}_{kl}-2C_m{}^n{}_r{}^s C^{mkrl}\tilde{g}^{\perp}_{nl}\tilde{g}^{\perp}_{ks})\,,$$

Although eq.(94) is derived in the case of Einstein gravity, Myers et al argue that it can be applied to the general case.

Now let us discuss the origin of the mismatch. First of all, as argued by Myers et al, the holographic results are the correct ones. Thus, something goes wrong with the CFT results. As we shall show below, some contributions are ignored in the CFT calculations. Following the assumption eq.(92), we focus on the cone metric (41) with $K_{aij}=V_{ai}=0$. According our formula eq.(52), in addition to the Wald entropy, a new term proportional to Q^2 also contribute to HEE

$$S \;=\; 2\pi\int d^dy\sqrt{g}\,[\frac{\delta L}{\delta R_{z\bar{z}z\bar{z}}}+64(\frac{\partial^2 L}{\partial\nabla_z R_{zizl}\partial\nabla_{\bar{z}}R_{\bar{z}k\bar{z}l}})_{\alpha_1}\frac{Q_{zzij}Q_{\bar{z}\bar{z}kl}}{\beta_{\alpha_1}}], \tag{95}$$

when the derivative of curvature is included in the action. Since only I_3 (89) contains such terms $C^{ijkl}\nabla^2C_{ijkl}$, so the mismatch ΔS should be proportional to B_3. This explains the proposal of Myers et al that $\Delta S\sim B_3$. Now let us calculate the contribute from $C^{ijkl}\nabla^2C_{ijkl}\cong-\nabla_mC_{ijkl}\nabla^mC^{ijkl}$ exactly. Applying the formula (95), we can derive the contribution ignored in eq.(90) as

$$\Delta S_1=128\pi B_3\log\delta\int_\Sigma d^4x\sqrt{h}(\bar{Q}_{zzij}\bar{Q}_{\bar{z}\bar{z}}{}^{ij}), \tag{96}$$

where $\bar{Q}_{abij}=Q_{abij}-\frac{Q_{ab}}{4}g_{ij}$ is the traceless part of Q_{abij}.

Substituted the cone metric (41) with $K_{aij}=V_{ai}=0$ into eq. (94), we obtain

$$\Delta S=128\pi B_3\log\delta\int_\Sigma d^4x\sqrt{h}(\bar{Q}_{zzij}\bar{Q}_{\bar{z}\bar{z}}{}^{ij}), \tag{97}$$

which is exactly the same as eq.(96). Thus taking into account the contributions from the higher-derivative term $C^{ijkl}\nabla^2C_{ijkl}$, the CFT results exactly match the holographic ones.

6 Conclusions

In this paper, we investigate HEE for the most general higher derivative gravity. In particular, we find a new class of generalized Wald entropy. It appears in the general higher derivative gravity and reduces to Wald entropy for stationary black holes. We also find all the possible would-be logarithmic terms which contribute to the anomaly of entropy. Combining the generalized Wald entropy and the anomaly of entropy together, we obtain a formal formula of HEE for the most general higher derivative gravity. We work out this formula exactly for $2n$-derivative gravity for some interesting cone metrics. We prove that our formula yields the correct universal term of entanglement entropy for 4d CFTs. This is a strong support of our results. As another check, we use the FPS method to derive HEE of six-derivative gravity and get full agreement. As an important application of our formulae, we solve the HMS puzzle that the logarithmic term of entanglement entropy derived from Weyl anomaly of CFTs does not match the holographic result even if the extrinsic curvature vanishes. We find that such mismatch comes from the contributions of the derivative of the curvature. Taking into account such contributions carefully, we find that the CFT result match the holographic one exactly.

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A Useful formulas

In the section, we list some formulas which would be useful for the calculations of HEE for six-derivative gravity. Applying these formulas, we compute HEE for some toy models of six-derivative gravity.

$$\begin{aligned}\bigtriangledown_{\bar{z}} R_{zizj} &= 2K_{zij}\partial_{\bar{z}}\partial_z A - \Big[2\partial_z A K_{\bar{z}i}{}^{k} K_{zkj} \\ &- 2\bar{z}\partial_z A(K_{zi}{}^{k}(Q_{\bar{z}\bar{z}kj} - 2K_{\bar{z}k}{}^{n}K_{\bar{z}nj}) + (i\leftrightarrow j)\Big],\end{aligned} \tag{98}$$

$$\begin{aligned}\bigtriangledown_{s} R_{zizj} &= 2\partial_z A(\bigtriangledown^{y}_{s} K_{zij} + 4iU_s K_{zij}) + 12i\bar{z}\partial_z A V_{\bar{z}s}K_{zij} \\ &+ \Big[2\bar{z}\partial_z A K_{zkj}\Big(2iU^k K_{\bar{z}si} + g^{kl}(\partial_i K_{\bar{z}ls} + \partial_s K_{\bar{z}lt} - \partial_l K_{\bar{z}si}) - 2\gamma^{n}_{si}K_{\bar{z}n}{}^{k}\Big) + (i\leftrightarrow j)\Big],\end{aligned} \tag{99}$$

$$\begin{aligned}\bigtriangledown_{z} R_{z\bar{z}zi} &= -3ie^{2A}\partial_z A V_{zi} \\ &+ 2ie^{2A}\bar{z}\partial_z A(K_{\bar{z}i}{}^{k}V_{zk} - K_{zi}{}^{k}V_{\bar{z}k}),\end{aligned} \tag{100}$$

$$\bigtriangledown_{\bar{z}} R_{z\bar{z}zi} = -2iU_i e^{2A}\partial_z\partial_{\bar{z}} A, \tag{101}$$

$$\begin{aligned}\bigtriangledown_{i} R_{z\bar{z}zj} &= -2K_{zij}\partial_z\partial_{\bar{z}} A - 2K_{\bar{z}i}{}^{k}K_{zkj}\partial_z A \\ &- 2\bar{z}\partial_z A K_{zkj}(Q_{\bar{z}\bar{z}li}g^{lk} - 2K_{\bar{z}}{}^{kn}K_{\bar{z}ni}),\end{aligned} \tag{102}$$

$$\begin{aligned}\bigtriangledown_{z} R_{zi\bar{z}j} &= -2\partial_z A K_{\bar{z}i}{}^{n}K_{znj} \\ &- 2\bar{z}\partial_z A(K_{zj}{}^{n}Q_{z\bar{z}ni} - 2K_{\bar{z}}{}^{mn}K_{\bar{z}ni}K_{zmj}),\end{aligned} \tag{103}$$

$$\begin{aligned}\bigtriangledown_{z} R_{z\bar{z}ij} &= 2\partial_z A K_{\bar{z}j}{}^{n}K_{zni} \\ &- \bar{z}\partial_z A(K_{zi}{}^{n}Q_{\bar{z}\bar{z}nj} + K_{z}{}^{nl}K_{\bar{z}nj}K_{zli}) - (i\leftrightarrow j)\end{aligned} \tag{104}$$

$$\begin{aligned}\bigtriangledown_{z} R_{zijk} &= 2\partial_z A(\partial_j K_{zki} + 4iU_j K_{zik} + K_{zlj}\gamma^{l}_{ik}) \\ &+ 2\bar{z}\partial_z A\Big[3iK_{ij}V_{\bar{z}k} + 2iU^l K_{zlk}K_{\bar{z}ij} - (i\leftrightarrow j) - 2K_{zlj}K_{\bar{z}}{}^{l}{}_{m}\gamma^{m}_{ik}\end{aligned}$$

$$+ \quad K_{\bar{z}j}{}^{m}(\partial_i K_{zkm} + \partial_k K_{zim} - \partial_m K_{zik})\Big] - (j \leftrightarrow k), \tag{105}$$

$$\begin{aligned} \bigtriangledown_l R_{zijk} &= 4e^{-2A}\partial_z A(K_{\bar{z}lj}K_{zik} - K_{\bar{z}lk}K_{zij}) \\ &+ 4e^{-2A}\bar{z}\partial_z A(Q_{\bar{z}\bar{z}lk}K_{zij} - Q_{\bar{z}zlj}K_{zik}). \end{aligned} \tag{106}$$

$$\begin{aligned} \bigtriangledown_z R_{ikjl} &= 4e^{-2A}\partial_z A(K_{zij}K_{\bar{z}kl} + K_{\bar{z}ij}K_{zkl}) \\ &- 4e^{-2A}\bar{z}\partial_z A(K_{zij}Q_{\bar{z}\bar{z}kl} + K_{zkl}Q_{\bar{z}\bar{z}ij}) - (j \leftrightarrow k), \end{aligned} \tag{107}$$

$$\begin{aligned} \bigtriangledown_z R_{zizj} &= 2K_{zij}\partial_z\partial_z A + \partial_z A(4Q_{zzij} - 8K_{zi}{}^{l}K_{zlj}) \\ &- e^{2A}\bar{z}\partial_z A\Big[-24TK_{zij} + i(\partial_j V_{zi} + \partial_i V_{zj}) \\ &- 2iU^l(\partial_j K_{zli} + \partial_i K_{zlj} - \partial_l K_{zji}) - 2iV_{zk}\gamma^k_{ij} \\ &- 8U^k(U_j K_{zki} + U_i K_{zkj}) + 4e^{-2A}(k_{zj}{}^{k}Q_{z\bar{z}ik} + K_{zi}{}^{l}Q_{z\bar{z}lj}) \\ &- 2Q_{zz\bar{z}ij} - 16K_{zik}K_{z}{}^{lk}K_{zlj}\Big]. \end{aligned} \tag{108}$$

Applying the above formulas and eqs.(27,29), let us compute the anomaly of entropy for some toy models.

For $L_1 = \bigtriangledown_\mu R \bigtriangledown^\mu R$, we obtain

$$S_{\text{Anomaly}} = 16\pi \int dy^d \sqrt{g}\Big[3[tr(K_z K_{\bar{z}})]^2 + K_z^2 K_{\bar{z}}^2 - 2K_z K_{\bar{z}} tr(K_z K_{\bar{z}})\Big], \tag{109}$$

For $L_2 = \bigtriangledown_\mu R_{\rho\nu} \bigtriangledown^\mu R^{\rho\nu}$, we get

$$\begin{aligned} S_{\text{Anomaly}} = \quad & -4\pi \int dy^d \sqrt{g}\Big[40TK_zK_{\bar{z}} - 6K_z^2 trK_{\bar{z}}^2 + 2K_zK_{\bar{z}}tr(K_zK_{\bar{z}}) - 2Q_{z\bar{z}}K_zK_{\bar{z}} \\ & -K_z^2K_{\bar{z}}^2 - 8trK_z^2 trK_{\bar{z}}^2 - 8tr(K_zQ_{z\bar{z}})K_{\bar{z}} + 18tr(K_{\bar{z}}^2K_z)K_z \\ & -8K_z tr(Q_{\bar{z}\bar{z}}K_z) + 8tr(K_z^2)Q_{\bar{z}\bar{z}} + 4K_z^2Q_{\bar{z}\bar{z}} - 4Q_{zz}Q_{\bar{z}\bar{z}} \\ & +4Q_{zz\bar{z}}K_{\bar{z}} - 8(trK_zK_{\bar{z}})^2 - 8tr(K_zK_zK_{\bar{z}}K_{\bar{z}}) - 8tr(K_zK_{\bar{z}}K_zK_{\bar{z}}) \\ & -\bigtriangledown_i^{(y)} K_z \bigtriangledown^{(y)i} K_{\bar{z}} - 2R_{zkij}R_{zk'i'j'}g^{kj}g^{k'j'}g^{ii'} + (z \leftrightarrow \bar{z})\Big], \end{aligned} \tag{110}$$

where $K_a \equiv g^{ij}K_{aij}$, $tr(K_aK_b) = g^{ij}K_{ail}K_{bj}{}^{l}$, $Q_{ab} = g^{ij}Q_{abij}$, etc.

For $L_3 = \nabla_\mu R_{\nu\alpha\beta\gamma}\nabla^\mu R^{\nu\alpha\beta\gamma}$, we have

$$\begin{aligned} S_{\text{Anomaly}} &= 32\pi \int d^d y \sqrt{g}\Big[4Q_{\bar{z}\bar{z}ij}Q_{zz}{}^{ij} + 8K_{\bar{z}}{}^{ij}K_{zj}{}^{k}Q_{z\bar{z}ki} - 2K_{\bar{z}p}{}^{p}K_{\bar{z}}{}^{ij}Q_{zzij} \\ &- 2K_{zp}{}^{p}K_{z}{}^{ij}Q_{\bar{z}\bar{z}ij} + 2K_{zij}K_{\bar{z}}{}^{ij}Q_{z\bar{z}p}{}^{p} + (K_{zij}K_{\bar{z}}{}^{ij})^2 + K_{zp}{}^{p}K_{\bar{z}q}{}^{q}K_{zij}K_{\bar{z}}{}^{ij} \\ &- 4K_{z}{}^{ij}K_{\bar{z}jk}K_{\bar{z}}{}^{kl}K_{zli} + \bigtriangledown_s^{(y)}K_{zij} \bigtriangledown^{(y)s} K_{z}{}^{ij} - 40TK_{zij}K_{\bar{z}}{}^{ij} \\ &+ K_{zij}K_{z}{}^{ij}K_{\bar{z}kl}K_{\bar{z}}{}^{kl} + R_{zijk}R_{\bar{z}}{}^{ijk} - 6Q_{zz\bar{z}ij}K_{\bar{z}}{}^{ij} - 6Q_{\bar{z}\bar{z}zij}K_{z}{}^{ij}\Big]. \end{aligned} \tag{111}$$

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