AEI-2004-122

On the existence of initial data containing isolated black holes

Sergio Dain,[1,*] José Luis Jaramillo,[2,†] and Badri Krishnan[1,‡]
[1] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany
[2] Laboratoire de l'Univers et de ses Théories, UMR 8102 du C.N.R.S., Observatoire de Paris, F-92195 Meudon Cedex, France
(Dated: DRAFT VERSION; Date: 2004/12/13 18:05:54)

We present a general construction of initial data for Einstein's equations containing an arbitrary number of black holes, each of which is instantaneously in equilibrium. Each black hole is taken to be a marginally trapped surface and plays the role of the inner boundary of the Cauchy surface. The black hole is taken to be instantaneously isolated if its outgoing null rays are shear-free. Starting from the choice of a conformal metric and the freely specifiable part of the extrinsic curvature in the bulk, we give a prescription for choosing the shape of the inner boundaries and the boundary conditions that must be imposed there. We show rigorously that with these choices, the resulting non-linear elliptic system always admits solutions.

PACS numbers: 04.80.Nn, 04.30.Db, 95.55.Ym, 07.05.Kf

I. INTRODUCTION

In this paper we consider the problem of specifying quasi-equilibrium multi-black hole initial data for the vacuum Einstein equations. This problem is important as a starting point for the numerical simulations of binary black hole spacetimes. The initial data should be such that the black holes are in equilibrium with time independent horizon geometries. Furthermore, the entire geometry of the spatial slice should, in an appropriate sense, be in quasi-equilibrium with minimal spurious radiation content. If both these criteria are satisfied, and if the black holes are in a roughly circular orbit around each other, we could reasonably expect the calculated gravitational waveforms to correspond to observations by gravitational wave detectors.

The problem of finding such initial data has received a lot of attention in the numerical relativity literature and significant progress has been made in the last few years. The original numerical work addressing this issue was due to Cook in 2002 [1] who, working in the so-called conformal thin sandwich (CTS) decomposition [2] of the initial data, proposed a set of conditions for solving the initial value problem subject to the quasi-equilibrium conditions. More recent developments in this direction can be found in [3, 4]. See Cook [5] and Gourgoulhon *et al.* [6, 7] for an earlier numerical study of binary black hole initial data which is in quasi-equilibrium in the bulk. The case of quasi-equilibrium configurations in presence of matter is discussed in e.g. [8]. See also [9] for an approach to this problem based on the post-Newtonian expansion.

Independently of the numerical work, on the analytical side, a quasi-local approach to black hole physics was developed by Ashtekar *et al.* (see e.g. [10, 11, 12, 13]). This work lead to the notion of an isolated horizon which models a black hole in equilibrium in an otherwise dynamical spacetime. While isolated horizons are defined in the full four dimensional spacetime, it would seem natural that this framework should also have implications for the construction of quasi-equilibrium initial data. This issue was studied in some detail by Jaramillo *et al.* [14]. They used the isolated horizon formalism, also working in the CTS decomposition, to arrive at a set of boundary conditions for the constrained parameters of the initial data which is similar to that obtained by Cook, but with certain additional constraints on the boundary values of the otherwise free data. While the initial results are promising in both approaches, it is not yet settled if they will finally succeed.

The aim of this work is to point out some potential mathematical difficulties with the approaches of [1, 3, 14] and to suggest a resolution of these difficulties. Our aim is less ambitious than [1, 3, 14] in the sense that we only consider the issue of the appropriate boundary conditions at the horizon: i) we do not make any statement about the conditions that must be satisfied in the bulk to ensure that the entire data set is in quasi-equilibrium and ii) we do not discuss gauge conditions for the evolution equations. Because we do not address i) and ii), we will work with the standard conformal transverse-traceless (CTT) decomposition of the initial data [15] and not with the CTS as in [1, 3, 14], because the former simplify the discussion. However we expect our main results to be relevant also in other decompositions including the CTS.

The conceptual issues have to do with the boundary conditions for the momentum constraint. All the decompositions involve an elliptic equation for a vector β^a. The references [1, 3, 14] all use a Dirichlet condition for β^a which adapts the time evolution vector to the properties of the horizon. This requirement becomes intertwined with the initial data construction in the CTS decomposition since the latter can be interpreted to involve also the notion of time evolution. However, strictly speaking, the initial data construction is distinct from the choice of gauge for time evolution. More importantly, as al-

*Electronic address: dain@aei.mpg.de
†Electronic address: jose-luis.jaramillo@obspm.fr
‡Electronic address: badri.krishnan@aei.mpg.de

ready pointed out in [14], a Dirichlet condition for β^a entails potential problems for the very existence of solutions for the Hamiltonian constraint. We will show that a new type of geometric boundary condition for β^a can be used, which allows us to apply the theorems proved by Dain and Maxwell [16, 17] in order to get a rigorous proof of existence of solutions for the resulting nonlinear equations under appropriate assumptions. These assumptions, which involve restrictions on the shape of the inner boundaries and some inequalities the free data should satisfy, can be checked *a priori*. The advantage of this analysis is that the solution is guaranteed to exist in more general cases than the ones studied numerically so far; in particular, in non conformally flat cases. This is important for two reason: first, the conformal geometry of a quasi-equilibrium black hole initial data is still unknown (though it is clear that it cannot be conformally flat because the Kerr metric does not admit conformally flat slices [18]). Second, one is usually interested not just in a single solution, but in families of solutions which depend smoothly on the relevant parameters of the problems, like separation distances, individual spins and linear momentum, etc. For example, one notion of quasi-equilibrium is defined as a variational problem for a family of solutions (cf. [1] and also [19] and references therein). Thus, we would like to ensure that the initial value equations admit solutions for the widest possible range of parameters and free data.

Section II sets up notation and states the mathematical problem we want to solve. Section III gives the main result and compares it to earlier work, section IV consists of a mathematical proof of the main result, and section V summarizes our results and suggests directions for future work.

II. THE CONFORMAL METHOD AND NON-EXPANDING HORIZONS

The problem we want to solve is to find initial data on a spatial slice M for the vacuum Einstein's equations. Thus, we want to find a 3-metric $\tilde{h}_{ab}$ and a second fundamental form

$$\tilde{K}_{ab} := -\tilde{h}_a^c \tilde{h}_b^b \nabla_c \tau_d = -\frac{1}{2}\mathcal{L}_\tau \tilde{h}_{ab}\,, \tag{2.1}$$

such that the constraints are satisfied:

$$\tilde{D}_a(\tilde{K}^{ab} - \tilde{K}\tilde{h}^{ab}) \;=\; 0\,, \tag{2.2}$$

$$\tilde{\mathcal{R}} + \tilde{K}^2 - \tilde{K}_{ab}\tilde{K}^{ab} \;=\; 0\,. \tag{2.3}$$

Here τ^a is the unit timelike normal to M, and ∇_a is the four dimensional covariant derivative. Our sign convention for $\tilde{K}_{ab}$ corresponds to what is commonly used in the numerical relativity literature. $\tilde{D}_a$ is the derivative operator compatible with $\tilde{h}_{ab}$, and $\tilde{\mathcal{R}}$ is its scalar curvature. We restrict ourselves to maximal slices, i.e. we always take $\tilde{K} = 0$. It is important to note that very little is known about solutions of the constraint equations (with or without inner boundaries) in the case when $\tilde{K}$ is not nearly constant (see the recent review [20] and references therein).

The metric and the traceless part of the extrinsic curvature are conformally rescaled

$$\tilde{h}_{ab} = \psi^4 h_{ab}\,, \qquad \tilde{K}^{ab} = \psi^{-10} K^{ab}\,. \tag{2.4}$$

As a rule, physical tensors on M will be denoted with a tilde to distinguish them from the corresponding conformally rescaled quantities (note the opposite convention with respect to Refs. [1, 3, 14]). In terms of the conformally rescaled quantities (h_{ab}, K_{ab}), the constraint equations become

$$D_a K^{ab} \;=\; 0, \tag{2.5}$$

$$L_h \psi \;=\; -\frac{1}{8} K_{ab} K^{ab} \psi^{-7}, \tag{2.6}$$

where L_h is the conformally invariant Laplacian operator: $L_h = \Delta - \mathcal{R}/8$; $\Delta := D_a D^a$ is the Laplacian, D_a is the derivative operator compatible with h_{ab}, and $\mathcal{R}$ is its scalar curvature. To solve the momentum constraint, we decompose (the traceless) K_{ab} according to York's prescription [15]:

$$K_{ab} = (\mathcal{L}\beta)_{ab} - Q_{ab}, \tag{2.7}$$

where β^a is a vector field on M, $\mathcal{L}$ is the conformal Killing operator defined as

$$(\mathcal{L}\beta)_{ab} \equiv 2D_{(a}\beta_{b)} - \frac{2}{3} h_{ab} D_c \beta^c, \tag{2.8}$$

and Q_{ab} is a freely specifiable symmetric and traceless tensor. The decomposition of K_{ab} given by equation (2.7) is known as the conformal transverse-traceless decomposition (CTT) but it is not the only possibility. In fact, the currently more commonly used decomposition in numerical relativity is the so called conformal thin-sandwich (CTS) decomposition also proposed originally by York [2].

Using the CTT decomposition, the momentum constraint (2.5) becomes

$$\Delta_L \beta^a = J^a\,. \tag{2.9}$$

Here we have defined

$$\Delta_L \beta^a := D_b(\mathcal{L}\beta)^{ab} = \Delta\beta^a + \frac{1}{3} D^a D_b \beta^b + \mathcal{R}_b^a \beta^b \tag{2.10}$$

and $J^a := D_b Q^{ab}$. Thus, we have to solve the system of elliptic equations (2.6) and (2.9) on a domain $M \subset \mathbb{R}^3$ subject to certain boundary conditions which will occupy us for the rest of this paper.

We wish to solve equations (2.6) and (2.9) on a domain M which is $\mathbb{R}^3$ with an arbitrary finite number of compact sets excised from it. The inner boundary will be denoted by S, and in general, it is allowed to have

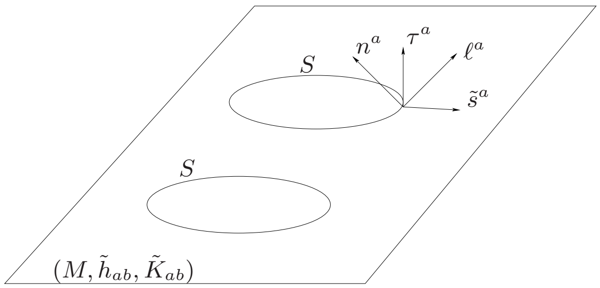

FIG. 1: The inner boundary S of the Cauchy surface M. In this figure the inner boundary consists of two disconnected spherical surfaces.

many disconnected components. The excision surface S serves as an inner boundary for our problem and it represents the surface of the black hole. This is understood to mean that each connected component of S is an outer marginally trapped surface, i.e. when it is viewed as a closed two surface in the full four-dimensional spacetime manifold, the congruence of future directed outgoing null geodesics orthogonal to it has zero expansion. This is depicted in figure 1 which shows the Cauchy surface M with inner boundary S. The unit spacelike normal to S (with respect to the physical metric $\tilde{h}_{ab}$ and pointing towards spatial infinity) is $\tilde{s}^a$, the unit timelike normal to M is τ^a, and

$$\ell^a := \tau^a + \tilde{s}^a \tag{2.11}$$

is a fiducial outgoing null normal to S. The ingoing null normal to S is

$$n^a = \tau^a - \tilde{s}^a \,. \tag{2.12}$$

The conformally rescaled normal s^a is defined as

$$\tilde{s}^a = \psi^{-2} s^a \,. \tag{2.13}$$

The induced physical 2-metric on S is denoted by $\tilde{q}_{ab}$ and it is conformally rescaled as

$$\tilde{q}_{ab} = \psi^4 q_{ab} \,. \tag{2.14}$$

The projection operator onto S is $\tilde{q}_a^b = q_a^b$. The second fundamental form of the inner boundary is defined by

$$\tilde{k}_{ab} = \tilde{q}_a^c \tilde{q}_b^d \tilde{D}_c \tilde{s}_d. \tag{2.15}$$

We denote by $\tilde{k}$ the trace and by $\tilde{k}^0_{ab}$ the trace-free part of $\tilde{k}_{ab}$, that is

$$\tilde{k}_{ab} = \tilde{k}^0_{ab} + \frac{1}{2}\tilde{k}\tilde{q}_{ab}, \quad \tilde{k} = \tilde{q}^{ab}\tilde{k}_{ab}. \tag{2.16}$$

With this notation, we can write down the expression for the expansion Θ_+ of ℓ^a:

$$\begin{aligned} \Theta_+ &= \tilde{q}^{ab}\nabla_a \ell_b \\ &= \tilde{K}_{ab}\tilde{s}^a\tilde{s}^b - \tilde{K} + \tilde{k}. \end{aligned} \tag{2.17}$$

In terms of the conformally rescaled fields and using again $\tilde{K} = 0$, this is equivalent to

$$\Theta_+ = \psi^{-3}\left(4s^a D_a\psi + \psi k + \psi^{-3} K_{ab}s^a s^b\right) \,. \tag{2.18}$$

Our first requirement on S is that the expansion Θ_+ must vanish. This is also known as the apparent horizon boundary condition in the literature. The problem of constructing initial data satisfying this boundary condition has already been studied [16, 17]. This boundary condition does not place any restrictions on the physical parameters of the black hole. We wish to impose the condition that the black hole is isolated in a certain specific sense.

Our notion of S being isolated is based on the isolated horizon framework [10, 11, 12, 21]. This is a quasi-local framework to study black holes without relying on the global notion of an event horizon; see [13] for a general review. It has found many applications in black hole mechanics, mathematical relativity, quantum gravity, and also in numerical relativity [14, 21]. An isolated horizon can be viewed as the world tube $\mathcal{T}$ of marginally trapped surfaces obtained by time evolution, in the case when $\mathcal{T}$ is null. In this case, it can be shown that the area of the black holes is constant in time and also the flux of gravitational waves falling into the black hole vanishes [22, 23, 24]. In the general case, $\mathcal{T}$ is expected to be spacelike, thereby representing a dynamical black hole. This is described by a dynamical horizon [23, 24]. Trapping horizons [22, 25] can describe the null and spacelike cases in a unified framework.

For our purposes, we do not need the details regarding isolated horizons. We only need to know that in vacuum, the condition for the world tube being null is equivalent to the vanishing of the *shear* σ_{ab} of ℓ^a [21, 22]. This is essentially a consequence of the Raychaudhuri equation whose expression, for a future directed geodesic ℓ^a with affine parameter λ, is

$$\frac{d\Theta_+}{d\lambda} = -\frac{1}{2}\Theta_+ - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}\ell^c\ell^d \,. \tag{2.19}$$

Here ω_{ab} is the twist of ℓ^a and R_{ab} is the spacetime Ricci tensor; note that $R_{ab}\ell^a\ell^b = 0$ in vacuum. If Θ_+ is initially zero, and taking the null rays to be surface forming ($\omega_{ab} = 0$), then a non-zero shear would imply that Θ_+ will be non-zero at a later time. Thus the apparent horizon cannot evolve along ℓ^a, and the black hole will not be isolated; see [22, 24] for further details and discussion. Conversely, if we take S to evolve strictly along ℓ^a so that $\mathcal{T}$ is null and Θ_+ is zero at all times, we get the condition that the shear vanishes identically, which in turn can be shown to imply that the rate of change of the area of S is zero, and S can be considered to be isolated. In the presence of matter, we must impose, say, the null energy condition to get analogous results. The condition $\sigma_{ab} = 0$ has also been considered independently by Cook [1]. In the language of the isolated horizon framework, this condition would guarantee that S is a cross-section of an infinitesimal *non-expanding horizon* [11].

The shear of ℓ^a is defined as the tracefree part of the projection of $\nabla_{(a}\ell_{b)}$ onto S:

$$\sigma_{ab} = \tilde{q}_a^c \tilde{q}_b^d \nabla_{(c}\ell_{d)} - \frac{1}{2}\Theta_+ \tilde{q}_{ab} \,. \tag{2.20}$$

In terms of the physical fields it is given by

$$\sigma_{ab} = -\tilde{q}_a^c \tilde{q}_b^d \tilde{K}_{cd} + \frac{1}{2}\tilde{q}_{ab}\tilde{q}^{cd}\tilde{K}_{cd} + \tilde{k}^0_{ab} \,, \tag{2.21}$$

and, in terms of the conformally rescaled fields, the shear is written as

$$\sigma_{ab} = \psi^{-2}\left(-q_a^c q_b^d K_{cd} + \frac{1}{2}q_{ab}q^{cd}K_{cd}\right) + \psi^2 k^0_{ab} \,. \tag{2.22}$$

Here k_{ab} is the extrinsic curvature of S embedded in M with respect to the conformal metric h_{ab}

$$k_{ab} = q_a^c q_b^d D_c s_d , \tag{2.23}$$

and, as before,

$$k_{ab} = k^0_{ab} + \frac{1}{2}k q_{ab}, \quad k = q^{ab}k_{ab}. \tag{2.24}$$

Our task is to prescribe appropriate boundary conditions for (ψ, β^a) so that equations (2.6) and (2.9) have a regular solution with ψ everywhere positive, and $\Theta_+ = 0$, $\sigma_{ab} = 0$ at S.

III. THE MAIN RESULT

A. The sign of $K_{ab}s^a s^b$

Let us start with the condition $\Theta_+ = 0$ which, from equation (2.18), can be written as

$$4s^a D_a\psi = -\psi k - \psi^{-3}K_{ab}s^a s^b \,. \tag{3.1}$$

We would like to use this as the boundary condition for solving equation (2.6) for ψ, together with the standard condition at infinity

$$\lim_{r\to\infty}\psi = 1. \tag{3.2}$$

Let us first discuss some general properties of Eqs. (2.6), (3.1) and (3.2).

We begin with the physical meaning of the sign of $K_{ab}s^a s^b$. In a realistic collapse, the boundary is expected to not only satisfy $\Theta_+ = 0$, but also to be a future marginally trapped surface, that is $\Theta_- \leq 0$, where Θ_- is the expansion of the ingoing null-normal n^a defined in Eq. (2.12). For example, a surface with $\Theta_+ = 0$ and $\Theta_- > 0$ is not expected to be present in a realistic gravitational collapse of matter; these surfaces are located on the inner null boundary of the left quadrant of the Kruskal diagram (region IV of Fig. 6.9 in [26]). The expansion Θ_- is given by

$$\begin{aligned}\Theta_- &= -\tilde{K} + \tilde{K}_{ab}\tilde{s}^a\tilde{s}^b - \tilde{k} && (3.3)\\ &= \psi^{-3}\left(-4s^a D_a\psi - \psi k + \psi^{-3}K_{ab}s^a s^b\right), && (3.4)\end{aligned}$$

where the second line applies if the data is maximal ($\tilde{K} = 0$).

Under this assumption, we get that $\Theta_+ + \Theta_- = 2\tilde{K}_{ab}\tilde{s}^a\tilde{s}^b$, then for a future marginally trapped surface on a maximal slice we always have

$$K_{ab}s^a s^b \leq 0. \tag{3.5}$$

Also, as mentioned below, it turns out that we also need to control the sign of $K_{ab}s^a s^b$ in order to provide necessary conditions for the existence of solutions of Eqs. (2.6), (3.1) and (3.2).

On the other hand, if we were to impose purely Dirichlet boundary conditions on β^a, then we could not guarantee that $K_{ab}s^a s^b$ has a definite sign at the boundary. This is simply because equation (2.7) implies that $K_{ab}s^a s^b$ involves radial derivatives of β^a which cannot be controlled by Dirichlet conditions:

$$K_{ab}s^a s^b = (\mathcal{L}\beta)_{ab}s^a s^b - Q_{ab}s^a s^b \,. \tag{3.6}$$

This conclusion is also valid for the conformal thin-sandwich decomposition. This is because also in the CTS decomposition, the extrinsic curvature is written in terms of the derivative of β^a and $\tilde{K}_{ab}\tilde{s}^a\tilde{s}^b$ will again involve radial derivatives of β^a.

We would like to prescribe $(\mathcal{L}\beta)_{ab}s^a s^b$ as free data on S and, at the same time, be able to enforce $\sigma_{ab} = 0$ on S. The main result of this paper is that this is indeed possible.

B. Existence of solutions to the Hamiltonian constraint

Let us now review the condition telling us about the existence of solutions to equation (2.6) with boundary conditions (3.1) and (3.2). The conformal factor ψ appears in the denominator of both equations (3.1) and (2.6), hence these equations are singular if the conformal factor vanishes, which is in agreement with the fact that the physical metric $\tilde{h}_{ab}$ does not make sense if the conformal factor is zero at some point. For general conformal metrics and boundaries, these equations will have no solutions. To illustrate this, let us consider the time symmetric case ($K_{ab} = 0$). For the case with no inner boundaries, necessary and sufficient conditions on the conformal metric h_{ab} for the solvability of Eq. (2.6) and (3.2) has been studied in [27] and [16]. This condition can be written in term of the following quantity known as the Yamabe invariant[1]

$$\lambda_h = \inf_{\varphi \in C_c^\infty(M),\, \varphi \not\equiv 0} \frac{\int_M (8 D_a\varphi D^a\varphi + \mathcal{R}\varphi^2)}{||\varphi||^2_{L^6}}, \tag{3.7}$$

where $C_c^\infty(M)$ denotes functions with compact support in M, $||\varphi||^2_{L^6} = (\int_M |\varphi|^6)^{1/6}$, and $\varphi \not\equiv 0$ means that φ cannot be identically zero everywhere. The number λ_h is conformally invariant in the following sense: let θ be any smooth and positive function, and let $\hat{h}_{ab} = \theta h_{ab}$; then $\lambda_{\hat{h}} = \lambda_h$. The Yamabe invariant has a long history. It was discovered by Yamabe for compact manifolds and later studied by many authors; see, for example, the review [28].

A solution of (2.6)–(3.2) (with $K_{ab} = 0$) exists if and only if $\lambda_h > 0$. Every metric with $\mathcal{R} \geq 0$ (the flat metric, for example) satisfies this condition. It is obvious from (3.7) that $\mathcal{R} \geq 0$ implies $\lambda_h \geq 0$, however, to prove that in fact $\lambda_h > 0$ is non trivial. However, there exist metrics which do not satisfy $\lambda_h > 0$ (see [27] for an explicit, axisymmetric example). For these conformal metrics there are no solutions of (2.6)–(3.2).

If we include inner boundaries, there exists a generalization of this condition (cf. [16]) in terms of the following conformal invariant which is a generalization of the Yamabe invariant:

$$\lambda_{h,S} = \inf_{\varphi \in C_c^\infty(M),\, \varphi \not\equiv 0} \frac{\int_M (8 D_a\varphi D^a\varphi + \mathcal{R}\varphi^2) - 2\oint_S k\varphi^2}{||\varphi||^2_{L^6}}. \tag{3.8}$$

This invariant has been studied also for the compact case in [29]. We have the following result proved in [16]. A solution of Eqs. (2.6), (3.1) and (3.2), with $K_{ab} = 0$, exists if and only if $\lambda_{h,S} > 0$. Note that now $\lambda_{h,S}$ depends on the choice of the boundary. In particular, note that the boundary term in (3.8) has a minus sign.[2] It can be proved that any metric with $\mathcal{R} \geq 0$ and boundary with $k \leq 0$ have $\lambda_{h,S} > 0$ [16]. As before, in this case it follows directly from the definition that $\lambda_{h,S} \geq 0$, but to prove that it is strictly positive requires extra work. Note that this does not apply to the flat metric with spheres as boundaries, since for spheres of radius r_0 we have $k = 2/r_0 > 0$. However, the flat metric with an arbitrary number of non-intersecting spheres as inner boundaries satisfies $\lambda_{h,S} > 0$, provided they are not too close to each other. This can be seen as follows. Assume that we have only one sphere of radius r_0 centered at the origin. Take the conformal factor $\theta = 1 + r_0/r$, and define the rescaled metric $\hat{h}_{ab} = \theta^4\delta_{ab}$ (where δ_{ab} is the flat metric). This is, of course, the Schwarzschild initial data. This metric satisfies $\hat{R} = 0$ and the boundary $r = r_0$ is a minimal surface with respect of the metric $\hat{h}_{ab}$, i.e, $\hat{k} = 0$. From the previous discussion it then follows that $\lambda_{\hat{h},S} > 0$, and since it is conformally invariant we have $\lambda_{\delta,S} = \lambda_{\hat{h},S} > 0$. In order to generalize for more spheres we will use the well known Misner initial data: Misner shows in Lemma 3 of [30] that there exists a conformal factor θ such that $\hat{h}_{ab} = \theta^4\delta_{ab}$, $\hat{R} = 0$, with the boundaries of the spheres being minimal surfaces with respect to the metric $\hat{h}_{ab}$, i.e, $\hat{k} = 0$. Then we have also in this case $\lambda_{\delta,S} = \lambda_{\hat{h},S} > 0$. For two spheres, the condition that the spheres are not too close used in this Lemma, just means that they do not touch each other.

It is easy to construct more general examples of metrics which satisfy $\lambda_{h,S} > 0$. As pointed out in [16], if we take any metric on $\mathbb{R}^3$ which satisfies $\mathcal{R} \geq 0$ and we excise appropriate small spheres on it, we get $\lambda_{h,S} > 0$ on M (see [16] for details).

For the non-time symmetric case, if we assume the maximal condition $K = 0$ and no inner boundaries, then is clear that every maximal initial data satisfies $\lambda_h > 0$ because $\tilde{\mathcal{R}} = \tilde{K}_{ab}\tilde{K}^{ab} \geq 0$. This is also a sufficient condition (with appropriate fall-off conditions for K_{ab} at infinity) for the conformal metric to ensure the existence of solutions to the non linear equation (2.6) with boundary conditions (3.2) (see [20], [31] and references therein).

If we include inner boundaries, still requiring the condition $K = 0$, it is not clear if the condition $\lambda_{h,S} > 0$ is necessary. However, it has been proved in [16, 17] that it is a *sufficient* condition together with additional assumptions on k and $K_{ab}s^as^b$. Moreover in these references the condition $\lambda_{h,S} > 0$ plays an essential role. Even if the condition $\lambda_{h,S} > 0$ turns out not to be necessary in generic situations, it is very likely that it will still be relevant for the black hole problem. The reason being that for these data, time symmetry arises as a limit when the linear and angular momentum of the black holes are zero, and in this case, as discussed above, $\lambda_{h,S} > 0$ is a necessary and sufficient condition.

The assumptions on k and $K_{ab}s^as^b$ are always made in some representative metric in the class $\lambda_{h,S} > 0$. That is, these assumptions are not conformally invariant. The two main examples are the following. If we assume that the conformal metric satisfies $\mathcal{R} \geq 0$ and $k \leq 0$ (this automatically implies $\lambda_{h,S} > 0$) and we assume $K_{ab}s^as^b \geq 0$ at the boundary S, then there is always a solution of Eqs. (2.6), (3.1) and (3.2) (with $K = 0$ and appropriate fall-off conditions for K_{ab}). This case can be obtained from [17] making a time reversion (that is, $t^a \to -t^a$, $\Theta_+ \to -\Theta_-$, $\Theta_- \to -\Theta_+$, $K_{ab} \to -K_{ab}$). This example seems to be physically relevant only if $K_{ab}s^as^b = 0$ at the boundary

[1] In this article we always assume the dimension of M is 3, however all of the following discussion can be generalized to arbitrary dimensions, see [16].

[2] Unfortunately, there exists in the literature different conventions for the signs of K_{ab}, k, and s^a. In this article we have used what seems to be the standard conventions in numerical relativity. The relation of our present convention with the ones in [16] and [17] is the following. The second fundamental form K_{ab} is denoted by σ_{ab} in [16], and let us denote by $\bar{K}_{ab}$ the one in [17]. Then we have $K_{ab} = \sigma_{ab} = -\bar{K}_{ab}$. The mean curvature of the boundary (in our notation k) is denoted by h in [16] and by H in [17]. We have $k = -2h = -H$. The normals are denoted by ν^a in both [16] and [17] (they use the same choice for it). We have $s^a = -\nu^a$.

since, if this is not the case, then $\Theta_- > 0$. The second example was studied in [16]. The conformal metric is assumed to satisfy $\lambda_{h,S} > 0$ and, in addition, $\mathcal{R} = 0$, $k > 0$ (the flat metric with spheres as boundaries satisfies these conditions). It is assumed also that $-k \leq K_{ab}s^a s^b \leq 0$. In this case we get future trapped surfaces.

It is important to note that for an arbitrary metric which satisfies $\lambda_{h,S} > 0$, it is possible to calculate (solving a linear equation) a conformal factor θ such that the rescaled metric $\hat{h}_{ab} = \theta^4 h_{ab}$ satisfies $\hat{R} = 0$ and $\hat{k} > 0$ or (with, of course, a different θ) $\hat{R} \geq 0$ and $\hat{k} \leq 0$ (cf. [16]).

We have seen that in both cases the existence theorems assume a definite sign for $K_{ab}s^a s^b$ at the boundary. This is essential in order to control the positivity of the conformal factor using the boundary condition (3.1) and the maximum principle. In order to be able to control $K_{ab}s^a s^b$ at the boundary, it is necessary to use a boundary condition for the momentum constraint (2.5) that allows us to prescribe $K_{ab}s^a s^b$ as free data. In [16, 17] this has been achieved using the following Neumann type boundary condition for β^a on S:

$$(\mathcal{L}\beta)_{ab}s^a = 0\,. \tag{3.9}$$

However, we will see in the next section that with this boundary condition, we cannot enforce $\sigma_{ab} = 0$ at the boundary. In order to do this a new kind of boundary condition will be needed.

C. Boundary conditions for the momentum constraint

Let us now focus on the condition $\sigma_{ab} = 0$ with σ_{ab} given by equation (2.22). First note that the two terms in (2.22) involve different powers of the conformal factor ψ. Since ψ is the unknown solution of the nonlinear equation (2.6), it will be very difficult to get $\sigma_{ab} = 0$ without requiring that both these terms vanish independently. In particular, this requirement will imply that the shear of the ingoing null geodesics along n^a is also zero. Then, the second term of (2.22) suggests that we impose $k^0_{ab} = 0$. This is a condition on the shape of the boundary S, and is also known as the umbilical condition. For example a sphere in the flat metric satisfies this. Note that this condition is conformally invariant, that is $k^0_{ab} = 0 \iff \tilde{k}^0_{ab} = 0$.

Having made the above choice for k_{ab}, we are then left with the first term of (2.22). Before discussing the general case, let us mention some important examples of initial data which not only satisfy $\sigma_{ab} = 0$ at S, but also have $\tilde{k}_{ab} = 0$.

The first example is provided by the Misner solution, already discussed above. The fact that these data satisfy $\sigma_{ab} = 0$ at S can be directly verified from (2.21): due to the reflection isometry of these data we automatically have $\tilde{k}_{ab} = 0$ [32], and $\tilde{K}_{ab} = 0$ because the data is time symmetric.

The second important example is the Kerr initial data for the Boyer-Lindquist slicing. Since Kerr is stationary, it is clear that $\sigma_{ab} = 0$. Moreover, these data also have an isometry which leaves the horizon fixed, then we also have $\tilde{k}_{ab} = 0$ in this case. Note that although the Kerr data are not time symmetric, the terms with $\tilde{K}_{ab}$ in (2.21) still vanish.

Finally, some of the Bowen-York data [33] also satisfy these conditions. These are the data for one black hole with spin (defined by Eq. (10) in [33]) and for the so called "negative inversion" single black hole case with linear momentum (defined by K^-_{ij} in Eq. 9 of [33]). For these cases, we have an isometry which leaves the horizon fixed, so that $\tilde{k}_{ab} = 0$. Moreover, one can explicitly check that the first term in (2.22) vanishes for these conformal second fundamental forms.

Let us now return to the general case. We make a decomposition of β into its normal and tangential parts with respect to S

$$\beta^a = bs^a + \beta^a_{||}, \tag{3.10}$$

where $\beta^a_{||}s_a = 0$, $b = \beta^a s_a$. We insert (2.7) in (2.22), we use the decomposition (3.10) and after some computations we find

$$\sigma_{ab} = (\psi^2 - 2\psi^{-2}b)k^0_{ab} + \psi^{-2}\left(q^c_a q^d_b Q_{cd} - \frac{1}{2}q_{ab}q^{cd}Q_{cd} - (l\beta_{||})_{ab}\right). \tag{3.11}$$

Here l is the conformal Killing operator on the surface S, that is

$$(lm)_{ab} = d_{(a}m_{b)} - q_{ab}d_c m^c, \tag{3.12}$$

where d is the covariant derivative with respect to q_{ab}, m^a is any tangential vector to S ($s^a m_a = 0$) and $m_a = q_{ab}m^b$.

The important point is that Eq. (3.11), if we assume $k^0_{ab} = 0$, only contains tangential derivatives of $\beta_{||}$[3]. These derivatives can be controlled using Dirichlet boundary conditions for $\beta_{||}$. Thus, if we choose $(l\beta_{||})_{ab}$ and Q_{ab} appropriately, we can ensure the vanishing of the shear at S. It turns out that this is in fact a valid boundary condition for β^a.

The complete set of boundary conditions on β^a for solving equation (2.9) is

$$s^a s^b(\mathcal{L}\beta)_{ab} = f, \tag{3.13}$$

$$\beta^a_{||} = \varphi^a\,, \tag{3.14}$$

where f and φ^a are to be prescribed a priori, subject to certain restrictions mentioned below.

[3] Incidentally, this is also the case in [3, 14], where it is achieved by canceling the factor multiplying k^0_{ab} by means of a Dirichlet boundary condition on b. The price is the loss of the control on $K_{ab}s^a s^b$.

With these choices, the problem is solved as follows. Chose any φ^a and let Q_{ab} be such that

$$(l\varphi)_{ab} - q_a^c q_b^d Q_{cd} + \frac{1}{2} q_{ab} q^{cd} Q_{cd} = 0, \tag{3.15}$$

(alternatively, one could prescribe Q_{ab} and attempt to solve for φ^a). Solve equation (2.9) for β^a using the boundary conditions (3.13)- (3.14). Then (provided $k^0_{ab} = 0$) we will have $\sigma_{ab} = 0$. Furthermore, $K_{ab}s^a s^b = f - Q_{ab}s^a s^b$ is a free data which can be chosen to be non-positive by an appropriate choice of the quantities $Q_{ab}s^a s^b$ and f which have, up to now, not been constrained at all. The proof that it is always possible to solve equation (2.9) with boundary conditions (3.13)-(3.14) is given in section IV.

Having solved the momentum constraint and having retained $K_{ab}s^a s^b$ as a free data, we can now solve the Hamiltonian constraint (2.6) for the conformal factor subject to the boundary conditions (3.1) and (3.2). The possible choices of this free data have been given earlier in section III C. Recall that to solve the momentum constraint, we only have fixed the tracefree part k^0_{ab}; the trace k is still free. The physically most relevant case is when we want to obtain future trapped surfaces. As discussed in section III C the two possibilities are

i) $\mathcal{R} \geq 0$, $k \leq 0$ and $K_{ab}s^a s^b = 0$. Note that all the examples discussed above satisfy $K_{ab}s^a s^b = 0$. This condition implies $\Theta_+ = \Theta_- = 0$ on S.

ii) The more general choice $\mathcal{R} = 0$, $k > 0$, and $K_{ab}s^s s^b$ such that

$$-k \leq K_{ab}s^a s^b \leq 0\,. \tag{3.16}$$

In ii) the case $K_{ab}s^a s^b = 0$ is also included. However, in practice, for non conformally flat metrics, it is perhaps more convenient to work with i) because in this case the conditions $\mathcal{R} \geq 0$, $k \leq 0$ involve only inequalities which are easier to achieve (for example, by a small perturbation of a given metric) than the condition $\mathcal{R} = 0$.

Just like the Dirichlet and Neumann conditions, the boundary conditions (3.13)–(3.14) are natural for the operator Δ_L. This is easily seen by considering the Green formula for the non flat operator Δ_L:

$$\int_\Omega (\mathcal{L}\beta)^{ab}(\mathcal{L}\xi)_{ab} = -\int_\Omega \beta^a \Delta_L \xi_a + \oint_{\partial\Omega} (\mathcal{L}\beta)_{ab} s^a \xi^b\,. \tag{3.17}$$

Taking ξ^a to be a solution of the homogeneous problem:

$$\Delta_L \xi^a = 0\,, \tag{3.18}$$

with boundary conditions

$$s^a s^b (\mathcal{L}\xi)_{ab} \ = \ 0\,, \tag{3.19}$$

$$\xi^a_{||} \ = \ 0\,, \tag{3.20}$$

and $\beta^a = \xi^a$, the boundary term in equation (3.17) can be seen to vanish. Thus we see immediately that ξ^a must satisfy $(\mathcal{L}\xi)_{ab} = 0$. Therefore, as for the Dirichlet and Neumann (i.e. equation (3.9)) cases, the kernel of the homogeneous problem with our new boundary conditions consists only of conformal Killing vectors. The only difference is that in the Dirichlet case the kernel is empty because there are no conformal Killing vectors which vanish at the boundary. In the Newmann case every conformal Killing vector is in the kernel because $(\mathcal{L}\xi)_{ab}s^b$ vanishes identically. For our present conditions, the only conformal Killing vectors that are in the kernel are the ones which are normal to the boundary. This suggests that these boundary conditions are well posed, and we shall show in section IV that this is indeed so.

Finally, we want to point out that the new boundary conditions (3.13)– (3.14) can be used in both references [16, 17] in replacement of (3.9), all the results presented there will hold without any essential change.

D. The 2+1 decomposition

While the above description is a complete specification of the problem, in order to get a better feeling of the nature of this elliptic system, we decompose the elliptic system into its radial and tangential parts, and show that the above boundary conditions translate into a Robin-type condition for the function b appearing in equation (3.10). Using the 2+1 decomposition (3.10) of the shift to express the boundary conditions (3.13) and (3.14) on the compact excised surface S, we find

$$2s^a D_a b - kb = \frac{3}{2} f + d_a \varphi^a - 2\varphi^a d_a \ln N\,, \tag{3.21}$$

$$\beta^a_{||} = \varphi^a\,. \tag{3.22}$$

Here S is characterized as the inverse image of a constant value r_o by the height function r, and N is the normalizing factor of s^a such that $s_a = N D_a r$, and $s^a s_a = 1$.

This decomposition suggests a general 2+1 splitting of M, provided by the slicing defined by r, as a procedure to solve the elliptical equation (2.9) for the shift. Extending (3.10) to the whole of M, we find for $\beta^a_{||}$:

$$\Delta_L \beta^a_{||} = J^a - \Delta_L(bs^a) \equiv J^a_{||}\,. \tag{3.23}$$

We can enforce the *radial* part of $J^a_{||}$ to vanish, if we impose the following elliptic equation on b

$$J^a_{||} s_a = 0 \Leftrightarrow \Delta_L(bs^a)s_a = J^a s_a \equiv J_\perp\,. \tag{3.24}$$

Expanding the differential operator we find

$$\begin{aligned} J_\perp = {} & D_a D^a b + \frac{1}{3} s^a s^c D_a D_c b \\ & + \frac{1}{3} D_a b\,(k s^a - d^a \ln N) \\ & + b\left(\mathcal{R}_{ac}s^a s^c + \frac{1}{3}D^a k - k_{ac}k^{ac} - (d_a \ln N)(d^a \ln N)\right). \end{aligned} \tag{3.25}$$

Note that the principal part of the operator defined by (3.25) is not the Laplacian, but it is nevertheless elliptic. In this scheme, we calculate the scalar source $J_\perp$ from the original source J^a by contracting with the normal vector s^a. Then we solve Eq. (3.25) for b by imposing boundary condition (3.21). With the resulting b we calculate the source $J^a_{||}$ and solve Eq. (3.23) for $\beta^a_{||}$ by using the Dirichlet boundary condition (3.22).

We conclude with some remarks. First, there is a remarkable choice for φ^a, namely to prescribe it as a conformal Killing vector on S (this is condition (49) in [3] or constraint (57) in [14]), that is $(l\varphi)_{ab} = 0$. This is a particular case of condition (3.15). However, this choice seems to play no fundamental role in our analysis.

Secondly, it is interesting to remark on a generalization of the above procedure. In [17], the free data is chosen in such a way that it is possible to control the size of both $|\Theta_-|$ and $|\Theta_+|$. In other words, it is possible to control, in some sense, how trapped the boundary is. The same is possible here with the shear. This question is of interest if we want to control the amount of radiation falling into the black hole [22, 23, 24]. In equation (3.15), if we choose Q_{ab} such that the right hand side is some Σ_{ab} instead of zero, then the elliptic equations for β^a and ψ can still be solved and the final solution will have an inner boundary with shear $\sigma_{ab} = -\psi^{-2}\Sigma_{ab}$. Since $\psi \geq 1$ we get $|\sigma_{ab}| \leq |\Sigma_{ab}|$. It is also possible to get a lower bound for $|\sigma_{ab}|$ using the upper bound for ψ obtained in the existence theorem in [17].

IV. ELLIPTIC BOUNDARY CONDITIONS FOR THE MOMENTUM CONSTRAINT

It is well known that the operator Δ_L defined by Eq. (2.10) is elliptic. For a given elliptic operator, a set of boundary conditions are called elliptic if they satisfy the Lopatinski-Schapiro conditions, also known as the covering conditions. For the definition of these conditions as well as the other concepts used in this section see, for example, the review [34] and references therein. These conditions are important because an elliptic operator with elliptic boundary conditions will always have solutions provided the sources and the boundary values satisfy a finite number of conditions.

The operator Δ_L is (for a three dimensional manifold) of degree 3, and we therefore need to prescribe three equations as boundary conditions. Let Ω be a bounded domain in $\mathbb{R}^3$. The most important example of an elliptic boundary condition for Δ_L is the Dirichlet one

$$\beta^a = \varphi^a \text{ on } \partial\Omega, \tag{4.1}$$

where φ^a is an arbitrary vector on the boundary. This condition has been extensively used in numerical relativity. The analog to the Neumann condition for Δ_L is given by

$$(\mathcal{L}\beta)_{ab}s^b = \varphi_a \text{ on } \partial\Omega, \tag{4.2}$$

where s^a is the normal to $\partial\Omega$. These conditions are elliptic.

In this section we want to prove that the following boundary conditions for Δ_L are also elliptic

$$s^a s^b(\mathcal{L}\beta)_{ab} = \varphi_1 \tag{4.3}$$

$$\beta_a m_1^a = \varphi_2, \tag{4.4}$$

$$\beta_a m_2^a = \varphi_3, \tag{4.5}$$

where m_1^a and m_2^a are tangential and linearly independent vectors at the boundary. Note that Eqs. (4.3)–(4.5) are three linear conditions for β^a at the boundary. In order to write the Lopatinski-Schapiro conditions we need to define the principal part of the both the operator and the boundary conditions at an arbitrary point x_0 on the boundary. For the operator Δ_L, the principal part is given by the standard definition. That is, it is given by the terms which contains only two derivatives. If we choose coordinates at x_0 such that $h_{ab}(x_0) = \delta_{ab}$, then the principal part of Δ_L is given by

$$\Delta_L^0 = \Delta^0\beta^a + \frac{1}{3}\partial^a\partial_b\beta^b, \tag{4.6}$$

where Δ^0 is the flat Laplacian and ∂ denotes partial derivatives.

For the boundary conditions we need to be careful in the definition of the principal part. If we choose only the terms which contain the highest order derivatives, then only Eq. (4.3) will survive and this will lead to an ill posed problem. In order to take into account the fact that terms of mixed order appear in the boundary operator, we need to use the general definition of the principal part given by [35]. This definition involves integer weights that, with the notation of [34] and for the operator Δ_L, are given by $t_\nu = s_\nu = 1$ ($\nu = 1, 2, 3$). For the boundary condition, again in the notation of [34], we have $r_0 = 0$, $r_1 = r_2 = -1$, where r_0 corresponds to Eq. (4.3), r_1 to Eq. (4.4), and r_2 to Eq. (4.5). With this choice the principal part will include all the equations (4.3)–(4.5). That is, the principal part at x_0 is given by

$$s^a s^b(\mathcal{L}^0\beta)_{ab}, \quad \beta_a m_1^a, \quad \beta_a m_2^a, \tag{4.7}$$

where

$$(\mathcal{L}^0\beta)_{ab} = 2\partial_{(a}\beta_{b)} - \frac{2}{3}\delta_{ab}\partial_c\beta^c, \tag{4.8}$$

and the tangential vectors m are evaluated at x_0.

For a given point x_0 at the boundary, we choose coordinates (x_1, x_2, x_3) such that the normal is given by $s = \partial/\partial x_3$, and (x_1, x_2) are coordinates on the tangential plane at x_0; see figure 2. Consider the homogeneous constant coefficient problem, on the half plane $x_3 < 0$ with boundary $x_3 = 0$

$$\Delta_L^0\beta^a = 0 \tag{4.9}$$

$$s^a s^b(\mathcal{L}^0\beta)_{ab} = 0 \tag{4.10}$$

$$\beta_a m_1^a = 0 \tag{4.11}$$

$$\beta_a m_2^a = 0. \tag{4.12}$$

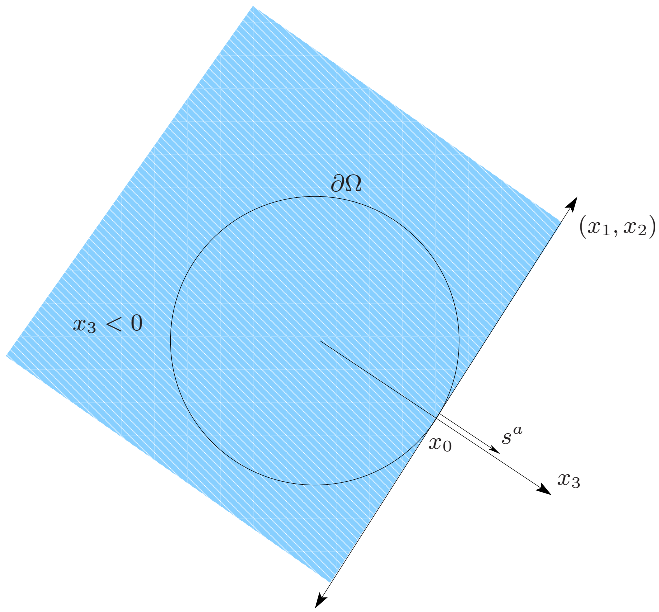

FIG. 2: The set-up of the Lopatinski-Schapiro condition. x_0 is a point on the boundary $\partial\Omega$, (x_1, x_2) span the tangent plane to $\partial\Omega$ at x_0, and x_3 is orthogonal to $\partial\Omega$; s^a is the unit normal to $\delta\Omega$. The shaded region is $x_3 < 0$.

The boundary conditions are said to satisfy the Lopatinski-Schapiro conditions if there are no nontrivial solution of (4.9)–(4.12) of the form

$$\beta^j = v^j(x_3)e^{i(\xi_1 x_1 + \xi_2 x_2)}, \tag{4.13}$$

where ξ_1 and ξ_2 are arbitrary real numbers, and $v^i(x_3)$ tends to zero exponentially as $x_3 \to -\infty$. To prove this, the key will be the following Green equation valid for any ξ^a:

$$\int_\Omega (\mathcal{L}^0\beta)^{ab}(\mathcal{L}^0\xi)_{ab} = -\int_\Omega \beta^a \Delta_L^0 \xi_a + \oint_{\partial\Omega} (\mathcal{L}^0\beta)_{ab} s^a \xi^b \,. \tag{4.14}$$

The proof is very similar to the one given in example 10 of [34] for the boundary conditions (4.1) and (4.2). Let us assume that there exists a solution of the form (4.13) to equations (4.9)–(4.12) in the half plane $x_3 \le 0$. Let $L_1 = 2\pi/\xi_1$ and $L_2 = 2\pi/\xi_2$. Consider the following sub-domain contained in the half plane $x_3 \le 0$: the infinite cubic region $x_3 \le 0$, $0 \le x_1 \le L_1$, $0 \le x_2 \le L_2$. For this sub-domain we use equation (4.14) for $\beta^a = \xi^a$. We want to prove that, on this domain, the boundary integral in (4.14) vanishes for a solution of the form (4.13). This is clear for the face $x_3 = -\infty$ because the solution, by hypothesis, decay exponentially at infinity. Take the face $x_3 = 0$. On this face the normal s^a is also the normal to Ω. Eqs. (4.11)–(4.12) imply that $\beta^a = \alpha s^a$ on this face for some function α. We use Eq. (4.10) to conclude that the integrand in the boundary integral vanishes on this face. On the other faces, the integrand does not vanish. However, because of the choice of L_1 and L_2, we have that the integrand of opposite faces are identical. Then, the sum of the boundary integrals vanishes because the normal is always outwards. We conclude that $(\mathcal{L}^0\beta)_{ab} = 0$. But there are no conformal Killing vectors in flat space which decay to zero at infinity. Hence the Lopatinski-Schapiro conditions are satisfied, and the boundary conditions are therefore elliptic.

From the previous discussion, using standard results in elliptic theory, we deduce that a solution β^a of the boundary value problem

$$\Delta_L \beta^a = J^a \text{ on } \Omega \tag{4.15}$$

$$s^a s^b (\mathcal{L}\beta)_{ab} = f \text{ on } \partial\Omega \tag{4.16}$$

$$\beta_a m_1^a = \varphi_1, \text{ on } \partial\Omega \tag{4.17}$$

$$\beta_a m_2^a = \varphi_2, \text{ on } \partial\Omega. \tag{4.18}$$

exists if and only if

$$\oint_{\partial\Omega} bf = \int_\Omega J^a \xi_a \tag{4.19}$$

for all conformal Killing vectors ξ^a of the metric h_{ab} which are normal to the boundary of Ω, and where b is defined by $\xi^a = bs^a$. This can be shown by considering the Green equation (3.17) from which we deduce

$$\int_\Omega (\xi^a \Delta_L \beta_a - \beta^a \Delta_L \xi_a) = \oint_{\partial\Omega} \left((\mathcal{L}\beta)_{ab} s^a \xi^b - (\mathcal{L}\xi)_{ab} s^a \beta^b \right) . \tag{4.20}$$

In this equation, set $(\mathcal{L}\xi)_{ab} = 0$ (which implies $\Delta_L \xi^a = 0$) and $\xi^a = bs^a$ at the boundary, to immediately get Eq. (4.19).

If the metric admits no conformal Killing vectors, then there are no restrictions on J^a. An example of a metric with conformal Killing vectors is the flat metric. If the boundary is a sphere centered at the origin, then $\xi^a = x^a$ is a conformal Killing vector which is normal to the boundary (here we have assumed coordinates such that $h_{ab} = \delta_{ab}$).

In our case, $J^a = D_b Q^{ab}$ for some trace-free tensor Q^{ab}. Then condition 4.19 can be written as

$$\oint_{\partial\Omega} \left(bf - Q_{ab} s^b \xi^a \right) = 0. \tag{4.21}$$

for the exterior region M discussed in this article, the boundary integral (4.21) contains two terms, one is the inner boundary S and the other is an integral over the sphere at infinity.

For simplicity we have considered in this section only bounded domains, since the new part here is given by the boundary conditions on the inner boundary. Conditions at infinity (i.e., fall-off) for β^a are the standard ones, that is $\beta^a \to 0$ at infinity. See, for example [16] where weighted spaces have been used and [17] where a compactification of the exterior region has been employed. With this fall-off condition, the kernel is always trivial because there are no conformal Killing vectors which decays to zero at infinity. However, in the presence of a conformal Killing vector normal to the boundary, equation (4.21) still plays a role. In this case it relates some asymptotic components of the solution with a boundary integral; this is the analog of the restrictions studied in [17, 36, 37] for the momentum constraint.

V. CONCLUSIONS

In this paper, we have explained why a Dirichlet condition on β^a may be potentially problematic for solving the Hamiltonian constraint. This is essentially because with a Dirichlet condition it is not possible to control the sign of $K_{ab}s^a s^b$ at the boundary, since this function depends on *normal* derivatives of β^a. To control the sign of this function is important for two reason: the first one is that for physically interesting solutions, i.e. ones which contain marginally future trapped surfaces, on maximal slices $K_{ab}s^a s^b$ is always non-positive. The second one, is that with a definite sign of $K_{ab}s^a s^b$ it is possible to prescribe a priori conditions that guarantee that the solutions of the non linear equations will always exist. It seems to be very hard to obtain such conditions without controlling the sign of this function. We have shown that this can be achieved imposing a Neumann (oblique) boundary condition on the radial part of β^a and Dirichlet conditions on its tangential parts. Using the theory of elliptic systems, we have shown that these boundary conditions are well posed.

Acknowledgements

We are grateful to Abhay Ashtekar for valuable discussions and for suggesting this problem to us, and to Helmut Friedrich for illuminating discussions regarding the invariant (3.7). We are also grateful to Marcus Ansorg, Eric Gourgoulhon, Francois Limousin, and Guillermo Mena Marugán for valuable discussions. We also acknowledge the support of the Albert Einstein Institute and of the Observatoire de Paris. SD acknowledges support from the Sonderforschungsbereich SFB/TR 7 of the Deutsche Forschungsgemeinschaft. JLJ acknowledges the support of a Marie Curie Intra-European Fellowship within the 6th European Community Framework Programme, and the hospitality of the Albert Einstein Institute.

[1] G. Cook, Phys. Rev. D **65**, 084003 (2002), gr-qc/0108076.
[2] J. W. York, Jr., Phys. Rev. Lett. **82**, 1350 (1999).
[3] G. Cook and H. Pfeiffer, Phys. Rev. D **70**, 104016 (2004), gr-qc/0407078.
[4] H.-J. Yo, J. N. Cook, S. L. Shapiro, and T. W. Baumgarte (2004), gr-qc/0406020.
[5] G. B. Cook, Phys. Rev. D **50**, 5025 (1994), gr-qc/9404043.
[6] S. B. E. Gourgoulhon, P. Grandclement, Phys. Rev. D **65**, 044020 (2002).
[7] S. B. E. Gourgoulhon, P. Grandclement, Phys. Rev. D **65**, 044021 (2002).
[8] T. Baumgarte, M. Skoge, and S. Shapiro, Phys. Rev. D **70**, 064040 (2004), gr-qc/0405077.
[9] G. Schäfer and A. Gopakumar, Phys. Rev. D **69**, 021501 (2004), gr-qc/0310041.
[10] A. Ashtekar, C. Beetle, and S. Fairhurst, Class. Quantum. Grav. **17**, 253 (2000), gr-qc/9907068.
[11] A. Ashtekar, S. Fairhurst, and B. Krishnan, Phys. Rev. D **62**, 104025 (2000), gr-qc/0005083.
[12] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski, and J. Wisniewski, Phys. Rev. Lett. **85**, 3564 (2000), gr-qc/0006006.
[13] A. Ashtekar and B. Krishnan (2004), gr-qc/0407042.
[14] J. Jaramillo, E. Gourgoulhon, and G. Mena-Marugán, to be published in Phys. Rev. D (2004), gr-qc/0407063.
[15] J. W. York, Jr., J. Math. Phys. **14**, 456 (1973).
[16] D. Maxwell, to be published in Commun. Math. Phys. (2003), gr-qc/0307117.
[17] S. Dain, Class. Quantum. Grav. **21**, 555 (2004), gr-qc/0308009.
[18] J. A. Valiente Kroon, Class. Quantum. Grav. **21**, 3237 (2004), gr-qc/0402033.
[19] G. B. Cook, Living Rev. Relativity **3** (2001), http://www.livingreviews.org/Articles/Volume3/2000-5cook/.
[20] R. Bartnik and J. Isenberg, in *The Einstein equations and large scale behavior of gravitational fields*, edited by P. T. Chruściel and H. Friedrich (Birhuser Verlag, Basel Boston Berlin, 2004), pp. 1–38, gr-qc/0405092.
[21] O. Dreyer, B. Krishnan, E. Schnetter, and D. Shoemaker, Phys. Rev. D **67**, 084019 (2002), gr-qc/0206008.
[22] S. Hayward, Phys. Rev. D **49**, 6467 (1994), gr-qc/9306006.
[23] A. Ashtekar and B. Krishnan, Phys. Rev. Lett. **89**, 261101 (2002), gr-qc/0207080.
[24] A. Ashtekar and B. Krishnan, Phys. Rev. D **68**, 104030 (2003), gr-qc/0308033.
[25] S. Hayward, Phys. Rev. D **70**, 104027 (2004), gr-qc/0408008.
[26] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
[27] M. Cantor and D. Brill, Compositio Mathematica **43**, 317 (1981).
[28] J. M. Lee and T. H. Parker, Bull. Amer. Math. Soc. **17**, 37 (1987).
[29] J. F. Escobar, J. Differential Geom. **35**, 21 (1992).
[30] C. W. Misner, Ann. Phys. (N.Y.) **24**, 102 (1963).
[31] D. Maxwell (2004), gr-qc/0405088.
[32] G. W. Gibbons, Commun. Math. Phys. **27**, 87 (1972).
[33] J. M. Bowen and J. W. York, Jr., Phys. Rev. D **21**, 2047 (1980).
[34] S. Dain (2004), to appear in the Proceeding of the March-2004 Heraeus Seminar in Bad Honnef, gr-qc/0411081.

[35] S. Agmon, A. Douglis, and L. Nirenberg, Comm. Pure App. Math. **17**, 35 (1964).

[36] R. Beig and N. O. Murchadha, Commun. Math. Phys. **176**, 723 (1996).

[37] S. Dain and H. Friedrich, Commun. Math. Phys. **222**, 569 (2001), gr-qc/0102047.