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MONTE CARLO SIMULATION OF SU(2) GAUGE THEORY WITH FERMIONS ON A FOUR-DIMENSIONAL LATTICE

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After integration over the fermions in an SU(2) lattice gauge theory, the effective fermionic action may be expressed as a sum over all possible closed gauge field loops with corresponding weight factors. We approximate this sum and perform a Monte Carlo simulation of a coupled fermion-gauge system on a 4^4 lattice. We compare our results for $\langle S_{\rm eff}\rangle$ and $\langle \bar{\psi}\psi\rangle$ for different values of the gauge field coupling β and fermion coupling κ with the free fermion theory on a lattice. $\langle S_{\rm eff}\rangle$ turns out to be quite small for $\kappa \leqslant \frac{1}{8}$.

1. Introduction

In the last few years, Monte Carlo (MC) techniques have been developed to deal with pure gauge theories on a euclidean lattice [1, 2]. More recently, the problem of incorporating fermions in this treatment has come into focus and has been discussed by several authors [3, 4]. In order to evaluate expectation values, the MC algorithm generates configurations of the field variables with statistical weight given by a Boltzmann factor. In the case when fermions are included, this Boltzmann factor is

$$\exp\left[S_0(U) + S_{\rm F}(\bar{\psi}, \psi, U)\right], \tag{1.1}$$

i.e., "Grassmann-valued" and not implementable on a computer. Since the action is quadratic in the fermions, one may integrate out the fermions [5] and one arrives at the well-known Matthews-Salam determinant [6]; this formalism is reviewed in sect. 2. Hence, (1.1) is replaced by the effective Boltzmann factor

$$\exp\left[S_0(U) + S_{\rm eff}(U)\right], \tag{1.2}$$

where $S_{\rm eff}(U)$ results from the Grassmann integration. As has been pointed out in refs. [3, 4], the main difficulty resides in the numerical calculation of the Matthews-Salam determinant which is obtained from a rather huge matrix; on the other hand, this matrix has only very few off-diagonal elements, and this fact will be of crucial importance in any calculation.

In this paper, we propose a method to deal with this problem which is based on an expansion of the logarithm of the fermionic determinant, i.e., the effective fermionic action. Similar ideas have been independently developed in refs. [7, 8]. We will work on a 4^4 lattice and approximate SU(2) by its largest subgroup: these and other details are discussed in sect. 3. In addition to an outline of the general strategy, we present some numerical results in sect. 4. The final section of this article is devoted to a discussion of the possible implications of our results and possible further improvements of our method.

2. Review of Matthews-Salam formalism

To begin with, we briefly recall a few facts about the MS formalism [6]. In any conventional gauge theory, the fermionic part of the action may be written as a quadratic form,

$$S_{\mathrm{F}}(\psi,\bar{\psi}) = \bar{\psi}Q\psi \equiv \sum_{i,j} \bar{\psi}_i Q_{ij} \psi_j, \tag{2.1}$$

in the Grassmann variables ψ_i. Here, the index i stands for the triple (x, α, A): x is the configuration space (= lattice) index, α the Dirac index and A the index of the gauge group. Flavour indices may of course be appended. Integrating out the fermions [5] leads to the well-known formulae [6]

$$\int \mathrm{d}\psi \mathrm{d}\bar{\psi} \exp[\bar{\psi}Q\psi] = \det Q,$$

$$\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\ \bar{\psi}_i \psi_j \exp[\bar{\psi}Q\psi] = Q^{-1}_{ji} \det Q,$$

$$\int \mathrm{d}\psi \mathrm{d}\bar{\psi}\ \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \exp[\bar{\psi}Q\psi] = \left(Q^{-1}_{ji} Q^{-1}_{lk} - Q^{-1}_{jk} Q^{-1}_{li}\right) \det Q, \tag{2.2}$$

etc.

In a lattice gauge theory, the matrix Q depends on the link variables $U_{x\mu}$ and may be taken as

$$Q_{ij} \equiv Q_{x\alpha A, y\beta B} = \delta_{xy}\delta_{\alpha\beta}\delta_{AB} - \kappa M_{x\alpha A, y\beta B}, \tag{2.3}$$

with

$$M_{x\alpha A, y\beta B} \equiv (1+\gamma^{\mu})_{\alpha\beta}(U_{x\mu})_{AB}\delta_{x,y-\hat{\mu}} + (1-\gamma^{\mu})_{\alpha\beta}(U^{+}_{x-\hat{\mu},\mu})_{AB}\delta_{x,y+\hat{\mu}}. \tag{2.4}$$

Following Wilson [9] we have introduced an explicit chiral symmetry breaking through the replacement of $\pm\gamma_\mu$ by $(1 \pm \gamma_\mu)$. It is a well-known problem that there appears to be no chirality preserving lattice formulation which avoids the notorious species doubling of Dirac fermions on the lattice [10]. In Wilson's prescription, the superfluous fermions decouple in the naive continuum limit $a \to 0$ where chiral invariance is restored.

The effective fermionic action $S_{\text{eff}}(U)$ is defined through*

$$\exp S_{\text{eff}}(U) \equiv \det(1 - \kappa M), \tag{2.5}$$

and the other quantity we are interested in is the fermionic propagator. Our MC method is based on an expansion of these quantities in the parameter κ according to the formulae

$$S_{\text{eff}}(U) = -\operatorname{Tr} \sum_{L=1}^{\infty} \frac{\kappa^{2L}}{2L} M(U)^{2L}, \tag{2.6}$$

$$Q^{-1}(U) = [1 - \kappa M(U)]^{-1} = \sum_{L=0}^{\infty} \kappa^L M(U)^L. \tag{2.7}$$

For their actual evaluation, we use the fact that, owing to the peculiar form of the matrix M (2.4), S_{eff} may be re-expressed as a sum of traces (Dirac + internal) along all closed paths on the lattice, such that a factor $(1 + \gamma^\mu)U$ is associated with a positively oriented link and $(1 - \gamma^\mu)U^+$ with a negatively oriented one; in this way, we exploit the circumstance that M is "almost diagonal" to our maximal advantage. That is, we get

$$S_{\text{eff}}(U) = -\sum_{L=2}^{\infty} \frac{\kappa^{2L}}{L} \sum_{\text{links } \ell} \sum_{\mathrm{P}_{2L}(\ell)}{}' T(U_{\mathrm{P}_{2L}}, \gamma_{\mathrm{P}_{2L}}), \tag{2.8}$$

where the sum runs over L, all links ℓ of the lattice and all positively oriented paths $\mathrm{P}_{2L}(\ell)$ of length $2L$ through a given link ℓ. $T(U, \gamma)$ denotes the trace**

$$T(U_{\mathrm{P}_{2L}}, \gamma_{\mathrm{P}_{2L}}) = \operatorname{Tr} \prod_{\ell \in \mathrm{P}_{2L}} (1 + \gamma_\ell) U_\ell \tag{2.9}$$

along the path P_{2L}. In the updating, we only need to know ΔS_{eff}, the change of the

* It can be shown that this determinant is strictly positive for $0 \leqslant \kappa < \frac{1}{8}$.

** Of course, $\gamma_{-\ell} \equiv -\gamma_\ell$ and $U_{-\ell} \equiv U_\ell^+$.

action under the change of the link variable U_ℓ; this quantity is exactly given by

$$\Delta S_{\text{eff}} = S_{\text{eff}}(U') - S_{\text{eff}}(U)$$
$$= -\sum_{L=2}^{\infty} 2\kappa^{2L} \sum_{P_{2L}(\ell)}{}' \left[T\left(U'_{P_{2L}}, \gamma_{P_{2L}}\right) - T\left(U_{P_{2L}}, \gamma_{P_{2L}}\right) \right], \qquad (2.10)$$

where the gauge field configurations U' and U differ only in the one-link variable U_ℓ. Note that we have taken into account that paths of length two do not contribute because $(1+\gamma^\mu)(1-\gamma^\mu)=0$. The lowest order ($\kappa^4$) term of ΔS_{eff} is proportional to ΔS_0 and constitutes a renormalization of the gauge coupling constant. Similar considerations apply to (2.7), and we are actually only interested in the (truncated) expectation values of string operators which, for simplicity, we take as

$$S_n \equiv \bar{\psi}_x U_{x\mu} U_{x+\hat{\mu},\mu} \cdots U_{x+(n-1)\hat{\mu},\mu} \psi_{x+n\hat{\mu}}. \qquad (2.11)$$

For the purposes of this article, we have computed the expectation values $\langle S_{\text{eff}} \rangle$, $\langle \bar{\psi}_0 \psi_0 \rangle$ and $\langle S_n \rangle$ as a function of the two parameters β and κ. Using (2.2) and the explicit form of M again, (2.11) leads once more to a sum completely analogous to (2.8). The only difference is that now the sum ranges over all closed paths containing the straight line from x to $x+n\hat{\mu}$ which is kept fixed, and that the Dirac trace is only over that piece of the closed path which is the complement of the line $x \to x+n\hat{\mu}$. There is no problem of principle in extending this analysis to more complicated structures such as meson operators.

3. MC procedure: description and justification

Our MC procedure follows the usual lines [2, 11]: we use the modified Metropolis method which has been widely applied before [12]. Instead of working with the full gauge group SU(2), we approximate it by its largest subgroup, the icosahedral group which has 120 elements. Except for the free case, where the gauge fields are frozen to unity, we always stay below the critical value of the gauge coupling $\beta_c = 6.05$, where the discrete group has a phase transition that is absent in the continuous group. It has been demonstrated that below this value of β the results obtained from the full group and its maximal discrete subgroup, respectively, are indistinguishable [12].

The pure gauge field action

$$S_0(U) = -\tfrac{1}{2}\beta \sum_{\square} \operatorname{Tr}(\mathbb{1} - U_\square), \qquad \beta \equiv 4/g^2 \qquad (3.1)$$

(the sum is over all plaquettes and $U_\square$ denotes the ordered product of link variables around the plaquette), is replaced by the full action

$$S(U) = S_0(U) + S_{\mathrm{eff}}(U) \tag{3.2}$$

[with $S_{\mathrm{eff}}(U)$ as defined in (2.8)] in the updating. The crucial point here is that at each MC step one needs to know only the change of the action ΔS that results from a change of a certain link variable. For the effective fermion action, ΔS_{eff} is given in (2.10) in terms of a sum over all closed paths through the given link, and we emphasize again that this formula is exact.

The essential difficulty is now the numerical computation of ΔS_{eff}, and one must resort to some approximation which, however, must be sufficiently reliable in order not to play havoc with the updating procedure. In our scheme, there are three approximations involved which we state below.

(i) We truncate the expansions (2.6) and (2.7) at order κ^{12}, i.e., we consider only paths of maximal length 12.

(ii) We evaluate only a limited number of paths (typically 20 per link update; i.e., 5 per length) which are randomly selected with equal probability from a table of all possible paths; the result is then scaled up by the corresponding factor. This amounts to a MC integration over all paths.

(iii) We use a hybrid between periodic and free boundary conditions by treating the gauge fields as periodic variables, but discarding all those paths which are closed only due to periodicity (e.g., straight lines running through the lattice). This is the least severe of our approximations and has the additional advantage of strongly reducing "finite size" effects of the expectation values of fermionic quantities. For a lattice of linear extension N there are no closed periodic paths of length $2L_{\max} < N$.

Let us now discuss the possible range of validity of our approximations (i) and (ii).

(i) It is fairly obvious that on a four-dimensional lattice the number N_{2L} of all closed paths of length $2L$ will increase as 8^{2L} asymptotically. However, due to $(1+\gamma^{\mu})(1-\gamma^{\mu}) = 0$ all closed paths with spikes do not contribute as has also been discussed by Stamatescu [7] and, therefore, the expected asymptotic growth of the number N'_{2L} of all closed paths without spikes is

$$N'_{2L} = \exp(2L\log 7 + \mathrm{O}(\log L)). \tag{3.3}$$

In the free fermion theory (see sect. 4) all expansions converge up to $\kappa = \frac{1}{8}$. Hence (to leading order)

$$N'_{2L}\left\langle \mathrm{Tr} \prod_{\mathrm{P}_{2L}} (1 \pm \gamma) \right\rangle = \mathrm{O}(1) \cdot 8^{2L}, \tag{3.4}$$

where $\langle \cdots \rangle$ here denotes the mean value of the Dirac traces. From (3.3) and (3.4)

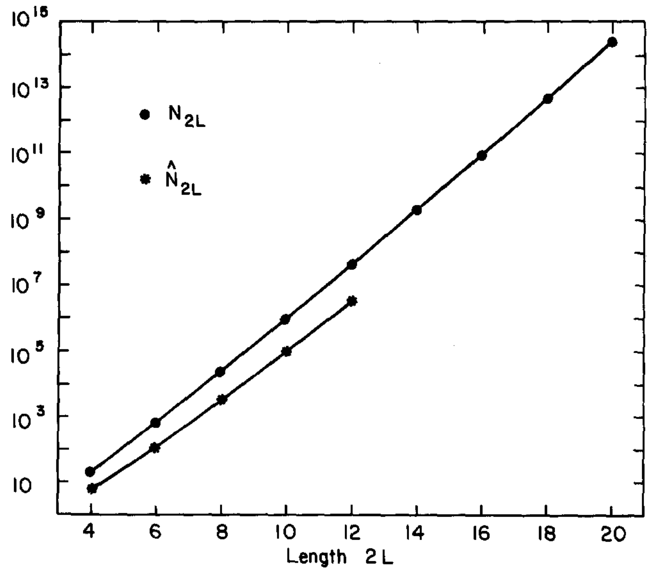

Fig. 1a. The number N_{2L} of closed paths of length $2L$ through a given link and the number $\hat{N}_{2L}$ of closed paths with non-vanishing γ traces.

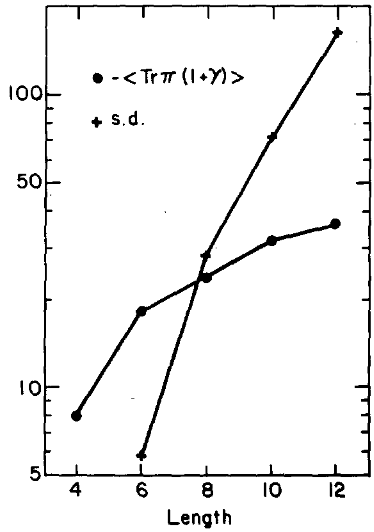

Fig. 1b. Negative average values of the γ traces for non-vanishing paths and their standard deviation as a measure of their fluctuation.

we infer the asymptotic behavior of the γ traces*

$$\left\langle \mathrm{Tr} \prod_{\mathrm{P}_{2L}} (1 \pm \gamma) \right\rangle = \mathrm{O}(1) \cdot \left(\tfrac{8}{7}\right)^{2L} \tag{3.5}$$

In fig. 1a, we compare the number of all closed paths N_{2L} with those of non-vanishing γ trace $\hat{N}_{2L}$; fig. 1b shows the slow increase of $-\langle \mathrm{Tr}\Pi_{\mathrm{P}_{2L}}(1 \pm \gamma)\rangle$ and its standard deviation.

Since the free theory corresponds to setting $\beta = \infty$, the rate of convergence will improve for decreasing β due to the expected perimeter decay of the Wilson loop

$$\left| \left\langle \mathrm{Tr} \prod_{\mathrm{P}_{2L}} U \right\rangle \right| \leqslant \mathrm{O}(1) \cdot c^{2L}, \tag{3.6}$$

with $c < 1$ for $\beta < \infty$. Furthermore, for $\beta > 4$ and $\kappa = 0$, we know that the correlation length is substantially larger than the extension of our 4^4 lattice [12, 13] and contributions from paths with length greater than 4 will be spurious. We therefore expect this approximation to be at least as good for $\beta < \infty$ as for $\beta = \infty$ in which case it is certainly justified up to the critical value $\kappa_c = \frac{1}{8}$ as will be shown in sect. 4.

(ii) Taking only a comparatively small number of paths introduces an error that decreases with the inverse square root of this number. We have checked that 5–10 paths per length are sufficient to obtain rather accurate results up to $\kappa = \frac{1}{8}$ and even beyond. Any approximation of ΔS_{eff} leads to a (quasi-) random error in the updating procedure, and we have simulated this effect by considering a pure SU(2) gauge theory and adding a small equidistributed error ε to ΔS_0 in the updating procedure. Investigating the thermal cycles, we found no effect for values of $|\varepsilon| < 0.1$ ($\langle \varepsilon \rangle = 0$). This provides an additional justification for our approximation scheme since the errors introduced here are typically much smaller for $\kappa \leqslant \frac{1}{8}$. One could further improve the accuracy by selecting different numbers of paths for different lengths according to their known weights $\kappa^{2L}\hat{N}_{2L}$ ("stratified sampling" [14]).

Finally, we mention a few technical details. In order to save computer time we prefabricated tables of pseudo-random sequences of paths on mass storage coded together with their γ trace values. In the updating procedure and the evaluation of averages, a set of path shapes was used for every 64 out of a total of 1024 links. As the γ trace depends only on the shape of the path, only the values of $\mathrm{Tr}\Pi_{\mathrm{P}_{2L}}U$ had to be recalculated for each link. In our MC iterations each link variable was allowed six subsequent changes; this number is optimal as has been discussed in refs. [2, 12]. On a CDC7600 one MC iteration of the pure gauge field system takes about 50 ms; taking into account 20 paths per link, the coupled fermion gauge field system requires about 0.7 seconds per iteration.

* Among the "spikeless" paths there are still some with vanishing γ-traces; thus, the actual number $\hat{N}_{2L}$ of non-vanishing paths is smaller than N'_{2L}.

4. Results

In this section, we present and discuss our numerical results, as well as some purely theoretical arguments in their support. The two main quantities of physical interest that we have computed by the MC simulation are the expectation value $\langle S_{\text{eff}} \rangle$ of the effective fermionic action normalized per link, and, secondly, the expectation value $\langle \bar{\psi}_0 \psi_0 \rangle$ which is relevant for the question of spontaneous chiral symmetry breaking [15]. We have also considered the expectation values $\langle S_n \rangle$ of string operators S_n but refrain from a detailed discussion as our results permit no clear-cut conclusions.

In fig. 2a, we display $\langle S_{\text{eff}} \rangle$ as a function of κ for several values of β and for the free theory. The most striking feature of our computation is the somewhat unexpected smallness of $\langle S_{\text{eff}} \rangle$ as compared to the average value of the gauge field action per link: the effective fermionic action is almost irrelevant to the updating procedure right up to the expected critical value $\kappa_c = \frac{1}{8}$ (for the free fermion theory). This result is confirmed by a computation of $\langle S_0 \rangle$ in the presence of fermions where no marked dependence on κ is observed.

The smallness of $\langle S_{\text{eff}} \rangle$ is less surprising in view of the diamagnetic inequality [16] which states that, for antiperiodic fermionic boundary conditions*,

$$S_{\text{eff}}(\kappa, U) \leqslant S_{\text{eff}}(\kappa, U = \mathbb{1}), \tag{4.1}$$

i.e., even before integrating over the gauge fields, and hence

$$\langle S_{\text{eff}}(\kappa) \rangle_\beta \leqslant S_{\text{eff}}(\kappa, U = \mathbb{1}). \tag{4.2}$$

Next, we observe that in the expansion (2.6) all powers of M^2 appear with a minus sign while, from fig. 1b**, the average value of the Dirac traces is also negative. Moreover, for $0 \leqslant \beta \leqslant \infty$, the average $\langle \mathrm{Tr}\, U \rangle$ varies between 0 and +2, and thus, we may improve (4.2)

$$0 \leqslant \langle S_{\text{eff}}(\kappa) \rangle_\beta \leqslant S_{\text{eff}}(\kappa, U = \mathbb{1}) \tag{4.2'}$$

where the right inequality is rigorous and the left strongly supported by our numerical evidence***. Consequently, we have quite stringent bounds on the average effective fermionic action, and the smallness of $\langle S_{\text{eff}} \rangle$ follows from the smallness of $S_{\text{eff}}(U = \mathbb{1})$.

As a further check on these results, we have computed S_{eff} for the free case in two entirely different ways: first by employing the approximation method described in

* The necessity of antiperiodic boundary conditions for fermions was first point out by Lüscher [17].

** Up to the order considered.

*** There is another argument in favour of $\langle S_{\text{eff}} \rangle \geqslant 0$ in the MC procedure: all those configurations with extremely negative fermionic action are strongly suppressed by the Boltzmann factor $\exp S_{\text{eff}}$.

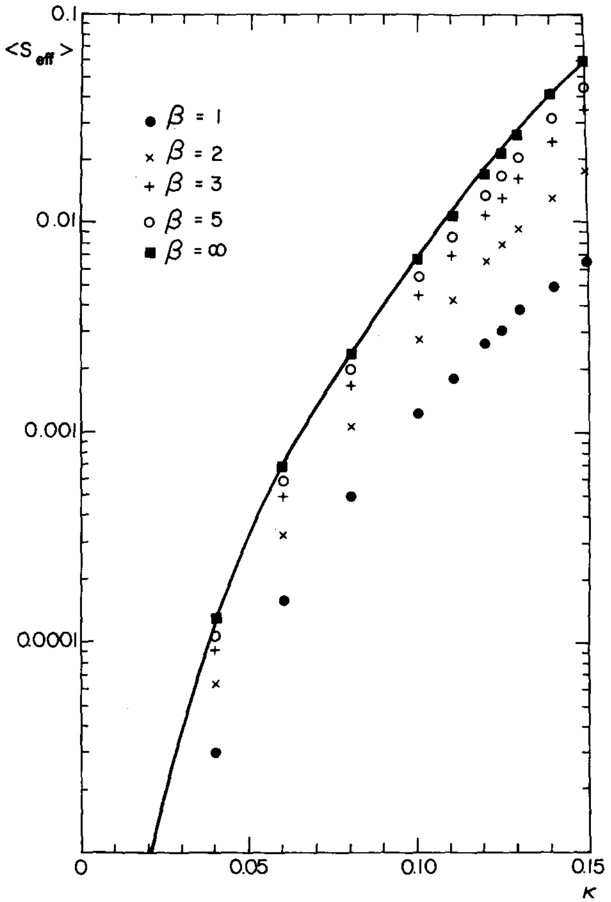

Fig. 2a. $\langle S_{\text{eff}} \rangle$ as a function of κ for $\beta = 1$ (●), 2 (×), 3 (+), 5 (○) and ∞ (■, all gauge fields frozen to unity). The full line depicts the exact solution on a 16^4 lattice with antiperiodic boundary conditions for the free fermion lattice theory.

sect. 3 (boxes in fig. 2a), and, secondly, by using the exact expression on an N^4 lattice (solid curve in fig. 2a)*

$$S_{\text{eff}}(\kappa, U = \mathbb{1}) = \frac{1}{2N^4} \sum_{p_\mu = 1/2, \ldots, N-1/2} \log\left[\left(1 - 8\kappa + 4\kappa \sum_\mu \sin^2 \frac{p_\mu}{2}\right) + 4\kappa^2 \sum_\mu \sin^2 p_\mu\right], \tag{4.3}$$

for $N = 16$. Notice that on account of the antiperiodic boundary conditions, the zero modes never appear in (4.3). Although we are effectively using free boundary conditions for the fermions in our first method, the two calculations are in excellent agreement as is evident from fig. 2a.

* Up to 12th order, $S_{\text{eff}}(\kappa, U = \mathbb{1}) = 48\kappa^4 + 1408\kappa^6 + 40\,992\kappa^8 + 1\,228\,800\kappa^{10} + 37\,538\,816\kappa^{12}$.

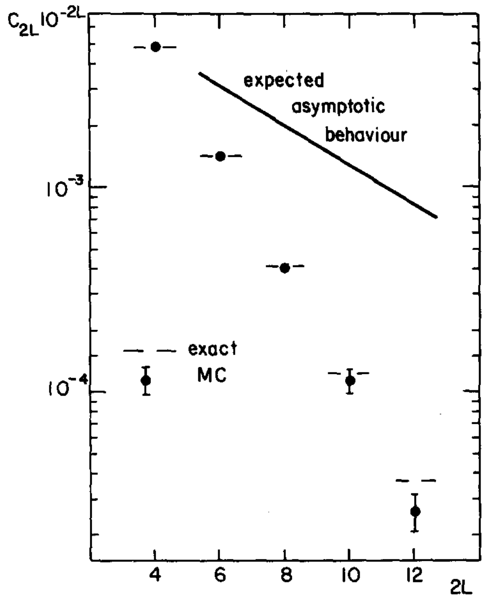

Fig. 2b. Coefficients C_{2L} of the expansion in κ^{2L} for the free theory as determined by our MC method (dots with error bars) in comparison with the exact values (vertical bars).

As a further indication of the efficiency of our method, we have plotted in fig. 2b the exact values of the first five coefficients of κ^{2L} in the expansion (2.6) for the free theory versus the ones determined from our MC simulation. The latter were obtained by applying the same procedure as before and setting $U=\mathbb{1}$, i.e., after averaging over ten lattices (for the free theory, this procedure is, of course, redundant; note also that the standard deviation increases in the coupled case due to the fluctuations in $\langle \mathrm{Tr}\Pi_\ell U_\ell \rangle$).

From these curves, there seems to be no indication of a phase transition at $\kappa=\frac{1}{8}$. Yet, the free theory does have a phase transition at this value of κ as may be seen from the thermodynamic limit of (4.3):

$$\lim_{N\to\infty} S_{\mathrm{eff}}(\kappa, U=\mathbb{1}) = \frac{1}{2(2\pi)^4} \times \int_{|p_\mu|\leq\pi} \mathrm{d}^4p \log\left[\left(1-8\kappa+4\kappa\sum_\mu \sin^2\frac{p_\mu}{2}\right)^2 + 4\kappa^2\sum_\mu \sin^2 p_\mu\right]. \tag{4.4}$$

The integrand has a logarithmic branch point singularity at $\kappa=\frac{1}{8}$ which is, however, softened by the four-dimensional integration*. The weakness of this singularity will

* In particular, all expansions converge in the *closed* interval $0\leq\kappa\leq\frac{1}{8}$.

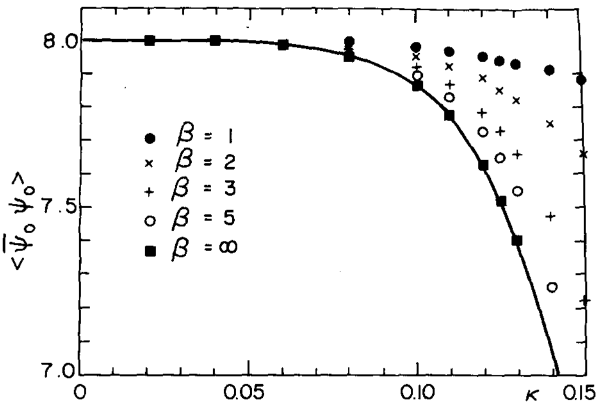

Fig. 3a. $\langle \bar{\psi}_0 \psi_0 \rangle$ as a function of κ for different values of β.

make it extremely difficult to detect by a numerical evaluation of $\langle S_{\rm eff} \rangle$ alone even in the free case. We expect this feature to persist for $\beta < \infty$, and indeed our curves do not show any evidence of a phase transition. Other and hopefully more efficient ways to determine the existence and location of a fermionic phase transition will be discussed below.

From fig. 2b, it is obvious that the expected asymptotic behaviour sets in at much higher values of $2L$. This is in accordance with the discussion of the entropy factors for the closed paths in sect. 3. Nevertheless, the coefficients decrease sufficiently fast to justify our approximation for $S_{\rm eff}$.

In fig. 3a, we have depicted the expectation value $\langle \bar{\psi}_0 \psi_0 \rangle$ as a function of κ for several values of β and, again, for the free theory. As the Wilson action explicitly breaks chiral invariance, we have $\langle \bar{\psi}_0 \psi_0 \rangle \neq 0$ even at $\kappa = \frac{1}{8}$ which corresponds to the massless case in the free lattice theory. Formally, chiral invariance is recovered for free fermions in the continuum limit $a \to 0$ by setting $1 - 8\kappa = 2\kappa a m_{\rm phys}$ and putting $m_{\rm phys} = 0$ afterwards [9]. However, in configuration space, $\langle \bar{\psi}_0 \psi_0 \rangle_{\rm free}$ is ill-defined in the continuum limit because of ultraviolet divergences. In fig. 3b, we have also plotted the difference $\Delta \langle \bar{\psi}_0 \psi_0 \rangle_\beta \equiv \langle \bar{\psi}_0 \psi_0 \rangle_\beta - \langle \bar{\psi}_0 \psi_0 \rangle_{\rm free}$ as a function of the variable

$$\beta_{1/2}(\beta) \equiv \frac{\int {\rm d}U \frac{1}{2} {\rm Tr}\, U \exp[\beta {\rm Tr}\, U]}{\int {\rm d}U \exp[\beta {\rm Tr}\, U]}, \qquad \big(\beta_{1/2}(0) = 0, \beta_{1/2}(\infty) = 1\big), \tag{4.5}$$

at $\kappa = \frac{1}{8}$. The difference vanishes for $\beta \to \infty$ as was to be expected from the scaling behaviour (we are grateful to J. Smit for a discussion on this point).

Finally, we have also considered the expectation values of string operators $\langle S_n \rangle$. As before, there are no sizeable modifications in the interval $0 \leqslant \kappa \leqslant \frac{1}{8}$. Presumably, the most suitable quantity which might signal a phase transition is $\log \langle S_{n+1} \rangle / \langle S_n \rangle$

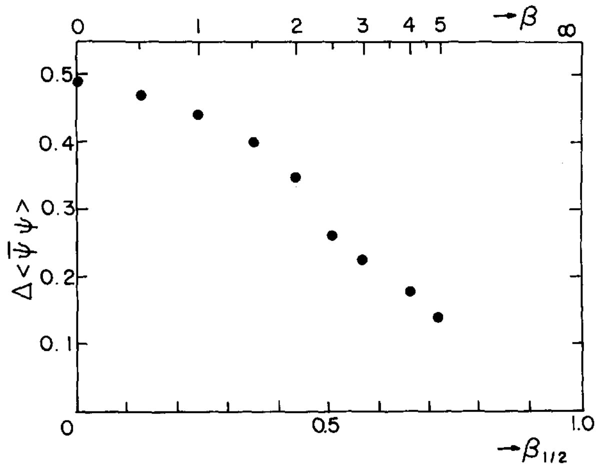

Fig. 3b. $\Delta\langle\bar{\psi}_0\psi_0\rangle \equiv \langle\bar{\psi}_0\psi_0\rangle_\beta - \langle\bar{\psi}_0\psi_0\rangle_{\text{free}}$ at $\kappa=\frac{1}{8}$ as a function of $\beta_{1/2}$.

which should be proportional to the inverse fermion correlation length and therefore become zero on the phase transition point. We have indeed computed this quantity for $n=1$ and 2, but in our present set-up with a 4^4 lattice, the available data do not permit any definite conclusions because a larger lattice will be needed for precise measurements of slowly decaying correlations.

5. Conclusions and outlook

The method proposed in this paper appears to be capable of incorporating fermions in MC simulations. One of its main advantages is its simple physical interpretation. For small κ, only small loops are relevant; with increasing κ, the fermionic correlation length increases too, and larger loops become important. In our first attempt on a 4^4 lattice, the truncation of paths of length $\geqslant 14$ seems quite justified. An extension to larger lattices or more flavours is straightforward. The necessary computer time is directly proportional to the lattice volume.

The free fermion lattice theory agrees remarkably well with our results for gauge fields frozen to unity. $\langle S_{\text{eff}}\rangle$ is rather small over the whole range $0 \leqslant \kappa \leqslant \frac{1}{8}$ which explains the almost negligible influence of the fermionic action on the updating of the gauge field configuration. Our choice of "hybrid" boundary conditions effectively reduces the finite size effects of the fermionic quantities.

On the other hand, our approximation did not allow reliable estimates on the asymptotic behaviour of the different expansions in κ, which presumably sets in at much higher orders of κ. These contributions, however, will be strongly interwoven with the finite size effects. In any approach, this will necessitate (maybe considerably) larger lattices.

For $\kappa > 0$, one expects asymptotic perimeter decay for the expectation value of the Wilson loop. From (2.8) it is clear that the effective fermionic action favours the ordering of rectangular Wilson loops for which the γ traces give the biggest contribution -2^{2L-1}. But in MC calculations, one is naturally restricted to finite loops and can never really distinguish between asymptotic area or perimeter law. The commonly adopted strategy is to look at the lower bound of quantities defined for finite rectangular loops that would become the string tension asymptotically. We expect that one has to go to quite large loops in order to find evidence for the perimeter decay.

Thus, an improvement of the proposed method will require larger lattices and the possibility of investigating larger loops. This would allow for a better understanding of our approximation and its merits and limitations. For this purpose, it would be attractive to have a reliable unbiased and fast random generator of closed path shapes.

Another interesting avenue for future research would be the determination of the gauge field expectation values of the coefficients of various fermionic quantities at $\kappa = 0$ by means of our MC method. In this way, one would combine a MC simulation for the pure gauge theory with a strong coupling expansion in κ.

It is a pleasure to thank D. Amati, B. Berg, D. Foerster, E. Marinari and C. Rebbi for many fruitful discussions.

Note added in proof

After submission of this paper we learnt of related work by A. and P. Hasenfratz (Central Res. Inst. f. Physics Budapest preprint KFKI-1981-47). In the meantime our results for S_{eff} have been confirmed up to sixth order in κ by Z. Kunszt. We want to thank both authors for stimulating discussions.

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