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The Structure of Nonchiral Topological Order

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Zusammenfassung

Diese Doktorarbeit untersucht die Struktur von Phasen kondensierter Materie mit nichtchiraler topologischer Ordnung. Es wird gezeigt, dass bekannte Fixpunkt-Modelle solcher Phasen dieselbe Universalitätsklasse beschreiben. Neue Modelle werden aus Quantengruppen konstruiert, um die Klasse derjenigen Fixpunkte zu erweitern, deren Quasiteilchen-Anregungen eine vollständige lokale Klassifikation erlauben. Zugleich wird eine nicht-Abelsche, elektrisch-magnetische Dualität in diesen nichtchiralen topologischen Phasen entdeckt. Außerdem stellt diese Doktorarbeit neue, effiziente Beschreibungen für Quantenzustände mit nichtchiraler topologischer Ordnung vor. Dazu werden Fixpunkt-Wellenfunktionen direkt aus der algebraischen Beschreibung dieser Phasen mit Hilfe eines neuartigen graphischen Kalküls entwickelt.

Abstract

This thesis studies the structure of condensed matter phases with nonchiral topological order. It is shown that previously known fixed point models of such phases describe the same underlying universality class. New models are constructed from quantum groups in order to extend the class of those fixed points amenable to a complete local classification of their quasiparticle excitations. At the same time a non-Abelian electric-magnetic duality among these nonchiral topological phases is discovered. Furthermore, this thesis devises novel efficient descriptions of many-body quantum states with nonchiral topological order. In particular, fixed point wavefunctions are developed directly from the algebraic treatment of these phases by employing a novel diagrammatic calculus.

Publications

Publications and preprints this thesis is based on:

1. Oliver Buerschaper, Juan Martín Mombelli, Matthias Christandl, and Miguel Aguado. A hierarchy of topological tensor network states. 2010. arXiv:1007.5283 [cond-mat.str-el].

2. Oliver Buerschaper, Matthias Christandl, Liang Kong, and Miguel Aguado. Electric-magnetic duality and topological order on the lattice. 2010. arXiv:1006.5823 [cond-mat.str-el].

3. Oliver Buerschaper and Miguel Aguado. Mapping Kitaev's quantum double lattice models to Levin and Wen's string-net models. *Phys. Rev. B* 80.15 (2009), p. 155136.

4. Oliver Buerschaper, Miguel Aguado, and Guifré Vidal. Explicit tensor network representation for the ground states of string-net models. *Phys. Rev. B* 79.8 (2009), p. 085119.

Other publications I contributed to:

1. Robert N. C. Pfeifer, Philippe Corboz, Oliver Buerschaper, Miguel Aguado, Matthias Troyer, and Guifré Vidal. Simulation of anyons with tensor network algorithms. *Phys. Rev. B* 82.11 (2010), p. 115126.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Topological order

In condensed matter physics a phase of matter commonly refers to a class of materials (or Hamiltonians) which exhibit the same macroscopic properties although their microscopic details of interactions, constituents etc. may well differ. For a long time physical phases have been classified very successfully by LANDAU's paradigm of local symmetry breaking. It states that all Hamiltonians belonging to the same phase obey a certain symmetry and transitions between different phases entail a change of this symmetry pattern. Moreover these symmetries can be detected locally. A quantum magnet (whose interactions obey rotation symmetry) constitutes the classical example of this paradigm: while in the unordered phase the (local) magnetization vanishes, in the ordered phase it aquires a non-zero value along a particular spatial direction, thus changing the pattern of symmetry. However, during the last decades it turned out that physical phases of matter cannot always be explained by this local symmetry breaking theory. Instead, some phases which do not break any local symmetry may exhibit topological order [76]. Here the effective quantum field theory describing the material's long-range properties is a topological quantum field theory (TQFT) [80].

One of the most characteristic properties of a physical phase is the kind of emerging particles it supports. In conventional, local symmetry breaking phases only bosonic or fermionic quasiparticle excitations are known to occur. As is well known their statistics is derived from the one-dimensional representations of the symmetric group through either symmetrizing or antisymmetrizing the wavefunction in first quantization. However, in two space dimensions a surprising alternative becomes thinkable: interpreting particle exchange as an actual physical operation one discovers that the wordlines describing the particles' trajectories form intertwined tangles in spacetime. This is because the wordline of a particle poses an obstruction to the worldline of a second particle winding around the first one—in two dimensions the wordlines can never be unwound continuously to yield a pair of parallel strands in spacetime. Naturally, particle exchange in two space dimensions should then be described by the braid group (which extends the symmetric group) and particles themselves would be characterized by its representations. Taking this approach seriously one will arrive at rather unconventional particles with fractional spin called anyons [77, 78].

It would be tempting to regard this phenomenon as a mere mathematical curiosity,

however, it is far from that. In fact, it was the discovery of the fractional quantum Hall effect [69, 46] which sparked the exploration of physical phases beyond local symmetry breaking. Nowadays many more examples of topologically ordered phases are known and have been studied intensely, such as chiral p-wave superconductors [64, 33], an exactly solvable model on the honeycomb lattice [38], an extended Hubbard model [27] as well as certain quantum magnets [81], just to name a few.

Besides its obvious importance for condensed matter physics topological order also has a profound impact on quantum information. It turned out that certain topological phases of matter would be capable of supporting universal quantum computation by creating, braiding and annihilating their anyonic quasiparticle excitations [39, 24, 25]. In particular, this idea relies on so called non-Abelian anyons whose statistics are given by higher dimensional representations of the braid group. The great advantage of this approach to quantum computation lies in the fact that only non-local properties of the system are used for information processing and hence resilience to local noise and other local imperfections would be built intrinsically into the hardware. Besides processing quantum information its storage also appears to benefit tremendously from cross-fertilization with vital concepts found in topological order. For example, one hopes that a self-correcting quantum memory can be built from the intuition that in a topologically ordered system with a degenerate ground state subspace quantum information would be protected from all local error operators whose support is not on the order of the system size. While it is unclear whether the shortcomings of existing models at finite temperature [15, 59, 32, 37] can be resolved satisfactorily these models remain appealing as quantum error correcting codes. Among the most relevant examples one finds the toric code and quantum double models based on finite groups [39], surface codes [9, 26], colour codes [6] or the string-net models [48]. For example, Mochon proved that a quantum double model based on the smallest non-Abelian group S_3 is universal for quantum computation provided that certain "magic" states can be prepared [54].

The bridges between topological order, condensed matter and quantum information have turned out so fruitful that often it is not clear which discipline has seen the greatest impact from these ideas. For example, Kitaev's quantum double models were conceived as a stage for fault-tolerant quantum computation but borrow heavily from gauge theory concepts and are most relevant in condensed matter as universality classes of nonchiral topological phases. String-net models were originally invented for the purpose of describing fixed points of topological phases, too, but are now applied even to quantum error correction [44], for example.

The classification of topological phases of matter would not be possible without substantial use of methods and tools from mathematics. Just as the paradigm of local symmetry breaking relies on (ordinary) group theory in order to describe a conventional system's symmetries topologically ordered materials rely on quantum groups and category theory in order to describe their emergent non-local properties. From this perspective one might even say that understanding topological order boils down to the question of how to obtain non-trivial representations of the braid group

from the microscopic dynamics of a physical system. Of course, the devil lies in the detail of implementing the mathematically possible within the tight constraints nature imposes.

Perhaps the clearest meeting point of mathematics and physics in the study of topological order is the collection of fixed points of topological phases. Here the universal properties of a physical phase are clearly visible because the short-range details of the interactions found in a material have already been integrated out and correlation lengths can be neglected. However, in general there is no straightforward, *canonical* procedure to obtain a renormalization group flow that would yield the fixed point of a topological phase. Still there are fixed point examples of nonchiral topological phases such as the quantum double [39] and the string-net models [48]. Unfortunately a full classification of topological phases is unavailable today, even for these *nonchiral* topological phases, i.e. phases invariant under parity and time reversal. This thesis contributes to filling parts of this knowledge gap by studying the previously known fixed point models of nonchiral topological phases and explaining how they merely provide different *perspectives* onto the *same* underlying universality class. In addition we construct new models based on quantum groups in order to extend the class of such fixed points amenable to a complete local classification of their quasiparticle excitations.

Unravelling the structure of topological order and physical materials in general ultimately also requires deep knowledge about the quantum states of a many-body system, typically ground, thermal and low-lying excited states. In recent years it has become increasingly evident that the traditional ways of describing such quantum states in Hilbert space may not be the most adequate formulation. This is because only a small corner of Hilbert space is actually physically accessible to a system with less than exponential resources [63]. Results like area laws [19] bear witness to this fact.

In order to describe the physically relevant quantum states efficiently a number of ingeniously crafted trial wavefunctions have been developed which are capable of reproducing the essential physics very accurately. Prominent examples include the Laughlin wavefunction for the fractional quantum Hall effect at filling factor $\nu = 1/3$ [46] or the BCS wavefunctions for various types of superconductors [5, 64]. Furthermore, in the context of real space renormalization several remarkably successful tensor network approaches to wavefunctions emerged. Among them one finds matrix product states (MPS) [21], projected entangled pair states (PEPS) [70, 71] or the multi-scale entanglement renormalization ansatz (MERA) [74, 73]. All these wavefunctions share the property that the number of parameters involved scales only polynomially with the number of particles constituting the system, instead of exponentially for a generic wavefunction.

As far as topological order in spin systems is concerned, a number of efficient representations of topologically ordered quantum states have been found, including a PEPS wavefunction of the toric code [72], MERA wavefunctions of all quantum double [1] and string-net models [45] and a *double* line tensor network of the toric code and the semion model [29]. This thesis expands on this collection and devises

new tensor network methods for describing efficiently those parts of Hilbert space with nonchiral topological order. In particular, we develop fixed point wavefunctions directly from the algebraic treatment of these phases in terms of quantum groups and categories and introduce a novel diagrammatic calculus for this task.

Finally, it is well known that duality ranks among the deepest ideas in physics. Dualities serve as powerful tools to gain understanding of mathematical and physical theories, including quantum many-body systems and quantum field theories [65]. Frequently a duality relates strong and weak coupling regimes of a physical theory and provides unique insights beyond the reach of perturbative methods. Many physical instances of duality involve the exchange of electric and magnetic degrees of freedom. Such a symmetry for the Maxwell equations *in vacuo* led Dirac to the introduction of pointlike sources of magnetic field in order to extend this electric-magnetic duality to matter, providing a unique argument for the quantisation of electric charge [17]. Dirac's insight has had a profound influence in quantum field theory and beyond [68].

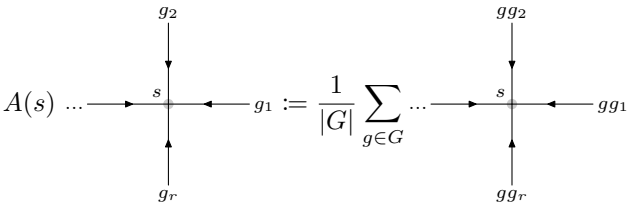
Instances of duality in topologically ordered systems are scarce in number unfortunately. As already mentioned the quantum double models build on an analogy to gauge theories with finite gauge groups [75, 79]. In the gauge context the topological degrees of freedom are Wilson loops. Violations of gauge invariance and magnetic fluxes are described analogously, hinting at electric-magnetic duality. In the toric code, the simplest quantum double model based on the group $\mathbb{Z}_2$, and in all quantum double models based on finite Abelian groups one finds the first instances of duality in a topologically ordered system: a self-duality which exchanges direct and dual lattice, in gauge theory language this is called an electric-magnetic duality. Its importance has been stressed by Fendley [22]. In string-net models unpublished work by Kitaev and Kong indicates that dualities, i.e. equivalences of unitary modular tensor categories, can be identified with invertible domain walls and, furthermore, excitations of string-net models can be described as superselection sectors of certain local operator algebras [40]. However, the *local* structure and properties of both these dualities and excitations are far from being understood. This thesis helps to gain a better understanding of dualities in topological phases by extending electric-magnetic duality to quantum double models based on non-Abelian groups and certain quantum groups as well as to a large class of string-net models.

Since this thesis studies the structure of nonchiral topological phases we will briefly review the known lattice models for their fixed points in the following.

1.2 Quantum double models from finite groups

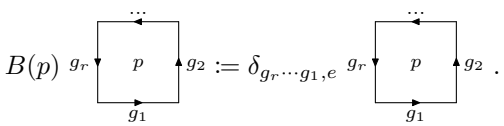
The quantum double model presented in [39] is a quantum spin model defined on any finite oriented graph $\Gamma = (V, E, F)$ with vertices V, edges E and faces F which can be embedded in a two-dimensional surface. Throughout this thesis we will denote the set of vertices surrounding a face $p \in F$ by $V(p)$, the set of all edges connected to a vertex $s \in V$ by $E(s)$ and similarly $E(p)$ will stand for the set of edges that

form the boundary of the face $p \in F$. In order to construct the Hilbert space $\mathcal{L}_\Gamma$ of the quantum double model one assigns to each oriented edge the local Hilbert space $\mathcal{L}_e = \mathbb{C}G$ whose canonical, orthonormal basis is $\mathcal{B}_G(\mathbb{C}G) = \{|g\rangle \mid g \in G\}$. Reversing the orientation of an edge corresponds to the map $|g\rangle \mapsto |g^{-1}\rangle$. The Hamiltonian is assembled from two sets of operators that act on vertices or faces of Γ respectively. At each vertex $s \in V$ one may orient the adjacent edges such that they point inwards. Then the vertex operator $A(s)$, which acts on $E(s)$, is given by



$$= \frac{1}{|G|} \sum_{g \in G} L^g_+(g_1) \otimes \cdots \otimes L^g_+(g_r) \tag{1.1}$$

where $L^g_+ \colon \mathbb{C}G \to \mathbb{C}G$, $|h\rangle \mapsto |gh\rangle$ denotes the left regular representation of G. Hence the action of $A(s)$ is a simultaneous left multiplication at each incident edge averaged over the group G. In fact, it projects onto the trivial representation. Using the orientation reversal isomorphism one may orient the edges around a face $p \in F$ in counterclockwise fashion. One then defines the face operator $B(p)$ by its action on basis states $|g_1, \ldots, g_r\rangle$:



(1.2)

Again, this is a projector which projects onto configurations with a trivial product of group elements around the face. Finally, the Hamiltonian reads:

$$\mathcal{H} = -\sum_{s \in V} A(s) - \sum_{p \in F} B(p). \tag{1.3}$$

This is a sum of mutually commuting projectors and therefore exactly solvable.

Note that from the gauge theory point of view edge degrees of freedom correspond to parallel transport operators with values in the gauge group and vertex operators project onto gauge invariant configurations. Face projectors minimise the Wilson action in that they project onto configurations with trivial magnetic flux through the face.

Since the Hamiltonian (1.3) is frustration-free its entire spectrum can be obtained by solving it *locally*. This means that excitations of the D($\mathbb{C}G$)-model correspond to local violations of vertex or face constraints and hence can be attributed a quasiparticle character. Moreover, elementary quasiparticle excitations can be conveniently classified algebraically. There are four classes of superselection sectors

or so called topological charges: (a) the vacuum, i.e. the ground state subspace, (b) magnetic excitations which violate a face constraint, (c) electric excitations which violate a vertex constraint and (d) dyonic excitations which violate both a vertex constraint and a face constraint at an adjacent face. Magnetic excitations are classified by conjugacy classes C of the group G, electric excitations by irreducible representations ρ of G and dyonic excitations by a conjugacy class C *and* an irreducible representation ρ_C of the centralizer subgroup $N_C \subset G$ of an (arbitrary) element $g \in C$.

Topological charge cannot only be attributed to vertices and faces but also to regions by lumping together all quasiparticle excitations in its interior. The total topological charge of a region can be measured from its boundary and even be defined for any closed loop on the underlying surface of the model, not necessarily the boundary of a region. For definiteness, projectors on the superselection sectors can be constructed using Kitaev's ribbon operators [39] which the vertex (1.1) and face projectors (1.2) are just examples of.

The data (C, ρ_C) describing the quasiparticle excitations of the spin model happen to coincide with the irreducible representations of a certain quasi-triangular Hopf algebra, Drinfeld's quantum double $\mathrm{D}(\mathbb{C}G)$ [18] of the group algebra $\mathbb{C}G$. In other words, quasiparticles are classified by their response to the action of a symmetry algebra, without doubt a very familiar and successful pattern in modern physics. Since this Hopf algebra symmetry extends beyond group symmetry the quasiparticles it entails turn out to be anyons, not merely bosons or fermions. Moreover, they obey the statistics of non-Abelian anyons precisely if the group G is non-Abelian.

Most importantly, in the $\mathrm{D}(\mathbb{C}G)$-models these quasiparticle excitations carry an internal space in addition to their topological charge (C, ρ_C). In the case of magnetic excitations the internal space consists of the elements of the conjugacy class C. For electric excitations the internal space equals the vector space the irreducible representation ρ_e acts in. For dyonic excitations one encounters a combination of both of the above. In sharp contrast to the categorical treatment of quasiparticles in topologically ordered systems this allows one to peek inside the "black box" given by the label (C, ρ_C). In fact, this lies at the heart of the various dualities we will discover in Chapters 3 and 6.

1.3 String-net models

String-net models were introduced in [48] in order to encode the universal physical properties of nonchiral $(2+1)D$ topological phases of matter in quantum lattice models with few-body interactions. The intuition behind their construction was that they should correspond to fixed points of some (abstract) renormalization procedure. Later this guiding principle was indeed confirmed using entanglement renormalization techniques [45]. As a result of their fixed-point property the Hamiltonians of string-net models take a fairly simple, frustration-free structure and are exactly solvable. In the following, we consider those models in [48] which exhibit a well-defined continuum limit.

The models are defined on a trivalent graph Γ, for simplicity we will assume a honeycomb lattice in the following. Local degrees of freedom are associated with oriented edges of Γ and elements of the computational basis are labelled by $i \in \{1, 2, \ldots, N\}$. These labels may be interpreted as particle species propagating along the edges. For each label i there is a unique label i^* denoting its antiparticle, and reversing the orientation of an edge corresponds to the mapping $i \mapsto i^*$. The distinguished label 1 stands for the absence of any particle (vacuum) and one has $1^* \coloneqq 1$ by definition. Furthermore, the physical Hilbert space of a string-net model is defined by a set of fusion rules δ_{ijk} specifying allowed ($\delta_{ijk} = 1$) and forbidden ($\delta_{ijk} = 0$) configurations of labels incident to a vertex. Given the set of labels and their fusion rules, one can build a fusion category (see [38] and below for a physical motivation) which includes recoupling relations encapsulated in the quantum $6j$-symbols F^{ijm}_{kln} and an assignment of quantum dimensions d_i to the labels. The total quantum dimension is given by

$$\mathcal{D} \coloneqq \sqrt{\sum_{i=1}^{N} d_i^2}\,. \tag{1.4}$$

Furthermore there is a natural correspondence between physical configurations on Γ and configurations of string-nets in a continuum model defined on the so called fat lattice. The latter is constructed from the physical lattice Γ by puncturing the underlying surface at the center of each face. String-nets in this continuum description consist of oriented strings carrying labels in the set $\{1, 2, \ldots, N\}$, joined at trivalent branch points in a way that respects the fusion rules δ_{ijk}, and avoiding the punctures. Fixed-point wavefunctions are constructed using local constraints [48] in order to define equivalence classes of configurations in the fat lattice. These constraints are crafted so as to enforce topological invariance of the wavefunction (which will eventually be identified with a quantum state in the physical, discrete lattice Γ) and are assembled from the objects d_i, F^{ijm}_{kln} introduced above. For a quantum state $|\psi\rangle = \sum_X \psi(X)\,|X\rangle$, where $|X\rangle$ is a basis configuration in the discrete lattice Γ and $\psi(X)$ its associated amplitude, they explicitly read:

$$\psi\Big(\overset{i}{\longrightarrow}\Big) = \psi\Big(\overset{\imath}{\smile}\Big)\,, \tag{1.5}$$

$$\psi\Big(\circlearrowright i\Big) = d_i\,\psi(\ldots)\,, \tag{1.6}$$

$$\psi\Big(i \to \overset{k}{\underset{l}{\circ}} \to j\Big) = \delta_{ij}\,\psi\Big(i \to \overset{k}{\underset{l}{\circ}} \to i\Big)\,, \tag{1.7}$$

$$\psi\Big(\overset{i}{j} \searrow m \swarrow \overset{l}{k}\Big) = \sum_n F^{ijm}_{kln}\,\psi\Big(\overset{i}{j} \to \boxed{n} \to \overset{l}{k}\Big)\,. \tag{1.8}$$

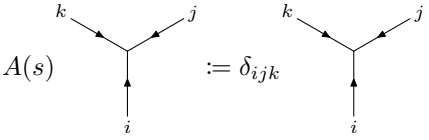
Intuitively, (1.5) ensures invariance under continuous deformations of strings, (1.6) describes trading isolated loops for quantum dimensions, (1.7) imposes charge conservation and finally (1.8) introduces a recoupling tensor F which generalizes the $6j$-symbols found e.g. in the theory of angular momentum.

String-net configurations are defined to be equivalent if they can be transformed into each other using the local relations. Equivalence classes are identified with physical configurations. Note that the physical configuration itself can be regarded as a particular string-net identical with the physical lattice. We will refer to this particular string-net as the canonical representative of the equivalence class, and its uniqueness is ensured by Mac Lane's coherence theorem [49]. As a consequence of this equivalence relation we may add or remove strings with the vacuum label 1 at will in the fat lattice picture, thus changing both isolated loops and branch points.

The Hamiltonian on the physical lattice reads

$$\mathcal{H} = -\sum_{s\in V} A(s) - \sum_{p\in F} B(p)\,, \tag{1.9}$$

where the sums range over the vertices and faces of the lattice. Vertex terms are projectors given by

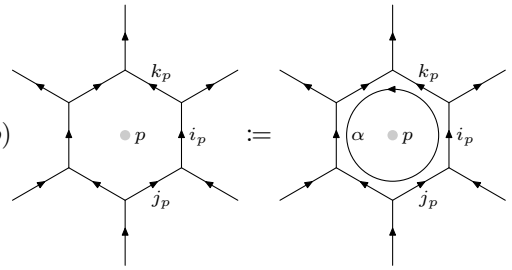
$A(s)$ 

(1.10)

enforcing the fusion rules while face projectors represent the kinetic part of the Hamiltonian and are defined by

$$B(p) = \sum_{\alpha=1}^{N} \frac{d_\alpha}{\mathcal{D}^2}\, B_\alpha(p)\,, \tag{1.11}$$

where $B_\alpha(p)$ acts on the edges of the enlarged face p, i.e. on $E(p)$ and the remaining edges connected to the vertices $V(p)$. Its precise definition is given in [48], as well as the following simple graphical interpretation on the fat lattice:

$B_\alpha(p)$ 

(1.12)

Reducing this expression in the fat lattice to its canonical representative in the physical lattice one finds that $B_\alpha(p)$ indeed acts on the entire enlarged face p. This action is nontrivial on the edges $E(p)$ and diagonal on the remaining outer edges.

From a more abstract perspective string-net models can be viewed as a procedure to construct a complete anyon model, including braiding, from a unitary fusion

category $\mathcal{C}$ the defining data of which we sketched above, i.e. fusion rules δ_{ijk}, quantum dimensions d_i, and quantum $6j$-symbols F^{ijm}_{kln}.

First of all, a fusion category is physically relevant since it encodes the essential properties of (topological) charge measurements, namely how to deduce the total charge of many particles given the individual charges of each particle. These charges are nothing but the particle labels introduced above which correspond to the (equivalence classes of) simple objects $\widehat{\mathcal{C}}$ in the fusion category $\mathcal{C}$. In a fusion process

$$a \times b = \sum_{c \in \widehat{\mathcal{C}}} N_{ab}^{c}\, c \tag{1.13}$$

it may happen that there are several (in fact $N_{ab}^{c} \in \mathbb{N}$) inequivalent ways of combining two given charges to a particular total charge[1]. In string-net models one usually restricts to multiplicity-free fusion and identifies $\delta_{abc} = N_{ab}^{c^*} \in \{0, 1\}$ although a slight generalization of the models would allow to incorporate general fusion multiplicities N_{ab}^{c} [48]. The antiparticle mapping $a \mapsto a^*$ mentioned earlier corresponds to a notion of duality in $\mathcal{C}$ which encodes the physical circumstances under which one should measure the neutral total charge 1. Furthermore, the quantum $6j$-symbols F^{abe}_{cdf} encode the physical assumption that measuring the total charge of three (or more) particles should not depend on which particles are measured first and for this reason are also called an associativity constraint. Finally, the quantum dimension d_a is an asymptotic statement about the number of inequivalent ways many particles of identical charge a can fuse to neutral total charge. Due to its asymptotic nature it is quite possible that a quantum dimension is non-integer and even irrational as is the case in the Fibonacci fusion category, for example.

As much as a fusion category abstracts physical charge measurement it is deficient in describing the statistics of the particles. In particular, what is lacking is a notion of braiding, i.e. exchanging particle positions while keeping track of (the topological details) of the wordline trajectories involved. The string-net construction can be seen as a method of adding precisely this missing notion of braiding by "doubling" the initial fusion category in a certain sense in order to obtain a new category. This resulting category is then a unitary modular tensor category [38] which is capable of describing the true quasiparticle excitations of the string-net model.

In [48] these quasiparticle excitations are introduced via *closed* string operators and analyzed by considering expectation values of virtual processes starting and ending in the vacuum sector, i.e. the subspace of ground states. However, so far there exists no satisfactory classification of actual quasiparticle excitations which would have to appear at the end of *open* string operators. This is in part because the quantum $6j$-symbols have no definite meaning for "fusion" processes which contradict the fusion rules of $\mathcal{C}$. In this situation one commonly sets the quantum $6j$-symbols to zero with the result that string-net configurations violating a vertex constraint automatically violate the constraints of all adjacent faces. If there were

[1] This should be familiar from Clebsch-Gordan decompositions in group theory, e.g. from the tensor product of two octet representations of $SU(3)$.

a consistent way of adding an internal structure to the "black box" labels of the fusion category one would gain a complete understanding of the entire excitation spectrum of string-net models. This will become most relevant in Chapters 3 and 6.

1.4 Reading guide

We begin our study of nonchiral topological phases in Chapter 2 by constructing ground state wavefunctions for all string-net models as *local* tensor networks which are built directly from the algebraic data defining the string-net models. In Chapter 3 we provide a unification of spin models describing the fixed points of nonchiral topological phases: we identify quantum double models based on finite groups with a subclass of string-net models via a completion of the local Hilbert spaces of the latter. In Chapter 4 we extend the quantum double models based on finite groups to generalized ones based on finite-dimensional Hopf C^*-algebras. In particular, we study how the Drinfeld quantum double of a Hopf algebra can be represented on a lattice. In Chapter 5 we develop a diagrammatic calculus for a new class of tensor network wavefunctions based on Hopf algebras. We then exploit this calculus to introduce a hierarchy of quantum many-body states among which many examples of topological order can be identified by construction and argue that the condensation of topological charges is the mechanism underlying this hierarchy. In Chapter 6 we uncover electric-magnetic duality for quantum double models based on non-Abelian groups (and beyond) and show that its natural stage is the generalized models based on Hopf algebras from Chapter 4. Furthermore we interpret these as extended string-net models certain projections of which recover the original string-net models. A net of dualities among fixed points of nonchiral topological phases emerges.

Chapter 2

Tensor network states for string-net models

In this chapter we provide a simple expression for the ground states of arbitrary string-net models in the form of *local* tensor networks. These tensor networks encode the data of the fusion category underlying a string-net model and thus represent all doubled topological phases of matter in the infrared limit according to [48]. Furthermore, our construction highlights the importance of the fat lattice equivalence between lattice and continuum descriptions of string-net models.

This construction expresses the ground state of an arbitrary string-net model as a tensor network built from F-symbols. The starting point is the form of the Hamiltonian $\mathcal{H} = -\sum_i P_i$ where the P_i are commuting projectors, which leads to the realisation of the ground level as the $+1$ eigenspace[1] of the product $\prod_i P_i$. The tensor network follows in a remarkably straightforward way. Its form is reminiscent of a classical statistical mechanical partition function with local (albeit possibly complex) weights, which is why we call it a Boltzmann weight tensor network[2]. The needed ingredients are the data of the underlying fusion category as explained in [48], i. e. the fusion rules and associated F-tensors. The construction is most appropriately understood from the fat lattice perspective.

Our Boltzmann weight tensor network differs from the MERA representation which is also based on F-symbols [45] in that it is much simpler than the latter—e. g. it is a two-dimensional network, while the MERA spans three dimensions. On the other hand, although conceived independently, for the toric code and semion models our construction coincides with the double line tensor network of [29], where the authors also hint at an unpublished result for generic string-net models.

2.1 Ground states from the fat lattice

In order to construct an explicit graphical expression for a ground state of a string-net model we start with the state $|1 \dots 1\rangle$ on the physical lattice where all edges carry the vacuum label 1. Note that this can be represented by a completely empty fat lattice. Obviously, this state is an eigenstate of all the $A(s)$ operators with

[1] Contrary to some treatments of stabilizer Hamiltonians note that in our case the other eigenvalue of each P_i is 0 rather than -1.

[2] For an early account of the link between partition functions and stabilizer Hamiltonians the reader is referred to [56, 57].

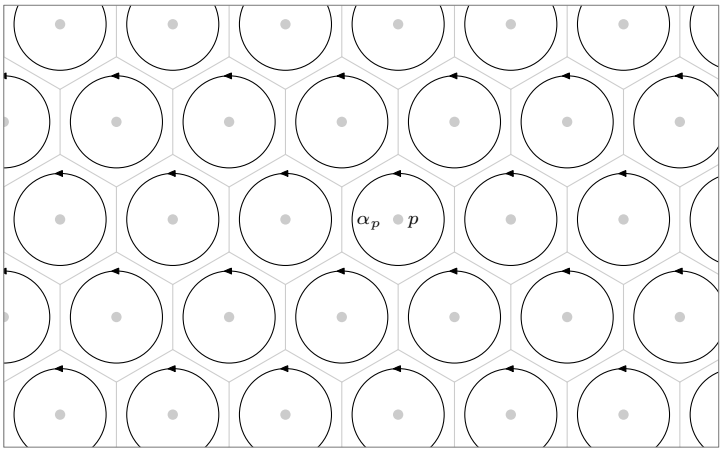


Figure 2.1: Initial fat lattice. An isolated loop at face p carries a label α_p and all edges of the physical lattice are labelled by the vacuum 1 (grey). The corresponding quantum state is denoted by $|\{\alpha_p\}\rangle$.

eigenvalue $+1$. Since the Hamiltonian is frustration-free we end up in the ground level by applying the projection $\prod_p B(p)$. Thus, up to an overall factor, this ground state on the physical lattice is represented by the following string-net state on the fat lattice:

$$|\psi\rangle = \sum_{\alpha_p=1}^{N} \left(\prod_p d_{\alpha_p} \right) |\{\alpha_p\}\rangle \tag{2.1}$$

where $|\{\alpha_p\}\rangle$ denotes the string-net configuration shown in Figure 2.1.

From now on we will use the local relations (1.5), (1.6), (1.7) and (1.8) of the string-net model in order to reduce (2.1) to its canonical representative, which can be directly translated into a configuration on the physical lattice. Here we only present the intermediate results of this reduction process, for details see Appendix A.

After applying three rounds of recouplings involving F-symbols (F-moves) to the strings on the fat lattice one has:

$$|\psi\rangle = \sum_{\alpha_p=1}^{N} \left(\prod_p d_{\alpha_p} \right) \sum_{i_p,j_p,k_p=1}^{N} \left(\prod_{(p,q)\in E_1} F^{\alpha_p^* \alpha_p 1}_{\alpha_q^* \alpha_q i_p} \right) \left(\prod_{(p,q)\in E_2} F^{\alpha_p^* \alpha_p 1}_{\alpha_q^* \alpha_q j_p} \right) \times \left(\prod_{(p,q)\in E_3} F^{\alpha_p^* \alpha_p 1}_{\alpha_q^* \alpha_q k_p} \right) |\{\alpha_p, i_p, j_p, k_p\}\rangle \,, \tag{2.2}$$

where $|\{\alpha_p, i_p, j_p, k_p\}\rangle$ denotes the state of the fat lattice as shown in Figure 2.2. Note that we have decomposed the edge set of the dual lattice Γ^* as $E(\Gamma^*) = \bigcup_{i=1}^{3} E_i$ where E_1 denotes the set of horizontal edges, E_2 one set of parallel diagonal edges, and E_3 the other one according to Figure 2.3.

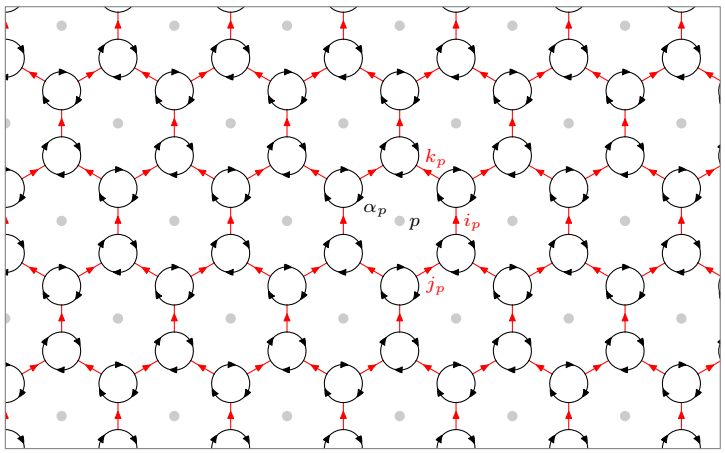


Figure 2.2: Intermediate fat lattice. This fat lattice configuration carries both the final physical labels i_p, j_p and k_p as well as three different labels α_p at each vertex loop. The corresponding quantum state is denoted by $|\{\alpha_p, i_p, j_p, k_p\}\rangle$.

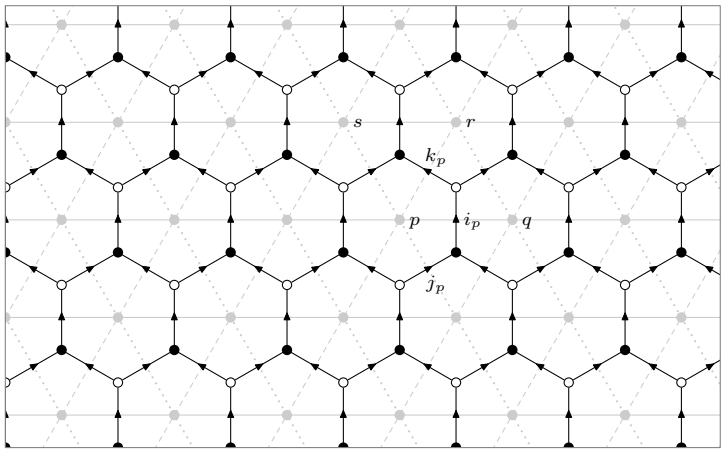


Figure 2.3: Honeycomb lattice Γ and its dual Γ^*. Horizontal edges (solid grey) of Γ^* belong to E_1, diagonal edges (dotted grey) to E_2 and the remaining diagonal edges (dashed grey) to E_3. The directed edges of Γ are labelled by uniquely associating them to a face. Circled vertices of Γ belong to the even sublattice $\Gamma_\circ$, filled ones to the odd sublattice $\Gamma_\bullet$.

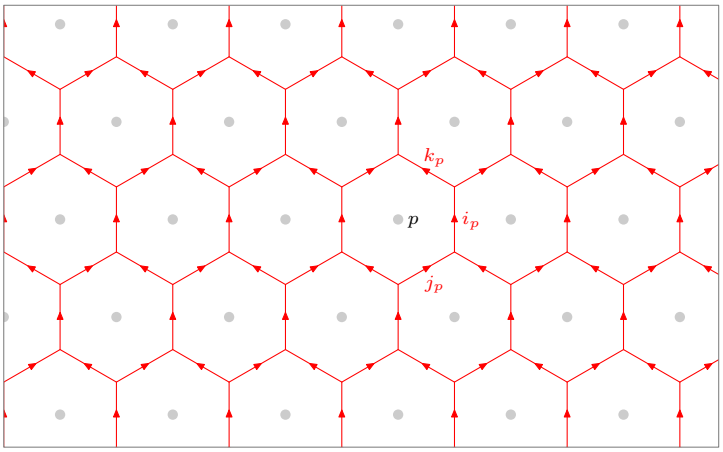


Figure 2.4: Physical lattice. The corresponding quantum state is given by $|\{i_p, j_p, k_p\}\rangle$.

Using the normalization

$$F^{ii^*1}_{j^*jk} = F^{ijk}_{j^*i^*1} = \sqrt{\frac{d_k}{d_i d_j}}\,\delta_{ijk} \tag{2.3}$$

this expression can be simplified in the case of an infinite or periodic lattice to yield:

$$|\psi\rangle = \sum_{\{\alpha_p, i_p, j_p, k_p\}_\star} \left(\prod_p \frac{\sqrt{d_{i_p} d_{j_p} d_{k_p}}}{d^2_{\alpha_p}} \right) |\{\alpha_p, i_p, j_p, k_p\}\rangle \,. \tag{2.4}$$

Note that we have omitted the δ-symbols and rather restricted the sum to configurations $\{\alpha_p, i_p, j_p, k_p\}_\star$ that respect the branching rules of the particular string-net model.

For a full reduction to the physical lattice we eventually need to remove the loops at the vertices. This can be done by applying two F-moves at each vertex:

$$|\{\alpha_p, i_p, j_p, k_p\}\rangle = \left(\prod_p \frac{d^2_{\alpha_p}}{\sqrt{d_{i_p} d_{k_p}}} \right) \prod_{s \in V(\Gamma_\circ)} f_\circ(s) \prod_{t \in V(\Gamma_\bullet)} f_\bullet(t)\, |\{i_p, j_p, k_p\}\rangle \,, \tag{2.5}$$

where $|\{i_p, j_p, k_p\}\rangle$ denotes a basis state of the physical lattice as shown in Figure 2.4 and $\Gamma_\circ$ ($\Gamma_\bullet$) denotes the even (odd) sublattice of Γ respectively. Furthermore one has:

$$f_\circ(s) = F^{\alpha_p^* j_p \alpha_q}_{k_q \alpha_r i_p} \,, \tag{2.6}$$

$$f_\bullet(t) = F^{\alpha_s^* i_s \alpha_p}_{j_r^* \alpha_r k_s} \tag{2.7}$$

where the faces $\{p, q, r\}$ of Γ surround an even vertex s and $\{p, r, s\}$ an odd vertex t as indicated in Figure 2.3.

At this point the ground state of the string-net model can be written in terms of the physical lattice only:

$$|\psi\rangle = \sum_{\alpha_p, i_p, j_p, k_p = 1}^{N} \left(\prod_p \sqrt{d_{j_p}} \right) \prod_{s \in V(\Gamma_\circ)} f_\circ(s) \prod_{t \in V(\Gamma_\bullet)} f_\bullet(t) \, |\{i_p, j_p, k_p\}\rangle \,. \tag{2.8}$$

Note that because of the normalization (2.3) of the F-symbols the branching rules at each vertex are automatically satisfied and we no longer need to restrict the sum. This allows one to isolate the wavefunction amplitude:

$$\psi(\{i_p, j_p, k_p\}) = \left(\prod_p \sqrt{d_{j_p}} \right) \sum_{\alpha_p = 1}^{N} \prod_{s \in V(\Gamma_\circ)} f_\circ(s) \prod_{t \in V(\Gamma_\bullet)} f_\bullet(t) \,. \tag{2.9}$$

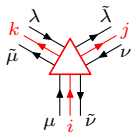
It is this very expression that we are now going to write in a graphical fashion as a contracted tensor network.

2.2 Tensor network representations

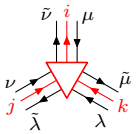
In order to write the wavefunction amplitude of the string-net model ground state given by (2.9) in a graphical fashion it is instructive to proceed locally. Let us therefore consider an arbitrary face a of Γ together with its next neighbours $b, \ldots, g$. Obviously, the sum over α_a can now be carried out immediately and we obtain the following local expression:

$$\begin{aligned} \psi(\{i_p, j_p, k_p\}) \sim \sqrt{d_{j_a} d_{j_d}} \sum_{\alpha_a = 1}^{N} & F^{\alpha_a^* j_a \alpha_g}_{k_g \alpha_b i_a} \, F^{\alpha_d^* j_d \alpha_a}_{k_a \alpha_c i_d} \, F^{\alpha_e^* j_e \alpha_f}_{k_f \alpha_a i_e} \\ & \times F^{\alpha_a^* i_a \alpha_b}_{j_c^* \alpha_c k_a} \, F^{\alpha_e^* i_e \alpha_a}_{j_d^* \alpha_d k_e} \, F^{\alpha_f^* i_f \alpha_g}_{j_a^* \alpha_a k_f} \,. \end{aligned} \tag{2.10}$$

Now define a vertex tensor for the even and odd sublattices of Γ respectively by



$$:= T^{[ijk]}_{\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}} := F^{\mu^* i \nu}_{j^* \lambda k} \, \delta_{\mu\tilde{\mu}} \, \delta_{\nu\tilde{\nu}} \, \delta_{\lambda\tilde{\lambda}} \,, \tag{2.11}$$



$$:= \tilde{T}^{[ijk]}_{\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}} := \sqrt{d_j} \, F^{\nu^* j \lambda}_{k \mu i} \, \delta_{\mu\tilde{\mu}} \, \delta_{\nu\tilde{\nu}} \, \delta_{\lambda\tilde{\lambda}} \tag{2.12}$$

and contract them according to the tensor network given in Figure 2.5. If we cut out a single face of this network it can easily be verified that it exactly reproduces the local form of our wavefunction amplitude (2.10), up to the factor $\sqrt{d_{j_e}}$ (which can be absorbed once the summation is extended to the adjacent faces).

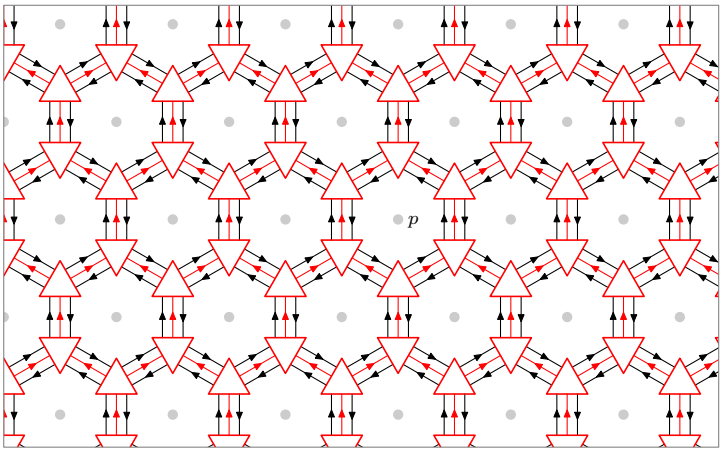
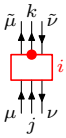


Figure 2.5: Tensor network describing the ground state of an arbitrary string-net model. Only virtual indices (black) are summed over while physical indices (red) are left uncontracted. As indicated these physical indices are associated with the edges of the honeycomb lattice and are shared between the adjacent vertex tensors.

Thus we have obtained a simple graphical notation that describes the ground state of an arbitrary string-net model and involves local terms only. In fact, following the arguments of [48], our graphical calculus encompasses the ground states of all "doubled" topological phases in the infrared limit.

Of course, we may also pull out the indices from the vertex tensors and collect physical indices denoting particle and antiparticle into a single physical index at the edge. This can be done by defining the tensors



$$\coloneqq A^{[i]}_{jk\mu\tilde{\mu}\nu\tilde{\nu}} \coloneqq \delta_{ij}\delta_{jk}\delta_{\mu\tilde{\mu}}\delta_{\nu\tilde{\nu}} \tag{2.13}$$

as well as the triangular ones

$$B_{\alpha\beta\gamma\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}} \coloneqq T^{[\alpha\beta\gamma]}_{\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}}\,, \tag{2.14}$$

$$\tilde{B}_{\alpha\beta\gamma\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}} \coloneqq \tilde{T}^{[\alpha\beta\gamma]}_{\mu\tilde{\mu}\nu\tilde{\nu}\lambda\tilde{\lambda}} \tag{2.15}$$

and contracting them according to Figure 2.6. Note that the vertex tensors T ($\tilde{T}$) and B ($\tilde{B}$) only differ in how their indices are regarded: what used to be a physical index of T ($\tilde{T}$) has been changed into a virtual one of B ($\tilde{B}$).

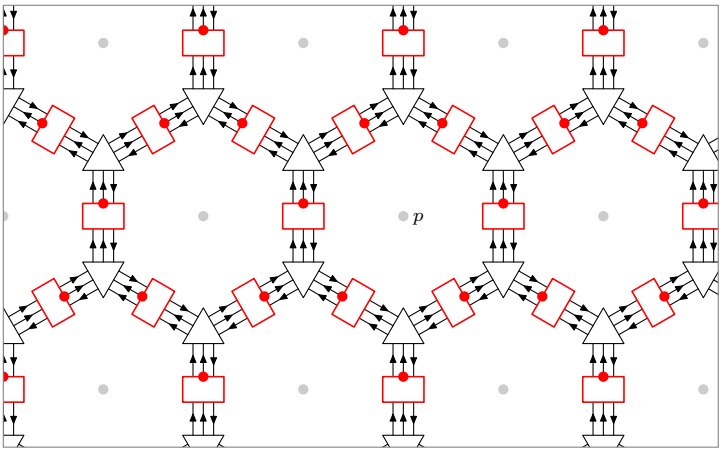


Figure 2.6: Alternative tensor network for ground states of arbitrary string-net models. Again all virtual indices (black) are contracted. Physical indices are now represented by their own edge tensors (red) while the vertex tensors (black) merely serve as a "metric" for the contraction.

2.3 Discussion

In conclusion, in this chapter we have derived a remarkably simple tensor network representation for Levin and Wen's string-net ground states. This construction follows directly from the characterisation of these states as simultaneous +1 eigenstates of the projectors in the Hamiltonian. It also heavily relies on the notion of the fat lattice. Understanding string-net models in terms of the mapping from the fat lattice to the physical lattice thus leads to insight and useful results. The tensor network is built from the fusion rules and F-tensors of the fusion category underlying the string-net model.

Note that from our Boltzmann weight tensor network one can trivially build a PEPS representation. In the case of quantum double models which can be explicitly written as string-net models (see Chapters 3 and 6) dramatic simplifications to this PEPS representation are possible due to their representation-theoretical properties (see Chapter 5). Also, for a general string-net model excited states may be expressed by absorbing their corresponding open string operators into a ground state tensor network representation. By the same token tensor network representations for other degenerate ground states on topologically nontrivial surfaces can be derived since in the fat lattice picture these correspond to an almost empty lattice with a single (nontrivial) string wrapping around a fundamental loop of the surface.

Chapter 3

Quantum double string-net duality

It is desirable to understand the relations among quantum double and string-net models which both describe fixed points of nonchiral topological phases of matter. As argued in [48] $2D$ string-net models are conceived to encompass all topological phases of this type, and it is implied that the discrete gauge theory phases described by quantum double models should be contained in the class of string-net models. In this chapter we prove that quantum double models indeed coincide with a subclass of string-net models. The essential step of our argument is a completion of the local Hilbert spaces in the string-net models considered. In more abstract terms, this is an example of Morita equivalence[1] (the origin of this concept can be found in [53]; see for instance [2]), whereby the local degrees of freedom in the lattice may be seen as objects in a category, and the physical excitations, equivalent in both cases, correspond to a representation category.

3.1 Revisiting string-net models

Let us first set

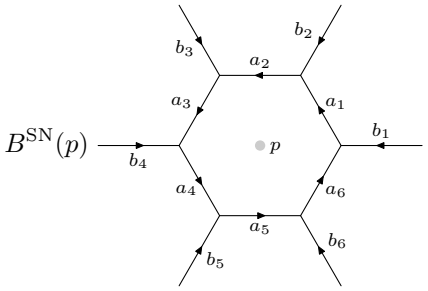
$$\begin{Bmatrix} i & j & m \\ k & l & n \end{Bmatrix} := F^{ijm}_{kln} \tag{3.1}$$

for the sake of notational clarity.

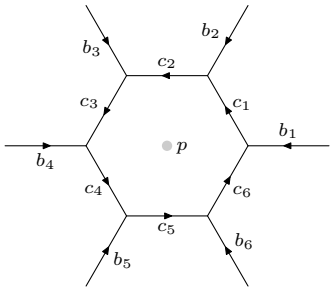
The explicit action of the $B^{\mathrm{SN}}_c(p)$ operators is spelled out in [48]. Without loss of generality we can restrict to an extended face in the honeycomb lattice and obtain

[1] We are indebted to Zhenghan Wang for pointing this out.

the following net result for (1.11):



$$= \sum_{c \in \widehat{\mathcal{C}}} \frac{d_c}{\mathcal{D}^2} \sum_{c_n \in \widehat{\mathcal{C}}} \left(\prod_{n=1}^{6} \begin{Bmatrix} b_n & c_n^* & c_{n-1} \\ c & a_{n-1} & a_n^* \end{Bmatrix} \right)$$



. (3.2)

As can be seen the action of $B^{\mathrm{SN}}(p)$ is nontrivial only on the (inner) edges $E(p)$ of the face p, however, it still depends on the outer edges where it acts diagonally. All in all, $B^{\mathrm{SN}}(p)$ constitutes a twelve-body operator.

All face and vertex constraints can be seen to commute with each other. This allows for a quite explicit treatment of the models. Levin and Wen studied the properties of physical excitations (which constitute a complete anyon model, or a unitary *braided* tensor category in the language of [38]) by looking at loop operators commuting with the Hamiltonian. However, there is no general representation-theoretic classification of excitations as that for quantum double models.

Note in addition that the definition of the face projectors in (3.2) uses F-symbols, and in principle these are defined only for processes with legal vertices, i.e. satisfying the fusion rules. As in [48] one may set them to zero whenever one of the involved vertices is illegal, however, this looks artificial. In Section 3.3 we will see how the definition of $B^{\mathrm{SN}}(p)$ for legal vertices agrees with and is naturally generalised by that of the quantum double models for the subclass of string-net models in the range of our mapping.

3.2 Fourier basis and "physical" subspace

Consider the $\mathrm{D}(\mathbb{C}G)$-model defined on a planar graph Γ, and perform a basis change at each oriented edge to the Fourier basis $\mathcal{B}_{\mathcal{F}}(\mathbb{C}G) = \{|b_{\mu,ij}\rangle\}$ defined by

$$|b_{\mu,ij}\rangle := \sqrt{\frac{d_\mu}{|G|}} \sum_{g \in G} D^{\mu}_{ij}(g) \, |g\rangle \,, \tag{3.3}$$

where $\mu \in \widehat{G}$ is an irreducible representation of G and D^μ is an arbitrary, but fixed matrix realisation of that representation μ with dimension d_μ. Standard representation-theoretical orthogonality relations imply that $\mathcal{B}_\mathcal{F}(\mathbb{C}G)$ is an orthonormal basis.

The inverse change of basis is given by

$$|g\rangle = \sum_{\mu\in\widehat{G}} \sqrt{\frac{d_\mu}{|G|}} \sum_{i,j=1}^{d_\mu} \overline{D^\mu_{ij}(g)}\, |b_{\mu,ij}\rangle\,, \tag{3.4}$$

and the isomorphism corresponding to edge reversal reads

$$|b_{\mu,ij}\rangle \mapsto |b_{\mu^*,ji}\rangle\,. \tag{3.5}$$

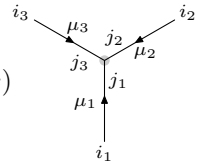
The change to the Fourier basis $\mathcal{B}_\mathcal{F}(\mathbb{C}G)$ can be interpreted loosely as splitting the local degrees of freedom into three subspaces, one labelled by the irreducible representations of G and the other two labelled by matrix elements of these representations. (This is not a rigorous interpretation because the dimensions of the latter subspaces depend on the irreducible representation. The rigorous statement is the Peter-Weyl theorem.)

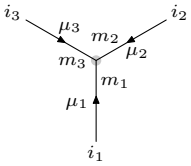
We now argue that the matrix indices are naturally associated with the beginning and end of an oriented edge as in

$$i \xrightarrow{\ \mu\ } j \coloneqq |b_{\mu,ij}\rangle\,, \tag{3.6}$$

and that the effect of vertex projectors in the Hamiltonian (1.3) is to determine the contraction of these indices at each vertex. Thus the degrees of freedom remaining after imposing vertex projectors are just the irreducible representations of G, for which the model can be interpreted as a string-net model. More precisely, this will turn out to be the string-net model based on the fusion category $\mathrm{Rep}(G)$ (the category of representations of the group G) with fusion rules determined from Clebsch-Gordan decomposition of tensor products of irreducible representations.

Indeed, using the inverse change of basis (3.4) it is easy to check that

$A^{\mathrm{QD}}(s)$ 

$= \sum_{m_n=1}^{d_{\mu_n}} (W_1^{\mu_1\mu_2\mu_3})_{m_1m_2m_3,j_1j_2j_3}$ 

(3.7)

which we wrote for a trivalent vertex s without loss of generality. Here

$$(W_1^{\mu_1\mu_2\mu_3})_{i_1i_2i_3,j_1j_2j_3} \coloneqq \frac{1}{|G|} \sum_{g\in G} D^{\mu_1}_{i_1j_1}(g)\, D^{\mu_2}_{i_2j_2}(g)\, D^{\mu_3}_{i_3j_3}(g) \tag{3.8}$$

is the matrix element of the projector $W_1^{\mu_1\mu_2\mu_3}$ onto the trivial isotypic subspace of $\mu_1 \otimes \mu_2 \otimes \mu_3$. It is clear that this generalizes in the obvious way to $W_1^{\mu_1\cdots\mu_r}$ for

vertices of arbitary degree r as does the action (3.7) of $A^{\mathrm{QD}}(s)$. Thus the effect of the vertex projector $A^{\mathrm{QD}}(s)$ is to project out vertex configurations in which the irreducible representations μ_n in the tensor product $\mu_1 \otimes \cdots \otimes \mu_r$ are not coupled to yield the trivial representation. This corresponds to the fusion rules in the string-net model to be identified below.

Moreover, since $W_1^{\mu_1\mu_2\mu_3}$ is a projector it can be split into a direct sum of orthogonal rank-one projectors. If the group G is multiplicity-free this decomposition simply reads

$$(W_1^{\mu_1\mu_2\mu_3})_{i_1 i_2 i_3, j_1 j_2 j_3} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ i_1 & i_2 & i_3 \end{bmatrix} \overline{\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ j_1 & j_2 & j_3 \end{bmatrix}} \tag{3.9}$$

where

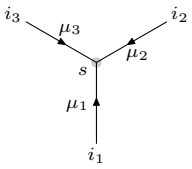
$$\begin{bmatrix} \mu & \nu & \lambda \\ i & j & k \end{bmatrix} \tag{3.10}$$

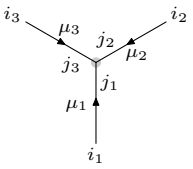
is the $3j$-symbol of G. More generally, if G is not multiplicity-free or one considers tensor products $\mu_1 \otimes \cdots \otimes \mu_r$ with $r > 3$ there may be several inequivalent ways of fusing $\mu_1 \times \cdots \times \mu_r \to 1$. This can be accounted for by introducing another index for the fusion multiplicity $N^1_{\mu_1 \ldots \mu_r} \coloneqq \operatorname{tr} W_1^{\mu_1 \cdots \mu_r}$ at each vertex.

Altogether we have shown that the action of $A^{\mathrm{QD}}(s)$ fixes how the indices j_n of a vertex configuration $|\{b_{\mu_n, i_n j_n}\}\rangle$ are contracted. Remember that we have defined this action assuming that all edges incident to the vertex s point towards s. Therefore, these indices correspond naturally to the ends of the oriented edges.

Now consider the action of the entire set of vertex projectors on the lattice. Then all matrix indices are contracted according to the annihilation channels of the incoming representations. Hence, if we consider the "physical" Hilbert space to be the surviving subspace after application of all $A^{\mathrm{QD}}(s)$, the only degrees of freedom left are precisely the irreducible representations of G living on oriented edges, with the constraint that representations incident on a given vertex can fuse to the vacuum.

More precisely, we refer to the system in which just irrep labels are associated with oriented edges, obeying fusion rules, as the string-net lattice (with corresponding "physical" Hilbert space $\mathcal{L}_\Gamma^{\mathrm{SN}}$) and we identify a configuration $|\{\mu_n\}\rangle_{\mathrm{SN}}$ there with the simultaneous $+1$ eigenvectors $|\{\mu_n\}\rangle_{\mathrm{QD}} \in \mathcal{L}_\Gamma^{\mathrm{QD}}$ of all vertex operators $A^{\mathrm{QD}}(s)$ of the quantum double model. We obtain these eigenvectors by inductively contracting the matrix indices associated with the vertices like so

$|\{\mu_n, i_n\}\rangle_{\mathrm{QD}} \coloneqq$ 

$\coloneqq \sum_{j_n=1}^{d_{\mu_n}} \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ j_1 & j_2 & j_3 \end{bmatrix}$ 

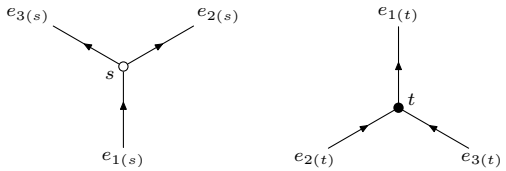
(3.11)

and define the state $|\{\mu_n\}\rangle_{\mathrm{QD}}$ as the result of this induction process which eliminates all matrix indices i_n eventually. Then the map $\mathcal{L}_\Gamma^{\mathrm{SN}} \hookrightarrow \mathcal{L}_\Gamma^{\mathrm{QD}}$ given by

$$|\{\mu_n\}\rangle_{\mathrm{SN}} \mapsto |\{\mu_n\}\rangle_{\mathrm{QD}} \tag{3.12}$$

is the desired embedding of the "physical" Hilbert space of the string-net model into the Hilbert space of the quantum double model.

For example, for a honeycomb lattice with the following oriented edges e_n around even and odd vertices



(3.13)

one has explicitly

$$|\{\mu_n\}\rangle_{\mathrm{QD}} = \sum_{i_n,j_n=1}^{d_{\mu_n}} \prod_{s\in V(\Gamma_\circ)} \begin{bmatrix} \mu_{1(s)} & \mu^*_{2(s)} & \mu^*_{3(s)} \\ j_{1(s)} & i_{2(s)} & i_{3(s)} \end{bmatrix} \times \prod_{t\in V(\Gamma_\bullet)} \begin{bmatrix} \mu^*_{1(t)} & \mu_{2(t)} & \mu_{3(t)} \\ i_{1(t)} & j_{2(t)} & j_{3(t)} \end{bmatrix} \bigotimes_{n\in E} |b_{\mu_n,i_nj_n}\rangle\,. \quad (3.14)$$

3.3 Face projectors and identification

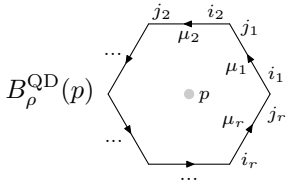
In order to compute the action of the face projectors $B^{\mathrm{QD}}(p)$ in the Fourier basis $\mathcal{B}_{\mathcal{F}}(\mathbb{C}G)$ note that

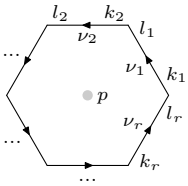
$$\delta_{g,e} = \sum_{\rho\in\widehat{G}} \frac{d_\rho}{|G|}\,\chi_\rho(g)\,, \quad (3.15)$$

with $\chi_\rho = \operatorname{tr} D^\rho$ the character of the irreducible representation ρ and $e \in G$ the identity element. Then we can write the face projector of the quantum double model as

$$B^{\mathrm{QD}}(p) = \sum_{\rho\in\widehat{G}} \frac{d_\rho}{|G|}\,B^{\mathrm{QD}}_\rho(p)\,, \quad (3.16)$$

where we defined the operator $B_\rho^{\mathrm{QD}}(p)$ as

$B_\rho^{\mathrm{QD}}(p)$ 

$$:= \sum_{\nu_n} \sum_{k_n,l_n=1}^{d_{\nu_n}} \sum_{m_n} \prod_{n=1}^{r} \sqrt{d_{\mu_n} d_{\nu_n}}\, (W_1^{\mu_n \nu_n^* \rho})_{i_n k_n m_n, j_n l_n m_{n+1}}$$ 

, (3.17)

or equivalently in terms of its matrix elements:

$$\langle \{b_{\nu_n,k_n l_n}\} | B_\rho^{\mathrm{QD}}(p) | \{b_{\mu_n,i_n j_n}\} \rangle = \sum_{m_n=1}^{d_\rho} \prod_{n=1}^{r} \sqrt{d_{\mu_n} d_{\nu_n}}\, (W_1^{\mu_n \nu_n^* \rho})_{i_n k_n m_n, j_n l_n m_{n+1}} \,. \tag{3.18}$$

In both cases the index n is understood to be cyclic.

We assert that this is the correct action of face projectors in the associated string-net model, i.e. the string-net model based on the fusion category $\mathrm{Rep}(G)$. Remember from [48] and Section 3.1 that this action is best understood in the fat lattice picture, namely via the loop insertion operators $B_a^{\mathrm{SN}}(p)$:

$$B^{\mathrm{SN}}(p) = \sum_{a \in \widehat{\mathcal{C}}} \frac{d_a}{\mathcal{D}^2} B_a^{\mathrm{SN}}(p) = \sum_{\rho \in \widehat{G}} \frac{d_\rho}{|G|} B_\rho^{\mathrm{SN}}(p) \,. \tag{3.19}$$

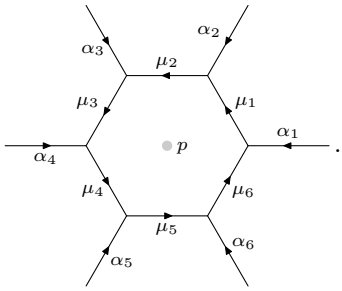
Here we have identified the label set $\widehat{\mathcal{C}}$ of the fusion category $\mathcal{C}$ the string-net model is based on with the set $\widehat{G}$ of irreducible representations of G, the quantum dimension d_a of label a with the (ordinary) dimension d_ρ of the irreducible representation ρ of G and finally the total quantum dimension as

$$\mathcal{D}^2 = \sum_{a \in \widehat{\mathcal{C}}} d_a^2 = \sum_{\rho \in \widehat{G}} d_\rho^2 = |G| \,. \tag{3.20}$$

As mentioned in Section 3.1 the operator $B_\rho^{\mathrm{SN}}(p)$ is equivalent to creating a loop of label ρ around the puncture of face p in the fat lattice and then absorbing it into the physical lattice by means of F-moves. Now in the case of group representations F-symbols are just $6j$-symbols[2] which can be written entirely in terms of $3j$-symbols via the projectors $W_1^{\mu_1 \mu_2 \mu_3}$ from (3.8). This is explained in Appendix B.

[2] This is also true for more general algebraic objects with a reasonable representation theory, such as Hopf C^*-algebras (and beyond).

In order to identify the quantum double with a string-net model we restrict to a trivalent graph, without loss of generality to a honeycomb lattice. We need to show that the action of the $B_\rho^{\mathrm{QD}}(p)$ operators on the reduced QD states $|\{\mu_n\}\rangle_{\mathrm{QD}}$ is the same as the action of $B_\rho^{\mathrm{SN}}(p)$ on states $|\{\mu_n\}\rangle_{\mathrm{SN}}$ in the SN lattice. To this end we consider a hexagonal face together with its outer edges and associate with it the following state

$|\{\mu_n\},\{\alpha_n\}\rangle_{\mathrm{SN}} :=$ 

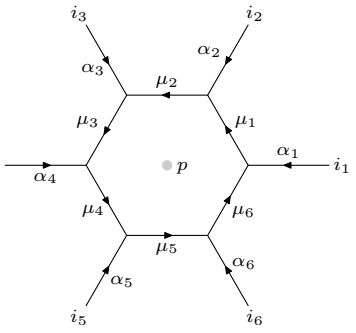
. (3.21)

By (3.2) the action of the loop insertion operator of the string-net model based on the fusion category $\mathrm{Rep}(G)$ is

$$
\begin{aligned}
&\langle\{\nu_n\},\{\alpha_n\}|B_\rho^{\mathrm{SN}}(p)|\{\mu_n\},\{\alpha_n\}\rangle_{\mathrm{SN}} \\
&\quad=\prod_{n=1}^{6}\begin{Bmatrix}\alpha_n & \nu_n^* & \nu_{n-1}\\ \rho & \mu_{n-1} & \mu_n^*\end{Bmatrix} \\
&\quad=\sum_{m_n=1}^{d_\rho}\sum_{i_n,j_n=1}^{d_{\mu_n}}\sum_{k_n,l_n=1}^{d_{\nu_n}}\sum_{a_n=1}^{d_{\alpha_n}}\prod_{n=1}^{6}\sqrt{d_{\mu_n}d_{\nu_n}}\,\overline{\begin{bmatrix}\alpha_n & \nu_n^* & \nu_{n-1}\\ a_n & k_n & l_{n-1}\end{bmatrix}} \\
&\qquad\times\overline{\begin{bmatrix}\rho & \mu_{n-1} & \nu_{n-1}^*\\ m_n & j_{n-1} & l_{n-1}\end{bmatrix}}\begin{bmatrix}\mu_{n-1} & \alpha_n & \mu_n^*\\ j_{n-1} & a_n & i_n\end{bmatrix}\begin{bmatrix}\nu_n^* & \rho & \mu_n\\ k_n & m_n & i_n\end{bmatrix}
\end{aligned}
\tag{3.22}
$$

where (B.8) has been used to express F-symbols in terms of $3j$-symbols.

Extending (3.11) we define states in the quantum double model by

$|\{\mu_n\},\{\alpha_n,i_n\}\rangle_{\mathrm{QD}} :=$ 

(3.23)

for the same extended face. The local Hilbert spaces are full group algebras $\mathbb{C}G$ labelled by irrep and matrix indices, but the latter are contracted with the $3j$-symbols. The action of the quantum double model face operator on these states is

$$\begin{aligned}
&\langle\{\nu_n\},\{\alpha_n,b_n\}|B_\rho^{\mathrm{QD}}(p)|\{\mu_n\},\{\alpha_n,b_n\}\rangle_{\mathrm{QD}} \\
&\quad=\sum_{i_n,j_n=1}^{d_{\mu_n}}\sum_{k_n,l_n=1}^{d_{\nu_n}}\sum_{a_n,\tilde{a}_n=1}^{d_{\alpha_n}}\prod_{n=1}^{6}\overline{\begin{bmatrix}\nu_n^* & \nu_{n-1} & \alpha_n\\ k_n & l_{n-1} & \tilde{a}_n\end{bmatrix}}\begin{bmatrix}\mu_n^* & \mu_{n-1} & \alpha_n\\ i_n & j_{n-1} & a_n\end{bmatrix} \\
&\qquad\times\langle\{b_{\nu_n,k_n,l_n}\},\{b_{\alpha_n,b_n,\tilde{a}_n}\}|B_\rho^{\mathrm{QD}}(p)|\{b_{\mu_n,i_n,j_n}\},\{b_{\alpha_n,b_n,a_n}\}\rangle \\
&\quad=\sum_{i_n,j_n=1}^{d_{\mu_n}}\sum_{k_n,l_n=1}^{d_{\nu_n}}\sum_{a_n,\tilde{a}_n=1}^{d_{\alpha_n}}\prod_{n=1}^{6}\overline{\begin{bmatrix}\nu_n^* & \nu_{n-1} & \alpha_n\\ k_n & l_{n-1} & \tilde{a}_n\end{bmatrix}}\begin{bmatrix}\mu_n^* & \mu_{n-1} & \alpha_n\\ i_n & j_{n-1} & a_n\end{bmatrix} \\
&\qquad\times\langle\{b_{\nu_n,k_n,l_n}\}|B_\rho^{\mathrm{QD}}(p)|\{b_{\mu_n,i_n,j_n}\}\rangle\,\langle\{b_{\alpha_n,b_n,\tilde{a}_n}\}|\{b_{\alpha_n,b_n,a_n}\}\rangle \\
&\quad=\sum_{m_n=1}^{d_\rho}\sum_{i_n,j_n=1}^{d_{\mu_n}}\sum_{k_n,l_n=1}^{d_{\nu_n}}\sum_{a_n=1}^{d_{\alpha_n}}\prod_{n=1}^{6}\sqrt{d_{\mu_n}d_{\nu_n}}\,\overline{\begin{bmatrix}\nu_n^* & \nu_{n-1} & \alpha_n\\ k_n & l_{n-1} & a_n\end{bmatrix}}\begin{bmatrix}\mu_n^* & \mu_{n-1} & \alpha_n\\ i_n & j_{n-1} & a_n\end{bmatrix} \\
&\qquad\times(W_1^{\mu_n\nu_n^*\rho})_{i_nk_nm_n,j_nl_nm_{n+1}}
\end{aligned}\tag{3.24}$$

since $B_\rho^{\mathrm{QD}}(p)$ only acts on the (inner) edges of the face p according to (3.18).

Noting that the above expression is in fact independent of the matrix indices b_n and comparing with (3.22) we conclude that

$$\langle\{\nu_n\},\{\alpha_n\}|B_\rho^{\mathrm{QD}}(p)|\{\mu_n\},\{\alpha_n\}\rangle_{\mathrm{QD}}=\langle\{\nu_n\},\{\alpha_n\}|B_\rho^{\mathrm{SN}}(p)|\{\mu_n\},\{\alpha_n\}\rangle_{\mathrm{SN}}\,.\tag{3.25}$$

Let us comment on the structure of this mapping. The SN definition of face operators relies on F-symbols, whose extension to configurations violating vertex conditions is somewhat arbitrary. Enlarging the local Hilbert spaces $\mathcal{L}_e^{\mathrm{SN}}=\{|\mu\rangle\mid\mu\in\widehat{G}\}$ of the string-net model by means of additional matrix degrees of freedom and thus moving over to the Hilbert spaces $\mathcal{L}_e^{\mathrm{QD}}=\mathbb{C}G$ of the quantum double model, we are able to express both face and vertex operators in a way that recovers the SN definition for the reduced states in the "physical" subspace $\mathcal{L}_\Gamma^{\mathrm{SN}}\subset\mathcal{L}_\Gamma^{\mathrm{QD}}$, but carries over to the entire Hilbert space $\mathcal{L}_\Gamma^{\mathrm{QD}}$ of the quantum double model. In more concrete terms, we can write

$$B^{\mathrm{SN}}(p)\sim B^{\mathrm{QD}}(p)\otimes\left(\bigotimes_{s\in V(p)}A^{\mathrm{QD}}(s)\right),\tag{3.26}$$

in the sense that $B^{\mathrm{SN}}(p)$ needs the vertices surrounding the face to fulfil the fusion rules, i.e. needs the state of the extended face to lie in the "physical" subspace $\mathcal{L}_\Gamma^{\mathrm{SN}}$.

In this subspace its action can be identified with that of $B^{\mathrm{QD}}(p)$. Incidentally, this accounts for the fact that the face operators of the string-net model are 12-local while the face operators of the quantum double model are 6-local.

String-net models obtained from quantum double models by the Fourier mapping can be defined naturally on general planar graphs, and not only on trivalent graphs like a generic SN model. The reason is that the vertex projectors have a natural interpretation in group representation theory, which generalises to n-valent vertices: a vertex configuration is allowed if the tensor product of the incident irreducible representations contains the trivial representation.

Moreover, group theory also provides us with a natural splitting of the F-symbols according to (B.8), implying that face projectors act effectively only on the (inner) edges of the faces, since the parts associated with the outer edges have the form of vertex projectors and act trivially on states in the "physical" subspace $\mathcal{L}_{\Gamma}^{\mathrm{SN}}$.

More generally, we have an identification of the superselection sectors as irreducible representations of (in this case) the quasi-triangular Hopf algebra $\mathrm{D}(\mathbb{C}G)$. Note that the matrix degrees of freedom i and j which must be added to the string-net lattice to fill the local Hilbert spaces with basis $\{|b_{\mu,ij}\rangle\}$ of the quantum double model allow us to keep track of the internal degrees of freedom within the different irreps of $\mathrm{D}(\mathbb{C}G)$ (e.g. the group element labels for the conjugacy classes defining the magnetic fluxes, the different vectors for the irreps of the group in electric charges).

From a more abstract point of view both quantum double and their corresponding string-net models can be seen as a procedure to obtain an anyon model, that of the physical excitations, which is a unitary braided tensor category (UMTC). This has as simple objects the superselection sectors, i.e. the quasiparticle excitations classified by irreducible representations of $\mathrm{D}(\mathbb{C}G)$. This UMTC is both obtained via the QD route, i.e. starting with a basis labelled by group elements (objects of a category G), and via the SN route, i.e. starting with a basis labelled by irreps (objects of the category $\mathrm{Rep}(G)$ of representations of G). These categories are equivalent in the sense that they contain the same quasiparticle excitations.

3.4 Discussion

We have shown explicitly how to identify Kitaev's quantum double models [39] with a subclass of the string-net models of Levin and Wen [48]. The general construction for string-net models can be further simplified in this case due to the interpretation of the fusion rules in terms of group theory.

As a result the subclass of string-net models corresponding to quantum double models can be extended naturally to arbitrary planar lattices. Their excitations can be given a representation-theoretic interpretation at the price of introducing auxiliary degrees of freedom necessary to keep track of the internal spaces of the different representations. And the appealing symmetry of electric and magnetic excitations is recovered, in that face projectors can be given a natural definition that does not depend on the completion of F-symbols outside the space of recouplings with legal vertices. This provides a local characterisation of excitations which we

find satisfactory. Not least this restored symmetry hints at a rigorous notion of electric-magnetic duality to be discussed in Chapter 6.

Interestingly, from the point of view of category theory the construction can be seen as an instance of Morita equivalence, which stresses the practical importance of these models as laboratories to provide simple examples of abstract mathematical notions which, in spite of their importance, are only in their way to become everyday tools of theoretical physicists.

Let us stress the significance of this construction. On the one hand, it is a nontrivial mapping relating the physics of two different classes of topological models. We have tried to emphasise the interplay of physical degrees of freedom which is needed to show this relationship, and how the smaller local Hilbert space for the string-net lattice can be naturally enlarged to the local Hilbert space of the corresponding quantum double model. On the other hand, it allows for a clearer picture of the anyons appearing as physical excitations of the particular class of string-net models obtained from our mapping, and this picture can be extended to more general string-net models (see Chapter 6).

Chapter 4

Quantum double models from Hopf algebras

In this chapter we aim at extending KITAEV's quantum double models [39] from the case of the group algebra of a finite group G to a finite-dimensional Hopf algebra with certain properties, as anticipated by KITAEV in [39]. This sheds more light on the Hopf algebra point of view KITAEV outlined originally, in particular, it clarifies how the structure maps of the Hopf algebra enter in the definition of the model and how the quantum double of the Hopf algebra arises in the description of the model's excitations. It turns out that an involutory Hopf algebra (i.e. a Hopf algebra whose antipode map squares to the identity) with an additional $*$-algebra structure is sufficient in order to define the generalized quantum double model properly. As we will show both requirements are satisfied if one chooses a finite-dimensional Hopf C^*-algebra (also known as a finite-dimensional Kac algebra). In order to illustrate the construction of our generalized model we give an example of a finite-dimensional Hopf C^*-algebra which is nontrivial in the sense that it is neither a group algebra nor the dual of a group algebra. This extends well-known instances of the family like the toric code for $G = \mathbb{Z}_2$ or the model based on $G = S_3$ which is universal for topological quantum computation [54].

In Section 4.1 we will provide a guide to the language of Hopf algebras, introducing the necessary notation needed in order to extend the quantum double models based on finite groups to Hopf algebras. In Section 4.2 we then generalize KITAEV's quantum double construction from group algebras to Hopf algebras, thereby introducing the physical model which stands in the centre of this work.

4.1 The language of Hopf algebras

4.1.1 Hopf algebras

If one sets out to find an algebraic structure for symmetries of linear spaces with tensor product structure, such as encountered in many-body quantum physics, one is naturally led to the concept of a Hopf algebra. This is because the axioms of a Hopf algebra directly allow for duals and tensor products of representations. As general reference for this section we recommend [36].

First and foremost a Hopf algebra H is a vector space (over some field k which we will take to be $\mathbb{C}$) equipped with some additional structure. On the one hand

there is an associative multiplication μ of vectors which is linear in each argument, hence one may define it as the linear map $\mu\colon H \otimes H \to H$ with

$$\mu(x \otimes y) = xy \tag{4.1}$$

such that

$$(xy)z = x(yz). \tag{4.2}$$

This multiplication is accompanied by a unit η which can be defined as the linear injection of scalars into H, hence $\eta\colon k \to H$ where

$$\eta(\lambda) = \lambda 1_H. \tag{4.3}$$

The element $1_H \in H$ is a left and right unit for the multiplication μ. At this stage H has the structure of an algebra. The multiplication encodes the composition of symmetry transformations in a representation space.

On the other hand there is a dual notion to multiplication. This dual linear map is the comultiplication $\Delta\colon H \to H \otimes H$ and is usually written in Sweedler notation[1] as

$$\Delta(x) = \sum_{(x)} x' \otimes x'' = \sum_{(x)} x^{(1)} \otimes x^{(2)}. \tag{4.4}$$

This serves as a shorthand for sums of the form $\sum_i x_i^{(1)} \otimes x_i^{(2)}$. It is required to be coassociative:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta. \tag{4.5}$$

Elements which satisfy $\Delta(x) = x \otimes x$ are called grouplike and collected in the set $G(H)$. This comultiplication has a companion called the counit $\epsilon\colon H \to k$ which is required to satisfy the axiom:

$$\sum_{(x)} \epsilon(x')\, x'' = \sum_{(x)} x' \epsilon(x'') = x. \tag{4.6}$$

In other words, it neutralizes the comultiplication. With this structure alone, the vector space H is called a coalgebra. The comultiplication encodes how symmetry transformations act on tensor products of representation spaces. The counit provides the appropriate notion of a trivial representation, or invariance under symmetry transformations.

If both the algebra and coalgebra structure are compatible then H is called a bialgebra. The compatibility axioms read:

$$\Delta(xy) = \Delta(x)\,\Delta(y), \tag{4.7}$$
$$\Delta(1_H) = 1_H \otimes 1_H, \tag{4.8}$$
$$\epsilon(xy) = \epsilon(x)\,\epsilon(y), \tag{4.9}$$
$$\epsilon(1_H) = 1_k. \tag{4.10}$$

[1] If there are more than three tensor factors in iterated coproducts such as $\Delta^{(3)}(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}$ we will consistently use superscripts instead of primes.

Finally, if there is a linear map $S\colon H \to H$ satisfying the axiom

$$\sum_{(x)} x' S(x'') = \epsilon(x)\, 1_H = \sum_{(x)} S(x')\, x'' \tag{4.11}$$

then $H = (H; \mu, \eta; \Delta, \epsilon; S)$ is called a Hopf algebra. The map S is called the antipode and has the following properties:

$$S(xy) = S(y)\, S(x), \tag{4.12}$$

$$S(1_H) = 1_H, \tag{4.13}$$

$$\sum_{(S(x))} S(x)' \otimes S(x)'' = \sum_{(x)} S(x'') \otimes S(x'), \tag{4.14}$$

$$\epsilon\big(S(x)\big) = \epsilon(x). \tag{4.15}$$

In any finite-dimensional Hopf algebra the order of the antipode S is finite [66], hence S is an invertible map. We will always make this assumption in the sequel. The antipode is used to define dual (or conjugate) representations.

We may also define opposite multiplication μ^{op} and comultiplication Δ^{cop} by

$$\mu^{\mathrm{op}}(x \otimes y) = yx, \tag{4.16}$$

$$\Delta^{\mathrm{cop}}(x) = \sum_{(x)} x'' \otimes x' \tag{4.17}$$

relative to the multiplication (4.1) and comultiplication (4.4). Then the sets

$$Z(H) = \{x \in H \mid \forall y \in H\colon \mu^{\mathrm{op}}(x \otimes y) = \mu(x \otimes y)\} \tag{4.18}$$

$$\mathrm{Cocom}(H) = \{x \in H \mid \Delta^{\mathrm{cop}}(x) = \Delta(x)\} \tag{4.19}$$

are called the centre of H and the cocommutative elements of H respectively. If $Z(H) = H$ then H is a commutative Hopf algebra, in the case $\mathrm{Cocom}(H) = H$ it is a cocommutative Hopf algebra. Furthermore it turns out that $H^{\mathrm{op}} = (H; \mu^{\mathrm{op}}, \eta; \Delta, \epsilon; S^{-1})$ is again a Hopf algebra, called the opposite Hopf algebra of H.

One can now show that for a given Hopf algebra $H = (H; \mu, \eta; \Delta, \epsilon; S)$ with underlying vector space H the dual vector space H^* has again the structure of a Hopf algebra. More precisely,

$$H^* = (H^*; \Delta^T, \epsilon^T; \mu^T, \eta^T; S^T) \tag{4.20}$$

with the structure maps[2] as indicated is called the dual Hopf algebra of H. By the same token, the opposite Hopf algebra H^{op} has the natural dual

$$X = (H^{\mathrm{op}})^* = \big(H^*; \Delta^T, \epsilon^T; (\mu^{\mathrm{op}})^T, \eta^T; (S^{-1})^T\big). \tag{4.21}$$

[2] For a linear map $f\colon U \to V$ the transpose map $f^T\colon V^* \to U^*$ is defined as usual by $\langle f^T(\alpha), x\rangle := \langle \alpha, f(x)\rangle$ for all $\alpha \in V^*$ and $x \in U$.

Finally, there is the notion of a (two-sided) integral Λ in a Hopf algebra H. Such an element is defined via

$$x\Lambda = \Lambda x = \epsilon(x)\Lambda \tag{4.22}$$

for all $x \in H$. In view of the role of the counit as a trivial representation, integrals are thus invariant elements under multiplication. The dual definition of an integral $\Gamma \in H^*$ can be phrased as

$$\sum_{(x)} x'\Gamma(x'') = \sum_{(x)} \Gamma(x')\, x'' = \Gamma(x)\, 1_H \tag{4.23}$$

for all $x \in H$. This turns out to be equivalent to (4.22) once applied to H^*.

Particularly important is the notion of a Haar integral. This is the normalized version of an integral which is guaranteed to exist uniquely for finite-dimensional Hopf C^*-algebras. Those are the Hopf algebras we will be occupied with mostly in the discussion of the physical model. We will supply a precise definition of Haar integrals in Section 4.2.2.

4.1.2 Quantum doubles as bicrossed products

In a general Hopf algebra H nothing can be said about how Δ and Δ^{cop} are related to each other. In other words, H can be non-cocommutative in the most unpleasant way. Therefore we seek Hopf algebras which are *almost* cocommutative[3] in a certain sense: they are equipped with a so-called quasitriangular structure which controls the extent to which the comultiplication fails to be cocommutative. In [18] Drinfeld gave his celebrated quantum double construction which produces such a quasitriangular Hopf algebra from a given Hopf algebra.

At the heart of his construction lies the idea of a bicrossed product. For example, in the case of the semidirect product of two groups one has an action of one group on the other and this action is used to define the multiplication in the product group. Similarly, one may define a bicrossed product of groups where there is an additional backaction of the second group on the first one. It is this concept that can be generalized to Hopf algebras and will yield the quantum double.

So given a Hopf algebra H with invertible antipode S (in particular, any finite-dimensional Hopf algebra) one can construct a new Hopf algebra $\mathrm{D}(H) = X \bowtie H$, the quantum double of H, as the bicrossed product of $X = (H^{\mathrm{op}})^*$ and H. This means that as a vector space $\mathrm{D}(H)$ simply equals $H^* \otimes H$. In order to define its multiplication we need an action of H on X and another one vice versa. This bicrossed structure is given by the actions

$$\triangleright\colon H \times X \to X, \qquad a \triangleright f = \sum_{(a)} f\big(S^{-1}(a'')\, ?\, a'\big), \tag{4.24}$$

$$\triangleleft\colon H \times X \to H, \qquad a \triangleleft f = \sum_{(a)} f\big(S^{-1}(a''')\, a'\big)\, a'', \tag{4.25}$$

[3] In physical applications this is related to the exchange statistics of particles in two dimensions.

where the question mark denotes the argument of the function $a \triangleright f$, i.e. in this particular instance

$$(a \triangleright f)(x) = \sum_{(a)} f\big(S^{-1}(a'')\, x a'\big) \tag{4.26}$$

holds for any $x \in H$. We will use this notation frequently later on.

One can then show that

$$(f \otimes a)(g \otimes b) = \sum_{(a)} f\, g\big(S^{-1}(a''')\, ?a'\big) \otimes a''b, \tag{4.27}$$

together with the canonical comultiplication on the tensor product of Hopf algebras, defines a valid Hopf algebra structure on the vector space $\mathrm{D}(H)$. Since both the embeddings $i_X \colon X \to \mathrm{D}(H)$, $f \mapsto f \otimes 1_H$ and $i_H \colon H \to \mathrm{D}(H)$, $a \mapsto 1_X \otimes a$ are algebra morphisms this multiplication formula is actually already determined by the so-called straightening formula[4]

$$af \coloneqq (1_X \otimes a)(f \otimes 1_H) = \sum_{(a)} f\big(S^{-1}(a''')\, ?a'\big)\, a''. \tag{4.28}$$

It should be noted that both underlying actions (4.24) and (4.25) are derived from the (left) adjoint representation which for $a, x \in H$ is defined by

$$\mathrm{ad}(a)(x) = \sum_{(a)} a' x\, S(a''). \tag{4.29}$$

4.1.3 Hopf $*$-algebras

If $k = \mathbb{C}$ then a Hopf algebra H may sometimes be equipped with a so-called $*$-structure that will become most important for physical applications. Namely, it will allow us to define Hilbert spaces and unitarity. For a general reference on Hopf $*$-algebras see [42].

Firstly, a conjugate-linear map $* \colon H \to H$ which satisfies

$$(x^*)^* = x \tag{4.30}$$
$$(xy)^* = y^* x^* \tag{4.31}$$

is called an involution and turns H into a $*$-algebra. It follows naturally that

$$1_H^* = 1_H. \tag{4.32}$$

If an involution $*$ is compatible with comultiplication

$$\Delta(x^*) = \Delta(x)^* \tag{4.33}$$

[4] This should be contrasted with the canonical (uncrossed) multiplication in the Hopf algebra $X \otimes H$. In this case we simply have $af = fa$ for any $f \in X$ and $a \in H$ since multiplication is defined componentwise.

then H is called a $*$-coalgebra. Here the involution of $H \otimes H$ is defined by $(x \otimes y)^* = x^* \otimes y^*$. In a $*$-coalgebra one always has

$$\epsilon(x^*) = \overline{\epsilon(x)} \tag{4.34}$$

where the bar denotes complex conjugation.

A bialgebra H with an involution for which it is both a $*$-algebra and a $*$-coalgebra is called a $*$-bialgebra.

Interestingly, if a Hopf algebra H also has the structure of a $*$-bialgebra then the interplay between antipode and involution is already determined:

$$S\big(S(x^*)^*\big) = x. \tag{4.35}$$

Consequently, H is called a Hopf $*$-algebra. Note that its antipode is always invertible (even if H is not finite-dimensional).

Finally, one can show that the dual of a Hopf $*$-algebra H is again a Hopf $*$-algebra with the involution given by

$$f^*(x) = \overline{f\big(S(x)^*\big)}. \tag{4.36}$$

4.2 Constructing quantum double models from Hopf algebras

The goal of this section is to construct a two-dimensional quantum spin model whose microscopic degrees of freedom are given by a finite-dimensional Hopf algebra H, such that its emerging degrees of freedom are characterized by the quantum double $\mathrm{D}(H)$. This illuminates Kitaev's insight in [39]. The model obtained in this fashion dynamically implements the quantum double construction. This means that its Hilbert space aquires additional structure, namely that the graph underlying the spin model can be interpreted as a $\mathrm{D}(H)$-module, or representation of $\mathrm{D}(H)$. State vectors describing (elementary) quasiparticle excitations above the ground state then live in the irreducible representations of $\mathrm{D}(H)$ the Hilbert space decomposes into. In fact, for every irreducible representation of the quantum double there will be a corresponding type of quasiparticle excitation. These quasiparticle excitations can then naturally be braided via the quasitriangular structure of $\mathrm{D}(H)$ and, in general, exhibit exchange statistics beyond bosons or fermions. In order to render their topological nature manifest we would also like these quasiparticle excitations to be agnostic to the details of the microscopic background. In particular, they should only feel the topological properties of the underlying surface rather than the precise shape of the embedded graph. In other words, the quantum spin model we construct should be insensitive to the particular discretization chosen. It goes without saying that any discretization will typically have a fine granularity in order to make the condensed matter character of the model evident.

Since there is a natural action of the quantum double $\mathrm{D}(H)$ on the Hopf algebra H itself one can nevertheless regard a minimal graph consisting of precisely one edge as

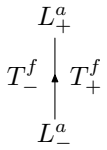


Figure 4.1: A graph edge representing the Hopf algebra H via the H-module structures $L_\pm$. At the same time it represents the Hopf algebra $X = (H^{\mathrm{op}})^*$ via the X-module structures $T_\pm$. These are related to each other by means of the antipode in (4.42) and (4.43).

a representation of $\mathrm{D}(H)$ by identifying the edge with H. However, it is clear that this system cannot contain all irreducible representations of the quantum double. While this restriction can already be overcome by considering two graph edges associated with $H \otimes H$ one still needs to go to a macroscopic regime from either of these small graphs. It is precisely the process of spatially extending the action of $\mathrm{D}(H)$ from one edge to many edges which will eventually yield the Hamiltonian and thus define the quantum double model completely.

In the following we will first construct a representation of $\mathrm{D}(H)$ from a minimal graph with just one edge, then from a small graph with two edges. Next we will endow any larger graph obtained from a surface cellulation with a local $\mathrm{D}(H)$-module structure. Finally we will introduce the Hilbert space for our generalized quantum double model and assemble the Hamiltonian from particular operators in the local representations of $\mathrm{D}(H)$.

4.2.1 Graph representations of quantum doubles

Given a finite-dimensional Hopf algebra H and an oriented graph $\Gamma = (V, E, F)$ we define our representation vector space as $\mathcal{L}_\Gamma = \bigotimes_{e\in E} H$. Our goal is to find local operators A_a and B_f acting on $\mathcal{L}_\Gamma$ which represent both parts H and $X = (H^{\mathrm{op}})^*$ of the quantum double $\mathrm{D}(H)$ separately via the embeddings i_X and i_H. Additionally we want these two operators to commute in such a way that they implement the straightening formula (4.28) of $\mathrm{D}(H)$, hence:

$$A_a B_f = \sum_{(a)} B_{f(S^{-1}(a''')?a')} A_{a''}. \tag{4.37}$$

Put differently, we require the operators A_a and B_f to *establish* the bicrossed multiplication of the quantum double by interacting nontrivially on the intersection of their supports. Loosely speaking, this amounts to assembling the adjoint representation from smaller building blocks. We will spend the rest of this subsection with obtaining these operators A_a and B_f on graphs of increasing size.

Before we actually turn to representations of the quantum double $\mathrm{D}(H)$ we can interpret any graph edge e and its associated vector space $\mathcal{L}_e = H$ as a representation of either the Hopf algebra H itself or its dual $X = (H^{\mathrm{op}})^*$ in several useful ways. In the spirit of [39] we define the following module structures on H for all $a, x \in H$

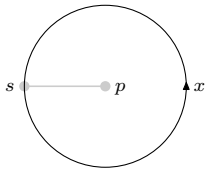


Figure 4.2: The minimal graph Γ_1 which carries the Schrödinger representation of $\mathrm{D}(H)$ on the associated vector space $\mathcal{L}_{\Gamma_1} = H$.

and $f \in X$:

$$L^a_+(x) \coloneqq ax, \tag{4.38}$$
$$L^a_-(x) \coloneqq x\, S(a), \tag{4.39}$$
$$T^f_+(x) \coloneqq \sum_{(x)} \langle f, x'' \rangle\, x', \tag{4.40}$$
$$T^f_-(x) \coloneqq \sum_{(x)} \langle f, S^{-1}(x') \rangle\, x''. \tag{4.41}$$

More precisely, the operators $L_\pm$ define actions of H on itself while $T_\pm$ define actions of X on H. Furthermore these actions are intimately related: one may start with the left multiplication L_+ and then canonically derive all other actions. As a consequence one has for example

$$L^a_- = S \circ L^a_+ \circ S^{-1}, \tag{4.42}$$
$$T^f_- = S \circ T^f_+ \circ S^{-1}. \tag{4.43}$$

This means that if one fixes an arbitrary pattern of edge orientations one may relate any other pattern to the original one using these relations. Unfortunately this does not treat all possible orientations on equal footing. We will resolve this issue in Section 4.2.2. Also note that

$$[L^a_+, L^b_-] = 0, \tag{4.44}$$
$$[T^f_+, T^g_-] = 0 \tag{4.45}$$

for arbitrary $a, b \in H$ and $f, g \in X$.

Now we focus on representations of $\mathrm{D}(H)$. As we hinted at in Section 4.1.2 the quantum double construction is a clever variation on the adjoint representation. Interestingly, one can turn the adjoint representation into the operator A_a "as is" and obtain a representation of $\mathrm{D}(H)$ on a minimal graph Γ_1 with just one edge as seen in Figure 4.2. In the mathematical literature this is known as the Schrödinger representation and translates into our picture as follows.

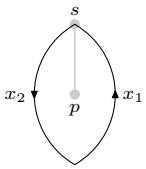


Figure 4.3: The graph Γ_2 associated with $\mathcal{L}_{\Gamma_2} = H \otimes H$. It affords a $\mathrm{D}(H)$-module structure at the site (s, p) via the operators $A_a(s, p)$ and $B_f(s, p)$.

Lemma 1. *The operators*

$$A_a \coloneqq \mathrm{ad}(a) \tag{4.46}$$

$$B_f \coloneqq T^f_- \tag{4.47}$$

form a representation of $\mathrm{D}(H)$ *on* $\mathcal{L}_{\Gamma_1} = H$.

Proof. We just need to show that A_a and B_f obey the straightening formula (4.37). Indeed, consider

$$\begin{aligned}
&\sum_{(a)} B_{f(S^{-1}(a'''){?}a')} A_{a''}(x) \\
&\quad = \sum_{(a)} B_{f(S^{-1}(a^{(4)}){?}a^{(1)})}\big(a^{(2)} x\, S(a^{(3)})\big) \\
&\quad = \sum_{(a)} \sum_{(a^{(2)} x S(a^{(3)}))} \big\langle f, S^{-1}(a^{(4)})\, S^{-1}\big[\big(a^{(2)} x\, S(a^{(3)})\big)'\big] a^{(1)} \big\rangle \, \big(a^{(2)} x\, S(a^{(3)})\big)'' \\
&\quad = \sum_{(a)} \sum_{(a^{(2)})} \sum_{(S(a^{(3)}))} \sum_{(x)} \big\langle f, S^{-1}\big((a^{(2)})' x'\, S(a^{(3)})' a^{(4)}\big) a^{(1)} \big\rangle \, (a^{(2)})'' x'' S(a^{(3)})'' \\
&\quad = \sum_{(a)} \sum_{(S(a^{(4)}))} \sum_{(x)} \big\langle f, S^{-1}\big(a^{(2)} x' S(a^{(4)})' a^{(5)}\big) a^{(1)} \big\rangle \, a^{(3)} x'' S(a^{(4)})'' \\
&\quad = \sum_{(a)} \sum_{(x)} \big\langle f, S^{-1}\big(a^{(2)} x'\, S(a^{(5)})\, a^{(6)}\big) a^{(1)} \big\rangle \, a^{(3)} x'' S(a^{(4)}) \\
&\quad = \sum_{(a)} \sum_{(x)} \langle f, S^{-1}(x')\, S^{-1}(a^{(2)})\, a^{(1)} \rangle \, a^{(3)} x'' S(a^{(4)}) \\
&\quad = A_a B_f(x).
\end{aligned}$$

Note that in the third line we used (4.7) as well as (4.12). In the fourth line we used coassociativity and in the fifth line we used (4.14). In the final two lines we used (4.11). □

We continue with the graph Γ_2 shown in Figure 4.3 and its associated vector space $\mathcal{L}_{\Gamma_2} = H \otimes H$ and attribute operators A_a and B_f to its sites (s, p). Following [39], a site is defined as a pair consisting of a vertex s and one of its adjacent

faces p. For the particular site (s,p) shown in Figure 4.3 we define the operators in terms of the above actions:

$$A_a(s,p) \coloneqq \sum_{(a)} L_+^{a'} \otimes L_-^{a''}, \tag{4.48}$$

$$B_f(s,p) \coloneqq \sum_{(f)} T_-^{f'} \otimes T_-^{f''}. \tag{4.49}$$

It is worth emphasizing that this definition only holds for the edge orientation chosen in the figure. However, using the conventions set forth by Figure 4.1 we can extend this definition to arbitrarily oriented edges. Also note an important detail: our seemingly similar notation refers to two rather different comultiplications, namely, the one in H and the other one in X.

We are now prepared to analyze the commutation properties of these operators which we take both at the *same* site (s,p) for the moment. It is clear that the support of both operators coincides on this small graph, hence we can study the interaction of A_a and B_f without any additional complications. The following lemma shows that indeed the two operators define a representation of $\mathrm{D}(H)$ on $H \otimes H$.

Lemma 2. *For the operators $A_a(s,p)$ and $B_f(s,p)$ the straightening formula (4.37) holds.*

Proof. The statement is proven by evaluating both sides of the straightening formula on arbitrary elements $x_1, x_2 \in H$. To this end consider first:

$$\begin{aligned}
A_a B_f(x_1 \otimes x_2) &= A_a \sum_{(f)} T_-^{f'}(x_1) \otimes T_-^{f''}(x_2) \\
&= A_a \sum_{(x_i)} \sum_{(f)} \langle f', S^{-1}(x_1') \rangle \langle f'', S^{-1}(x_2') \rangle \, x_1'' \otimes x_2'' \\
&= \sum_{(x_i)} \Big\langle \sum_{(f)} f' \otimes f'', S^{-1}(x_1') \otimes S^{-1}(x_2') \Big\rangle \sum_{(a)} L_+^{a'}(x_1'') \otimes L_-^{a''}(x_2'') \\
&= \sum_{(a)(x_i)} \langle f, S^{-1}(x_2') \, S^{-1}(x_1') \rangle \, a' x_1'' \otimes x_2'' \, S(a'') \\
&= \sum_{(a)(x_i)} \langle f, S^{-1}(x_1' x_2') \rangle \, a' x_1'' \otimes x_2'' \, S(a'').
\end{aligned}$$

Note that in the third line we have used the opposite multiplication μ^{op} which is the appropriate dual to swap the comultiplication

$$\sum_{(f)} f' \otimes f'' = (\mu^{\mathrm{op}})^T(f)$$

in X from f onto its argument in H.

In a second step compare this with:

$$\begin{aligned}\sum_{(a)} B_{f(S^{-1}(a'''){?}a')} A_{a''}(x_1 \otimes x_2) &= \sum_{(a)} B_{f(S^{-1}(a^{(4)}){?}a^{(1)})}\big(a^{(2)}x_1 \otimes x_2\, S(a^{(3)})\big) \\ &= \sum_{(a)} B_{\tilde{f}}\big(a^{(2)}x_1 \otimes x_2\, S(a^{(3)})\big)\end{aligned}$$

where we used the temporary abbreviation $\tilde{f} = f\big(S^{-1}(a^{(4)})\,{?}\,a^{(1)}\big)$. Now we have for the right-hand side

$$\begin{aligned}&\sum_{(a)(\tilde{f})} T_-^{\tilde{f}'}(a^{(2)}x_1) \otimes T_-^{\tilde{f}''}\big(x_2\, S(a^{(3)})\big) \\ &\quad= \sum_{(a)(\tilde{f})} \sum_{(a^{(2)}x_1)} \big\langle \tilde{f}', S^{-1}\big((a^{(2)}x_1)'\big)\big\rangle\, (a^{(2)}x_1)'' \otimes \sum_{(x_2 S(a^{(3)}))} \big\langle \tilde{f}'', S^{-1}\big[\big(x_2\, S(a^{(3)})\big)'\big]\big\rangle\, \big(x_2\, S(a^{(3)})\big)'' \\ &\quad= \sum_{(a)} \sum_{(a^{(2)}x_1)} \sum_{(x_2 S(a^{(3)}))} \big\langle \tilde{f}, S^{-1}\big[\big(x_2\, S(a^{(3)})\big)'\big]\, S^{-1}\big((a^{(2)}x_1)'\big)\big\rangle\, (a^{(2)}x_1)'' \otimes \big(x_2\, S(a^{(3)})\big)'' \\ &\quad= \sum_{(a)} \sum_{(a^{(2)}x_1)} \sum_{(x_2 S(a^{(3)}))} \big\langle \tilde{f}, S^{-1}\big[(a^{(2)}x_1)'\big(x_2\, S(a^{(3)})\big)'\big]\big\rangle\, (a^{(2)}x_1)'' \otimes \big(x_2\, S(a^{(3)})\big)''\end{aligned}$$

and upon restoring f this becomes

$$\begin{aligned}&\sum_{(a)} \sum_{(a^{(2)}x_1)} \sum_{(x_2 S(a^{(3)}))} \big\langle f, S^{-1}(a^{(4)})\, S^{-1}\big[(a^{(2)}x_1)'\big(x_2\, S(a^{(3)})\big)'\big]\, a^{(1)}\big\rangle\, (a^{(2)}x_1)'' \otimes \big(x_2\, S(a^{(3)})\big)'' \\ &\quad= \sum_{(a)} \sum_{(a^{(2)}x_1)} \sum_{(x_2 S(a^{(3)}))} \big\langle f, S^{-1}\big[(a^{(2)}x_1)'\big(x_2\, S(a^{(3)})\big)' a^{(4)}\big]\, a^{(1)}\big\rangle\, (a^{(2)}x_1)'' \otimes \big(x_2\, S(a^{(3)})\big)'' \\ &\quad= \sum_{(a)(x_i)} \sum_{(a^{(2)})} \sum_{(S(a^{(3)}))} \big\langle f, S^{-1}\big((a^{(2)})'x_1'x_2'\, S(a^{(3)})'a^{(4)}\big)\, a^{(1)}\big\rangle\, (a^{(2)})''x_1'' \otimes x_2''\, S(a^{(3)})'' \\ &\quad= \sum_{(a)(x_i)} \sum_{(S(a^{(4)}))} \big\langle f, S^{-1}\big(a^{(2)}x_1'x_2'\, S(a^{(4)})'a^{(5)}\big)\, a^{(1)}\big\rangle\, a^{(3)}x_1'' \otimes x_2''\, S(a^{(4)})'' \\ &\quad= \sum_{(a)(x_i)} \sum_{(a^{(4)})} \big\langle f, S^{-1}\big[a^{(2)}x_1'x_2'\, S\big((a^{(4)})''\big)\, a^{(5)}\big]\, a^{(1)}\big\rangle\, a^{(3)}x_1'' \otimes x_2''\, S\big((a^{(4)})'\big) \\ &\quad= \sum_{(a)(x_i)} f\big[S^{-1}\big(a^{(2)}x_1'x_2'\, S(a^{(5)})\, a^{(6)}\big)\, a^{(1)}\big]\, a^{(3)}x_1'' \otimes x_2''\, S(a^{(4)}) \\ &\quad= \sum_{(a)(x_i)} f\big[S^{-1}\big(a^{(2)}x_1'x_2'\, \epsilon(a^{(5)})\big)\, a^{(1)}\big]\, a^{(3)}x_1'' \otimes x_2''\, S(a^{(4)}) \\ &\quad= \sum_{(a)(x_i)} f\big(S^{-1}(x_1'x_2')\, S^{-1}(a^{(2)})\, a^{(1)}\big)\, a^{(3)}x_1'' \otimes x_2''\, S(a^{(4)})\end{aligned}$$

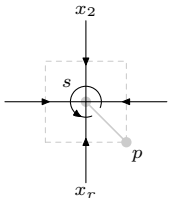
$$= \sum_{(a)(x_i)} f\big(S^{-1}(x_1' x_2')\,\epsilon(a')\big)\, a'' x_1'' \otimes x_2''\, S(a''')$$
$$= \sum_{(a)(x_i)} \langle f, S^{-1}(x_1' x_2')\rangle\, a' x_1'' \otimes x_2''\, S(a'').$$

Note that in the third line we have employed the fact that Δ is an algebra morphism. Since the above is true for all $x_1, x_2 \in H$ we have just proven the claim. □

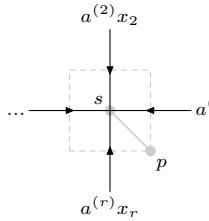
In order to obtain local $\mathrm{D}(H)$-representations at the sites of arbitrary graphs we need to extend the above actions of A_a and B_f to larger vertices and faces. It will inevitably happen that the two operators A_a and B_f act on different sets of edges which do *not* fully coincide. We need to ensure that a) both operators continue to represent their respective parts H and X individually and b) the commutation relation arising from common edges still implements the bicrossed structure of the quantum double $\mathrm{D}(H)$.

The extension can be done as follows for general vertices s and faces p.

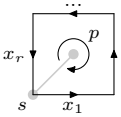
Definition 1. Let (s,p) be a site of the graph Γ, $a \in H$ and $f \in X$. Then we define vertex and face operators via

$A_a(s,p)$ 

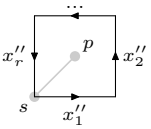
$:= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r)$

$= \sum_{(a)}$ 

, (4.50)

$B_f(s,p)$ 

$:= \sum_{(f)} T_-^{f^{(r)}}(x_1) \otimes \cdots \otimes T_-^{f^{(1)}}(x_r)$

$= \sum_{(x_i)} f\big(S^{-1}(x_r' \cdots x_1')\big)$ 

. (4.51)

Note that for $A_a(s,p)$ the vertex s denotes the centre of the loop on the dual graph and p denotes the starting point within the loop for comultiplication. Analogously, for $B_f(s,p)$ the starting point of the loop around the face p is marked by s. For

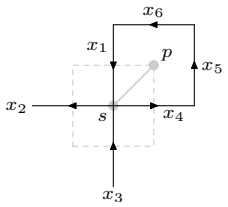


Figure 4.4: Graph used in the proof of Theorem 1. The commutation relation and hence the structure of $\mathrm{D}(H)$ is determined on the intersection of the supports of $A_a(s,p)$ and $B_f(s,p)$.

different edge orientations the actions L_+ and T_- in (4.50) and (4.51) may need to be replaced per edge by L_- and T_+ according to Figure 4.1.

The idea behind this is: first one fixes the action of $a \in H$ such that the coproduct of a is applied to the edges of the vertex s in *counterclock*wise order. This is indicated by the arrow winding around s. When defining the action of $f \in X$ on a face it then appears at first there might be two choices for the orientation of the coproduct of f. It turns out that only one of them, namely distributing the coproduct of f in *clock*wise orientation on the graph, will yield a meaningful theory. The orientation of the coproduct is again indicated by the arrow winding around p.

At this point the reader may wonder why one does not take the function f directly from H^* (the dual of H *without* flipped comultiplication) and the following more symmetric definition of the action (4.51):

$$\tilde{B}_f(s,p) \;\; \begin{array}{c}\cdots \\ x_r \;\; p \;\; x_2 \\ s \quad x_1\end{array} \;\; := \sum_{(f)} T_-^{f^{(1)}}(x_1) \otimes \cdots \otimes T_-^{f^{(r)}}(x_r)$$

$$= \sum_{(x_i)} f\big(S^{-1}(x'_r \cdots x'_1)\big) \;\; \begin{array}{c}\cdots \\ x''_r \;\; p \;\; x''_2 \\ s \quad x''_1\end{array} \;\; . \tag{4.52}$$

Note that here the arrow winds around p in *counterclock*wise orientation since $f \in H^*$ has a different coproduct now as compared to (4.51). Clearly, both (4.51) and (4.52) define the same (algebra) action on the boundary edges of the face. However, as cumbersome as taking $f \in X$ in the definition of $B_f(s,p)$ may seem it will make the next task tremendously easier: making contact with the Drinfeld double $\mathrm{D}(H) = X \bowtie H$ which these operators A_a and B_f represent.

Theorem 1. *Let H a finite-dimensional Hopf algebra. Then each site (s,p) of the graph Γ supports a $\mathrm{D}(H)$-module structure given by the operators $A_a(s,p)$ and $B_f(s,p)$ from Definition 1.*

Proof. In order to keep the notation simple we only consider sites such as the one depicted in Figure 4.4. The generality of the argument will not be affected.

In order to prove the $\mathrm{D}(H)$-module structure at site (s,p) it is again enough to show that the straightening formula (4.37) holds. Let $a \in H$, $f \in X$ and evaluate $A_a B_f$ on arbitrary edges $x_i \in H$:

$$\begin{aligned}
&A_a B_f(x_1 \otimes \cdots \otimes x_6) \\
&\quad = A_a \sum_{(x_i)} f\big(S^{-1}(x_1' x_6' x_5' x_4')\big)\, x_1'' \otimes x_2 \otimes x_3 \otimes x_4'' \otimes x_5'' \otimes x_6'' \\
&\quad = \sum_{(x_i)} f\big(S^{-1}(x_1' x_6' x_5' x_4')\big) \sum_{(a)} a^{(1)} x_1'' \otimes x_2\, S(a^{(2)}) \otimes a^{(3)} x_3 \otimes x_4''\, S(a^{(4)}) \otimes x_5'' \otimes x_6'' \\
&\quad = \sum_{(a)(x_i)} \langle f, S^{-1}(x_1' x_6' x_5' x_4') \rangle\, a^{(1)} x_1'' \otimes x_2\, S(a^{(2)}) \otimes a^{(3)} x_3 \otimes x_4''\, S(a^{(4)}) \otimes x_5'' \otimes x_6''.
\end{aligned}$$

Compare this with the case:

$$\begin{aligned}
&\sum_{(a)} B_{f(S^{-1}(a'''){?}a')} A_{a''}(x_1 \otimes \cdots \otimes x_6) \\
&\quad = \sum_{(a)} B_{f(S^{-1}(a^{(6)}){?}a^{(1)})}\, a^{(2)} x_1 \otimes x_2\, S(a^{(3)}) \otimes a^{(4)} x_3 \otimes x_4\, S(a^{(5)}) \otimes x_5 \otimes x_6.
\end{aligned}$$

Again, abbreviating $\tilde{f} = f\big(S^{-1}(a^{(6)})\, {?}\, a^{(1)}\big)$ we obtain for the right-hand side

$$\begin{aligned}
&\sum_{(a)} \sum_{(x_4 S(a^{(5)}))} \sum_{(x_5)(x_6)} \sum_{(a^{(2)} x_1)} \langle \tilde{f}, S^{-1}\big[(a^{(2)} x_1)' x_6' x_5' \big(x_4\, S(a^{(5)})\big)'\big] \rangle\, (a^{(2)} x_1)'' \otimes x_2\, S(a^{(3)}) \otimes a^{(4)} x_3 \otimes \big(x_4\, S(a \\
&\quad = \sum_{(a)} \sum_{(x_i)} \sum_{(a^{(2)})} \sum_{(a^{(5)})} \langle \tilde{f}, S^{-1}\big[(a^{(2)})' x_1' x_6' x_5' x_4'\, S\big((a^{(5)})''\big)\big] \rangle\, (a^{(2)})'' x_1'' \otimes x_2\, S(a^{(3)}) \otimes a^{(4)} x_3 \otimes x_4''\, S\big((a^{(5)} \\
&\quad = \sum_{(a)(x_i)} \sum_{(a^{(2)})} \sum_{(a^{(5)})} \langle \tilde{f}, S^{-1}\big[(a^{(2)})' x_1' x_6' x_5' x_4'\, S\big((a^{(5)})''\big)\big] \rangle\, (a^{(2)})'' x_1'' \otimes x_2\, S(a^{(3)}) \otimes a^{(4)} x_3 \otimes x_4''\, S\big((a^{(5)})'
\end{aligned}$$

and now restoring f this becomes

$$\begin{aligned}
&\sum_{(a)(x_i)} \big\langle f\big(S^{-1}(a^{(8)})\, {?}\, a^{(1)}\big), S^{-1}\big(a^{(2)} x_1' x_6' x_5' x_4'\, S(a^{(7)})\big) \big\rangle\, a^{(3)} x_1'' \otimes x_2\, S(a^{(4)}) \otimes a^{(5)} x_3 \otimes x_4''\, S(a^{(6)}) \otimes x_5'' \otimes x_6' \\
&\quad = \sum_{(a)(x_i)} \langle f, S^{-1}(a^{(8)})\, a^{(7)} S^{-1}(x_1' x_6' x_5' x_4')\, S^{-1}(a^{(2)})\, a^{(1)} \rangle\, a^{(3)} x_1'' \otimes x_2\, S(a^{(4)}) \otimes a^{(5)} x_3 \otimes x_4''\, S(a^{(6)}) \otimes x_5'' \\
&\quad = \sum_{(a)(x_i)} \langle f, S^{-1}(x_1' x_6' x_5' x_4')\, S^{-1}(a^{(2)})\, a^{(1)} \rangle\, a^{(3)} x_1'' \otimes x_2\, S(a^{(4)}) \otimes a^{(5)} x_3 \otimes x_4''\, S(a^{(6)}) \otimes x_5'' \otimes x_6'' \\
&\quad = \sum_{(a)(x_i)} \langle f, S^{-1}(x_1' x_6' x_5' x_4') \rangle\, a^{(1)} x_1'' \otimes x_2\, S(a^{(2)}) \otimes a^{(3)} x_3 \otimes x_4''\, S(a^{(4)}) \otimes x_5'' \otimes x_6''.
\end{aligned}$$

This proves the statement. □

Remark 1. For certain choices of $a \in H$ and $f \in X$ the full specification of a site (s,p) becomes redundant for the operators A_a and B_f. More specifically,

suppose that $\langle fg, a\rangle$ for any $f, g \in X$ and $f(xy) = f(yx)$ for any $x, y \in H$ or, equivalently, that $a \in \mathrm{Cocom}(H)$ and $f \in \mathrm{Cocom}(X)$[5]. Since coproducts can now be permuted cyclically, the starting point p of the dual loop used to define the vertex operator $A_a(s,p)$ loses any significance. All that matters for the action is the vertex s itself. In order to reflect this fact we set for all $a \in \mathrm{Cocom}(H)$:

$$A_a(s) \coloneqq A_a(s,p). \tag{4.53}$$

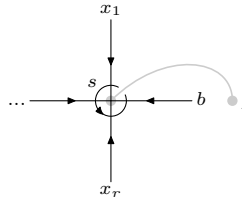
By the same token a face operator $B_f(s,p)$ only cares about the face p once we choose $f \in \mathrm{Cocom}(X)$:

$$B_f(p) \coloneqq B_f(s,p). \tag{4.54}$$

□

It remains to study the action of the quantum double $\mathrm{D}(H)$ on the graph in the presence of degenerate vertices and faces. More specifically, we are interested in graphs with spikes (edges connected to a vertex of degree 1) and/or loops. In such cases it may happen that a site (s,p) is not well defined, rather there are two inequivalent ways $(s,p)^+$ and $(s,p)^-$ to pair up a vertex s with a neighbouring face p. In light of the preceding remark this distinction only matters if the action of a vertex or face operator depends on the full site (and not merely on the vertex or face alone). Hence assume $a \notin \mathrm{Cocom}(H)$ and $f \notin \mathrm{Cocom}(X)$ for the following discussion.

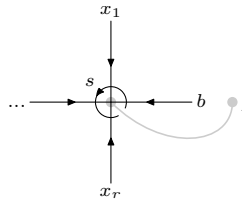
Indeed, for $b \in H$ arbitrary the operators

$A_a(s,p)^+$ 

$= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r) \otimes L_+^{a^{(r+1)}}(b)$

(4.55)

and

$A_a(s,p)^-$ 

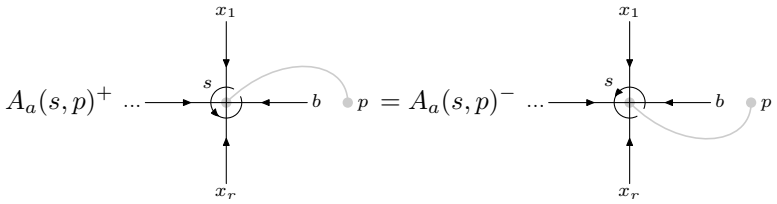
$= \sum_{(a)} L_+^{a^{(2)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r+1)}}(x_r) \otimes L_+^{a^{(1)}}(b)$

(4.56)

act differently.

[5] Clearly, $\mathrm{Cocom}(H)$ and $\mathrm{Cocom}(X)$ are defined relative to the respective coproducts in H and X, which are fundamentally different.

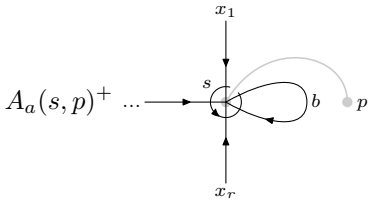
However, if b is an integral we have



$$= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r) \otimes b, \tag{4.57}$$

thus the spike b decouples from the H-part of the $\mathrm{D}(H)$-action.

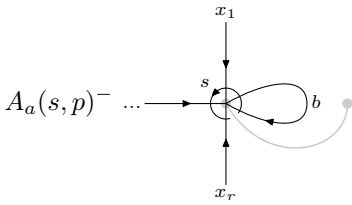
Similarly, the actions of the operators



$$= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r) \otimes L_-^{a^{(r+2)}}\big(L_+^{a^{(r+1)}}(b)\big) \tag{4.58}$$

$$= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r) \otimes \mathrm{ad}(a^{(r+1)})(b) \tag{4.59}$$

and



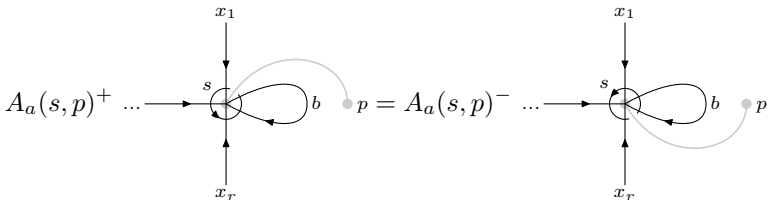
$$= \sum_{(a)} L_+^{a^{(2)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r+1)}}(x_r) \otimes \mathrm{ad}(a^{(1)})(b) \tag{4.60}$$

do not coincide for arbitrary $b \in H$.

However, for any $b \in H$ invariant under the adjoint H-action (4.29), i.e.

$$\mathrm{ad}(a)\, b = \epsilon(a)\, b, \tag{4.61}$$

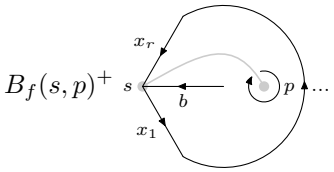
we have the equality



$$= \sum_{(a)} L_+^{a^{(1)}}(x_1) \otimes \cdots \otimes L_+^{a^{(r)}}(x_r) \otimes b, \tag{4.62}$$

thus the loop b decouples from the H-part of the $\mathrm{D}(H)$-action, too.

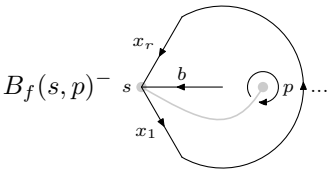
Now also consider the operator actions



$$= \sum_{(f)} T_+^{f^{(r+2)}}\big(T_-^{f^{(r+1)}}(b)\big) \otimes T_-^{f^{(r)}}(x_1) \otimes \cdots \otimes T_-^{f^{(1)}}(x_r) \tag{4.63}$$

$$= \sum_{(b)(x_i)} f\big(b''' S^{-1}(b')\, S^{-1}(x_r' \cdots x_1')\big)\, b'' \otimes x_1'' \otimes \cdots \otimes x_r'' \tag{4.64}$$

and



$$= \sum_{(f)} T_+^{f^{(2)}}\big(T_-^{f^{(1)}}(b)\big) \otimes T_-^{f^{(r+2)}}(x_1) \otimes \cdots \otimes T_-^{f^{(3)}}(x_r) \tag{4.65}$$

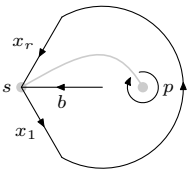
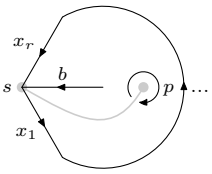
$$= \sum_{(b)(x_i)} f\big(S^{-1}(x_r' \cdots x_1')\, b''' S^{-1}(b')\big)\, b'' \otimes x_1'' \otimes \cdots \otimes x_r'' \tag{4.66}$$

which are not equivalent for arbitrary $b \in H$.

However, if b is invariant under the X-action

$$\sum_{(b)} f\big(b''' S^{-1}(b')\big)\, b'' = f(1_H)\, b \tag{4.67}$$

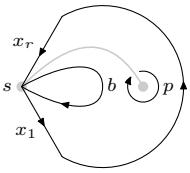
for any $f \in X$ then we have the equality

$B_f(s,p)^+$  $= B_f(s,p)^-$ 

$$= \sum_{(x_i)} f\big(S^{-1}(x_r' \cdots x_1')\big)\, b \otimes x_1'' \otimes \cdots \otimes x_r'', \qquad (4.68)$$

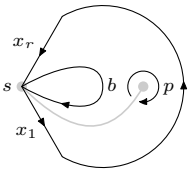
thus the spike b decouples from the X-part of the $\mathrm{D}(H)$-action.

Furthermore, the actions of the operators

$B_f(s,p)^+$ 

$$= \sum_{(b)(x_i)} f\big(S^{-1}(x_r' \cdots x_1' b')\big)\, b'' \otimes x_1'' \otimes \cdots \otimes x_r'' \qquad (4.69)$$

and

$B_f(s,p)^-$ 

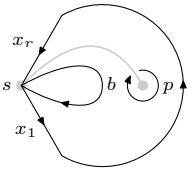
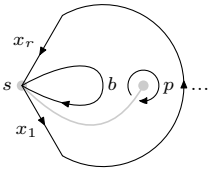
$$= \sum_{(b)(x_i)} f\big(S^{-1}(b' x_r' \cdots x_1')\big)\, b'' \otimes x_1'' \otimes \cdots \otimes x_r'' \qquad (4.70)$$

do not coincide either for arbitrary $b \in H$.

Yet, if b is invariant under the X-action (4.41), i.e.

$$T_-^f(b) = f(1_H)\, b \qquad (4.71)$$

for any $f \in X$, then we have

$B_f(s,p)^+$  $= B_f(s,p)^-$ 

$$= \sum_{(x_i)} f\big(S^{-1}(x_r' \cdots x_1')\big)\, b \otimes x_1'' \otimes \cdots \otimes x_r'', \qquad (4.72)$$

thus the loop b decouples from the X-part of the $\mathrm{D}(H)$-action, too.

In summary we state the following

Proposition 1 (Vertex sector)**.** *Let $a \in H$, $b_1 \in H$ an integral and $b_2 \in H$ invariant under the adjoint H-action (4.29).*

Then any spike b_1 and any loop b_2 attached at vertex s decouple from all vertex operators $A_a(s,p)$.

Proposition 2 (Face sector)**.** *Let $f \in X$, $b_3 \in H$ invariant under the X-action (4.41) and $b_4 \in H$ invariant under the X-action (4.67).*

Then any loop b_3 and any spike b_4 attached at vertex s decouple from all face operators $B_f(s,p)$.

Remark 2. If there were a process that inserted spikes and loops as in Proposition 1 into the graph Γ then at most the action of face operators could be modified. Conversely, any process introducing spikes and loops as in Proposition 2 can at most modify the action of vertex operators. We will again turn to spikes and loops in Section 5.2.1 where we properly define these insertion processes as isometric maps. □

Remark 3. Since the subspace of integrals of a Hopf algebra is one-dimensional (and similarly for the elements invariant under the action (4.41) of the dual) only the elements b_2 and b_4 invariant under their respective actions from the preceding Propositions will turn out nontrivial. □

Example 1. Let $H = \mathbb{C}G$. Since any group algebra is cocommutative vertex operators $A_a(s,p) = A_a(s)$ only depend on vertices rather than full sites. This has been noted in [39].

As far as decoupling from the vertex operators is concerned we have

$$b_1 \propto h, \tag{4.73}$$

$$b_2 = \sum_{g \in G} \lambda(g)\, g \tag{4.74}$$

with $\lambda\colon G \to \mathbb{C}$ a class function. This reflects the fact that purely magnetic charges are classified by conjugacy classes of the group G [39].

Loops and spikes decouple from the face operators if

$$b_3 \propto e, \tag{4.75}$$

$$b_4 \in \mathbb{C}G. \tag{4.76}$$

This reflects the fact that purely electric charges are classified by irreducible representations of the group G [39] which are realized as subspaces of the group algebra. □

4.2.2 Hilbert space

In order to obtain a physical system from the above discussion we need to define both a Hilbert space and a Hamiltonian for our topological lattice model. In particular, we need to find Hopf $*$-algebras H that allow for an inner product and $*$-representations. This means that adjoint operators (which as usual are defined relative to the inner product) in a representation of H are compatible with the $*$-structure of H itself. It turns out we can reach both goals by requiring H to be a finite-dimensional Hopf C^*-algebra. Such a Hopf algebra comes endowed with a

unique element called the Haar integral which will naturally define both the inner product and the Hamiltonian.

In order to define the Hilbert space we begin with the following proposition which is obtained from [58, 43].

Proposition 3. *Let H be a finite-dimensional Hopf C^*-algebra. Then $S^2 = \mathrm{id}$ and there exists a unique two-sided integral $h \in H$ with the following properties:*

1. $h^2 = h$,
2. $h^* = h$,
3. $S(h) = h$,
4. $h \in \mathrm{Cocom}(H)$.

Furthermore, H^* is a Hopf C^*-algebra again and its unique integral $\phi \in H^*$ satisfying 1–4 is a faithful positive functional, or trace, on H.

As a first consequence we can easily resolve the issue of edge orientation: since the antipode is now involutive we define the reversal of an edge e simply by

$$x_e \mapsto S(x_e). \tag{4.77}$$

This is obviously compatible with the actions (4.38), (4.39), (4.40) and (4.41). At the same time we no longer need to pick a distinguished pattern of edge orientations, rather all patterns are equivalent.

Although the preceding proposition tells us that ϕ is a positive trace on H, we need to know its precise relationship with the usual trace $\mathrm{tr}_H(a) = \mathrm{tr}(L^a_+)$ found in the literature on Hopf algebras. Setting $|H| \coloneqq \dim H$ we have

Lemma 3. *Let H a finite-dimensional Hopf C^*-algebra. Then*

$$\mathrm{tr}_H = |H| \cdot \phi \tag{4.78}$$

holds where $\phi \in H^*$ is the Haar functional on H.

Proof. By the above Proposition we know that $S^2 = \mathrm{id}$. In this case we have

$$\mathrm{tr}_H = \epsilon(\Lambda)\, \phi \tag{4.79}$$

for the integral $\Lambda \in H$ which is normalized such that $\phi(\Lambda) = 1$ [66]. Since $\phi(h) = |H|^{-1}$ for the Haar integral $h \in H$ [43] we actually have that $\Lambda = |H|\, h$. Finally, $\epsilon(h) = 1$ concludes the proof. □

Since ϕ is a faithful positive trace on H we can derive a Hermitian inner product on H from it by setting

$$(a, b)_H = \phi(a^* b). \tag{4.80}$$

This inner product now turns both the module structures $L_\pm$ and $T_\pm$ on H into $*$-representations. Indeed, relative to (4.80) the adjoint map of $L^a_\pm$ is given by $(L^a_\pm)^\dagger = L^{a^*}_\pm$ because of

$$\big(x, L^a_+(y)\big) = \phi(x^* a y) = \langle \phi, (a^* x)^* b\rangle = \big(L^{a^*}_+(x), y\big). \tag{4.81}$$

An easy but tedious calculation shows that $(T^f_\pm)^\dagger = T^{f^*}_\pm$ holds, too. Remember that f^* is given by (4.36).

This means that for the operators A_a and B_f which represent the Drinfeld double $\mathrm{D}(H)$ the adjoint operators are given by:

$$A^\dagger_a(s,p) = A_{a^*}(s,p), \tag{4.82}$$

$$B^\dagger_f(s,p) = B_{f^*}(s,p). \tag{4.83}$$

4.2.3 Hamiltonian

It remains to specify a Hamiltonian for the model. In analogy to [39] we would like to get a frustration-free Hamiltonian, i.e. a sum of commuting terms, and we would like to derive it from the local operators A_a and B_f defined previously. Hence we need to identify a subset of these operators such that they mutually commute with each other.

Before anything else it is natural to analyze the commutation relation between A_a and B_f at the *same* site (s,p) of the graph Γ. Suppose in the following that $a \in \mathrm{Cocom}(H)$ and $f \in \mathrm{Cocom}(X)$. Then at the level of $\mathrm{D}(H)$ we have

$$\begin{aligned} af &= \sum_{(a)} f\big(S^{-1}(a''')\,?\,a'\big)\,a'' \\ &= \sum_{(a)} f\big(a'\,S^{-1}(a''')\,?\big)\,a'' \\ &= \sum_{(a)} f\big(a'''S^{-1}(a'')\,?\big)\,a' \\ &= \sum_{(a)} f\big(\epsilon(a'')\,?\big)\,a' \\ &= fa, \end{aligned}$$

where we used the cocommutativity of f (or a respectively) in the second (third) line and the skew-antipode S^{-1} in the fourth one. In other words, the straightening formula (4.28) becomes trivial for such elements. Since for a fixed site (s,p) the operators $A_a(s,p)$ and $B_f(s,p)$ form a representation of $\mathrm{D}(H)$ this commutation relation immediately carries over to

$$A_a(s,p)\,B_f(s,p) = B_f(s,p)\,A_a(s,p) \tag{4.84}$$

which in fact reduces to

$$A_a(s)\,B_f(p) = B_f(p)\,A_a(s). \tag{4.85}$$

Having restricted the possible candidates for the terms in the Hamiltonian by exploiting the $\mathrm{D}(H)$-module structure at a single fixed site we need to ensure subsequently that *all* vertex operators $A_a(s)$ commute among themselves, too. Observe first that

$$[A_a(s), A_b(t)] = 0 \tag{4.86}$$

for any $a, b \in H$ whenever two vertices s and t do not coincide. Indeed, if $A_a(s)$ acts on a common edge via L_+ then $A_b(t)$ acts on the same edge via L_- and vice versa, hence both operators commute by (4.44). If there are no common edges then there is nothing to show. On the other hand if $s = t$ then the edge set $E(s)$ has an H-module structure (inherited from the $\mathrm{D}(H)$-module structure at any site that contains s) which implies

$$[A_a(s), A_b(s)] = A_{[a,b]}(s). \tag{4.87}$$

This suggests to further narrow down our set of candidate vertex operators $A_a(s)$ by additionally requiring $a \in \mathrm{Z}(H)$. By analogy, we are naturally led to assume $f \in \mathrm{Z}(X) \cap \mathrm{Cocom}(X)$ in the following if we want all face operators $B_f(p)$ to commute.

Finally we would like to remark that $A_a(s)$ and $B_f(p)$ trivially commute if the pair (s, p) is not a site. In fact, the sets of edges they act on are even disjoint in this case.

Recalling the properties of the Haar integral from Proposition 3 we now state the main result of this section.

Theorem 2 (Generalized quantum double model)**.** *Let H a finite-dimensional Hopf C^*-algebra with Haar integral h and Haar functional ϕ and let Γ a graph. Furthermore for each $s \in V$ and $p \in F$ define the projectors*

$$A(s) \coloneqq A_h(s, p), \tag{4.88}$$
$$B(p) \coloneqq B_\phi(s, p). \tag{4.89}$$

Then

$$\mathcal{H} = -\sum_{s \in V} A(s) - \sum_{p \in F} B(p) \tag{4.90}$$

is a local, frustration-free Hamiltonian defining the $\mathrm{D}(H)$-model.

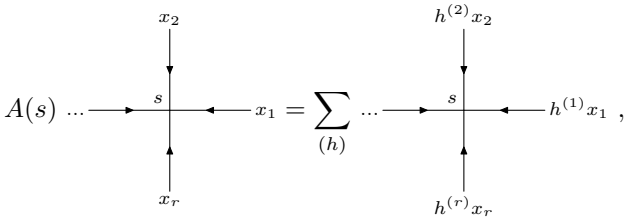
Proof. First observe that

$$A(s)^2 = A_{h^2}(s, p) = A(s) \tag{4.91}$$

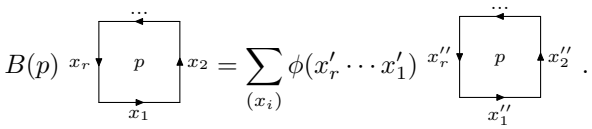
by Proposition 3. The same argument shows that $B(p)$ is a projector, too.

By the preceding discussion and Proposition 3 it is clear that all local terms $A(s)$ and $B(p)$ commute with each other. Furthermore, (4.82) and (4.83) imply that they are Hermitian. □

For the reader's convenience we give a short summary of how these operators act on the graph:

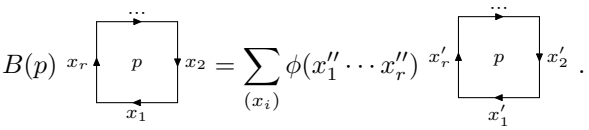


(4.92)



(4.93)

Alternatively, for a different orientation of the graph edges the face operator acts as follows:



(4.94)

It is no coincidence that already at this stage do the Haar integrals of H and H^* reveal themselves as the crucial ingredients for the generalized quantum double models. Since we set out to construct a quantum spin model whose (elementary) quasiparticle excitations are characterized by irreducible representations of $\mathrm{D}(H)$ the ground state sector with no quasiparticles present necessarily has the structure of a trivial representation of $\mathrm{D}(H)$ locally. Precisely this is what the element $\phi \otimes h \in \mathrm{D}(H)$ embodies. As the Hamiltonian intrinsically encodes information about the ground state sector it should not surprise the reader that the Haar integrals h and ϕ play such a prominent role.

4.3 Discussion

We would now like to briefly comment on the relation between our generalized quantum double models and Kitaev's original construction in [39].

First note that in the case $H = \mathbb{C}G$ the terms given by (4.92) and (4.93) in the above Hamiltonian reduce to the operators of (1.1) and (1.2) which are the ones Kitaev employed for his quantum double models based on a group G. This becomes clear from Section C.1 where the relevant expressions for h and ϕ are listed.

Secondly, the entire theory of ribbon operators readily carries over from [39] to our generalized quantum double models. This is because ribbon operators are constructed from certain elementary operators associated with two types of triangles any given ribbon path decomposes into. These operators are nothing but the $L_\pm$ and $T_\pm$ that implement the H- and X-module structures. Furthermore patching together a ribbon operator from those elementary pieces only involves the structure

maps of the Hopf algebra H itself. Actually, we even used ribbon paths in order to define the local $\mathrm{D}(H)$-module structures (and hence the Hamiltonian) without saying so.

Finally, our generalized quantum double models inherit all the beautiful topological properties of the original since these follow exclusively from the algebraic structure of ribbon operators. In particular, these features include the degeneracy of the ground state sector as well as the exotic statistics of the quasiparticle excitations whose anyonic nature is revealed via braiding and fusion operations.

These generalized quantum double models are physically relevant for at least two reasons. First, they give a broader perspective on the algebraic structures used to construct the original quantum double models based on groups and thus emphasize the role of Hopf symmetry which is so vital to topologically ordered phases in two dimensions. In doing so they pave the way for the novel class of Hopf tensor network states to be introduced in Chapter 5. Second, the generalized quantum double models based on Hopf algebras comprise the smallest class of models admitting a non-Abelian electric-magnetic duality in topological phases. This will be discussed in detail in Chapter 6.

Chapter 5

Hierarchy of Hopf tensor network states

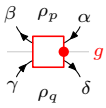
In this chapter we present a hierarchy of quantum many-body states among which many examples of topological order can be identified by construction. We define these states in terms of a general, basis-independent framework of tensor networks based on finite-dimensional Hopf C^*-algebras. At the top of the hierarchy we identify ground states of the generalized quantum double models we introduced in Chapter 4. For these states we exhibit the mechanism responsible for their non-zero topological entanglement entropy by constructing a renormalization group flow. Furthermore it is shown that those states of the hierarchy associated with KITAEV's original quantum double models are related to each other by the condensation of topological charges. We conjecture that charge condensation is the physical mechanism underlying the hierarchy in general.

As efficient representations of ground states serve as a valuable tool in mapping out phase diagrams of physical models and determining phase transitions between topologically and conventionally ordered phases in particular it is crucial to first understand the properties of tensor networks describing states of the model deep within the different phases. In order to derive such tensor network representations for quantum double model ground states we introduce a novel diagrammatic technique. It turns out that the tensor networks can be formulated basis-independently and that furthermore a single distinguished element of the Hopf C^*-algebra, namely its unique Haar integral, plays the key role in the construction. Other than that only the structure maps of the Hopf C^*-algebra are employed in the definition of the tensor network states.

Additionally we are able to extend these ground states to a hierarchy of tensor network states for each finite-dimensional Hopf C^*-algebra. This hierarchy of states is characterized by different values of the topological entanglement entropy γ as defined in [41, 47], and hence these states represent different instances of topological order, in other words different unitary modular tensor categories (UMTCs). The way this hierarchy depends on the Hopf subalgebras of the original Hopf C^*-algebra points towards condensation of topological charges [3, 4]. Furthermore we show how different isomorphism classes of finite-dimensional Hopf C^*-algebras exhibit different mechanisms for the non-zero γ. In particular, we explain how the boundary configurations of a region differ between models based on a group algebra, the dual of a group algebra and non-trivial finite-dimensional Hopf C^*-algebras.

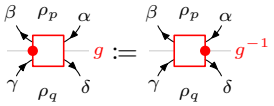
Let us begin by briefly reviewing how tensor network representations for particular states occurring in the quantum double models based on finite groups can be

obtained. It is instructive to start from their inherent (group) symmetries. We can assign a representation ρ_p of G to each face $p \in F$ and associate the tensor



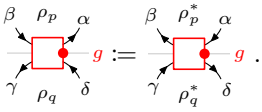
$$= A(\rho_p, \rho_q)^g_{\alpha\beta\gamma\delta} \coloneqq \rho_p(g)_{\alpha\beta}\, \rho_q(g^{-1})_{\gamma\delta} \tag{5.1}$$

with each oriented edge. Here $\rho_p(g)_{\alpha\beta}$ denotes a matrix element of the representation ρ_p and the red dot represents the *physical* index g with its orientation inherited from the underlying graph edge. The Greek letters attached to black arrows are called *virtual* indices. The tensor for a reversed graph edge naturally reads



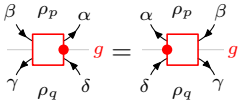
(5.2)

Reversing virtual arrows is defined by



(5.3)

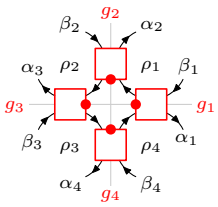
where the dual representation ρ^* of a representation ρ is given by the matrix equation $\rho^*(g) \coloneqq \rho^T(g^{-1})$. Note that this is equivalent to a reflection of the tensor about its vertical axis:



(5.4)

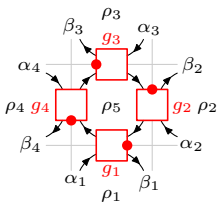
The tensor network obtained by contracting all virtual indices around each face represents what we will call a *group tensor network state* in the following.

For example, on a square lattice the tensor representing a vertex with a particular edge orientation is given by



$$= \prod_{j=1}^{4} \rho_j(g_j^{-1} g_{j+1})_{\beta_j \alpha_{j+1}} \tag{5.5}$$

where the index j labelling graph edges is assumed to be cyclic. Quite similarly, the tensor corresponding to a particular face reads



$$= \chi_{\rho_5}(g_4 \cdots g_1) \prod_{j=1}^{4} \rho_j(g_j^{-1})_{\alpha_j \beta_j} \tag{5.6}$$

where χ_ρ is the character of the representation ρ.

In fact, both this vertex and face tensor are *partially* contracted tensor networks and represent linear maps from the uncontracted virtual indices to the physical ones. So a piece of a tensor network with open indices can be regarded as a PEPS projection map [70, 55]. If V_p is the vector space associated with the representation ρ_p then this linear map $P\colon V_p \otimes V_p \otimes V_q \otimes V_q \to \mathbb{C}G$ is given by

$$P = \sum_{g,\alpha,\ldots,\delta \in G} A(\rho_p, \rho_q)^g_{\alpha\beta\gamma\delta} |g\rangle\langle\alpha, \beta, \gamma, \delta| \tag{5.7}$$

for a single edge tensor $A(\rho_p, \rho_q)$.

On the contrary, a *fully* contracted tensor network is a complex number which equals the inner product between a particular basis state and the group tensor network state. If viewed as a linear map from basis configurations to amplitudes we will call it a *tensor trace*.

Clearly, a group tensor network state only depends on the isomorphism class of the representations ρ_p assigned to the faces. In general, there is vastly more gauge freedom for a tensor network to represent the same physical state, however, this does not necessarily respect the G-action.

If we choose all representations ρ_p to coincide with the left regular one the resulting group tensor network state will be a ground state of the D($\mathbb{C}G$)-model. Since its Hamiltonian is frustration free (i.e. a sum of mutually commuting terms) this can easily be shown by considering independently the action of $A(s)$ and $B(p)$ on the partially contracted tensor networks in (5.5) and (5.6).

As an example, a ground state of the toric code is given by the group tensor network for the group $\mathbb{Z}_2 = \{e, a\}$ with all faces carrying the regular representation. In that case the PEPS projection map reads $P = |e\rangle\langle\phi^+|\langle\phi^+| + |a\rangle\langle\psi^+|\langle\psi^+|$ with the Bell states

$$|\phi^+\rangle = \sum_{\alpha,\beta \in \mathbb{Z}_2} L(e)_{\alpha\beta} |\alpha, \beta\rangle = |e, e\rangle + |a, a\rangle, \tag{5.8}$$

$$|\psi^+\rangle = \sum_{\alpha,\beta \in \mathbb{Z}_2} L(a)_{\alpha\beta} |\alpha, \beta\rangle = |e, a\rangle + |a, e\rangle. \tag{5.9}$$

We would like to remark that this projection map coincides (up to a trivial isomorphism) with the one given in [72].

Later in this chapter we will turn to the physical significance of group tensor network states arising from representations of G other than the regular one.

In order to build some more intuition we briefly review the structure of a local tensor in a group tensor network state. It is easily seen that the tensor in (5.1) factorizes with respect to partitioning the virtual indices into the sets $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. This implies that the PEPS projection map (5.7) can be rewritten as:

$$P = \sum_{g \in G} |g\rangle\langle\phi_p(g)|\langle\phi_q(g^{-1})| \tag{5.10}$$

where a state associated to the virtual indices in face p is given by:

$$|\phi_p(g)\rangle = \sum_{\alpha,\beta\in G} \overline{\rho_p(g)_{\alpha\beta}}|\alpha,\beta\rangle. \tag{5.11}$$

This means that one can carry out the contraction of the tensor network in two steps. First one contracts each virtual loop separately and then one needs to glue these pieces together. In a group tensor network this glueing process is simply given by

$$|g\rangle \otimes |g^{-1}\rangle \mapsto |g\rangle. \tag{5.12}$$

It turns out that this step becomes rather nontrivial once we generalize the quantum double model to finite-dimensional Hopf C^*-algebras. In fact, we anticipate that this glueing step is one of the crucial ingredients for deriving the hierarchy of tensor network states in this chapter.

In Section 5.1, we will solve this model by providing a tensor network representation of its ground state. This representation only involves the canonical structures associated to the underlying Hopf algebra and leads us to propose a novel hierarchy of tensor network states based on Hopf subalgebras. In Section 5.2 we calculate both the entanglement entropy and the topological entanglement entropy for distinguished states in the hierarchy. In Section 5.3 we provide concluding remarks and an outlook on future work.

5.1 Diagrammatic calculus

In this section we develop a general diagrammatic language for tensor network states built from finite-dimensional Hopf C^*-algebras. Underlying surfaces both with and without boundaries are considered and we show how to naturally describe subsystems. Using this framework we solve the generalized quantum double model introduced in the preceding section by providing a tensor network representation for one of its ground states. Any other energy eigenstate can be obtained from there by an appropriate ribbon operator. The tensor network representation for that ground state only involves the canonical structures of the underlying Hopf C^*-algebra H and its dual: multiplication, comultiplication, antipode and Haar integral. This insight leads us to propose a novel hierarchy of tensor network states based on Hopf subalgebras. For $H = \mathbb{C}G$ and $H = \mathbb{C}^G$ we are able to completely classify this hierarchy of states in terms of charge condensation. Finally we describe the relation between our Hopf tensor network language and the usual formulation of PEPS.

Unless otherwise noted, from now on H will be a finite-dimensional Hopf C^*-algebra with Haar integral $h \in H$ and Haar functional $\phi \in H^*$.

5.1.1 Tensor traces

As we discussed in the introduction, the fully contracted tensor network (which is a complex number) for a certain ground state of the D($\mathbb{C}G$)-model on the oriented

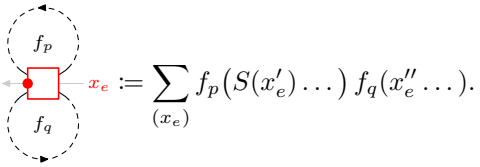
graph Γ can be interpreted as a collection of virtual loops in the faces of Γ that are suitably glued together to form the physical degrees of freedom.

We can extend this idea to the case of any finite-dimensional Hopf C^*-algebra H now. In each face $p \in F$ place a virtual loop and associate a function $f_p \in X$ to this loop. For the moment we may restrict to virtual loops oriented in counterclockwise direction as well as to $f_p \in \mathrm{Cocom}(X)$. With each oriented edge $e \in E$ we associate an algebra element $x_e \in H$ which splits into two parts as follows:

$$\big((S \otimes \mathrm{id}) \circ \Delta\big)(x_e) = \sum_{(x_e)} S(x'_e) \otimes x''_e. \tag{5.13}$$

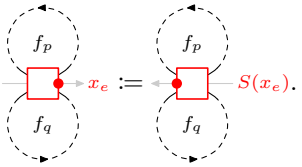
Subsequently, we attribute x''_e to the left adjacent face of e and $S(x'_e)$ to the right one. A virtual loop in face p is then evaluated by taking the *clockwise* product of all elements thus associated with the loop from the surrounding edges $E(p)$. The result is then fed to the function f_p.

In order to simplify arguments we introduce a diagrammatic notation which encodes calculations with these tensor networks. In diagrammatic language the evaluation rule just described reads



$$:= \sum_{(x_e)} f_p\big(S(x'_e)\dots\big)\, f_q(x''_e\dots). \tag{5.14}$$

Here the red dot indicates the orientation of the underlying graph edge. Since f_p is assumed cocommutative its argument can be permuted cyclically and we may start both virtual loops around p and q at the edge e without loss of generality. The clockwise order of the product remains important though. Both the dashed lines and the ellipses denote the remaining degrees of freedom of the faces p and q respectively. Note that the glueing procedure generalizing (5.12) is implemented by the coproduct. We will introduce a full description of this later in (5.24) and (5.26). Finally, a reversed edge is resolved via



(5.15)

In summary we have the following

Definition 2 (Hopf tensor trace without boundary)**.** Let $\Gamma = (V, E, F)$ an oriented graph embedded in a surface M without boundary.

Then the *Hopf tensor trace* associated with Γ is the linear function $\mathrm{ttr}_\Gamma \colon H^{\otimes|E|} \otimes \mathrm{Cocom}(X)^{\otimes|F|} \to \mathbb{C}$,

$$\bigotimes_{e\in E} x_e \bigotimes_{p\in F} f_p \mapsto \mathrm{ttr}_\Gamma(\{x_e\};\{f_p\}) \tag{5.16}$$

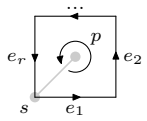
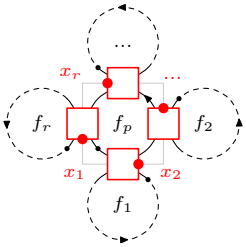


Figure 5.1: Decorated face.

which is defined via diagrams and the evaluation rules (5.14) and (5.15).

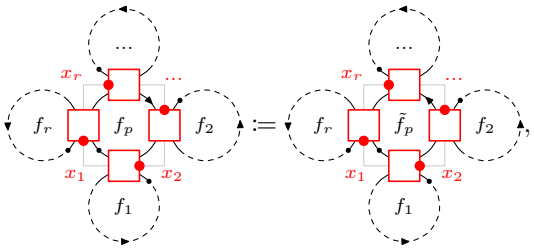
If we want to allow for arbitrary orientation of the virtual loops and relax the condition $f_p \in \mathrm{Cocom}(X)$ then we need to generalize the notion of an oriented graph slightly. Namely, we need faces to be *oriented* and their boundary edges to be *ordered*. This means that each face p has either clock- or anticlockwise orientation and a distinguished site (s, p) with $s \in V(p)$. This site together with the orientation of the face induces an ordering of $E(p)$ (see Figure 5.1). We will call such a face *decorated* subsequently.

We can again associate a Hopf tensor network to an oriented graph with decorated faces. In diagrammatic language this means



$$:= \sum_{(x_i)} f_p\big(x_r'' \cdots x_1''\big) \prod_{i=1}^{r} f_i\big(\ldots S(x_i')\big) \tag{5.17}$$

where the orientation of the virtual loop reflects the orientation of the underlying graph face. The black dot indicates the origin of the virtual loop such that it visits the surrounding edges of the face p in the order given by the ordering of $E(p)$. This order is *reversed* for the product of all elements associated with that loop. A reversed edge is resolved as before via (5.15) and a reversed face via



$$\tilde{f}_p = S^T(f_p) \tag{5.18}$$

Note that in contrast to edge and face reversal the virtual loop origin cannot be moved in general.

In summary we have the following

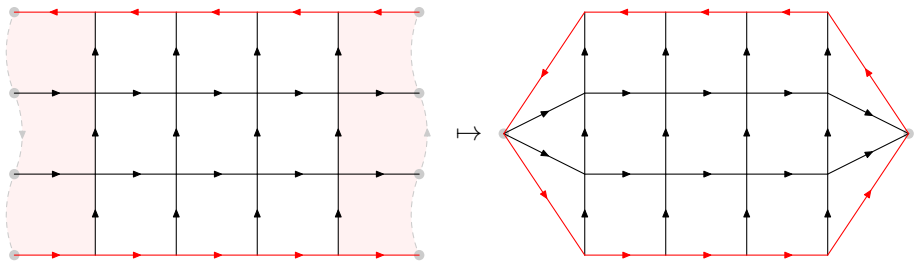


Figure 5.2: Proper faces $F = F_{M^\circ}$ (white) vs. boundary "faces" $F_{\partial M}$ (light red). Upon continuous deformation of those pieces of the boundary ∂M which do not coincide with graph edges (dashed grey) the vertices marked in grey on the left and on the right are identified with each other respectively. The boundary faces $F_{\partial M}$ are precisely those which are completed to proper faces by this process. Furthermore, both the boundary edges $E_{\partial M}$ (red) and interior edges E_{M° (black) are shown.

Definition 3 (General Hopf tensor trace without boundary)**.** Let $\Gamma = (V, E, F)$ an oriented graph with decorated faces F embedded in a surface M without boundary.

Then the (general) *Hopf tensor trace* associated with Γ is the linear function $\mathrm{ttr}_\Gamma \colon H^{\otimes|E|} \otimes X^{\otimes|F|} \to \mathbb{C}$,

$$\bigotimes_{e\in E} x_e \bigotimes_{p\in F} f_p \mapsto \mathrm{ttr}_\Gamma(\{x_e\};\{f_p\}) \tag{5.19}$$

which is defined via diagrams and the evaluation rules (5.17), (5.15) and (5.18).

Note that this definition reduces to Definition 2 if all $f_p \in \mathrm{Cocom}(X)$ and all faces are oriented counterclockwise.

It remains to generalize Definition 2 in another direction, namely to general Hopf tensor networks on graphs *with* boundaries. These arise either from (external) boundaries of the surface M itself which the graph $\Gamma = (V, E, F)$ is embedded in or from (internal) boundaries of certain regions $R \subset M$.

It is clear that the surface boundary ∂M induces a notion of boundary for the graph Γ by decomposing its edge set $E_M := E$ into the disjoint union of interior edges E_{M° and boundary edges $E_{\partial M}$ (see Figure 5.2). At the same time ∂M introduces the concept of boundary "faces" $F_{\partial M}$ which themselves do not belong to the proper faces F of the graph but could be completed to proper ones by continuously deforming M and identifying two vertices on the external boundary ∂M in the process (see Figure 5.2). Together with the (proper) interior faces $F_{M^\circ} := F$ they form the set of generalized faces $F' := F_{M^\circ} \cup F_{\partial M}$. Naturally we can then regard $\partial\Gamma := E_{\partial M} \cup F_{\partial M}$ as the boundary of the graph with an orientation inherited from the orientation of ∂M.

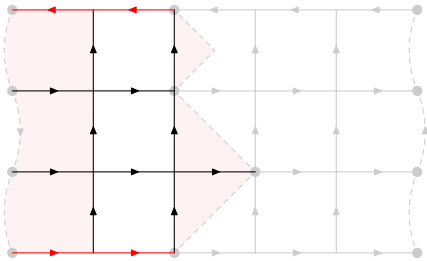


Figure 5.3: A region R whose boundary ∂R never crosses any interior edges E_{M° cleanly partitions the edges into the set E_R (black, red) which belongs to R and the rest (grey). The internal boundary $(\partial R)^\circ$ consists of those pieces which do not coincide with the boundary of the ambient surface M and gives rise to the internal boundary faces $F_{(\partial R)^\circ}$ (light red, middle).

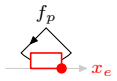
Similarly, it is clear that a region R with the *internal boundary* $(\partial R)^\circ \coloneqq \partial R \setminus (\partial R \cap \partial M)$ in M induces a notion of internal boundary for the graph Γ provided that $(\partial R)^\circ$ does not cross any edges of Γ except at vertices[1] (see Figure 5.3). We will only consider regions that meet this requirement in the following. Indeed, a partition of M into such a region R and its complement $\bar{R}$ naturally divides the set of edges E_M into the disjoint sets E_R and $E_{\bar{R}}$. At the same time it divides the set of proper faces F into the disjoint sets of proper faces F_R and $F_{\bar{R}}$ as well as new internal boundary faces $F_{(\partial R)^\circ}$ and $F_{(\partial \bar{R})^\circ}$. The original boundary faces $F_{\partial M}$ are attributed to either the region or its complement in the obvious way. Hence given a generalized graph $\Gamma = (V, E, F')$ embedded in M a region R induces a subgraph $\Gamma_R = (V_R, E_R, F'_R)$ whose oriented boundary $\partial\Gamma_R$ we define analogously to the above case of external boundaries. In particular, its internal boundary reads $(\partial\Gamma_R)^\circ = F_{(\partial R)^\circ}$. It is not difficult to incorporate the notion of decorated faces into both cases.

On the other hand, these (sub)graphs *define* the boundaries of surfaces (regions) up to continuous deformations. In the first case this means that connected segments of the surface boundary ∂M are classified into *smooth* or *rough* boundaries depending on whether they correspond to the boundary type $E_{\partial M}$ or $F_{\partial M}$ in the boundary $\partial\Gamma$ of the graph. In the second case a subgraph Γ_R defines the region R itself. Not surprisingly, the internal boundary $(\partial R)^\circ$ of the region will always be of the rough type. Furthermore we can measure the length of any boundary ∂M or ∂R simply by counting the elements in the respective graph boundaries $\partial\Gamma$ or $\partial\Gamma_R$. In particular, we define $|(\partial R)^\circ| \coloneqq |F_{(\partial R)^\circ}|$.

In order to take these boundaries into account we need to refine our diagrammatic notation for tensor networks. Pick an arbitrary edge $e \in E$. Since it belongs to the boundary of at least one face p in either F_{M° or $F_{\partial M}$ (i.e. p is either an interior or a boundary face) we may define the following elementary diagram for

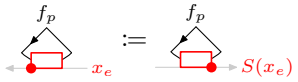
[1] The remaining boundary $\partial R \setminus (\partial R)^\circ$ may contain boundary edges $E_{\partial M}$, of course.

any elements $f_p \in X$ and $x_e \in H$ simply as their canonical pairing:

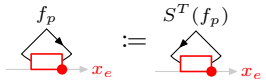


$$:= f_p(x_e). \tag{5.20}$$

Different orientations of graph edges and virtual loops are resolved via

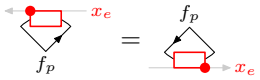


(5.21)



(5.22)

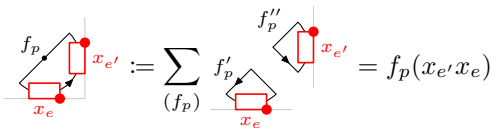
where all these elementary diagrams are assumed to be invariant under arbitrary rotations, for instance:



(5.23)

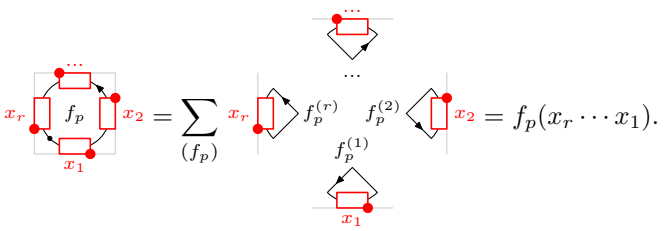
Note also that $S^T = (S^{-1})^T$ is the correct antipode of $X = (H^{\mathrm{op}})^*$. As a consequence, any of the above elementary diagrams has the same value as its mirror image under reflection about a vertical axis. In fact, (5.21) and (5.22) are mirror images of each other in that sense. Furthermore for some f_p (such as the dual Haar integral ϕ) it may happen that $S^T(f_p) = f_p$ and the loop orientation may become unimportant.

If the face p has edges other than e in its boundary we may extend the above diagrams as follows. Pick another edge $e' \in E(p)$ which shares a common vertex with e. Then for any $x_{e'} \in H$ we define a "virtual" glueing operation by



$$= f_p(x_{e'} x_e) \tag{5.24}$$

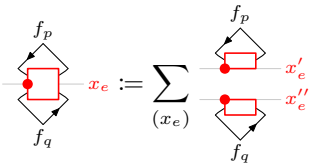
where the arrows indicate the order in which the coproduct of $f_p \in X$ is applied to the elementary diagrams. The black dot denotes the origin for this comultiplication. Both this origin and the order of comultiplication are determined from the decoration of the face p as implied in Definition 3. Here we uncover the connection between the face decoration and Hopf algebra structures. For example, for the decorated face in Figure 5.1 we have



$$= f_p(x_r \cdots x_1). \tag{5.25}$$

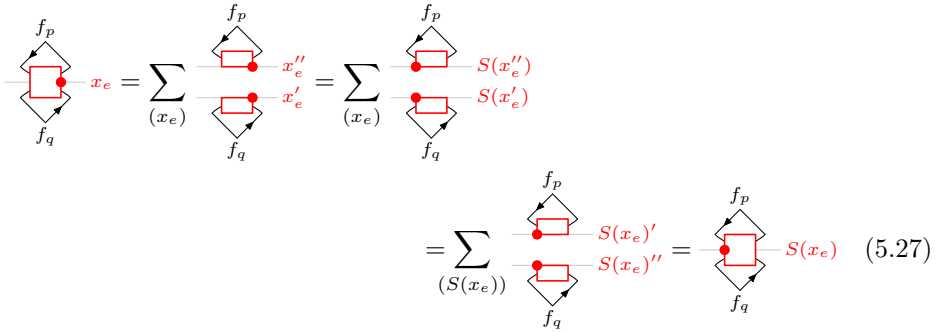
While the origin of the virtual loop is very important in general it can be neglected if f_p is cocommutative, for instance. In such a case we will simply omit the corresponding dot from the diagram as we did in Definition 2. In any case one needs to pay attention to the correct comultiplication in X which causes the product around the edges of p to be taken in clockwise order.

Finally for any interior edge $e \in E_{M^\circ}$ with adjacent interior or boundary faces $p, q \in F'$ we pick $x_e \in H$ and $f_p, f_q \in X$ arbitrarily and define a "physical" glueing operation by



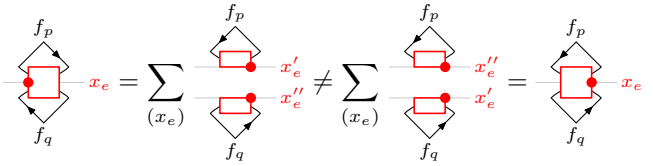
(5.26)

where the order of comultiplication is determined by the orientation of the underlying graph edge. Consequently one has for instance



(5.27)

which is perfectly compatible with (4.77) as expected. Note that in general a simultaneous reversal of both virtual loops is *not* given by reflecting a composite diagram about its vertical axis:



(5.28)

Rather, equality holds if $x_e \in \mathrm{Cocom}(H)$, for instance. This should be compared with (5.4).

Thus by starting from the elementary diagram (5.20) we have reexpressed the evaluation rule (5.14) for an interior edge entirely in terms of "virtual" (5.24) and "physical" (5.26) glueing operations. These are given by comultiplication in X and H respectively. This means that an interior edge is formed by appropriately

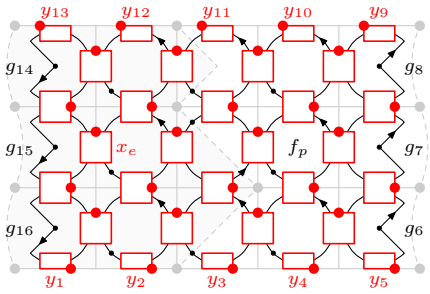


Figure 5.4: Diagram encoding the tensor trace $\mathrm{ttr}_\Gamma(\{x_e\};\{f_p\};\{y_e\};\{g_q\})$. While the interior degrees of freedom $\{x_e\} \subset H$ and $\{f_p\} \subset X$ are only shown partially the boundary degrees of freedom $\{y_e\} \subset H$ and $\{g_q\} \subset X$ are labelled in such a way that the ordering of the boundary is evident.

glueing two boundary edges together via comultiplication. Alternatively, one may regard this as glueing together two virtual interior loops. Virtual loops themselves are assembled via glueing together smaller loop pieces.

In summary we have the following general

Definition 4 (Hopf tensor trace)**.** Let $\Gamma = (V, E, F')$ an oriented graph with decorated generalized faces F' embedded in a surface M with boundary.

The *Hopf tensor trace* associated with Γ is the linear function $\mathrm{ttr}_\Gamma \colon H^{\otimes|E_{M^\circ}|} \otimes X^{\otimes|F_{M^\circ}|} \otimes H^{\otimes|E_{\partial M}|} \otimes X^{\otimes|F_{\partial M}|} \to \mathbb{C}$,

$$\bigotimes_{e\in E_{M^\circ}} x_e \bigotimes_{p\in F_{M^\circ}} f_p \bigotimes_{e\in E_{\partial M}} y_e \bigotimes_{q\in F_{\partial M}} g_q \mapsto \mathrm{ttr}_\Gamma(\{x_e\};\{f_p\};\{y_e\};\{g_q\}) \tag{5.29}$$

which is defined via diagrams and the evaluation rules (5.20), (5.21), (5.22), (5.24) and (5.26).

Note that this generalized Hopf tensor trace reduces to the Hopf tensor trace for a surface without boundary (as in Definition 3) in the natural way:

$$\mathrm{ttr}_\Gamma(\{x_e\};\{f_p\}) \coloneqq \mathrm{ttr}_\Gamma(\{x_e\};\{f_p\};\emptyset;\emptyset). \tag{5.30}$$

5.1.2 Quantum states

So far we have merely defined a particular, fully contracted tensor network (which is a complex number) with *no* reference to a quantum many-body state whatsoever. We now take the next step and use the tensor trace above to generate actual quantum states in a remarkably straightforward fashion:

Definition 5 (Hopf tensor network state)**.** Let $x_e, y_e \in H$ and $f_p, g_q \in X$ as in Definition 4. Let Γ the graph embedded in the surface M.

1. If $\partial M \neq \emptyset$ then

$$|\psi_\Gamma(\{x_e\};\{f_p\};\{y_e\};\{g_q\})\rangle \coloneqq \sum_{(x_e)}\sum_{(y_e)} \mathrm{ttr}_\Gamma(\{x''_e\};\{f_p\};\{y''_e\};\{g_q\}) \bigotimes_{e\in E_{M^\circ}} |x'_e\rangle \bigotimes_{e\in E_{\partial M}} |y'_e\rangle. \quad (5.31)$$

2. If $\partial M = \emptyset$ then

$$|\psi_\Gamma(\{x_e\};\{f_p\})\rangle \coloneqq \sum_{(x_e)} \mathrm{ttr}_\Gamma(\{x''_e\};\{f_p\}) \bigotimes_{e\in E_M} |x'_e\rangle. \quad (5.32)$$

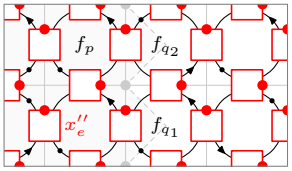
In both cases we call the resulting state a *Hopf tensor network state* on the graph Γ.

Given a region $R \subset M$ meeting our previous requirements it is straightforward to partition such a quantum state into subsystems corresponding to R and its complement $\bar{R}$. In the language of Hopf tensor network states this simply amounts to cutting virtual loops by comultiplication.

We will get particularly nice partitions if the internal boundary $(\partial R)^\circ$ is *compatible* with the decoration of the faces it cuts, i.e. after suitable deformation $(\partial R)^\circ$ intersects each virtual loop at its origin (and one other point). Not surprisingly, only the internal boundary $(\partial R)^\circ$ will matter for the decomposition into subsystems hence without loss of generality we state the following

Proposition 4 (Partitions)**.** *Let $\Gamma = (V, E, F)$ an oriented graph with decorated faces F embedded in a surface M without boundary. Let the subgraph $\Gamma_R = (V_R, E_R, F_R) \subset \Gamma$ define a simply connected region $R \subset M$ whose boundary ∂R is compatible with the decoration of F. Let*

$$|\psi_\Gamma(\{x_e\};\{f_p\})\rangle = \sum_{(x_e)} \quad \bigotimes_{e\in E_M} |x'_e\rangle \quad (5.33)$$



a Hopf tensor network state on this graph. Let $g_q \in X$ and define

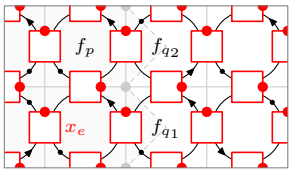
$$|\psi_R(\{g_q\})\rangle \coloneqq |\psi_{\Gamma_R}(\{x_e\}_R;\{f_p\}_R;\emptyset;\{g_q\})\rangle \quad (5.34)$$

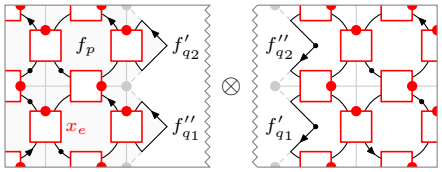
where $\{x_e\}_R = \{x_e \mid e \in E_R\}$ and $\{f_p\}_R = \{f_p \mid p \in F_R\}$ are the natural restrictions to Γ_R.

Then

$$|\psi_\Gamma(\{x_e\};\{f_p\})\rangle = \sum_{\substack{(f_q)\\ q\in F_{\partial R}}} |\psi_R(\ldots, f''_{q_1}, f'_{q_2}, \ldots)\rangle \otimes |\psi_{\bar{R}}(\ldots, f'_{q_1}, f''_{q_2}, \ldots)\rangle. \quad (5.35)$$

Proof. Obvious from the complete diagrams and evaluation rule (5.24) which needs to be applied backwards. Indeed, we have

$\mathrm{ttr}_\Gamma(\{x_e\};\{f_p\}) =$ 

$= \sum_{\substack{(f_q)\\ q\in F_{\partial R}}}$ 

where the left part corresponds to the region R and the right part to its complement $\bar{R}$. □

Remark 4. It is not difficult to see that a Hopf tensor network state can be partitioned in this fashion in the most general situation: this includes surfaces with external boundaries and arbitrary regions which are neither simply connected nor compatible with the decorated faces they cut. One simply decomposes the diagram defining the Hopf tensor trace all the way to its atomic pieces (5.20) and then regroups them into diagrams corresponding to R and $\bar{R}$ respectively. Since the partition never cuts any edges the Hilbert space vectors associated to them can be mated uniquely with these regrouped diagrams, yielding two quantum states glued together along the internal boundary via comultiplication. □

If the origin of the cut virtual loops can be neglected we get

Corollary 1 (Standard partitions)**.** *Let $\Gamma = (V, E, F)$ an oriented graph embedded in a surface M without boundary. Let the subgraph $\Gamma_R = (V_R, E_R, F_R) \subset \Gamma$ define a simply connected region $R \subset M$ and assume $f_p \in \mathrm{Cocom}(X)$.*

Then

$$|\psi_\Gamma(\{x_e\};\{f_p\})\rangle = \sum_{\substack{(f_q)\\ q\in F_{\partial R}}} |\psi_R(\{f'_q\})\rangle \otimes |\psi_{\bar{R}}(\{f''_q\})\rangle. \tag{5.36}$$

Building on this general framework of Hopf tensor network states we can solve the generalized quantum double model now. Namely, we identify a particular Hopf tensor network state as a ground state of the model. As such this state is directly seen to be topologically ordered. Note that we will only make use of the structure maps of H as well as the Haar integral and its dual to describe the state. Again we assume a surface without boundaries.

Theorem 3 (Ground state of the generalized quantum double model)**.** *Let $h \in H$ and $\phi \in X$ the respective Haar integrals. The state*

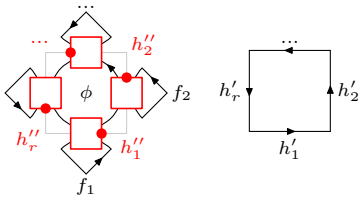
$$|\psi_\Gamma\rangle \coloneqq |\psi_\Gamma(h,\dots,h;\phi,\dots,\phi)\rangle \tag{5.37}$$

is a ground state of the D(H)*-model.*

Proof. Since the Hamiltonian of the D(H)-model is a sum of local, commuting terms by Theorem 2 it is enough to show that each operator $A(s)$ and $B(p)$ leaves the state $|\psi_\Gamma\rangle$ invariant individually. In order to do so we may partition $|\psi_\Gamma\rangle$ into an interior part corresponding to the support of such an operator and an exterior part. Both parts are glued via comultiplication in X. It will then suffice to prove that this interior part remains unchanged by either $A(s)$ or $B(p)$ respectively.

Hence consider a face $p \in F$ with a boundary consisting of r edges. According to (5.34) the interior part of $|\psi_\Gamma\rangle$ is given by

$$|\psi_p(f_1,\dots,f_r)\rangle = \sum_{(h_i)}$$



(5.38)

$$= \sum_{(h_i)} \phi(h_r'''\cdots h_1''') \prod_{j=1}^{r} f_j\big(S(h_j'')\big)|h_1'\rangle \otimes \cdots \otimes |h_r'\rangle. \tag{5.39}$$

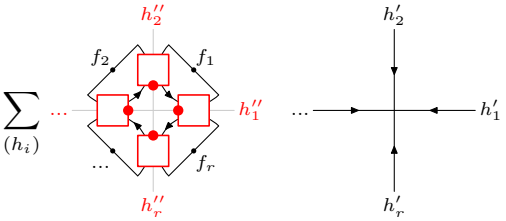
with the Haar integrals $h_i = h \in H$ and arbitrary $f_i \in X$. It is invariant under the action of $B(p)$ as can be seen from

$$\begin{aligned}
&B(p)|\psi_p(f_1,\dots,f_r)\rangle \\
&\quad= \sum_{(h_i)} \phi(h_r^{(4)}\cdots h_1^{(4)})\,\phi(h_r^{(1)}\cdots h_1^{(1)}) \prod_{j=1}^{r} f_j\big(S(h_j^{(3)})\big)|h_1^{(2)}\rangle \otimes \cdots \otimes |h_r^{(2)}\rangle \\
&\quad= \sum_{(h_i)} \phi(h_r^{(3)}\cdots h_1^{(3)})\,\phi(h_r^{(4)}\cdots h_1^{(4)}) \prod_{j=1}^{r} f_j\big(S(h_j^{(2)})\big)|h_1^{(1)}\rangle \otimes \cdots \otimes |h_r^{(1)}\rangle \\
&\quad= \sum_{(h_i)} \phi^2(h_r'''\cdots h_1''') \prod_{j=1}^{r} f_j\big(S(h_j'')\big)|h_1'\rangle \otimes \cdots \otimes |h_r'\rangle \\
&\quad= |\psi_p(f_1,\dots,f_r)\rangle.
\end{aligned}$$

Next consider a vertex $s \in V$ with r attached edges. In this case the interior part

of $|\psi_\Gamma\rangle$ reads

$$|\psi_s(f_1,\dots,f_r)\rangle = \sum_{(h_i)} \quad (5.40)$$



$$= \sum_{(h_i)} \prod_{j=1}^{r} f_j\big(S(h''_j)\, h'''_{j+1}\big)|h'_1\rangle \otimes \cdots \otimes |h'_r\rangle. \quad (5.41)$$

Note that the orientation of the graph edges can always be reduced to the above setting using (4.77) for kets and (5.21) for diagrams. Now we obtain

$$\begin{aligned}
&A(s)|\psi_s(f_1,\dots,f_r)\rangle \\
&\quad= \sum_{(h_i)} \prod_{j=2}^{r-1} f_j\big(S(h''_j)\, h'''_{j+1}\big) \sum_{(h)} f_1\big(S(h''_1)\, h'''_2\big)\, f_r\big(S(h''_r)\, h'''_1\big)|h^{(1)}h'_1\rangle \otimes \cdots \otimes |h^{(r)}h'_r\rangle \\
&\quad= \sum_{(h_i)} \prod_{j=3}^{r-1} f_j\big(S(h''_j)\, h'''_{j+1}\big) \sum_{(h)} f_1\big(S(h''_1)\, h^{(2)}h'''_2\big)\, f_2\big(S(h''_2)\, h'''_3\big)\, f_r\big(S(h^{(1)}h''_r)\, h'''_1\big)|h'_1\rangle \otimes |h^{(3)}h'_2\rangle \otimes \cdots \otimes |h^{(r+1)}h'_r\rangle \\
&\quad= \sum_{(h_i)} f_1\big(S(h''_1)\, h'''_2\big) \prod_{j=3}^{r-1} f_j\big(S(h''_j)\, h'''_{j+1}\big) \sum_{(h)} f_2\big(S(h''_2)\, h^{(2)}h'''_3\big)\, f_r\big(S(h^{(1)}h''_r)\, h'''_1\big)|h'_1\rangle \otimes |h'_2\rangle \otimes |h^{(3)}h'_3\rangle \otimes \cdots \otimes |h^{(r} \\
&\quad= \sum_{(h_i)} \prod_{j=1}^{r-2} f_j\big(S(h''_j)\, h'''_{j+1}\big) \sum_{(h)} f_{r-1}\big(S(h''_{r-1})\, h^{(2)}h'''_r\big)\, f_r\big(S(h_1^{(1)}h''_r)\, h'''_1\big)|h'_1\rangle \otimes \cdots \otimes |h'_{r-1}\rangle \otimes |h^{(3)}h'_r\rangle \\
&\quad= |\psi_s(f_1,\dots,f_r)\rangle
\end{aligned}$$

where we repeatedly used Lemma 6. This concludes the proof. □

In fact, the proof of Theorem 3 implies that $|\psi_\Gamma\rangle$ is also invariant under each local action of $\mathrm{D}(H)$ at any site (s,p), not just under the operators constituting the Hamiltonian. More precisely, one has

$$B_f(s,p)\, A_a(s,p)|\psi_\Gamma\rangle = \epsilon(a)\, f(1_H)|\psi_\Gamma\rangle \quad (5.42)$$

for any $f \otimes a \in \mathrm{D}(H)$. This can be easily seen from the local $\mathrm{D}(H)$-module structure, the fact that $A(s) = A_h(s,p)$ and $B(p) = B_\phi(s,p)$ leave $|\psi_\Gamma\rangle$ strictly invariant and the properties of the Haar integrals. In other words, *the quantum state* $|\psi_\Gamma\rangle$ *is nothing but a trivial representation of the quantum double* $\mathrm{D}(H)$. Comparing with the comment after Theorem 2 one realizes that $|\psi_\Gamma\rangle$ should be viewed as a spatially distributed version of the integral $\phi \otimes h \in \mathrm{D}(H)$. For these reasons one may call $|\psi_\Gamma\rangle$ the vacuum of the model and as such it has trivial topological charge everywhere.

Let us emphasize again that one really needs just a single datum, i.e. the finite-dimensional Hopf C^*-algebra H, to produce this topological ground state since the Haar integral h (and the Haar functional ϕ) is uniquely defined. In particular, the construction is fully basis-independent.

Furthermore the construction is symmetric[2] in the algebras H and X with their respective integrals h and ϕ, hence it has a natural dual notion. In fact, this foreshadows electric-magnetic duality as shown in Chapter 6.

While we excluded surface boundaries explicitly for Theorem 3 the following example shows what one can learn from the presence of boundaries imposed by the underlying surface.

Example 2. Consider the graph Γ underlying the diagram shown in Figure 5.4 and let $H = \mathbb{C}\mathbb{Z}_2$. Then the state

$$|\psi_0\rangle \coloneqq |\psi_\Gamma(\{h\};\{\phi\};\{h\};\{\phi\})\rangle \tag{5.43}$$

is a codeword of the surface code defined in [9]. It encodes the logical state $|+\rangle$. The other codeword $|\psi_1\rangle$ can be obtained by acting on this Hopf tensor network state with the appropriate string operator connecting the two rough boundaries. This generalizes to any finite-dimensional Hopf C^*-algebra by using the appropriate ribbon operators.

Remark 5. From the proof of Theorem 3 it is clear that any Hopf tensor network state of the form

$$|\psi_\Gamma(h,\ldots,h;f_1,\ldots,f_{|F|})\rangle \tag{5.44}$$

is invariant under all vertex operators $A(s)$. From the perspective of lattice gauge theory this means that deforming the ground state $|\psi_\Gamma\rangle$ of the $\mathrm{D}(H)$-model by changing the functions $\{\phi\} \mapsto \{f_i\}$ only will never break the gauge symmetry. These deformations in the state might therefore well correspond to a local perturbation of the Hamiltonian and thus preserve topological order provided the strength of the perturbation is limited to a certain finite threshold [8, 7]. Some initial work towards this direction has been conducted in [16] which is concerned with tensor network deformations of the toric code. □

5.1.3 Hierarchy

Apart from ground states of the $\mathrm{D}(H)$-models the framework of Hopf tensor network states based on an arbitrary Hopf C^*-algebra H comprises more intriguing examples of quantum many-body states. These can be obtained from certain other choices of elements in H and X which are motivated both by the algebraic structure itself as well as by ideas of charge condensation [3, 4]:

[2] Strictly speaking, symmmetry holds up to a flip in the comultiplication, which is precisely the difference between H^* and X.

Definition 6 (Hierarchy)**.** Let $A \subset H$ and $B \subset X$ Hopf subalgebras with Haar integrals $h_A \in A$ and $\phi_B \in B$ respectively. Set

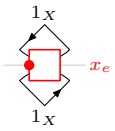
$$|\psi_\Gamma^{A,B}\rangle := |\psi_\Gamma(h_A, \dots, h_A; \phi_B, \dots, \phi_B)\rangle. \tag{5.45}$$

Obviously, with the choice $A = H$ and $B = X$ we recover the state $|\psi_\Gamma\rangle$ which is topologically ordered as a ground state of the $\mathrm{D}(H)$-model. On the other hand, if $B = \{1_X\}$ is the trivial Hopf subalgebra of X then the resulting Hopf tensor network state is a product state for any Hopf subalgebra $A \subset H$. This can be seen from (4.8). Indeed, suppose a face p has r edges in its boundary and $f_p = 1_X$. Then iterating (4.8) yields

$$\sum_{(f_p)} f_p^{(1)} \otimes \cdots \otimes f_p^{(r)} = 1_X \otimes \cdots \otimes 1_X \tag{5.46}$$

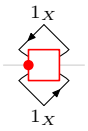
and consequently one has

$$\mathrm{ttr}_\Gamma(\{x_e\}; 1_X, \dots, 1_X) = \prod_{e \in E_M}$$



(5.47)

up to a possible application of the antipode S on each graph edge depending on its orientation. Since $1_X = \epsilon^T(1_{\mathbb{C}})$ is the analogue of (4.3) for the Hopf C^*-algebra X we deduce



$$= \sum_{(x_e)} \langle \epsilon^T(1_{\mathbb{C}}), S(x_e')\rangle \, \langle \epsilon^T(1_{\mathbb{C}}), x_e''\rangle = \sum_{(x_e)} \epsilon\big(S(x_e')\big)\, \epsilon(x_e'') = \epsilon(x_e) \tag{5.48}$$

and therefore

$$\mathrm{ttr}_\Gamma(\{x_e\}; 1_X, \dots, 1_X) = \prod_{e \in E_M} \epsilon(x_e) \tag{5.49}$$

which no longer depends on the orientation of Γ. Now by virtue of (5.32) the quantum state

$$|\psi_\Gamma(\{x_e\}; 1_X, \dots, 1_X)\rangle = \sum_{(x_e)} \bigotimes_{e \in E_M} \epsilon(x_e'')|x_e'\rangle = \bigotimes_{e \in E_M} |x_e\rangle \tag{5.50}$$

is seen to factor into a simple product state.

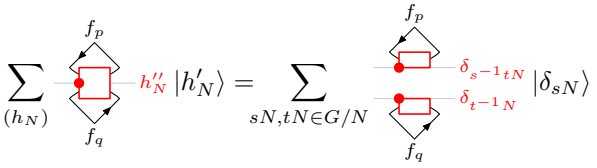
In between these two extremes a hierarchy of quantum states unfolds which are indexed by different choices of A and B. However, depending on the Hopf algebra H in question the interior of this hierarchy may collapse partially. This means that different pairs (A, B) of Hopf subalgebras may actually define identical quantum states. Unfortunately we do not know how to characterize the surviving equivalence classes of states in closed form without additional assumptions on H.

However, if $H = \mathbb{C}G$ we have a clear picture of the above hierarchy. It turns out that the classes of Hopf tensor network states emerging from the partial collapse are isomorphic to ground states of certain quantum double models based on groups smaller than G. Indeed, the relevant Hopf subalgebras in this case exactly read $A = \mathbb{C}K$ and $B = \mathbb{C}^{G/N}$ where $K \subset G$ is a subgroup and $N \lhd G$ a normal subgroup (see Section C.3). For simplicity we abbreviate such a pair of Hopf subalgebras by (K, N). Then it is not difficult to see that both (K, N) and $(K, K \cap N)$ yield identical Hopf tensor network states. Furthermore, if $|k_1, \ldots, k_{|E|}\rangle$ with $k_e \in K$ for each edge $e \in E$ is a basis state then one has

$$\langle k_1, \ldots, k_{|E|} | \psi_\Gamma^{K,K\cap N} \rangle = \langle k_1 l_1, \ldots, k_{|E|} l_{|E|} | \psi_\Gamma^{K,K\cap N} \rangle$$

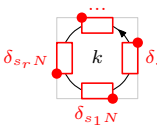
for arbitrary elements $l_e \in K \cap N$, so all amplitudes are actually constant on the cosets $k_e(K \cap N)$. This means that one may apply the canonical projection $\pi\colon K \to K/(K\cap N)$ at each edge and regard $\phi_{K\cap N}$ as the Haar integral of $\mathbb{C}^{K/(K\cap N)}$. Hence the resulting state coincides with the ground state $|\psi_\Gamma\rangle$ of the quantum double model based on the group algebra of the group $K/(K \cap N) \simeq KN/N$.

Also, for $H = \mathbb{C}^G$ we can describe the hierarchy explicitly. Now the Hopf subalgebras are precisely given by $A = \mathbb{C}^{G/N}$ and $B = \mathbb{C}K$ where again $K \subset G$ is a subgroup, $N \lhd G$ a normal subgroup and we abbreviate this pair of Hopf subalgebras by (N, K). Locally, the state $|\psi_\Gamma^{N,K}\rangle$ then looks like



(5.51)

at an edge and for $k \in K$ diagrams at a face evaluate as[3]



$$= \langle k, \delta_{s_1 N} \cdots \delta_{s_r N} \rangle = \prod_{i=2}^{r} \delta_{s_i, s_1} \sum_{n \in N} \delta_{s_1}(kn) \tag{5.52}$$

and vanish unless $s_1 = \cdots = s_r \in KN$. This implies that in (5.51) we can assume $s^{-1}t \in KN$, $t^{-1} \in KN$ and therefore $s \in KN$, in other words, we can regard h_N as the Haar integral of $\mathbb{C}^{KN/N}$ rather than of $\mathbb{C}^{G/N}$ without changing the Hopf tensor network state. Effectively, this corresponds to a projection $\mathbb{C}^{G/N} \to \mathbb{C}^{KN/N}$ at each

[3] Note that we implicitly identify A with the functions on G which are constant on cosets of N. In particular we set $\delta_{sN} = \sum_{n\in N} \delta_{sn}$ for $\{s\}$ a transversal of the cosets.

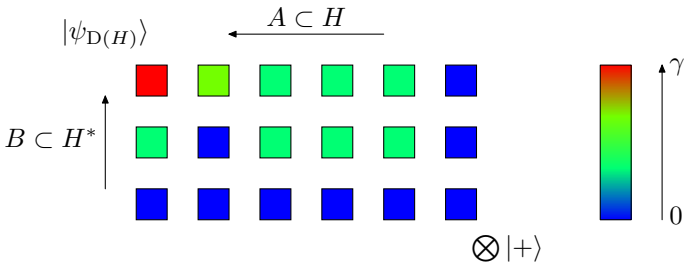
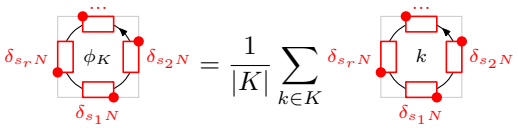
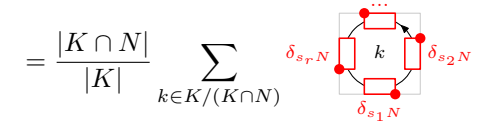


Figure 5.5: Hierarchy of Hopf tensor network states $|\psi_\Gamma^{A,B}\rangle$ for $H = \mathbb{C}S_3$ before the partial collapse. Subalgebras $A = \mathbb{C}K$ and $B = \mathbb{C}^{G/N}$ are ordered by dimension, not necessarily by inclusion. Surviving equivalenve classes of quantum states share the same value of the topological entanglement entropy γ, see Section 5.2 for a detailed discussion.

edge. Also, the value of the diagram (5.52) is constant on cosets $k(K \cap N)$ hence by





(5.53)

and abuse of notation we may replace the Haar integral ϕ_K by $\phi_{K/(K\cap N)}$. This shows that the hierarchy state indexed by (N, K) coincides with the ground state $|\psi_\Gamma\rangle$ of the generalized quantum double model based on the algebra of functions on the group $K/(K \cap N) \simeq KN/N$.

We conclude that the hierarchy of Hopf tensor network states arranges generalized quantum double models based on (functions on) *different* groups in one coherent picture (see Figure 5.5). These groups are precisely isomorphic to KN/N so the equivalence classes of states in the hierarchy are the trivial representations of the quantum doubles $\mathrm{D}\big(\mathbb{C}(KN/N)\big)$ (or $\mathrm{D}(\mathbb{C}^{KN/N})$ respectively). These comprise all possibilities for the residual symmetry algebra after the full $\mathrm{D}(\mathbb{C}G)$-symmetry has been partially broken down to a smaller symmetry algebra [3, 4]. For that reason the quantum states in our hierarchy realize the condensation of topological charges within the *same* underlying Hilbert space.

5.1.4 PEPS

At this stage it is important to make contact with one of the usual formulations of tensor network states. In the PEPS approach [71] a quantum state is represented by choosing some fixed basis and encoding the wavefunction amplitude for each basis element in a fully contracted tensor network. Contrastingly, our Hopf tensor

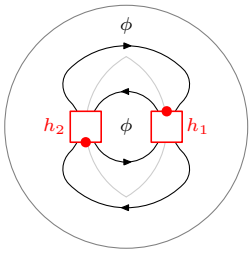


Figure 5.6: Hopf tensor network state $|\psi_\Gamma\rangle$ on a small graph Γ embedded in S^2. The outer circle is identified with the north pole of S^2.

network states are defined without reference to any basis. It is rather the choice of particular, often canonical, elements $\{x_e\} \subset H$ and $\{f_p\} \subset X$ which determines the properties of the quantum state. If the need arises one can still obtain an explicit wavefunction expansion in a straightforward manner. As the next example shows, those distinguished elements naturally generate all relevant sums and amplitudes.

Example 3. Let $H = \mathbb{C}^G$. We would like to explicitly construct the Hopf tensor network state $|\psi_\Gamma\rangle$ for the graph Γ shown in Figure 5.6 and obtain an expansion in terms of the canonical basis $\{\delta_g \mid g \in G\}$ of $\mathbb{C}^G$. We assume the underlying surface to be homeomorphic to S^2 as indicated in Figure 5.6. Using the Haar integral $h_i = \delta_e$ of the dual group algebra $\mathbb{C}^G$ and its corresponding Haar functional $\phi \in H$ we get by Theorem 3

$$\begin{aligned}
|\psi_\Gamma\rangle &= \sum_{(h_i)} \phi(h_1'''h_2''')\,\phi\big(S(h_1'')\,S(h_2'')\big)|h_1'\rangle \otimes |h_2'\rangle \\
&= \sum_{(h_i)} \langle \phi \otimes \phi, \Delta(h_1''h_2'')\rangle |h_1'\rangle \otimes |h_2'\rangle \\
&= \sum_{(h_i)} \phi(h_1''h_2'')|h_1'\rangle \otimes |h_2'\rangle \\
&= \sum_{u,v\in G} \phi(\delta_{u^{-1}}\delta_{v^{-1}})|\delta_u\rangle \otimes |\delta_v\rangle \\
&= \frac{1}{|G|}\sum_{g\in G} |\delta_g\rangle \otimes |\delta_g\rangle.
\end{aligned}$$

Interestingly, while the Haar integral δ_e merely represents a single element of the basis, it is actually the act of comultiplication which produces the sum over the entire basis from δ_e. □

As outlined in the introduction PEPS are defined in terms of local tensors with *open* virtual indices, for a particular example see (5.1). Contrastingly, our diagrammatic notation as given by (5.24) uses the comultiplication of the Hopf

algebra to disconnect and separate local objects on the virtual level, hence it appears there are *no* virtual indices at all in our formalism.

This is not entirely true. If the functions $\{f_p\}$ for evaluating virtual loops belong to a particular class[4], then indeed one may regard the evaluation of a virtual loop as tracing over a product of certain matrices, once a particular basis of H has been chosen. Consequently, each such matrix naturally constitutes part of a local tensor with open virtual indices and evaluating the loop corresponds to a cyclic contraction of those indices. Note that this applies to all Hopf tensor network states of our hierarchy (but is not limited to these). For example, for the state $|\psi_\Gamma\rangle$ at the top of the hierarchy one has the local projector

$$P = \|h\|^{-1} \sum_{(h)} \sum_{\alpha,\beta,\gamma,\delta\in\mathcal{B}} \left(L_+^{S(h'')}\right)_{\alpha\beta} \left(L_+^{h'''}\right)_{\gamma\delta} |h'\rangle\langle\alpha,\beta,\gamma,\delta| \tag{5.54}$$

where $(L_+^a)_{\alpha\beta}$ denotes a matrix element of the action (4.38) with respect to some basis $\mathcal{B}$.[5] Note that in this setting the local matrices encode both information about the spin state on the graph edge as well as about the function used to evaluate the virtual loop.

Furthermore, if a Hopf tensor trace admits a representation in terms of local tensors with open indices properties like *injectivity* may be studied. More precisely, a PEPS is called *injective* [61] if there is a partition of the underlying graph in disjoint regions R_i such that for every region R_i the linear map from open virtual indices at the boundary to physical indices in the interior is injective. Whether or not a particular tensor network representation of a quantum state is injective has important consequences for e.g. the existence of a parent Hamiltonian that has the given PEPS as its unique ground state [61]. One can show that all Hopf tensor network states $|\psi_\Gamma^{A,B}\rangle$ in the hierarchy are *not* injective for any finite-dimensional Hopf C^*-algebra H. However, (at least) the state $|\psi_\Gamma\rangle$ at the top of the hierarchy obeys a relaxed version of injectivity which one may call *H-injectivity*. This is a certain generalization of the G-injectivity condition defined in [67].

5.2 Calculating the topological entanglement entropy

Topological order is commonly associated with *non-local* order parameters. It is believed that among these resides the topological entanglement entropy γ [30, 41, 47] which is a universal additive correction to the area law for the entanglement entropy of a bipartition of the system into a region R and its complement. Given a ground state of a topologically ordered system, the entanglement entropy S_R for a

[4] This is the so-called character ring $R_{\mathbb{C}}(H) = \sum_{i=1}^n \mathbb{C}\chi_i$ where the χ_i are the irreducible characters of H.

[5] Furthermore one may express the (physical) ket in the same basis, too, and appreciate the similarities and differences as compared to the trivial case (5.7).

region R is argued to scale as

$$S_R = \alpha \cdot |\partial R| - \gamma \tag{5.55}$$

in the limit of infinitely large regions. Here α is a parameter which encodes non-universal behaviour on short length scales. In fact, the topological entanglement entropy γ also contains partial information about the types of quasiparticle excitations that may occur in the low-energy sector of the system.

For these reasons we would like to show in this section how the non-local order parameter γ can be understood naturally in the context of Hopf tensor network states. In particular, we are going to show that different classes of finite-dimensional Hopf C^*-algebras yield fundamentally different mechanisms for the emergence of a non-vanishing topological entanglement entropy.

To this end, we will exploit the generic decomposition of Hopf tensor network states into interior and exterior parts as stated in Proposition 4. We will then compute the block entropy directly from certain properties of the reduced density operator. In order to render its simple structure evident we will make use of isometries and completely clear out the interior of both the region R and its complement $\bar{R}$. Effectively, we concentrate all topological information contained in the ground state into the (internal) boundaries of the system. Furthermore, the distillation process will be crafted such that all intermediate quantum states can be kept track of conveniently via Hopf tensor traces.

It should be noted that our scheme of applying isometries implements entanglement renormalisation [74, 73] both for the $\mathrm{D}(H)$-models as well as for states of the hierarchy. In fact, it directly extends the work in [1] which is concerned with states at the top of the hierarchy for $H = \mathbb{C}G$. At the same time our scheme provides a complementary view on entanglement renormalisation for string-net models [45], namely from the perspective of a local symmetry algebra.

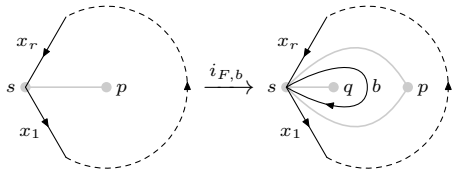
5.2.1 Isometries

We begin by developing the distillation process. For that we interpret the process of inserting the spikes and loops of Section 4.2.1 into a given graph as "little" isometries. Certain such isometries commute with the entire local $\mathrm{D}(H)$-action and thus leave all sectors of quasiparticle excitations invariant. Subsequently we introduce general unitary maps which allow for reconnecting edges. Together with the little isometries they give rise to isometries which split faces and vertices by inserting edges appropriately. A particular subset of those forms a hierarchy of isometries which is a perfect match for our hierarchy of Hopf tensor network states as we will see in the next section.

As far as notation is concerned we will always denote the original graph by Γ_1 and the modified one by Γ_2. Whenever it is appropriate to talk about actual quantum double models, $\mathcal{H}_i$ will denote the Hamiltonian of the $\mathrm{D}(H)$-model on the graph Γ_i.

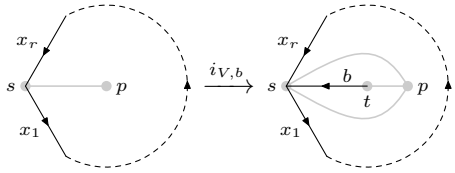
Proposition 5. *Let $b \in H$ such that $\|b\| = 1$.*

Then the maps $i_{F,b}$ and $i_{V,b}$ defined via



(5.56)

and

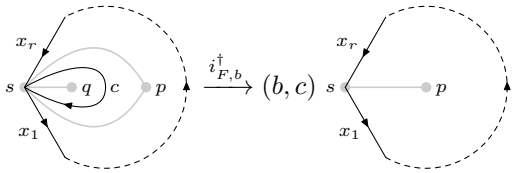


(5.57)

are isometries.

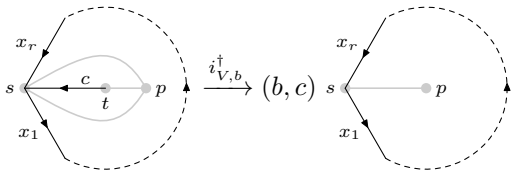
Proof. Immediate. □

For the sake of completeness we also provide the adjoint maps which are given by



(5.58)

and



(5.59)

Proposition 6 (Little isometries)**.** *Let $A \subset H$ and $B \subset X$ Hopf subalgebras with the respective Haar integrals $h_A \in A$ and $\phi_B \in B$. Additionally, let*

$$\lambda_{A,B} \coloneqq \sum_{(h_A)} \phi_B(h_A'')\, h_A', \tag{5.60}$$

$$1_{A,B} \coloneqq \frac{\lambda_{A,B}}{\|\lambda_{A,B}\|}, \tag{5.61}$$

$$\Lambda_A \coloneqq \frac{h_A}{\|h_A\|} = \frac{1}{\sqrt{\phi(h_A)}}\, h_A. \tag{5.62}$$

Then the linear maps

$$i_F^{A,B} \coloneqq i_{F,1_{A,B}}, \tag{5.63}$$

$$i_V^{A} \coloneqq i_{V,\Lambda_A} \tag{5.64}$$

are isometries.

Furthermore, the maps $i_F \coloneqq i_F^{H,X}$ *and* $i_V \coloneqq i_V^{H}$ *satisfy*

$$i_F A_a(s,p) = A_a(s,p)\, i_F, \tag{5.65}$$

$$i_F B_f(s,p) = B_f(s,p)\, i_F, \tag{5.66}$$

$$i_V A_a(s,p) = A_a(s,p)\, i_V, \tag{5.67}$$

$$i_V B_f(s,p) = B_f(s,p)\, i_V \tag{5.68}$$

for all $a \in H$, $f \in X$ *and sites* (s,p) *with* $s \neq t$ *and* $p \neq q$.

Proof. For the first claim, it is enough to show the invariance of the inner product or equivalently that $\|1_{A,B}\| = \|\Lambda_A\| = 1$. Indeed, from

$$\|h_A\|^2 = (h_A, h_A) = \phi(h_A^* h_A) = \phi(h_A^2) = \phi(h_A).$$

this is easily seen to be true.

As far as the second claim is concerned it is clear we only need to study sites which contain the face p as indicated in (5.56) and (5.57). Since $1_{H,X} = 1_H$ is invariant under the adjoint H-action (4.29) Proposition 1 implies (5.65). Since 1_H is also invariant under the X-action (4.41) the commutation relation (5.66) follows directly from Proposition 2. Furthermore $\Lambda_H \propto h$ is an integral hence (5.67) follows from Proposition 1. Since Λ_H is also invariant under the X-action (4.67) by

$$\sum_{(h)} f\big(h''' S(h')\big)\, h'' = \sum_{(h)} f\big(h' S(h'')\big)\, h''' = \sum_{(h)} f\big(\epsilon(h')\, 1_H\big)\, h'' = f(1_H)\, h$$

the commutation relation (5.68) follows from Proposition 2. □

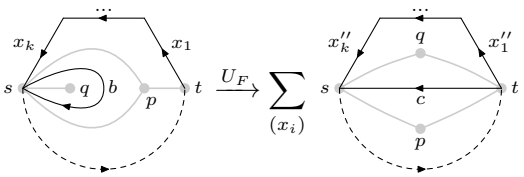
Corollary 2 (Isometric intertwiners)**.** *The isometries* i_F *and* i_V *satisfy*

$$i_F \mathcal{H}_1 = \mathcal{H}_2 i_F, \tag{5.69}$$

$$i_V \mathcal{H}_1 = \mathcal{H}_2 i_V. \tag{5.70}$$

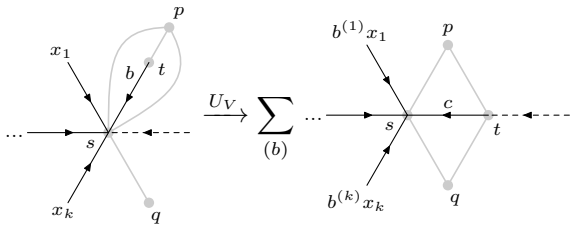
Next we define the unitary maps that allow for reconnecting edges of the underlying graph.

Lemma 4 (Unitaries)**.** *The linear maps* U_F *and* U_V *defined via*



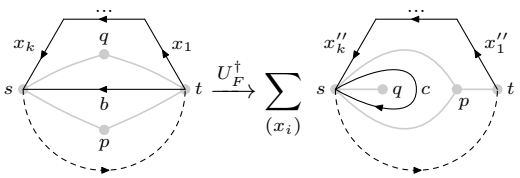
$$c = x_k' \cdots x_1' b \tag{5.71}$$

and



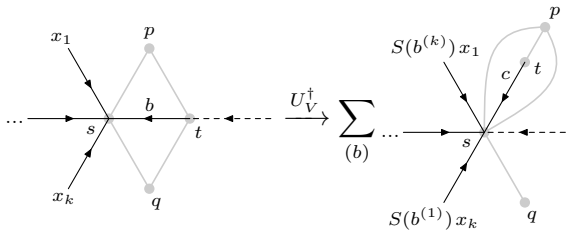
$$c = b^{(k+1)} \tag{5.72}$$

are unitary. Their inverses read



$$c = S(x'_k \cdots x'_1)\, b \tag{5.73}$$

and



$$c = b^{(k+1)} \tag{5.74}$$

respectively.

Before we turn to the proof let us briefly describe the meaning of the above. By reconnecting the loop edge the unitary U_F trades the old site $(s,p)^+$ for the new one (t,q). Similarly, U_V trades $(s,p)^-$ for (t,q). Interestingly, the reconnecting patterns given by U_F and U_V are precisely dual to each other!

Proof. It is easy to check that $U_F^\dagger$ ($U_V^\dagger$) as defined above is both a left and a right inverse of the map U_F (U_V). Indeed:

$$\begin{aligned} U_F^\dagger U_F(x_1 \otimes \cdots \otimes x_k \otimes b) &= U_F^\dagger \Bigg(\sum_{(x_i)} x''_1 \otimes \cdots \otimes x''_k \otimes x'_k \cdots x'_1 b \Bigg) \\ &= \sum_{(x_i)} x'''_1 \otimes \cdots \otimes x'''_k \otimes S(x''_k \cdots x''_1)\, x'_k \cdots x'_1 b \\ &= \sum_{(x_i)} x'''_1 \otimes \cdots \otimes x'''_k \otimes S(x''_1) \cdots S(x''_k)\, x'_k \cdots x'_1 b \end{aligned}$$

$$
\begin{aligned}
&= \sum_{(x_i)} x_1'' \otimes \cdots \otimes x_k'' \otimes \epsilon(x_1') \cdots \epsilon(x_k')\, b \\
&= x_1 \otimes \cdots \otimes x_k \otimes b.
\end{aligned}
$$

Similarly, one shows $U_F U_F^\dagger = \mathrm{id}$. Now

$$
\begin{aligned}
&U_V^\dagger U_V (x_1 \otimes \cdots \otimes x_k \otimes b) \\
&\quad = U_V^\dagger \sum_{(b)} b^{(1)} x_1 \otimes \cdots \otimes b^{(k)} x_k \otimes b^{(k+1)} \\
&\quad = \sum_{(b)} \sum_{(b^{(k+1)})} S\big((b^{(k+1)})^{(k)}\big)\, b^{(1)} x_1 \otimes \cdots \otimes S\big((b^{(k+1)})^{(1)}\big)\, b^{(k)} x_k \otimes (b^{(k+1)})^{(k+1)} \\
&\quad = \sum_{(b)} S(b^{(2k)})\, b^{(1)} x_1 \otimes \cdots \otimes S(b^{(k+1)})\, b^{(k)} x_k \otimes b^{(2k+1)} \\
&\quad = \sum_{(b)} S(b^{(2k-2)})\, b^{(1)} x_1 \otimes \cdots \otimes S(b^{(k)})\, b^{(k-1)} x_{k-1} \otimes x_k \otimes b^{(2k-1)} \\
&\quad = \sum_{(b)} S(b'')\, b' x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes b''' \\
&\quad = x_1 \otimes \cdots \otimes x_k \otimes b.
\end{aligned}
$$

and by the same token one proves $U_V U_V^\dagger = \mathrm{id}$.

It remains to show that for all $x_i, y_i, b, c \in H$ the inner product (4.80) is invariant under U_F (U_V):

$$
\begin{aligned}
&\big(U_F(x_1 \otimes \cdots \otimes x_k \otimes b), U_F(y_1 \otimes \cdots \otimes y_k \otimes c)\big) \\
&\quad = \Big(\sum_{(x_i)} x_1'' \otimes \cdots \otimes x_k'' \otimes x_k' \cdots x_1' b, \sum_{(y_i)} y_1'' \otimes \cdots \otimes y_k'' \otimes y_k' \cdots y_1' c\Big) \\
&\quad = \sum_{(x_i)(y_i)} (x_k' \cdots x_1' b, y_k' \cdots y_1' c) \prod_{j=1}^{k} (x_j'', y_j'') \\
&\quad = \sum_{(x_i)(y_i)} \phi\big(b^* (x_1^*)' \cdots (x_k^*)' y_k' \cdots y_1' c\big)\, \phi\big((x_k^*)'' y_k''\big) \prod_{j=1}^{k-1} \phi\big((x_j^*)'' y_j''\big) \\
&\quad = (x_k, y_k) \sum_{(x_i)(y_i)} \phi\big(b^* (x_1^*)' \cdots (x_{k-1}^*)' y_{k-1}' \cdots y_1' c\big) \prod_{j=1}^{k-1} \phi\big((x_j^*)'' y_j''\big) \\
&\quad = (b, c) \prod_{j=1}^{k} (x_j, y_j) \\
&\quad = (x_1 \otimes \cdots \otimes x_k \otimes b, y_1 \otimes \cdots \otimes y_k \otimes c).
\end{aligned}
$$

Note that we used property (4.23) of the Haar integral ϕ in the fifth line. Furthermore

we have

$$
\begin{aligned}
&\big(U_V(x_1 \otimes \cdots \otimes x_k \otimes b), U_V(y_1 \otimes \cdots \otimes y_k \otimes c)\big) \\
&\quad = \Big(\sum_{(b)} b^{(1)} x_1 \otimes \cdots \otimes b^{(k)} x_k \otimes b^{(k+1)}, \sum_{(c)} c^{(1)} y_1 \otimes \cdots \otimes c^{(k)} y_k \otimes c^{(k+1)}\Big) \\
&\quad = \sum_{(b)(c)} (b^{(k+1)}, c^{(k+1)}) \prod_{j=1}^{k} (b^{(j)} x_j, c^{(j)} y_j) \\
&\quad = \sum_{(b)(c)} \phi\big((b^{(k+1)})^* c^{(k+1)}\big) \prod_{j=1}^{k} \phi\big(x_j^* (b^{(j)})^* c^{(j)} y_j\big) \\
&\quad = \sum_{(b^*c)} \phi\big((b^*c)^{(k+1)}\big) \prod_{j=1}^{k} \phi\big(x_j^* (b^*c)^{(j)} y_j\big) \\
&\quad = (x_k, y_k) \sum_{(b^*c)} \phi\big((b^*c)^{(k)}\big) \prod_{j=1}^{k-1} \phi\big(x_j^* (b^*c)^{(j)} y_j\big) \\
&\quad = (b, c) \prod_{j=1}^{k} (x_j, y_j) \\
&\quad = (x_1 \otimes \cdots \otimes x_k \otimes b, y_1 \otimes \cdots \otimes y_k \otimes c).
\end{aligned}
$$

This concludes the proof. □

Proposition 7 (Unitary intertwiners)**.** *The unitaries U_F and U_V from Lemma 4 satisfy*

$$U_F \mathcal{H}_1 = \mathcal{H}_2 U_F, \tag{5.75}$$
$$U_V \mathcal{H}_1 = \mathcal{H}_2 U_V. \tag{5.76}$$

Proof. We will only verify the claim for U_F and leave the proof of the intertwining property of U_V to the reader. We need to show that U_F commutes appropriately with the terms $A(s)$ and $B(p)$ in the Hamiltonians $\mathcal{H}_i$.

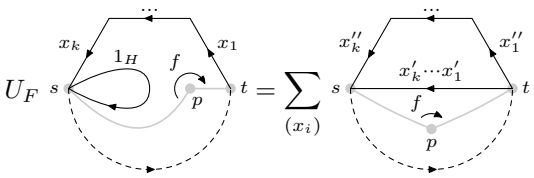
Starting with the face operators we see that this can be done directly for the face q in (5.71):

$$
\begin{aligned}
&U_F B(q)(x_1 \otimes \cdots \otimes x_k \otimes b) \\
&\quad = U_F\Big(x_1 \otimes \cdots \otimes x_k \otimes \sum_{(b)} \phi(b')\, b''\Big) \\
&\quad = \sum_{(x_i)(b)} x_1'' \otimes \cdots \otimes x_k'' \otimes x_k' \cdots x_1' b' \phi(b'') \\
&\quad = \sum_{(x_i)(b)} \phi\big(S(b'')\, S(x_1^{(2)}) \cdots S(x_k^{(2)})\, x_k^{(3)} \cdots x_1^{(3)}\big)\, x_1^{(4)} \otimes \cdots \otimes x_k^{(4)} \otimes x_k^{(1)} \cdots x_1^{(1)} b'
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{(x_i)(b)} \phi\big(S(x_k^{(2)} \cdots x_1^{(2)} b'')\, x_k^{(3)} \cdots x_1^{(3)}\big)\, x_1^{(4)} \otimes \cdots \otimes x_k^{(4)} \otimes x_k^{(1)} \cdots x_1^{(1)} b' \\
&= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes S) \sum_{(x_i)(b)} \sum_{(S(x_k' \cdots x_1' b))} \phi\big(S(x_k' \cdots x_1' b)' x_k'' \cdots x_1''\big)\, x_1''' \otimes \cdots \otimes x_k''' \otimes S(x_k' \cdots x_1' b)'' \\
&= B(q)(x_1'' \otimes \cdots \otimes x_k'' \otimes x_k' \cdots x_1' b) \\
&= B(q)\, U_F(x_1 \otimes \cdots \otimes x_k \otimes b).
\end{aligned}
$$

Note that in the fourth line we repeatedly used the definition of the antipode. Since $B(q)$ projects b onto the value $\phi(b)\, 1_H$ we may assume in the following that b is replaced by 1_H.

In order to verify the intertwining property for the face p it is then enough to show



for any $f \in X$, that is,

$$
\begin{aligned}
U_F\Big(\sum_{(f)} T_-^{f^{(k+1)}}(x_1) \otimes \cdots \otimes T_-^{f^{(2)}}(x_k) \otimes T_-^{f^{(1)}}(1_H)\Big) \\
= (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes T_-^f)\, U_F(x_1 \otimes \cdots \otimes x_k \otimes 1_H).
\end{aligned}
$$

This should be compared with (4.51). Indeed, we have:

$$
\begin{aligned}
&U_F\Big(\sum_{(f)} T_-^{f^{(k+1)}}(x_1) \otimes \cdots \otimes T_-^{f^{(2)}}(x_k) \otimes T_-^{f^{(1)}}(1_H)\Big) \\
&\quad = U_F\Big(\sum_{(x_i)} f\big(S(x_k' \cdots x_1')\big)\, x_1'' \otimes \cdots \otimes x_k'' \otimes 1_H\Big) \\
&\quad = \sum_{(x_i)} f\big(S(x_k' \cdots x_1')\big)\, x_1''' \otimes \cdots \otimes x_k''' \otimes x_k'' \cdots x_1'' \\
&\quad = \sum_{(x_i)} x_1'' \otimes \cdots \otimes x_k'' \otimes T_-^f(x_k' \cdots x_1') \\
&\quad = (\mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes T_-^f)\, U_F(x_1 \otimes \cdots \otimes x_k \otimes 1_H).
\end{aligned}
$$

For all other faces having a boundary edge on which U_F acts nontrivially one can similarly show that the corresponding operator T_+^f commutes with U_F for any $f \in X$.

Turning to the vertex operators we see that for the vertex s in (5.71) it suffices

to show

$$U_F\Big(\sum_{(a)} x_1 \otimes \cdots \otimes x_{k-1} \otimes L_+^{a''}(x_k) \otimes \mathrm{ad}(a')(1_H)\Big) = \Big(\sum_{(a)} \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes L_+^{a''} \otimes L_+^{a'}\Big) U_F(x_1 \otimes \cdots \otimes x_{k-1} \otimes x_k \otimes 1_H)$$

for any $a \in H$. This should be compared with (4.60). Indeed, we now have

$$\begin{aligned} &U_F\Big(\sum_{(a)} x_1 \otimes \cdots \otimes x_{k-1} \otimes L_+^{a''}(x_k) \otimes \mathrm{ad}(a')(1_H)\Big) \\ &\quad = U_F(x_1 \otimes \cdots \otimes x_{k-1} \otimes a x_k \otimes 1_H) \\ &\quad = \sum_{(x_i)(a)} x_1'' \otimes \cdots \otimes x_{k-1}'' \otimes a'' x_k'' \otimes a' x_k' \cdots x_1' \\ &\quad = \Big(\sum_{(a)} \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes L_+^{a''} \otimes L_+^{a'}\Big) U_F(x_1 \otimes \cdots \otimes x_{k-1} \otimes x_k \otimes 1_H). \end{aligned}$$

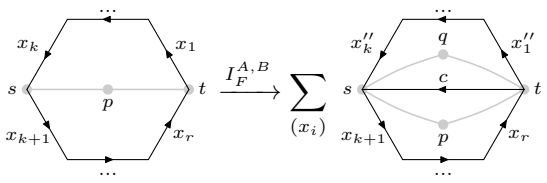
For the vertex t and any $a \in H$ the following holds by a similar argument:

$$U_F\big(L_-^a(x_1) \otimes x_2 \otimes \cdots \otimes x_k \otimes 1_H\big) = \Big(\sum_{(a)} L_-^{a'} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes L_-^{a''}\Big) U_F(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes 1_H).$$

Finally, for the remaining vertices in (5.71) whose edges are affected by U_F one can show that $\sum_{(a)} L_+^{a'} \otimes L_-^{a''}$ (which acts on $x_i \otimes x_{i+1}$ only) and U_F commute. □

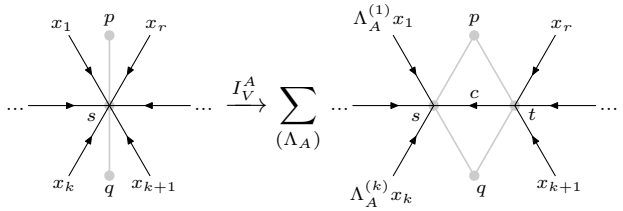
Remark 6. The unitaries U_F and U_V do *not* commute with the full local $\mathrm{D}(H)$-action, only with the Hamiltonians. For example, the minimal face q does not decouple from arbitrary $A_a(s,p)$ operators in the sense of Proposition 1 for most values b hence it will carry a nontrivial topological charge. Its electric component will be associated with the vertex s the loop is attached to. By reconnecting the loop edge the unitary U_F then splits this electric component between the vertices s and t so naturally $A_a(s,p)$ cannot be unaffected. □

Corollary 3 (Isometries)**.** *The linear maps* $I_F^{A,B} := U_F \circ i_F^{A,B}$,



$$c = x_k' \cdots x_1' 1_{A,B} \qquad (5.77)$$

and $I_V^A \coloneqq U_V \circ i_V^A$,



(5.78)

are isometries. Furthermore, for $I_F \coloneqq I_F^{H,H^*}$ *and* $I_V \coloneqq I_V^H$ *one has*

$$I_F \mathcal{H}_1 = \mathcal{H}_2 I_F, \tag{5.79}$$

$$I_V \mathcal{H}_1 = \mathcal{H}_2 I_V. \tag{5.80}$$

Put differently, the collection of these maps allows to move between the D(H)-models defined on two arbitrary graphs which are embedded in the *same* surface. Furthermore, all maps are local, i.e. they cannot change nonlocal topological quantum numbers.

5.2.2 Transforming hierarchy states

In the preceding section we have learned that the set $\{I_F, I_V\}$ of isometries maps the ground state subspaces of generalized quantum double models on related graphs onto each other. We would like to prove that this not only holds for the entire subspace but also for individual ground states. Namely, the isometries $\{I_F, I_V\}$ precisely identify the Hopf tensor network states $|\psi_{\Gamma_i}\rangle$ as given by Theorem 3 for the different graphs Γ_i.

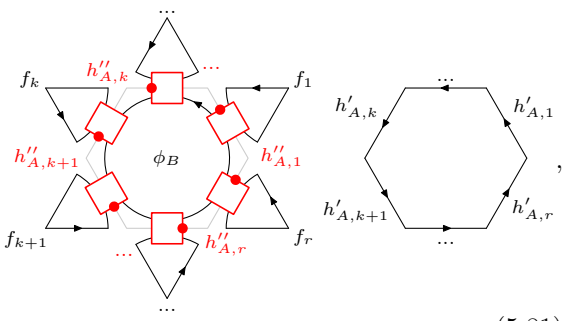
Keeping our focus on states we will furthermore show (for a special case) that the hierarchy of isometries $\{I_F^{A,B}, I_V^A\}$ has the very same effect on the hierarchy of states $|\psi_{\Gamma_i}^{A,B}\rangle$ as given by Definition 6. In other words, identifying these states in an isometric fashion is as simple as adding or removing edges from the underlying graph and adjusting the tensor trace canonically.

Once we act with the isometries on our hierarchy of Hopf tensor network states it will become clear that the quantum states defined in Section 5.1.2 are not properly normalized relative to each other. We can establish correct relative normalization by choosing integrals of unit norm instead of the Haar integrals h_A and ϕ_B. In other words, we need to include a factor of $\|h_A\|^{-1}$ for each edge and a factor of $\|\phi_B\|^{-1}$ for each (proper) face of a Hopf tensor network state. It is understood that from now on all Hopf tensor network states will be normalized in this fashion without change of notation unless otherwise stated.

Starting from (5.34) we now fix some notation so that we can easily refer to pieces of Hopf tensor network states later on.

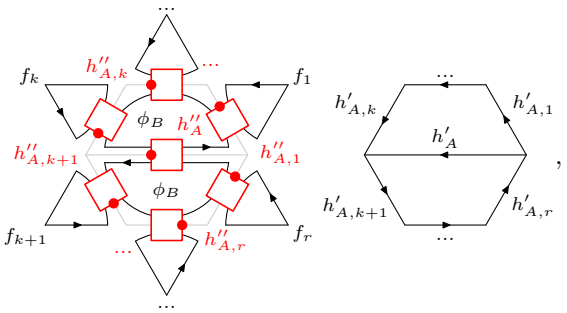
Definition 7. Let $A \subset H$, $B \subset X$ Hopf subalgebras, $h_A \in A$, $\phi_B \in B$ their respective Haar integrals and let $f_1, \ldots, f_r \in B$. Set

$$|\psi_p^{A,B}(\{f_i\})\rangle \coloneqq \frac{1}{\|h_A\|^r \, \|\phi_B\|} \sum_{(h_{A,i})}$$



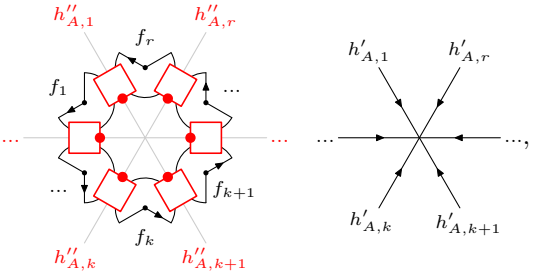
(5.81)

$$|\psi_{p\cup q}^{A,B}(\{f_i\})\rangle \coloneqq \frac{1}{\|h_A\|^{r+1}\|\phi_B\|^2} \sum_{(h_{A,i})(h_A)}$$



(5.82)

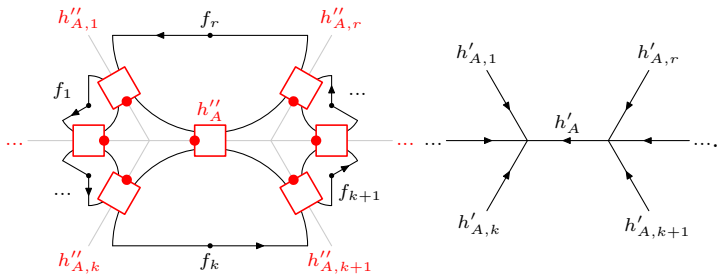
$$|\psi_s^{A,B}(\{f_i\})\rangle \coloneqq \frac{1}{\|h_A\|^r} \sum_{(h_{A,i})}$$



(5.83)

and

$$|\psi_{s\cup t}^{A,B}(\{f_i\})\rangle \coloneqq \frac{1}{\|h_A\|^{r+1}} \sum_{(h_{A,i})(h_A)}$$



(5.84)

As before we simply write $|\psi_R(f_1,\ldots,f_r)\rangle$ whenever $A=H$ and $B=X$. Note that in this case one deals with a fragment of a quantum double model ground state!

Proposition 8.

$$I_F|\psi_p(f_1,\ldots,f_r)\rangle = |\psi_{p\cup q}(f_1,\ldots,f_r)\rangle, \tag{5.85}$$

$$I_V^A|\psi_s^{A,B}(f_1,\ldots,f_r)\rangle = |\psi_{s\cup t}^{A,B}(f_1,\ldots,f_r)\rangle. \tag{5.86}$$

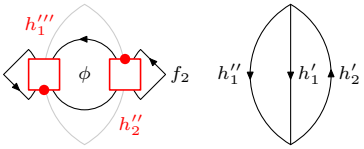
Proof. First let us prove the relation involving the face isometry. It is enough to consider a face which is bounded by two edges. In this case we first transform the right hand side of the equation in order to eliminate one of the virtual loops:

$$\begin{aligned}
&|\psi_{p\cup q}(f_1,f_2)\rangle \\
&\quad = |H|^{\frac{5}{2}} \sum_{(h_i)(h)} \phi(h_1'''h''')\,\phi\big(h_2'''\,S(h'')\big) \prod_{j=1}^{2} f_j\big(S(h_j'')\big)|h_1'\rangle\otimes|h'\rangle\otimes|h_2'\rangle \\
&\quad = |H|^{\frac{3}{2}} \sum_{(h_i)} \phi(h_2'''h_1^{(3)})\, f_1\big(S(h_1^{(2)})\big)\, f_2\big(S(h_2'')\big)|h_1^{(1)}\rangle\otimes|S(h_1^{(4)})\rangle\otimes|h_2'\rangle \\
&\quad = |H|^{\frac{3}{2}} \sum_{(h_i)} \phi(h_2'''h_1^{(4)})\, f_1\big(S(h_1^{(3)})\big)\, f_2\big(S(h_2'')\big)|h_1^{(2)}\rangle\otimes|S(h_1^{(1)})\rangle\otimes|h_2'\rangle
\end{aligned}$$

where in the second line we employed Lemma 7.

In diagrammatic notation the above amounts to

$$|\psi_{p\cup q}(f_1,f_2)\rangle = |H|^{\frac{3}{2}} \sum_{(h_i)}$$



which proves the claim about the face isometry.

As for the vertex isometries we can in fact assume $A=H$ without loss of generality. By the same token we will never make use of $f_i\in B$ in the following argument, hence it suffices to assume $B=X$, too. We begin with the state on the original graph:

$$|\psi_s(f_1,\ldots,f_r)\rangle = \|h\|^{-r} \sum_{(h_i)} \prod_{j=1}^{r} f_j\big(S(h_j'')\,h_{j+1}'''\big)|h_1'\rangle\otimes\cdots\otimes|h_r'\rangle.$$

Applying I_V yields:

$$\begin{aligned}
&\|h\|^{r+1}\, I_V|\psi_s(f_1,\ldots,f_r)\rangle \\
&\quad = \sum_{(h)(h_i)} \prod_{j=1}^{r} f_j\big(S(h_j'')\,h_{j+1}'''\big) \bigotimes_{l=1}^{k} |h^{(l)}h_l'\rangle\otimes|h^{(k+1)}\rangle\otimes|h_{k+1}'\rangle\otimes\cdots\otimes|h_r'\rangle \\
&\quad = \sum_{(h_i)(h)} f_r\big(S(h_r'')\,h_1'''\big) \prod_{j=1}^{k} f_j\big(S(h_j'')\,h_{j+1}'''\big) \bigotimes_{l=1}^{k} |h^{(l)}h_l'\rangle\otimes|h^{(k+1)}\rangle\otimes\cdots
\end{aligned}$$

where in the last line we have abbreviated the remaining tensor factor

$$\prod_{j=k+1}^{r-1} f_j\big(S(h_j'')\,h_{j+1}'''\big) \bigotimes_{l=k+1}^{r} |h_l'\rangle$$

by the trailing ellipsis. In order to simplify notation we will continue to do so in the following. Hence:

$$\begin{aligned}
&\|h\|^{r+1}\, I_V|\psi_s(f_1,\dots,f_r)\rangle \\
&\quad= \sum_{(h_i)(h)} f_r\big(S(h_r'')\,h_1'''\big)\, f_1\big(S(h_1'')\,h_2'''\big) \prod_{j=2}^{k} f_j\big(S(h_j'')\,h_{j+1}'''\big) |h^{(1)}h_1'\rangle \otimes\cdots\otimes |h^{(k)}h_k'\rangle \otimes |h^{(k+1)}\rangle\otimes\cdots \\
&\quad= \sum_{(h_i)(h)} f_r\big(S(h_r'')\,S(h^{(1)})\,h_1'''\big)\, f_1\big(S(h_1'')\,h^{(2)}h_2'''\big)\, f_2\big(S(h_2'')\,h_3'''\big) \prod_{j=3}^{k} f_j\big(S(h_j'')\,h_{j+1}'''\big) \\
&\qquad |h_1'\rangle \otimes |h^{(3)}h_2'\rangle \otimes\cdots\otimes |h^{(k+1)}h_k'\rangle \otimes |h^{(k+2)}\rangle\otimes\cdots \\
&\quad= \sum_{(h_i)(h)} f_r\big(S(h_r'')\,S(h^{(1)})\,h_1'''\big)\, f_1\big(S(h_1'')\,h_2'''\big)\, f_2\big(S(h_2'')\,h^{(2)}h_3'''\big) \prod_{j=3}^{k} f_j\big(S(h_j'')\,h_{j+1}'''\big) \\
&\qquad |h_1'\rangle \otimes |h_2'\rangle \otimes |h^{(3)}h_3'\rangle \otimes\cdots\otimes |h^{(k)}h_k'\rangle \otimes |h^{(k+1)}\rangle\otimes\cdots \\
&\quad= \sum_{(h_i)(h)} f_r\big(S(h_r'')\,S(h^{(1)})\,h_1'''\big)\, f_k\big(S(h_k'')\,h^{(2)}h_{k+1}'''\big) \prod_{j=1}^{k-1} f_j\big(S(h_j'')\,h_{j+1}'''\big) |h_1'\rangle \otimes\cdots\otimes |h_k'\rangle \otimes |h^{(3)}\rangle\otimes\cdots \\
&\quad= \sum_{(h_i)(h)} f_k\big(S(h_k'')\,h'''h_{k+1}'''\big) f_r\big(S(h_r'')\,S(h'')\,h_1'''\big) \prod_{j\neq k,r} f_j\big(S(h_j'')\,h_{j+1}'''\big) |h_1'\rangle \otimes\cdots\otimes |h_k'\rangle \otimes |h'\rangle \otimes |h_{k+1}'\rangle\otimes\cdots\otimes |h_r'\rangle \\
&\quad= \|h\|^{r+1} |\psi_{s\cup t}(f_1,\dots,f_r)\rangle.
\end{aligned}$$

Note that from the third line on we repeatedly applied Lemma 6. □

Unfortunately, we do not know how *arbitrary* hierarchy states $|\psi_\Gamma^{A,B}\rangle$ are affected by the associated face isometries $I_F^{A,B}$ for a general finite-dimensional Hopf C^*-algebra. However, for $H = \mathbb{C}G$ we can state the following and leave the proof to the reader.

Lemma 5. *Let $K \subset G$ a subgroup and $N \lhd G$ a normal subgroup. Furthermore let $f_i \in \mathbb{C}^{G/N}$. Then*

$$I_F^{K,N}|\psi_p^{K,N}(f_1,\dots,f_r)\rangle = \sqrt{\frac{|KN|}{|G|}}\,|\psi_{p\cup q}^{K,N}(f_1,\dots,f_r)\rangle. \tag{5.87}$$

The above lemma shows that unless one draws the quantum state from the top of the hierarchy one needs to include an additional factor of $\sqrt{|KN|/|G|}$ for each face to insure proper *relative* normalization of the Hopf tensor network states. In order

to fix the normalization *absolutely* we may calculate the norm of a single (preferably small) Hopf tensor network state on a given surface. Since all other states on the same surface can be reached from this initial one via isometries their norm will be determined automatically. It turns out that the absolute normalization factor is entirely a property of the surface the Hopf tensor network state is embedded in. For example, absorbing the additional factor per face into the definition of the Hopf tensor network states one has for arbitrary graphs Γ on S^2

$$\| |\psi_\Gamma^{K,N}\rangle \| = \sqrt{|KN/N|} \tag{5.88}$$

while on T^2 one has

$$\| |\psi_\Gamma^{K,N}\rangle \| = \frac{1}{\sqrt{|KN/N|}} \sqrt{\sum_{g \in KN/N} |C_{KN/N}(g)|}. \tag{5.89}$$

Here $C_G(g)$ denotes the centralizer of the element g in the group G. In order to keep the following discussion as general as possible we will stick to relative normalization unless otherwise noted.

5.2.3 Entanglement entropy

Having defined the isometries I_F and I_V we finally embark on an exact calculation of the entanglement entropy for a simply connected region R on S^2. We focus on the Hopf tensor network state $|\psi_\Gamma\rangle$.

In the following we show how the inner product between two face pieces of the Hopf tensor network state $|\psi_\Gamma\rangle$ depends on the boundary degrees of freedom.

Proposition 9 (Face-like boundaries)**.** *Let $f_i, g_i \in X$ and $r = |\partial R|$. Then*

$$\langle \psi_p(f_1, \dots, f_r) | \psi_p(g_1, \dots, g_r) \rangle = |H| \sum_{(h_i)} \phi(h_1' \cdots h_r') \prod_{j=1}^{r} (g_j f_j^*)(h_j''). \tag{5.90}$$

Proof. Let $\lambda_i = h_i = h$ the Haar integral of H. Then from Definition 7 we get

$$\begin{aligned}
&\langle \psi_p(f_1, \dots, f_r) | \psi_p(g_1, \dots, g_r) \rangle \\
&\quad = \|h\|^{-2r} \, \|\phi\|^{-2} \sum_{(h_i)(\lambda_i)} \overline{\phi(h_r''' \cdots h_1''')} \, \phi(\lambda_r''' \cdots \lambda_1''') \prod_{j=1}^{r} \overline{f_j\big(S(h_j'')\big)} \, g_j\big(S(\lambda_j'')\big) \, \langle h_j' | \lambda_j' \rangle \\
&\quad = |H|^{r+1} \sum_{(h_i)(\lambda_i)} \phi(h_1''' \cdots h_r''') \, \phi(\lambda_r''' \cdots \lambda_1''') \prod_{j=1}^{r} f_j^*(h_j'') \, g_j\big(S(\lambda_j'')\big) \, \phi(h_j' \lambda_j')
\end{aligned}$$

where we used (4.80), the properties of h and the involution $*$ as well as (4.36) for

the second line. Employing Lemma 8 we can simplify this to

$$\begin{aligned}
&\langle\psi_p(f_1,\dots,f_r)|\psi_p(g_1,\dots,g_r)\rangle \\
&\quad= |H|^r \sum_{(h_i)(\lambda_i)} \phi(h_1'''\cdots h_r''')\,\phi\big(\lambda_r'''\cdots\lambda_2'''\,S(h_1')\big)(g_1 f_1^*)(h_1'')\prod_{j=2}^{r} f_j^*(h_j'')\,g_j\big(S(\lambda_j'')\big)\,\phi(h_j'\lambda_j') \\
&\quad= |H|\sum_{(h_i)} \phi(h_1'''\cdots h_r''')\,\phi\big(S(h_1'\cdots h_r')\big)\prod_{j=1}^{r}(g_j f_j^*)(h_j'').
\end{aligned}$$

Finally, the claim follows from $\phi^2=\phi$. □

Analogously, we are interested in the inner product between vertex pieces of the same Hopf tensor network state.

Proposition 10 (Vertex-like boundaries)**.** *Let $f_i, g_i \in X$ and $r=|\partial R|$. Then*

$$\langle\psi_s(f_1,\dots,f_r)|\psi_s(g_1,\dots,g_r)\rangle = \sum_{(h_i)}\prod_{j=1}^{r} f_j^*\big(h_j^{(3)}S(h_{j+1}^{(4)})\big)\,g_j\big(h_j^{(2)}S(h_{j+1}^{(1)})\big). \tag{5.91}$$

Proof. Let again $\lambda_i = h_i = h$ the Haar integral of H. From Definition 7 one has

$$\begin{aligned}
&\langle\psi_s(f_1,\dots,f_r)|\psi_s(g_1,\dots,g_r)\rangle \\
&\quad= \|h\|^{-2r}\sum_{(h_i)(\lambda_i)}\prod_{j=1}^{r}\overline{f_j\big(S(h_j'')\,h_{j+1}'''\big)}\,g_j\big(S(\lambda_j'')\,\lambda_{j+1}'''\big)\,\langle h_j'|\lambda_j'\rangle \\
&\quad= |H|^r\sum_{(h_i)(\lambda_i)}\prod_{j=1}^{r} f_j^*\big(h_j''\,S(h_{j+1}''')\big)\,g_j\big(S(\lambda_j'')\,\lambda_{j+1}'''\big)\,\phi(h_j'\lambda_j').
\end{aligned}$$

Additionally, Lemma 8 yields

$$\sum_{(\lambda_i)}\prod_{j=1}^{r}\phi(a_j\lambda_j')\,g_j\big(S(\lambda_j'')\,\lambda_{j+1}'''\big) = |H|^{-r}\sum_{(a_i)}\prod_{j=1}^{r} g_j\big(a_j''\,S(a_{j+1}')\big)$$

where $a_j\in H$ is arbitrary. Armed with this identity it is easy to verify the claim. □

We now state the main result of this section.

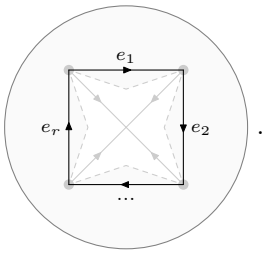
Theorem 4. *Let $R\subset M$ a simply connected region and let $\Gamma_R\subset\Gamma$ the corresponding subgraph. Then for any $\alpha\geq 0$ the state $|\psi_\Gamma\rangle$ has the Rényi entanglement entropies*

$$S_\alpha(\rho_R) = |\partial R|\log|H| - \log|H|. \tag{5.92}$$

In particular, the topological entanglement entropy reads

$$\gamma = \log|H|. \tag{5.93}$$

Proof. Without loss of generality we restrict to $M \simeq S^2$. Acting with the isometries I_F and I_V on $|\psi_\Gamma\rangle$ we may wipe out the bulk of both R and $\bar{R}$ completely and reduce Γ to the graph Γ_0 given by



.

For the resulting Hopf tensor network state on this graph Proposition 4 yields the natural splitting

$$\begin{aligned}|\psi_{\Gamma_0}\rangle &= \|\phi\|^{-r} \sum_{(\phi_i)} |\psi_R(\phi'_1,\dots,\phi'_r)\rangle \otimes |\psi_{\bar{R}}(\phi''_1,\dots,\phi''_r)\rangle \\ &= |H|^{r/2} \sum_{(\phi_i)} |\psi_p(\phi'_1,\dots,\phi'_r)\rangle \otimes |\psi_s(\phi''_1,\dots,\phi''_r)\rangle\end{aligned}$$

where $r = |\partial R|$ denotes the length of the boundary.

We determine the reduced density operator $\rho_R = N^{-1}\tilde{\rho}_R$ of the region R as follows. Let $\phi_i = \varphi_i = \phi$ the Haar integral of X and $h_i = \lambda_i = \kappa_i = h$ the Haar integral of H. Ignoring the normalization factor N for a moment one has

$$\begin{aligned}\tilde{\rho}_R &= \mathrm{tr}_{\bar{R}}\left(|\psi_{\Gamma_0}\rangle\langle\psi_{\Gamma_0}|\right) \\ &= |H|^r \sum_{(\phi_i)(\varphi_i)} |\psi_p(\phi'_1,\dots,\phi'_r)\rangle\langle\psi_p(\varphi'_1,\dots,\varphi'_r)| \cdot \langle\psi_s(\varphi''_1,\dots,\varphi''_r)|\psi_s(\phi''_1,\dots,\phi''_r)\rangle \\ &= |H|^r \sum_{(\phi_i)(\varphi_i)} |\psi_p(\phi'_1,\dots,\phi'_r)\rangle\langle\psi_p(\varphi'_1,\dots,\varphi'_r)| \sum_{(h_i)} \prod_{j=1}^{r} (\varphi''_{r+1-j})^*\big(h_j^{(3)} S(h_{j+1}^{(4)})\big)\, \phi''_{r+1-j}\big(h_j^{(2)} S(h_{j+1}^{(1)})\big) \\ &= |H|^{2r+1} \sum_{(\phi_i)(\varphi_i)} \Big(\sum_{(\lambda_i)} \phi(\lambda'''_r \cdots \lambda'''_1) \prod_{k=1}^{r} \phi'_k\big(S(\lambda''_k)\big) |\lambda'_1,\dots,\lambda'_r\rangle\Big)\Big(\sum_{(\kappa_i)} \overline{\phi(\kappa'''_r \cdots \kappa'''_1)} \prod_{l=1}^{r} \overline{\varphi'_l\big(S(\kappa''_l)\big)}\, \langle\kappa'_1,\dots \\ &\quad \sum_{(h_i)} \prod_{j=1}^{r} (\varphi''_j)^*\big(h_j^{(3)} S(h_{j-1}^{(4)})\big)\, \phi''_j\big(h_j^{(2)} S(h_{j-1}^{(1)})\big) \\ &= |H|^{2r+1} \sum_{(h_i)} \sum_{(\kappa_i)} \sum_{(\lambda_i)} \phi(\lambda'''_r \cdots \lambda'''_1)\, \phi(\kappa'''_1 \cdots \kappa'''_r) \prod_{j=1}^{r} \phi\big(h_j^{(2)} S(h_{j-1}^{(1)})\, S(\lambda''_j)\big)\, \phi\big(h_j^{(3)} S(h_{j-1}^{(4)})\, \kappa''_j\big) \\ &\quad |\lambda'_1\rangle \otimes \cdots \otimes |\lambda'_r\rangle \otimes \phi(\kappa'_1 ?) \otimes \cdots \otimes \phi(\kappa'_r ?)\end{aligned}$$

where we used Proposition 10 in the third line.

Furthermore, using Lemma 9 we may derive

$$\sum_{(\kappa_i)} \phi(\kappa_1''' \cdots \kappa_r''') \prod_{j=1}^{r} \phi(b_j \kappa_j'') \bigotimes_{k=1}^{r} \phi(\kappa_k' ?) = |H|^{-r} \sum_{(b_i)} \phi(b_r'' \cdots b_1'') \bigotimes_{j=1}^{r} \phi\big(S(b_j')\, ?\big).$$

With this under our belt we arrive at

$$\begin{aligned}\tilde{\rho}_R = |H|^{r+1} \sum_{(h_i)(\lambda_i)} & \phi(\lambda_r''' \cdots \lambda_1''')\, \phi\big(h_r^{(4)} S(h_{r-1}^{(5)}) \cdots h_2^{(4)} S(h_1^{(5)})\, h_1^{(4)} S(h_r^{(5)})\big) \prod_{j=1}^{r} \phi\big(h_j^{(2)}\, S(h_{j-1}^{(1)})\, S(\lambda_j'')\big) \\ & |\lambda_1', \ldots, \lambda_r'\rangle \bigotimes_{k=1}^{r} \phi\big[S\big(h_k^{(3)} S(h_{k-1}^{(6)})\big)\, ?\big] \\ = |H|^{r+1} \sum_{(h_i)(\lambda_i)} & \phi(\lambda_r''' \cdots \lambda_1''') \prod_{j=1}^{r} \phi\big(h_j^{(2)} S(h_{j-1}^{(1)})\, S(\lambda_j'')\big) |\lambda_1', \ldots, \lambda_r'\rangle \bigotimes_{k=1}^{r} \phi\big(h_{k-1}^{(4)}\, S(h_k^{(3)})\, ?\big).\end{aligned}$$

Again some more preparation, namely Lemma 10 allows us to find

$$\sum_{(\lambda_i)} \phi(\lambda_r''' \cdots \lambda_1''') \prod_{j=1}^{r} \phi(a_j \lambda_j'') \bigotimes_{k=1}^{r} |\lambda_k'\rangle = |H|^{-r} \sum_{(a_j)} \phi(a_1'' \cdots a_r'') \bigotimes_{j=1}^{r} |S(a_j')\rangle$$

and subsequently we have for the reduced density operator:

$$\tilde{\rho}_R = |H| \sum_{(h_i)} \bigotimes_{j=1}^{r} |h_j^{(2)}\, S(h_{j-1}^{(1)})\rangle \bigotimes_{k=1}^{r} \phi\big(h_{k-1}^{(4)}\, S(h_k^{(3)})\, ?\big).$$

Finally, it can be shown that the reduced density operator of the region R takes the simple form

$$\begin{aligned}\tilde{\rho}_R = |H| \sum_{(h_i)} & |h_1^{(1)}\rangle \otimes \cdots \otimes |h_{r-1}^{(1)}\rangle \otimes |S(h_{r-1}^{(4)} \cdots h_1^{(4)})\rangle \\ & \otimes \phi\big(S(h_1^{(2)})\, ?\big) \otimes \cdots \otimes \phi\big(S(h_{r-1}^{(2)})\, ?\big) \otimes \phi(h_{r-1}^{(3)} \cdots h_1^{(3)} ?).\end{aligned}$$

up to normalization. It is easy to see that

$$\begin{aligned}\operatorname{tr}(\tilde{\rho}_R) &= |H| \sum_{(h_i)} \prod_{j=1}^{r-1} \phi\big(S(h_j^{(2)})\, h_j^{(1)}\big)\, \phi\big(h_{r-1}^{(3)} \cdots h_1^{(3)} S(h_{r-1}^{(4)} \cdots h_1^{(4)})\big) \\ &= |H|\end{aligned}$$

and hence we can fix the normalization by setting $N = |H|$.

Furthermore it is not difficult to show that ρ_R is proportional to a projector. Indeed, consider

$$\begin{aligned}\rho_R^2 = \sum_{(h_i)(\lambda_i)} & \prod_{j=1}^{r-1} \phi\big(S(h_j^{(2)})\, \lambda_j^{(1)}\big)\, \phi\big(h_{r-1}^{(3)} \cdots h_1^{(3)} S(\lambda_{r-1}^{(4)} \cdots \lambda_1^{(4)})\big) \\ & |h_1^{(1)}\rangle \otimes \cdots \otimes |h_{r-1}^{(1)}\rangle \otimes |S(h_{r-1}^{(4)} \cdots h_1^{(4)})\rangle \otimes \phi\big(S(\lambda_1^{(2)})\, ?\big) \otimes \cdots \otimes \phi\big(S(\lambda_{r-1}^{(2)})\, ?\big) \otimes \phi(\lambda_{r-1}^{(3)} \cdots \lambda_1^{(3)} ?) \\ = |H|^{1-r} & \rho_R.\end{aligned}$$

This means that the spectrum of ρ_R is flat.

Finally one can prove that the rank of the reduced density operator ρ_R is given by $|H|^{r-1}$ hence each non-zero eigenvalue equals $|H|^{1-r}$. Then the Rényi entropies read

$$S_\alpha(\rho_R) = \frac{1}{1-\alpha}\log\big(\mathrm{tr}(\rho_R^\alpha)\big) = |\partial R|\log|H| - \log|H|. \tag{5.94}$$

independently of α. □

Remark 7. We would like to stress that any reduced density operator ρ_R obtained from the states $|\psi_{K,N}\rangle$ is proportional to a projector as has been noted before for the case of group algebras at the top of the hierarchy [23]. In other words, the entanglement spectrum is flat and all Rényi entropies are equal. □

5.2.4 Boundary configurations

We would like to illustrate the topological constraint on the (internal) boundary of a Hopf tensor network state in more detail. This topological constraint is responsible for the emergence of the universal correction γ to the area law.

To this end we analyze the boundary configuration $(f_1,\ldots,f_r) \in X^r$ of the state $|\psi_R(f_1,\ldots,f_r)\rangle$ in particular canonical bases $\mathcal{B}_\mathcal{F}(X)$ or $\mathcal{B}_G(X)$ of X which are derived via a generalized Fourier construction (see Chapters 3 and 6). The dependence of the interior state $|\psi_R(f_1,\ldots,f_r)\rangle$ on this boundary configuration is considerably different if H is a group algebra, the dual of a group algebra or a non-trivial Hopf C^*-algebra like H_8.

For example, if H is a group algebra one has $\mathcal{B}_\mathcal{F}(\mathbb{C}^G) = \{\delta_g \mid g \in G\}$ and the topological constraint is a parity constraint obtained from group multiplication along the boundary as the following theorem shows.

Theorem 5. *Let $H = \mathbb{C}G$ and $g_i \in G$. Then*

1. *$|\psi_R(\delta_{g_1},\ldots,\delta_{g_r})\rangle \neq 0$ iff $g_1\cdots g_r = e$ and*
2. *all such non-zero interior states are pairwise orthogonal with identical norm.*

Proof. It is enough to consider purely face- and vertex-like regions R since a general simply connected region works similarly. From Propositions 9 and 10 one can easily show that

$$\begin{aligned}\langle\psi_p(\delta_{s_1},\ldots,\delta_{s_r})|\psi_p(\delta_{t_1},\ldots,\delta_{t_r})\rangle &= \langle\psi_s(\delta_{s_1},\ldots,\delta_{s_r})|\psi_s(\delta_{t_1},\ldots,\delta_{t_r})\rangle \\ &= |G|^{1-r}\delta_{s_1\cdots s_r,e}\prod_{j=1}^{r}\delta_{s_j,t_j}\end{aligned}$$

which proves the claim. □

If H is the dual of a (non-Abelian) group algebra then we use the canonical basis $\mathcal{B}_G(\mathbb{C}G) = G$. As the following theorem shows the topological constraint is an equivalence relation between boundary configurations differing by a global offset.

Theorem 6. *Let $H = \mathbb{C}^G$ and $g_i \in G$. Then*

1. $\| |\psi_R(g_1, \ldots, g_r)\rangle \| = 1$,
2. $|\psi_R(s_1, \ldots, s_r)\rangle = |\psi_R(t_1, \ldots, t_r)\rangle$ *iff* $t_i = g s_i$ *for some* $g \in G$ *and*
3. *all non-identical interior states are pairwise orthogonal.*

Proof. Again it is enough to consider purely face- and vertex-like regions R. From Proposition 9 we get

$$\langle \psi_p(s_1, \ldots, s_r) | \psi_p(t_1, \ldots, t_r) \rangle = \sum_{g \in G} \prod_{j=1}^{r} \delta_{g s_j, t_j}$$

for all $s_i, t_i \in G$ and from Proposition 10

$$\langle \psi_s(s_1, \ldots, s_r) | \psi_s(t_1, \ldots, t_r) \rangle = \prod_{j=1}^{r} \delta_{s_j^{-1} s_{j-1}, t_j^{-1} t_{j-1}} \ .$$

It is not difficult to see that these conditions are in fact the same and establish an equivalence relation $(s_1, \ldots, s_r) \sim (t_1, \ldots, t_r) \Leftrightarrow t_i = g s_i$ among boundary configurations. It follows that if two boundary configurations are not equivalent their corresponding interior states are orthogonal.

In particular, we have

$$\| |\psi_p(g_1, \ldots g_r)\rangle \| = \| |\psi_s(g_1, \ldots g_r)\rangle \| = 1$$

for any boundary configuration $(g_1, \ldots, g_r) \in \mathcal{B}^r$. □

Finally, if H is a nontrivial Hopf C^*-algebra like H_8 one will observe a combination of the two mechanisms just described. More precisely, there exists no basis of H_8^* for which the topological constraint on the boundary configuration is *purely* the mechanism of either Theorem 5 or Theorem 6.

5.3 Discussion

First of all, we have developed a tensor network language, based on the formalism of finite-dimensional Hopf C^*-algebras, which is both more flexible and more natural than conventional descriptions for topological states. We have given rules to evaluate tensor network diagrams by means of tensor traces and to construct quantum states on the lattice, as well as to perform operations related to the spatial lattice structure, such as cutting and joining along region boundaries.

This tensor network language has been shown to directly lead to the construction of well-known topologically ordered states, namely ground states of Kitaev's quantum double models based on groups [39]. All these states are written, in a basis-independent way, in the form of tensor networks involving the intrinsic Hopf C^*-algebra structure *only*.

Relaxing this property gently leads us to a *hierarchy* of states defined from different subalgebras of a given Hopf C^*-algebra. We study the classes obtained from group algebras, i.e. KITAEV's original models, and show that the hierarchy arises from the mechanism of condensation of topological charges [3, 4], and so we conjecture that this mechanism can be described in general in our language.

Furthermore the hierarchy states can be regarded as ground states of certain frustration-free Hamiltonians beyond the D(H)-model. These are obtained from Theorem 2 by replacing the Haar integrals $h \in H$ and $\phi \in X$ with $h_A \in A$ and $\phi_B \in B$ respectively where $A \subset H$ and $B \subset X$ are Hopf C^*-subalgebras.

Not least, we have established isometric mappings defining *entanglement renormalization* [74, 73] extending the work of [1] for the states at the top of the hierarchy. This is a systematic procedure to thin out degrees of freedom keeping the topological nature of the states, and hence their nonlocal properties, intact. Our computation of the topological entanglement entropy is an application of this general scheme.

Chapter 6

Electric-magnetic duality

So far the extension of the electric-magnetic duality of the toric code to non-Abelian models has remained an open problem, and thus a deep insight into the structure of topological order has been only available in the very restricted Abelian setting. As mentioned in the introduction the natural place to look for a well defined electric-magnetic duality are KITAEV's quantum double models, since the excitations there are clearly understood and can be assigned electric and magnetic quantum numbers, while this characterisation is absent in string-net models.

In this chapter we uncover electric-magnetic duality for all of KITAEV's D($\mathbb{C}G$)-models, both Abelian and non-Abelian, and beyond. We do this by identifying its natural setting as the quantum double models based on Hopf algebras, a class of models anticipated in [39] and defined and studied in Chapters 4 and 5. Extending the methods employed in Chapter 3 we then use Fourier techniques to write these generalised quantum double models as *extended* string-net (ESN) models, a variant of LEVIN and WEN's SN models with a canonically enlarged Hilbert space, such that anyons can be understood in representation theoretic language. ESN models can be defined independently, in analogy to LEVIN and WEN's SN models, by data from unitary tensor categories, equipped in this case with a so-called fibre functor [11]. By EM duality, each ESN model maps to two SN models, its electric and magnetic projections (as depicted in Figure 6.2). We conjecture that *all* SN models are obtained in such a way from parent ESN models with a generalised algebraic structure, in which EM duality continues to hold. These results lead us to propose the duality landscape of Figure 6.1.

6.1 Toric code and self-duality

Let us begin by reviewing the well-known self-duality in the toric code [39], the quantum double model based on the group $\mathbb{Z}_2$. This is a model of qubits along the edges of a graph Γ embedded in a two-dimensional surface, with a Hamiltonian of the form

$$\mathcal{H} = -\sum_{s\in V} A(s) - \sum_{p\in F} B(p)\,, \tag{6.1}$$

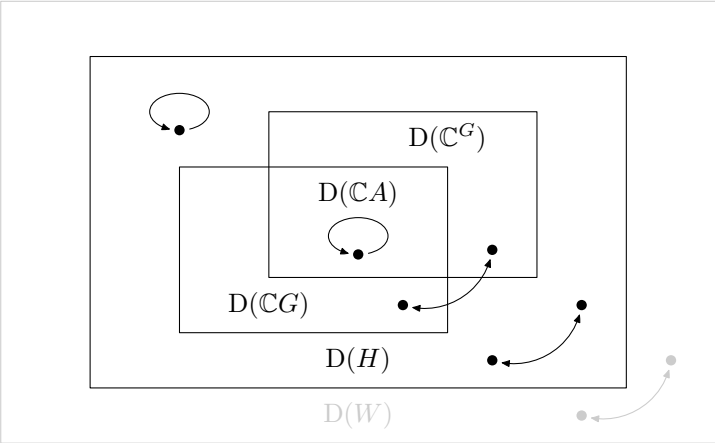


Figure 6.1: Duality landscape. D($\mathbb{C}A$) denotes quantum double models based on Abelian groups (for example, the toric code). D($\mathbb{C}G$) denotes quantum double models based on arbitrary (finite) groups (for example, $G = S_3$). By electric-magnetic duality these are related to quantum double models D($\mathbb{C}^G$) based on algebras of functions. D(H) denotes quantum double models based on Hopf C^*-algebras. D(W) denotes quantum double models based on weak quasi-Hopf C^*-algebras which we conjecture to correspond to ESN models.

where s runs over vertices and p runs over faces of Γ. Mutually commuting vertex and face operators are projectors involving Pauli operators

$$A(s) = \frac{1}{2}\left(\mathrm{id} + \bigotimes_{i\in E(s)} \sigma_i^x\right), \tag{6.2}$$

$$B(p) = \frac{1}{2}\left(\mathrm{id} + \bigotimes_{j\in E(p)} \sigma_j^z\right), \tag{6.3}$$

with support on the edges around the corresponding vertex s or face p. Ground states $|\psi\rangle$ of the frustration-free Hamiltonian (6.1) minimize each individual term in the sum, i.e. they satisfy all vertex and face constraints $A(s)|\psi\rangle = |\psi\rangle$, $B_p|\psi\rangle = |\psi\rangle$. Breakdown of any one such constraint means that the state is in an excited level and can be interpreted as the presence of a quasiparticle excitation located at the vertex or the face whose constraint is broken. From the gauge theory interpretation, these are called electric and magnetic excitations, respectively.

Self-duality means that the global unitary map

$$\mathcal{U}_\Gamma = \bigotimes_{i\in E} \mathcal{U}_i \tag{6.4}$$

built from the local Hadamard maps

$$\mathcal{U}_i = \frac{1}{\sqrt{2}}(\sigma_i^x + \sigma_i^z) \tag{6.5}$$

at each edge i maps the toric code on Γ to a toric code on the dual graph Γ^*

$$\mathcal{U}_\Gamma A(s)\mathcal{U}_\Gamma^\dagger = \tilde{B}(s)\,, \tag{6.6}$$

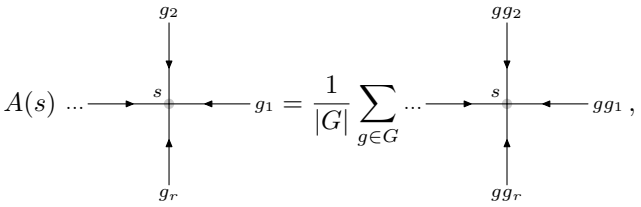
$$\mathcal{U}_\Gamma B(p)\mathcal{U}_\Gamma^\dagger = \tilde{A}(p) \tag{6.7}$$

where now $\tilde{B}(s)$ corresponds to a dual face s and $\tilde{A}(p)$ to a dual vertex p in Γ^*. The electric-magnetic nature of this duality comes from the interchange of vertices and faces in going from the original to the dual graph.

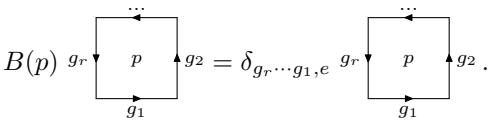
Hidden in the action of the global map $\mathcal{U}_\Gamma$ are (a) the mapping from Γ to Γ^* with the corresponding reinterpretation of vertex and face operators, and (b) a mapping of the elements of the group $\mathbb{Z}_2$ to the functions from this group to $\mathbb{C}$. We obtain again a toric code on the dual graph because, for an *Abelian* group A, the space of functions $\mathbb{C}^A$ has the same structure as the group algebra $\mathbb{C}A$. One can exploit this algebraic fact to extend the self-duality of the toric code to all quantum double models based on Abelian groups in a straightforward way: this is the $\mathrm{D}(\mathbb{C}A)$ region of Figure 6.1.

6.2 Revisiting quantum double models

Recall from Section 1.2 that for Kitaev's quantum double models based on a group G the operators constituting the Hamiltonian are the following projectors:



(6.8)



(6.9)

The obvious parallelism of faces and vertices in the Abelian case is lost if G is non-Abelian. Upon switching to the dual graph we cannot reinterpret the operators $A(s)$ and $B(p)$ as face and vertex operators in a quantum double model based on a *group* since the algebra of functions $\mathbb{C}^G$ on a non-Abelian group G no longer has the same structure as the group algebra $\mathbb{C}G$ itself. Hence, the extension of the EM duality to non-Abelian $\mathrm{D}(\mathbb{C}G)$-models is not possible within the class of quantum double models based on groups. Yet dual models *can* be constructed—these are quantum double models based no longer on groups, but rather on algebras of functions. This duality is indicated by the arrow between the regions $\mathrm{D}(\mathbb{C}G)$ and $\mathrm{D}(\mathbb{C}^G)$ in Figure 6.1.

In order to make sense of this one has to extend the construction of quantum double lattice models beyond the group case. As we will show the natural habitat for

the EM duality of D($\mathbb{C}G$)-models is the class of quantum double models based on Hopf algebras from Chapter 4. This corresponds to the region D(H) in Figure 6.1. This is the smallest class which contains all the D($\mathbb{C}G$)-models *and* is closed under EM duality. The duality, moreover, takes a remarkably simple form in the language of Hopf algebras.

While the generalized quantum double models based on Hopf C^*-algebras are discussed in detail in Chapter 4 let us briefly recall their essential features and our motivation for considering more general symmetries than those furnished by group theory in the first place. Perhaps the most important reason for why these more general symmetries emerge in quantum many-body systems is the fact that linear transformations in tensor products of vector spaces almost naturally lead to Hopf algebras.

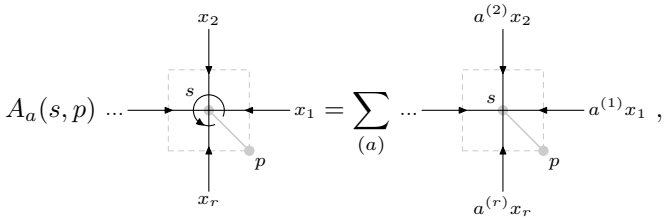
Consequently, let us regard a Hopf algebra H as a space of transformations on a many-body Hilbert space. First of all, the linear nature of the target is naturally extended to its transformations, so Hopf algebras are vector spaces. We must be able to compose transformations and to include the identity transformation, so H has a multiplication of vectors, and a unit, making it into an algebra. Most importantly, we must have a rule to distribute the action of an element of H into a tensor product of target spaces. This is the so called comultiplication. Additionally H has a trivial representation ϵ, called counit, making precise the notion of spaces invariant under the action of H. As for groups, the representation theory of Hopf algebras includes a notion of dual representation, implemented via an antipode mapping S, which for groups is just the inversion $g \mapsto g^{-1}$.

To be able to construct Hilbert spaces we use finite-dimensional Hopf C^*-algebras where an inner product can be defined. In addition they come equipped with a canonical, normalized, highly symmetric element, the Haar integral $h \in H$, which is invariant under multiplication in the sense that $ah = \epsilon(a)\, h$ for all elements $a \in H$. As emphasized before this canonical element is crucial for the construction of the lattice model and its ground states.

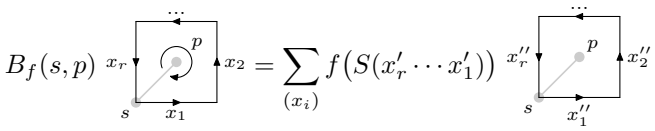
The root of the EM duality to be unveiled shortly is the following algebraic fact: Hopf C^*-algebras are closed under dualisation. In other words, given a finite-dimensional Hopf C^*-algebra H its dual space H^* (the functions from H to $\mathbb{C}$) is again a Hopf C^*-algebra whose structure is determined by the structure of H (see Section 4.1.1 and Appendix C for details and examples). This closure property is shared by the class of Abelian group algebras, which are all self-dual, but not by the whole class of group algebras. The landscape of Figure 6.1 reflects these statements at the physical level of topologically ordered spin models.

Quantum double models based on Hopf algebras, D(H)-models for short, are a class of topological models defined on two-dimensional graphs (lattices) Γ whose local Hilbert space along oriented edges is a finite-dimensional Hopf C^*-algebra H.

Recall from Section 4.2.1 that there are two kinds of operators



(6.10)



(6.11)

acting on the vertices and faces of Γ. These operators depend on elements $a \in H$ and $f \in X = (H^{\mathrm{op}})^*$ respectively. When these elements are taken as the canonical Haar integrals $h \in H$ and $\phi \in X$ the resulting operators

$$A(s) = A_h(s,p)\,, \tag{6.12}$$

$$B(p) = B_\phi(s,p) \tag{6.13}$$

are mutually commuting projectors defining via (6.1) the Hamiltonian of the $\mathrm{D}(H)$-model. The Hamiltonian is thus entirely constructed from the canonical elements h and ϕ as well as the algebraic structure of H.

Kitaev's original models constitute a subclass of the $\mathrm{D}(H)$-models which is recovered once H is the group algebra of a finite group G. Both this algebra $\mathbb{C}G$ and its dual $\mathbb{C}^G$ have particularly simple Hopf C^*-algebra structures summarised in Appendix C. In particular, their Haar integrals read

$$h = \frac{1}{|G|}\sum_{g\in G} g \qquad \in \mathbb{C}G \tag{6.14}$$

$$\phi = \delta_e \qquad \in \mathbb{C}^G\,. \tag{6.15}$$

From here it is not difficult to deduce (6.8) and (6.9) from their general counterparts (6.12) and (6.13).

6.3 Electric-magnetic duality

We now define EM duality for general $\mathrm{D}(H)$-models. Consider the following unitary map $U\colon H \to H^*$:

$$a \overset{U}{\longmapsto} f_a\,, \qquad f_a(x) \coloneqq \sqrt{|H|}\,\phi(ax)\,, \tag{6.16}$$

where $|H|$ is the dimension of H, and the function f_a is constructed via the dual Haar integral $\phi \in H^*$. For example, it is easy to check that in the case of a group G one has

$$U(g) = \sqrt{|G|}\,\delta_{g^{-1}}\,. \tag{6.17}$$

Indeed, the map U is unitary because

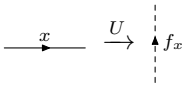
$$\begin{aligned}(f_a, f_b)_{H^*} &= \sum_{(h)} f_a^*(h')\, f_b(h'') \\ &= |H| \sum_{(h)} \phi\big(S(a^*)\, h'\big)\, \phi(bh'') \\ &= |H| \sum_{(h)} \phi(h')\, \phi(ba^* h'') \\ &= |H|\, \phi(h)\, \phi(ba^*) \\ &= \phi(a^* b) \\ &= (a, b)_H\end{aligned}$$

where we used $f_a^* = f_{S(a^*)}$ in the second line. Furthermore, its adjoint $U^\dagger \colon H^* \to H$ is given by

$$U^\dagger(f) = \sqrt{|H|}\, T_-^f(h) = \sqrt{|H|} \sum_{(h)} f\big(S(h')\big)\, h'' \tag{6.18}$$

as one may easily verify.

Next we associate this map with the transformation of an edge in Γ into its dual edge in Γ^* as follows:



(6.19)

The global map

$$U_\Gamma \coloneqq \bigotimes_{i \in E(\Gamma)} U_i \tag{6.20}$$

then sends the vertex (6.10) and face operators (6.11) precisely to the face and vertex operators associated with the $\mathrm{D}(H^*)$-model on the dual graph Γ^*. Note that in the following p is both a face of Γ and a vertex of Γ^* while s is both a vertex of Γ and a face of Γ^*.

Theorem 7 (Duality of local symmetry algebra)**.** *Let $a \in H$ and $f \in H^*$.*

$$U_\Gamma B_f(s,p)\, U_\Gamma^\dagger = \tilde{A}_f(p,s)\,, \tag{6.21}$$

$$U_\Gamma A_a(s,p)\, U_\Gamma^\dagger = \tilde{B}_a(p,s)\,. \tag{6.22}$$

Proof. See Appendix F. □

Focussing on the projectors which constitute the Hamiltonian we naturally obtain from Theorem 7:

$$U_\Gamma B(p)\, U_\Gamma^\dagger = \tilde{A}(p)\,, \tag{6.23}$$

$$U_\Gamma A(s)\, U_\Gamma^\dagger = \tilde{B}(s)\,. \tag{6.24}$$

It is this very expression which generalises (6.6), singling out U_Γ as the actual EM duality map.

Thus U_Γ *identifies* the $\mathrm{D}(H)$-model on Γ with the $\mathrm{D}(H^*)$-model on Γ^* as it transforms the Hilbert spaces and Hamiltonian of the former into those of the latter. We write this symbolically as

$$\mathrm{D}(H)_\Gamma = \mathrm{D}(H^*)_{\Gamma^*} \,. \tag{6.25}$$

But EM duality is deeper than a mere identification of energy levels. The Hamiltonian involves just the projectors $A(s)$ and $B(p)$ but according to Theorem 7 the duality map U_Γ induces a transformation of *all* local symmetry operators (6.10) and (6.11). Most importantly, this transformation respects their algebra structure and thus relates the anyonic excitations of both models [11].

6.4 Net of dualities

We now make the connection with Levin and Wen's string-net models [48]. Recall from Chapter 3 that the $\mathrm{D}(\mathbb{C}G)$-model is identified, via a Fourier mapping, with an *extended* string-net model on the same graph, with edge degrees of freedom given by irreducible representations of G and auxiliary matrix indices depending on each such representation (see also [35]). String-net vertex conditions amount to knitting the matrix indices together by means of $3j$-symbols, effectively mapping the local Hilbert spaces isometrically to those of a Levin-Wen SN model, whose degrees of freedom and underlying category theoretical structure are given solely by irreducible representations of G and their fusion properties. The ground levels of SN and their extended versions are identical.

Now the representation theory of finite-dimensional Hopf C^*-algebras is essentially identical to that of finite groups, the only real difference being that the fusion of representations need not be commutative. Therefore the construction of Chapter 3 generalises to any $\mathrm{D}(H)$-model. That is, if H has irreducible representations μ with particular matrix realizations D^μ of dimension d_μ we define a Fourier basis $\mathcal{B}_\mathcal{F}(H) = \{b_{\mu,ij}\}$ in H by

$$b_{\mu,ij} \coloneqq \sqrt{|H|\, d_\mu} \sum_{(h)} D^\mu_{ij}(h')\, h'' \tag{6.26}$$

where $1 \leq i, j \leq d_\mu$ are matrix indices for the irreducible representation D^μ. This is an orthonormal basis for each edge, and the analysis of Chapter 3 carries through intact to show that the $\mathrm{D}(H)_\Gamma$-model can be written as an extended string-net model $\mathrm{ESN}(H)_\Gamma$ with edge degrees of freedom labelled by triplets (μ, i, j). The extended string-net model reduces to a Levin-Wen string-net model $\mathrm{mSN}(H)_\Gamma$, whose degrees of freedom are only the irreducible representations μ of H and whose ground level is identical to that of the extended model. This we call the *magnetic projection* of the original model on Γ.

The same construction can be applied to the $\mathrm{D}(H^*)_{\Gamma^*}$-model which we have shown to be exactly equivalent to the original $\mathrm{D}(H)_\Gamma$-model via EM duality. By

using the Fourier basis $\mathcal{B}_{\mathcal{F}}(H^*)$ for H^* we write the $\mathrm{D}(H^*)_{\Gamma^*}$-model as an equivalent extended string-net model $\mathrm{ESN}(H^*)_{\Gamma^*}$ on the dual graph. Its magnetic projection is the (original) string-net model $\mathrm{mSN}(H^*)_{\Gamma^*}$ on the dual graph, with irreducible representations of H^* as edge degrees of freedom. We regard this as the *electric projection* of the original extended string-net model $\mathrm{ESN}(H)_\Gamma$ and write

$$\mathrm{eSN}(H)_{\Gamma^*} \coloneqq \mathrm{mSN}(H^*)_{\Gamma^*} \tag{6.27}$$

These relations between quantum double models, extended string-net models and their electric and magnetic projections are conveniently summarized in Figure 6.2.

From here it becomes clear that in Chapter 3 we analyzed the magnetic SN projection of the $\mathrm{D}(\mathbb{C}G)$-model. Its *electric* SN projection on the other hand has the same local degrees of freedom as the $\mathrm{D}(\mathbb{C}G)$-model since group elements are the (one-dimensional) irreducible representations of the dual $\mathbb{C}^G$ of the group algebra[1].

Thus, by the (Morita) duality between quantum double and string-net models EM duality extends to those string-net models which are obtained by reduction of ESN models. This leads us to the following

Conjecture 1. *All string-net models can be extended into quantum double models based on a sufficiently general algebraic structure, the class of weak Hopf C^*-algebras*[2] *[31, 50]. Moreover, electric-magnetic duality can be defined for all extended string-net models so that the pattern of Figure 6.2 holds.*

6.5 Topological invariants from tensor network states

Some physical consequences of EM duality are immediate. First of all, EM duality allows us to measure the topology of the surface underlying the lattice models by using *locally* defined states only.

As shown in Chapter 5 each $\mathrm{D}(H)_\Gamma$-model has one canonical tensor network ground state

$$|\psi(H,\Gamma)\rangle \coloneqq |\psi_\Gamma^{H,X}\rangle \tag{6.28}$$

constructed from *identical* tensors at each site and these are defined solely by the structure of H.

From the corresponding canonical state $|\psi(H^*,\Gamma^*)\rangle$ of the dual $\mathrm{D}(H^*)_{\Gamma^*}$-model on the dual graph we can obtain another ground state of the original $\mathrm{D}(H)_\Gamma$-model by using the EM duality map (6.16):

$$|\tilde{\psi}(H,\Gamma)\rangle \coloneqq U_\Gamma^\dagger\, |\psi(H^*,\Gamma^*)\rangle\,. \tag{6.29}$$

[1] This connection of SN and $\mathrm{D}(\mathbb{C}G)$-models had been recognised independently by Héctor Bombín.

[2] The excitations in such models would be classified by representations of the double $\mathrm{D}(W)$ of the weak Hopf C^*-algebra W, which is expected to be a weak quasi-Hopf algebra in the sense of [50].

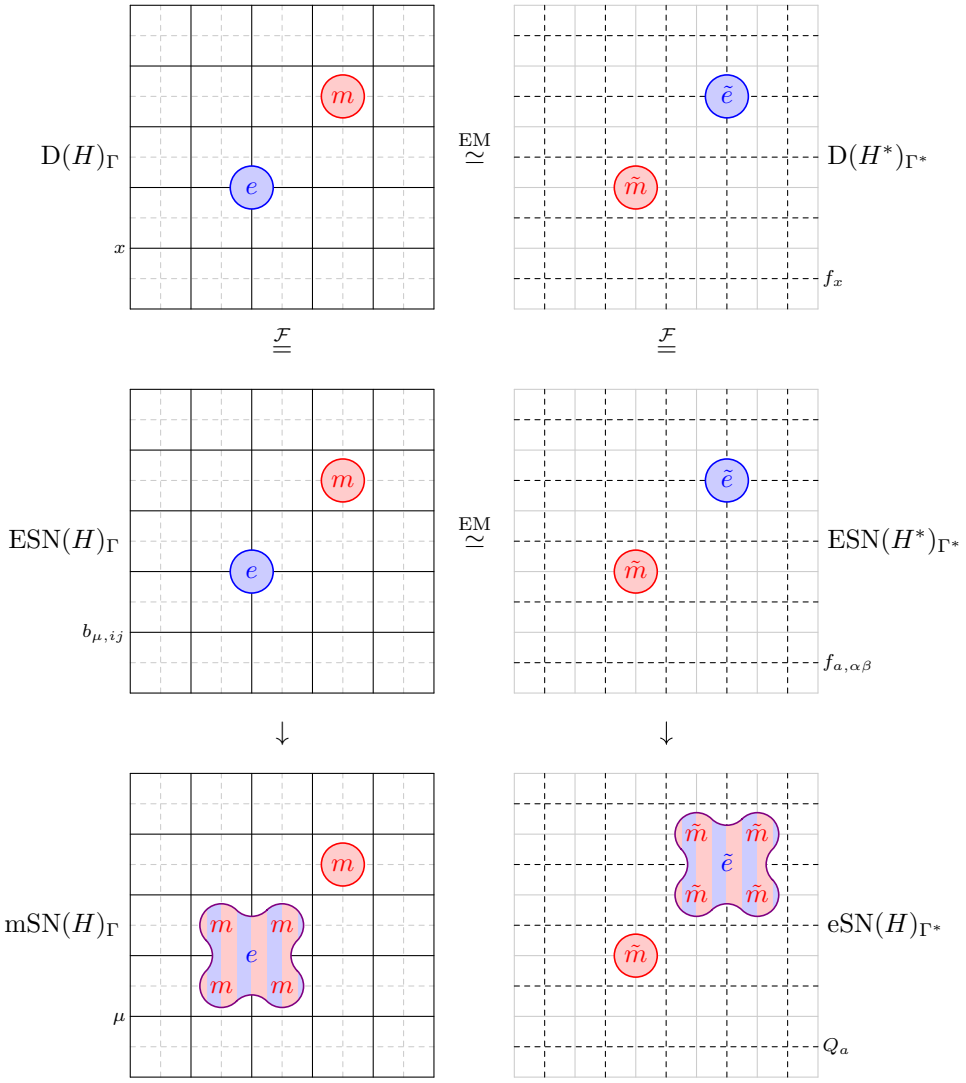


Figure 6.2: Relations between quantum double models, extended string-net models and their electric and magnetic projections. The D(H)-model on the graph Γ is equivalent to the D(H^*)-model on the dual graph Γ^* by electric-magnetic duality. Furthermore, the D(H)-model is identical to the extended string-net model ESN(H) on the same graph by Morita duality. This Morita duality is transported to the right column by electric-magnetic duality. Magnetic projection reduces the degrees of freedom of ESN$(H)_\Gamma$ and yields the string-net model mSN$(H)_\Gamma$. Electric projection in turn produces the string-net model eSN$(H)_{\Gamma^*}$ on the dual graph. Small labels at edges denote the respective local degrees of freedom of the models. Note that quasiparticle excitations are much more symmetric in ESN models than in string-net models: in the latter violating a vertex constraint implies violating all surrounding face constraints.

The relation between $|\psi(H,\Gamma)\rangle$ and $|\tilde{\psi}(H,\Gamma)\rangle$ depends on the topology of the surface underlying Γ. On the sphere, for instance, the ground level is nondegenerate hence both ground states coincide. On a topologically nontrivial surface such as the torus these ground states can be shown to always be linearly independent.

For instance, in the toric code the tensor network construction of the canonical state $|\psi(H,\Gamma)\rangle$ coincides with the projected entangled-pair state (PEPS) ansatz in [72]. On the torus this yields the logical $|{+}{+}\rangle$ state. The dual ground state $|\tilde{\psi}(H,\Gamma)\rangle$ is the logical $|00\rangle$ state. For details including the non-Abelian case see [11].

On the other hand, the correspondence between string-net and quantum double models can be used to relate these PEPS to the tensor network descriptions of string-net ground states put forward in Chapter 2 (also see [28]). These come from the $|\psi(H,\Gamma)\rangle$ construction of the corresponding $\mathrm{D}(H)_\Gamma$-model, while the $|\tilde{\psi}(H,\Gamma)\rangle$ can be seen to yield a *new* TN given simply by the knitting together of $3j$-symbols at vertices in the ESN degrees of freedom. More details will be given in [11]. The Fourier construction is also expected to relate the entanglement renormalization analyses of generalized quantum double models from Chapter 5 (see [1] for an earlier result) and string-net models [45].

6.6 Discussion

We have defined EM duality for non-Abelian $\mathrm{D}(\mathbb{C}G)$-models, showing how it arises naturally in the context of $\mathrm{D}(H)$-models. The connection to SN models comes from the Fourier construction of extended string-nets models and their electric and magnetic projections. We conjecture that all SN models have extended versions, and that EM duality can be defined for all the extended models. EM duality offers nonlocal information about the systems, unveiling global characteristics of space, from tensor networks defined locally. Beyond this, it serves as an *organising principle* for topological models. Indeed, we envisage a net of dualities for topological phases. This then reflects on the field theories underlying topological systems and on their mathematical structures, such as category theory.

We wish to stress the importance of the *whole* class of models based on Hopf algebras. This is indeed *the* natural class of models to study to understand electric-magnetic duality for lattice models based on non-Abelian groups. We thus shift the focus away from lattice gauge theories based on groups, which are a primary concern in high energy theory, even when the Hopf algebra language is applied [60]. Ours is, moreover, the first proposal for a general EM duality of topological phases.

The Hopf algebraic language renders the EM duality transparent and easy to grasp, instead of obscuring it. In addition, this language can be learnt here hands-on from the standpoint of quantum many-body systems, which can even be built in a laboratory. We also provide, with the Fourier construction, a way to understand some aspects of category theory in terms of representations, more familiar to many physicists.

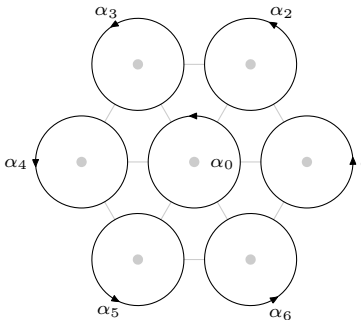
Appendix A

Fat lattice reduction

Here we show how a part of the fat lattice configuration (2.1) is eventually reduced to the canonical representative in several steps. In the following we omit the vacuum label 1 on all grey edges. For the sake of clarity we also set

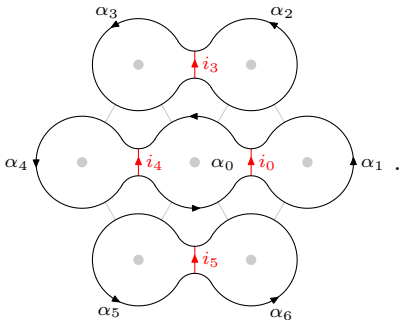
$$\begin{Bmatrix} i & j & m \\ k & l & n \end{Bmatrix} \coloneqq F^{ijm}_{kln} \,. \tag{A.1}$$

The first step consists in applying F-moves to all horizontal vacuum edges that we may add to the state $|\{\alpha_p\}\rangle$ on the fat lattice:



$$= \sum_{i_p=1}^{N} \begin{Bmatrix} \alpha_0^* & \alpha_0 & 1 \\ \alpha_1^* & \alpha_1 & i_0 \end{Bmatrix} \begin{Bmatrix} \alpha_3^* & \alpha_3 & 1 \\ \alpha_2^* & \alpha_2 & i_3 \end{Bmatrix}$$

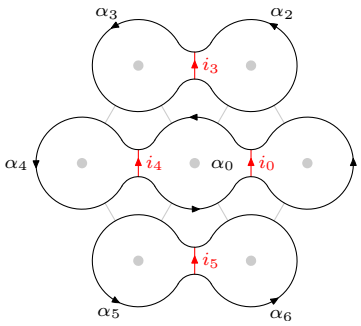
$$\times \begin{Bmatrix} \alpha_4^* & \alpha_4 & 1 \\ \alpha_0^* & \alpha_0 & i_4 \end{Bmatrix} \begin{Bmatrix} \alpha_5^* & \alpha_5 & 1 \\ \alpha_6^* & \alpha_6 & i_5 \end{Bmatrix}$$



. (A.2)

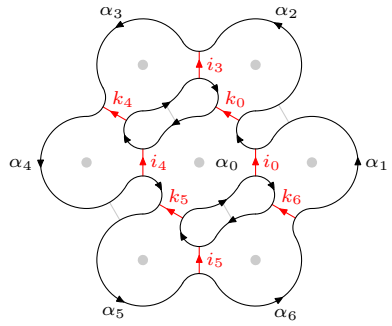
Note that the labels α_p do not change in the process. In particular, all (black) loop segments whose nearest puncture pierces the face p carry the same label α_p throughout. This is because they originate from the same isolated loop.

We repeat the same procedure for the diagonal vacuum edges:



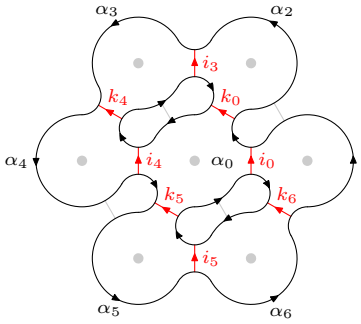
$$= \sum_{k_p=1}^{N} \begin{Bmatrix} \alpha_0^* & \alpha_0 & 1 \\ \alpha_2^* & \alpha_2 & k_0 \end{Bmatrix} \begin{Bmatrix} \alpha_4^* & \alpha_4 & 1 \\ \alpha_3^* & \alpha_3 & k_4 \end{Bmatrix}$$

$$\times \begin{Bmatrix} \alpha_5^* & \alpha_5 & 1 \\ \alpha_0^* & \alpha_0 & k_5 \end{Bmatrix} \begin{Bmatrix} \alpha_6^* & \alpha_6 & 1 \\ \alpha_1^* & \alpha_1 & k_6 \end{Bmatrix}$$



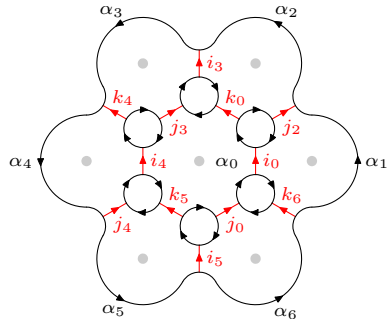
, (A.3)

and



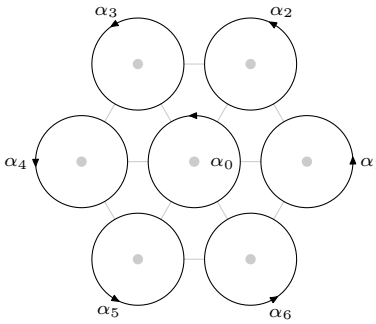
$$= \sum_{j_p=1}^{N} \begin{Bmatrix} \alpha_0^* & \alpha_0 & 1 \\ \alpha_6^* & \alpha_6 & j_0 \end{Bmatrix} \begin{Bmatrix} \alpha_2^* & \alpha_2 & 1 \\ \alpha_1^* & \alpha_1 & j_2 \end{Bmatrix}$$

$$\times \begin{Bmatrix} \alpha_3^* & \alpha_3 & 1 \\ \alpha_0^* & \alpha_0 & j_3 \end{Bmatrix} \begin{Bmatrix} \alpha_4^* & \alpha_4 & 1 \\ \alpha_5^* & \alpha_5 & j_4 \end{Bmatrix}$$



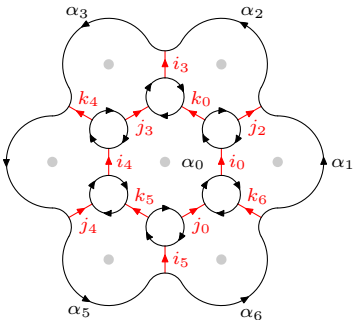
. (A.4)

Using the normalization (2.3) of the F-symbols we obtain at this point



$$= \frac{1}{d_{\alpha_0}^3 \sqrt{\prod_{p=1}^{6} d_{\alpha_p}^3}} \sum_{\{i_p, j_p, k_p\}_\star} \sqrt{d_{i_0} d_{i_3} d_{i_4} d_{i_5}}$$

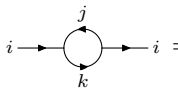
$$\times \sqrt{d_{j_0} d_{j_2} d_{j_3} d_{j_4}} \sqrt{d_{k_0} d_{k_4} d_{k_5} d_{k_6}}$$



(A.5)

where we only sum over configurations $\{i_p, j_p, k_p\}_\star$ which together with the $\{\alpha_p\}$ satisfy the fusion rules at each branch point.

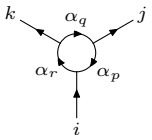
Before we proceed let us state that by using two F-moves (1.8), relation (1.7), normalization (2.3) and relation (1.6) one may fully evaluate



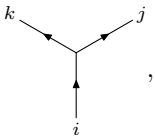
$$= \sqrt{\frac{d_j d_k}{d_i}}\, \delta_{ijk^*}$$

. (A.6)

Furthermore, by employing two F-moves (1.8), relation (1.7), normalization (2.3) and twice the previous result we can remove the loop segments at a vertex as follows

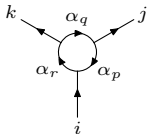


$$= \sqrt{\frac{d_{\alpha_p} d_{\alpha_r}}{d_i}} \begin{Bmatrix} k^* & \alpha_r & \alpha_q^* \\ \alpha_p^* & j^* & i \end{Bmatrix}$$

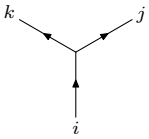


, (A.7)

or alternatively as



$$= \sqrt{\frac{d_{\alpha_p} d_{\alpha_q}}{d_j}} \begin{Bmatrix} \alpha_q^* & k^* & \alpha_r \\ i & \alpha_p & j \end{Bmatrix}$$

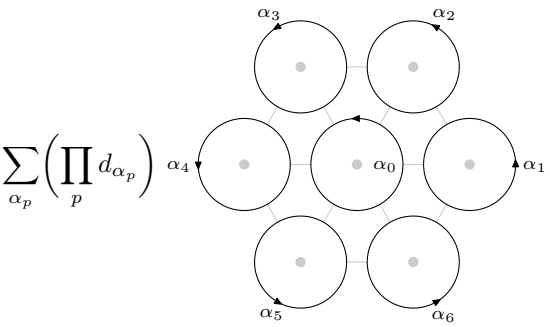


(A.8)

which follows from the symmetries of the F-symbols [48]. Note that fusion rules are implicitly contained in these expressions through the convention for the F-symbols.

In summary we obtain the following local reduction from the state $|\{\alpha_p\}\rangle$ on the fat lattice to its canonical representative on the physical lattice:

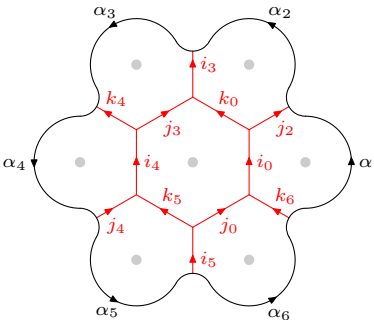
$$\sum_{\alpha_p}\Big(\prod_p d_{\alpha_p}\Big)$$



$$= \sum_{\alpha_p} \sqrt{d_{\alpha_3} d_{\alpha_6}} \sum_{i_p,j_p,k_p=1}^{N} \sqrt{\frac{d_{i_0}\cdots d_{i_5} d_{k_0}\cdots d_{k_6}}{d_{j_0} d_{j_3}}} \begin{Bmatrix} \alpha_2^* & k_0^* & \alpha_0 \\ i_0 & \alpha_1 & j_2 \end{Bmatrix}$$

$$\times \begin{Bmatrix} \alpha_0^* & k_0 & \alpha_2 \\ i_3^* & \alpha_3 & j_3^* \end{Bmatrix} \begin{Bmatrix} \alpha_3^* & k_4^* & \alpha_4 \\ i_4 & \alpha_0 & j_3 \end{Bmatrix} \begin{Bmatrix} \alpha_5^* & k_5 & \alpha_0 \\ i_4^* & \alpha_4 & j_4^* \end{Bmatrix} \begin{Bmatrix} \alpha_0^* & k_5^* & \alpha_5 \\ i_5 & \alpha_6 & j_0 \end{Bmatrix}$$

$$\times \begin{Bmatrix} \alpha_6^* & k_6 & \alpha_1 \\ i_0^* & \alpha_0 & j_0^* \end{Bmatrix}$$



. (A.9)

Appendix B

Classical F-symbols from $3j$-symbols

It can be shown that the projector onto the trivial representation subspace of the product $\mu \otimes \nu \otimes \lambda \otimes \rho$ of irreducible representations of the group G splits into a sum of orthogonal projectors associated with the internal channel σ in the coupling scheme $\mu \otimes \nu \xrightarrow{\sigma} \lambda \otimes \rho$, say, as

$$W^{\mu\nu\lambda\rho} = \bigoplus_{\sigma \in \widehat{G}} \Pi_{\sigma}^{\mu\nu,\lambda\rho} \,, \tag{B.1}$$

where the projectors

$$\left(\Pi_{\sigma}^{\mu\nu,\lambda\rho}\right)_{mnlr,\bar{m}\bar{n}\bar{l}\bar{r}} = d_{\sigma} \sum_{s,\bar{s}} W^{\mu\nu\sigma}_{mns,\bar{m}\bar{n}\bar{s}} W^{\lambda\rho\sigma^*}_{lrs,\bar{l}\bar{r}\bar{s}} \tag{B.2}$$

are expressed in terms of W connecting three irreducible representations[1]. This leads to the definition of the $\hat{F}$ operation as the change of basis, within the range of $W^{\mu\nu\lambda\rho}$, from the states associated with $\Pi_{\sigma}^{\mu\nu,\lambda\rho}$ to those of the alternative coupling scheme $\Pi_{\tau}^{\rho\mu,\nu\lambda}$. Explicitly, the operators read

$$\hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau} = \Pi_{\sigma}^{\mu\nu,\lambda\rho} \, \Pi_{\tau}^{\rho\mu,\nu\lambda} \tag{B.3}$$

and obviously commute with $W^{\mu\nu\lambda\rho}$. From here it is immediate to check, for instance, that

$$\sum_{\tau} \hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau} \, \hat{F}^{\rho\mu\tau}_{\nu\lambda\xi} = \delta_{\sigma\xi^*} \Pi_{\sigma}^{\mu\nu,\lambda\rho} \,. \tag{B.4}$$

In components, taking into account (3.9) for rank one projectors, one has

$$\left(\hat{F}^{\mu\nu\sigma}_{\lambda\rho\tau}\right)_{mnlr,\bar{m}\bar{n}\bar{l}\bar{r}} = \left(p_{\sigma}^{\mu\nu,\lambda\rho}\right)_{rmnl} F^{\mu\nu\sigma}_{\lambda\rho\tau} \left(p_{\tau}^{\rho\mu,\nu\lambda}\right)^{*}_{\bar{m}\bar{n}\bar{l}\bar{r}} \,, \tag{B.5}$$

where

$$\left(p_{\sigma}^{\mu\nu,\lambda\rho}\right)_{mnlr} = \sqrt{d_{\sigma}} \sum_{s} \begin{bmatrix} \mu & \nu & \sigma \\ m & n & s \end{bmatrix} \begin{bmatrix} \lambda & \rho & \sigma^* \\ l & r & s \end{bmatrix} \tag{B.6}$$

are $+1$ eigenvectors of the (rank one) projectors in (B.2),

$$\left(\Pi_{\sigma}^{\mu\nu,\lambda\rho}\right)_{mnlr,\bar{m}\bar{n}\bar{l}\bar{r}} = \left(p_{\sigma}^{\mu\nu,\lambda\rho}\right)_{mnlr} \left(p_{\sigma}^{\mu\nu,\lambda\rho}\right)^{*}_{\bar{m}\bar{n}\bar{l}\bar{r}} \tag{B.7}$$

[1] It can be seen easily that $\operatorname{tr} \Pi_{\sigma}^{\mu\nu,\lambda\rho} = \Delta^{\mu\nu\sigma} \Delta^{\lambda\rho\sigma^*}$, so these are rank one projectors as long as the three-irrep W's are.

and

$$F^{\mu\nu\sigma}_{\lambda\rho\tau} = \sqrt{d_\sigma d_\tau} \sum_{mnlrst} \overline{\begin{bmatrix} \mu & \nu & \sigma \\ m & n & s \end{bmatrix}} \overline{\begin{bmatrix} \lambda & \rho & \sigma^* \\ l & r & s \end{bmatrix}} \begin{bmatrix} \rho & \mu & \tau \\ r & m & t \end{bmatrix} \begin{bmatrix} \nu & \lambda & \tau^* \\ n & l & t \end{bmatrix} \tag{B.8}$$

for which, for instance,

$$\sum_\tau F^{\mu\nu\sigma}_{\lambda\rho\tau} F^{\rho\mu\tau}_{\nu\lambda\xi} = \Delta_{\mu\nu\sigma}\Delta_{\lambda\rho\xi}\delta_{\sigma\xi^*} \tag{B.9}$$

and

$$F^{\mu\nu 1}_{\lambda\rho\tau} = \sqrt{\frac{d_\tau}{d_\mu d_\lambda}}\, \delta_{\mu\nu^*}\delta_{\lambda\rho^*}\Delta_{\mu\lambda^*\tau} \tag{B.10}$$

(up to a phase from the square root).

Now the effect of the $\hat{F}$ operators can be interpreted directly in the string-net lattice by forgetting about the p tensors, whose role is enforcing physical constraints throughout. The $F^{\mu\nu\sigma}_{\lambda\rho\tau}$ are the same as the F-symbols in [48].

Appendix C

Trivial Hopf algebras and their quantum doubles

Here we briefly discuss some cases of how Hopf algebras can arise from groups. In particular we focus on how a group itself can be directly understood in the language of Hopf algebras.

C.1 Trivial Hopf algebras

A finite-dimensional Hopf algebra is said to be trivial if it is a group algebra or the dual of a group algebra for some finite group G. Let $\{g \mid g \in G\}$ be a basis of the group algebra $\mathbb{C}G$ with multiplication

$$\mu(g \otimes h) = gh \tag{C.1}$$

and unit $\eta(1_{\mathbb{C}}) = e$. Its comultiplication and counit are given by

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1_{\mathbb{C}} \tag{C.2}$$

and extended by linearity. Similarly, the antipode map is defined by

$$S(g) = g^{-1}. \tag{C.3}$$

It is easy to see that every group algebra $\mathbb{C}G$ is *co*commutative, but in general *non*-commutative. Furthermore, one may endow $\mathbb{C}G$ with the involution map

$$g^* = g^{-1} \tag{C.4}$$

which is extended by conjugate linearity. This turns $\mathbb{C}G$ into a Hopf C^*-algebra. Its Haar integral reads

$$h = \frac{1}{|G|} \sum_{g \in G} g. \tag{C.5}$$

The dual $(\mathbb{C}G)^*$ of a group algebra coincides with the space $\mathbb{C}^G$ of linear functions from the group to the field of complex numbers. For the dual basis $\{\delta_g \mid g \in G\}$ multiplication and unit are defined by

$$\delta_g \delta_h = \delta_{g,h} \delta_g, \qquad \eta(1_{\mathbb{C}}) = \sum_{g \in G} \delta_g = 1 \tag{C.6}$$

where $\delta_{g,h}$ denotes the Kronecker delta. The comultiplication, counit and antipode maps are given by

$$\Delta(\delta_g) = \sum_{uv=g} \delta_u \otimes \delta_v, \qquad \epsilon(\delta_g) = \delta_g(e) = \delta_{g,e}, \qquad S(\delta_g) = \delta_{g^{-1}} \tag{C.7}$$

for each $g \in G$. One may check that every function algebra $\mathbb{C}^G$ is commutative, but in general *non-co*commutative. When contrasted with the corresponding group algebra, this is not too surprising: since $\mathbb{C}G$ and $\mathbb{C}^G$ are dual to each other, one obtains the multiplication of one algebra from the comultiplication of the other one and vice versa. Again, $\mathbb{C}^G$ is a Hopf C^*-algebra via

$$\delta_g^* = \delta_g \tag{C.8}$$

and conjugate-linear extension[1] . Finally,

$$\phi = \delta_e \tag{C.9}$$

is the Haar integral of $\mathbb{C}^G$.

C.2 Quantum doubles of trivial Hopf algebras

As a vector space the quantum double of a group algebra is nothing but $\mathrm{D}(\mathbb{C}G) = \mathbb{C}^G \otimes \mathbb{C}G$ with basis $\{\delta_x \otimes g \mid (x, g) \in G \times G\}$. The fact that the quantum double is a crossed product is reflected in the multiplication map

$$(\delta_x \otimes g)(\delta_y \otimes h) = \delta_{x,gyg^{-1}} \delta_x \otimes gh \tag{C.10}$$

where the algebra acts on itself by conjugation. Actually, this is fully determined by the straightening formula

$$g\delta_x \coloneqq (1 \otimes g)(\delta_x \otimes e) = \delta_{gxg^{-1}} \otimes g \tag{C.11}$$

(and the embeddings $\delta_x \mapsto \delta_x \otimes e$ and $g \mapsto 1 \otimes g$ being algebra morphisms). Here $g\delta_x$ is a frequently used shorthand notation and should be clear from context whenever it appears. The unit of $\mathrm{D}(\mathbb{C}G)$ is $1 \otimes e$. Comultiplication and counit are given by

$$\Delta(\delta_x \otimes g) = \sum_{uv=x} (\delta_v \otimes g) \otimes (\delta_u \otimes g), \tag{C.12}$$

$$\epsilon(\delta_x \otimes g) = \epsilon(g)\, \delta_x(e) = \delta_{x,e} \tag{C.13}$$

on the basis elements. Note that the comultiplication in $\mathrm{D}(\mathbb{C}G)$ is derived from the comultiplication in $\mathbb{C}G$ and $(\mathbb{C}^G)^{\mathrm{cop}} = ((\mathbb{C}G)^{\mathrm{op}})^*$ rather than from $\mathbb{C}^G$ itself! This

[1] If G is an Abelian group then $\delta_g^\star = \delta_{g^{-1}}$ defines another Hopf $*$-structure for the standard convention. In fact, for G non-Abelian the choice of $*$ or $\star$ corresponds to a choice of either the standard convention or the convention $(a \otimes b)^\star = b^\star \otimes a^\star$ for the action on tensor products.

is because as an algebra the quantum double is the bicrossed product $\mathrm{D}(\mathbb{C}G) = \left((\mathbb{C}G)^{\mathrm{op}}\right)^* \bowtie \mathbb{C}G$.

The antipode map is

$$S(\delta_x \otimes g) = \delta_{g^{-1}x^{-1}g} \otimes g^{-1}. \tag{C.14}$$

C.3 Hopf subalgebras of trivial Hopf algebras

The Hopf subalgebras of a Hopf algebra will be important for the construction of our hierarchy of Hopf tensor network states.

If $H = \mathbb{C}G$, all the Hopf subalgebras A and B can be characterized easily [3, 4]. One finds that the Hopf subalgebras of $\mathbb{C}G$ are precisely the group algebras $\mathbb{C}K$ for $K \subset G$ a subgroup. Furthermore the Hopf subalgebras of $\mathbb{C}^G$ are exactly given by $\mathbb{C}^{G/N}$ with $N \lhd G$ a normal subgroup. It is also not difficult to find the Haar integrals of these Hopf subalgebras. In the first case one has

$$h_K = \frac{1}{|K|} \sum_{g \in K} g. \tag{C.15}$$

For the second case note that $\mathbb{C}^{G/N}$ can be identified with the functions in $\mathbb{C}^G$ which are constant on cosets of N. Now consider the character χ_N of G that factors through N to the regular character of G/N. It is easy to see that when multiplied with the above functions χ_N acts as a two-sided integral. Hence taking normalization into account, one may convince oneself that $|G/N|^{-1}\chi_N$ is precisely the Haar integral of $\mathbb{C}^{G/N}$. By abuse of notation we may write it as:

$$\phi_N = \sum_{n \in N} \delta_n. \tag{C.16}$$

Appendix D

An example of a nontrivial finite-dimensional Hopf C^*-algebra

We are primarily interested in finite-dimensional nontrivial Hopf C^*-algebras, i.e. finite-dimensional nontrivial Hopf algebras over $\mathbb{C}$ with a compatible involution. In order to find an example of such a Hopf algebra it is enough to investigate semisimple Hopf algebras in low dimensions and check whether these support a compatible $*$-structure. It turns out that any semisimple Hopf algebra H of dimension p or pq with prime numbers $p < q$ is trivial [20]. The same applies to dimensions p^2 [52]. Hence one may expect nontrivial semisimple Hopf algebras only for $\dim H \geq 8$.

D.1 The Hopf algebra H_8

In fact, we need to look no further than $\dim H = 8$ for an example. At this dimension there exists a unique nontrivial semisimple Hopf algebra (up to isomorphism) which is commonly denoted by H_8 and is due to Kac and Paljutkin [34]. Interestingly, it can also be endowed with a compatible involution which turns H_8 into a Hopf C^*-algebra.

Following [51] this algebra can be presented by generators x, y, z and the relations

$$x^2 = y^2 = 1, \qquad z^2 = \frac{1}{2}(1 + x + y - xy), \tag{D.1}$$

$$xy = yx, \qquad zx = yz, \qquad zy = xz. \tag{D.2}$$

It is now easy to see that $\mathcal{B} = \{1, x, y, xy, z, zx, zy, zxy\}$ is a basis of H_8. We will denote the corresponding dual basis by $\{\delta_a \mid a \in \mathcal{B}\}$. The coalgebra structure is determined by

$$\Delta(x) = x \otimes x, \qquad \Delta(y) = y \otimes y, \tag{D.3}$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + y \otimes 1 + 1 \otimes x - y \otimes x)(z \otimes z), \tag{D.4}$$

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \tag{D.5}$$

and from this it is evident that $G(H_8) = \{1, x, y, xy\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The antipode reads

$$S(x) = x, \qquad S(y) = y, \qquad S(z) = z. \tag{D.6}$$

Finally, the involution is given by

$$x^* = x, \qquad y^* = y, \qquad z^* = z^{-1} = \frac{1}{2}(z + zx + zy - zxy) = z^3. \tag{D.7}$$

This shows in particular that antipode and involution act differently on the generators of H_8.

The Haar integrals $h \in H_8$ and $\phi \in H_8^*$ are given by:

$$h = \frac{1}{8}(1 + x + y + xy + z + zx + zy + zxy), \tag{D.8}$$

$$\phi = \delta_1. \tag{D.9}$$

Remarkably they look like the Haar integrals of a group algebra and its dual! The dual Haar integral ϕ and the complex involution automatically yield a Hermitian inner product on H_8 as discussed in Section 4.2.2. Explicitly, we have for all $a, b \in H_8$

$$(a, b) = \delta_1(a^* b). \tag{D.10}$$

It follows that $\mathcal{B}$ is an orthonormal basis of H_8.

D.2 Hopf subalgebras of H_8 and its dual

The Hopf subalgebras of $H = H_8$ read $\{1\}$, $\{1, x\} \simeq \mathbb{C}\mathbb{Z}_2$, $\{1, y\} \simeq \mathbb{C}\mathbb{Z}_2$, $\{1, xy\} \simeq \mathbb{C}\mathbb{Z}_2$, $\{1, x, y, xy\} \simeq \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and H_8. This is because Hopf subalgebras of a Hopf algebra H correspond bijectively to sets of irreducible representations of H^* which close under fusion and dualisation [3, 4]. Now H_8^* has four one-dimensional irreducible representations Q_1, Q_x, Q_y and Q_{xy} derived from $G(H_8)$ and the two-dimensional representation given by

$$Q_2 = \frac{1}{2}\begin{pmatrix} z + zx & z - zx \\ zy - zxy & zy + zxy \end{pmatrix}. \tag{D.11}$$

From the fusion rules

$$Q_g \times Q_{g'} = Q_{gg'} \tag{D.12}$$

$$Q_g \times Q_2 = Q_2 \tag{D.13}$$

$$Q_2 \times Q_2 = Q_1 + Q_x + Q_y + Q_{xy} \tag{D.14}$$

for all $g, g' \in G(H_8)$ one concludes that precisely the sets $\{Q_1\}$, $\{Q_1, Q_x\}$, $\{Q_1, Q_y\}$, $\{Q_1, Q_{xy}\}$, $\{Q_1, Q_x, Q_y, Q_{xy}\}$ and $\{Q_1, Q_x, Q_y, Q_{xy}, Q_2\}$ close under fusion (and dualization) and therefore are the ones that correspond bijectively to Hopf subalgebras of H_8.

The Haar integrals of the Hopf subalgebras read

$$h_{\{1\}} = 1, \tag{D.15}$$

$$h_{\langle x \rangle} = \frac{1}{2}(1 + x), \tag{D.16}$$

$$h_{\langle y\rangle} = \frac{1}{2}(1+y), \tag{D.17}$$

$$h_{\langle xy\rangle} = \frac{1}{2}(1+xy), \tag{D.18}$$

$$h_{\langle x,y\rangle} = \frac{1}{4}(1+x+y+xy), \tag{D.19}$$

$$h = \frac{1}{8}(1+x+y+xy+z+zx+zy+zxy). \tag{D.20}$$

Since $H_8^* \simeq H_8$ as Hopf algebras the Hopf subalgebras $B \subset H_8^*$ are given as the images of the Hopf subalgebras of H_8 under the isomorphism. A particular such Hopf isomorphism can be obtained as follows. From the dual basis $\{\delta_a\}$ of the basis $\mathcal{B} \subset H_8$ define another basis $\{f_a\} \subset H_8^*$ via

$$f_1 \coloneqq \delta_1 + \delta_x + \delta_y + \delta_{xy} + \delta_z + \delta_{zx} + \delta_{zy} + \delta_{zxy}, \tag{D.21}$$

$$f_x \coloneqq \delta_1 - \delta_x - \delta_y + \delta_{xy} + i\delta_z - i\delta_{zx} - i\delta_{zy} + i\delta_{zxy}, \tag{D.22}$$

$$f_y \coloneqq \delta_1 - \delta_x - \delta_y + \delta_{xy} - i\delta_z + i\delta_{zx} + i\delta_{zy} - i\delta_{zxy}, \tag{D.23}$$

$$f_{xy} \coloneqq \delta_1 + \delta_x + \delta_y + \delta_{xy} - \delta_z - \delta_{zx} - \delta_{zy} - \delta_{zxy}, \tag{D.24}$$

$$f_z \coloneqq \delta_1 - i\delta_x + i\delta_y - \delta_{xy} - i\sqrt{2}(\delta_{zx} - \delta_{zy}), \tag{D.25}$$

$$f_{zx} \coloneqq \delta_1 + i\delta_x - i\delta_y - \delta_{xy} + \sqrt{2}(\delta_z - \delta_{zxy}), \tag{D.26}$$

$$f_{zy} \coloneqq \delta_1 + i\delta_x - i\delta_y - \delta_{xy} - \sqrt{2}(\delta_z - \delta_{zxy}), \tag{D.27}$$

$$f_{zxy} \coloneqq \delta_1 - i\delta_x + i\delta_y - \delta_{xy} + i\sqrt{2}(\delta_{zx} - \delta_{zy}). \tag{D.28}$$

Then the map $H_8 \to H_8^*$ given by $a \mapsto f_a$ for all $a \in \mathcal{B}$ and linear extension is a Hopf isomorphism.

Consequently, the Haar integrals ϕ_B read:

$$\phi_{\{f_1\}} = \delta_1 + \delta_x + \delta_y + \delta_{xy} + \delta_z + \delta_{zx} + \delta_{zy} + \delta_{zxy} = \epsilon, \tag{D.29}$$

$$\phi_{\langle f_x\rangle} = \delta_1 + \delta_{xy} + \frac{1+i}{2}(\delta_z + \delta_{zxy}) + \frac{1-i}{2}(\delta_{zx} + \delta_{zy}), \tag{D.30}$$

$$\phi_{\langle f_y\rangle} = \delta_1 + \delta_{xy} + \frac{1-i}{2}(\delta_z + \delta_{zxy}) + \frac{1+i}{2}(\delta_{zx} + \delta_{zy}), \tag{D.31}$$

$$\phi_{\langle f_{xy}\rangle} = \delta_1 + \delta_x + \delta_y + \delta_{xy}, \tag{D.32}$$

$$\phi_{\langle f_x, f_y\rangle} = \delta_1 + \delta_{xy}, \tag{D.33}$$

$$\phi = \delta_1. \tag{D.34}$$

Appendix E

Hopf singlets

We collect here some lemmas, with their proofs, that are used throughout the text.

Lemma 6. *Let $h \in H$ the Haar integral, $a, b, c \in H$ and $f, g \in X$. Then one has*

$$\sum_{(h)} f\big(S(h'')\, b\big)\, g(ch''')\, ah' = \sum_{(a)(h)} f\big(S(h'')\, a''b\big)\, g\big(c\, S(a')\, h'''\big)\, h'. \tag{E.1}$$

Proof. It is clear that the statement will follow from

$$\sum_{(h)} ah' \otimes S(h'') \otimes h''' = \sum_{(a)(h)} h' \otimes S(h'')\, a'' \otimes S(a')\, h'''. \tag{E.2}$$

Indeed, we have

$$\begin{aligned}
\sum_{(h)} ah' \otimes S(h'') \otimes h''' &= \sum_{(a)(h)} a'h' \otimes S(h'')\, \epsilon(a'') \otimes h''' \\
&= \sum_{(a)(h)} a'h' \otimes S(h'')\, S(a'')\, a''' \otimes h''' \\
&= \sum_{(a)(h)} a^{(1)}h' \otimes S(a^{(2)}h'')\, a^{(4)} \otimes \epsilon(a^{(3)})\, h''' \\
&= \sum_{(a)(h)} a^{(1)}h' \otimes S(a^{(2)}h'')\, a^{(5)} \otimes S(a^{(4)})\, a^{(3)}h''' \\
&= \sum_{(a)} \sum_{(a'h)} (a'h)' \otimes S\big((a'h)''\big)\, a''' \otimes S(a'')\, (a'h)''' \\
&= \sum_{(a)(h)} \epsilon(a')\, h' \otimes S(h'')\, a''' \otimes S(a'')\, h''' \\
&= \sum_{(a)(h)} h' \otimes S(h'')\, a'' \otimes S(a')\, h'''.
\end{aligned}$$

□

Lemma 7. *Let $a, b \in H$ and $h \in H$ as well as $\phi \in X$ the respective Haar integrals. Then*

$$\sum_{(h)} \phi(ah''')\, \phi\big(b\, S(h'')\big)\, h' = |H|^{-1} \sum_{(a)} \phi(ba')\, S(a''). \tag{E.3}$$

Proof. Similarly to the proof of Lemma 6 one can first show that

$$\sum_{(h)} h' \otimes S(h'') \otimes a h''' = \sum_{(a)(h)} S(a'')\, h' \otimes S(h'')\, a' \otimes h'''$$

holds for any $a \in H$. Then we use the defining property (4.23) of the dual (Haar) integral:

$$\begin{aligned}
\sum_{(h)} \phi(ah''')\, \phi\big(b\, S(h'')\big)\, h' &= \sum_{(a)(h)} \phi(h''')\, \phi\big(b\, S(h'')\, a'\big)\, S(a'')\, h' \\
&= \sum_{(a)(h)} \phi\big[b\, S\big(h''\, \phi(h''')\big)\, a'\big]\, S(a'')\, h' \\
&= \sum_{(a)(h)} \phi\big(b\, S(1_H)\, a'\big)\, S(a'')\, h'\, \phi(h'') \\
&= \phi(h) \sum_{(a)} \phi(ba')\, S(a'').
\end{aligned}$$

□

Lemma 8. *Let $a, b, c, d \in H$ and $f, g \in X$. Furthermore let $h \in H$ and $\phi \in X$ the respective Haar integrals. Then*

$$\sum_{(h)} \phi(ah')\, f\big(b\, S(h'')\, c\big)\, g(dh''') = |H|^{-1} \sum_{(a)} f(ba''c)\, g\big(d\, S(a')\big). \tag{E.4}$$

Proof. From (E.2) we get

$$\begin{aligned}
\sum_{(h)} \phi(ah')\, f\big(b\, S(h'')\, c\big)\, g(dh''') &= \sum_{(a)(h)} \phi(h')\, f\big(b\, S(h'')\, a''c\big)\, g\big(d\, S(a')\, h'''\big) \\
&= \sum_{(a)(h)} f(ba''c)\, g\big(d\, S(a')\, \phi(h')\, h''\big) \\
&= \phi(h) \sum_{(a)} f(ba''c)\, g\big(d\, S(a')\big)
\end{aligned}$$

which proves the claim. □

Lemma 9. *Let $a, b, c, d \in H$ and $h \in H$ and $\phi \in X$ the respective Haar integrals. Then*

$$\sum_{(h)} \phi(h'a)\, \phi(bh'')\, \phi(ch'''d) = |H|^{-1} \sum_{(b)} \phi\big(S(b')\, a\big)\, \phi\big(c\, S(b'')\, d\big). \tag{E.5}$$

Proof. We first show that

$$\sum_{(h)} h' \otimes bh'' \otimes h''' = \sum_{(b)(h)} S(b')\, h' \otimes h'' \otimes S(b'')\, h'''.$$

Indeed, we have

$$\begin{aligned}
\sum_{(h)} h' \otimes bh'' \otimes h''' &= \sum_{(b)(h)} \epsilon(b')\, h' \otimes b''h'' \otimes h''' \\
&= \sum_{(b)(h)} S(b')\, b''h' \otimes b'''h'' \otimes h''' \\
&= \sum_{(b)(h)} S(b^{(1)})\, b^{(2)}h' \otimes b^{(3)}h'' \otimes \epsilon(b^{(4)})\, h''' \\
&= \sum_{(b)(h)} S(b^{(1)})\, b^{(2)}h' \otimes b^{(3)}h'' \otimes S(b^{(5)})\, b^{(4)}h''' \\
&= \sum_{(b)} \sum_{(b''h)} S(b')\, (b''h)' \otimes (b''h)'' \otimes S(b''')\, (b''h)''' \\
&= \sum_{(b)(h)} S(b')\, \epsilon(b'')\, h' \otimes h'' \otimes S(b''')\, h''' \\
&= \sum_{(b)(h)} S(b')\, h' \otimes h'' \otimes S(b'')\, h'''.
\end{aligned}$$

Again by property (4.23) of the dual (Haar) integral we deduce:

$$\begin{aligned}
\sum_{(h)} \phi(h'a)\, \phi(bh'')\, \phi(ch'''d) &= \sum_{(b)(h)} \phi\big(S(b')\, h'a\big)\, \phi(h'')\, \phi\big(c\, S(b'')\, h'''d\big) \\
&= \sum_{(b)(h)} \phi\big(S(b')\, a\big)\, \phi\big(c\, S(b'')\, \phi(h')\, h''d\big) \\
&= \phi(h) \sum_{(b)} \phi\big(S(b')\, a\big)\, \phi\big(c\, S(b'')\, d\big).
\end{aligned}$$

□

Lemma 10. *Let $a, b, c \in H$ and $h \in H$ as well as $\phi \in X$ the respective Haar integrals. Then*

$$\sum_{(h)} \phi(h''a)\, \phi(bh'''c)\, h' = |H|^{-1} \sum_{(a)} \phi\big(b\, S(a'')\, c\big)\, S(a'). \tag{E.6}$$

Proof. Along the same lines as the proof of Lemma 9 one can show that

$$\sum_{(h)} h' \otimes h''a \otimes h''' = \sum_{(a)(h)} h'\, S(a') \otimes h'' \otimes h'''\, S(a'').$$

As before the rest then follows from the properties of the dual Haar integral. □

Appendix F

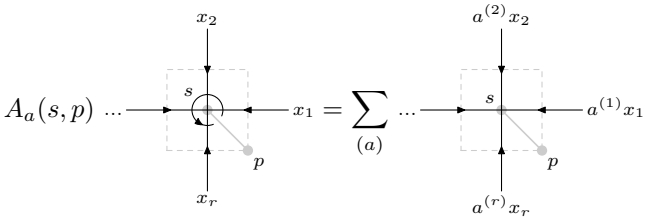
Proof of Theorem 7

First recall from Section 4.2.1 that both the face operator $B_f(s,p)$ and the alternative face operator $\tilde{B}_f(s,p)$ yield the same algebra action on a face of the graph. In that sense we may simple state

$$B_{f\in X}(s,p) = \tilde{B}_{f\in H^*}(s,p)\,. \tag{F.1}$$

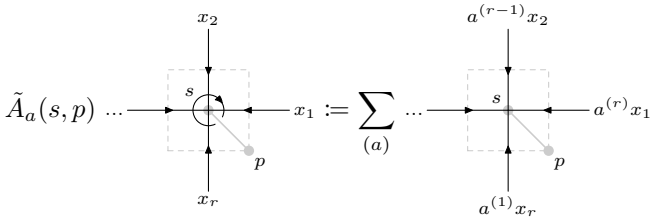
As indicated the subtle difference between the two lies in how the function f is interpreted: in the former case we have $f \in X = (H^{\mathrm{op}})^* = (H^*)^{\mathrm{cop}}$ while in the latter we have $f \in H^*$. This is reflected in the different orientations in which the arrows wind around the face p in (4.51) and (4.52) because the Hopf algebra X has exactly the *opposite* comultiplication of the Hopf algebra H^*. Needless to say, this will become important in any computation involving the coproduct of f.

By analogy, let us define an alternative vertex operator, too. While for $a \in H$ we have the vertex operator



(F.2)

as introduced in (4.50) assume now that $a \in H^{\mathrm{cop}}$ and set



(F.3)

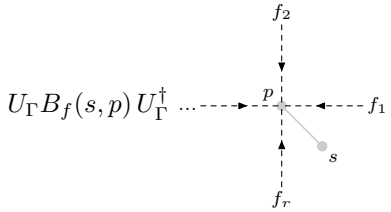
As far as only the algebra action is concerned there is no difference between the two operators so we may write

$$A_{a\in H}(s,p) = \tilde{A}_{a\in H^{\mathrm{cop}}}(s,p)\,. \tag{F.4}$$

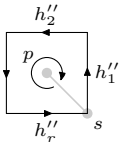
Now we turn to the actual proof of Theorem 7.

Proof. Evaluate for $f \in X$ the expression[1]

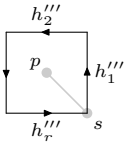
$$U_\Gamma B_f(s,p)\, U_\Gamma^\dagger$$

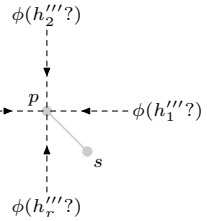


$$= |H|^{\frac{r}{2}}\, U_\Gamma B_f(s,p) \sum_{(h_i)} \langle f_1, S(h_1')\rangle \cdots \langle f_r, S(h_r')\rangle$$



$$= |H|^{\frac{r}{2}}\, U_\Gamma \sum_{(h_i)} \langle f_1, S(h_1')\rangle \cdots \langle f_r, S(h_r')\rangle \langle f, S(h_r'' \cdots h_1'')\rangle$$





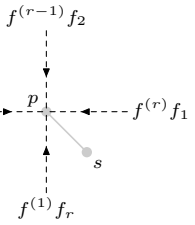
$$= |H|^{r} \sum_{(h_i)} \langle f_1, S(h_1')\rangle \cdots \langle f_r, S(h_r')\rangle \langle f, S(h_r'' \cdots h_1'')\rangle$$

$$= |H|^{r} \sum_{(h_i)} \sum_{(f)} \langle f_1, S(h_1')\rangle \langle f^{(r)}, S(h_1'')\rangle \cdots \langle f_r, S(h_r')\rangle \langle f^{(1)}, S(h_r'')\rangle$$
$$\times \phi(h_1'''?) \otimes \cdots \otimes \phi(h_r'''?)$$

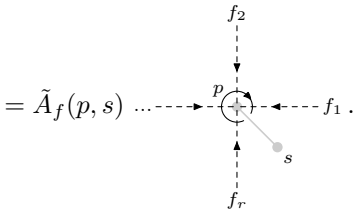
$$= |H|^{r} \sum_{(f)} \sum_{(h_i)} \langle f_1, h_1'''\rangle \langle f^{(r)}, h_1''\rangle \cdots \langle f_r, h_r'''\rangle \langle f^{(1)}, h_r''\rangle$$
$$\times \phi\big(S(h_1')\,?\big) \otimes \cdots \otimes \phi\big(S(h_r')\,?\big)$$

$$= |H|^{r} \sum_{(f)} \sum_{(h_i)} \langle f^{(r)} \cdot f_1, h_1''\rangle \cdots \langle f^{(1)} \cdot f_r, h_r''\rangle\, \phi\big(S(h_1')\,?\big) \otimes \cdots \otimes \phi\big(S(h_r')\,?\big)$$

$$= \sum_{(f)}$$



[1] Note that it does not make any difference whether we take $f_i \in X$ or $f_i \in H^*$ since comultiplication is never used on the f_i.



Here the penultimate line follows from

$$\sum_{(h)} f(h'')\,\phi\big(S(h')\,?\big) = |H|^{-1} f$$

as can one verify by evaluation on an arbitrary $x \in H$.

In summary we have proven that

$$U_\Gamma B_{f\in X}(s,p)\,U_\Gamma^\dagger = \tilde{A}_{f\in X}(p,s) = A_{f\in H^*}(p,s)\,, \tag{F.5}$$

i.e. the duality map takes any face operator of the $\mathrm{D}(H)$-model on the graph Γ to a vertex operator of the $\mathrm{D}(H^*)$-model on the dual graph Γ^*.

Similarly one proves the other assertion. □

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