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CONFORMALLY INVARIANT SUPERSYMMETRIC FIELD THEORIES ON $S^p \times S^1$ AND SUPER p-BRANES

Hermann NICOLAI

Institut für Theoretische Physik, Universität Karlsruhe, Karlsruhe, FRG

Ergin SEZGIN and Yoshiaki TANII

International Centre for Theoretical Physics, Trieste, Italy

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We construct supersymmetric field theories on $S^p \times S^1$ for $p = 1, \dots, 5$. These spaces are the boundaries of the anti-de Sitter space in $(p+2)$-dimensions. These theories are invariant under the super extension of the anti-de Sitter group in $(p+2)$-dimensions, whose bosonic subalgebra is $SO(p+1,2) \oplus SO(N)$, for $N = 1,2,4,8$. The anti-de Sitter group $SO(p+1,2)$ acts as the conformal group on $S^p \times S^1$. We conjecture that these theories describe the super p-branes propagating in $AdS_{p+2} \times S^{N-1}$. The internal space is always one of the parallelizable spheres, namely S^1, S^3 or S^7.

1. Introduction

Some time ago [1] we derived the $N = 1$ supersymmetric singleton field theory from the Wess–Zumino model in AdS_4 [2]. (We recall that singletons are the most fundamental representation of the anti-de Sitter group $SO(3,2)$ [3].) This theory was formulated on the boundary of AdS_4 which is $S^2 \times S^1$, in accordance with earlier results of refs. [4, 5]. However, in the second half of the previous paper [1], a mass term for the bosonic singleton and a subtlety in the truncation of the supersymmetry transformation rules were overlooked. The purpose of this paper is to correct these errors and further extend the results of ref. [1]. Moreover, we are motivated by a recent surge of interest in singletons due to their possible occurrence in supermembranes [6, 7].

One of the advantages of our treatment is that the theory is formulated directly on $S^2 \times S^1$ from the outset, and the action of $SO(3,2)$ or $OSp(N|4)$ can be exhibited explicitly. Therefore, we need not address the problem of how to extend the singleton actions into the whole of AdS_4. In this paper we also apply our techniques to construct a supersymmetric doubleton field theory. The doubletons are the most fundamental representations of the anti-de Sitter group $SO(6,2) \equiv O^*(8)$ which were

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found in ref. [8]. It was pointed out in ref. [8] that if the $N = 4$ supersymmetric doubleton field theory existed in six dimensions it would contain a two-index antisymmetric tensor field with a self-dual field strength. Therefore, this theory would not admit a covariant action. However, we construct the $N = 4$ supersymmetric covariant field equations, formulated directly on the boundary of AdS_7 which is $S^5 \times S^1$. The $N = 2$ truncation of this theory no longer contains the two-index antisymmetric tensor field. Therefore, the $N = 2$ case does admit a covariant action which we will exhibit.

So far we have discussed supersymmetric field theories on $S^2 \times S^1$ and $S^5 \times S^1$. In fact, in this paper we will construct supersymmetric field theories on $S^p \times S^1$ for $p = 1, \ldots, 5$. These theories are invariant under the super extension of the anti-de Sitter group in $(p + 2)$-dimensions whose bosonic subalgebra is $SO(p + 1, 2) \oplus SO(N)$ for $N = 1, 2, 4, 8$. The anti-de Sitter group $SO(p + 1, 2)$ acts as the conformal group on $S^p \times S^1$. We conjecture that these theories describe the super p-branes [9,10] propagating in $AdS_{p+2} \times S^{N-1}$. Note the interesting feature that the internal space is always one of the parallelizable spheres, namely S^1, S^3 or S^7.

In order to put the results of the following sections in perspective, here we list the hierarchy of the supersymmetric singleton field theories we have constructed, together with their symmetries, and the conjectured relation with super p-branes

Extended object	Spacetime	Bosonic symmetries	Superalgebra	N
String	$AdS_3 \times S^{N-1}$	$SO(2,2) \oplus SO(N)$	$OSp(N\|2)$	1, 2, 4, 8
Membrane	$AdS_4 \times S^{N-1}$	$SO(3,2) \oplus SO(N)$	$OSp(N\|4)$	1, 2, 4, 8
3-brane	$AdS_5 \times S^{2N-1}$	$SO(4,2) \oplus SO(2N)$	$SU(2,2\|N)$	1, 2
4-brane	$AdS_6 \times S^{2N-1}$	$SO(5,2) \oplus SO(2N)$	F(4)	2
5-brane	$AdS_7 \times S^{2N-1}$	$SO(6,2) \oplus SO(2N)$	$OSp(6,2\|N)$	2

F(4) is the exceptional subalgebra whose bosonic subalgebra is $SO(5,2) \oplus SU(2)$. (For a discussion of the super extension of anti-de Sitter groups, see [11].) Note that for $N = 2$, $p = 3, 4, 5$, there is an SU(2) symmetry which commutes with the superalgebra. In sects. 2, 3 and 4 we shall describe the supersymmetric field theories on $S^2 \times S^1$, $S^3 \times S^1$ and $S^5 \times S^1$, respectively. The cases for $S^1 \times S^1$ [12] and $S^4 \times S^1$ are given in appendix B. General formula for conformal transformations, lagrangians, Killing vectors and spinors on $S^p \times S^1$ for any p are given in appendix A. The metric on $S^p \times S^1$ will always be denoted by h_{ij}. In our conventions $\{\gamma_i, \gamma_j\} = 2h_{ij}$ $(i = 0, 1, \ldots, p)$ and the signature of h_{ij} is $(- + + \ldots +)$.

2. OSp(N|4) invariant singleton field theory on $S^2 \times S^1$ and interactions

The $N = 1$ singleton supermultiplet consists of a real scalar ϕ and a four-component spinor λ_+. These fields live on $S^2 \times S^1$ and therefore depend on coordinates (t, θ, φ). In addition to the four-dimensional Majorana condition $\bar{\lambda}_+ = \lambda^T_+ C$, λ_+ satisfies the following chirality condition

$$\gamma^3 \lambda_+ = \lambda_+ , \tag{2.1}$$

which, unlike the usual chirality condition, is compatible with the Majorana condition (the matrix γ^3 corresponds to $\hat{x}^i \gamma^i$ of ref. [1]). The correct free lagrangian is given by

$$\mathscr{L}_0 = -\tfrac{1}{2}\sqrt{-h}\left(h^{ij}\,\partial_i \phi\,\partial_j \phi + \tfrac{1}{4}\phi^2 - i\bar{\lambda}_+ \gamma^i\,\nabla_i \lambda_+\right), \tag{2.2}$$

where h_{ij} is the metric and ∇_i is the covariant derivative on $S^2 \times S^1$. The *mass term* was not given in ref. [1] but must be present because of SO(3,2) invariance. The bosonic and fermionic part of eq. (2) are separately invariant under the transformations [7]

$$\delta\phi = \xi^i_{AB}\,\partial_i \phi + \Omega_{AB}\phi , \tag{2.3}$$

$$\delta\lambda_+ = \xi^i_{AB}\,\nabla_i \lambda_+ + \tfrac{1}{4}\nabla_i \xi_{jAB}\gamma^{ij}\lambda_+ + 2\Omega_{AB}\lambda_+ , \tag{2.4}$$

where the indices A, B assume the values 0, 1, 2, 3, 5 and ξ^i_{05} and ξ^i_{ab} $(a, b = 1, 2, 3)$ are the Killing vectors generating SO(2) × SO(3) transformations while ξ^i_{0a}, ξ^i_{5a} are conformal Killing vectors on $S^2 \times S^1$ generating the remaining transformations of SO(3,2). The explicit expressions for these may be obtained from the SO(3,2) Killing vectors of AdS_4 given in ref. [2] by restricting them to the boundary [7]. In obtaining the functions Ω_{AB} one has to rescale the component of the AdS Killing vector in the radial direction by a factor of $\cos\rho$, where $\rho = \frac{1}{2}\pi$ corresponds to the boundary of AdS. The result is given in appendix C. To prove invariance, one only needs the relations [7]

$$\nabla_i \xi_{jAB} + \nabla_j \xi_{iAB} = 4\Omega_{AB} h_{ij}, \qquad \nabla^i \partial_i \Omega_{AB} = -\Omega_{AB}, \qquad (2.5),(2.6)$$

where only Ω_{0a}, Ω_{5a} are different from zero, and eq. (2.6) is a consequence of eq. (2.5).

The combined action (2.2) is invariant under the supersymmetry transformations

$$\delta_0 \phi = -i\bar{\eta}_- \lambda_+, \qquad \delta_0 \lambda_+ = \gamma^i \partial_i \phi \eta_- + \tfrac{1}{2}\phi\eta_+ . \tag{2.7}$$

Here, η_+ and η_- obey $\gamma^3 \eta_\pm = \pm\eta_\pm$ as well as the Killing spinor equations

$$\left(\partial_0 - \tfrac{1}{2}\gamma_0\right)\eta = 0, \qquad \left(\nabla_\alpha - \tfrac{1}{2}\gamma_\alpha\gamma_3\right)\eta = 0, \quad (\alpha = 1, 2), \tag{2.8}$$

which, again, may be derived from the Killing spinors of AdS_4 given in ref. [2]*. Eq. (2.8) implies the following $S^2 \times S^1$ covariant equations

$$\nabla_i \eta_- - \tfrac{1}{2}\gamma_i \eta_+ = 0, \qquad \gamma^i \nabla_i \eta_+ + \tfrac{1}{2}\eta_- = 0. \tag{2.9}$$

One can show that the commutator of two supersymmetry transformations (2.7) yields the SO(3,2) transformations, and thus the theory is OSp(4|1) invariant. (This is shown in detail for the $N = 8$ singleton in ref. [7].) We emphasize this remarkable property of singletons: although they are formulated on the boundary of AdS, they exhibit the *full* symmetry of the AdS spacetime itself.

We now turn to the problem of introducing interactions. We first observe that the interactions, if any, cannot involve derivative couplings. This is because h_{ij}, ϕ and λ_+ have conformal weights -4, 1 and 2, respectively, as can be seen from eqs. (2.3), (2.4) and (2.5), and no derivative coupling terms can be written down with correct scaling properties; although the $S^p \times S^1$-coordinates, and therefore derivatives, have zero conformal weight, introduction of each derivative introduces an inverse vielbein with conformal weight 2. Let us consider, then, an arbitrary function $F(\phi)$, which must be separately invariant under eq. (2.3), as eq. (2.2) is already invariant by itself. This immediately gives

$$\begin{aligned} g\left[\sqrt{-h}\,F(\phi)\right] &= \sqrt{-h}\,F'(\phi)\left[\xi^i_{AB}\,\partial_i\phi + \Omega_{AB}\phi\right] \\ &= \sqrt{-h}\left[-F(\phi)\nabla_i\xi^i_{AB} + F'(\phi)\phi\Omega_{AB}\right], \end{aligned} \tag{2.10}$$

where we have dropped a total derivative. Substituting eq. (2.5), we get $F'(\phi)\phi = 6F(\phi)$ or $F(\phi) = c\phi^6$, where c is an arbitrary constant. In a similar way, one establishes that the only allowed interaction involving the fermions is $\phi^2\lambda_+\gamma_5\lambda_+$. The supersymmetric extension is obtained by adding the following interaction lagrangian

$$\mathscr{L}_1 = \tfrac{1}{2}\sqrt{-h}\left(-c^2\phi^6 - 3c\phi^2\bar{\lambda}_+\gamma_5\lambda_+\right), \tag{2.11}$$

to the free lagrangian $\mathscr{L}_0$, and

$$\delta_1\lambda_+ = ic\phi^3\gamma_5\eta_-, \tag{2.12}$$

to the transformation rules given in eq. (2.7). These results agree with those obtained by Fronsdal [5] by a different method. For the invariance of the interacting theory,

* The precise relation is

$$\varepsilon = \sqrt{2}\,(\cos\rho)^{-1/2}\left[\left(\cos\tfrac{1}{2}\rho + \sin\tfrac{1}{2}\rho\right)\eta_- + \left(\cos\tfrac{1}{2}\rho - \sin\tfrac{1}{2}\rho\right)\eta_+\right].$$

In ref. [1], the second term, which is nonsingular in the limit $\rho \to \tfrac{1}{2}\pi$, was incorrectly dropped.

the following identity is crucial

$$\bar{\lambda}_+\gamma_5\lambda_+\bar{\lambda}_+\eta_- = 0\,, \tag{2.13}$$

which follows from the Fierz rearrangement and the Majorana and chirality properties of λ_+.

In the remaining part of this section we consider the construction of singleton actions with extended supersymmetry and focus on the case $N=8$; the case $N<8$ is completely analogous. The fields ϕ^A and $\lambda_+^{\dot{A}}$ now belong to the inequivalent spinor representations of 8_s and 8_c of SO(8). The free lagrangian (2.2) is generalized to

$$\mathscr{L}_0 = -\tfrac{1}{2}\sqrt{-h}\left(h^{ij}\,\partial_i\phi^A\,\partial_j\phi^A + \tfrac{1}{4}\phi^A\phi^A - i\bar{\lambda}_+^{\dot{A}}\gamma^i\,\nabla_i\lambda_+^{\dot{A}}\right), \tag{2.14}$$

while the transformation rules (2.7) now read

$$\delta_0\phi^A = -i\Gamma^I_{A\dot{A}}\bar{\eta}^I_-\lambda_+^{\dot{A}}\,, \qquad \delta_0\lambda^{\dot{A}} = \Gamma^I_{A\dot{A}}\left(\gamma^i\partial_i\phi^A\eta^I_- + \tfrac{1}{2}\phi^A\eta^I_+\right), \tag{2.15}$$

where $\Gamma^I_{A\dot{A}}$ are the SO(8) Clebsch–Gordan coefficients. One can show that eq. (2.14), in contrast to eq. (2.2), does *not* admit interactions which preserve $N=8$ supersymmetry. By $\mathrm{SO}(3,2)\times\mathrm{SO}(8)$ invariance the only possibilities are $(\phi^A\phi^A)^3$ and $\phi^A\phi^A\bar{\lambda}_+^{\dot{A}}\gamma_5\lambda_+^{\dot{A}}$ ★. Supersymmetry will fail, however, because

$$\bar{\lambda}_+^{\dot{A}}\gamma_5\lambda_+^{\dot{A}}\bar{\lambda}_+^{\dot{B}}\eta^I_- \neq 0\,, \tag{2.16}$$

for $N=8$. One can easily check that the underlying algebra of eq. (2.15) is OSp(8|4) [7].

For $N=2$, remarkably, the following identity holds

$$\bar{\lambda}_+^A\gamma_5\lambda_+^B\bar{\lambda}_+^C\eta^I_- = 0\,, \qquad A=1,2\,. \tag{2.17}$$

Consequently, it turns out that self-interactions are possible for the $N=2$ case as well – a result first obtained by Sokatchev [15]. In that case, the superspace action of ref. [15] in components reads as

$$\begin{aligned}\mathscr{L} = -\tfrac{1}{2}\sqrt{-h}\Big[&h^{ij}\,\partial_i\phi^A\,\partial_j\phi^A + \tfrac{1}{4}\phi^2 - i\bar{\lambda}_+^A\gamma^i\nabla_i\lambda_+^A \\ &+ c^2\left(\phi^2\right)^3 - 6c\left(\phi^A\phi^B - \tfrac{1}{2}\phi^2\delta^{AB}\right)\bar{\lambda}_+^A\gamma_5\lambda_+^B\Big],\end{aligned} \tag{2.18}$$

★ This can be seen as follows. First, we recall that the derivative couplings are not allowed due to their wrong scaling properties. Next, we observe that, due to the SO(8) Fierz identity, we only need to consider terms of the type $(\phi\phi)$, $(\phi\Gamma^{IJ}\phi)$ and $(\phi\Gamma^{IJKL}\phi)$. The second term is vanishing because Γ^{IJ} is antisymmetric. The last term has the correct symmetry property, but it vanishes due to the duality property of Γ^{IJKL} (modulo a chirality operator) combined with the fact that ϕ^A and $\lambda^{\dot{A}}$ have opposite SO(8) chirality.

which has the following symmetry

$$\delta\phi^A = -i\bar{\eta}^I_- \sigma^I_{AB} \lambda^B_+ ,$$
$$\delta\lambda^A_+ = \sigma^I_{AB}\left(\gamma^i \partial_i \phi^B \eta_- + \tfrac{1}{2}\phi^B \eta_-\right) - 2ic\left(\phi^A\phi^B - \tfrac{1}{2}\phi^2\delta^{AB}\right)\phi^C \sigma_{BC}{}^I \gamma_5 \eta^I_- , \quad (2.19)$$

where c is an aribtrary constant, $\phi^2 \equiv \phi^A\phi^B\delta_{AB}$, $A = 1,2$ and $\sigma_{AB}{}^I$ $(I = 1,2)$ are two of the Pauli matrices.

For $N > 2$, the analog of eq. (2.17) is not satisfied. Therefore it is difficult to write down SO(N) invariant, polynomial interactions for $N > 2$. Recently, however, it was shown [15] that for the $N = 4$ case, nonpolynomial interactions which are not SO(4) invariant are possible.

The conformal transformations (2.3) and (2.4), the Killing vector equations (2.5) and (2.6), the Killing spinor equations (2.8) and (2.9) have the natural generalization to $S^p \times S^1$ for any p. The general formulae are given in appendix A.

3. (SU(2,2|N)-invariant doubleton field theories on $S^3 \times S^1$

The super anti-de Sitter group in five dimensions is SU(2,2|4). It was pointed out in ref. [13] that this group has a doubleton representation which could be described by an $N = 4$ supersymmetric Yang–Mills theory in four dimensions. Here we will show that, it is indeed possible to construct an SU(2,2|4) invariant Yang–Mills theory on $S^3 \times S^1$. The theories which we believe to have a connection with 3-branes are those which are the $N = 2$ truncations of this theory. We will describe these truncations in this section. The fields of the $N = 4$ supersymmetric Yang–Mills multiplet in four dimensions and their reality properties are

$$\phi^{ab} : \phi^{ab} = -\phi^{ba} , \qquad \phi^{ab} = \tfrac{1}{2}\varepsilon^{abcd}\phi_{cd} , \qquad \phi_{cd} \equiv (\phi^{cd})^* ,$$
$$\lambda^a_+ : \text{complex spinor}, \quad \gamma_5 \lambda^a_+ = \lambda^a_+ , \qquad \gamma_5 = i\gamma_{0123} ,$$
$$A_i : \text{Maxwell field}, \qquad i = 0,1,2,3 ; \qquad a, b = 1,\ldots,4 . \quad (3.1)$$

It is an interesting exercise to show that the SU(2,2|4) invariant Yang–Mills action is

$$\mathscr{L} = \sqrt{-h}\left(-\tfrac{1}{4}\,\partial_i\phi_{ab}\,\partial^i\phi^{ab} - \tfrac{1}{4}\phi_{ab}\phi^{ab} + i\bar{\lambda}_{+a}\gamma^i \nabla_i \lambda^a_+ - \tfrac{1}{4}F_{ij}F^{ij}\right). \quad (3.2)$$

Apart from the SO(4,2) ≈ SU(2,2) and the SU(4) symmetry, this action also possesses the following $N = 4$ supersymmetry

$$\delta\phi^{ab} = \sqrt{2}\, i\left(\bar{\eta}^{[a}_- \lambda^{b]}_+ - \tfrac{1}{2}\varepsilon^{abcd}\bar{\lambda}_{+c}\eta_{-d}\right),$$
$$\delta\lambda^a_+ = \sqrt{\tfrac{1}{2}}\,\gamma^i \partial_i \phi^{ab} \eta_{-b} + \sqrt{\tfrac{1}{2}}\,\phi^{ab}\eta_{+b} + \tfrac{1}{4} i \gamma^{ij} F_{ij} \eta^a_+ ,$$
$$\delta A_i = \tfrac{1}{2}\left(\bar{\eta}_{+a}\gamma_i \lambda^a_+ + \bar{\lambda}_{+a}\gamma_i \eta^a_+\right). \quad (3.3)$$

The proof of the SU(2, 2) invariance requires the (conformal) Killing vector equations, while the proof of supersymmetry requires the use of the Killing spinor equations. These equations are those given in appendix A, for $p = 3$. It is important to recall, however, that the Killing spinors obey the following reality properties

$$\bar{\eta}_{+a} = \eta^{\mathrm{T}}_{-a} C , \qquad \bar{\eta}_{-a} = \eta^{\mathrm{T}}_{+a} C . \tag{3.4}$$

For completeness, we record the nontrivial part of the commutator algebra which can be computed from eq. (3.3)

$$\begin{aligned}
\left[\delta_{\mathrm{Q}}(\eta^{(1)}), \delta_{\mathrm{Q}}(\eta^{(2)})\right] &= \delta_{\mathrm{SO}(4,2)}(\xi, \Omega) + \delta_{\mathrm{SU}(4)}(\Lambda) + \delta_{\mathrm{g}}(v) , \\
\xi^i &= -\tfrac{1}{2} i \bar{\eta}^{(2)a}_{-} \gamma^i \eta^{(1)}_{-a} - (1 \leftrightarrow 2) , \\
\Omega &= \tfrac{1}{8} i \bar{\eta}^{(2)a}_{-} \eta^{(1)}_{+a} - \tfrac{1}{8} i \bar{\eta}^{(2)a}_{+} \eta^{(1)}_{-a} - (1 \leftrightarrow 2) , \\
\Lambda^a_b &= -\tfrac{1}{2} i \bar{\eta}^{(2)a}_{-} \eta^{(1)}_{+b} - \tfrac{1}{2} i \bar{\eta}^{(2)a}_{+} \eta^{(1)}_{-b} - \text{trace} - (1 \leftrightarrow 2) , \\
v &= \tfrac{1}{2} i \bar{\eta}^{(2)a}_{-} \gamma^i \eta^{(1)}_{-a} A_i + \sqrt{\tfrac{1}{8}}\, \bar{\eta}^{(2)}_{+a} \eta^{(1)}_{-b} \phi^{ab} - \sqrt{\tfrac{1}{8}}\, \bar{\eta}^{(2)a}_{-} \eta^{(1)b}_{+} \phi_{ab} .
\end{aligned} \tag{3.5}$$

An $N = 2$ invariant consistent truncation of this theory is obtained by imposing the following conditions

$$\begin{aligned}
&\phi^{12} = \phi^{34} = 0 , \qquad A_i = 0 , \qquad \lambda^1_{+} = \lambda^2_{+} = 0 , \\
&\eta^3_{+} = \eta^4_{+} = \eta_{+3} = \eta_{+4} = 0 .
\end{aligned} \tag{3.6}$$

Using the indices $A = 1, 2$ and $\dot{A} = 3, 4$, we obtain from eq. (3.2) the following $N = 2$ supersymmetric lagrangian for the hypermultiplet on $\mathrm{S}^3 \times \mathrm{S}^1$

$$\mathscr{L} = \sqrt{-h} \left(-\tfrac{1}{2} \partial_i \phi^{A\dot{A}} \partial^i \phi_{A\dot{A}} - \tfrac{1}{2} \phi^{A\dot{A}} \phi_{A\dot{A}} + i \bar{\lambda}_{+\dot{A}} \gamma^i \nabla_i \lambda^{\dot{A}}_{+} \right) . \tag{3.7}$$

The scalar fields obey the reality condition

$$\phi^{A\dot{A}} = \Omega^{AB} \Omega^{\dot{A}\dot{B}} \phi_{B\dot{B}} , \qquad \Omega_{AB} = \Omega^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \text{idem } \Omega^{\dot{A}\dot{B}} . \tag{3.8}$$

The supersymmetry transformation rules are easily obtained from eq. (3.3) by using the truncation rules (3.6). The result is

$$\begin{aligned}
\delta \phi^{A\dot{A}} &= \sqrt{\tfrac{1}{2}}\, \bar{\eta}^A_{-} \lambda^{\dot{A}}_{+} + \sqrt{\tfrac{1}{2}}\, \Omega^{AB} \Omega^{\dot{A}\dot{B}} \bar{\lambda}_{+\dot{B}} \eta_{-B} , \\
\delta \lambda^{\dot{A}}_{+} &= -\sqrt{\tfrac{1}{2}}\, \gamma^i \partial_i \phi^{A\dot{A}} \eta_{-A} - \sqrt{\tfrac{1}{2}}\, \phi^{A\dot{A}} \eta_{+A} .
\end{aligned} \tag{3.9}$$

The interesting feature of the truncation in the algebra of commutators given in eq. (3.5) is that the automorphism group SU(4) breaks down to SU(2) × U(1). Finally, we give the consistent $N=1$ truncation. To this end, we impose the conditions

$$\phi^{1\dot{4}} = \phi^{2\dot{3}} = 0\,, \qquad \lambda_{+}^{\dot{4}} = 0\,, \qquad \eta_{-2} = \eta_{+2} = 0\,. \tag{3.10}$$

Let us denote the nonvanishing fields and parameters as follows

$$\phi^{1\dot{3}} \equiv \phi\,, \qquad \lambda_{+}^{\dot{3}} \equiv \lambda_{+}\,, \qquad \eta_{-1} \equiv \alpha_{-}\,, \qquad \eta_{+1} \equiv \beta_{+}\,. \tag{3.11}$$

Employing eqs. (3.10) and (3.11) in the lagrangian (3.7), we obtain the following $N=1$ supersymmetric action for the Wess–Zumino multiplet on $S^3 \times S^1$

$$\mathscr{L} = \sqrt{-h}\left(-\partial_i\phi\,\partial^i\phi^* - \phi\phi^* + i\bar{\lambda}_{+}\gamma^i\nabla_i\lambda_{+}\right). \tag{3.12}$$

The supersymmetry transformation rules of this lagrangian is obtained from eq. (3.9) by using eqs. (3.10) and (3.11). The result is

$$\delta\phi = \sqrt{\tfrac{1}{2}}\,\bar{\alpha}_{-}\lambda_{+}\,, \qquad \delta\lambda_{+} = -\sqrt{\tfrac{1}{2}}\,\gamma^i\partial_i\phi\alpha_{-} - \sqrt{\tfrac{1}{2}}\,\phi\beta_{+}\,. \tag{3.13}$$

The automorphism group is now only a U(1).

The supersymmetric doubleton field theories described above do not allow interactions. This is simply due to the scaling properties of the fields *and* the chirality of the spinor.

4. OSp(6, 2|2N)-invariant doubleton field theory on $S^5 \times S^1$

The anti-de Sitter group in seven dimensions, SO(6, 2), has a doubleton representation [8]. The super extension of this anti-de Sitter group is OSp(6, 2|2N). It was pointed out in ref. [8] that if the $N=4$ supersymmetric dobleton field theory existed in six dimensions it would contain five real scalars, four Majorana-Weyl spinors and a two-index antisymmetric tensor field with a self-dual field strength. We will show that such a field theory indeed exists, and it lives on the boundary of AdS_7 which is $S^5 \times S^1$.

The fields of the $N=4$, $d=7$ doubleton supermultiplet which lives on $S^5 \times S^1$ are

$$\begin{aligned}
&\phi^a\ (a=1,2,\dots,5)\,,\\
&\lambda_{+}^{A}\ (A=1,\dots,4)\,, \qquad \gamma^7\lambda_{+}^{A} = A_{+}^{A}\,,\\
&B_{ij}\ (i,j=0,1,\dots,5) \qquad (\text{self-dual})\,.
\end{aligned} \tag{4.1}$$

For the field equations of these fields we have found the following result

$$\begin{aligned}
&\left(\nabla^i\partial_i - 4\right)\phi^a = 0, \qquad\qquad \gamma^i\nabla_i\lambda_{+}^{A} = 0\,,\\
&H_{ijk} = \frac{1}{3!}\sqrt{-h}\,\varepsilon_{ijklmn}H^{lmn}\,, \qquad H_{ijk} \equiv 3\partial_{[i}B_{jk]}\,.
\end{aligned} \tag{4.2}$$

These equations of motion transform into each other under the action of the following supersymmetry transformations

$$\delta\phi^a = i\bar{\eta}_-\Gamma^a\lambda_+ ,$$
$$\delta\lambda_+ = \gamma^i\partial_i\phi^a\Gamma^a\eta_- + 2\phi^a\Gamma^a\eta_+ + \tfrac{1}{12}i\gamma^{ijk}H_{ijk}\eta_- ,$$
$$\delta B_{ij} = \bar{\eta}_-\gamma_{ij}\lambda_+ , \tag{4.3}$$

where $\eta_\pm$ are the Killing spinors on $S^5 \times S^1$ which satisfy the equations provided in appendix A, for the case $p = 5$. In addition, the theory has an SO(6,2) invariance whose transformations can also be found in appendix A. In particular, the field B_{ij} is inert under the conformal transformations. The theory is also manifestly SO(5) invariant.

The commutator of two supersymmetry transformations (4.3) yields SO(6,2) $\times$ SO(5) and antisymmetric tensor gauge transformations as follows

$$\left[\delta_Q(\eta^{(1)}), \delta_Q(\eta^{(2)})\right] = \delta_{SO(6,2)}(\xi, \Omega) + \delta_{SO(5)}(\Lambda) + \delta_g(v),$$
$$\xi^i = 2i\bar{\eta}_-^{(2)}\gamma^i\eta_-^{(1)}, \qquad \Omega = \tfrac{1}{2}i\bar{\eta}_-^{(2)}\eta_+^{(1)} - 1 \leftrightarrow 2,$$
$$\Lambda^{ab} = 2i\bar{\eta}_-^{(2)}\Gamma^{ab}\eta_+^{(1)} - 1 \leftrightarrow 2,$$
$$v^i = 4\eta_-^{(2)}\gamma^i\Gamma^a\eta_-^{(1)}\phi^a + 4i\bar{\eta}_-^{(2)}\gamma^j\eta_-^{(1)}B_{ji} . \tag{4.4}$$

The above result admits an $N = 2$ supersymmetric consistent truncation given by

$$\phi^5 = 0, \qquad \Gamma^5\lambda_+ = \lambda_+ , \qquad B_{ij} = 0, \qquad \Gamma^5\eta_\pm = -\eta_\pm . \tag{4.5}$$

Now that the B-field is absent, we can write down a covariant lagrangian. We find that the $N = 2$ supersymmetric doubleton lagrangian is

$$\mathscr{L} = -\tfrac{1}{2}\sqrt{-h}\left(\partial_i\phi^a\partial^i\phi^a + 4\phi^a\phi^a - i\bar{\lambda}_+\gamma^i\nabla_i\lambda_+\right), \tag{4.6}$$

which is invariant under

$$\delta\phi^a = i\bar{\eta}_-\Gamma^a\lambda_+ , \qquad \delta\lambda_+ = \gamma^i\partial_i\phi^a\Gamma^a\eta_- + 2\phi^a\Gamma^a\eta_+ . \tag{4.7}$$

Since only the Killing spinors with a negative chirality with respect to the SO(4) symmetry are maintained, the automorphism group is now SU(2), i.e. the commutator of two $N = 2$ supersymmetry transformations now yields only an SU(2) as an internal symmetry. The theory has additional SU(2) invariance whose transformations commute with the superalgebra.

Finally, we note that both the $N = 2$ and $N = 4$ doubleton theories do not admit interactions, i.e. the SO(6,2) invariant ϕ^3 interaction does not admit a supersymmetric extension. The reason is as for the case of $S^3 \times S^1$, namely the scaling properties of the fields and the chirality of the spinor.

5. Comments

All theories described above, with the exception of the OSp(1|4) and OSp(2|4) singleton theories, are free. In the OSp(1|4) theory, it has been argued that the singleton states by themselves are very difficult to observe, and shown that the two singleton states give rise to infinitely many massless states of all possible integer and half integer spin [4, 14]. Thus, the interacting singleton theories could be singled out as the only relevant ones among all the other singleton or doubleton theories, *if* one insists on interpreting them as sort of preon theories. However, there is another possible interpretation of these theories which make the *free* ones useful. This possibility can be stated as the conjecture that the supersymmetric field theories on $S^p \times S^1$ for $p = 1, \ldots, 5$ describe the super p-branes propagating in $\mathrm{AdS}_{p+2} \times S^{N-1}$. If this conjecture is true, these theories should be capable of describing interacting field theories in AdS_{p+2} without introducing interactions in the world volume of the super p-brane, in much the same way that string theory as a 2-dimensional conformally invariant field theory can describe an interacting quantum field theory in a higher dimensional spacetime.

Our conjecture is motivated by the following observations. First, the super extensions of the anti-de Sitter algebras $\mathrm{SO}(p+2, 2)$ exists only for $p = 1, \ldots, 5$, which are precisely the values for which a super p-brane exists. Second, the internal symmetry groups of the super AdS algebras are in correspondence with the isometry groups of the parallelizable spheres, i.e. S^1, S^3 and S^7, in such a way that the sum of the dimension of the anti-de Sitter spacetime and the allowed parallelizable spheres is always precisely the critical dimension in which a super p-brane can exist.

It would be interesting to investigate the possible anomalies in the rigid super anti-de Sitter symmetries of these theories (which are the analogs of the super-Poincaré symmetries of the usual superstring theories), in an attempt to narrow down the list of candidate theories for a consistent unified description of elementary particles at the *Planck scale*. At present, it is known that the $p = 2$, $N = 8$ case is anomaly free [7].

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Note added in proof

After this paper was completed we received a paper by Blencowe and Duff [16] where results similar to these of sect. 2 of this paper are obtained, and a possible connection between super anti-de Sitter algebras and super p-branes is conjectured.

Appendix A

In this appendix, we shall give the general formulae for the (conformal) Killing vectors, spinors, lagrangians and transformation rules for field theories formulated on $S^p \times S^1$, for any p.

It is well known that S^p admits $(p+1)$ conformal Killing vectors. On $S^p \times S^1$, this number is doubled, and all together there are $2(p+1)$ conformal Killing vectors generating motions in AdS_{p+2}. They obey (the radius of S^p is set equal to 1 for simplicity)

$$\nabla_i \xi_j + \nabla_j \xi_i = 4\Omega h_{ij}, \qquad \nabla^i \partial_i \Omega = (1-p)\Omega. \qquad \text{(A.1, A.2)}$$

It is not difficult to show that eq. (A.2) follows from eq. (A.1).

In order to write down an $SO(p+1,2)$-invariant kinetic term for a scalar field living on $S^p \times S^1$, the scalar field must be assigned a conformal weight $(p-1)$, i.e.

$$\delta\phi = \xi^i \partial_i \phi + (p-1)\Omega\phi. \qquad \text{(A.3)}$$

The $SO(p+1,2)$-invariant (up to a total derivative) lagrangian for a free scalar field then reads

$$\mathscr{L}_0 = -\tfrac{1}{2}\sqrt{-h}\left[h^{ij}\,\partial_i\phi\,\partial_j\phi + \tfrac{1}{4}(p-1)^2\phi^2\right]. \qquad \text{(A.4)}$$

Given eq. (A.3), the only possible $SO(p+1,2)$-invariant interactions are easily found to be

$$\mathscr{L}_1 = \sqrt{-h}\,\phi^{2(p+1)/(p-1)}. \qquad \text{(A.5)}$$

The analogs of eqs. (A.3), (A.4) and (A.5) for a spinor living on $S^p \times S^1$ are

$$\delta\lambda = \xi^i \nabla_i \lambda + \tfrac{1}{4}\nabla_i \xi_j \gamma^{ij}\lambda + p\Omega\lambda, \qquad \text{(A.6)}$$

$$\mathscr{L}_0 = \sqrt{-h}\, i\bar{\lambda}\gamma^i \nabla_i \lambda, \qquad \mathscr{L}_1 = \sqrt{-h}\,\bar{\lambda}\lambda\phi^{2/(p-1)}. \qquad \text{(A.7, A.8)}$$

We now turn to a general description of the Killing spinors on $S^p \times S^1$. There are $2^{[(p+2)/2]}$ of them. They can be obtained from the Killing spinors of AdS_{p+2} given in ref. [2] by restricting them to the boundary. From the usual Killing spinor equation on AdS_{p+2} one finds that on $S^p \times S^1$ they obey the following equations

$$\left(\partial_0 - \tfrac{1}{2}\gamma_0\right)\eta = 0, \qquad \left(\nabla_\alpha - \tfrac{1}{2}\gamma_\alpha\gamma\right)\eta = 0 \qquad (\alpha = 1,2,\ldots,p), \qquad \text{(A.9, A.10)}$$

where γ is defined as follows

$$p = \text{odd}: \quad \gamma \equiv i^{(p-1)/2}\gamma_0\gamma_1\ldots\gamma_p \equiv \gamma_{p+2}, \qquad \gamma^2 = 1, \qquad \text{(A.11)}$$

$$p = \text{even}: \quad \gamma \equiv \gamma_{p+1}, \qquad \gamma^2 = 1. \qquad \text{(A.12)}$$

Although for all p the matrix γ is the Dirac matrix in the spatial radial direction in the anti-de Sitter space, for odd p, γ is the usual criticality operator, while for even p, it is an unusual one.

A weaker version of eqs. (A.11) and (A.12) can be cast into an $S^p \times S^1$ covariant form which reads

$$\nabla_i \eta_- - \tfrac{1}{2}\gamma_i \eta_+ = 0, \qquad \gamma^i \nabla_i \eta_+ + \tfrac{1}{2}(p-1)\eta_- = 0. \qquad \text{(A.13, A.14)}$$

Specific internal symmetry representations and the reality properties of Killing spinors should be considered case by case for each p, and can be found for $p = 2, 3, 5$ in the text, and $p = 1, 4$ in appendix B.

Appendix B

B.1. OSp(N|2) INVARIANT STRING THEORY ON $S^1 \times S^1$

For simplicity, let us focus our attention on the case $N = 8$. The other cases work in precisely analogous manner. Using the formulae of appendix A, for $p = 1$, and the results of sect. 2 for the $N = 8$ supersymmetric singleton, we readily find the following lagrangian on $S^1 \times S^1$ [12]

$$\mathscr{L} = -\tfrac{1}{2}\sqrt{-h}\left(h^{ij}\,\partial_i\phi^A\,\partial_j\phi^A - i\bar{\lambda}_+^{\dot{A}}\gamma^i\nabla_i\lambda_+^{\dot{A}}\right), \qquad \text{(B.1)}$$

which is invariant (up to a total derivative) under

$$\delta\phi^A = -i\Gamma^I_{A\dot{A}}\bar{\eta}_-^I\lambda_+^{\dot{A}}, \qquad \delta\lambda_+^{\dot{A}} = \Gamma^I_{A\dot{A}}\gamma^i\partial_i\phi^A\eta_-^I. \qquad \text{(B.2)}$$

The indices A, $\dot{A}$, I label the 8_s, 8_c and 8_v of SO(8), respectively. The Killing spinor η_- is constant. The conformal weights of ϕ^A and $\lambda^{\dot{A}}$ are 0 and 1, respectively. Note that the lagrangian (B.1) is just the usual heterotic string lagrangian in ten dimensions without the 32 heterotic fermions. The latter can be added, if desired, without spoiling the OSp(8|2) symmetry.

B.2. F(4) INVARIANT DOUBLETON THEORY ON $S^4 \times S^1$

The doubleton multiplet living on $S^4 \times S^1$ consists of four scalar fields as in eq. (3.8), and two 8-component spinors satisfying the following "chirality" and symplectic Majorana conditions

$$\gamma_5\lambda_+^{\dot{A}} = \lambda_+^{\dot{A}}, \qquad \bar{\lambda}_+^{\dot{A}} = \lambda_+^{\dot{A}\mathrm{T}}C, \qquad \lambda_{+\dot{A}} \equiv \lambda_+^{\dot{B}}\Omega_{\dot{B}\dot{A}}, \qquad C^{\mathrm{T}} = -C. \qquad \text{(B.3)}$$

Note that the γ matrices are 8×8. With the aid of the formulae of appendix A, for $p = 4$, and the results of sect. 3, which are very similar to the present case, we find the following doubleton lagrangian on $S^4 \times S^1$

$$\mathscr{L} = -\tfrac{1}{2}\sqrt{-h}\left(h^{ij}\,\partial_i\phi^{A\dot{A}}\,\partial_j\phi_{A\dot{A}} + \tfrac{9}{4}\phi^{A\dot{A}}\phi_{A\dot{A}} - i\bar{\lambda}_{+\dot{A}}\gamma^i\nabla_i\lambda_{+\dot{A}}\right), \qquad \text{(B.4)}$$

which is up to a total derivative invariant under

$$\delta\phi^{A\dot{A}} = \sqrt{2}\,\bar{\eta}_{-}^{A}\lambda_{+}^{\dot{A}}\,,$$

$$\delta\lambda_{+}^{\dot{A}} = \sqrt{\tfrac{1}{2}}\,\gamma^{i}\partial_{i}\phi^{A\dot{A}}\,\eta_{-A} - \sqrt{\tfrac{1}{2}}\,\phi^{A\dot{A}}\eta_{+A}\,. \tag{B.5}$$

The Killing spinors satisfy eqs. (A.9) and (A.10) with the chirality operator $\gamma = \gamma_5$. In addition, we must impose the symplectic Majorana conditions

$$\bar{\eta}_{\pm A} = \eta^{\mathrm{T}}_{\pm A}C\,, \qquad \eta^{A}_{\pm} = \Omega^{AB}\eta_{\pm B}\,. \tag{B.6}$$

In the commutator of two supersymmetric transformations, one now finds an SU(2) as an internal symmetry. This is in accordance with the fact that the exceptional F(4) superalgebra has SO(5,2) ⊗ SU(2) as a bosonic subalgebra. The action also possesses an SU(2) symmetry which commutes with this superalgebra. Finally, we note that the action does not admit interactions which preserve the supersymmetry because of the scaling properties of the fields, the "chirality" of the spinor, *and* the symmetry of γ_7.

Appendix C

(CONFORMAL) KILLING VECTORS ON $S^2 \times S^1$

$$\xi_{05} = \frac{\partial}{\partial t}\,, \qquad \xi_{12} = \frac{\partial}{\partial\varphi}\,, \qquad \xi_{23} + i\xi_{31} = \mathrm{e}^{i\varphi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\varphi}\right),$$

$$\xi_{15} + i\xi_{10} = \mathrm{e}^{-it}\left(-i\sin\theta\cos\varphi\frac{\partial}{\partial t} + \cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right),$$

$$\xi_{25} + i\xi_{20} = \mathrm{e}^{-it}\left(-i\sin\theta\sin\varphi\frac{\partial}{\partial t} + \cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right),$$

$$\xi_{35} + i\xi_{30} = \mathrm{e}^{-it}\left(-i\cos\theta\frac{\partial}{\partial t} - \sin\theta\frac{\partial}{\partial\theta}\right),$$

$$\Omega_{05} = 0\,, \qquad \Omega_{12} = 0\,, \qquad \Omega_{23} + i\Omega_{31} = 0\,,$$

$$\Omega_{15} + i\Omega_{10} = -\tfrac{1}{2}\mathrm{e}^{-it}\sin\theta\cos\varphi\,,$$

$$\Omega_{25} + i\Omega_{20} = -\tfrac{1}{2}\mathrm{e}^{-it}\sin\theta\sin\varphi\,,$$

$$\Omega_{35} + i\Omega_{30} = -\tfrac{1}{2}\mathrm{e}^{-it}\cos\theta\,.$$

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