

INTEGRABLE CLASSICAL AND QUANTUM GRAVITY*

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1 INTRODUCTION

In these lectures we report recent work on the exact quantization of dimensionally reduced gravity [1, 2, 3, 4]. Assuming the presence of commuting Killing symmetries which effectively eliminate the dependence on all but two space-time coordinates allows us to cast the models into the form of $2d$ non-linear (G/H) -coset space σ -models coupled to gravity and a dilaton. This construction includes a variety of models described by different coset spaces — ranging from pure $4d$ Einstein gravity with two commuting Killing vector fields to dimensionally reduced maximal supergravity with the coset space $E_{8(+8)}/SO(16)$ [5]. Although these models superficially resemble the $2d$ dilaton-gravity models considered more recently in the context of string theory and Liouville theory (see [6] and references therein), the latter generically admit more general couplings of the dilaton and the Liouville field while restricting the matter sector essentially to a set of free fields. By contrast, the matter sector of the models considered here is governed by highly non-linear interactions; its underlying group theoretical structure is a crucial ingredient in our analysis.

In terms of the unified formulation presented in these lectures the models allow consistent quantization by use of methods developed over many years in the context of flat space integrable systems [7, 8]. More specifically, we will show that the Wheeler-DeWitt (WDW) equation can be reduced to a modified version of the Knizhnik-Zamolodchikov (KZ) equations from conformal theory [9], the insertions being given by singularities in the spectral parameter plane. This basic result in principle permits the explicit construction of solutions, i.e. physical states of the quantized theory. In this way, we arrive at integrable models of quantum gravity with infinitely many self-interacting propagating degrees of freedom. We here would like to emphasize not just the technical aspects

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of the construction, but also the fact that our results may serve to investigate various conceptual issues of quantum gravity (see [10, 11, 12] for introductory reviews with many further references).

The lectures are divided into four parts. We first describe how the dimensional reduction of $3d$ gravity-coupled non-linear σ -models gives rise to the particular $2d$ models studied in these lectures, of which pure Einstein gravity with two commuting Killing vectors is the simplest example. Chapter 3 deals with the classical integrability of the model, that is manifest in the existence of a linear system. The spectral parameter current is introduced as new fundamental quantity and the dynamics studied in these terms. Restriction to an isomonodromic ansatz results in a set of decoupled differential equations completely describing the dynamics; the relaxation of these restrictions is briefly sketched in the end. In Chapter 4 we set up a Hamiltonian framework for the model, such that translations along the light-cone are governed by two independent Hamiltonians and the phase space is essentially attached to one point in space-time. We compare the new scheme to the conventional Hamiltonian approach and discuss some open questions. Exploiting the new Hamiltonian structure, we develop the quantization of the model in Chapter 5. The Wheeler-DeWitt equations that identify physical states in the quantum theory are explicitly stated. Physical states solving these equations are built from solutions of a modified version of the Knizhnik-Zamolodchikov system. Finally, the appendix sketches the extension of the formalism to supersymmetric models.

2 NONLINEAR σ -MODELS COUPLED TO GRAVITY

After dimensional reduction, the gravity and supergravity models mentioned in the introduction lead to $3d$ gravity-coupled coset σ -model with various coset spaces G/H (see [13] for a systematic discussion). We start by describing these models and further reducing them to two dimensions.

2.1 Nonlinear σ -Models in Three Dimensions

The models we are going to consider in these lectures are most conveniently obtained by dimensional reduction of the following non-linear σ -model in three dimensions coupled to gravity

$$\mathcal{L} = -\frac{1}{2}eR(e) + \frac{1}{2}eh^{mn} \text{tr}P_m P_n. \quad (2.1)$$

The first term is the usual Einstein-Hilbert action for the $3d$ metric $h_{mn} = e_m^a e_n^b \eta_{ab}$ with $e \equiv \det e_m^a$ and η_{ab} the flat (Minkowski) metric, where indices m, n, \dots and a, b, \dots label curved and flat space vectors in three dimensions. The matter sector of this model, governed by the second term in (2.1), is based on a set of scalar fields, which are combined into a matrix $\mathcal{V}(x)$ taking values in a non-compact Lie group G with the maximal compact subgroup H . This subgroup can be characterized by means of a symmetric space involution $\eta : G \rightarrow G$ as

$$H = \{h \in G \mid \eta(h) = h\}$$

The involution extends naturally to the Lie algebras $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{h} = \text{Lie } H$, respectively, such that $\mathfrak{g} \equiv \mathfrak{h} \oplus \mathfrak{k}$ is a direct sum of the eigenspaces of η , orthogonal with respect to the Cartan-Killing form. The maximality of the coset space is expressed by

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$$

In terms of a basis Z^a of \mathfrak{g} ($a, b, \dots = 1, \dots, \dim \mathfrak{g}$)¹, the commutation relations are given by

$$[Z^a, Z^b] = f^{ab}{}_c Z^c \quad (2.2)$$

Whenever necessary we will distinguish the subgroup and coset generators by writing Z^α and Z^A , respectively, where $\alpha, \beta, \dots = 1, \dots, \dim \mathfrak{h}$ and A, B, \dots run over the remaining indices. As an example take $G = SL(n, \mathbb{R})$: its maximal compact subgroup $H = SO(n)$ is characterized by the involution $\eta(g) \equiv (g^t)^{-1}$ for $g \in G$ or $\eta(X) \equiv -X^t$ for $X \in \mathfrak{g}$, respectively.

The decomposition

$$\mathcal{V}^{-1} \partial_m \mathcal{V} \equiv Q_m + P_m \in \mathfrak{g} \quad (2.3)$$

defines the quantities $Q_m \equiv Q_m^\alpha Z_\alpha \in \mathfrak{h}$ and $P_m \equiv P_m^A Z_A \in \mathfrak{k}$ appearing in the Lagrangian (2.1). Due to their definition they are subject to the compatibility relations (valid in any dimension)

$$\begin{aligned} \partial_m Q_n - \partial_n Q_m + [Q_m, Q_n] &= -[P_m, P_n] \\ D_m P_n - D_n P_m &= 0 \end{aligned} \quad (2.4)$$

with $D_m P_n \equiv \partial_m P_n + [Q_m, P_n]$.

The Lagrangian (2.1) is invariant under the transformations

$$\mathcal{V}(x) \mapsto g^{-1} \mathcal{V}(x) h(x) \quad (2.5)$$

where $g \in G$ is constant and $h(x) \in H$ is a local (gauge) transformation. Consequently, the composite fields P_m and Q_m are inert w.r.t. to the rigid G invariance, but do transform under H according to

$$P_m \mapsto h^{-1} P_m h \quad , \quad Q_m \mapsto h^{-1} Q_m h + h^{-1} \partial_m h \quad (2.6)$$

The transformations (2.5) are analogous to the transformation properties of the vierbein (tetrad) in general relativity, where the rigid transformations represent the freedom of constant linear coordinate transformations whereas the local Lorentz transformations correspond to the invariance of the metric under different decompositions into the vielbein. The latter (local) freedom may be used to parametrize the physical fields and the coset space G/H by a fixed system of representative elements of G . E.g., for $SL(n, \mathbb{R})/SO(n)$ it is often convenient to choose a triangular form of the matrices.

The equations of motion derived from (2.1) read

$$D_m (e h^{mn} P_n) = 0 \quad (2.7)$$

together with Einstein's equations for the metric h_{mn} .

As the simplest example for these models, let us briefly recall how the dimensional reduction of pure Einstein gravity from four dimensions to three dimensions leads to a Lagrangian of the type (2.1). The original vierbein may be brought into triangular form by means of a local Lorentz transformation:

$$E_M^A = \begin{pmatrix} \Delta^{-\frac{1}{2}} e_m^a & \Delta^{\frac{1}{2}} B_m \\ 0 & \Delta^{\frac{1}{2}} \end{pmatrix} \quad (2.8)$$

¹Since flat 3d Minkowski indices will not play any role after this chapter, we hope that this double usage of indices will not cause undue confusion.

With this parametrization and dropping the dependence on the fourth (spatial) coordinate, the equations of motion can be equivalently obtained from the Lagrangian:

$$\mathcal{L}^{(3)} = -\frac{1}{2}eR^{(3)}(e) + \frac{1}{4}eh^{mn}\Delta^{-2}(\partial_m\Delta\partial_n\Delta + \partial_m B\partial_n B), \quad (2.9)$$

where B is dual to the Kaluza-Klein vector B_m :

$$\Delta^2(\partial_m B_n - \partial_n B_m) \equiv \epsilon_{mnp}\partial^p B,$$

The scalar fields Δ and B descending from the vierbein (2.8) now build the matter part of the $3d$ model (2.9), and are coupled to $3d$ gravity (which carries no propagating degrees of freedom any more). They correspond to the two helicity states of the graviton, with $SO(2)$ as the helicity subgroup of $SO(1, 3)$. The Lagrangian (2.9) is a nonlinear σ -model of the type (2.1) with the triangular matrices

$$\mathcal{V} = \begin{pmatrix} \Delta^{\frac{1}{2}} & \Delta^{-\frac{1}{2}}B \\ 0 & \Delta^{-\frac{1}{2}} \end{pmatrix} \in SL(2, \mathbb{R}) \quad (2.10)$$

which build a representative system of the coset space $SL(2, \mathbb{R})/SO(2)$.

2.2 Reduction to Two Dimensions

The reduction of the $3d$ model to two dimensions is achieved dropping the dependence on another coordinate. Depending on the norm of the corresponding Killing vector, the $2d$ model will live on an Euclidean or Lorentzian worldsheet, respectively. While the former reduction of Einstein's theory corresponds to stationary axisymmetric solutions, the latter can describe physically inequivalent solutions, namely (in the free field truncation) Einstein-Rosen gravitational waves [14], or colliding plane waves [15]. In these lectures we will concentrate on the second case, i.e. Lorentzian signature worldsheets. However, in a very formal sense, the two cases are related by a Wick rotation from real to imaginary time.

Consider again a triangular form of the dreibein e_m^a which we parametrize as

$$e_m^a = \begin{pmatrix} e_\mu^\alpha & \rho A_\mu \\ 0 & \rho \end{pmatrix} \quad (2.11)$$

where e_μ^α is the zweibein (dyad), and Greek indices label the remaining two space-time dimensions. By the field equations, the Kaluza-Klein vector field A_μ carries no dynamical degrees of freedom; assuming absence of a cosmological constant we can thus ignore it. The reduced Lagrangian is given by

$$\mathcal{L}^{(2)} = -\frac{1}{2}\rho e R^{(2)}(e) + \frac{1}{2}\rho e h^{\mu\nu} \text{tr} P_\mu P_\nu \quad (2.12)$$

where e is now the zweibein determinant. The appearance of the dilaton field ρ is a typical feature of Kaluza-Klein type dimensional reduction. Namely, this field ‘‘measures’’ the size of the compactified dimensions of the higher-dimensional space-time: for the direct reduction of the vielbein E_M^A from d to two dimensions (as opposed to the detour via three dimensions we are taking here)

$$E_M^A = \begin{pmatrix} e_\mu^\alpha & * \\ 0 & E_{M'}^{A'} \end{pmatrix}$$

where M' and A' label the “internal” dimensions, we have $\rho = \det E_{M'}^{A'}$. This justifies the name “dilaton” for ρ .

By coordinate reparametrizations the $2d$ metric can be brought into the conformal gauge at least locally:

$$e_\mu^\alpha = \lambda \delta_\mu^\alpha \equiv \exp(\sigma) \delta_\mu^\alpha \quad (2.13)$$

In terms of the light-cone (isothermal) coordinates²

$$x^\pm := x^0 \pm x^1, \quad \partial_\pm := \frac{1}{2}(\partial_0 \pm \partial_1)$$

the metric takes the form

$$ds^2 = \exp(2\sigma) dx^+ dx^- \quad (2.14)$$

It still admits conformal reparametrizations $x^\pm \mapsto \tilde{x}^\pm(x^\pm)$, which preserve the diagonal form of the metric. Note that the Liouville degree of freedom σ does not transform as a genuine scalar. Rather, it is the following expression

$$\hat{\sigma} \equiv \ln \lambda - \frac{1}{2} \ln(\partial_+ \rho \partial_- \rho), \quad (2.15)$$

which behaves as a scalar under conformal reparametrization and which appears naturally in the equations of motion [16].

Let us now state the equations of motion for all fields:

- The dilaton field ρ obeys a free field equation independently of the coset G/H :

$$\square \rho = 0, \quad (2.16)$$

whose general solution is given by

$$\rho(x) \equiv \frac{1}{2}(\rho^+(x^+) + \rho^-(x^-))$$

The dual field (“axion”) is defined by

$$\tilde{\rho}(x) \equiv \frac{1}{2}(\rho^+(x^+) - \rho^-(x^-))$$

- The conformal factor satisfies two (compatible) first order equations:

$$\partial_\pm \rho \partial_\pm \hat{\sigma} = \frac{1}{2} \rho \operatorname{tr} P_\pm P_\pm, \quad (2.17)$$

with $\hat{\sigma}$ defined above. The equations (2.17) determine the conformal factor up to a constant, since they are of first degree. Rather than equations of motion of the usual type, they should be regarded as constraints; actually they descend from variation of the two unimodular degrees of freedom of the $2d$ metric $h_{\mu\nu}$, that appear as Lagrangian multipliers in (2.12). The second order equation of motion for the conformal factor results from variation of the Lagrangian w.r.t. ρ :

$$\partial_+ \partial_- \hat{\sigma} \equiv \partial_+ \partial_- \hat{\sigma} = -\frac{1}{2} \operatorname{tr}(P_+ P_-) \quad (2.18)$$

²We similarly define $V^\pm := V^0 \pm V^1$ and $V_\pm := \frac{1}{2}(V_0 \pm V_1)$ for any vector V^μ .

- The matter fields \mathcal{V} obey

$$D^\mu(\rho P_\mu) = 2\{D_+(\rho P_-) + D_-(\rho P_+)\} = 0 \quad (2.19)$$

where the covariant derivative $D_\mu = \partial_\mu + \text{ad}_{Q_\mu}$ was introduced in the previous section. Similar equations appear in the flat space σ -models (i.e. (2.12) without coupling to gravity) except for the appearance of ρ here. The equations for the conformal factor (2.17) further show that ρ may not be chosen constant without trivializing the matter part of the solution. This difference in (2.19) accounts for the essentially new features of these models in comparison with the flat space models.

Notice that all the equations of motion are consistent as (2.18) is a consequence of (2.16), (2.17) and (2.19), taking into account the relations (2.4).

There is an equivalent form of (2.19), frequently used in the literature, in terms of the combination

$$g \equiv \mathcal{V}\eta(\mathcal{V}^{-1}) \in G, \quad (2.20)$$

(so e.g. $g = \mathcal{V}\mathcal{V}^t$ for $SL(n, \mathbb{R})$) which we state for later reference:

$$\partial^\mu(\rho g^{-1}\partial_\mu g) = 2\{\partial_+(\rho g^{-1}\partial_-g) + \partial_-(\rho g^{-1}\partial_+g)\} = 0 \quad (2.21)$$

In the models of dimensionally reduced gravity the variables g essentially build the compactified part of the former higher-dimensional metric. Their main technical advantage, which we will eventually exploit, is the fact, that in contrast to (2.19) the equations of motion (2.21) can be formulated without Q_μ (this is no longer true in the presence of fermionic couplings). Indeed,

$$g^{-1}\partial_\mu g = \eta(\mathcal{V})\left(Q_\mu + P_\mu - \eta(Q_\mu + P_\mu)\right)\eta(\mathcal{V}^{-1}) = 2\eta(\mathcal{V})P_\mu\eta(\mathcal{V}^{-1}) \quad (2.22)$$

As an illustration let us once more consider the example $G = SL(2, \mathbb{R})$. This model is commonly formulated in terms of the Ernst-potential \mathcal{E} , defined by

$$\mathcal{E} = \Delta + iB \quad (2.23)$$

in the parametrization of (2.10). The equations of motion for the matter part (2.19) then take the form

$$\Delta\partial_\mu(\rho\partial^\mu\mathcal{E}) = \rho\partial_\mu\mathcal{E}\partial^\mu\mathcal{E}, \quad (2.24)$$

which is the celebrated Ernst equation [17]. A further reduction corresponding to $B=0$ leads to the collinearly polarized so-called Einstein-Rosen gravitational waves. In this case the Ernst equation reduces to the classical linear Euler-Darboux equation. These solutions, discovered in [14] have already served as testing ground for technical and conceptual issues of quantum gravity [18, 19].

3 CLASSICAL INTEGRABILITY

3.1 The Linear System

The main part of the equations of motion stated in the previous section are the equations (2.19) for the matter fields $\mathcal{V}(x)$. Once these equations are solved, the equations (2.17) for the conformal factor can be integrated with (2.19) ensuring their integrability.

The matter field equations may be obtained from a linear system; this means that they are expressed as compatibility equations of a linear system of differential equations. These techniques have been common in the theory of flat space integrable systems [7, 8], where the existence of a family of linear systems parametrized by the spectral parameter in particular gives rise to the construction of an infinite set of conserved charges, reflecting the integrability of the model. For the axisymmetric stationary solutions of Einstein's equations the linear system was constructed by Belinskii and Zakharov [20] and by Maison [21]. The generalization to arbitrary non-linear σ -models is discussed in [22, 16]. Alternatively, the integrability of these models can be derived on the basis of the (anti)self-dual Yang-Mills equations in four dimensions. This approach, which is quite different from the one taken here in that it emphasizes the twistor geometrical aspects has been extensively studied in [23, 24, 25].

The linear system for a function $\hat{\mathcal{V}}(x, \gamma)$, where γ denotes the spectral parameter, is given by:

$$\hat{\mathcal{V}}^{-1} \partial_\mu \hat{\mathcal{V}} = Q_\mu + \frac{1 + \gamma^2}{1 - \gamma^2} P_\mu + \frac{2\gamma}{1 - \gamma^2} \epsilon_{\mu\nu} P^\nu \quad (3.1)$$

or, in light-cone coordinates,

$$\hat{\mathcal{V}}^{-1} \partial_\pm \hat{\mathcal{V}} = Q_\pm + \frac{1 \mp \gamma}{1 \pm \gamma} P_\pm \quad (3.2)$$

This is formally almost the same linear system as for the flat space σ -model. The essential difference lies in the fact, that in order to obtain $D^\mu(\rho P_\mu) = 0$ as compatibility equations rather than $D^\mu P_\mu = 0$, the spectral parameter must depend on the space-time coordinates according to the differential equation

$$\begin{aligned} \gamma^{-1} \partial_\mu \gamma &= \frac{1 + \gamma^2}{1 - \gamma^2} \rho^{-1} \partial_\mu \rho + \frac{2\gamma}{1 - \gamma^2} \epsilon_{\mu\nu} \rho^{-1} \partial^\nu \rho \\ \iff \gamma^{-1} \partial_\pm \gamma &= \frac{1 \mp \gamma}{1 \pm \gamma} \rho^{-1} \partial_\pm \rho \end{aligned} \quad (3.3)$$

and is therefore no longer a constant as in the flat space integrable systems. This equation can be explicitly solved with two solutions γ, γ^* (due to the invariance of (3.3) with respect to the involution $\gamma \mapsto 1/\gamma$):

$$\gamma(x, w) = \frac{\sqrt{w + \rho^+(x^+)} - \sqrt{w - \rho^-(x^-)}}{\sqrt{w + \rho^+(x^+)} + \sqrt{w - \rho^-(x^-)}} = \frac{1}{\gamma^*(x, w)} \quad (3.4)$$

The free integration constant w may be regarded as the hidden ‘‘constant spectral parameter’’ whereas γ will be referred to as the ‘‘variable spectral parameter’’. The reader should check that compatibility of the linear system indeed requires (2.19).

The linear system (3.1) determines the function $\hat{\mathcal{V}}(x, \gamma)$ only up to constant left multiplication by an arbitrary matrix depending on w :

$$\hat{\mathcal{V}}(x, \gamma) \mapsto S(w) \hat{\mathcal{V}}(x, \gamma) \quad (3.5)$$

We will restrain this freedom by further assumptions below.

To reconstruct the physical fields from $\hat{\mathcal{V}}$, we adopt a generalized “triangular gauge” for $\hat{\mathcal{V}}(x, \gamma)$ [26, 22], demanding regularity for $\gamma \rightarrow 0$.³ This allows the expansion

$$\begin{aligned}\hat{\mathcal{V}}(x, \gamma) &= \exp[\varphi^{(0)}(x) + \gamma\varphi^{(1)}(x) + \mathcal{O}(\gamma^2)] \\ &= \mathcal{V}(x) + \mathcal{O}(\gamma)\end{aligned}$$

Substituting this into the linear system (3.1) yields the original (non-linear) equations of motion for $\mathcal{V}(x) = \exp[\varphi^{(0)}(x)]$, i.e. at $\gamma = 0$. The next order in γ leads to

$$\partial_\mu \varphi^{(1)} = \epsilon_{\mu\nu} \partial^\nu \varphi^{(0)} + \text{nonlinear terms},$$

such that the fields $\varphi^{(1)}(x), \varphi^{(2)}(x), \dots$ form an infinite hierarchy of dual potentials, analogous to the one originally introduced in [27] for the investigation of the action of the Geroch group on stationary axisymmetric solutions. However, in order to properly implement the infinite dimensional affine symmetries of the model, another infinity of potentials $\varphi^{(-n)}, n \geq 1$ is required; these symmetries are extensively discussed in [26, 22, 16, 28].

The involution η defining the symmetric space G/H and acting on \mathcal{V} may be extended to an involution η^∞ acting on $\hat{\mathcal{V}}(\gamma)$ by

$$\eta^\infty(\hat{\mathcal{V}}(\gamma)) \equiv \eta\left(\hat{\mathcal{V}}\left(\frac{1}{\gamma}\right)\right)$$

and leaving the linear system (3.1) invariant. E.g. for $G = SL(n, \mathbb{R})$, this means:

$$\eta^\infty(\hat{\mathcal{V}}(\gamma)) = \left(\hat{\mathcal{V}}\left(\frac{1}{\gamma}\right)^t\right)^{-1}$$

The existence of this generalized involution motivates the following definition [22]:

$$\mathcal{M} \equiv \hat{\mathcal{V}}\eta^\infty(\hat{\mathcal{V}}^{-1}) \quad \left(= \hat{\mathcal{V}}(\gamma)\hat{\mathcal{V}}\left(\frac{1}{\gamma}\right)^t \right) \quad (3.6)$$

is called the *monodromy matrix* associated with $\hat{\mathcal{V}}(\gamma)$. Due to the invariance of (3.1) under η^∞ , this matrix depends on the constant spectral parameter only:

$$\partial_\mu \mathcal{M} = 0 \quad \implies \quad \mathcal{M} = \mathcal{M}(w)$$

Obviously, \mathcal{M} is not invariant w.r.t. (3.5), but transforms as

$$\mathcal{M}(w) \mapsto S(w)\mathcal{M}(w)\eta(S^{-1}(w)),$$

s.t. the form of \mathcal{M} may be restricted in various ways. Let us comment on the two preferred choices (“pictures”) eliminating the freedom left by (3.5).

- The freedom (3.5) may be invoked to demand holomorphy of $\hat{\mathcal{V}}(\gamma)$ inside a domain containing the unit disc $|\gamma| \leq 1$. Roughly speaking, the invariance $w(\gamma) = w(\gamma^{-1})$ allows to reflect all singularities at the unit circle by multiplication with a suitable $S(w)$. This picture has been introduced in [22]. As a consequence, the matrix $\mathcal{M}(w(\gamma))$ is non-singular as a function of γ in an annular region containing the unit circle $|\gamma| = 1$ and contains the complete information about $\hat{\mathcal{V}}$. The linear system matrix $\hat{\mathcal{V}}(\gamma)$ may then be recovered by solving a Riemann-Hilbert factorization problem on this annulus. The absence of singularities in the disk in particular permits us to recover the original field via $\mathcal{V}(x) = \hat{\mathcal{V}}(x, \gamma)|_{\gamma=0}$.

³Demanding regularity just at $\gamma = 0$ is, however, not quite sufficient to fix $\hat{\mathcal{V}}$ uniquely, see below.

- Since the matrix $\mathcal{M}(w)$ obeys:

$$\mathcal{M}(w) = \eta^\infty(\mathcal{M}^{-1}(w)) \implies \mathcal{M}(w) = \eta(\mathcal{M}^{-1}(w)) \quad (3.7)$$

we can represent it as

$$\mathcal{M}(w) = S(w) \eta(S^{-1}(w))$$

Hence, by exploiting (3.5) we can require

$$\mathcal{M}(w) \equiv I, \quad (3.8)$$

for *all* solutions. This picture was introduced in [20] and will be used in the sequel. Note however, that it still allows the freedom of left multiplication (3.5) by H -valued matrices $S(w)$ (for which $\eta(S) = S$). The main difference with the previous picture is that, while still regular at $\gamma = 0$, the linear system matrix $\hat{\mathcal{V}}$ is now allowed to have poles inside the unit disk in the γ -plane as well.

For later use, let us state an alternative version of (3.1). Define $\Psi(\gamma) \equiv \hat{\mathcal{V}}(\gamma)\eta(\mathcal{V}^{-1})$, then

$$\begin{aligned} \Psi^{-1}\partial_\pm\Psi &= \eta(\mathcal{V}) \left(Q_\pm + \frac{1 \mp \gamma}{1 \pm \gamma} P_\pm - \eta(Q_\pm + P_\pm) \right) \eta(\mathcal{V}^{-1}) \\ &= \frac{2}{1 \pm \gamma} \eta(\mathcal{V}) P_\pm \eta(\mathcal{V}^{-1}) \end{aligned}$$

This form of the linear system is tailored for the variables g introduced in (2.20), in terms of which it becomes (cf. (2.22))

$$\Psi^{-1}\partial_\pm\Psi = \frac{1}{1 \pm \gamma} g^{-1}\partial_\pm g \quad (3.9)$$

The main advantage of this version of the linear system has already been stressed above: the Q_\pm do not appear explicitly. However, we trade this technical simplification for the drawback of hiding part of the group theoretical structure. This may already be seen from the transformation behavior of \mathcal{V} (2.5), that clearly exposes the combined left and right action of the groups $G^{\text{rigid}} \times H^{\text{local}}$. The variables g on the other hand and consequently the Ψ -function transform under the adjoint action of G and remain invariant under H . For a proper treatment of the coset structure, it will hence be necessary to return to (3.1). This is essential when the action of the solution-generating Geroch group is implemented by an infinite-dimensional extension G^∞/H^∞ of the coset G/H where the ‘‘maximal compact subgroup’’ H^∞ is defined by means of the involution η^∞ [26, 22, 16]. The original form of the linear system is also required for the extension of the linear system to supergravity, since the Q_\pm are indispensable for the coupling to fermions. We will sketch this generalization in the appendix.

Finally, we would like to stress that the conformal gauge (2.14) has been adopted for convenience only. It is, in fact, possible to generalize the linear system to arbitrary (non-singular) $2d$ metrics by means of the Beltrami differentials parametrizing the inequivalent conformal structures of the world-sheets [29]. These constitute extra physical albeit global degrees of freedom of the theory.

3.2 Spectral Parameter Current

So far, from the function $\hat{\mathcal{V}}$ we have considered only the defining expressions $\hat{\mathcal{V}}^{-1}\partial_{\pm}\hat{\mathcal{V}}$ which appear in the linear system (3.1). As it turns out, however, the crucial quantity for our subsequent considerations is the spectral parameter current

$$B(x; \gamma) := (\hat{\mathcal{V}}^{-1}\partial_{\gamma}\hat{\mathcal{V}})(x, \gamma) \equiv \sum_{j=1}^N \frac{B_j(x)}{\gamma - \gamma_j(x)} \in \mathfrak{g}_{\mathbb{C}} \quad (3.10)$$

The form of the right hand side defines the so-called isomonodromic ansatz [1]; the $\gamma_j(x)$ are given by (3.4): $\gamma_j(x) \equiv \gamma(x, w_j)$. For later use we record the residue at infinity

$$B_{\infty} := \sum_{j=1}^N B_j = \lim_{\gamma \rightarrow \infty} \gamma B(\gamma) \quad (3.11)$$

which governs the behavior of $\hat{\mathcal{V}}$ near $\gamma = \infty$. Observe that local analyticity of the spectral parameter current as a function of γ already follows from the linear system. The extra assumption in (3.10) is that $B(\gamma)$ should be single-valued in the whole γ -plane and possess only simple poles. These restrictions may be justified by the fact that almost all known and physically interesting axisymmetric stationary or colliding plane wave solutions of Einstein's equations are of this type.⁴ We will sketch the generalization to arbitrary spectral parameter currents at the end of this chapter. This extension of the formalism is also inspired by the treatment of the isomonodromic solutions; naively the first step may be understood as replacing the sum in (3.10) by an integral. Nevertheless, the hope is, that the isomonodromic solutions as defined above are (in some sense) dense in the "space of all solutions".

In the following, the spectral parameter current will be considered as the fundamental object of the theory. Thus, in particular we should be able to reconstruct the original currents $\mathcal{V}^{-1}\partial_{\pm}\mathcal{V}$ from knowledge of $B(\gamma)$. Indeed, writing out the derivatives ∂_{\pm} and using (3.3) we get

$$\hat{\mathcal{V}}^{-1}\partial_{\pm}\hat{\mathcal{V}} = \hat{\mathcal{V}}^{-1}\partial_{\pm}\hat{\mathcal{V}}\Big|_{\gamma} + \rho^{-1}\partial_{\pm}\rho \frac{\gamma(1 \mp \gamma)}{1 \pm \gamma} \hat{\mathcal{V}}^{-1}\partial_{\gamma}\hat{\mathcal{V}} \quad (3.12)$$

Comparing this with (3.2), taking $\gamma = \mp 1$ and assuming regularity of $\hat{\mathcal{V}}^{-1}\partial_{\pm}\hat{\mathcal{V}}\Big|_{\gamma}$ at $\gamma = \mp 1$ according to (3.10), we infer that

$$P_{\pm}(x) = \mp \rho^{-1}\partial_{\pm}\rho \hat{\mathcal{V}}^{-1}\partial_{\gamma}\hat{\mathcal{V}}\Big|_{\gamma=\mp 1} = \rho^{-1}\partial_{\pm}\rho \sum_{j=1}^N \frac{B_j(x)}{1 \pm \gamma_j(x)} \quad (3.13)$$

In the presence of fermions, regularity of the spectral parameter current at $\gamma = \mp 1$ is no longer valid and the argument must be slightly modified, cf. appendix. Fixing the local H -gauge freedom (2.5), we can determine Q_{\pm} as functions of P_{\pm} . In this fashion, the complete current $\mathcal{V}^{-1}\partial_{\pm}\mathcal{V}$ may be recovered from (2.3). The (nontrivial) coset constraints ensuring $P_{\pm} \in \mathfrak{k}$ will be discussed in section 3.4 below.

⁴Higher order poles in (3.10) would not only imply essential singularities in $\hat{\mathcal{V}}(\gamma)$ but also rather nasty singularities in the actual solutions.

Thus, we can now proceed to formulate the theory entirely in terms of the new quantities $B(\gamma)$. The former matter field equations of motion (2.19) have been completely absorbed into the linear system, i.e. they are implicitly part of the definition of the spectral parameter current $B(\gamma)$. This already suggests that the coordinate dependence of this current should be nothing but a consequence of its definition and its γ -dependence. We shall explicitly work this out in the next section where we will in particular study the x^\pm -dynamics of the residues B_j .

3.3 Deformation Equations

Let us investigate the consequences of the isomonodromic ansatz (3.10). For the derivation we make use of the alternative form of the linear system given in (3.9) for which an analogous spectral current may be defined:

$$A(x; \gamma) \equiv \Psi^{-1} \partial_\gamma \Psi(x, \gamma) \equiv \sum_j \frac{A_j(x)}{\gamma - \gamma_j(x)}, \quad (3.14)$$

The residues A_j are obviously related to the B_j by

$$A_j = \eta(\mathcal{V}) B_j \eta(\mathcal{V}^{-1}) \quad (3.15)$$

In analogy with (3.11) we define

$$A_\infty = \eta(\mathcal{V}) B_\infty \eta(\mathcal{V}^{-1}) = \sum_{j=1}^N A_j \quad (3.16)$$

The current (3.14) contains the original currents in a way similar to (3.13):

$$g^{-1} \partial_\pm g = \mp 2\rho^{-1} \partial_\pm \rho \Psi^{-1} \partial_\gamma \Psi|_{\gamma=\mp 1} = 2\rho^{-1} \partial_\pm \rho \sum_{j=1}^N \frac{A_j(x)}{1 \pm \gamma_j(x)} \quad (3.17)$$

Next we demonstrate in detail how the definition of the spectral parameter current together with the isomonodromic ansatz uniquely determines the x^\pm -dynamics of the residues A_j and B_j . The definition (3.14) requires validity of the compatibility equations:

$$\Psi^{-1} \left[\partial_\pm, \frac{\partial}{\partial \gamma} \right] \Psi = \partial_\pm \left(\Psi^{-1} \frac{\partial \Psi}{\partial \gamma} \right) - \frac{\partial}{\partial \gamma} (\Psi^{-1} \partial_\pm \Psi) + \left[\Psi^{-1} \partial_\pm \Psi, \Psi^{-1} \frac{\partial \Psi}{\partial \gamma} \right] \quad (3.18)$$

Due to the explicit coordinate dependence of the variable spectral parameter γ , the l.h.s. does not vanish, but is

$$\left[\partial_\pm, \frac{\partial}{\partial \gamma} \right] = \partial_\pm \left(\frac{\partial w}{\partial \gamma} \right) \frac{\partial}{\partial w} = \rho^{-1} \partial_\pm \rho \left(1 - \frac{2}{(1 \pm \gamma)^2} \right) \frac{\partial}{\partial \gamma}$$

Together with the linear system (3.9) and (3.17) this permits us to express the compatibility relations (3.18) entirely in terms of the spectral parameter current $A(\gamma)$:

$$\underbrace{\partial_\pm A(\gamma)}_{\text{l.h.s.}} = \rho^{-1} \partial_\pm \rho \left\{ \underbrace{\frac{2A(\mp 1)}{(1 \pm \gamma)^2}}_a + \underbrace{\left(1 - \frac{2}{(1 \pm \gamma)^2} \right) A(\gamma)}_b \pm \underbrace{\frac{2}{1 \pm \gamma} [A(\mp 1), A(\gamma)]}_c \right\} \quad (3.19)$$

We evaluate the l.h.s. and the three terms of the r.h.s. separately:

$$\begin{aligned}
\text{l.h.s.} &= \partial_{\pm} \left(\sum_j \frac{A_j}{\gamma - \gamma_j} \right) \\
&= \sum_j \frac{1}{\gamma - \gamma_j} \partial_{\pm} A_j + \rho^{-1} \partial_{\pm} \rho \sum_j \frac{A_j}{(\gamma - \gamma_j)^2} \left\{ \frac{\gamma_j(1 \mp \gamma_j)}{1 \pm \gamma_j} - \frac{\gamma(1 \mp \gamma)}{1 \pm \gamma} \right\} \\
&= \sum_j \frac{\partial_{\pm} A_j}{\gamma - \gamma_j} + \rho^{-1} \partial_{\pm} \rho \sum_j \frac{A_j}{\gamma - \gamma_j} \left\{ 1 - \frac{2}{(1 \pm \gamma)(1 \pm \gamma_j)} \right\} \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
a &= \mp \frac{2}{(1 \pm \gamma)^2} \left(\sum_j \frac{A_j}{1 \pm \gamma_j} \right) \\
&= \frac{2}{1 \pm \gamma} \rho^{-1} \partial_{\pm} \rho \sum_j \frac{A_j}{\gamma - \gamma_j} \left\{ \frac{1}{1 \pm \gamma} - \frac{1}{1 \pm \gamma_j} \right\} \tag{3.21}
\end{aligned}$$

$$b = \left(1 - \frac{2}{(1 \pm \gamma)^2} \right) \sum_j \frac{A_j}{\gamma - \gamma_j} \tag{3.22}$$

$$\begin{aligned}
c &= -\frac{2}{1 \pm \gamma} \left[\sum_j \frac{A_j}{1 \pm \gamma_j}, \sum_k \frac{A_k}{\gamma - \gamma_k} \right] \\
&= -\frac{2}{1 \pm \gamma} \sum_{j,k} \frac{[A_j, A_k]}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \\
&= \frac{1}{1 \pm \gamma} \sum_{j,k} [A_j, A_k] \frac{(1 \pm \gamma)(\gamma_j - \gamma_k)}{(1 \pm \gamma_j)(1 \pm \gamma_k)(\gamma - \gamma_j)(\gamma - \gamma_k)} \\
&= 2 \sum_{j,k} \frac{[A_j, A_k]}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \frac{1}{\gamma - \gamma_j} \tag{3.23}
\end{aligned}$$

Combining (3.20)–(3.23) now yields the deformation equations for the residues A_j [1]

$$\partial_{\pm} A_j = 2\rho^{-1} \partial_{\pm} \rho \sum_k \frac{[A_j, A_k]}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \tag{3.24}$$

The corresponding deformation equations for the residues B_j are easily derived from this by use of (3.15)

$$\begin{aligned}
D_{\pm} B_j &= 2\rho^{-1} \partial_{\pm} \rho \sum_k \frac{[B_j, B_k]}{(1 \pm \gamma_j)(1 \pm \gamma_k)} - [B_j, P_{\pm}] \tag{3.25} \\
&= \rho^{-1} \partial_{\pm} \rho \sum_k \frac{1 \mp \gamma_j}{(1 \pm \gamma_j)(1 \pm \gamma_k)} [B_j, B_k]
\end{aligned}$$

Again, we note the explicit appearance of the Q_{\pm} , which are part of the covariant derivative $D_{\pm} = \partial_{\pm} + \text{ad}_{Q_{\pm}}$, whereas P_{\pm} has been expressed in terms of $B(\gamma)$ according

to (3.13). We may also translate the deformation equations (3.19) into the spectral parameter current $B(\gamma)$:

$$D_{\pm}B(\gamma) = \rho^{-1}\partial_{\pm}\rho \left\{ \frac{2B(\mp 1)}{(1 \pm \gamma)^2} + \left(1 - \frac{2}{(1 \pm \gamma)^2} \right) B(\gamma) \pm \frac{2}{1 \pm \gamma} [B(\mp 1), B(\gamma)] \right\} - [B(\gamma), P_{\pm}] \quad (3.26)$$

Together with the equations for the γ_j following from (3.3), the dynamics of the spectral parameter currents $A(\gamma)$ and $B(\gamma)$ in the x^{\pm} directions is completely given from (3.25). Remarkably, these deformation equations are automatically compatible; we leave this computation as a little exercise to the reader.

In summary, we have reduced the Einstein equations for the stationary axisymmetric or the colliding plane wave truncation and their generalizations to arbitrary σ -models to a system of compatible ordinary first order matrix differential equations. Any solution of (3.24) induces a solution of the matter field equations of motion (2.19) from which the conformal factor may be determined by integration of (2.17). The simplest class of solutions is obtained if the A_j are restricted to the Cartan-subalgebra of \mathfrak{g} ; then, they are in fact constant owing to (3.24). This special case includes for instance the multi-Schwarzschild solutions and illustrates the remarkable fact that the coordinate dependence of these solutions arises solely through the coordinate dependence of the spectral parameters $\gamma_j(x)$.

The form of (3.24) gives rise to several constants of motion. From (3.24) one immediately verifies the conservation laws [3]

$$\partial_{\pm}A_{\infty} = \partial_{\pm}(\text{tr}A_j^n) = 0 \quad (n = 1, 2, \dots) \quad (3.27)$$

which have obvious analogs in terms of the B_j variables. We call A_{∞} the ‘‘Ehlers charge’’ because it generates (the analog of) Ehlers transformations in the canonical approach [3]. The constancy of the traces tells us that the eigenvalues of the matrices A_j (or B_j) are likewise constant (note that $\text{tr}A_j$ vanishes for semisimple G). Later on we shall also describe a more general construction of conserved charges non-local in the spectral parameter plane.

The above reduction of the original equations shows a remarkable general feature: the number of dimensions has been effectively reduced from two to one. Recall that the initial values of the physical fields are usually given on a spacelike hypersurface, whereas their evolution in the time direction is described by the equations of motion. Here, on the contrary we have evolution equations for the time direction as well as for the space direction and the two flows commute. The knowledge of the initial values of the new fields B_j and γ_j at one space-time *point* is sufficient to reconstruct the whole solution by means of (3.25) and (3.3).

This at first sight puzzling feature may be understood as follows: the space dimension which previously provided the initial data has been traded for an additional dimension parametrized by the spectral parameter. In fact, given the spectral parameter current $B(\gamma)$ at fixed $\gamma = \pm 1$ on a spacelike hypersurface (which according to (3.13) are nothing but the original currents) allows us to evolve it into time direction by means of the equations of motion and into the γ -direction via the compatibility equations (3.26). Vice versa, given $B(\gamma)$ at fixed space-time point but for all γ one can deduce its space evolution from the compatibility equations. The isomonodromic ansatz is finally employed to parametrize the behavior of the spectral parameter current

in the γ -plane by a discrete (even finite) set of variables, such that the original field theory reduces to an “ N -particle” problem (note however, that these “particles” are localized only in the spectral parameter plane, but *not* in the actual space-time and should accordingly be referred to as “ N -waves”). In this way we have arrived at an effectively one-dimensional description of the $2d$ theory without sacrificing the nontriviality of the solutions. In particular, the quantization of this system will resemble that of an ordinary quantum mechanical system. From the point of view of quantum field theory, one may think of it as a kind of collective quantization.

3.4 Reality and Coset Constraints

We have introduced the new variables $A(\gamma)$ or equivalently $B(\gamma)$ as fundamental objects. However, in the way they have been defined, these variables contain too many degrees of freedom. In order to restore the original fields g and \mathcal{V} , respectively, from the spectral parameter currents, additional constraints must be imposed in order to ensure the reality and coset properties of the original variables, such that e.g. $\mathcal{V} \in G$ rather than $\mathcal{V} \in G_{\mathbb{C}}$. For the sake of clarity of presentation we will only consider the spectral parameter current $B(\gamma)$, in terms of which the reality and coset conditions can be stated most clearly. These conditions are direct consequences of (3.13) and the corresponding constraints on P_{\pm} .

The first important observation is that, for complex $\gamma \in \mathbb{C}$, $B(\gamma)$ lives in the *complexified* Lie algebra $\mathfrak{g}_{\mathbb{C}}$. To ensure reality of the physically relevant quantities (in particular of $B(\pm 1)$) we impose the constraint

$$\overline{B(\gamma)} = B(\bar{\gamma}) \quad (3.28)$$

Here the bar denotes complex conjugation on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ defined by

$$\overline{\sum_a y_a Z^a} := \sum_a \bar{y}_a Z^a \quad (y_a \in \mathbb{C}) \quad (3.29)$$

where the generators Z^a span the chosen real form \mathfrak{g} . Let us remark already here that the appearance of the complexified Lie algebra is very important for the quantum theory since the structure of unitary irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ is quite different from that of \mathfrak{g} .

From (3.13) we see that for $P_{\pm} \in \mathfrak{k}$, we must demand in addition that

$$B(\pm 1) \in \mathfrak{k} \subset \mathfrak{g}, \quad (3.30)$$

Thus, $B(\pm 1)$ is constrained to take values in the non-compact part of the real Lie algebra \mathfrak{g} . A priori this is not a consequence of the isomonodromic ansatz since the generic solution of the deformation equations will not satisfy this constraint.

To further analyze the coset constraint recall that we work in the BZ-picture where $\mathcal{M} = I$ (3.8), whence

$$0 = \frac{\partial \mathcal{M}}{\partial w} \frac{\partial w}{\partial \gamma} = \hat{\mathcal{V}}(\gamma) \left\{ B(\gamma) + \frac{1}{\gamma^2} \eta \left(B \left(\frac{1}{\gamma} \right) \right) \right\} \eta \left(\hat{\mathcal{V}}^{-1} \left(\frac{1}{\gamma} \right) \right),$$

i.e.

$$B(\gamma) + \frac{1}{\gamma^2} \eta \left(B \left(\frac{1}{\gamma} \right) \right) = 0. \quad (3.31)$$

This constraint immediately leads to

$$B(\pm 1) + \eta(B(\pm 1)) = 0 \implies B(\pm 1) \in \mathfrak{k} \quad (3.32)$$

Moreover, multiplying (3.31) by γ and taking $\gamma \rightarrow \infty$ we see from (3.11) that (3.31) implies

$$B_\infty = -\lim_{\gamma \rightarrow 0} \gamma \eta(B(\gamma)) \quad (3.33)$$

which vanishes if $B(\gamma)$ is regular at $\gamma = 0$ (i.e., in the triangular gauge). Conversely, for $B_\infty \neq 0$, we see that $B(\gamma)$ must have a first order pole at $\gamma = 0$ with residue $-\eta(B_\infty)$.

The reality and coset constraints (3.28) and (3.31) effectively reduce the number of degrees of freedom of the spectral parameter current. In terms of the isomonodromic ansatz (3.10) we can satisfy them for instance by taking $N = 2n$ and real poles γ_j with:

$$\gamma_j = \frac{1}{\gamma_{j+n}} \in \mathbb{R} \quad , \quad B_j = \overline{B_{j+n}} = \eta(B_{j+n}) = \overline{\eta(B_{j+n})} \quad (j = 1, \dots, n) \quad (3.34)$$

and

$$\sum_{j=1}^n (B_j + \eta(B_j)) = 0 \iff \sum_{j=1}^n B_j \in \mathfrak{k} \quad (3.35)$$

In this case all residues belong to the real form of the Lie algebra, i.e. $B_j \in \mathfrak{g}$. Alternatively, and more generally, we can take the number of (complex) poles to be a multiple of four, i.e. $N = 4n$ with

$$\gamma_j = \bar{\gamma}_{j+n} = \frac{1}{\gamma_{j+2n}} = \frac{1}{\bar{\gamma}_{j+3n}} \in \mathbb{C} \quad (j = 1, \dots, n) \quad (3.36)$$

and

$$B_j = \overline{B_{j+n}} = \eta(B_{j+2n}) = \eta(\overline{B_{j+3n}}) \quad (3.37)$$

Now (3.33) is ensured by

$$\sum_{j=1}^{2n} (B_j + \eta(B_j)) = 0 \iff \sum_{j=1}^{2n} B_j \in \mathfrak{k} \quad (3.38)$$

Consequently, we now have $B_j \in \mathfrak{g}_{\mathbb{C}}$. Of course, we can also consider both possibilities together, when only some of the poles and residues are subject to (3.34), and the remaining ones satisfy (3.36) together with (3.37).

We leave it as an exercise to the reader to check that the deformation equations (3.25) are indeed compatible with all these constraints (taking into account that η and complex conjugation are automorphisms). Furthermore, the reader may reformulate the complete set of constraints in terms of the alternative spectral parameter current $A(\gamma)$ to convince himself that the coset constraints take a far less convenient form there due to the explicit appearance of the original fields \mathcal{V} or g in the constraints.

3.5 τ -Function and Conformal Factor

Having resolved the matter part of the model, we are still left with the first order equations of motion (2.17) for the conformal factor. These formulas relating the gravitational to the matter sector have a well known analog in the theory of integrable

models. Though there is no conformal factor in the flat space models, an additional function can be introduced by these equations, which contains the complete information about the system in the sense that it serves as generating function for the Hamiltonians of the corresponding deformation dynamics. This is the so-called τ -function.

Let us explain this in more detail. Define the one-form

$$\omega_0 \equiv \frac{1}{2} \sum_{j \neq k} \text{tr}(A_j A_k) \, d \ln(\gamma_j - \gamma_k), \quad (3.39)$$

which was originally introduced in [30, 31]. It is usually understood as a function of the parameters γ_j and turns out to be closed

$$d\omega_0 = 0,$$

if the residues A_j (considered as functions of the γ_k) are subject to the so-called Schlesinger equations

$$\frac{\partial A_k}{\partial \gamma_j} = \frac{[A_j, A_k]}{\gamma_j - \gamma_k}, \quad \frac{\partial A_j}{\partial \gamma_j} = - \sum_{k \neq j} \frac{[A_j, A_k]}{\gamma_j - \gamma_k}, \quad (3.40)$$

which describe a γ_j -dependence induced by certain isomonodromy conditions. This gives rise to the definition of the so-called τ -function:

$$d\tau = \omega_0 \quad (3.41)$$

This function generates the Hamiltonians governing the γ_j -dynamics of (3.40) [30].

We will now show how to apply this concept to our model, treating all quantities as functions of the coordinates x^\pm . In particular, we will assume the dependence $A_j = A_j(\{\gamma_k(x^\pm)\})$, i.e. the residues depend on the coordinates x^\pm only via the γ_k . It will turn out, that the corresponding τ -function equals the conformal factor up to some explicit factor. In the next section we will further see that the Hamiltonians for the x^\pm -dynamics arise in the same way.

Let us again define the one-form ω_0 by (3.39), but now with respect to the space-time coordinates x^\pm , taking into account the coordinate-dependence of the γ_j :

$$\omega_0 = \frac{1}{2} \sum_{j \neq k} \frac{\text{tr}(A_j A_k)}{\gamma_j - \gamma_k} \left[\partial_+(\gamma_j - \gamma_k) dx^+ + \partial_-(\gamma_j - \gamma_k) dx^- \right] \quad (3.42)$$

We leave it as an exercise to show, that again ω_0 is closed if the residues A_j satisfy the deformation equations (3.24). Thus, the τ -function may be defined as in (3.41), the differential again taken w.r.t. x^\pm . Notice that it does not matter for this definition which of the spectral parameter currents A and B we employ, as they only differ by matrix conjugation which drops out in the trace.

Next we observe that the equations of motion for the conformal factor take the following form upon substitution of (3.13) into (2.17)

$$\partial_\pm \rho \partial_\pm \hat{\sigma} = \frac{1}{2} \rho \, \text{tr} P_\pm P_\pm = \frac{1}{2} \rho^{-1} (\partial_\pm \rho)^2 \sum_{j,k} \frac{\text{tr}(A_j A_k)}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \quad (3.43)$$

which we rewrite as

$$\begin{aligned} \partial_{\pm} \hat{\sigma} = \rho^{-1} \partial_{\pm} \rho & \left\{ \frac{1}{2} \sum_{j \neq k} \text{tr}(A_j A_k) \left[\frac{2}{(1 \pm \gamma_j)(1 \pm \gamma_k)} - 1 \right] \right. \\ & \left. + \sum_j \frac{\text{tr} A_j^2}{(1 \pm \gamma_j)^2} + \frac{1}{2} \sum_{j \neq k} \text{tr}(A_j A_k) \right\} \end{aligned}$$

The first term on the r.h.s. is just $\partial_{\pm} \ln \tau$. For the second line we use

$$\sum_{k \neq j} \text{tr}(A_j A_k) = \text{tr} A_{\infty}^2 - \text{tr} A_j^2,$$

which is constant by (3.27). Thus, the corresponding terms in the above equation can be integrated explicitly; we have

$$\begin{aligned} \partial_{\pm} \hat{\sigma} &= \partial_{\pm} \tau + \rho^{-1} \partial_{\pm} \rho \left\{ \sum_j \text{tr} A_j^2 \left(\frac{1}{(1 \pm \gamma_j)^2} - \frac{1}{2} \right) + \frac{1}{2} \text{tr} A_{\infty}^2 \right\} \\ &= \partial_{\pm} \tau + \sum_j \frac{1}{2} \text{tr} A_j^2 \partial_{\pm} \left(\ln \frac{\partial \gamma_j}{\partial w_j} \right) + \frac{1}{2} \text{tr} A_{\infty}^2 \partial_{\pm} (\ln \rho) \end{aligned}$$

which may be integrated up to give

$$\hat{\lambda} = \exp \hat{\sigma} = \rho^{\frac{1}{2} \text{tr} A_{\infty}^2} \prod_{j=1}^N \left(\frac{\partial \gamma_j}{\partial w_j} \right)^{\frac{1}{2} \text{tr} A_j^2} \cdot \tau \quad (3.44)$$

Thus the conformal factor coincides with the τ -function up to an explicit factor. Note that for solutions satisfying the coset constraints the first factor is unity since $A_{\infty} = 0$; this factor would, however, play a role for axisymmetric stationary solutions which are not asymptotically flat, but have a non-vanishing Kasner parameter. If moreover the whole set of coset constraints (3.34) is fulfilled, the relation (3.44) takes the form:

$$\hat{\lambda} = \prod_{j=1}^n \left(\gamma_j^{-1} \frac{\partial \gamma_j}{\partial w_j} \right)^{\text{tr} A_j^2} \cdot \tau \quad (3.45)$$

In the next chapter we will show that the conformal factor in these models does have all the requisite properties of a τ -function in that it serves as a generating function for the various Hamiltonians. The relation (3.44) will reappear in the quantized theory and reveal an intriguing link between our model and conformal field theory.

3.6 Beyond the Isomonodromic Sector

The restrictions implied by the isomonodromic ansatz (3.10) can actually be relaxed. In this section, we sketch how to generalize the treatment to arbitrary non-singular classical solutions. One of the main advantages of the isomonodromic ansatz was the possibility to parametrize the spectral parameter current $B(\gamma)$ by a complete set of quantities with rather simple evolution equations (3.25), whereas the deformation equations (3.26) in terms of a general $B(\gamma)$ took a rather complicated form. The task

at hand is then to identify a proper analog of the variables B_j in the general case with deformation equations similar to (3.25).

Let us assume that the function $B(x; \gamma(x, w))$ is holomorphic as a function of w in a small striplike region parallel to the real axis of the complex w -plane on both sheets of the Riemann surface of the function $\sqrt{(w + \rho^+(x^+))(w - \rho^-(x^-))}$, which has a “movable” (i.e. x -dependent) branch cut extending from $-\rho^+(x^+)$ to $\rho^-(x^-)$ along the real axis. We will denote this double strip by D and its (oriented) boundaries by $\ell \cup \ell^* \equiv \partial D$; by assumption there exists $\varepsilon > 0$ such that the distance from ∂D to the real axis is always greater than ε . The image of the double strip D in the γ -plane

$$\mathcal{D}(x) := \{\gamma \in \mathbb{C} \mid \gamma = \gamma(x; w), w \in D\} \quad (3.46)$$

is a “movable” annular region containing the unit circle $|\gamma| = 1$. Since the exchange of the two sheets in the w -plane corresponds to the involution $\gamma \rightarrow 1/\gamma$ in the γ -plane, the two contours ℓ_γ and ℓ_γ^* bounding \mathcal{D} are related in the same way. These contours are illustrated in figures 1 and 2.

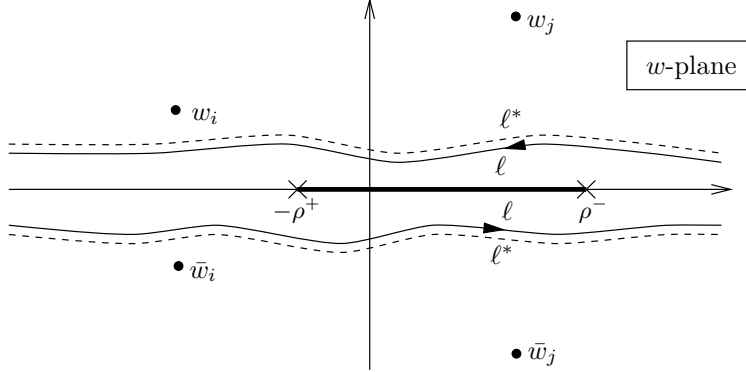


Figure 1. Paths ℓ and ℓ^* in the w -plane

The assumed regularity of $B(x; \gamma(x, w))$ in the whole double strip D implies that the singularities never meet the branch cut, and hence the corresponding solutions of the original field equations are non-singular in space-time (as is, for instance the case for Einstein-Rosen waves). In the γ -plane this means that the movable singularities always stay away from the annulus $\mathcal{D}(x)$, and hence from the circle $|\gamma| = 1$.

In order to generalize the ansatz (3.10) to non-isomonodromic solutions we observe that with the above assumptions we can write the former as

$$B(x; \gamma) = \oint_{\partial D} \frac{\mathcal{F}(x, \tilde{\gamma})}{\tilde{\gamma} - \gamma} \frac{d\tilde{\gamma}}{2\pi i} \quad (3.47)$$

by Cauchy’s formula where \mathcal{F} coincides with the boundary values of B . We can now exploit the freedom of adding any function holomorphic outside of \mathcal{D} to demand that

$$\mathcal{F}(x, \tilde{\gamma}) = 2\pi i \mathcal{B}(x; w(x, \tilde{\gamma})) \frac{\partial w(x; \tilde{\gamma})}{\partial \tilde{\gamma}} \quad (3.48)$$

where \mathcal{B} is defined to have the x -dependence

$$D_{\pm} \mathcal{B}(w) = 2\rho^{-1} \partial_{\pm} \rho \oint_{\ell \cup \ell^*} \frac{[\mathcal{B}(w), \mathcal{B}(v)]}{(1 \pm \gamma(w))(1 \pm \gamma(v))} dv - [\mathcal{B}(w), P_{\pm}], \quad (3.49)$$

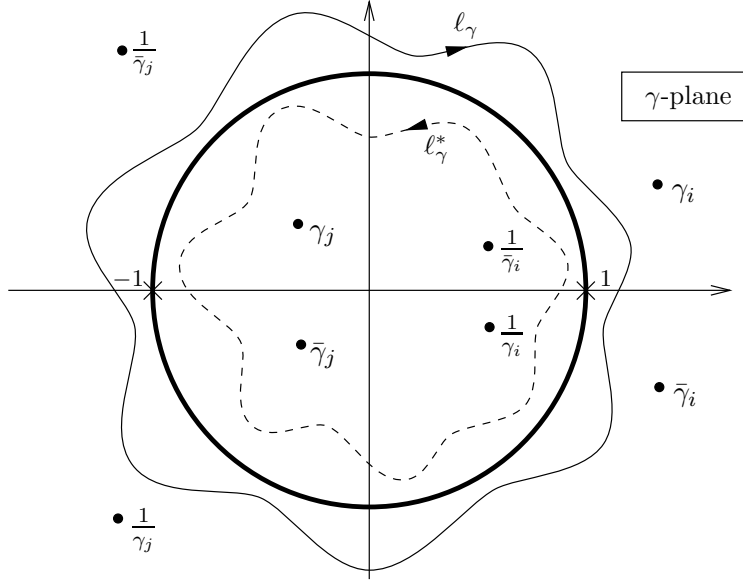


Figure 2. Paths ℓ_γ and ℓ_γ^* in the γ -plane

such that in particular \mathcal{F} does explicitly not coincide with the boundary values of B any longer. For consistency (as we could of course not demand any form of (3.49)), it must be ensured, that (3.49) via (3.47) still implies the deformation equations (3.26), which indeed it does. Summarizing, we have

$$B(x; \gamma) = \oint_{\partial\mathcal{D}(x)} \frac{\mathcal{B}(x; w(x, \tilde{\gamma}))}{\tilde{\gamma} - \gamma} \frac{\partial w(x; \tilde{\gamma})}{\partial \tilde{\gamma}} d\tilde{\gamma} = \oint_{\partial D} \frac{\mathcal{B}(x; w)}{\gamma(x; w) - \gamma} dw \quad (3.50)$$

Comparing (3.49) with the deformation equations of the residues (3.25), we recognize \mathcal{B} as the proper generalization of the B_j from the isomonodromic sector. In the limiting case, where the singularities γ_j just lie on the contour $\partial\mathcal{D}$, we can embed the isomonodromic sector into the general framework by setting

$$\mathcal{B}(w) = - \sum_{j=1}^N B_j \delta(w - w_j) \quad w, w_j \in \ell \quad (3.51)$$

where

$$\delta(w - w_j) := \frac{ds}{dw} \delta(s - s_j) \quad (3.52)$$

with some affine parameter s along the curve ∂D (note that this definition is independent of the chosen parametrization).

It remains to generalize the reality and coset constraints. This is easily done: the reality constraint (3.28) reads

$$\overline{\mathcal{B}(x; w)} = \mathcal{B}(x; \bar{w}) \quad (3.53)$$

The coset constraint (3.31), on the other hand is ensured by demanding

$$\mathcal{B}(x; w) = \eta(\mathcal{B}(x; w^*)) \quad \text{for } w \in \partial D \quad (3.54)$$

where w^* is the point lying over w in the second sheet.

4 HAMILTONIAN FORMULATION

The form of the deformation equations (3.25) suggests that we should seek a Hamiltonian formulation of the theory which exploits the decoupling exhibited by these equations and look for *two* Hamiltonians (or more precisely for two Hamiltonian constraints) describing the evolutions in x^\pm -directions, respectively. Evidently, this would not be quite the usual Hamiltonian formalism where time is singled out as the direction in which “something happens” and where the canonical variables depend on the coordinate x^1 of some fixed space-like (one-dimensional) hypersurface.⁵ Rather, the canonical variables in our formulation depend on the spectral parameter γ , such that the brackets are given as “equal point brackets” at a fixed space-time reference point instead of the usual equal time brackets containing spatial δ -functions. One obvious advantage of this formulation is its $2d$ covariance which is manifest at every stage.

Although the relation of these ideas with the standard formalism remains to be fully elucidated, we will first review the conventional Hamiltonian treatment of the models in the following section.⁶ For the truncation corresponding to Einstein-Rosen waves this program can be carried to completion [18, 19]. However, for the full theory with (non-polynomially) self-interacting matter fields the conventional treatment gets bogged down in technical difficulties which have not yet been completely overcome even for the simpler flat space non-linear σ -model. To obviate this impasse new methods are evidently needed, and our “two-time formalism” addresses precisely these difficulties, exploiting many results of the theory of flat space integrable systems [7, 8]. A further motivation for presenting both formulations side by side is to underline the similarities between the present model and string theory, which can be viewed as the simplest example of a solvable model of matter coupled quantum gravity in two dimensions. In particular, the notorious “time problem” of quantum gravity [35] was already considered (and solved) in this context by string theorists a long time ago. The choice of Weyl canonical coordinates for axisymmetric stationary gravity, or its Lorentzian analog, precisely corresponds to the light-cone gauge in string theory.

In canonical quantum gravity it is customary to make certain assumptions about the global structure of space-time (existence of a global time coordinate, asymptotic flatness, etc.) in order to disentangle gauge from dynamics and to extract from the WDW equation a Schrödinger-type equation for the matter degrees of freedom (see also [36] for a discussion and further references on this topic). To be sure, such assumptions are indispensable if one wants the quantum theory to possess standard Fock space properties (distinguishability of positive and negative frequencies, existence of a vacuum). However, we would prefer to avoid them here because Fock space concepts are clearly not of much use in the presence of non-polynomial interactions, and furthermore do not exploit the underlying symmetry structures of the model. For this reason, we will set up the theory locally and restrict attention to a particular coordinate patch, separating the gauge part of the evolution (bubble time evolution) from the dynamics as a local observer would do.⁷ An extra bonus is that this description can be natu-

⁵We note that reservations with regard to the special role of time in the usual canonical formalism were expressed already long ago by Dirac [32].

⁶See also [33] for a recent discussion of symmetry reduced Einstein gravity in the framework of Ashtekar’s variables and [34] for the quantization of Levi-Civita spacetimes.

⁷The intuitive idea behind this procedure is that an earth based quantum gravity practitioner should not need to make any assumptions about the structure of space-time for an observer in Andromeda

rally extended to more complicated space-time topologies, i.e. arbitrary (Lorentzian) Riemann surfaces.

4.1 Conventional Approach

We will not use the original Lagrangian (2.12), because for the canonical formulation it is somewhat more convenient to work with an equivalent one presented in [28] that treats the dilaton ρ and the axion $\tilde{\rho}$ as independent fields, and which reads

$$\mathcal{L} = \tilde{\varepsilon}^{\mu\nu} \omega_\mu \partial_\nu \rho + \tilde{h}^{\mu\nu} \omega_\mu \partial_\nu \tilde{\rho} + \frac{1}{2} \rho \tilde{h}^{\mu\nu} \text{Tr} P_\mu P_\nu \quad (4.1)$$

with $\tilde{\varepsilon}^{\mu\nu} := e e_\alpha^\mu e_\beta^\nu \epsilon^{\alpha\beta}$ and $\tilde{h}^{\mu\nu} := e e_\alpha^\mu e_\beta^\nu \eta^{\alpha\beta}$. Since $\tilde{\varepsilon}^{\mu\nu}$ is a density and the conformal factor drops out from the unimodular metric $\tilde{h}_{\mu\nu}$, (4.1) is manifestly Weyl invariant. Let us first check that this Lagrangian gives rise to the desired equations of motion. Varying ω_μ yields

$$\tilde{\varepsilon}^{\mu\nu} \partial_\nu \rho + \tilde{h}^{\mu\nu} \partial_\nu \tilde{\rho} = 0 \quad (4.2)$$

which implies (the generally covariant analog of) (2.16). The variation of $\tilde{\rho}$ tells us that ω_μ is a curl, i.e. there exists a scalar $\hat{\sigma}$ such that

$$\omega_\mu = \tilde{\varepsilon}_\mu{}^\nu \partial_\nu \hat{\sigma} \quad (4.3)$$

Subsequent variation of ρ and use of (4.3) leads to

$$\partial_\mu (\tilde{h}^{\mu\nu} \partial_\nu \hat{\sigma}) = -\frac{1}{2} \tilde{h}^{\mu\nu} \text{Tr} P_\mu P_\nu \quad (4.4)$$

This is just our previous equation (2.18); therefore the field $\hat{\sigma}$ is indeed the same as the one introduced in (2.15). Finally, variation of $\tilde{h}_{\mu\nu}$ together with (4.3) reproduces the two first order equations for the conformal factor (2.17).

The formal disappearance of the conformal factor from the Lagrangian (4.1) can also be seen in the canonical framework. Let us first compute the canonical momenta of the non-matter degrees of freedom [28]

$$\Pi := \frac{\delta \mathcal{L}}{\delta \partial_0 \rho} = \omega_1 \quad , \quad \tilde{\Pi} := \frac{\delta \mathcal{L}}{\delta \partial_0 \tilde{\rho}} = \tilde{h}^{00} \omega_0 + \tilde{h}^{01} \omega_1 \quad , \quad \Omega^\mu := \frac{\delta \mathcal{L}}{\delta \partial_0 \omega_\mu} = 0 \quad (4.5)$$

Hence, by (4.3),

$$\Pi = -\partial_0 \hat{\sigma} \quad , \quad \tilde{\Pi} = -\partial_1 \hat{\sigma} \quad (4.6)$$

The four equations (4.5) constitute a set of second class constraints. Following standard procedures [37, 38] we can thus eliminate the variables ω_μ and Ω^μ , keeping only ρ , $\tilde{\rho}$ and their canonical momenta as independent phase space variables. The relevant Dirac brackets are then given by

$$\begin{aligned} \{\Pi(x), \rho(y)\} &= \{\tilde{\Pi}(x), \tilde{\rho}(y)\} = \delta(x - y) \\ \{\Pi(x), \tilde{\rho}(y)\} &= \{\Pi(x), \tilde{\rho}(y)\} = 0 \end{aligned} \quad (4.7)$$

where from now on in this section $x \equiv x^1, y \equiv y^1, \dots$ denote *spatial* coordinates at fixed time $x^0 = y^0$.

galaxy in order to be able to extract a Schrödinger equation from the WDW equation, if he assumes all his matter wave functions to vanish beyond (say) Pluto's orbit.

The non-linear σ -model sector requires a little more work and we here just quote the relevant results. The canonical momenta are

$$\tilde{P}^A := \frac{\delta \mathcal{L}}{\delta P_0^A} = \rho(\tilde{h}^{00} P_0^A + \tilde{h}^{01} P_1^A) \quad , \quad \phi^\alpha := \frac{\delta \mathcal{L}}{\delta Q_0^\alpha} = 0 \quad (4.8)$$

Clearly,

$$\phi^\alpha(x) \approx 0 \quad (4.9)$$

is the constraint implementing the local H invariance of the theory as these generators satisfy

$$\{\phi^\alpha(x), \phi^\beta(y)\} = f^{\alpha\beta}{}_\gamma \phi^\gamma(x) \delta(x-y) \quad (4.10)$$

The remaining non-vanishing brackets are given by

$$\begin{aligned} \{\tilde{P}^A(x), \tilde{P}^B(y)\} &= f^{AB}{}_\alpha \phi^\alpha(x) \delta(x-y) \\ \{\tilde{P}^A(x), P_1^B(y)\} &= f^{AB}{}_\alpha Q_1^\alpha(x) \delta(x-y) - \delta^{AB} \partial_1 \delta(x-y) \\ \{\tilde{P}^A(x), Q_1^\alpha(y)\} &= f^{A\alpha}{}_B P_1^B(x) \delta(x-y) \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} \{\phi^\alpha(x), \tilde{P}^A(y)\} &= f^{\alpha A}{}_B \tilde{P}^B(x) \delta(x-y) \\ \{\phi^\alpha(x), P_1^A(y)\} &= f^{\alpha A}{}_B P_1^B(x) \delta(x-y) \\ \{\phi^\alpha(x), Q_1^\beta(y)\} &= f^{\alpha\beta}{}_\gamma Q_1^\gamma(x) \delta(x-y) \end{aligned} \quad (4.12)$$

With the following parametrization of the worldsheet metric in terms of lapse N^0 and shift N^1

$$e_\mu{}^\alpha = \begin{pmatrix} N^0 & N^1 \\ 0 & e_1^1 \end{pmatrix} \implies \tilde{h}_{\mu\nu} = \begin{pmatrix} N^0 - \frac{(N^1)^2}{N^0} & -\frac{N^1}{N^0} \\ -\frac{N^1}{N^0} & -\frac{1}{N^0} \end{pmatrix} \quad (4.13)$$

(which differs slightly from the usual one because $\tilde{h}_{\mu\nu}$ is unimodular), a straightforward calculation leads to the expected result [37, 38]

$$\mathcal{L} = \Pi \partial_0 \rho + \tilde{\Pi} \partial_0 \tilde{\rho} + P_0^A \tilde{P}^A - H[N^0, N^1, Q_0^\alpha] \quad (4.14)$$

Here

$$H[N^0, N^1, Q_0^\alpha] := \int dx \left(N^0(x) \mathcal{H}(x) + N^1(x) \mathcal{P}(x) - Q_0^\alpha(x) \phi^\alpha(x) \right) \quad (4.15)$$

is the total Hamiltonian with the Hamiltonian (WDW) constraint

$$\mathcal{H}(x) := \Pi \partial_1 \tilde{\rho} + \tilde{\Pi} \partial_1 \rho + \frac{1}{2} \rho^{-1} \tilde{P}^A \tilde{P}^A + \frac{1}{2} \rho P_1^A P_1^A \approx 0 \quad (4.16)$$

and the (spatial) diffeomorphism constraint

$$\mathcal{P}(x) := \Pi \partial_1 \rho + \tilde{\Pi} \partial_1 \tilde{\rho} + \tilde{P}^A P_1^A \approx 0 \quad (4.17)$$

In the remainder we will mostly work with the combinations

$$\begin{aligned} T_{\pm\pm}(x) = \frac{1}{2}(\mathcal{H}(x) \pm \mathcal{P}(x)) &= \pm \Pi_\pm \partial_1 \rho^\pm + \rho P_\pm^A P_\pm^A \\ &= 2\Pi_\pm \partial_\pm \rho + \rho P_\pm^A P_\pm^A \end{aligned} \quad (4.18)$$

where

$$\Pi_{\pm} := \frac{1}{2}(\Pi \pm \tilde{\Pi}) \quad (4.19)$$

are canonically conjugate to ρ^{\pm} . Obviously, $T_{\pm\pm}$ generate infinitesimal conformal diffeomorphisms $x^{\pm} \rightarrow \tilde{x}^{\pm}(x^{\pm})$. The constraints $T_{\pm\pm}(x) \approx 0$ are the analogs in our model of the Virasoro constraints of string theory; besides, they are just the first order equations of motion (2.17) for the conformal factor. Note also that by Weyl invariance of (4.1) we automatically have $T_{+-}(x) \equiv 0$. With the above commutation relations it is straightforward to check that (4.18) generate two commuting classical Virasoro algebras (i.e., without central term).

To fix the gauge we identify the dilaton ρ and the axion $\tilde{\rho}$ with the world-sheet coordinates (x^0, x^1) in the given coordinate patch. For axisymmetric stationary gravity this is a well known trick [39] leading to Weyl canonical coordinates (which usually are assumed to be global coordinates in the upper half plane, although this is not necessary [40]). For Lorentzian world-sheets, there is an extra subtlety because we must distinguish whether the vector $\partial_{\mu}\rho$ is spacelike, timelike or null; these cases correspond to physically distinct situations.⁸ In fact, there are cosmological models [41] where the signature $\partial_{\mu}\rho\partial^{\mu}\rho$ varies over different regions, thus supporting a canonical treatment according to our local point of view. For definiteness we will from now on assume that

$$G[f] := \int dx f(x) \partial_1 \rho(x) = 0 \quad , \quad \tilde{G}[f] := \int dx f(x) (\partial_1 \tilde{\rho}(x) - 1) = 0 \quad (4.20)$$

for *all* smooth functions $f \in C_0^{\infty}(\text{I})$ with compact support in the given coordinate patch I (so that boundary terms arising in partial integration can be dropped). Hence, by the equations of motion (4.2)

$$\rho(x) = x^0 + \rho_0 \quad , \quad \tilde{\rho}(x) = x^1 + \tilde{\rho}_0 \quad (4.21)$$

since the zero modes $(\rho_0, \tilde{\rho}_0)$ cannot be gauge fixed. Thus the dilaton $\rho > 0$ serves as a “clock field” (i.e., time is measured by the size of the internal Kaluza Klein universe which has an initial singularity at $\rho = 0$), while the axion $\tilde{\rho}$ would have to be interpreted as a “measuring rod field”⁹.

Identifying the coordinates with ρ and $\tilde{\rho}$ in this way amounts to solving the constraints $\mathcal{H}(x) = 0$, $\mathcal{P}(x) = 0$ up to their zero (constant) modes. The remaining constraints $H = \int \mathcal{H} dx$ and $P = \int \mathcal{P} dx$ then generate the true dynamics of our model; in the quantum theory they will effectively lead to two “Schrödinger equations” in the two-time formalism. To check the admissibility of the gauge choice (4.20), we note that

$$\begin{aligned} \{H[N^0, N^1, Q_0^{\alpha}], G[f]\} &= \int dx f(x) \partial_1 N^0(x) \\ \{H[N^0, N^1, Q_0^{\alpha}], \tilde{G}[f]\} &= \int dx f(x) \partial_1 N^1(x) \end{aligned} \quad (4.22)$$

For any non-zero $(N^0, N^1) \in C_0^{\infty}(\text{I})$, this indeed cannot vanish for all $f \in C_0^{\infty}$. Therefore such (N^0, N^1) generate “bubble-like” deformations which we consider as pure gauge. On the other hand, the constraints H and P correspond to *constant* (non-zero) lapse and shift, and one easily sees that for them, the r.h.s. of (4.22) vanishes.

⁸Yet another inequivalent choice, appropriate for colliding plane wave solutions which we will not discuss here, is $\rho(x) = 1 - (x^+)^2 - (x^-)^2$.

⁹Or, as H. Weyl would have called it, “Maßstabfeld”.

This means that the above gauge choice is preserved by the evolution generated by H and P . In analogy with (4.18), we will now consider the mutually commuting constraint generators

$$C_{\pm} := \frac{1}{2}(H \pm P) \quad (4.23)$$

(for closed strings, we would have $C_+ \equiv L_0$, $C_- \equiv \bar{L}_0$.) These light-cone Hamiltonians generate the dynamics along x^{\pm} in the local patch, and form the basis of the “two-time formalism”. It is straightforward to check that commutation with $C_+ + C_-$ reproduces the equations of motion: for instance, we get

$$\partial_0 \tilde{\Pi} = \partial_1 \Pi \quad (4.24)$$

which is consistent with (4.6), and

$$\partial_0 \Pi = \partial_1 \tilde{\Pi} + \frac{1}{2} \rho^{-2} \tilde{P}^A \tilde{P}^A - \frac{1}{2} P_1^A P_1^A \quad (4.25)$$

which is just the second order equation (2.18) for the conformal factor.

A proper analysis would require that we treat the constraints $\mathcal{H}(x) = 0$ and $\mathcal{P}(x) = 0$ together with the gauge fixing conditions (4.20) as an infinite system of second class constraints, with only the zero mode parts of H and P remaining as first class constraints (since only for them, the r.h.s. of (4.22) vanishes). This procedure shrinks the full phase space of the covariant theory down to the so-called *reduced phase space*, which we will be mainly concerned with in the sequel. It not only eliminates the δ -functions in the canonical brackets but also leads to a replacement of the constraint functionals by ordinary functions. In particular, the zero mode constraints from (4.18) become¹⁰

$$\mathcal{C}_{\pm} = \Pi_{\pm} + \rho P_{\pm}^A P_{\pm}^A \approx 0 \quad (4.26)$$

where we have replaced the integrated constraints C_{\pm} by the corresponding densities \mathcal{C}_{\pm} at an arbitrary but fixed space-time reference point in the patch. This is permitted because after solving the non-zero mode part of the constraints, only the constant part of the densities remains, and consequently C_{\pm} and \mathcal{C}_{\pm} differ merely by an irrelevant volume (length) factor. Furthermore, we now have the reduced phase space brackets

$$\{\Pi_+, \rho^+\} = \{\Pi_-, \rho^-\} = 1 \quad , \quad \{\Pi_+, \rho^-\} = \{\Pi_-, \rho^+\} = 0 \quad (4.27)$$

where again all variables are to be taken at the chosen fixed reference point in space-time. These brackets result from (4.7) after removal of the non-zero mode contributions to the δ -function.

Upon quantization, (4.26) will become ordinary differential operators rather than functional differential operators unlike the original constraints (4.18); similarly, the WDW functional of the covariantly quantized theory will become a (Hilbert-space-valued) function of the coordinates x^{\pm} . We will return to this in chapter 5. However, we must first discuss the role of (4.26) for the matter variables.

4.2 Poisson Structure and Hamiltonians for the Matter Sector

For the matter sector, an *ab initio* derivation of the new Poisson structure is not yet available, and we will thus simply postulate the brackets in such a way that the correct

¹⁰For spacelike $\partial_{\mu} \rho$ we would have $\mathcal{C}_{\pm} = \pm \Pi_{\pm} + \rho P_{\pm}^A P_{\pm}^A$.

equations of motion are obtained. The new Poisson structure presented here amounts to a *de facto* resolution of certain technical problems of the corresponding flat space models (for instance having to do with the non-ultralocal term on the r.h.s. of (4.11) [42]), mainly because spatial δ -functions and their derivatives are altogether absent in our formalism. Before writing the brackets down, however, we give the light-cone Hamiltonians¹¹ in the Weyl gauge (4.20). They are

$$\begin{aligned} H_{\pm} &:= \rho^{-1} \sum_{k,l} \frac{\text{tr}(B_k B_l)}{(1 \pm \gamma_k)(1 \pm \gamma_l)} - \rho^{-1} \sum_{k,l} \frac{\text{tr}(B_k B_l)}{(1 \pm \gamma_l)} \\ &= \rho^{-1} \text{tr}(B(\mp 1))^2 - \text{tr}(B_{\infty} P_{\pm}) \end{aligned} \quad (4.28)$$

The first term on the r.h.s. can be directly deduced from (4.26) by substitution of (3.13) into the constraints (4.26), remembering that $\partial_{\pm} \rho = \frac{1}{2}$ in this gauge. The second term involving B_{∞} cannot be motivated in this way. As we will see in a moment, however, it vanishes once the coset constraint $B_{\infty} = 0$, which is not automatically included in the deformation equations unlike in (4.26), is properly taken into account. With this caveat, we can claim that the constraints

$$\mathcal{C}_{\pm} = \Pi_{\pm} + H_{\pm} \approx 0 \quad (4.29)$$

indeed coincide with (4.26).

The Poisson brackets with the requisite properties for the spectral parameter current $B(\gamma) \equiv B^a(\gamma) Z_a$ turn out to be

$$\{B^a(\gamma), B^b(\gamma')\} = -f^{ab}{}_c \frac{B^c(\gamma) - B^c(\gamma')}{\gamma - \gamma'} \quad (4.30)$$

where the space-time coordinate is kept fixed (hence “equal point brackets”). These brackets may be written in several equivalent ways. Depending on their background readers might prefer the explicit index notation

$$\{B_{\alpha\beta}(\gamma), B_{\gamma\delta}(\gamma')\} = \frac{1}{\gamma - \gamma'} \left(\delta_{\gamma\beta} [B(\gamma) - B(\gamma')]_{\alpha\delta} - \delta_{\alpha\delta} [B(\gamma) - B(\gamma')]_{\gamma\beta} \right) \quad (4.31)$$

for the matrices $B_{\alpha\beta}(\gamma)$, or the compact tensor notation of [7]

$$\{B(\gamma) \otimes B(\gamma')\} = \left[B(\gamma) \otimes I + I \otimes B(\gamma'), \frac{\Omega}{\gamma - \gamma'} \right] \quad (4.32)$$

with the Casimir operator $\Omega \equiv Z_a \otimes Z^a$. We emphasize that the Poisson structure (4.30), which is ubiquitous in integrable systems [7], cannot be “amended” by the introduction of spatial δ -functions, as otherwise it would be incompatible with the deformation equations (3.26). On the contrary, these equations *define* how to continue the bracket to other space-time points.

The brackets can be alternatively expressed in terms of the singularities and residues of the spectral parameter current. We leave it as an exercise to show that (4.30) in the parametrization of (3.10) is equivalent to:

$$\{B_j^a, B_k^b\} = \delta_{jk} f^{ab}{}_c B_k^c, \quad \{B_j^a, \gamma_k\} = 0, \quad \{\gamma_j, \gamma_k\} = 0 \quad (4.33)$$

¹¹By a slight abuse of language, we will sometimes refer to the matter part of (4.26) simply as the (matter) “Hamiltonians”, cf. (4.36).

We now have all the tools ready to check that that the deformation equations (3.25) admit a Hamiltonian formulation with respect to the Poisson structure (4.30) and the Hamiltonians (4.29). Indeed,

$$\begin{aligned} \{\text{tr}(B(\mp 1))^2, B(\gamma)\} &= 2 \frac{[B(\mp 1), B(\gamma)]}{\gamma \pm 1} \\ \implies \{\text{tr}(B(\mp 1))^2, B_j\} &= 2 \sum_k \frac{[B_j, B_k]}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \end{aligned} \quad (4.34)$$

and

$$\{\text{tr}(B_\infty P_\pm), B_j\} = [B_j, P_\pm] \quad (4.35)$$

In summary, we have formulated the model as a Hamiltonian system such that the deformation equations can be obtained from

$$D_\pm B_j = \{H_\pm, B_j\} \quad (4.36)$$

The compatibility of the deformation equations that was already emphasized above is equivalently expressed by the fact that the Hamiltonians H_\pm have vanishing mutual Poisson bracket; consequently, the corresponding flows commute. However, we still have to take care of the coset constraints of section 3.4. In a Hamiltonian formulation they must be properly treated à la Dirac [37, 38]. We briefly describe the result of this procedure. The nontrivial constraint is (3.31). The detailed analysis shows that only

$$B_\infty + \eta(B_\infty) \approx 0 \quad (4.37)$$

remains as a first-class constraint, whereas all other constraints (among them in particular $B_\infty - \eta(B_\infty)$) allow explicit resolution, such that after the Dirac procedure they vanish strongly. The Poisson structure of $B(\gamma)$ is modified to [4]

$$\begin{aligned} \{B^a(\gamma), B^b(\gamma')\}_{DB} &= -\frac{1}{2} f^{ab}{}_c \frac{B^c(\gamma) - B^c(\gamma')}{\gamma - \gamma'} \\ &\quad + \frac{1}{2} f^{a\eta(b)}{}_c \frac{B^c(\gamma)}{\gamma' - \frac{1}{\gamma}} + \frac{1}{2} f^{\eta(a)b}{}_c \frac{B^c(\gamma')}{\gamma - \frac{1}{\gamma'}}, \end{aligned} \quad (4.38)$$

with the notational convention (and choice of basis) $Z^{\eta(a)} \equiv \eta(Z^a)$.

In terms of the residues B_j this implies (if we for simplicity restrict to the case (3.34)):

$$\{B_j^a, B_k^b\} = \frac{1}{2} \delta_{jk} f^{abc} B_k^c \quad \text{for } j, k \leq n \quad (4.39)$$

as well as the strong identities

$$B_j = \eta(B_{j+n}), \quad (4.40)$$

For B_∞ this already implies, that

$$B_\infty = \sum_{j=1}^n (B_j + \eta(B_j)) \in \mathfrak{h},$$

whereas the remaining constraint

$$\sum_{j=1}^n (B_j + \eta(B_j)) = 0 \quad (4.41)$$

remains first class according to (4.37). Thus, the Poisson structure is defined on half of the variables B_j , whereas for the other half it is determined by the solution of the constraints. After the Dirac procedure, the second term of the Hamiltonian (4.28) may be dropped, since the relations $B_\infty = \eta(B_\infty) \in \mathfrak{h}$ and $P_\pm \in \mathfrak{k}$ hold strongly.

Finally, we would like to study the action of the surviving first class constraint (4.37). It is a simple exercise to show that this expression generates the gauge transformations

$$B(x; \gamma) \mapsto h^{-1}(x)B(x; \gamma)h(x) \quad \text{with} \quad h(x) \in H \quad (4.42)$$

Owing to (3.13), these are precisely the H -valued gauge transformations (2.5) of the original theory. The Hamiltonians

$$H_\pm = \rho^{-1} \operatorname{tr} B^2(\mp 1) \quad (4.43)$$

generate deformation only modulo H gauge transformations (as evident from the appearance of the covariant derivative in (4.36)). Thus, the first class constraint (4.37) can be added with impunity and moreover be employed to keep the dynamics in a fixed section of the gauge orbits.

4.3 Hamiltonian Formalism for Non-Isomonodromic Configurations

As explained in section 3.6, the natural generalization of the residues $B_j(x)$ for non-isomonodromic solutions is the density $\mathcal{B}(x; w)$, $w \in \ell$. The deformation equations (3.25) are replaced by their continuous analogs (3.49), such that the isomonodromic ansatz can be recovered in a special case (3.51). Inspection of (3.51) immediately suggests a Poisson structure for $\mathcal{B}(w)$: in the isomonodromic sector we can rewrite (4.33) as¹²

$$\begin{aligned} \{\mathcal{B}^a(w), \mathcal{B}^b(v)\} &= \left\{ \sum_{j=1}^N B_j^a \delta(w - w_j), \sum_{j=1}^N B_j^b \delta(v - w_j) \right\} \\ &= \sum_{j=1}^N f^{ab} {}_c B_j^c \delta(w - w_j) \delta(v - w_j) \\ &= \sum_{j=1}^N f^{ab} {}_c B_j^c \delta(w - w_j) \delta(w - v) = -f^{ab} {}_c \mathcal{B}^c(w) \delta(w - v) \end{aligned}$$

Regarding $\mathcal{B}(w)$ as the basic variables, we are thus led to postulate

$$\{\mathcal{B}^a(w), \mathcal{B}^b(v)\} = -f^{ab} {}_c \mathcal{B}^c(w) \delta(w - v) \quad v, w \in \ell \quad (4.44)$$

which has precisely the structure of an affine Lie algebra. The bracket for the spectral parameter currents $B(\gamma(w))$ is then a direct consequence of the representation (3.50):

$$\{B^a(\gamma(w)), B^b(\gamma(v))\} = \left\{ \oint_{\partial D} \frac{\mathcal{B}^a(w') dw'}{\gamma(w') - \gamma(w)}, \oint_{\partial D} \frac{\mathcal{B}^b(v') dv'}{\gamma(v') - \gamma(v)} \right\}$$

¹²We suppress the x -dependence from now on since all brackets are to be taken at a fixed reference point.

$$\begin{aligned}
&= \oint_{\partial D} \oint_{\partial D} \frac{\{\mathcal{B}^a(w'), \mathcal{B}^b(v')\} dw' dv'}{(\gamma(w') - \gamma(w))(\gamma(v') - \gamma(v))} \\
&= -f^{ab}{}_c \oint_{\partial D} \frac{\mathcal{B}^c(w') dw'}{(\gamma(w') - \gamma(w))(\gamma(w') - \gamma(v))} \\
&= -f^{ab}{}_c \frac{B^c(\gamma(w)) - B^c(\gamma(v))}{\gamma(w) - \gamma(v)}
\end{aligned}$$

which indeed coincides with (4.30). The Dirac bracket (4.38) would follow in the same fashion after the Dirac procedure of (3.54) at the level of $\mathcal{B}(w)$.

An obvious question at this point is whether we can include a central term in the bracket (4.44), i.e. replace (4.44) by

$$\{\mathcal{B}^a(w), \mathcal{B}^b(v)\} = -f^{ab}{}_c \mathcal{B}^c(w) \delta(w - v) + K \eta^{ab} \partial_w \delta(w - v) \quad v, w \in \ell \quad (4.45)$$

Remarkably, this is indeed possible because the holomorphic bracket (4.30) is insensitive to this modification, as the central term drops out in the above integral. It is not clear at the moment, whether this central extension is related to the central term appearing in the Geroch algebra [26, 22, 16].

The matter Hamiltonians H_{\pm} in (4.28) are re-expressed as follows:

$$H_{\pm} = \rho^{-1} \partial_{\pm} \rho \operatorname{tr} \left[\oint_{\ell \cup \ell^*} \frac{\mathcal{B}(w) dw}{1 \pm \gamma(w)} \right]^2 + \operatorname{tr} \left[\oint_{\ell \cup \ell^*} \mathcal{B}(w) dw P_{\pm} \right], \quad (4.46)$$

which shows that in the general scheme they are non-local in the spectral parameter plane. This is one indication how the non-locality in space-time is converted into non-locality in the spectral parameter plane in this formalism.

The Dirac procedure will again kill the second term of the Hamiltonian. Solving the coset constraint (3.54) here amounts to defining the Poisson structure on one sheet of the Riemann surface and transporting it to the other sheet by means of the constraint. The surviving first class constraint (4.37) takes the form

$$\mathcal{B}_{\infty} \equiv \oint_{\ell \cup \ell^*} \mathcal{B}(w) dw \equiv \oint_{\ell} \left(\mathcal{B}(w) + \eta(\mathcal{B}(w^*)) \right) dw = 0 \quad (4.47)$$

From (4.46), (4.44) and (3.49) we see that the total dynamics of $\mathcal{B}(w)$, $w \in \ell$ is generated by the matter Hamiltonians H_{\pm} . The variables $B((\gamma(w)))$ on the contrary also carry an explicit x^{\pm} -dependence via γ , whose dynamics is not generated by these Hamiltonians. This is in complete analogy with the isomonodromic sector, where an explicit x^{\pm} -dependence solely originated from the x -dependence of the locations of the poles γ_j .

4.4 Conserved Non-Local Charges

We have already encountered some conserved quantities in the isomonodromic sector. In this section, we shall construct an even larger set of observables that is complete in the sense that the solution in any isomonodromic sector with a fixed number of poles may (generically) be uniquely reconstructed from these data. With the additional structure of Poisson brackets at our disposal, we can proceed to study the algebraic structure of these observables.

Returning to the deformation equations (3.25), it is obvious, that the set of eigenvalues of the residue matrices B_j , encoded into the traces $\text{tr} B_j$, $\text{tr} B_j^2, \dots$ is independent of the coordinates x^\pm . However, according to (4.30) the Poisson structure of these eigenvalues turns out not to be too interesting because they all commute. This means that we have to fix all these quantities by hand in order to achieve non-degeneracy of the Poisson structure. A more “advanced” construct regarding the isomonodromic ansatz (3.10) are the monodromy matrices related to this current, which are defined by

$$\hat{\mathcal{V}}(\gamma) \mapsto M_j \hat{\mathcal{V}}(\gamma) \quad \text{for } \gamma \text{ encircling } \gamma_j \quad (4.48)$$

These monodromies indeed encapsulate the complete information about the spectral parameter current (3.10) itself (after resolution of the classical Riemann-Hilbert problem). As they are obviously x^\pm -independent, they build a promising object for observables of the model.

Their mutual Poisson brackets calculated from (4.30) exhibit the following quadratic structure [4]

$$\{M_i^1, M_i^2\} = i\pi \left(M_i^2 \Omega M_i^1 - M_i^1 \Omega M_i^2 \right) \quad (4.49)$$

$$\{M_i^1, M_j^2\} = i\pi \left(M_i^1 \Omega M_j^2 + M_j^2 \Omega M_i^1 - \Omega M_i^1 M_j^2 - M_i^1 M_j^2 \Omega \right), \quad (4.50)$$

for $i < j$

where we have introduced the shorthand notation $M_i^1 \equiv M_i \otimes I$ and $M_i^2 \equiv I \otimes M_i$. This algebra and its quantization have been extensively discussed in the lectures of Alekseev at this School [43]. We will therefore only comment on some of its features to our model.

- The algebra of monodromy matrices includes the first-class constraint

$$M_\infty \equiv \prod_{i=1}^N M_i = I, \quad (4.51)$$

which generates common conjugation. This is the analogue of (4.37) in terms of the spectral parameter current. Gauge invariant objects are built from traces of arbitrary products of monodromy matrices.

- The precise definition of the monodromy matrices depends on the normalization $\hat{\mathcal{V}}|_{\gamma=\infty} = \eta(\mathcal{V})$ (cf. (3.8)) — otherwise they are defined only up to gauge transformation. The distinguished path $[\gamma \rightarrow \infty]$ in the γ -plane gives rise to a cyclic ordering of the monodromy matrices, that defines (4.50) and also (4.51). It is a remnant of the so-called eyelash that enters the definition of the analogous Poisson structure in the combinatorial approach [44, 45, 43], being attached to every vertex and representing part of the freedom in this definition.
- An apparent obstacle of the structure (4.49), (4.50) is the violation of Jacobi identities. Actually, this can be traced back to the constraint (4.37) in the calculation of the Poisson brackets. As these brackets are valid only on the first-class constraint surface (4.51), Jacobi identities cannot be expected to hold in general. However, the same reasoning shows, that the structure (4.49), (4.50) restricts to a Poisson structure fulfilling the Jacobi identities on the space of gauge invariant

objects. On this space, the structure coincides with the restrictions of previously found and studied structures on the monodromy matrices [44, 43]:

$$\begin{aligned} \{M_i^1, M_i^2\} &= M_i^2 r_+ M_i^1 + M_i^1 r_- M_i^2 - r_- M_i^1 M_i^2 - M_i^1 M_i^2 r_+ \\ \{M_i^1, M_j^2\} &= M_i^1 r_+ M_j^2 + M_j^2 r_+ M_i^1 - r_+ M_i^1 M_j^2 - M_i^1 M_j^2 r_+ \\ &\text{for } i < j, \end{aligned} \quad (4.52)$$

where r_+ and $r_- := -\Pi r_+ \Pi$ are arbitrary solutions of the classical Yang-Baxter equation

$$[r^{12}, r^{23}] + [r^{12}, r^{13}] + [r^{13}, r^{23}] = 0. \quad (4.53)$$

From the quantum point of view, non-associativity of (4.49), (4.50) may be understood as a classical limit of the associated quasi-quantum groups [4].

- The previously found conserved quantities $\text{tr} B_j$, $\text{tr} B_j^2, \dots$ are embedded into the algebra of monodromy matrices via

$$M_j \sim e^{2\pi i B_j},$$

their trivial brackets corresponding to the trivial brackets of eigenvalues of monodromy matrices according to (4.49), (4.50).

- In fact, the structure (4.49), (4.50) has been calculated from the Poisson structure (4.30); the coset constraints (3.31) (except the first-class part (4.37)) have not yet been taken into account. Thus, an additional Dirac procedure is also required on the level of this algebra. The coset constraints in terms of monodromies translate into existence of a matrix C_0 with

$$M_j = C_0 \eta(M_{j+2n}) C_0^{-1}$$

The space of gauge invariant objects therefore admits a decomposition

$$M_S \oplus M_A$$

according to the involution η^∞ defined by $\eta^\infty(M_j) = C_0 \eta(M_{j+2n}) C_0^{-1}$. This involution is an automorphism of (4.49), (4.50), such that its invariant subspace M_S forms a closed subalgebra invariant also w.r.t. the Dirac procedure.

Abandoning our local point of view for the moment, let us briefly describe the relation of these observables found in the isomonodromic sector to the conserved charges that have been identified within the conventional formulation of the models. A simple consequence of the linear system (3.1) is that for asymptotically flat solutions the function $\hat{\mathcal{V}}$ becomes constant at spatial infinity:

$$\hat{\mathcal{V}}(w, x^0, x^1 \rightarrow \infty) \rightarrow \hat{\mathcal{V}}_\infty(w) \quad (4.54)$$

Its monodromies — for γ encircling γ_j or equivalently w encircling w_j — that we have discussed above, certainly survive this limit. We can now translate our quantities back into the Breitenlohner-Maison picture (cf. (3.8)) by $\hat{\mathcal{V}}_{BM} = S(w) \hat{\mathcal{V}}_{BZ}$. Making use of a result of [22] according to which this function tends to unity at spatial infinity, we arrive at the identification

$$\hat{\mathcal{V}}_\infty(w) = S(w)^{-1} \quad (4.55)$$

Thus, we have essentially succeeded in computing the algebra of the monodromies of the matrix $\mathcal{M}_{BM}(w)$, which plays the pivotal role in the approach of [22]. This suggests yet another route for comparing our Poisson structure to the canonical one: calculation of the canonical Poisson bracket between these conserved charges should lead to the same result. Unfortunately, the conventional approach got stuck before attaining this aim.

5 QUANTIZATION

5.1 General Remarks

The quantization of the models will be based on the reduced phase space description corresponding to the Weyl gauge (alias the light-cone gauge) (4.20) introduced in section 4.1. The WDW equation and diffeomorphism constraint of the covariant approach, which are functional differential equations, are thereby reduced to the partial differential equations (5.4) below, which we will simply refer to as the “quantum constraints”, and which encode the dynamical content of the model. There is, of course, the question whether by quantizing on the reduced phase space, possible anomalies of the constraint algebra of $2d$ diffeomorphisms (i.e. the Virasoro algebra) might have been swept under the carpet. However, just as in light-cone gauge string theory, such anomalies are not directly visible in this gauge, but would manifest themselves in some analog of the target space Lorentz algebra (whose identification in the present context would amount to a “stringy” interpretation of these models [3]).

Accordingly, we perform the textbook substitution

$$\{ \cdot, \cdot \} \longrightarrow \frac{1}{i\hbar} [\cdot, \cdot] \quad (5.1)$$

in all canonical brackets, converting all phase space variables into operators. Since $\rho^\pm = x^\pm + \rho_0^\pm$ in this gauge, we set

$$\Pi_\pm = i\hbar\partial_\pm \quad (5.2)$$

The constraints (4.26) are thereby transmuted into differential operator constraints acting on the quantum Hilbert space of the theory. The quantum states $\Phi \equiv \Phi(x^\pm)$ are space-time coordinate dependent elements of a Hilbert space \mathcal{H} to be specified below whose precise structure depends on the matter sector. *Physical states* are by definition those elements of \mathcal{H} which are annihilated by the constraint operators, i.e.

$$\Phi(x) \in \mathcal{H}_{phys} \iff \mathcal{C}_\pm(x)\Phi(x) = 0 \quad (5.3)$$

In other words, physical states must satisfy the differential equations

$$(i\hbar\partial_\pm + H_\pm)\Phi(x) = 0 \quad (5.4)$$

where we must now insert the Hamiltonians from (4.28). In addition, physical states may be subject to further constraints related to the local internal symmetries of the model. The two equations (5.4) resemble the time-dependent Schrödinger equation, as the matter Hamiltonians explicitly depend on the coordinates x^\pm . We stress that the operators on the l.h.s. of (5.4) are to be interpreted as total derivatives d/dx^\pm . In other

words, (5.4) is nothing but the statement that the solutions Φ should not depend of the coordinates; we thus have a rather simple realization of the idea that physical states in quantum gravity should be invariant under the full set of $2d$ coordinate transformations!

The difficult part in solving these equations is the analysis of the matter sector governed by the Hamiltonians H_{\pm} , which are non-trivial operators acting in the Hilbert space \mathcal{H} . By contrast, for the Einstein-Rosen waves and for the (closed) string the operators H_{\pm} are free field Hamiltonians, and the equations can be dealt with by standard Fock space methods; e.g. for the closed string, \mathcal{H}_{phys} would just be the Fock space of two sets of transverse oscillators, and the equations (5.4) would be solved by demanding $L_0 = \bar{L}_0$ and $M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} \bar{\alpha}_{-n}^i \bar{\alpha}_n^i - 2$ on the physical states. This path is obstructed here, because even in the flat space limit, the non-linear σ -model can at best be treated perturbatively with these methods. On the other hand, the methods borrowed from the theory of integrable systems, which we employ here, are essentially non-perturbative. Our results furthermore indicate that the underlying group theoretical structure will play an essential role in the further development of the subject. The rest of this section is therefore devoted to a discussion of the matter sector. As we will explicitly demonstrate, the central equations (5.4) can be reduced to the KZ equations. The recourse to techniques borrowed from conformal field theory thus leads to a substantial simplification of (5.4).

5.2 Canonical Quantization of the Matter Sector

As we explained before, the matter sector can be equivalently described in terms of $B(\gamma)$ or $\mathcal{B}(w)$, but for clarity of presentation we will restrict attention to the isomonodromic sectors (hence “isomonodromic quantization”). To this aim, let us immediately proceed to quantize the Poisson bracket (4.30). We arrive at the following commutation relations

$$[B^a(\gamma), B^b(\gamma')] = -i\hbar f^{ab}{}_c \frac{B^c(\gamma) - B^c(\gamma')}{\gamma - \gamma'} \quad (5.5)$$

or, equivalently,

$$[B_j^a, B_k^b] = i\hbar \delta_{jk} f^{ab}{}_c B_k^c \quad (5.6)$$

(remember that we suppress the x -dependence in these equations). The spectral parameter current and its residues have thus become operators on a Hilbert space, which for a given isomonodromic (“ N -wave”) sector is the direct product

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N \quad (5.7)$$

of N representation spaces of the Lie algebra \mathfrak{g} or its complexification $\mathfrak{g}_{\mathbb{C}}$, one for each set of operators $B_j \equiv B_j^a Z_a$. Note that every B_j is a matrix, all of whose entries are operators on \mathcal{H}_j .¹³ For obvious reasons we shall assume the spaces \mathcal{H}_j to be *unitary representation spaces*, since this will automatically lead to a positive definite scalar product in the Hilbert space associated with the reduced phase space quantization. As is well known the structure of unitary irreducible representations of the non-compact group G and its complexification $G_{\mathbb{C}}$ are very different. Moreover, if we are dealing with a given real form of a non-compact group, highest or lowest weight representations may or may not exist, depending on whether H contains a $U(1)$ factor [46, 47], with consequences for the spectrum of physical parameters.

¹³This is the poor man’s definition of a quantum group.

To illustrate these remarks we will mostly restrict attention in the remainder to the group $G = SL(2, \mathbb{R})$, whose representation theory is well understood [46]. Specializing the (isomonodromic) spectral parameter current to this case, we have

$$B(\gamma) = \frac{i\hbar}{2} \begin{pmatrix} \mathbf{h}(\gamma) & 2\mathbf{e}(\gamma) \\ 2\mathbf{f}(\gamma) & -\mathbf{h}(\gamma) \end{pmatrix} \iff B_j = \frac{i\hbar}{2} \begin{pmatrix} \mathbf{h}_j & 2\mathbf{e}_j \\ 2\mathbf{f}_j & -\mathbf{h}_j \end{pmatrix} \quad (5.8)$$

with

$$\mathbf{h}(\gamma) = \sum_{j=1}^N \frac{\mathbf{h}_j}{\gamma - \gamma_j} \quad , \quad \mathbf{e}(\gamma) = \sum_{j=1}^N \frac{\mathbf{e}_j}{\gamma - \gamma_j} \quad , \quad \mathbf{f}(\gamma) = \sum_{j=1}^N \frac{\mathbf{f}_j}{\gamma - \gamma_j} \quad (5.9)$$

The reality condition (3.28) translates into

$$\mathbf{h}(\gamma)^\dagger = -\mathbf{h}(\bar{\gamma}) \quad , \quad \mathbf{e}(\gamma)^\dagger = -\mathbf{e}(\bar{\gamma}) \quad , \quad \mathbf{f}(\gamma)^\dagger = -\mathbf{f}(\bar{\gamma}) \quad (5.10)$$

For real poles (cf. (3.34)), the entries of B_j become hermitean, and the operators $(\mathbf{h}_j, \mathbf{e}_j, \mathbf{f}_j)$ are just the anti-hermitean Chevalley generators of N mutually commuting $SL(2, \mathbb{R})$ groups, viz.

$$[\mathbf{h}_j, \mathbf{e}_k] = 2\delta_{jk}\mathbf{e}_j \quad , \quad [\mathbf{h}_j, \mathbf{f}_k] = -2\delta_{jk}\mathbf{f}_j \quad , \quad [\mathbf{e}_j, \mathbf{f}_k] = \delta_{jk}\mathbf{h}_k \quad (5.11)$$

Hence, each \mathcal{H}_j is a unitary representation space of $SL(2, \mathbb{R})$. For *complex* poles $\gamma_j(x)$ (cf. (3.36) and (3.37)), on the other hand, the operators $(\mathbf{h}_j, \mathbf{e}_j, \mathbf{f}_j)$ obey the same commutation relations as before, but

$$\mathbf{h}_j^\dagger = -\mathbf{h}_{j+n} \quad , \quad \mathbf{e}_j^\dagger = -\mathbf{e}_{j+n} \quad , \quad \mathbf{f}_j^\dagger = -\mathbf{f}_{j+n} \quad (5.12)$$

In this case, we are dealing with $\frac{1}{2}N$ mutually commuting $SL(2, \mathbb{C})$ groups. Readers may remember from the representation theory of the Lorentz group that $SL(2, \mathbb{C})$ possesses no discrete unitary representations at all. This is also evident from the fact that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(1, 3)$ does not decompose into a sum of two mutually commuting subalgebras, unlike $\mathfrak{so}(4)$ or $\mathfrak{so}(2, 2)$. Consequently, there is no escape from this difficulty.

To analyze the coset constraints we will only consider real poles to keep the discussion as simple as possible. We recall that $\eta(X) = -X^t$ for $SL(2, \mathbb{R})$; with the numbering of poles as in (3.34), we readily obtain

$$\mathbf{h}_j = -\mathbf{h}_{j+n} \quad , \quad \mathbf{e}_j = -\mathbf{e}_{j+n} \quad , \quad \mathbf{f}_j = -\mathbf{f}_{j+n} \quad (5.13)$$

The condition (3.35) reduces to

$$\sum_{j=1}^n (\mathbf{f}_j - \mathbf{e}_j) = 0 \quad (5.14)$$

In accord with our remarks in section 4.2, (5.13) should be regarded as second class constraints; indeed, (5.13) instructs us to eliminate all operators with index values $j > n$ in terms of the remaining ones. On the other hand, (5.14) is first class, in agreement with (4.37): it is just the canonical generator of the $SO(2)$ gauge transformations on the states. Therefore, physical states by definition must be annihilated by (5.14) and cannot carry any H charge.

For practical calculations it is oftentimes convenient to switch to the $SU(1,1)$ Chevalley basis

$$e_j := \frac{1}{2}(-i\mathbf{h}_j + \mathbf{e}_j + \mathbf{f}_j) \quad , \quad f_j := \frac{1}{2}(i\mathbf{h}_j + \mathbf{e}_j + \mathbf{f}_j) \quad , \quad h_j := i(\mathbf{f}_j - \mathbf{e}_j) \quad (5.15)$$

The main advantage of this basis is that the relations $h_j^\dagger = h_j, e_j^\dagger = -f_j$ allow us to diagonalize the operator h_j on the states and to interpret e_j and f_j as creation and annihilation operators. It should be emphasized that these operators have nothing to do with the conventional Fock space creation and annihilation operators of free particles, as they create and annihilate “collective excitations”. (5.13) now reads

$$h_j = -h_{j+n} \quad , \quad e_j = -e_{j+n} \quad , \quad f_j = -f_{j+n} \quad (5.16)$$

while (5.14) becomes

$$\left(\sum_{j=1}^n h_j \right) \Phi = 0 \quad \text{for} \quad \Phi \in \mathcal{H}_{phys} \quad (5.17)$$

5.3 Quantum Constraints and KZ Equations

We now return to section 3.5 where the classical constraints were solved for the conformal factor in terms of the τ -function, and show that this result has a precise quantum mechanical analog. This we do by reducing the quantum constraints (5.4) to a modified version of the KZ equations from conformal field theory [9]. In fact, disregarding the coset constraints (3.31) (i.e. here (5.16)), we would arrive precisely at the KZ equations, with the only difference that the worldsheet coordinates z_j labeling the insertions of conformal operators in the correlator are replaced by the movable singularities $\gamma_j(x)$ in the spectral parameter plane. In this fashion one can see that the quantum analog of the τ -function is just the physical state solving the quantum constraints (5.3); hence, it is quite appropriate to call $\Phi(x)$ the *quantum τ -function* (alternatively,¹⁴ one could reserve this name for the quantum mechanical evolution operator, which is a “matrix” whose columns consist of an orthonormal basis in the space of solutions of the KZ equations). However, due to the coset constraints, we obtain a slightly modified version of the KZ-system, that we refer to as the Coset-Knizhnik-Zamolodchikov (CKZ) system [4]. Since the techniques for solving these modified equations have not yet been elaborated, but might resemble the strategies followed in solving the usual KZ equations [48, 49], we first describe the quantization neglecting the coset constraints.

To prove the above assertions, we start from the ansatz

$$\Phi(x) = F(x)\tilde{\Phi}(\{\gamma_j(x)\}) \quad (5.18)$$

where $F(x)$ is an ordinary function and $\tilde{\Phi} \in \mathcal{H}$ by assumption depends on the coordinates only through the $\gamma_j(x)$. From (5.18) we get, using (3.3),

$$\partial_{\pm}\Phi = (\partial_{\pm}F)\tilde{\Phi} + \rho^{-1}\partial_{\pm}\rho F \sum_{j=1}^N \frac{\gamma_j(1 \mp \gamma_j)}{1 \pm \gamma_j} \frac{\partial\tilde{\Phi}}{\partial\gamma_j} \quad (5.19)$$

¹⁴As suggested to us by A.A. Morosov.

We next split this equation into two sets of equations, one for F and one for $\tilde{\Phi}$. To cut a long story short, we will assume the following equations to hold for $\tilde{\Phi}$

$$\frac{\partial \tilde{\Phi}}{\partial \gamma_j} = \sum_{k \neq j}^N \frac{\Omega_{jk}}{\gamma_j - \gamma_k} \tilde{\Phi} \quad (5.20)$$

which are just the famous KZ equations. To reconcile this ansatz with the original equations (5.4), we must make the identification

$$\Omega_{jk} = \frac{1}{i\hbar} \text{tr} B_j B_k \quad (5.21)$$

which e.g. for $SU(1, 1)$ leads to

$$\Omega_{jk} = i\hbar \left(\frac{1}{2} h_j \otimes h_k + e_j \otimes f_k + f_j \otimes e_k \right) \quad (5.22)$$

Substituting the ansatz (5.18) and (5.19) into (5.4), a little algebra shows that we can satisfy the constraint, provided that

$$\left\{ i\hbar \partial_{\pm} \log F + \rho^{-1} \partial_{\pm} \rho \text{tr} B_{\infty}^2 + \rho^{-1} \partial_{\pm} \rho \sum_j \text{tr} B_j^2 \left(\frac{1}{(1 \pm \gamma_j)^2} - \frac{1}{2} \right) \right\} \Phi = 0 \quad (5.23)$$

This equation still contains the operators $\text{tr} B_j^2$ and $\text{tr} B_{\infty}^2$, but it can be integrated in closed form if we assume that they act diagonally on the quantum state Φ . For the known solutions of the KZ equations, the validity of this assumption can be verified by explicit computation [3]. Designating the respective eigenvalues by $(i\hbar)^2 a_j$ and $(i\hbar)^2 a_{\infty}$ respectively, we arrive at the final result

$$\Phi(x) = \rho^{\frac{1}{2} i\hbar a_{\infty}} \prod_{j=1}^N \left(\frac{\partial \gamma_j}{\partial w_j} \right)^{\frac{1}{2} i\hbar a_j} \tilde{\Phi} \quad (5.24)$$

e.g. for $SL(2, \mathbb{R})$ we have $a_j = s_j(s_j - 2)$ where $s_j \in \{2, 3, 4, \dots\}$ is the (non-compact) spin of the j -th representation. Note the striking similarity of the formula (5.24) with its classical analog (3.44), which was already stressed at the beginning of this section.

Equation (5.24) thus expresses the physical state Φ solving (5.4) as a product of an explicitly computable function F and a solution of the KZ equation. There is a large body of literature on the KZ equations, and although most of this work is concerned with compact groups, explicit solutions based on the discrete representations of $SL(2, \mathbb{R})$ are known [48, 49]. It would thus appear that we simply have to insert these solutions into (5.24), and we would be done. However, we still have to take into account the coset constraints (5.16): as it turns out, unfortunately, the solutions given above (5.24) are not compatible with these extra conditions. In fact, because the constraints are second class the whole construction must be modified. Actually, the solutions of [48, 49] are already incompatible with the single constraint $B_{\infty} = 0$, as can be readily seen: to satisfy $B_{\infty} = 0$ or, equivalently, (5.17) (and thereby get rid of the first factor in (5.24)), we need $a_{\infty} = 0$; however, for the lowest weight unitary representations of $SL(2, \mathbb{R})$ used in [48, 49], we have $a_{\infty} = M + \sum_j s_j(s_j - 2)$ where M is a positive integer, and the constraint can never be satisfied because $s_j(s_j - 2) \geq 0$ (using highest weight representations instead just “inverts” the problem). The constraint

$B_\infty = 0$ could conceivably be satisfied with continuous representations, or alternatively, by simultaneous use of positive and negative (i.e. both highest *and* lowest weight) representations of $SL(2, \mathbb{R})$, but no solutions of the KZ equations of this type are presently known.¹⁵

To really solve the complete set of coset constraints, we must explicitly take into account the relations (5.16) (remember that we are only discussing the case of real poles and $B_j \in \mathfrak{g}$). This means that we express everything in terms of half of the variables. The quantum constraints (5.4) are modified accordingly. We just state the result, whose derivation is analogous to (5.24): if $\tilde{\Phi}$ is a solution to the following modified (Coset)-KZ system

$$\frac{\partial \tilde{\Phi}}{\partial \gamma_j} = \left\{ \sum_{k=1, k \neq j}^n \frac{1 + \gamma_k / \gamma_j}{\gamma_j - \gamma_k} \Omega_{jk} + \sum_{k=1}^n \frac{\gamma_k + 1 / \gamma_j}{\gamma_j \gamma_k - 1} \tilde{\Omega}_{jk} \right\} \tilde{\Phi} \quad (5.25)$$

with

$$\Omega_{jk} = \frac{1}{i\hbar} \text{tr} B_j B_k = i\hbar \left(\frac{1}{2} h_j \otimes h_k + e_j \otimes f_k + f_j \otimes e_k \right) \quad (5.26)$$

$$\tilde{\Omega}_{jk} = \frac{1}{i\hbar} \text{tr} \eta(B_j) B_k = -i\hbar \left(\frac{1}{2} h_j \otimes h_k + e_j \otimes e_k + f_j \otimes f_k \right), \quad (5.27)$$

then the quantum constraints (5.4) are solved by

$$\Phi(x) = \prod_{j=1}^n \left(\gamma_j^{-1} \frac{\partial \gamma_j}{\partial w_j} \right)^{i\hbar a_j} \tilde{\Phi} \quad (5.28)$$

The analogous formula was obtained in the classical theory (3.45). One obvious technical difficulty with (5.27) is that the operator $\tilde{\Omega}_{jk}$ does not preserve the excitation number unlike (5.22). This indicates that an ansatz of the type used in [48, 49] which starts from a state of fixed occupation number will no longer work, so entirely new techniques may be necessary to make progress with the above equations. We would also like to mention that for the “really interesting” higher rank non-compact groups (such as e.g. $E_{8(+8)}$ for $N = 16$ supergravity), the representation theory is much less developed [47]; for instance, although it is known that these groups do admit discrete unitary irreducible representations which are neither of highest nor lowest weight type, so far only preliminary results on their explicit form are available.

Finally, our results show that, in the reduction to two dimensions, the world-sheet itself has become a secondary object, while the complex spectral parameter γ emerges as the truly fundamental variable. In this sense the theory has become effectively one-dimensional. In fact, as recently shown in [28], the theory is not only generally covariant as a $2d$ world-sheet theory, but in addition admits a kind of general covariance w.r.t. to the spectral parameters γ and w as well. Its inherent quantum group structure has been mentioned several times in these lectures. At a more technical level it was discussed in section 4.4, where the quantization of the quadratic algebra of observables (4.50), (4.49) was shown to naturally imply a quantum group structure. Perhaps the emergence of a “quantum space time” (in the target space) might find a natural explanation in these theories.

¹⁵Let us mention, that for *compact* coset spaces such as $SU(2)/U(1)$ the constraint (5.17) can also be satisfied [2].

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APPENDIX: EXTENSION TO SUPERSYMMETRY

In this appendix we briefly describe the extension of the results given in the main body of these lectures to locally supersymmetric models, i.e. matter-coupled supergravities in two dimensions. In addition to the bosonic fields introduced in section 1, these models contain matter fermions as well as N gravitinos and dilatinos, which are the superpartners of the zweibein and the dilaton, where $N \leq 16$ is the number of local supersymmetries. The structure of the relevant supermultiplets has been discussed in [50, 51, 29], to which we refer the reader for further details. As shown there, the equations of motion and the associated linear system are considerably more complicated than for the bosonic theory. Our account will therefore be quite sketchy. To make life as simple as possible, we only consider the linear system of [50] where all terms containing gravitinos and dilatinos have been eliminated. The linear system can then be cast into a form analogous to (3.2):

$$\Psi^{-1}\partial_{\pm}\Psi = \left\{ \frac{2}{1 \pm \gamma} \widehat{P}_{\pm} + \frac{\gamma}{(1 \pm \gamma)^2} \widehat{R}_{\pm} \right\} \quad (\text{A.1})$$

where (cf. (2.22))

$$\widehat{P}_{\pm} \equiv \eta(\mathcal{V})P_{\pm}\eta(\mathcal{V}^{-1}) \quad , \quad \widehat{R}_{\pm} \equiv \eta(\mathcal{V})R_{\pm}\eta(\mathcal{V}^{-1}) \quad (\text{A.2})$$

with $P_{\pm} \equiv P_{\pm}^A Z_A \in \mathfrak{k}$ defined as before (cf. (2.3)), while $R_{\pm} \equiv R_{\pm}^{\alpha} Z_{\alpha} \in \mathfrak{h}$ is the following expression bilinear in the matter fermions

$$R_{\pm} := \bar{\chi}\gamma_{\pm}\Gamma(Z^{\alpha})\chi \cdot Z_{\alpha} \in \mathfrak{h} \quad (\text{A.3})$$

Here γ_{\pm} are the standard $2d$ γ -matrices, and the matter fermions χ belong to a spinorial representation of the gauge group H which is generated by the matrices $\Gamma(Z^{\alpha})$.

The presence of higher order poles in (A.1) necessitates a modification of the isomonodromic ansatz (3.10): in addition to the ‘‘movable’’ poles at $\gamma = \gamma_j(x)$ in (3.10) we must now allow for ‘‘rigid’’ poles at the branch points $\gamma = \pm 1$. The modified isomonodromic ansatz is given by

$$A(\gamma) := \Psi^{-1}\partial_{\gamma}\Psi = \sum_{j=1}^N \frac{A_j}{\gamma - \gamma_j} + \frac{A_+}{\gamma + 1} + \frac{A_-}{\gamma - 1} \quad (\text{A.4})$$

The residues will again be subject to certain reality and coset constraints which are implied by $P_{\pm} \in \mathfrak{k}$ and $R_{\pm} \in \mathfrak{h}$. We will not discuss these constraints here, save for remarking that the constraint

$$A_{\infty} \equiv \sum_{j=1}^N A_j + A_- + A_+ = 0 \quad (\text{A.5})$$

will be understood to hold.

Analyzing the singular terms at $\gamma = \pm 1$ a calculation completely analogous to the one leading to (3.13) now shows that we can again recover \widehat{R}_\pm and \widehat{P}_\pm in terms of the residues A_j and A_\pm ; the result is

$$\begin{aligned}\widehat{R}_\pm &= \pm 4\rho^{-1}\partial_\pm\rho A_\pm \\ \widehat{P}_\pm &= 2\rho^{-1}\partial_\pm\rho \left\{ \pm\frac{1}{2}(A_+ + A_-) + \sum_{j=1}^N \frac{A_j}{1 \pm \gamma_j} \right\}\end{aligned}\quad (\text{A.6})$$

The compatibility conditions of the linear system (A.1) with the supersymmetrized isomonodromic ansatz A.4 imply the following set of deformation equations:

$$\begin{aligned}\partial_\pm A_j &= \rho^{-1}\partial_\pm\rho \left\{ 2\sum_{k=1}^N \frac{[A_k, A_j]}{(1 \pm \gamma_k)(1 \pm \gamma_j)} + \frac{1}{1 \pm \gamma_j}[A_\mp, A_j] + \frac{1 \pm 3\gamma_j}{(1 \pm \gamma_j)^2}[A_\pm, A_j] \right\} \\ \partial_\pm A_\pm &= \rho^{-1}\partial_\pm\rho \left\{ \sum_j \frac{1 \pm 3\gamma_j}{(1 \pm \gamma_j)^2}[A_j, A_\pm] \pm [A_-, A_+] \right\} \\ \partial_\pm A_\mp &= \rho^{-1}\partial_\pm\rho \left\{ \sum_j \frac{1}{(1 \pm \gamma_j)^2}[A_j, A_\mp] \pm [A_+, A_-] \right\}\end{aligned}\quad (\text{A.7})$$

The Poisson structure can also be generalized. As in section 4.2 it is most conveniently written down in terms of

$$B(\gamma) := \eta(\mathcal{V}^{-1})A(\gamma)\eta(\mathcal{V}) \quad (\text{A.8})$$

where $A(\gamma)$ is given by (A.4). It turns out that bosonic brackets between B_j and B_k (4.33) remain unaltered, while for the new variables B_\pm we have to demand

$$\{B_\pm^a, B_\pm^b\} = f^{ab}{}_c B_\pm^c$$

Given this Poisson structure, the matter Hamiltonians H_\pm governing the evolution in the x^\pm -directions are

$$\begin{aligned}H_\pm &= \rho^{-1}\partial_\pm\rho \left\{ \sum_{k,j} \frac{\text{tr}B_j B_k}{(1 \pm \gamma_j)(1 \pm \gamma_k)} \right. \\ &\quad \left. + \sum_j \left[\frac{1}{1 \pm \gamma_j} \text{tr}B_j B_\mp + \frac{1 \pm 3\gamma_j}{(1 \pm \gamma_j)^2} \text{tr}B_j B_\pm \right] + \text{tr}B_+ B_- \right\}\end{aligned}\quad (\text{A.9})$$

The above results indicate that an exact quantization of the supersymmetric models may be possible. Generally one would expect that the best way to go about this task is to solve the supersymmetry constraints, i.e. the ‘‘square roots’’ of the bosonic constraints, but this may no longer be true if supersymmetry is ‘‘bosonized’’ as in (A.1). As usual, the first step of quantization would be the replacement of the above brackets by quantum commutators, such that in addition to the commutation relations (5.6) we get two more relations

$$[B_\pm^a, B_\pm^b] = i\hbar f^{ab}{}_c B_\pm^c \quad (\text{A.10})$$

The Hilbert space corresponding to a fixed isomonodromic sector would consequently contain two extra factors,

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N \otimes \mathcal{H}_+ \otimes \mathcal{H}_- \quad (\text{A.11})$$

where as before \mathcal{H}_j and \mathcal{H}_\pm are unitary representation spaces of \mathfrak{g} or $\mathfrak{g}_\mathbb{C}$. The bosonic quantum constraints are formally the same as in the bosonic theory, but with the new Hamiltonians (A.9).

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