

Electron-cyclotron absorption in high-temperature plasmas: quasi-exact analytical evaluation and comparative numerical analysis

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Abstract. On the basis of the electromagnetic energy balance equation, a quasi-exact *analytical* evaluation of the electron cyclotron (EC) absorption coefficient is performed for arbitrary propagation (with respect to the magnetic field) in a high-temperature (Maxwellian) plasma. The calculation makes use of Bateman's expansion for the product of two Bessel functions, retaining the lowest order contribution. The integration over electron momentum can be carried out analytically, fully accounting for finite Larmor radius effects in this approximation. On the basis of the analytical expressions for the EC absorption coefficients of both the extraordinary and ordinary mode thus obtained, (i) for the case of perpendicular propagation simple formulae are derived for both modes and (ii) a numerical analysis of the angular distribution of EC absorption is carried out, along with an assessment of the accuracy of asymptotic expressions existing in literature.

Introduction. To evaluate the electron cyclotron (EC) wave absorption coefficient is advantageous to make use of the electromagnetic energy balance, as this approach yields the absorption coefficient in a compact form wherefrom the effects of both the wave polarization and the finiteness of the (electron) Larmor radius are evidenced [1]. With reference, in particular, to the Larmor radius effects, described by Bessel functions the argument of which is just the Larmor parameter $b = \bar{\omega} N_{\perp} \bar{p}_{\perp}$ (the bar denotes that the frequency ω and the perpendicular (to the magnetic field) momentum p_{\perp} are normalized, respectively, to $\omega_c e mc$, and N_{\perp} is the perpendicular refractive index), one usually proceeds to a series expansion of the relevant Bessel functions in powers of b , retaining the lowest significant order terms [1]. Alternatively to this series expansion, one can make use of Bateman's expansion for the product of two Bessel functions [2], such a procedure having been employed to evaluate the EC absorption on the basis of the (numerical) solution of the dispersion relation [3]. Here, starting from the electromagnetic energy balance equation, we make use of Bateman's expansion to obtain quasi-exact *analytical* evaluation of the EC absorption coefficient. A numerical analysis of the analytical results thus obtained is also performed, along with an assessment of the accuracy of the asymptotic expressions for the EC absorption coefficient obtained by Robinson [4] and Trubnikov [5].

Quasi-exact analytical evaluation of the EC absorption coefficient. Starting from the expression of the absorption coefficient obtained on the basis of the

electromagnetic energy balance equation [1, Eq.(2.2.16)], we proceed as follows: i) the electron distribution function is assumed to be a (relativistic) Maxwellian; ii) the wave-polarization effects are evaluated in the cold plasma-limit [1]; iii) the integration over the perpendicular (to the magnetic field) component of the (electron) momentum \mathbf{p} is readily carried out by means of the δ -function occurring in the anti-Hermitian part of the dielectric tensor and connected with the (relativistic) EC resonance condition, $\gamma(\bar{p}_\perp, \bar{p}_\parallel) - N_\parallel \bar{p}_\parallel - n/\bar{\omega} = 0$ (where γ is the Lorentz factor; \mathbf{p} again is normalized to mc and the subscripts \perp and \parallel denote, respectively, perpendicular and parallel to the magnetic field; N_\parallel is the parallel refractive index; and n is the harmonic number); iv) as for the \bar{p}_\parallel -integration, on making the change of variable

$$\bar{p}_\parallel = \frac{\bar{\omega}}{n_0} \left\{ \frac{n}{n_0} N_\parallel + \left[\left(\frac{n}{n_0} \right)^2 - 1 \right]^{1/2} t \right\}, \quad n \geq n_0 \equiv \bar{\omega} \sqrt{1 - N_\parallel^2}, \quad N_\parallel^2 < 1, \quad (1)$$

($-1 \leq t \leq 1$), one can express the resulting t -integration in terms of derivatives of the function

$$i(x_n(N_\perp), y_n(N_\parallel)) \equiv \int_0^1 dt J_n^2(x_n(N_\perp) \sqrt{1-t^2}) \cosh(y_n(N_\parallel)t), \quad (2)$$

$$x_n(N_\perp) = \bar{\omega} N_\perp \left[(n/n_0)^2 - 1 \right]^{1/2}, \quad y_n(N_\parallel) \equiv (\mu/n_0)(N_\parallel/N_\perp)x_n(N_\perp) \quad (3)$$

with $\mu \equiv mc^2/T$, the variable $x_n(N_\perp)$ being related to the effects of finite Larmor radius (FLR); v) the integral (2) is carried out by making use of (the first form of) the Bateman's expansion [2],

$$x J_n^2(x) = g_n J_{2n+1}(2x) + O(J_{2n+5}(2x)), \quad g_n \equiv (2n+1)/(2^n n!)^2. \quad (4)$$

Retaining the first term of the expansion is accurate to within a few percent for $x \leq 6$, $n \geq 3$. In particular, one should note that the first term in (4) covers FLR effects up to the second order of a power series expansion for small argument of the Bessel functions, i.e., the FLR corrections in (4) have to do with FLR effects of fourth (and higher) order. The evaluation of the absorption coefficient based on using the first term of (4) is referred to as **quasi-exact** (QE). In this approximation the evaluation of the integral (2) yields

$$i(x, y) = \frac{\pi}{2} g_n \times \begin{cases} \frac{1}{x} |J_{n+1/2}(z)|^2, & z \equiv \frac{1}{2} (\sqrt{4x^2 - y^2} + iy), \\ \text{for } 4x^2 - y^2 > 0, \text{ i.e., (cf. 3), } 2n_0 > \mu N_\parallel / N_\perp \\ \\ \frac{1}{\sqrt{z^+ z^-}} I_{n+1/2}(z^+) I_{n+1/2}(z^-), & z^\pm \equiv \frac{1}{2} (y \pm \sqrt{y^2 - 4x^2}), \\ \text{for } y^2 - 4x^2 > 0, \text{ i.e., } 2n_0 < \mu N_\parallel / N_\perp. \end{cases} \quad (5)$$

The inequalities occurring in (5), respectively, characterize the propagation at large (upper entry) and small (lower entry) angles to the magnetic field.

As a result, one thus obtains the quasi-exact expression of the absorption coefficient

$$\alpha^{(i)}(N_{\parallel}) = \frac{\omega_p^2}{c\omega_c} \bar{\alpha}^{(i)}(N_{\parallel}), \text{ for both the ordinary } (i = O) \text{ and extraordinary } (i = X) \text{ mode,}$$

$$\bar{\alpha}^{(i)}(N_{\parallel}) = \sqrt{\frac{\pi}{2}} \frac{a(\mu)\mu^{5/2}}{n_0} \sum_{n \geq n_0} \left[\left(\frac{n}{n_0} \right)^2 - 1 \right]^{1/2} P_n^{(i)}(N_{\parallel}) e^{-\mu \left(\frac{n/n_0}{\sqrt{1-N_{\parallel}^2}} - 1 \right)}, \quad (6)$$

where $a(\mu) \equiv (\pi/2\mu)^{1/2} e^{-\mu} / K_2(\mu)$ is related to the normalization of the (relativistic) Maxwellian, while $P_n^{(i)}(N_{\parallel})$ is connected with the wave-polarization effects along with FLR effects and can be expressed as the sum of a number of integral terms each of which in turn can be written in terms of derivatives of the integral (2). For arbitrary propagation direction the expression for $P_n^{(i)}(N_{\parallel})$ is rather cumbersome; therefore, only its limiting form for **perpendicular** ($N_{\parallel} = 0$) propagation is given here, *viz.*,

$$P_n^{(O)}(N_{\parallel} = 0) = \frac{\pi}{2} g_n \frac{x_n}{\bar{\omega}^2 (N_{\perp}^{(O)})^3} (J_{n+1/2}^2 - J_{n-1/2} J_{n+3/2}) \quad (7)$$

for the O -mode, this result being the same as the one obtained on solving the corresponding dispersion relation [3], and

$$P_n^{(X)}(N_{\parallel} = 0) = \frac{\pi g_n}{(N_{\perp}^{(X)})^3 x_n} \left\{ (1 + A_{\perp})^2 \left(\frac{n}{\bar{\omega}} J_{n+1/2} \right)^2 - \frac{x_n}{\bar{\omega}} \left[2A_{\perp} \frac{n}{\bar{\omega}} J_{n+1/2} J_{n+3/2} + \right. \right. \\ \left. \left. + \frac{1}{2(n+1)} \frac{x_n}{\bar{\omega}} L_n \right] \right\} \quad (8)$$

$$L_n \equiv n \left(J_{n+1/2}^2 + J_{n-1/2} J_{n+3/2} \right) - \frac{2n+3}{n+2} \left(J_{n+3/2}^2 + J_{n+1/2} J_{n+5/2} \right) \quad (9)$$

for the X -mode. In (7), (8) and (9), the argument of the Bessel functions is $x_n = N_{\perp}^{(i)} (n^2 - \bar{\omega}^2)^{1/2}$, $N_{\perp}^{(i)}$ is the perpendicular (cold) refractive index of mode i , respectively, and A_{\perp} is given by the expression on the right-hand side of Eq.(3.1.14b) of [1] with $n \rightarrow \bar{\omega}$. In the limit of small FLR effects for which a lower-order series expansion of the Bessel functions is adequate, results (6)-(9) reduce to the well-known results [1].

Numerical analysis. For the case of **perpendicular** propagation, the quasi-exact (QE) absorption coefficient for both the O and X -mode (cf. Eqs.(6)-(9)) is shown in Fig.1a as a function of electron temperature T_e , for $\bar{\omega} = 5$, along with the asymptotic

results obtained by Robinson [4] and Trubnikov [5], as well as the exact result [6]. The relative error of the different approximations with respect to the corresponding exact value of the absorption coefficient is shown in Fig.1b. In particular, from Fig.1b it appears that (i) the QE result is quite accurate for both modes, underestimating the exact result by less than 1% ; (ii) for T_e between 25 and 90keV Robinson's asymptotic result underestimates the exact value of the absorption coefficient of the X -mode by less than 10% , whereas for the O -mode, it overestimates the exact value by less than about 25% for T_e between 15 and (more than) 100keV ; (iii) Trubnikov's asymptotic result is more (less) accurate than the corresponding Robinson's result for the O -mode (X -mode); (iv) both Trubnikov's and Robinson's asymptotic treatment tend to overestimate the absorption by more than 25% for temperature $T_e \lesssim 15keV$, i.e., for temperature for which the absorption is weak, cf. Fig.1a.

As for the propagation at an **arbitrary angle** with respect to the magnetic field, the QE result for the angular distribution of the absorption at $\bar{\omega} = 5$ is shown in Fig.2a for the X -mode, for electron temperatures $T_e = 30keV$ and $40keV$, along with the corresponding asymptotic results obtained by Robinson [4] and Trubnikov [5], (see also Ref. [7]). The curves labelled “QE” refer to the absorption coefficient $2\text{Im}k$ (normalized to $\omega_p^2 / c\omega_c$) as obtained from the numerical solution of the (quasi-exact) dispersion relation [3]. The relative deviation of both the QE result and the asymptotic results with respect to the “QE” result is shown in Fig.2b. The analogous numerical analysis for the O -mode is given in Fig.3. In particular, for the X -mode, from Fig.2b it appears that (i) the QE analytical absorption coefficient obtained from the electromagnetic energy balance is practically identical to the one obtained by solving numerically the dispersion relation; (ii) for almost perpendicular propagation, i.e., for the angular range $75^\circ \lesssim \theta \leq 90^\circ$ where the absorption is strongest, the accuracy of Robinson's asymptotic result is better than 10% for $T_e \geq 40keV$ and better than Trubnikov's approximation $\theta \gtrsim 83^\circ$ (see also Fig.1b); (iii) over most of the angular range, namely, for $10^\circ \lesssim \theta \lesssim 75^\circ$, Trubnikov's asymptotic result is better than Robinson's, the accuracy of the former being better than 10% , whereas Robinson's result becomes quite inaccurate specifically for propagation at small angles, typically, for $\theta \lesssim 20^\circ$. Turning now to the O -mode, from Fig.3a it appears that the characteristic non-monotonous angular distribution of the O -mode absorption is accounted for by both the QE and “QE” evaluation, as well as by Robinson's asymptotic treatment. Trubnikov's asymptotic result, instead, underestimates the O -mode absorption significantly, with the only exception of perpendicular propagation, cf. Fig.1b, and fails to reproduce the non-monotonous behaviour of the angular distribution [7]. As it appears from Fig.3b, (i) the QE absorption coefficient again agrees very well with the “QE” value obtained from solving the dispersion relation [3]; (ii) Robinson's asymptotic evaluation is accurate by better than about 20% over most of the angular range, with the exception of the propagation at small angles ($\theta \lesssim 25^\circ$), for which it overestimates the O -mode absorption by more than 30% .

Concluding remarks. In conclusion, the quasi-exact (QE) analytical result presented here provides an excellent approximation to the EC absorption coefficient of a Maxwellian plasma. Notwithstanding the lower accuracy of both Robinson's and Trubnikov's asymptotic results compared to the QE treatment, the noticeable advantage of these asymptotic expressions for the absorption coefficient is that they are free from the sum over harmonics [7], as present in the QE absorption coefficient. Since, overall, Robinson's asymptotic form is a quite reasonable approximation to the EC absorption coefficient in most of the parameter range of importance for evaluating the total EC wave power flux (effectively, a better one than Trubnikov's form), this form can be expected to be a useful starting point for calculating the effective EC wave power loss from a large hot plasma (cf. [8]). This fact has been confirmed by solving the radiative transfer equation for fusion plasma parameters, showing in particular that applying this approach calculation times are reduced typically by two orders of magnitude with respect to using the QE form (and to the "QE" approach) [9].

Figure captions

FIGURE 1A. The (normalized) quasi-exact (QE) absorption coefficient of both the X -mode (full line) and O -mode (dashed line) (QE-curves), for perpendicular propagation, Eqs. (6)-(9), as a function of the electron temperature T_e , for $(\omega_p / \omega_c)^2 = 0.1$ and $\bar{\omega}(\equiv \omega / \omega_c) = 5$. Also shown are the exact result (E-curves) and the asymptotic results of both Robinson (R-curve) and Trubnikov (T-curve).

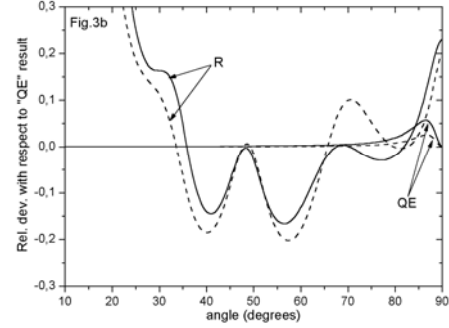
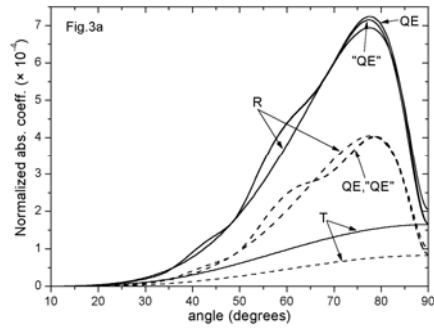
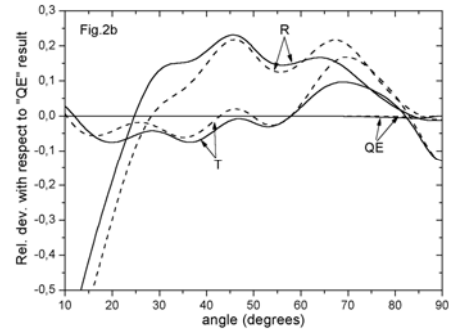
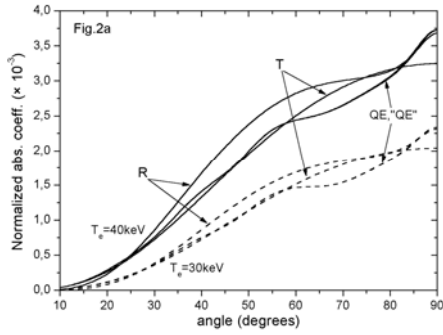
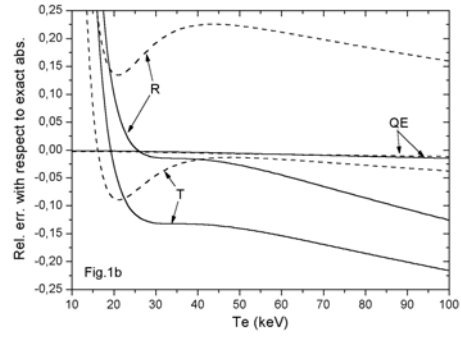
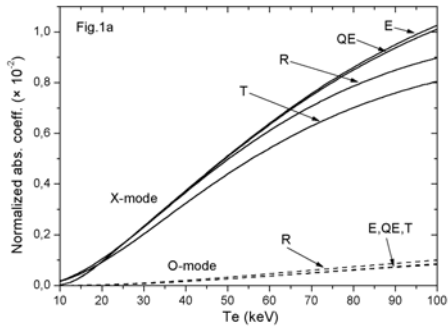
FIGURE 1B. Relative error $\Delta \equiv (\alpha_A - \alpha_E) / \alpha_E$ of the absorption coefficient of both X (full curve) and O (dashed curve) -mode (the subscripts "A" (=QE, R and T) and "E" are the same as the labels in Fig.1a).

FIGURE 2A. Quasi-exact (QE) absorption coefficient of the X -mode as a function of the angle of propagation with respect to the magnetic field, for $(\omega_p / \omega_c)^2 = 0.1$, $\bar{\omega}(\equiv \omega / \omega_c) = 5$ and electron temperatures $T_e = 30keV$ (dashed curve) and $40keV$ (full curve). Also shown is the absorption coefficient $2 \text{Im}k$ (normalized to $\omega_p^2 / c\omega_c$), "QE"-curve, as given by the solution of the dispersion relation, as well as the asymptotic results of both Robinson (R-curve) and Trubnikov (T-curve).

FIGURE 2B. Relative deviation of both the QE results and Robinson's and Trubnikov's asymptotic results for the absorption coefficient of the X -mode with respect to the "QE" result, for the same parameters as Fig.2a.

FIGURE 3A. The same as Fig.2a for the O -mode.

FIGURE 3B. The same as Fig.2b for the O -mode (Trubnikov's asymptotic result being omitted).



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