

The Extreme Distortion of Black Holes due to Matter

David Petroff*

Theoretisch-Physikalisches Institut, University of Jena, Max-Wien-Platz 1, 07743 Jena, Germany

Marcus Ansorg[†]

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, 14476 Golm, Germany

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A highly accurate computer program is used to study axially symmetric and stationary spacetimes containing a Black Hole surrounded by a ring of matter. It is shown that the matter ring affects the properties of the Black Hole drastically. In particular, the absolute value of the ratio of the Black Hole's angular momentum to the square of its mass not only exceeds one, but can be greater than ten thousand ($|J_c|/M_c^2 > 10^4$). Indeed, the numerical evidence suggests that this quantity is unbounded.

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One of the great contributions to general relativity was Roy Kerr's presentation in 1963 [1] of an axially symmetric, stationary solution to Einstein's vacuum equations that turned out to describe a rotating Black Hole – the Kerr Black Hole. The fact that this simple solution is fully described by specifying merely two parameters, the fact that there exist proofs (see for example [2]) showing it to be the only stationary (non-charged) Black Hole and the fact that recent observations show Black Holes to be important and ubiquitous objects in the cosmos, make the Kerr solution all the more important. The Black Holes known to exist in the centres of many galaxies play an important role in present day astronomy and compact binaries containing at least one Black Hole are expected to be important sources for the burgeoning field of gravitational wave astronomy. Such objects are, however, anything but isolated, whence the Kerr solution is not always going to be appropriate to model the corresponding Black Hole.

In this letter we consider a stationary and axisymmetric spacetime containing a Black Hole surrounded by a ring of matter. We write the line element in Lewis-Papapetrou coordinates

$$ds^2 = e^{2\mu}(d\varrho^2 + d\zeta^2) + \varrho^2 B^2 e^{-2\nu} (d\varphi - \omega dt)^2 - e^{2\nu} dt^2,$$

where the four metric functions μ , B , ν and ω depend only on ϱ and ζ . We make use of a coordinate freedom to choose the horizon to be a sphere in these coordinates and ensure that this surface is indeed an event horizon by requiring that

$$e^\nu = 0, \quad B = 0 \quad \text{and} \quad \omega = \text{constant} =: \Omega_c$$

hold there [3]. The functions e^ν and B tend to zero on the horizon in such a way that

$$e^u := e^\nu/B$$

remains finite and non-zero. Einstein's equations along with these boundary conditions, the correct asymptotic

behaviour, transition conditions and regularity make up a free-boundary problem (the surface of the ring is unknown *a priori*), which we solve using a multi-domain pseudo-spectral method described in detail in [4]. After specifying the equation of state used to describe the perfect fluid ring, the solution depends on four parameters. In this letter we choose the very simple model of a uniformly rotating ring of constant density.

It is well known that for the Kerr solution, the dimensionless quantity¹ $|J|/M^2$ describing the ratio of the angular momentum to the square of the mass of the Black Hole can range from zero in the static limit to one for the extreme Kerr Black Hole. Recently, steps have been taken toward showing that this quantity must be less than one for all axially symmetric, asymptotically flat, complete pure vacuum data [5, 6, 7]. In the case being considered here, it is possible to define the mass and angular momentum of the Black Hole and the ring separately. It follows from the field equations that the total angular momentum of the system can be represented as a surface integral over the horizon of the Black Hole plus a volume integral over the ring. The integrals are the same as those one finds for the case of a lone Black Hole or ring so that one can define this local object as 'belonging' to the body in question. The situation for the mass of the two objects is very similar (see [3] for more details). It was shown in [4] that the inequality $|J_c|/M_c^2 \leq 1$, where the subscript 'c' indicates that the objects refer to the Black Hole (i.e. the central object), no longer holds for a Black Hole surrounded by a ring of matter. There, an exemplary configuration in which $|J_c|/M_c^2 = 20/19$ was calculated to high accuracy. We now show that this quantity can become dramatically larger than one.

We consider a sequence of configurations for which the

¹ We choose natural units in which the speed of light c and gravitational constant G are both equal to one.

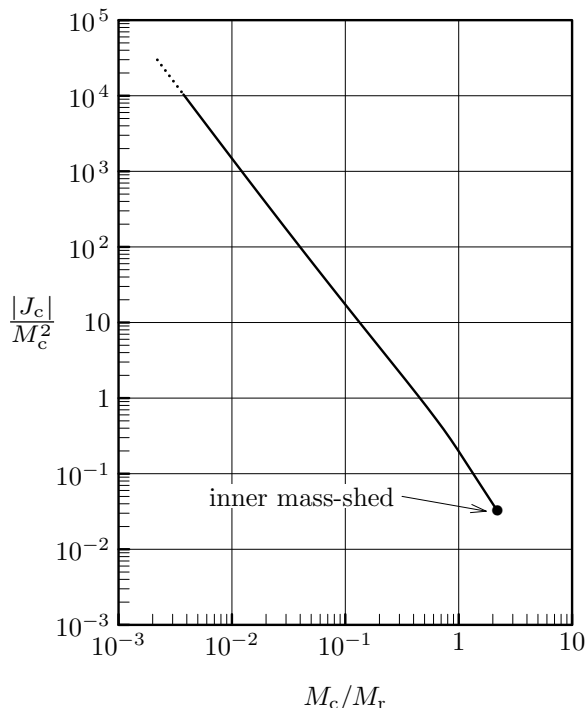


FIG. 1: The dimensionless quantity $|J_c|/M_c^2$ is plotted versus M_c/M_r for a sequence in which $\Omega_c = 0$, $\varrho_i/\varrho_o = 0.7$ and $\beta_o = 0.1$ were held constant. The dotted line indicates that the curve continues on further.

angular velocity of the horizon with respect to infinity is taken to be $\Omega_c = 0$, the ratio of the inner to outer coordinate radius of the ring is held constant at a value of $\varrho_i/\varrho_o = 0.7$ and a geometric parameter measuring the ‘distance’ of the outer edge of the ring to a mass-shedding limit is $\beta_o = 0.1$ ($\beta_o = 0$ corresponds to the outer mass-shedding limit)². We can then choose the ratio of the Black Hole mass to that of the ring M_c/M_r to parameterize the one dimensional sequence that results. The choice $\beta_o = 0.1$ ensures that the outer edge of the ring always remains close to the mass-shedding limit, which, in turn means that the influence of the Black Hole on the ring is never negligible, even in the limit $M_c/M_r \rightarrow 0$. This can be seen by examining the behaviour of a sequence of homogeneous rings without a central body, where $\varrho_i/\varrho_o = 0.7$ is held constant. For such a sequence, the outer mass-shedding parameter ranges from $\beta_o = 0.832$ in the Newtonian limit to $\beta_o = 0.463$ in the extreme Kerr limit. This sequence is a vertical line in Fig. 1 of [8] and can be seen to remain far away from the mass-shedding curve. Hence the choice $\beta_o = 0.1$ is only possible by virtue of the influence of the central Black

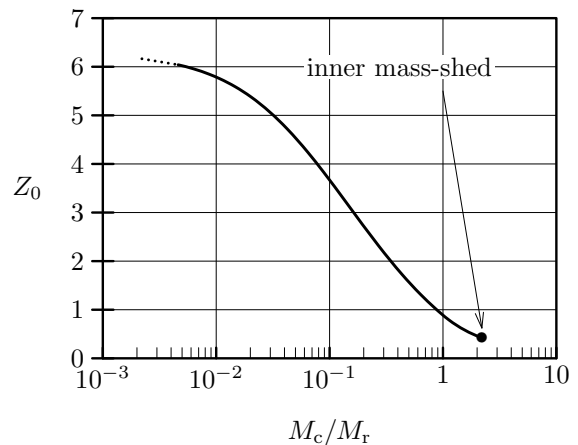


FIG. 2: The relative redshift of zero angular momentum photons emitted from the surface of the ring and observed at infinity is plotted versus M_c/M_r for the sequence plotted in Fig. 1. The dotted line indicates that the curve continues on further.

Hole.

Fig. 1 reveals the dramatic influence the matter ring can have on the Black Hole. Beginning with the right side of the curve, we see that for large values of M_c/M_r , $|J_c|/M_c^2$ grows small, just as expected when nearing a Schwarzschild Black Hole. The effect of the ring is no longer so great. As M_c/M_r increases, the Black Hole ‘pulls’ more and more on the ring until an inner mass-shedding limit is reached.

Now turning our attention to the left side of the curve in Fig. 1, we see that as $M_c/M_r \rightarrow 0$, $|J_c|/M_c^2$ grows dramatically. Indeed, there is no indication that there is any bound whatsoever to this quantity. This is one measure of the fact that the properties of axially symmetric, stationary Black Holes that we know so well from the Kerr solution can be affected drastically by the presence of matter. That such an extremely large influence is observed for the sequence in Fig. 1, presumably has to do with the fact that a situation was chosen in which the surrounding matter can become highly relativistic, as measured for example by the relative redshift Z_0 of a zero angular momentum photon leaving the surface of the ring and observed at infinity. In Fig. 2, Z_0 is plotted versus M_c/M_r .

In Table I various physical parameters are shown for a configuration near the left edge of Fig. 1 for which $|J_c|/M_c^2$ has already reached a very large value (all digits listed are valid). The symbols R_p , R_e , R_i and R_o refer to circumferential radii (proper circumference/ 2π) for the polar and equatorial radii of the horizon and the inner and outer radii of the ring respectively. The symbol κ refers to surface gravity and A_c to the surface area of the horizon. The subscript ‘c’ refers to the Black Hole (i.e. central body) and ‘r’ to the ring. The bar above the symbol indicates that the quantity is dimensionless through

² The definition of β_o is $\frac{2\varrho_o}{\varrho_o - \varrho_i} \frac{d(\zeta_B^2)}{d(\varrho^2)} \Big|_{\varrho=\varrho_o}$, where $\zeta_B = \zeta_B(\varrho)$ is a parametric representation of the surface of the ring.

TABLE I: Physical parameters for a configuration near the left edge of Fig. 1 (left column). The other two columns provide comparative values from the Kerr solution with the two prescribed parameters shown in bold. The last two rows show how well the identities $M_i = M_a$ and $J_i = J_a$ are fulfilled (see text for a description of the various symbols).

	BH-Ring	Kerr with same $\bar{J}_c, \kappa M_c$	Kerr with same \bar{J}_c, \bar{A}_c
$ J_c /M_c^2$	3.281340×10^4	$1 - 4.5 \times 10^{-10}$	0.9998
\bar{J}_c	-5.415253×10^{-2}	-5.42×10^{-2}	-5.42×10^{-2}
\bar{J}_r	0.4383768	–	–
\bar{M}_c	0.001284647	0.233	0.233
\bar{M}_r	0.6124967	–	–
$\bar{\Omega}_c$	0	-2.15	-2.11
$\bar{\Omega}_r$	0.6148853	–	–
R_p/R_e	0.6160728	0.608	0.616
\bar{R}_e	0.4654637	0.465	0.465
\bar{R}_i	1.251534	–	–
\bar{R}_o	1.518647	–	–
κM_c	1.494494×10^{-5}	1.49×10^{-5}	0.00951
\bar{A}_c	1.387660	1.36	1.39
$ \frac{M_i - M_a}{M_a} $	2.6×10^{-9}	–	–
$ \frac{J_i - J_a}{J_a} $	3.0×10^{-9}	–	–

multiplication with the appropriate power of energy density, which we have chosen to be the scaling parameter. The last two rows show that the identities $M_i = M_a$ and $J_i = J_a$ are satisfied to very high accuracy. Here M_i and J_i refer to the total mass and angular momentum of the system as calculated via a volume integral over the ring plus a surface integral over the horizon of the Black Hole whereas M_a and J_a refer to the total mass and angular momentum as read off at infinity from the asymptotic behaviour of the metric functions. For more details regarding the various quantities discussed here, see [4].

If one compares the Black Hole in the configuration of Table I with a Kerr Black Hole of either the same dimensionless parameters \bar{J}_c and κM_c or \bar{J}_c and \bar{A}_c , then

one finds that the first is very close to the extreme Kerr solution ($\kappa M_c = 0$ marks this solution after all), whereas the second is significantly farther away, but nonetheless quite close, as witnessed by the value $|J_c|/M_c^2 = 0.9998$ for example. In both cases, the difference between the pure vacuum case and the case containing the matter ring is dramatic.

It will be both interesting and important to study the stability of systems consisting of Black Holes surrounded by rings. If it turns out that they can be expected to exist for astrophysically relevant periods of time, then we are going to have to rethink what conclusions can be drawn from the observation of a densely filled region of space. Not only could normal matter make a significant contribution to the mass of such a region, but the properties of the central Black Hole (if one is expected to exist) are not necessarily going to be akin to those we know from the Kerr solution or indeed from a perturbation to the Kerr solution.

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* Electronic address: D.Petroff@tpi.uni-jena.de

† Electronic address: Marcus.Ansorg@aei.mpg.de

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