

# Chaos in Fermionic Many-Body Systems and the Metal-Insulator Transition

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We show that finite Fermi systems governed by a mean field and a few-body interaction generically possess spectral fluctuations of the Wigner-Dyson type and are, thus, chaotic. Our argument is based on an analogy to the metal-insulator transition. We construct a sparse random-matrix ensemble ScE that mimics that transition. Our claim then follows from the fact that the generic random-matrix ensemble modeling a fermionic interacting many-body system is much less sparse than ScE.

## I. INTRODUCTION

Finite fermionic many-body systems (atoms, molecules, nuclei) often display spectral fluctuation properties that agree with predictions of random-matrix theory (RMT), more precisely, with those of the Gaussian Orthogonal Ensemble (GOE) [1]. This fact is commonly taken as evidence for chaotic motion [2, 3]. In RMT, all pairs of states are coupled by independent matrix elements. In a many-body system that situation arises only in the presence of many-body interactions. But atoms, molecules and nuclei are governed by a mean field with residual interactions that are predominantly of two-body nature. Therefore, an important question is: Does a two-body interaction (or, more generally, a  $k$ -body interaction with  $k \geq 2$  integer but smaller than the number  $m$  of fermions) generically give rise to chaotic motion? The question has received much attention (see the review [4]). In condensed-matter physics it has recently been addressed as the problem of many-body localization [5].

While for small values of  $m$  the question can be answered by matrix diagonalization, a general answer is difficult to obtain both analytically and numerically because the Hamiltonian matrices become ever more sparse with increasing  $m$ : In every row and column the ratio of the number of non-zero matrix elements to the total number of matrix elements tends asymptotically ( $m \rightarrow \infty$ ) to zero. Therefore, analytical arguments like the ones in Ref. [6] (where that ratio is taken to be asymptotically finite) do not apply. And for the physically interesting cases  $k \geq 2$  matrices with dimensions that are numerically practicable are very far from the sparse limit.

The most determined effort so far to overcome these difficulties was made in Ref. [7]. The authors considered a fermionic many-body system governed by random one- and two-body interactions. For fixed parameter values of the one-body part, they determined the critical strength of the two-body interaction where the crossover from Poisson statistics to Wigner-Dyson statistics takes place. Their arguments are based on a combination of a perturbative approach and numerical results. Their re-

sult implies that a system governed by a pure two-body interaction is chaotic. Needless to say, the matrices studied numerically were far from the sparse limit.

In this paper we aim at a further clarification of the issue. We consider the matrix representation of a random Hamiltonian of the  $k$ -body type. We are interested in the way in which properties of the resulting ensemble of random matrices affect both, the shape of the average spectrum and the spectral fluctuation properties. Our approach differs from previous ones in that we are guided by an analogy to the metal-insulator transition (MIT). In disordered metals, the degree of disorder determines whether a system is an insulator (with Poissonian level statistics) or a metal (with Wigner-Dyson level statistics). As disorder increases, a transition from the metallic to the insulating regime takes place [8]. At the transition point, the spectral fluctuation properties are governed by a “critical statistic” which usually is characterized by three measures [9]: (i) The distribution of spacings  $s$  of nearest eigenvalues is linear in  $s$  for small  $s$  and falls off exponentially for large  $s$ ; (ii) the variance  $\Sigma^{(2)}(L)$  of the number of eigenvalues in an interval of length  $L$  is approximately logarithmic for small  $L$  and linear for large  $L$ ; (iii) the eigenfunctions display fractional statistics, and the distribution of the inverse participation ratios (sum of the fourth power of the expansion coefficients of the eigenfunctions in an arbitrary basis, see Eq. (6) below) scales with  $N$  like an inverse fractional power of  $N$ . Properties (i) and (ii) obviously interpolate between Wigner-Dyson statistics for short spectral distances and Poisson statistics for larger spectral distances.

Within the framework of random-matrix theory, an ensemble has been constructed [10] that simulates the critical statistic at the MIT. That ensemble is a special case of a power-law random band matrix (PLRBM). In such random matrix ensembles, the variances of the non-diagonal matrix elements  $H_{\mu\nu}$  fall off with some power  $2a$  of the distance  $|\mu - \nu|$  from the main diagonal. For  $a > 1$  the spectral statistics of the PLRBM is Poissonian, for  $a < 1$  it is of Wigner-Dyson type, and for  $a = 1$  it is critical [10]. We note that in contrast to Hamiltonian matrices for interacting Fermi systems the PLRBM is not sparse.

Motivated by the analogy to the MIT and by the PLRBM in Ref. [10], we construct a random-matrix ensemble (the “scaffolding ensemble” [ScE]) with the following properties. (i) ScE is more sparse than the Hamiltonian matrix of a fermionic many-body system with  $k$ -body interactions. (ii) The spectral fluctuation properties of ScE are those of the critical ensemble. This then suggests (and we show) that for all  $k \geq 2$ , the Hamiltonian matrix of the fermionic problem lies on the metallic side of the MIT and is, therefore, chaotic. The case  $k = 1$  is special and supplies additional arguments that support our reasoning.

Not surprisingly we cannot offer strict analytical proofs for some of these statements. Our arguments are based on a combination of analytical arguments, numerical evidence, and the application of a criterion due to Levitov [11] that is introduced below.

## II. EMBEDDED ENSEMBLE (EGOE( $k$ ))

Specifically we investigate a paradigmatic random-matrix model that simulates a fermionic many-body system: The Embedded Gaussian Orthogonal Ensemble (EGOE) [12]. We consider  $m$  spinless Fermions in  $l > m$  degenerate single-particle states labeled  $j = 1, \dots, l$  with associated creation and annihilation operators  $a_j^\dagger$  and  $a_j$ , respectively. The states carry no further quantum numbers. With  $1 \leq k \leq m$  the  $k$ -body Hamiltonian is

$$\mathcal{H}^{(k)} = \frac{1}{k!2} \sum_{j_1 \dots j_k : j'_1 \dots j'_k} v_{j'_1 \dots j'_k}^{j_1 \dots j_k} a_{j_1}^\dagger \dots a_{j_k}^\dagger a_{j'_k} \dots a_{j'_1} . \quad (1)$$

The matrix  $v$  is real-symmetric. The matrix elements are antisymmetric under the exchange of any pair of primed or unprimed indices and are uncorrelated random variables with a Gaussian probability distribution with zero mean value and a common second moment which without loss of generality is taken to be unity. The matrix representation of  $\mathcal{H}^{(k)}$  in the space of  $m$ -body Slater determinants labeled  $\mu$  or  $\nu$  with  $m$ -body matrix elements  $\langle \nu | \mathcal{H}^{(k)} | \mu \rangle = H_{\nu\mu}$  defines an ensemble of real random matrices of dimension  $N = \binom{l}{m}$  referred to as EGOE( $k$ ). We study the spectral properties of EGOE( $k$ ) in the limit  $N \gg 1$ . For fixed  $k$  that limit is reached by letting  $l, m \rightarrow \infty$ . Without distinction we consider both the dilute limit ( $m/l \rightarrow 0$ ) and the dense limit ( $m/l \rightarrow \text{constant} \neq 0$ ). In Dyson’s classification,  $\mathcal{H}^{(k)}$  has orthogonal symmetry. Our arguments apply likewise to the cases of unitary and symplectic symmetry. As mentioned above, for  $k < m$  EGOE( $k$ ) has withstood all attempts at a direct analytical treatment [4, 13].

## III. SCAFFOLDING ENSEMBLE (SCE)

We first describe the construction of the scaffolding ensemble. A criterion due to Levitov then suggests and nu-

merical simulations confirm that ScE possesses the critical behavior characteristic of the MIT.

We define ScE with the help of a real and symmetric scaffolding matrix  $A^{(n)}$  that has dimension  $N = 2^n$  with  $n$  positive integer. We recall the definition of the auxiliary diagonal for matrices of dimension  $N$ : The auxiliary diagonal has matrix elements with indices  $(\mu, N + 1 - \mu)$  and  $\mu = 1, \dots, N$ . We construct the matrix  $A^{(n)}$  by induction: For  $n = 1$ ,  $A^{(1)}$  has dimension 2, zero diagonal elements and unit entries in the auxiliary diagonal. Given  $A^{(n-1)}$ , the two diagonal blocks (dimension  $2^{(n-1)}$ ) of the matrix  $A^{(n)}$  are each occupied by  $A^{(n-1)}$ . The elements in the two off-diagonal blocks are all zero except for the auxiliary diagonal for which all elements have the value unity. We display the scaffolding matrix  $A^{(n)}$  for  $n = 3$  as an example,

$$A^{(3)} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} . \quad (2)$$

By construction, the matrices  $A^{(n)}$  have the important property  $\sum_{\nu=1}^N A_{\mu\nu}^{(n)} = \sum_{\mu=1}^N A_{\mu\nu}^{(n)} = n$  for all  $\mu = 1, \dots, N$ . Thus, in every row and column of  $A^{(n)}$ , the number of non-zero non-diagonal elements is  $n = \ln N / \ln 2$ . Hence, with increasing distance  $|\mu - \nu|$  from the main diagonal, the average density of non-diagonal elements of  $A^{(n)}$  falls off like  $1/|\mu - \nu|$ . The matrices  $A^{(n)}$  share that important property with the power-law random band matrices [10] that simulate the MIT. Moreover, the matrices  $A^{(n)}$  bear some similarity to the “ultrametric” matrices studied recently [14].

With the help of  $A^{(n)}$ , ScE is defined for every  $n$  as an ensemble  $H^{(n)}$  of random matrices. The non-zero elements of  $H^{(n)}$  reside on the unit elements of the matrix  $A^{(n)}$  (hence the name “scaffolding matrix” for  $A^{(n)}$ ) and on the main diagonal. Except for the symmetry condition  $H_{\mu\nu}^{(n)} = H_{\nu\mu}^{(n)}$  the matrix elements are uncorrelated real random variables with a Gaussian distribution and zero mean values. With  $\alpha$  positive and

$$B_{\mu\nu}^{(n)} = \alpha \delta_{\mu\nu} + A_{\mu\nu}^{(n)} , \quad (3)$$

the variances are given by

$$(1 + \delta_{\mu\nu}) \langle H_{\mu\nu}^{(n)} H_{\rho\sigma}^{(n)} \rangle = (\delta_{\mu\sigma} \delta_{\nu\rho} + \delta_{\mu\rho} \delta_{\nu\sigma}) B_{\mu\nu}^{(n)} . \quad (4)$$

In particular, the variance of every diagonal element is equal to  $\alpha$ .

We address the spectral properties of ScE. With  $E$  the energy and  $G(E) = 1/(E^+ - H^{(n)})$  the retarded Green’s function, the average level density is  $\rho(E) = -(1/\pi) \Im \langle G(E) \rangle$ . Here and in what follows, angular brackets denote the ensemble average. To calculate

$\langle G(E) \rangle$ , we expand  $G(E)$  in powers of  $H^{(n)}$  and use Wick contraction in each term of the sum. Following Ref. [12] we denote Wick-contracted pairs of matrix elements by the same letter and distinguish nested and cross-linked contributions. Among the sixth-order contributions, for instance,  $ABCCBA$  and  $ABBACC$  are nested while  $ABCABC$  and  $ABABCC$  are cross-linked. For  $n \gg \alpha$ , only nested contributions contribute to  $\langle G(E) \rangle$ . That rule (for the case of the GOE demonstrated in Ref. [12]) can be inferred by comparing the values of low-order terms like  $ABBA$ ,  $ABBACC$ , and  $ABCABC$ . Resummation of the nested contributions gives the Pastur equation  $\langle G(E) \rangle = (1/E) + (1/E)\langle H^{(n)}\langle G(E) \rangle H^{(n)} \rangle \langle G(E) \rangle$ . We use Eq. (4) and find

$$[E - \sum_{\rho} B_{\mu\rho}^{(n)} \langle G(E) \rangle_{\rho\rho}] \langle G(E) \rangle_{\mu\nu} = \delta_{\mu\nu}. \quad (5)$$

To solve Eq. (5) we observe that  $\langle G(E) \rangle$  is expected to be an analytic function in  $E$  with a finite number of branch points but without singularity at  $E = \infty$ . Therefore, we expand  $\langle G(E) \rangle$  for  $|E| \gg 1$  in a Laurent series,  $\langle G(E) \rangle_{\mu\nu} = \sum_{p=0}^{\infty} E^{-p} g_{\mu\nu}^{(p)}$ . Inserting that into Eq. (5) and comparing powers of  $E$  we find that non-vanishing solutions exist only for  $j = \pm 1$ . For both solutions we find that the coefficients  $g_{\mu\nu}^{(p)}$  are proportional to the unit matrix for all  $p$ . That conclusion hinges in an essential way on the fact that  $\sum_{\nu} A_{\mu\nu}^{(n)} = n$  for all  $\mu$  so that  $\sum_{\rho} B_{\mu\rho}^{(n)} = \alpha + n$  for all  $\mu$ , see Eq. (3). Thus  $\langle G(E) \rangle_{\mu\nu} = \delta_{\mu\nu} g(E)$ . To determine  $g(E)$  we use Eq. (5) and find with  $\lambda^{\text{sc}} = (n + \alpha)^{1/2}$  that  $\lambda^{\text{sc}} g(E) = (E/(2\lambda^{\text{sc}}) \pm i\sqrt{1 - (E/(2\lambda^{\text{sc}}))^2})$ . The two solutions with  $j = \pm 1$  correspond to the two signs in front of the square root. We conclude that the average spectrum has the shape of Wigner's semicircle, half the GOE radius  $\lambda^{\text{GOE}} \propto \sqrt{N}$  being replaced by  $\lambda^{\text{sc}} = \sqrt{n + \alpha} \approx \sqrt{n}$ .

We have not been able to establish the spectral fluctuation properties of ScE (and of EGOE( $k$ )) analytically. For the EGOE( $k$ ) we instead use a criterion established in Ref. [11]. To test the applicability of that criterion to sparse random matrices, we apply it to ScE. Levitov investigated a class of random matrices  $H_{\mu\nu}^{(L)}$  for which the variances  $\langle |H_{\mu\mu}^{(L)}|^2 \rangle$  of the diagonal elements are much larger than the variances  $\langle |H_{\mu\nu}^{(L)}|^2 \rangle$  ( $\mu \neq \nu$ ) of the non-diagonal elements. All of the latter differ from zero. Levitov considered the sum  $S = \sum_{\nu} (1 - \delta_{\mu\nu}) \langle |H_{\mu\nu}^{(L)}|^2 \rangle^{1/2} / \langle |H_{\mu\mu}^{(L)}|^2 \rangle^{1/2}$  in the limit of large matrix dimension  $N$ . Using renormalization-group arguments, he distinguished three cases: (i)  $S$  has a finite limit. Then, the spectral statistics of the ensemble is Poissonian. (ii)  $S$  diverges more strongly than  $\ln N$ . Then, the spectral statistics is of Wigner-Dyson type. (iii)  $S$  diverges like  $\ln N$ . Then the spectral statistics is that of the critical ensemble at the metal-insulator transition.

ScE and EGOE( $k$ ) are sparse random-matrix ensembles, and the renormalization-group argument used to establish Levitov's criterion is not readily applicable. Thus

it is not clear whether the criterion applies. If it does, ScE must for  $\alpha^{1/2} \gg 1$  possess the spectral statistics of the critical ensemble since  $\sum_{\nu} A_{\mu\nu}^{(n)} = n = \ln N / \ln 2$ . In a finite system the interesting regime is thus  $n \gg \sqrt{\alpha} \gg 1$ . Fortunately, this regime is just within reach of present-day numerical simulations.

We construct a number of realizations of the ScE and compute spectra via matrix diagonalization. We are limited to  $n \leq 13$ , and we consider the ScE for  $n = 10, \dots, 13$ . Our random-matrix ensembles consist of about 50 realizations for  $n = 13$  and of up to 30,000 realizations for  $n = 10$ . The wave-function statistics requires the largest ensembles. For both spectral statistics and wave-function statistics, we employ only the levels that are in the central 20% of the spectral densities. In this window, the average level density is maximal and nearly constant, and finite-size effects can be neglected. For the long-range  $\Sigma^{(2)}$  statistic, we verified that the results are insensitive to the degree of the polynomial fit of the average level density.

We first vary  $\alpha$  at fixed  $n = 10$  and study the nearest-neighbor spacing distribution. Figure 1 shows the transition from a delocalized Wigner-Dyson regime to a localized Poisson regime as  $\alpha$  is increased from  $\alpha = 3$  to  $\alpha = 100$ . (The ensembles consist of 300 realizations for  $\alpha = 3$  and of 3000 realizations for the other values of  $\alpha$ .) Figure 1 suggests that we are in the critical regime somewhere around  $\alpha \approx 10$ .

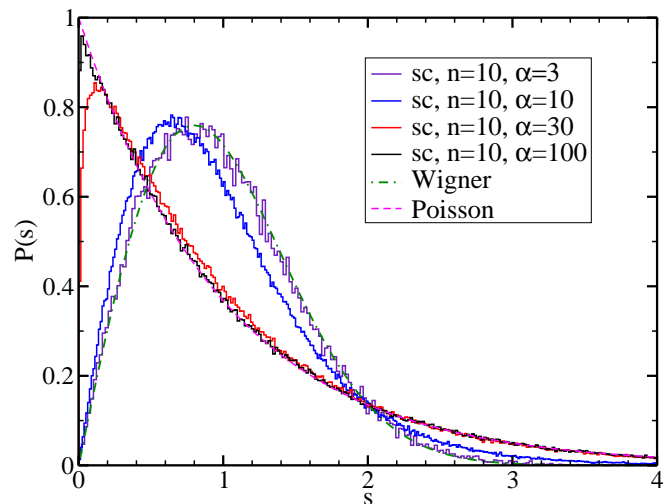


FIG. 1: (Color online) Nearest-neighbor spacing distribution  $P(s)$  of the scaffolding ensemble for  $n = 10$  and  $\alpha = 3, 10, 30, 100$  (histograms from right to left) versus  $s$ , the actual level spacing in units of the mean level spacing, compared to the Poisson distribution and Wigner's surmise (i.e., the GOE).

More detailed and extensive numerical calculations show that we approach the critical regime (the MIT) for  $\alpha \approx 17$  and  $n \geq 13$ . (Within the available computational resources, these parameter values optimally satisfy the conditions  $n \gg \sqrt{\alpha} \gg 1$ .) The nearest-neighbor spac-

ing distribution shown in Fig. 2 and the long-range  $\Sigma^{(2)}$  statistic shown in Fig. 3 support this claim. For these parameter values, the levels of the scaffolding ensemble are clearly strongly correlated at short distances and only weakly correlated at longer distances.

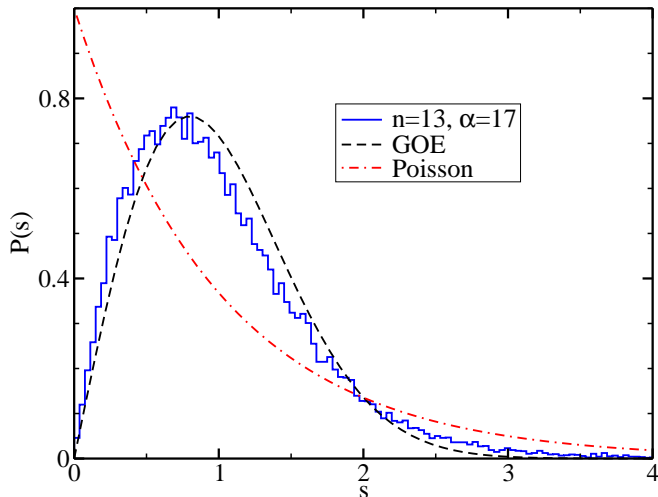


FIG. 2: (Color online) Nearest-neighbor spacing distribution  $P(s)$  as in Fig. 1 of the scaffolding ensemble for  $n = 13$  and  $\alpha = 17$  compared to the Poisson distribution and Wigner’s surmise for the GOE.

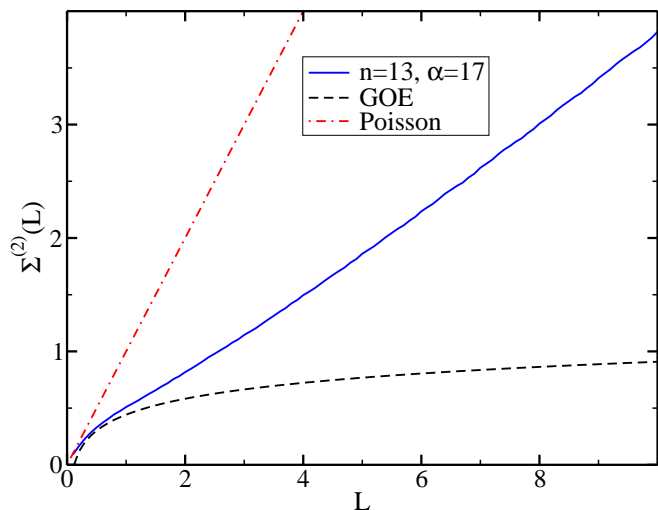


FIG. 3: (Color online) Long-range  $\Sigma^{(2)}$  statistic of the scaffolding ensemble versus the length  $L$  of the energy interval (in units of the mean level spacing) for  $n = 13$  and  $\alpha = 17$  compared to that of a Poissonian spectrum and the GOE.

The hallmark of the MIT, however, is a scale-invariant distribution of the inverse participation ratio (IPR) [15]. The IPR of a normalized state with expansion coefficients  $\psi_\nu$  is defined as

$$\text{IPR} = \sum_{\nu} |\psi_{\nu}|^4. \quad (6)$$

Figure 4 shows the distribution of the IPR for the eigenstates of the scaffolding ensemble. At fixed  $\alpha = 17$  and with increasing  $n$ , the distribution becomes scale invariant (the form of the distribution becomes independent of the dimension  $N = 2^n$  of the ensemble). Following Ref. [15], we determine the fractal dimension  $D_2$  from the shift of the IPR distribution that results from a doubling of the dimension as  $D_2 \approx 0.85$ . For the largest  $n$  shown, the IPR distribution exhibits a power-law tail with exponent  $x_2 \approx 1.5$ .

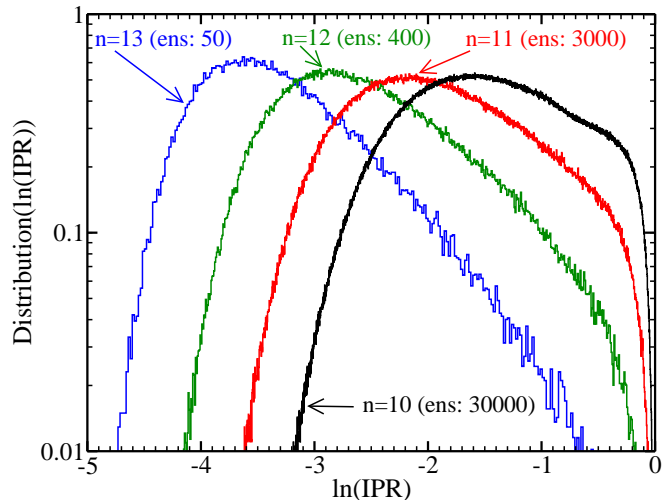


FIG. 4: (Color online) Distribution of inverse participation ratios (IPR) for  $\alpha = 17$  and  $n = 10, 11, 12, 13$ . As  $n$  is increased, the distribution approaches a scale-invariant form. The figures behind the symbol “ens” give the number of realizations.

Let us summarize the main results of this Section. We have shown analytically that the average spectrum of ScE has the shape of a semicircle. For the three fluctuation measures that characterize the MIT, our numerical results indicate that ScE possesses critical statistic for  $\alpha \gg 1$  and  $n \rightarrow \infty$ . This conclusion is on somewhat safer grounds than are statements for EGOE( $k$ ) simply because ScE is much more sparse: For  $n = 13$ , the fraction of non-zero matrix elements in every row and column is less than  $2 \times 10^{-3}$  while for EGOE( $k$ ) with  $k = 2$  and matrices of similar dimension, it is about two orders of magnitude bigger. For the ScE and  $n \gg \alpha^{1/2} \gg 1$ , Levitov’s criterion correctly indicates critical statistic. We conclude that Levitov’s criterion applies to sparse random matrices. We now use that fact to establish the spectral fluctuation properties of EGOE( $k$ ) for  $k > 1$ .

#### IV. SPECTRAL PROPERTIES OF EGOE( $k$ )

We compare the EGOE( $k$ ) defined in Section II with ScE as defined in Section III. We observe that EGOE( $k$ ) differs from ScE in three important ways. (i) Counting shows that for all  $k = 1, \dots, m$  the number of non-

vanishing non-diagonal matrix elements  $H_{\mu\nu}$ , equal in every row and every column, is given by  $\sum_{p=1}^k \binom{m}{p} \binom{l-m}{p}$  and, thus, for  $l, m \gg 1$  much larger than  $\ln N \approx m \ln l$ . Hence, for all  $k$  EGOE( $k$ ) is much less sparse than ScE. Therefore, we expect EGOE( $k$ ) to lie on the delocalized side of the MIT and to possess Wigner-Dyson spectral statistics. (ii) The number of  $k$ -body matrix elements  $v_{j_1 \dots j_k}^{j_1 \dots j_k}$  contributing to a fixed  $m$ -body matrix element  $H_{\mu\nu}$  is, in general, bigger than one. For  $k = 2$ , for instance, the  $m$ -body matrix element of two Slater determinants differing in the occupation numbers of orbitals 1 and 2 but both with occupied orbitals 3, 4,  $\dots$ , ( $m+1$ ) equals  $\sum_{j=3}^{m+1} v_{2j}^{1j}$ . In general, the matrix element connecting two Slater determinants that differ in the occupation numbers of  $p$  single-particle states is the sum of  $\binom{m-p}{k-p}$   $k$ -body matrix elements. We conclude that the variances of the diagonal elements are all equal and given by  $2 \binom{m}{k}$ ; those of the non-vanishing non-diagonal elements connecting two Slater determinants that differ in the occupation numbers of  $p$  single-particle states are  $\binom{m-p}{k-p}$ . It follows that for  $N \rightarrow \infty$  the variances of the diagonal elements are much bigger than those of all non-diagonal elements. That property is assumed by Levitov's criterion. (iii)  $m$ -body matrix elements occurring in different rows and columns may be correlated. For  $k = 2$ , for instance, the matrix element  $v_{34}^{12}$  contributes to all  $m$ -body matrix elements of pairs of Slater determinants for which the occupation numbers of orbitals (1 and 2) and (3 and 4) differ while all other occupation numbers agree. Correlations occur only among  $m$ -body matrix elements in different rows and columns because the same  $k$ -body matrix element cannot connect a given Slater determinant with two different ones.

*Neglect of Correlations.* We first consider EGOE( $k$ ) under neglect of all correlations, i.e., without property (iii), so that  $k$ -body matrix elements appearing in different locations of the  $m$ -body EGOE matrix are assumed to be uncorrelated Gaussian random variables. This assumption greatly increases the number of independent random variables and destroys the connection between the resultant random-matrix ensemble and the random  $k$ -body Hamiltonian of Eq. (1). Under this assumption we conclude immediately that for  $N \rightarrow \infty$ , the spectral properties of EGOE( $k$ ) are for all  $k$  the same as for GOE. The proof proceeds as for ScE. For the average level density we use the fact that  $\sum_{\nu} (1 - \delta_{\mu\nu}) \langle |H_{\mu\nu}|^2 \rangle = \sum_{p=1}^k \binom{m-p}{k-p} \binom{m}{p} \binom{l-m}{p}$  increases with  $N$  much more strongly than the variances  $2 \binom{m}{k}$  of the diagonal elements. That property guarantees that the Pastur equation holds. The analogue of the ScE relation  $\sum_{\rho} B_{\mu\rho}^{(n)} = \alpha + n$  for all  $\mu$  also holds: The scaffolding matrix of EGOE( $k$ ) has the same number of non-zero entries in every row and in every column. It follows as in Section III that the average spectrum has the shape of a semicircle. The radius  $2\lambda^{\text{EGOE}(k)}$  is given by  $(\lambda^{\text{EGOE}(k)})^2 = 2 \binom{m}{k} + \sum_{p=1}^k \binom{m-p}{k-p} \binom{m}{p} \binom{l-m}{p}$ .

For the spectral fluctuations we use Levitov's criterion. We avoid infinitely large variances for  $N \rightarrow \infty$  by rescaling the energy and all matrix elements of EGOE( $k$ ) by the factor  $(2 \binom{m}{k})^{-1/2}$ . Then the diagonal elements all have unit variance, and the variances  $\binom{m-p}{k-p} / [2 \binom{m}{k}]$  of the non-diagonal elements all become very small as  $N \rightarrow \infty$ . For all  $k \geq 1$  the critical sum  $S = \sum_{p=1}^k \binom{m-p}{k-p}^{1/2} \binom{m}{p} \binom{l-m}{p} / (2 \binom{m}{k})^{1/2}$  diverges more strongly with  $N \rightarrow \infty$  than  $\ln N \approx m \ln l$ , and Levitov's criterion implies that the spectral statistics is of Wigner-Dyson type.

*Influence of Correlations.* Correlations among  $m$ -body matrix elements occurring in different rows and columns influence the average level density  $\rho(E)$  and the spectral correlations in different ways. For  $\rho(E)$ , correlations cause deviations from the semicircular shape. Indeed, the Pastur equation is derived under the assumption that cross-linked contributions are negligible. That assumption fails in the presence of correlations, i.e., when  $\langle H_{\mu\nu} H_{\rho\sigma} \rangle \neq 0$  for  $\{\mu, \nu\} \neq \{\sigma, \rho\}$ . Such correlations, nonexistent for  $k = m$ , become stronger as  $k$  decreases, attaining a maximum at  $k = 1$ . For  $\rho(E)$  correlations cause cross-linked contributions to be as important as nested ones. Mon and French [12], calculating even moments of EGUE( $k$ ) in a basis of Slater determinants and using the representation of  $\mathcal{H}^{(k)}$  in Eq. (1), have shown that in the dilute limit and for  $k \ll m$  such contributions drive  $\rho(E)$  towards a Gaussian. Thus, the shape of the average spectrum of EGOE( $k$ ) is expected to change from Gaussian form for  $k = 1$  (where correlations are strongest) to semicircular shape for  $k = m$  (where correlations are absent).

We do not expect correlations between  $m$ -body matrix elements located in different rows and different columns of  $H_{\mu\nu}$  to influence the spectral fluctuations of EGOE( $k$ ). Wigner-Dyson statistics is a robust property of spectra caused by level repulsion. Such repulsion is caused by individual matrix elements connecting pairs of close-lying levels and is independent of the presence of other correlated matrix elements. Strong support for this qualitative argument comes from the study of EGOE(1).

## V. A SPECIAL CASE: EGOE(1)

The EGOE(1) is special: The real-symmetric matrix  $v_{j'j}^j$  can be diagonalized, and the eigenvalues follow Poisson statistics. The  $m$ -body matrix  $H_{\mu\nu}$  is then diagonal, too, each diagonal element being given by a sum of  $m$  such eigenvalues. For  $m \gg 1$  such sums are uncorrelated, and the spectrum is Poissonian. That symmetry of EGOE(1) is not obvious in the  $m$ -body matrix representation. In excluding such a hidden symmetry for  $k \geq 2$  we appeal to the results of numerical diagonalizations. Although done for matrices of small dimensions, such calculations should have revealed the existence of a symmetry.

Because of that special feature, the case  $k = 1$  can be used to support some of our arguments and conclusions very nicely. We compare three ensembles. (i) We consider the EGOE(1). As is well known [4, 12], the EGOE(1) has an average spectrum that is (nearly) Gaussian, and the eigenvalues have Poisson statistics. (ii) We consider an ensemble that has the same “scaffolding matrix” as EGOE(1) but for which all independent  $m$ -body matrix elements are uncorrelated Gaussian-distributed real random variables. In this ensemble, the connection with the EGOE(1) is severed, integrability is lost, and all correlations present in EGOE(1) are destroyed. Because of the lack of correlations between the elements of the random matrix, we expect the density  $\rho(E)$  to have semicircular shape. Fig. 5 shows that this is indeed the case. Because of the loss of integrability we also expect Wigner–Dyson statistics for the eigenvalues. This expectation is confirmed in Fig. 6. (iii) A third ensemble is generated by randomly redistributing the single-particle matrix elements  $v_j^i$  of the Hamiltonian (1) over the non-zero elements of the “scaffolding matrix” of EGOE(1). This random exchange of one-body matrix elements also destroys the connection of the resulting ensemble with the Hamiltonian (1) and, thereby, integrability. However, it retains the existence of correlations between matrix elements. For ensemble (iii) we, therefore, expect a (nearly) Gaussian form for the average level density  $\rho(E)$  but Wigner–Dyson statistics for the eigenvalues. Our numerical calculations for  $l = 12$  and  $m = 6$  confirm these expectations: Figure 7 shows that  $\rho(E)$  is close to a Gaussian, and Fig. 8 confirms the Wigner–Dyson statistics for the spacing distribution.

We have devoted particular attention to EGOE(1) not only because it is special but also because it is much closer to the sparse limit than EGOE( $k$ ) with  $k > 1$ . For practicable matrix dimensions it is difficult to draw valid conclusions about the spectral fluctuation properties of EGOE( $k$ ) with  $k \geq 2$  because the matrices are far from the sparse limit. Fortunately, correlations are strongest for  $k = 1$ . Our conclusions hold, therefore, *a fortiori* for  $k \geq 2$ . In particular, the results for EGOE(1) strongly support the conclusions drawn at the end of Section IV.

## VI. CONCLUSIONS

We have constructed a random-matrix ensemble (ScE) that is more sparse than EGOE( $k$ ) for all  $k$ . We have shown that ScE mimics the metal-insulator transition and possesses critical spectral statistics. Using ScE as a test case, we have verified that Levitov’s criterion applies to sparse random matrices.

Comparison with ScE suggests that for all  $k > 1$ , EGOE( $k$ ) is on the delocalized side of the metal-insulator transition and possesses Wigner–Dyson spectral statistics. We have presented a number of arguments which strongly support that expectation: Levitov’s criterion indicates chaos, and properties of the (modified) EGOE(1)

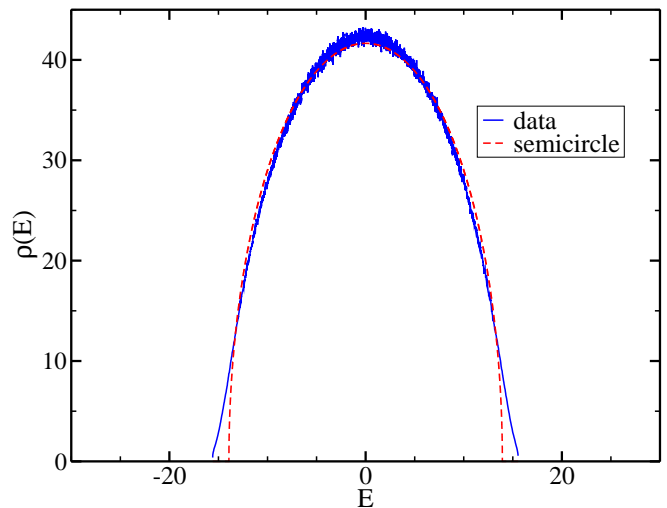


FIG. 5: (Color online) Level density  $\rho(E)$  (compared to a best-fit semicircular density) for a random-matrix ensemble with a scaffolding matrix of the EGOE(1) and random, uncorrelated matrix elements (case (ii) discussed in the text). The ensemble consists of 1000 realizations for  $l = 12$  orbitals and  $m = 6$  fermions.

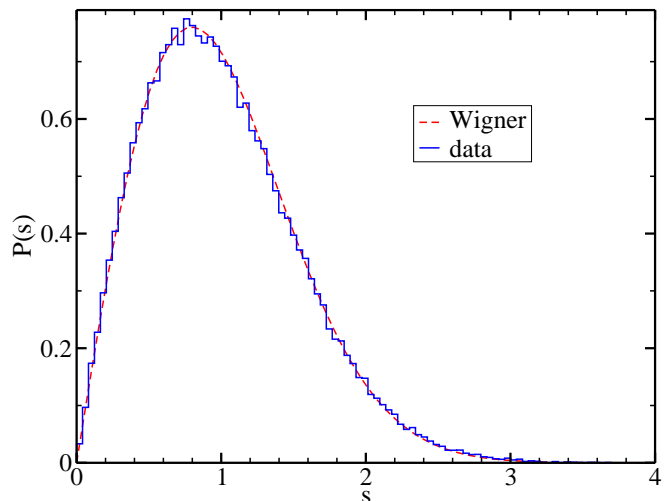


FIG. 6: (Color online) Nearest-neighbor spacing distribution  $P(s)$  compared to Wigner’s surmise for a random-matrix ensemble with a scaffolding matrix of the EGOE(1) and random, uncorrelated matrix elements (case (ii) discussed in the text). The ensemble consists of 1000 realizations for  $l = 12$  orbitals and  $m = 6$  fermions.

illustrate the different roles played by integrability of the underlying Hamiltonian on the one hand, and the existence of correlations between matrix elements on the other. We conclude that spectra in finite many-body systems governed by few-body interactions generically display Wigner–Dyson level statistics. By implication we conclude that in the limit of infinite matrix dimension, the distribution of the eigenfunctions is Gaussian. These conclusions hold for all three symmetry classes (orthogonal, unitary, symplectic).

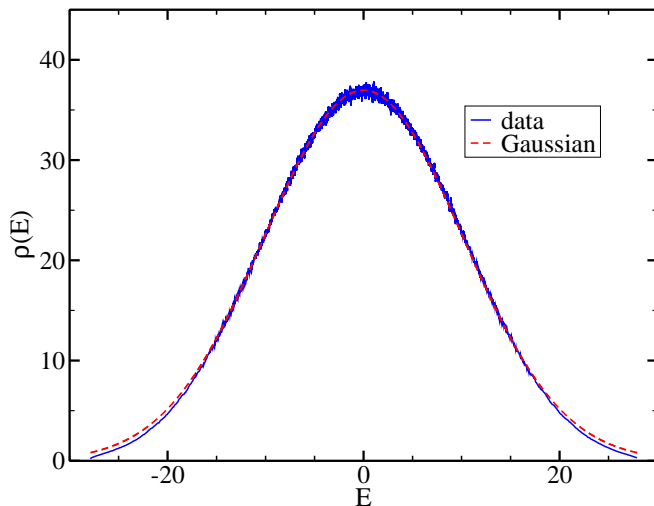


FIG. 7: (Color online) Average level density  $\rho(E)$  (compared to a best-fit Gaussian density) for a random-matrix ensemble with a scaffolding matrix of the EGOE(1) but with randomized correlated matrix elements from the EGOE(1) (case (iii) discussed in the text). The ensemble consists of 1000 realizations for  $l = 12$  orbitals and  $m = 6$  fermions.

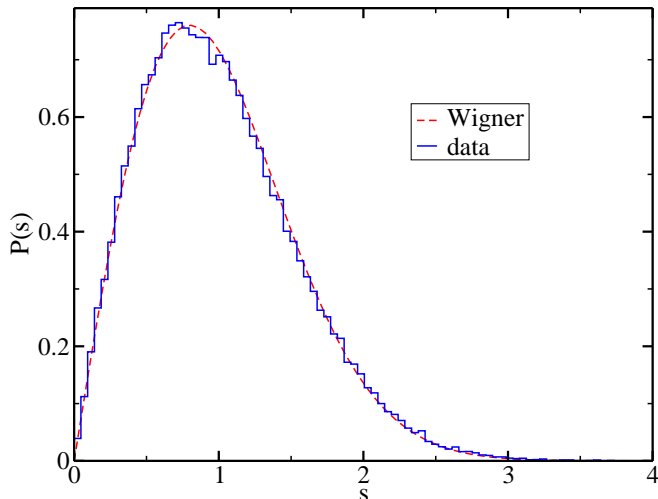


FIG. 8: (Color online) Nearest-neighbor spacing distribution  $P(s)$  compared to Wigner's surmise for a random-matrix ensemble with a scaffolding matrix of the EGOE(1) but with randomized correlated matrix elements from the EGOE(1) (case (iii) discussed in the text). The ensemble consists of 1000 realizations for  $l = 12$  orbitals and  $m = 6$  fermions.

EGOE( $k$ ) is based on the assumption that the single-particle states are degenerate. If that degeneracy is lifted, chaos may be reduced. We have not specifically addressed that case [7].

We believe that the results of this paper, although not entirely based on strict analytical arguments, convey new insight into the mechanisms that determine the spectral shape and the spectral fluctuation properties of fermionic many-body systems.

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[1] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, *Phys. Rep.* **299** (1998) 189.  
 [2] O. Bohigas, M. J. Giannoni, and C. Schmit, *Phys. Rev. Lett.* **52** (1984) 1.  
 [3] S. Heusler, S. Müller, A. Altland, P. Braun, F. Haake, *Phys. Rev. Lett.* **98** (2007) 044103.  
 [4] L. Benet and H. A. Weidenmüller, *L. Phys. A: Math. Gen.* **36** (2003) 3569.

[5] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, *On the problem of many-body localization*, in: *Problems of Condensed-Matter Physics*, A. L. Ivanov and S. G. Tikhodeev, editors, Oxford University Press, Oxford 2008.  
 [6] Y. V. Fyodorov and A. D. Mirlin, *Phys. Rev. Lett.* **67** (1991) 2049.  
 [7] Ph. Jacquod and D. L. Shepelyansky, *Phys. Rev. Lett.*

- 79** (1997) 1837.
- [8] P. W. Anderson, Phys. Rev. **109** (1958) 1492.
  - [9] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80** (2008) 1355.
  - [10] A. D. Mirlin, Y. V. Fyodorov, F.-M. Dittes, J. Quezada, and T. H. Seligman, Phys. Rev. E **54** (1996) 3221.
  - [11] L. S. Levitov, Phys. Rev. Lett. **64** (1990) 547.
  - [12] K. F. Mon and J. B. French, Ann. Phys. (N.Y.) **95** (1975) 90.
  - [13] M. Srednicki, Phys. Rev. E **66**, 046138 (2002).
  - [14] Y. V. Fyodorov, A. Ossipov, and A. Rodriguez, J. Stat. Mech. (2009) L12001.
  - [15] F. Evers and A. D. Mirlin, Phys. Rev. Lett. **84**, 3690 (2000).