The Singular Coagulation and Coagulation-Fragmentation Equations

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

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geb. am 13.12.1979 in Havanna, Kuba

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

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Eingereicht am: 10.10.2012
Verteidigung am: 10.07.2013

Acknowledgements

This thesis is the result of a very nice period of research in my life. But it was only possible with the help of many different persons whom I would like to thank.

First and foremost, I would like the express my sincere thanks to my supervisor, Prof. Dr. Gerald Warnecke for his guidance and support during my work at the Institute for Analysis and Numerics at the Otto-von-Guericke University in Magdeburg. He gave me the opportunity to reach one of my aims in life. He not only increased my mathematical knowledge, but also advised me about my life beside work.

I would like to thank the state of Saxony-Anhalt for funding my work through the International Max-Planck Research School (IMPRS) for Analysis, Design and Optimization in Chemical and Biochemical Process Engineering, Magdeburg. This thesis would not have been possible without that financial support.

Many thanks to my colleagues and friends from the Institute for Analysis and Numerics for all their help and support during this time. I especially would like to thank Robin Grpler and Ludwig Pulst for the interesting discussions about my dissertation topic and for their support and cooperation. A special thanks to my ex-colleague Bolaji James Adesokan, who was my guide and help for all the bureaucratic stuff at my arrival at Magdeburg. I am also very grateful for the support I received from Dr. Walfred Grambow and Stephanie Wernicke.

To Dr. Barbara Witter and Dr. Jrgen Koch my sincere thanks for the incredible work done as IMPRS coordinators. All the seminars, workshops and IMPRS activities have been very helpful for my formation.

A very special thanks goes out to my girlfriend Nadine Brunsendorf, who has been beside me in the most difficults moments regarding my adaptation to my new life in Germany. Your love and care is everything for me and there is no way to thank you for that. Thanks for your understanding of my work schedule and for pushing me to work harder, you have been my most valued support.

Thanks to my german teacher Kirsten Brcker for the first, second and third step in the German language. Thanks to my friends Dr.-Ing. Juan Miguel Vivar Peres and Dr. rer. nat. Jorge Ricardo Bergado Acosta for all the good times together.

I would like to give a huge thanks to my whole family. Especially to my parents Jos Manuel Cueto Daz and Hayde Camejo Moreno and my sisters Madeleyn and Patricia for all there support and for never stopping to believe in me. Whatever I am today is a result of my growing up with all of you togethers. Thanks to my parents for been my power and love, without you I would be nothing.

Finally, a big thanks to my grandparents Emelina Erotida Daz Lorenzo and Manuel Cueto Rivero for believing that through study is the only way to become rich. He used to say: "Study! That what you learn is the only thing nobody can ever take away from you." This work is dedicated to both of you.

Abstract

In this thesis we study the coagulation and the coagulation-fragmentation models, which were developed by Smoluchowski in 1917 as well as Blatz and Tobolsky in 1945 respectively. The models consist of an integro-differential equation.

The existence and uniqueness of solutions to the caogulation equation has been the target of much mathematical research. But, to our knowledge, the coagulation equation with singular kernels has been studied just in the case of self similar solutions. In this work we present an existence result to the coagulation equation with kernels, which have singularities on the axes. We cover in our work the non-singular kernels, which we have found in the previous literature. As a base of our proof, we use weighted L^1-spaces to deal with the singularities. The important Smoluchowski kernel is covered by our result. A weak L^1 compactness method is applied to a suitably chosen approximating equation as a base of our proof. The uniqueness of solutions question is also answered in our work. In order to do that, we need more restrictive conditions on the kernels. The uniqueness result can be obtained by taking the difference of two solutions and showing that this difference is equal to zero.

The existence and uniqueness of solutions to the coagulation-fragmentation equation has also been object of many studies. Nevertheless, the case of multifragmentation with singular kernels has not been approached. In this work we also present an existence and uniqueness result to the coagulation equation with multifragmentation, where the coagulation kernels have singularities on the axes. It is important to point out that there is no previous result concerning the coagulation-fragmentation equation with singular kernels. The Smoluchowski kernel is also covered by this approach. As above, as a base of our proof, we use weighted L^1-spaces to deal with the singularities. A weak L^1 compactness method is applied to a suitably chosen approximating equation as a base of our proof. We again need to impose more restrictive conditions on the kernels in order to get the uniqueness result.

Zusammenfassung

In dieser Doktorarbeit studieren wir die Koagulation- und Koagulation-Fragmentation-Modelle, die von Smoluchowski in 1917 beziehungsweise Blatz und Tobolsky in 1945 entwickelt wurden. Die Modelle bestehen aus einer Integrodifferentialgleichung.

Die Existenz und Eindeutigkeit von Lsungen der Koagulationsgleichung ist das Ziel von verschiedenen mathematischen Forschungen gewesen. Aber nach unserem Wissen wurde die Koagulationsgleichung mit singulren Kernen nur im Falle selbsthnlicher Lsungen untersucht. In dieser Arbeit prsentieren wir ein Existenzresultat der Koagulationsgleichung mit Kernen, die Singularitten auf den Achsen haben. Wir haben auch in unserer Arbeit die nicht-singulren Kerne umfasst, die in der bisherigen Literatur untersucht wurden. Als Basis unseres Beweises verwenden wir gewichtete L^1-Rume, um mit den Singularitten umzugehen. Der wichtige Smoluchowski-Kern wird durch unser Ergebnis abgedeckt. Eine schwache L^1-Kompaktheitsme- thode wird auf geeignet gewhlte angenherte Gleichungen angewendet als Basis unseres Beweises. Die Frage der Eindeutigkeit von Lsungen wird auch in unserer Arbeit beantwortet. Um dies zu tun, brauchen wir eine restriktivere Bedingung an die Kerne. Das Eindeutigkeitsresultat kann erhalten werden durch Bildung der Differenz der beiden Lsungen um zu zeigen, dass diese Differenz gleich null ist.

Die Existenz und Eindeutigkeit von Lsungen der Koagulation-Fragmentationsgleichung ist auch Gegenstand von vielen Studien gewesen. Dennoch wurde der Fall der mehrfachen Fragmentation noch nicht angegangen. In dieser Arbeit prsentieren wir auch ein Existenz- und Eindeutigkeitsresultat der Koagulationsgleichung mit mehrfacher Fragmentation, bei der die Koagulationskerne Singularitten auf den Achsen haben. Es ist wichtig, darauf hinzuweisen, dass es kein frheres Ergebnis bezglich der Koagulation-Fragmentationsgleichung mit singulren Kernen gibt. Der Smoluchowski-Kern wird auch durch diesen Ansatz abgedeckt. Wie oben erwhnt wurde, verwenden wir als Basis fr unseren Beweis gewichtete L^1-Rume, um mit den Singularitten umzugehen. Eine schwache L^1-Kompaktheitsmethode wird auf geeignet gewhlte angenherte Gleichungen als Basis unseres Beweises angewendet. Wir mssen wieder restriktivere Bedingungen an die Kerne stellen, um das Eindeutigkeitsresultat zu bekommen.

Contents

Chapter 1

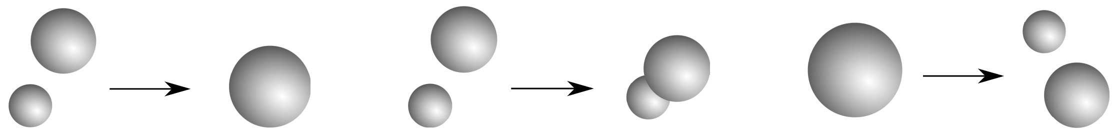
Introduction

In this chapter we give a general introduction to the coagulation equation and to the coagulation-fragmentation equation. We describe the models together with the terminologies and notation used and briefly mention the new results of our work. We review some of the existing results. At the end we outline the structure of the thesis.

1.1 Overview

Certain problems in the physical sciences are governed by the coagulation and the coagulation-fragmentation equations. These equations are a type of integro-differential equations which are also known as aggregation and aggregation-breakage equations respectively. The coagulation (aggregation) term describes the kinetics of particle growth where particles can coagulate (aggregate) to form larger particles via binary interaction. On the other side, the fragmentation (breakage) term describes how particles break into two or more fragments. The term aggregation covers two processes, the coagulation and agglomeration process. The coagulation process is when particles aggregate forming a new particle where it is not posible to define them in the new particle. The agglomeration process is when particles aggregate and it is posible to define them in the new particle. The coagulation and agglomeration processes are often found in liquid and solid substance respectively. Mathematically the two processes are described by the same equation, therefore we will refer to it as coagulation. In breakage and fragmentation process there is not difference, these are just two different ways to name the process, see Figure 1.1. In many applications, the size of a particle is considered as the only relevant particle property. If we consider the size of a particle by its mass, we have that during the coagulation process the total number of particles drecreases while by the fragmentation process the total number of particles increases. In the coagulation process as well as in the fragmentation process the total mass remains constant. Examples of these processes can be found e.g. in astrophysics [30], in chemical and process engineering [29], polymer science [43], and aerosol science [7], [32], [33].

Let the non-negative variables i and t represent the size of some particles and time respectively. By $u_i(t)$ we denote the number density of particles with size i at time t. The rate at which particles of size i coalesce with particles of size j is represented by the coagulation kernel $K_{i,j}$, which is assumed to be non-negative and symmetric, i.e. $K_{i,j} \geq 0$ and $K_{i,j} = K_{j,i}$ for $i, j \geq 1$.



(a) Coagulation process. (b) Agglomeration process. (c) Fragmentation process.

Figure 1.1: The aggregation and fragmentation process.

Now, we want to study how the number density $u_i(t)$ change. We have that when the particles $u_j(t)$ of size $j = 1, 2, \dots, i-1$ coalesce with the particles $u_{i-j}(t)$ of size $i-j$ at the rate $K_{i-j,j}$ we obtain new particles of size i, see Figure 1.2. In this way by the law of mass action the number density $u_i(t)$ increases by

$$\frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j} u_{i-j}(t) u_j(t).$$

The factor $\frac{1}{2}$ comes to avoid double counting.

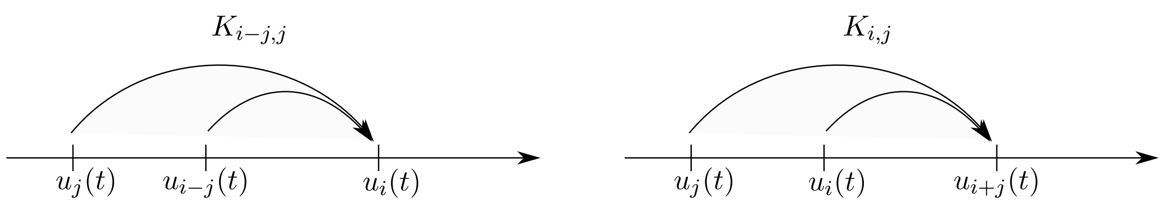


Figure 1.2: Agglomeration terms.

In the same way, if particles $u_i(t)$ of size i coalesce with particles $u_j(t)$ of size $j = 1, 2, \dots, \infty$ at the rate $K_{i,j}$ we get new particles with size $i + j$, see Figure 1.2, decreasing the number density $u_i(t)$ by

$$\sum_{j=1}^{\infty} K_{i,j} u_i(t) u_j(t).$$

The general coagulation equation is now given by

$$\frac{d}{dt} u_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j} u_{i-j}(t) u_j(t) - u_i(t) \sum_{j=1}^{\infty} K_{i,j} u_j(t), \tag{1.1}$$

where the first term is known as *birth term* and the second one as *death term*.

The first coagulation equation was formulated by Smoluchowski (1917) [34] in a discrete form in order to describe the coagulation of colloids moving according to a Brownian motion. This equation and its' integral form are also known as *Smoluchowski coagulation equation*.

In 1928, Mller [26] rewrote the Smoluchowski coagulation equation to the continuous integral form

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy, \tag{1.2}$$

where the non-negative variables x and t represent the size of some particles and time respectively. By $u(x,t)$ we denote the number density of particles with size x at time t. The rate at which particles of size x coalesce with particles of size y is represented by the coagulation kernel $K(x,y)$, which is assumed to be non-negative and symmetric, i.e. $K(x,y) \geq 0$ and $K(x,y) = K(y,x)$ for $x, y \geq 0$.

In 1945 Blatz and Tobolsky [5], in order to study the kinetic of systems manifesting simultaneous polymeration and depolymeration phenomena, extended the Smoluchowski coagulation equation to the following coagulation-fragmentation equation

$$\frac{d}{dt}u_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j}u_j(t)u_{i-j}(t) - u_i(t)\sum_{i=1}^{\infty} K_{ij}u_j(t) + \sum_{j=1}^{\infty} F_{ij}u_{i+j}(t) - \frac{1}{2}\sum_{j=1}^{i-1} F_{i-j,j}u_j(t). \tag{1.3}$$

The two first terms on the right hand side of (1.3) represent the coagulation terms as in (1.1). The third term represents the fragmentation birth term of a particle of size $i+j$ breaking into two particles of sizes i and j at a rate $F_{i,j}$ respectively. The last term represents the fragmentation death term of particles of size i breaking into two particles of size $i-j$ and j at a rate $F_{i-j,j}$ respectively, see Figure 1.3.

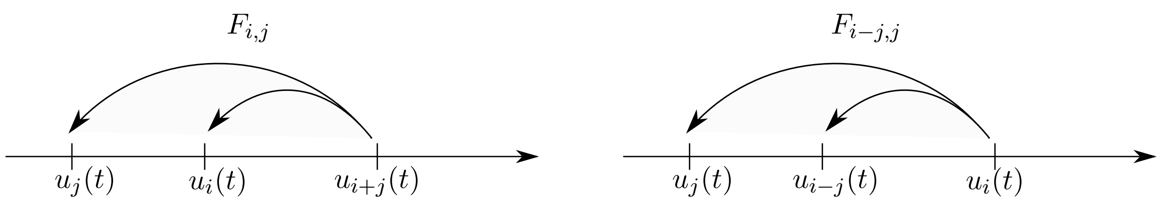


Figure 1.3: Fragmentation terms.

The equivalent continuous coagulation-fragmentation equation to (1.3) is given by

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = {} & \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy \\ & -\frac{1}{2}\int_0^x F(x-y,y)u(x,t)\,dy + \int_0^\infty F(x,y)u(x+y,t)dy.\end{aligned} \tag{1.4}$$

As in (1.2) the first two terms in the right hand side of (1.4) represent the birth and death terms due to coagulation. The last two terms are respectively the death and birth terms due to fragmentation, where $F(x,y)$ represents the binary fragmentation kernel. It is also considered to be symmetric and non-negative, i.e. $F(x,y) \geq 0$ and $F(x,y) = F(y,x)$ for $x, y \geq 0$.

The moments of the number density distribution are important quantities. The p-th moment of the number density distribution $u(x,t)$, if it exists, is defined by

$$M_p(t) = \int_0^\infty x^p u(x,t)\,dx.$$

The zeroth moment ($p = 0$) gives the total number of particles at time t and the first moment gives the total mass or total volume of particles at time t if our size property is mass or volume respectively.

Now, we consider the non-negative variables x and t representing the size of the particles and time respectively. The values $u(x,t)$ denote the number density of particles with size x at time t. The rate at which particles of size x coalesce with particles of size y is represented by the coagulation kernel $K(x,y)$. The rate at which particles of size x are selected to break is determined by the selection function $S(x)$. The breakage function $b(x,y)$ gives the number of particles of size x produced when a particle of size y breaks up. It is non-zero only for $x < y$. The formation rate of particles of mass y due to the fragmentation of particles of mass x is given by the multifragmentation kernel $\Gamma(x,y)$.

Then we can write the coagualtion equation with multifragmentation as follows

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy \\ &\quad + \int_x^\infty b(x,y)S(y)u(y,t)\,dy - S(x)u(x,t), \end{aligned} \tag{1.5}$$

where the multifragmentation kernel Γ defines the selection function S and the breakage function b by

$$S(x) = \int_0^x \frac{y}{x}\Gamma(x,y)dy, \qquad b(x,y) = \Gamma(y,x)/S(y), \tag{1.6}$$

or vice versa.

The breakage function is assumed here to have the following properties

$$\int_0^y x b(x,y)dx = y \quad \text{for all} \quad y > 0, \tag{1.7}$$

which is the conservation of mass and

$$\int_0^y b(x,y)dx = N < \infty \quad \text{for all} \quad y > 0, \quad \text{and} \quad b(x,y) = 0 \quad \text{for} \quad x > y, \tag{1.8}$$

where the parameter N represents the number of particles produced in fragmentation events. In this work N is assumed to be finite and independent of y. We consider $N > 1$. Equation (1.7) allows the system to conserve the total mass during the fragmentation events. It states that the total mass of the fragments is equal to the mass y of the particle that breaks.

Now, setting

$$S(x) = \int_0^x F(y, x-y)\,dy \quad \text{and} \quad b(x,y) = F(x, y-x)/S(y)$$

we can see the coagulation-fragmentation equation (1.4) as a particular case of the coagulation equation with multifragmentation (1.5). The binary fragmentation kernel F is assumed to be symmetric.

1.2 Previous and new results

When a model is studied from the mathematical point of view there are always at least two questions to answer: Does it have a solution? If it has, is the solution unique? The aim of this work is to present some results related to the existences and uniqueness of solutions to the coagulation and the coagulation equation with multifragmentation. Below, we recall some previous results and present briefly our new results concerning these two question.

1.2.1 The coagulation equation

There are many previous results related to the existence and uniqueness of solutions to the different forms of the coagulation equation for non-singular kernels, see e.g. [11], [23], [25]. But to our knowledge there are few works on Smoluchowski's coagulation equation with singular kernels, see e.g. [9], [10], [28]. Fournier and Laurençot [10] proved the existence of self similar solutions to the Smoluchowski coagulation equation with homogeneous kernels, while Escobedo and Mischler [9] gave some regularity and size properties of the self similar profiles. These special solutions are not a topic of this work. Norris [28] proved the existence of weak solutions in property space that are local in time to the Smoluchowski equation when the kernel is estimated by the product of sublinear functions, i.e.

$$K(x,y) \le \varphi(x)\varphi(y) \quad \text{with} \quad \varphi : E \to [0,\infty[,\ \varphi(\lambda x) \le \lambda\varphi(x) \quad \text{for all} \quad x \in E, \lambda \ge 1.$$

In this work we present a proof of an existence theorem of solutions to the Smoluchowski coagulation equation (2.1) for the following class of singular kernels

$$K(x,y) \le k(1+x+y)^{\lambda}(xy)^{-\sigma}, \qquad \lambda - \sigma \in [0,1[,\ \sigma \in [0,1/2]. \tag{1.9}$$

Our result is much stronger than the result of Norris because the solutions he obtained are weak solutions in space and time while our solutions are *strong* solutions that lie in the space $C_B\left([0,\infty[, L^1(]0,\infty[)\right)$.

We would also like to point out that the solutions obtained in Norris [28] are conservative if $\varphi(x) \geq \varepsilon x$ for all x and some $\varepsilon > 0$ and

$$\int_0^\infty \varphi^2(x)\mu_0(dx) < \infty. \tag{1.10}$$

These two conditions together mean that he needs at least to bound the second moment to have conservative solutions. We just need the ζ-moment bound, with $\zeta = 1 + \lambda - \sigma$ which is a lower moment. Further, we can prove for initial data in our weighted L^1 spaces, that the solution remains in such a space while the result of Norris gives a solution in the space of measures. So our existence result is in some sense less general but more precise.

A key ingredient for our existence theorem is the use of specific weighted L^1-spaces. In [10], Fournier and Laurençot obtained their existence result in the weighted space $L^1([0,\infty[; x\,dx)$. For our result we introduce the weighted space $L^1\left([0,\infty[; x^{-1} + x\,dx\right)$. We are giving a more general result than [10] since we do not restrict ourselves to self similar solutions. Also in [10] they just considered the equality for the kernels

$$\begin{aligned}
&K_1(x,y) = (x^\alpha + y^\alpha)(x^{-\beta} + y^{-\beta}), && \alpha \in [0,1[,\ \beta \in]0,\infty[,\\
&K_2(x,y) = (x^\alpha + y^\alpha)^\beta, && \alpha \in [0,\infty[,\ \beta \in]0,\infty[,\ \alpha\beta \in [0,1[,\\
&K_3(x,y) = x^\alpha y^\beta + x^\beta y^\alpha, && \alpha \in]0,1[,\ \beta \in]0,1[,\ \alpha+\beta \in [0,1[,
\end{aligned} \tag{1.11}$$

which are included in our class of kernels. The K_1 kernels case is covered by our result just for $\alpha \in [0,1[$ and $\beta \in [0,1/2]$. For the others two kernels we have a different parameter range. For details see the end of Section 2.1. The uniqueness problem is also studied and we obtain uniqueness for a more restricted class of kernels than (1.9).

Our result is obtained in a suitable weighted Banach space of L^1 function for kernels with singularities on the axes, covering in this way the important Smoluchowski coagulation kernel

$$K(x,y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3})$$

for Brownian motion, see Smoluchowski [34]. This kernel is one of the few kernels used in applications that is derived from fundamental physics and not just by ad hoc modeling. The equi-partition of kinetic energy (EKE) kernel

$$K(x,y) = (x^{1/3} + y^{1/3})^2 \sqrt{\frac{1}{x} + \frac{1}{y}},$$

and the granulation kernel

$$K(x,y) = \frac{(x+y)^a}{(xy)^b},$$

see Kapur [17], are also covered by our analysis. These kernels were not included in the results of Fournier and Laurençot [11]. Our paper can be viewed as a completion of this work of Founier and Laurençot.

It is important to point out that our result is also valid in the weighted space $L^1\left([0,\infty[; x^{-2\sigma} + x\,dx\right)$ which in the case of nonsingular kernels, i.e. $\sigma = 0$, becomes $L^1([0,\infty[; x\,dx)$.

Our approach is based on the well known method by Stewart [36] for non-singular kernels. However, it turned out that our modification using weighted L^1-spaces was not always straight forward. Stewart in his method defined a sequence of truncated problems. He proved the existence and uniqueness of solutions to them. Using weak compactness theory, he proved that this sequence of solutions converges to a certain function. Then it is shown that the limiting function solves the original problem. In our approach we redefine Stewart's truncated problem in order to eliminate the singularities of the kernels. Using the contraction mapping principle we prove that our truncated problems have a unique solution. We construct a singular sequence around the origin to deal with the singularities of the kernels and prove that this sequence and the sequence of solutions to the truncated problems are weakly relatively compact and equicontinuous in time by using the Dunford-Pettis and Arzela-Ascoli Theorem repectively. These properties of the sequence are later used to prove that the sequence of solutions to the truncated problem converges to a solution of our original problem. In that way we obtain the existence of solutions to the coagulation equation with singular kernels. The uniqueness result can be obtained as in Stewart [37] by taking the difference of two solutions and showing that this difference is equal to zero by appliying Gronwall's inequality. Unfortunately the equi-partition of kinetic energy kernel is not covered by our uniqueness result.

1.2.2 The coagulation equation with multifragmentation

The existence and uniqueness of solutions to the coagulation-fragmentation equation has already been the subject of several papers. The case of multifragmentation, that is, when the particules can break into two or more parts, has also been studied, see e.g. [19], [22], [24], and [39]. For more recent result see e.g. [2], [3], [13] and [14]. Giri et al.[13] studied the coagulation kernels of the form $K(x,y) = \phi(x)\phi(y)$ for some sublinear function ϕ under the growth restriction $\phi(x) \leq (1+x)^{\mu}$ for $0 \leq \mu < 1$, and the selection function $S(x)$ is there also considered under the same growth assumption. In [14], Giri et al. proved the existence of solutions to the coagulation equation with multifragmentation for a more general fragmentation kernel, in order to cover the fragmentation kernel $\Gamma(y,x) = (\alpha+2)x^{\alpha}y^{\gamma-(\alpha+1)}$ getting a result for $\alpha > -1$ and $\gamma \in]0, \alpha+2[$. The existence proofs in [14] and [13] are based on the well known basic method by Stewart [36], where the solution is obtained through the convergence of the solutions to a sequence of truncated problems. In [14] the uniqueness of the solutions was not studied. In [2] Banasiak and Lamb proved the existence and uniqueness of solutions to the coagulation-fragmentation equation when $K(x,y) \in L_{\infty}(\mathbb{R}_+ \times \mathbb{R}_+)$ while in [3] the authors proved existence and uniqueness of classical solutions for the class of coagulation kernels $K(x,y) \leq k\big((1+a(x))^{\alpha} + (1+a(y))^{\alpha}\big)$ where a is the fragmentation rate, $k > 0$, and $0 \leq \alpha < 1$.

To our knowledge there is just one result concerning the coagulation mutlifragmentation equation with singular coagulation kernels. Cañizo Rincón [6] proved the existence of $L^{\infty}\big([0,T[, M_1\big)$ solutions in the distribution sense for the coagulation kernels $a(y,y')$ such that

$$K_a\big(y^{\alpha}(y')^{\beta} + (y')^{\alpha}y^{\beta}\big) \leq a(y,y') \leq K_a'\big(y^{\alpha}(y')^{\beta} + (y')^{\alpha}y^{\beta}\big)$$

with $\alpha < \beta < 1$, $0 < \alpha+\beta < 1$, $\beta - \alpha < 1$, constants $K_a, K_a' > 0$, and where M_1 is the space of measures μ on $(0,\infty)$ with first bounded moment, see [6, Section 3.2] or [6, page 59]. His result is resticted to the kernels with order α and β in y and y' respectively, and being $\alpha \neq \beta$. The singularity is restricted to the case $\sigma \in]0, 1/2[$ translated into our terms. The result from [6] leaves out, for example, the cases of the Smoluchowski and the equi-partition of kinetic energy kernels. The uniqueness of the solutions was not studied in [6].

In the present article, our aim is to prove the existence and uniqueness of solutions to the coagulation equation with multifragmentation with singular coagulation kernels

$$K(x,y) \leq k(1+x)^{\lambda}(1+y)^{\lambda}(xy)^{-\sigma} \quad \text{with} \quad \lambda-\sigma \in [0,1[, \sigma \geq 0, \tag{1.12}$$

giving in this way an existence and uniqueness result for the case of the important Smoluchowki coagulation kernel

$$K(x,y) = (x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$$

for Brownian motion, see Smoluchowski [34]. The equi-partition of kinetic energy (EKE) kernel

$$K(x,y) = (x^{1/3}+y^{1/3})^2\sqrt{\frac{1}{x}+\frac{1}{y}},$$

is also covered by our analysis. We are giving a more general result than Cañizo Rincón [6] since we do not restrict our kernel to an specific order. We allow α to be equal β, the singularity can be as big as it is wished, and we obtain *strong* solutions in the space $C_B\left([0,\infty[, L^1[0,\infty[\right)$.

Our existence result is based on the proof of Stewart [36]. We extend the methods we developed in Chapter 2 for singular kernels in the pure coagulation problem.

For our existence and uniqueness result we consider the class of fragmentation kernels

$$\Gamma(y,x) \leq y^{\theta} b(x,y) \tag{1.13}$$

with

$$\int_0^y b(x,y)x^{-2\sigma} \leq Cy^{-2\sigma} \quad \text{for} \quad \theta \in [0,1[, \sigma \in [0,1/2], \text{ and a constant } C \tag{1.14}$$

and such that, there exist $q>1$ and $\tau_1, \tau_2 \in [-2\sigma-\theta, 1-\theta]$ such that

$$\int_0^y b^q(x,y) \leq B_1 y^{q\tau_1}, \text{ and } \int_0^y x^{-q\sigma} b^q(x,y) \leq B_2 y^{q\tau_2} \quad \text{for constant } B_1, B_2 > 0. \tag{1.15}$$

From (1.6) and (1.13) we have that $S(y) \leq y^{\theta}$. The case $S(y) = y^{\theta}$ with $\theta > 0$ was considered in [21], where McGuinness et al. studied the pure fragmentation equation with singular initial conditions. The selection function $S(y) = y^{\theta}$ has also been studied in [18], [20], and [41]. In [42] it has been considered for $\theta = 0$.

The class of frangmentation kernels (1.13) holding (1.14) and (1.15) includes the kernel $\Gamma(y,x) = (\alpha+2)x^{\alpha}y^{\gamma-(\alpha+1)}$ for $\alpha > 2\sigma+\epsilon-1$ and $\gamma \in]0,1[$ with $0<\epsilon<\theta$. This kernel was studied by Giri et al. [14], where they proved the existence of weak solutions to the coagulation equation with multifragmentation, but with nonsingular coagulation kernels.

1.3 Outline of contents

The thesis is organized as follows. In Chapter 2 we study the existence and uniqueness of solutions to the coagulation equation with singular kernels. In Section 2.1 we present the hypotheses for our problem and some necessary definitions. We prove in Theorem 2.2.4 the existence and uniqueness of solutions to the truncated problem and we extract a weakly convergent subsequence in L^1 from a sequence of unique solutions for truncated equations to (2.1)-(2.2). Next, we show that the solution of (2.1) is actually the limit function obtained from the weakly convergent subsequence of solutions of the truncated problem. At the end of the chapter, we prove the uniqueness, based on methods of Stewart [37], of the solutions to (2.1)-(2.2) for a modification of the class (1.9) of kernels. We obtain uniqueness for some kernels which are not covered by the existence result.

In Chapter 3 we study the existence and uniqueness of solutions to the singular coagulation equation with multifragmentation. We define the sequence of truncated problems and prove in Theorem 3.2.4 the existence and uniqueness of solutions to them. We extract a weakly convergent subsequence in L^1 from a sequence of unique solutions for truncated equations to (3.1)-(3.2). Next, we show that the solution of (3.1) is actually the limit function obtained from the weakly convergent subsequence of solutions of the truncated problem. In Section 3.4 we prove the uniqueness, based on the method of Stewart [37], of the solutions to (3.1)-(3.2) for a modification of the classes (1.12) and (1.13) of coagulation and fragmentation kernels respectively.

The work in Chapter 2 and the work in Chapter 3 are submitted to two journals for publication. These papers consist of excerpts of this thesis.

Chapter 2

The coagulation equations with singular kernels

In this chapter we deal with our result on existence and uniqueness of solutions to the continuous coagulation equations with singular kernels. Weak L^1 compactness methods are applied to suitably chosen approximating equations as a base of our proof. Our result is obtained in a suitable weighted Banach space of L^1 functions

$$Y^+ = \left\{ u \in L^1 : \int_0^\infty \left(x + x^{-1}\right) |u| dx < \infty,\, u > 0 \text{ a.e.} \right\}$$

for non-negative initial data $u_0 \in Y^+$. The main novelty of the result is that it includes kernels with singularities on the axes.

The chapter is organized as follows. In Section 2.1 we present the hypotheses for our problem and some necessary definitions. In Section 2.2 we prove in Theorem 2.2.4 the existence and uniqueness of solutions to the truncated problem and we extract a weakly convergent subsequence in L^1 from a sequence of unique solutions for truncated equations to (2.1)-(2.2). In Section 2.3 we show that the solution of (2.1) is actually the limit function obtained from the weakly convergent subsequence of solutions of the truncated problem. In Section 2.4 we prove the uniqueness, based on methods of Stewart [37], of the solutions to (2.1)-(2.2). We obtain uniqueness for some kernels which are not covered by the existence result.

2.1 Weak solutions and weighted L^1-spaces

Let the non-negative variables x and t represent the size of some particles and time respectively. By $u(x,t)$ we denote the number density of particles with size x at time t. The rate at which particles of size x coalesce with particles of size y is represented by the coagulation kernel $K(x,y)$. Then we recall the nonlinear continuous coagulation equation (1.2) from Chapter 1:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy. \tag{2.1}$$

The equation (2.1) is considered for some given initial data $u_0(x) \geq 0$, i.e. we consider the initial condition

$$u(x,0) = u_0(x) \geq 0 \quad \text{a.e.} \tag{2.2}$$

In order to study the existence of solutions of (2.1)-(2.2), we define Y to be the following space

$$Y = \big\{u \in L^1\big(]0,\infty[\big) : \|u\|_Y < \infty\big\}$$

with norm

$$\|u\|_Y = \int_0^\infty (x + x^{-1})|u(x,t)|dx. \tag{2.3}$$

Now, by taking the function $x\exp(-x)$ we find that

$$\int_0^\infty x\exp(-x)dx = 1 \quad \text{and} \quad \int_0^\infty (x + x^{-1})x\exp(-x)dx = 3,$$

which means that the space Y is not an empty space.

Lemma 2.1.1. *Y is a Banach space.*

Proof. In order to prove that Y is a Banach space, we need to show that every Cauchy sequence in Y converges to an element in Y.

Let u_n be a Cauchy sequence in Y. By defintion of the space Y we have that $(x^{-1} + x)u_n =: w_n$ is a Cauchy sequence in $L^1\big(]0,\infty[\big)$, but $L^1\big(]0,\infty[\big)$ is a Banach space and we get

$$w_n \to w \quad \text{in} \quad L^1\big(]0,\infty[\big).$$

Now, we define $u := (x^{-1} + x)^{-1}w$ and we show that u is in Y. As $w \in L^1\big(]0,\infty[\big)$ we find that

$$\|u\|_Y = \int_0^\infty (x^{-1} + x)|u|dx = \int_0^\infty (x^{-1} + x)(x^{-1} + x)^{-1}|w|dx < \infty.$$

By using $\int |fg| \leq \|f\|_\infty \|g\|_{L^1}$, $v \in L^1\big(]0,\infty[\big)$, and $(x^{-1} + x)^{-1} \in L^\infty\big(]0,\infty[\big)$ we show that $u \in L^1\big(]0,\infty[\big)$. Taking the L^1-norm of u we have

$$\int_0^\infty |u|\,dx = \int_0^\infty \big|(x^{-1} + x)^{-1}w\big|\,dx \leq \|(x^{-1} + x)^{-1}\|_\infty \|w\|_{L^1} < \infty.$$

Then, we have $u \in Y$.

Now, we prove that the sequence u_n converges to u in Y. Taking the norm of the difference between u_n and u we find that

$$\begin{aligned}\|u_n - u\|_Y &= \int_0^\infty (x^{-1}+x)|u_n - u|dx \\ &= \int_0^\infty \left|(x^{-1}+x)u_n - (x^{-1}+x)(x^{-1}+x)^{-1}w\right| dx \\ &= \int_0^\infty |w_n - w|dx \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence, we have that every Cauchy sequence in Y converge to an element in Y. □

We also write

$$\|u\|_x = \int_0^\infty xu(x,t)dx \quad \text{and} \quad \|u\|_{x^{-1}} = \int_0^\infty x^{-1}u(x,t)dx,$$

and set

$$Y^+ = \{u \in Y : u \geq 0 \quad a.e.\}.$$

We define a solution of problem (2.1)-(2.2) in the same way as Stewart [36]:

Definition 2.1.2. *Let $T \in]0,\infty[$. A solution $u(x,t)$ of (2.1)-(2.2) is a function $u : [0,T[\longrightarrow Y^+$ such that for a.e. $x \in]0,\infty[$ and $t \in [0,T[$ the following properties hold*

(i) *$u(x,t) \geq 0$ for all $t \in [0,\infty[$,*

(ii) *$u(x,\cdot)$ is continuous on $[0,T[$,*

(iii) *for all $t \in [0,T[$ the following integral is bounded*

$$\int_0^t \int_0^\infty K(x,y)u(y,\tau)\,dy\,d\tau < \infty,$$

(iv) *for all $t \in [0,T[$, u satisfies the following weak formulation of (2.1)*

$$u(x,t) = u(x,0) + \int_0^t \left[\frac{1}{2}\int_0^x K(x-y,y)u(x-y,\tau)u(y,\tau)\,dy - \int_0^\infty K(x,y)u(x,\tau)u(y,\tau)\,dy\right] d\tau.$$

In the next sections we make use of the following hypotheses

Hypotheses 2.1.3.

(H1) $K(x, y)$ *is a continuous non-negative function on* $]0, \infty[\times]0, \infty[$,

(H2) $K(x, y)$ *is a symmetric function, i.e.* $K(x, y) = K(y, x)$ *for all* $x, y \in]0, \infty[$,

(H3) $K(x, y) \leq \kappa(1 + x + y)^{\lambda}(xy)^{-\sigma}$ *for* $\sigma \in [0, 1/2]$, $\lambda - \sigma \in [0, 1[$, *and a constant* $\kappa > 0$.

In the rest of the chapter we consider $\kappa = 1$ for the simplicity.

We introduce now some easily derived inequalities that will be used throughout the chapter. The proof of these inequalities can be found in an appendix of Giri [12]. For any $x, y > 0$

$$2^{p-1}(x^p + y^p) \leq (x + y)^p \leq x^p + y^p \qquad \text{if} \qquad 0 \leq p \leq 1, \tag{2.4}$$
$$2^{p-1}(x^p + y^p) \geq (x + y)^p \geq x^p + y^p \qquad \text{if} \qquad p \geq 1, \tag{2.5}$$
$$2^{p-1}(x^p + y^p) \geq (x + y)^p \qquad \text{if} \qquad p < 0. \tag{2.6}$$

The inequalities (2.4) and (2.5) are also satified for $x, y \geq 0$. From the inequality (2.6) we have for $x, y \geq 0$

$$(x^p + y^p) \geq (x + y)^p \qquad \text{if} \qquad p < 0. \tag{2.7}$$

We show now, how the kernels (1.11) are included in the class of kernels we are considering. We rewrite the kernels K_1 in (1.11) as follows

$$K_1(x, y) = (x^{\alpha} + y^{\alpha})(x^{-\beta} + y^{-\beta}) = (x^{\alpha} + y^{\alpha})(x^{\beta} + y^{\beta})(xy)^{-\beta}.$$

Defining ϑ as

$$\vartheta = \begin{cases} 2^{1-\beta} & \text{if} \quad 0 \leq \beta \leq 1 \\ 1 & \text{if} \quad \beta \geq 1 \end{cases}$$

we have, by using (2.4) and (2.5)

$$\begin{aligned} K_1(x, y) = (x^{\alpha} + y^{\alpha})(x^{-\beta} + y^{-\beta}) &\leq 2^{1-\alpha}\vartheta(x + y)^{\alpha+\beta}(xy)^{-\beta} \\ &\leq 2^{1-\alpha}\vartheta(1 + x + y)^{\alpha+\beta}(xy)^{-\beta}. \end{aligned}$$

Then the kernels K_1 can be estimated as follows

$$K_1(x, y) = (x^{\alpha} + y^{\alpha})(x^{-\beta} + y^{-\beta}) \leq \kappa(1 + x + y)^{\lambda}(xy)^{-\sigma},$$

where $\kappa = 2^{1-\alpha}\vartheta$, $\alpha + \beta = \lambda$, and $\beta = \sigma$. In that way we can see that the kernels K_1 is considered in our study for $\alpha = \lambda - \sigma \in [0, 1[$ and $\beta = \sigma \in [0, 1/2]$. Working in a similar way with the kernels K_2 and K_3 we find that

$$K_2(x, y) = (x^{\alpha} + y^{\alpha})^{\beta} \leq 2^{1-\alpha}(1 + x + y)^{\alpha\beta} = \kappa(1 + x + y)^{\lambda}(xy)^{-\sigma} \tag{2.8}$$

for $\kappa = 2^{1-\alpha}$, $\lambda = \alpha\beta$, and $\sigma = 0$, and

$$\begin{aligned} K_3(x,y) = x^\alpha y^\beta + x^\beta y^\alpha &\leq x^{\alpha+\beta} + x^\alpha y^\beta + x^\beta y^\alpha + y^{\alpha+\beta} \\ &= (x^\beta + y^\beta)(x^\alpha + y^\alpha) \\ &\leq 2^{1-\beta} 2^{1-\alpha} (1+x+y)^{\alpha+\beta} \\ &= \kappa (1+x+y)^\lambda (xy)^{-\sigma}. \end{aligned} \tag{2.9}$$

for $\kappa = 2^{1-\beta}2^{1-\alpha}$, $\lambda = \alpha + \beta$, and $\sigma = 0$. From (2.8) and (2.9) we have that the kernels K_2 and K_3 are included in our result for $\alpha\beta \in [1/2, 3/2[$ and $\alpha + \beta \in [1/2, 3/2[$ respectively, which means that we cover partially the result of [10] for $\alpha\beta \in [1/2, 1[$ and $\alpha + \beta \in [1/2, 1[$. But our study includes the cases $\alpha\beta \in [1, 3/2[$ and $\alpha + \beta \in [1/2, 1[$ respectively which were not studied in [10] and we also do not restrict the values of α and β.

2.2 The truncated problem

We prove the existence of a solution to the problem (2.1)-(2.2), by taking the limit of the sequence of solutions of the equations given by replacing the kernel $K(x,y)$ by the 'cut-off' kernel $K_n(x,y)$ for any given $n \in \mathbb{N}$,

$$K_n(x,y) = \begin{cases} K(x,y) & \text{if} \quad x+y \leq n \quad \text{and} \quad x,y \geq 1/n \\ 0 & \text{otherwise.} \end{cases}$$

The resulting equations are written as

$$\frac{\partial u^n(x,t)}{\partial t} = \frac{1}{2}\int_0^x K_n(x-y,y)u^n(x-y,t)u^n(y,t)\,dy - \int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)\,dy, \tag{2.10}$$

with the truncated initial data

$$u_0^n(x) = \begin{cases} u_0(x) & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{otherwise,} \end{cases} \tag{2.11}$$

where u^n denotes the solution of the problem (2.10)-(2.11) for $x \in [0, n]$. Next, we rewrite our truncated problem (2.10)-(2.11) in an equivalent form. We prove three lemmas, which are used to show the existence and uniqueness of the solution to the truncated problem.

2.2.1 Existence and uniqueness of solutions of the truncated problem

We rewrite the truncated problem (2.10)-(2.11) in the equivalent form

$$\frac{\partial}{\partial t}\left[u^n(x,t)\exp\big(P(x,t,u^n)\big)\right] = \frac{1}{2}\exp\big(P(x,t,u^n)\big)\int_0^x K_n(x-y)u^n(x-y,t)u^n(y,t)\,dy \tag{2.12}$$

$$u_0^n(x) = \begin{cases} u_0(x) & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

where

$$P(x,t,u^n) = \int\limits_0^t \int\limits_0^{n-x} K_n(x,y)u^n(y,\tau)\,dy\,d\tau.$$

Let us define the operator G as

$$G(c)(x,t) = u_0^n(x)\exp\big(-P(x,t,c)\big) + \frac{1}{2}\int\limits_0^t \exp\big(-[P(x,t,c)-P(x,\tau,c)]\big) \cdot \int\limits_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy\,d\tau, \quad (2.14)$$

for $c \in C\big([0,T];L^1\big(]0,n[\big)\big)$. Using (2.12) and (2.14), we can easily check that a solution u^n to (2.10)-(2.11) satisfies

$$u^n(x,t) = G(u^n)(x,t). \quad (2.15)$$

The problem (2.10)-(2.11) is equivalent to the problem (2.13)-(2.15). Therefore, we prove the existence of solutions of the problem (2.13)-(2.15). In order to do that, we use the contraction mapping principle in some interval $[0,T]$.

We introduce some necessary definitions. Let us set

$$L = \|u_0^n\|_Y, \quad M = \sup\{K(x,y) : x,y \in [1/n,n]\}, \quad (2.16)$$

and choose $t_1, t_2 \geq 0$ such that

$$\exp\big(2nMLt_1\big)\big(1+n^3MLt_1\big) \leq 2 \quad (2.17)$$

$$\exp\big(2nMLt_2\big)nMLt_2\big(1+n^3MLt_2+n^2\big) < 1. \quad (2.18)$$

We set

$$t_0 = \min(t_1,t_2,T).$$

For those $c \in C\big([0,T];L^1\big(]0,n[\big)\big)$ for which $\int\limits_0^1 x^{-1}|c(x,t)|dx$ is finite for all $t \in [0,t_0]$ we define the norm $\|\cdot\|_D$ by

$$\|c\|_D = \sup_{t\in[0,t_0]} \int\limits_0^n x^{-1}|c(x,t)|\,dx.$$

Now we set

$$D = \big\{c \in C\big([0,t_0];L^1\big(]0,n[\big)\big) : \|c\|_D \leq 2L\big\}.$$

For $c \in D$ we have

$$P(x,t,c) \leq nMt\|c\|_D \leq 2nMLt \quad (2.19)$$

and for $\tau \in [0, t[$

$$P(x,t,c) - P(x,\tau,c) \le nM(t-\tau)\|c\|_D.$$

We present now some lemmas which are necessary for the proof of Theorem 2.2.4.

Lemma 2.2.1. *The functional G maps the set D into itself.*

Proof. Choose c such that $\|c\|_D \le 2L$. For $t \in [0, t_0]$, using (2.14), (2.18), (2.19) and Fubini's Theorem, we have

$$\begin{aligned}
\int_0^n |G(c)(x,t)| x^{-1} dx &= \int_0^n \Bigg| u_0^n(x) \exp\big(P(x,t,c)\big) + \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c) - P(x,\tau,c)]\big) \\
&\qquad \cdot \int_0^x K_n(x-y,y) c(x-y,\tau) c(y,\tau)\, dy\, d\tau \Bigg| x^{-1} dx \\
&\le \int_0^n |u_0^n(x)| \exp\big(P(x,t,|c|)\big) x^{-1} + \frac{1}{2}\int_0^t \int_0^n \exp\big(P(x,t,|c|)\big) \\
&\qquad \cdot \int_0^x K_n(x-y,y) |c(x-y,\tau)| |c(y,\tau)| x^{-1} dy\, dx\, d\tau \\
&\le \|u_0^n\|_Y \exp\big(2nMLt\big) + \frac{1}{2}\exp\big(2nMLt\big) \\
&\qquad \cdot \int_0^t \int_0^n \int_0^x K_n(x-y,y) |c(x-y,\tau)| |c(y,\tau)| x^{-1} dy\, dx\, d\tau.
\end{aligned}$$

Changing the order of integration, then a change of variable $x - y = z$ and then re-changing the order of integration while replacing z by x, see Appendix E, gives

$$\begin{aligned}
&\int_0^n |G(c)(x,t)| x^{-1} dx \\
&\le \exp\big(2nMLt\big) \left[\|u_0^n\|_Y + \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} K_n(x,y) |c(x,\tau)| |c(y,\tau)| (x+y)^{-1} dy\, dx\, d\tau \right] \\
&\le \exp\big(2nMLt\big) \left[\|u_0^n\|_Y + \frac{1}{2}\int_0^t \int_{1/n}^{n-1/n} \int_{1/n}^{n-x} K(x,y) |c(x,\tau)| |c(y,\tau)| (x+y)^{-1} dy\, dx\, d\tau \right]
\end{aligned} \tag{2.20}$$

Now, multiplying and dividing by xy and using the definition of M we arrive at

$$\begin{aligned}
&\int_0^n |G(c)(x,t)|x^{-1}dx \\
&\quad \leq \exp\big(2nMLt\big)\left[\|u_0^n\|_Y + \frac{1}{4}n^3M\int_0^t\int_{1/n}^{n-1/n}\int_{1/n}^{n-x}|c(x,\tau)||c(y,\tau)|(xy)^{-1}dy\,dx\,d\tau\right] \\
&\quad \leq \exp\big(2nMLt\big)\left(\|u_0^n\|_Y + \frac{1}{4}n^3Mt\|c\|_D^2\right) \\
&\quad \leq \exp\big(2nMLt\big)\big(1+n^3MLt\big)L \leq 2L. \qquad (2.21)
\end{aligned}$$

The later inequality is obtained using (2.17). Hence, we have $\|G\|_D \leq 2L$ and this completes the proof of the lemma. □

Lemma 2.2.2. *Let $B = \max\{\|c_1\|_D, \|c_2\|_D\}$ and*

$$H(x,\tau,t) = \exp\big(-[P(x,t,c_1)-P(x,\tau,c_1)]\big) - \exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big). \qquad (2.22)$$

For $c_1, c_2 \in C\big([0,t_0]; L^1\big(]0,n[\big)\big)$ we have for $0 \leq \tau \leq t \leq t_0$ and $0 \leq x \leq n$

$$|H(x,\tau,t)| \leq (t-\tau)nM\exp\big((t-\tau)nBM\big)\|c_1-c_2\|_D.$$

Proof: Suppose that for some arbitrary but fixed x, t and τ,

$$P(x,t,c_1) - P(x,\tau,c_1) \geq P(x,t,c_2) - P(x,\tau,c_2).$$

Then

$$\begin{aligned}
|H(x,\tau,t)| =& -H(x,\tau,t) \\
&= \exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big) \\
&\quad \cdot\big(1-\exp\big(-[P(x,t,c_1)-P(x,\tau,c_1)-[P(x,t,c_2)-P(x,\tau,c_2)]]\big)\big). \qquad (2.23)
\end{aligned}$$

Since $1-\exp(-x) \leq x$ for $x \geq 0$, (2.23), together with the definitions of B and M leads to

$$\begin{aligned}
&|H(x,\tau,t)| \\
&\quad \leq \exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big)\big(P(x,t,c_1)-P(x,\tau,c_1)-[P(x,t,c_2)-P(x,\tau,c_2)]\big) \\
&\quad = \exp\left(-\int_\tau^t\int_0^{n-x}K_n(x,y)c_2(y,s)\,dy\,ds\right)\int_\tau^t\int_0^{n-x}K_n(x,y)[c_1(y,s)-c_2(y,s)]dy\,ds \\
&\quad \leq (t-\tau)nM\exp\big((t-\tau)nBM\big)\|c_1-c_2\|_D.
\end{aligned}$$

If $P(x,t,c_1) - P(x,\tau,c_1) \leq P(x,t,c_2) - P(x,\tau,c_2)$ then inequality (2.22) can be derived analogously. □

Lemma 2.2.3. *For $c_1, c_2 \in D$ and t_0 as above there exists a constant $\gamma \in [0,1[$ such that*

$$\|G(c_1) - G(c_2)\|_D \le \gamma \|c_1 - c_2\|_D,$$

i.e. the operator G is a contraction.

Proof. Choose $c_1, c_2 \in D$. Then from

$$\begin{aligned}
&G(c_1) - G(c_2)\\
&\quad = u_0^n(x)\big[\exp\big(P(x,t,c_1)\big) - \exp\big(P(x,t,c_2)\big)\big]\\
&\qquad + \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_1) - P(x,\tau,c_1)]\big)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau\\
&\qquad - \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau\\
&\qquad - \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_0^x K_n(x-y,y)c_2(x-y,\tau)c_2(y,\tau)\,dy\,d\tau\\
&\qquad + \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau
\end{aligned}$$

and the definition (2.22) of H it follows that

$$\begin{aligned}
&G(c_1) - G(c_2)\\
&= u_0^n(x)H(x,0,t) + \frac{1}{2}\int_0^t H(x,\tau,t)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau\\
&- \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\left[\int_0^x K_n(x-y,y)c_2(x-y,\tau)\,[c_2(y,\tau) - c_1(y,\tau)]\right.\\
&\left. + \int_0^x K_n(x-y,y)c_1(x,\tau)\,[c_2(x-y,\tau) - c_1(x-y,\tau)]\right]dy\,d\tau.
\end{aligned}$$

Applyig the norm $\|\cdot\|_{x^{-1}}$ to $G(c_1) - G(c_2)$ we have

$$\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-1}}\\
&= \int_0^n \left| u_0^n(x)H(x,0,t) + \frac{1}{2}\int_0^t H(x,\tau,t)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau\right.\\
&- \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\left[\int_0^x K_n(x-y,y)c_2(x-y,\tau)\,[c_2(y,\tau) - c_1(y,\tau)]\right.\\
&\left.\left. + \int_0^x K_n(x-y,y)c_1(x,\tau)\,[c_2(x-y,\tau) - c_1(x-y,\tau)]\right]dy\,d\tau\right| x^{-1}dx.
\end{aligned}$$

Applying the triangle inequality we obtain

$$\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-1}} \\
&\le \int_0^n |u_0^n(x)|\, |H(x,0,t)|\, x^{-1} dx \\
&+ \frac{1}{2} \int_0^t |H(x,\tau,t)| \int_0^x K_n(x-y,y)\, |c_1(x-y,\tau)|\, |c_1(y,\tau)|\, x^{-1} dy\, dx\, d\tau \\
&+ \frac{1}{2} \int_0^t \exp\big(P(x,t,|c_2|)\big) \left[\int_0^x K_n(x-y,y)\, |c_2(x-y,\tau)|\, |c_2(y,\tau) - c_1(y,\tau)| \right. \\
&\left. + \int_0^x K_n(x-y,y)\, |c_1(x,\tau)|\, |c_2(x-y,\tau) - c_1(x-y,\tau)| \right] x^{-1} dy\, dx\, d\tau.
\end{aligned}$$

Changing the order of integration, then a change of variable $x - y = z$, then re-changing the order of integration while replacing z by x and making use of Lemma 2.2.2 gives

$$\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-1}} \\
&\le \|u_0^n\|_Y nMt \exp\big(2nMLt\big) \|c_1 - c_2\|_D + \frac{1}{2} nMt \exp\big(2nMLt\big) \|c_1 - c_2\|_D \\
&\cdot \int_0^t \int_0^n \int_0^{n-x} K_n(x,y) |c_1(x,\tau)| |c_1(y,\tau)| (x+y)^{-1} dy\, dx\, d\tau \\
&+ \frac{1}{2} \exp\big(2nMLt\big) \int_0^t \left[\int_0^n \int_0^{n-x} K_n(x,y) |c_2(x,\tau)| |c_1(y,\tau) - c_2(y,\tau)| (x+y)^{-1} dy\, dx \right. \\
&\left. + \int_0^n \int_0^{n-x} K_n(x,y) |c_1(y,\tau)| |c_1(x,\tau) - c_2(x,\tau)| (x+y)^{-1} dy\, dx \right] d\tau \\
&\le \exp\big(2nMLt\big) nMLt \|c_1 - c_2\|_D + \frac{1}{4} \exp\big(2nMLt\big) n^4 M^2 t^2 \|c_1\|_D^2 \|c_1 - c_2\|_D \\
&+ \frac{1}{4} \exp\big(2nMLt\big) n^3 Mt \left[\|c_2\|_D \|c_1 - c_2\|_D + \|c_1\|_D \|c_1 - c_2\|_D \right] \\
&\le \exp\big(2nMLt\big) nMLt \|c_1 - c_2\|_D + \exp\big(2nMLt\big) n^4 M^2 L^2 t^2 \|c_1 - c_2\|_D \\
&+ \exp\big(2nMLt\big) n^3 MLt \|c_1 - c_2\|_D \\
&= \exp\big(2nMLt\big) nMLt \left(1 + n^3 MLt + n^2\right) \|c_1 - c_2\|_D.
\end{aligned}$$

From where we can conclude that

$$\|G(c_1) - G(c_2)\|_D \le \gamma \|c_1 - c_2\|_D,$$

where $\gamma = \exp\big(2nMLt\big) nMLt \left(1 + n^3 MLt + n^2\right) < 1$, which completes the proof of the lemma. □

Theorem 2.2.4. *Suppose that **(H1)**, **(H2)**, **(H3)** hold and* $u_0 \in Y^+$. *Then for each* $n = 2, 3, 4, \ldots$ *the problem (2.10)-(2.11) has a unique solution* u^n *with* $u^n(x,t) \geq 0$ *for a.e.* $x \in [0,n]$ *and* $t \in [0,\infty[$. *Moreover, for all* $t \in [0,\infty[$

$$\int_0^n xu^n(x,t)\,dx = \int_0^n xu^n(x,0)\,dx. \tag{2.24}$$

Proof. From Lemmas 2.2.1, 2.2.3 and the contraction mapping theorem, it follows that there exists a unique solution $u^n(x,t)$ to (2.13)-(2.15) in $[0,t_0]$. We proceed now to check that these solutions are non-negative. If we set

$$c_0 = u_0^n \quad \text{and} \quad c_i = G(c_{i-1}),$$

for $i = 1, 2, 3, \ldots$, we find that fixed point iteration gives

$$c_i \to u^n \quad \text{in} \quad L^1\big(]0,\infty[\big) \quad \text{as} \quad i \to \infty,$$

and u^n is constructed by positivity preserving iterations, for G given in (2.14).

Let us check now that the mass conservation property (2.24) holds. Multiplying (2.10) by x and integrating with respect to x on $[0,n]$ we have by changes of variables and order of integration as in (2.21)

$$\begin{aligned}
&\frac{d}{dt}\int_0^n xu^n(x,t)\,dx \\
&= \frac{1}{2}\int_0^n\int_0^x xK_n(x-y,y)u^n(x-y,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx \\
&= \frac{1}{2}\int_0^n\int_0^{n-x} (x+y)K_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx \\
&= \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx = 0,
\end{aligned}$$

from where we have

$$\frac{d}{dt}\int_0^n xu^n(x,t)\,dx = 0 \qquad \Longrightarrow \qquad \int_0^n xu^n(x,t)dx = \int_0^n xu_0^n(x)\,dx.$$

Now we show that our solution for $t \in [0,t_0]$ extends to arbitrarily large times, changing variables as we did in (2.21) we proceed to obtain a uniform bound.

$$\begin{aligned}\int_0^n u^n(x,t)x^{-1}\,dx &= \int_0^t \Bigg[\frac{1}{2}\int_0^n\int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)x^{-1}dy\,dx \\ &\quad -\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-1}dy\,dx\Bigg]\,d\tau + \int_0^n u_0^n(x)x^{-1}dx \\ &= \int_0^t \Bigg[\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x+y)^{-1}dy\,dx \\ &\quad -\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-1}dy\,dx\Bigg]\,d\tau + \int_0^n u_0^n(x)x^{-1}dx.\end{aligned}$$

Making use of the inequality (2.6) and the symmetry of $K(x,y)$ we obtain

$$\begin{aligned}\int_0^n u^n(x,t)x^{-1}dx &\le \int_0^t \Bigg[\frac{1}{8}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x^{-1}+y^{-1})\,dy\,dx \\ &\quad -\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x^{-1}+y^{-1})\,dy\,dx\Bigg]\,d\tau + \int_0^n u_0^n(x)x^{-1}dx \\ &\le -\frac{3}{8}\int_0^t\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x^{-1}+y^{-1})\,dy\,dx + \int_0^n u_0^n(x)x^{-1}dx \\ &\le \int_0^n u_0^n(x)x^{-1}dx \le \|u_0^n\|_Y = L. \end{aligned} \tag{2.25}$$

To complete the proof of Theorem 2.2.4 we can now extend the solution interval from $[0,t_0]$ to $[0,\infty[$. By considering the operator

$$\begin{aligned}G_1(c)(x,t) = u^n(x,t_0)\exp\big(P_1(x,t,c)\big) + \frac{1}{2}\int_{t_0}^t \exp\big(-[P_1(x,t,c)-P_1(x,\tau,c)]\big) \\ \cdot\int_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy\,d\tau,\end{aligned}$$

with

$$P_1(x,t,c) = \int_{t_0}^t\int_0^{n-x} K(x,y)c(x,\tau)c(y,\tau)\,dx\,d\tau$$

we can repeat the above argument to show, that there is a unique non-negative solution u^n on $[t_0, t_1]$ where $t_1 = 2t_0$. We can extend the unique solution to $[0, t_j]$ $\quad j = 1, 2, 3, \ldots$, repeating this process by considering the operators

$$G_{j+1}(c)(x,t) = u^n(x,t_j)\exp\big(P_{j+1}(x,t,c)\big) + \frac{1}{2}\int_{t_j}^{t} \exp\big(-[P_{j+1}(x,t,c) - P_{j+1}(x,\tau,c)]\big) \cdot \int_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy\,d\tau,$$

with

$$P_{j+1}(x,t,c) = \int_{t_j}^{t}\int_0^{n-x} K(x,y)c(x,\tau)c(y,\tau)\,dy\,d\tau.$$

In that way we extend the solution to all of $[0,\infty[$. The argument used to get (2.24) for $[0, t_0]$ shows that (2.24) holds for $t \in [0, \infty[$ and thus we have completed the proof of Theorem 2.2.4 by the arbitrariness of n. □

2.2.2 Properties of the solutions of the truncated problem

In the rest of the chapter we consider for each u^n their zero extension on $\mathbb{R}$, i.e.

$$\hat{u}^n(x,t) = \begin{cases} u^n(x,t) & 0 \le x \le n, \;\; t \in [0,T], \\ 0 & x < 0 \;\text{ or }\; x > n. \end{cases}$$

For clarity we drop the notation $\hat{\ }$ for the remainder of the chapter.

Lemma 2.2.5. *Assume that **(H1)**, **(H2)** and **(H3)** hold. We take u^n to be the non-negative zero extension of the solution to the truncated problem found in Theorem 2.2.4. Then the following are true*

(i) *We have using L from (2.16) the bound*

$$\int_0^\infty (1 + x + x^{-2\sigma})u^n(x,t)\,dx \le 3L.$$

(ii) *Given $\epsilon > 0$ there exists an $R > 1$ such that for all $t \in [0,T]$*

$$\sup_n \left\{ \int_R^\infty (1 + x^{-\sigma})u^n(x,t)\,dx \right\} \le \epsilon.$$

(iii) *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n = 2, 3, \ldots$ and $t \in [0,T]$*

$$\int_A (1 + x^{-\sigma})u^n(x,t)\,dx < \epsilon \qquad \textit{whenever} \qquad \mu(A) < \delta.$$

Proof. Property (i) We split the following integral into three parts

$$\int_0^\infty (1+x+x^{-2\sigma})u^n(x,t)\,dx = \int_0^n u^n(x,t)\,dx + \int_0^n xu^n(x,t)\,dx + \int_0^n x^{-2\sigma}u^n(x,t)\,dx. \quad (2.26)$$

Working with the first integral of the right hand side of (2.26) and using that $\sigma \in \left[0, \frac{1}{2}\right]$

$$\begin{aligned}\int_0^n u^n(x,t)\,dx &= \int_0^1 x^{-1}xu^n(x,t)\,dx + \int_1^n x^{-1}xu^n(x,t)\,dx\\ &\leq \int_0^1 x^{-1}u^n(x,t)\,dx + \int_1^n xu^n(x,t)\,dx\\ &\leq \int_0^n x^{-1}u^n(x,t)\,dx + \int_0^n xu^n(x,t)\,dx.\end{aligned}$$

Using the mass conservation property (2.24) and $n > 1$ combined with (2.25) we obtain

$$\int_0^n u^n(x,t)\,dx \leq \int_0^n x^{-1}u_0^n(x)\,dx + \int_0^n xu_0^n(x)\,dx \leq \|u_0^n\|_Y = L. \quad (2.27)$$

Now let us consider the third integral on the right hand side of (2.26)

$$\begin{aligned}\int_0^n u^n(x,t)x^{-2\sigma}\,dx &= \int_0^1 u^n(x,t)x^{-2\sigma}\,dx + \int_1^n u^n(x,t)x^{-2\sigma}\,dx\\ &\leq \int_0^1 u^n(x,t)x^{-1}\,dx + \int_1^n xu^n(x,t)\,dx\\ &\leq \int_0^n u^n(x,t)x^{-1}\,dx + \int_0^n xu^n(x,t)\,dx\\ &\leq \|u_0^n\|_Y = L. \end{aligned} \quad (2.28)$$

Thus, by using (2.24) together with (2.27) and (2.28) we may estimate

$$\int_0^\infty (1+x+x^{-2\sigma})u^n(x,t)\,dx \leq 3\|u_0^n\|_Y = 3L.$$

Property (ii) Choose $\epsilon > 0$ and let $R > 1$ be such that $R > \frac{2\|u_0\|_Y}{\epsilon}$. Then we get using (2.24)

$$\begin{aligned}
\int_R^\infty (1 + x^{-\sigma}) u^n(x,t)\,dx &= \int_R^\infty (1 + x^{-\sigma}) \frac{x}{x} u^n(x,t)\,dx \\
&\le \frac{1}{R} \int_R^\infty (x + x^{1-\sigma}) u^n(x,t)\,dx \\
&\le \frac{1}{R} \int_R^\infty x u^n(x,t)\,dx + \frac{1}{R} \int_1^\infty x^{1-\sigma} u^n(x,t)\,dx \\
&\le \frac{1}{R} \int_R^\infty x u^n(x,t)\,dx + \frac{1}{R} \int_1^\infty x u^n(x,t)\,dx \\
&\le \frac{2}{R}\|u_0^n\|_Y \le \frac{2}{R}\|u_0\|_Y < \epsilon.
\end{aligned}$$

Property (iii) By property**(ii)** we can choose $r > 1$ such that for all n and $t \in [0,T]$

$$\int_r^\infty (1 + x^{-\sigma}) u^n(x,t)\,dx < \frac{\epsilon}{2}. \tag{2.29}$$

Let χ_A denote the characteristic function of a set A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Let us define for all $n = 1, 2, 3, \ldots$ and $t \in [0,T]$

$$f^n(A,t) = \sup_{0 \le z \le r} \int_0^\infty \chi_{A \cap [0,r]}(x+z)(1 + x^{-\sigma}) u^n(x,t)\,dx$$

and set

$$k(r) = \frac{1}{2} \max_{\substack{0 \le x \le r \\ 0 \le y \le r}} (1 + x + y)^\lambda (1 + y^\sigma).$$

Now, using $\|u_0^n\|_Y = L$ leads to

$$\int_0^\infty (1 + x^{-\sigma}) u_0^n(x)\,dx \le 2 \int_0^1 x^{-1} u_0^n(x)\,dx + 2 \int_1^\infty x u_0^n(x)\,dx \le 2\|u_0^n\|_Y = 2L.$$

By the absolute continuity of the Lebesgue integral, there exists a $\delta > 0$ such that

$$f^n(A,0) = \sup_{0 \le z \le r} \int_0^\infty \chi_{A \cap [0,r]}(x+z)(1 + x^{-\sigma}) u_0^n(x)\,dx < \frac{\epsilon}{2\exp\big(k(r)LT\big)}, \tag{2.30}$$

whenever $A \subset \mathbb{R}$ with $\mu(A) \leq \delta$. Now we multiply (2.10) by $\chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})$. This we integrate from 0 to t w.r.t. s and over $[0,\infty[$ w.r.t. x. Using the non-negativity of each u^n and $\mu(A) \leq \delta$ we obtain

$$\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u^n(x,t)\,dx$$
$$\leq \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \chi_{A\cap[0,r]}(x+z)\chi_{[0,x]\cap[0,r]}(y)(1+x^{-\sigma})K_n(x-y,y)u^n(x-y,s)u^n(y,s)\,dy\,dx\,ds$$
$$+\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u_0^n(x)\,dx.$$

Changing the order of integration, then making a change of variable $x-y=x'$ and replacing x' by x gives

$$\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u^n(x,t)\,dx$$
$$\leq \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \chi_{A\cap[0,r]}(x+y+z)\chi_{[0,x+y]\cap[0,r]}(y)\left[1+(x+y)^{-\sigma}\right]$$
$$\cdot K_n(x,y)u^n(x,s)u^n(y,s)\,dx\,dy\,ds$$
$$+\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u_0^n(x)\,dx.$$

Using the estimate **(H3)** of $K(x,y)$ we find that

$$\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u^n(x,t)\,dx$$
$$\leq \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \chi_{A\cap[0,r]}(x+y+z)\chi_{[0,x+y]\cap[0,r]}(y)\left[1+(x+y)^{-\sigma}\right]$$
$$\cdot(1+x+y)^\lambda(xy)^{-\sigma}u^n(x,s)u^n(y,s)\,dx\,dy\,ds+f^n(A,0)$$
$$\leq \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \chi_{A\cap[0,r]}(x+y+z)\chi_{[0,x+y]\cap[0,r]}(y)(1+y^{-\sigma})$$
$$\cdot(1+x+y)^\lambda(1+x^{-\sigma})y^{-\sigma}u^n(x,s)u^n(y,s)\,dx\,dy\,ds+f^n(A,0)$$
$$\leq \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty \chi_{A\cap[0,r]}(x+y+z)\chi_{[0,x+y]\cap[0,r]}(y)(1+y^{\sigma})$$
$$\cdot(1+x+y)^\lambda(1+x^{-\sigma})y^{-2\sigma}u^n(x,s)u^n(y,s)\,dx\,dy\,ds+f^n(A,0).$$

We use now the definition of $k(r)$ and (2.28)

$$
\begin{aligned}
&\int_0^\infty \chi_{A\cap[0,r]}(x+z)(1+x^{-\sigma})u^n(x,t)\,dx \\
&\quad \le k(r)\int_0^t\int_0^r u^n(y,s)y^{-2\sigma}\int_0^\infty \chi_{A\cap[0,r]}(x+y+z)(1+x^{-\sigma})u^n(x,s)\,dx\,dy\,ds + f^n(A,0) \\
&\quad \le k(r)\int_0^t\int_0^r u^n(y,s)y^{-2\sigma}\sup_{0\le\omega\le r}\int_0^\infty \chi_{A\cap[0,r]}(x+\omega)(1+x^{-\sigma})u^n(x,s)\,dx\,dy\,ds + f^n(A,0) \\
&\quad \le k(r)L\int_0^t f^n(A,s)\,ds + f^n(A,0).
\end{aligned} \tag{2.31}
$$

Since the right hand side is independent of z we may take $\sup_{0\le z\le r}$ on the left hand side to obtain

$$
f^n(A,t) \le k(r)L\int_0^t f^n(A,s)ds + \epsilon/\left(2\exp\left(k(r)LT\right)\right).
$$

By Gronwall's inequality, see e.g. Appendix A[Theorem A.0.6]

$$
f^n(A,t) \le \epsilon\exp\left(k(r)LT\right)/\left(2\exp\left(k(r)LT\right)\right) = \frac{\epsilon}{2}. \tag{2.32}
$$

By (2.29) and (2.32) follows that

$$
\begin{aligned}
\int_A (1+x^{-\sigma})u^n(x,t)dx &= \int \chi_{A\cap[0,r]}(x)(1+x^{-\sigma})u^n(x,t)dx + \int \chi_{A\cap[r,\infty[}(x)(1+x^{-\sigma})u^n(x,t)dx \\
&\le f^n(A,t) + \int_r^\infty (1+x^{-\sigma})u^n(x,t)dx \\
&\le \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

whenever $\mu(A) < \delta$.

This completes the proof of Lemma 2.2.5. □

Let us define $v^n(x,t) = x^{-\sigma}u^n(x,t)$. Due to the Lemma 2.2.5 above and the Dunford-Pettis Theorem, see Appendix A[Theorem A.0.4], we can conclude that for each $t \in [0,T]$ the sequences $\left(u^n(t)\right)_{n\in\mathbb{N}}$ and $\left(v^n(t)\right)_{n\in\mathbb{N}}$ are weakly relatively compact in $L^1\left(]0,\infty[\right)$.

2.2.3 Equicontinuity in time

Now we proceed to show that the sequences $\left(u^n(t)\right)_{n\in\mathbb{N}}$ and $\left(v^n(t)\right)_{n\in\mathbb{N}}$ are equicontinuous in time.

Lemma 2.2.6. *Assume that **(H1)**, **(H2)**, and **(H3)** hold. Take $\left(u^n\right)$ now to be the sequence of extended solutions to the truncated problems (2.10)-(2.13) found in Theorem 2.2.4 and $v^n(x,t) = x^{-\sigma}u^n(x,t)$. Then there exists a subsequences $\left(u^{n_k}(t)\right)$ and $\left(v^{n_l}(t)\right)$ of $\left(u^n(t)\right)_{n\in\mathbb{N}}$ and $\left(v^n(t)\right)_{n\in\mathbb{N}}$ respectively such that*

$$\begin{aligned} u^{n_k}(t) &\rightharpoonup u(t) \quad in \quad L^1\left(]0,\infty[\right) \quad as \quad n_k\to\infty \\ v^{n_l}(t) &\rightharpoonup v(t) \quad in \quad L^1\left(]0,\infty[\right) \quad as \quad n_l\to\infty \end{aligned}$$

for any $t\in[0,T]$. Giving $u,v\in C_B\left([0,T];\Omega_1\right) = \{\eta : [0,\infty[\to\Omega_1, \eta$ continuous and $\eta(t)$ bounded for all $t\geq 0\}$, where Ω_1 is $L^1\left(]0,\infty[\right)$ equipped with the weak topology. This convergence is uniform for all $t\in[0,T]$.

Proof: Choose $\epsilon>0$ and $\phi\in L^\infty\left(]0,\infty[\right)$. Let $s,t\in[0,T]$ and assume that $t\geq s$. Choose $a>1$ such that

$$\frac{6L}{a}\|\phi\|_{L^\infty(]0,\infty[)} \leq \epsilon/2. \tag{2.33}$$

Using Lemma 2.2.5**(i)**, for each n, we have

$$\int_a^\infty |u^n(x,t)-u^n(x,s)|\,dx \leq \frac{1}{a}\int_a^\infty x\,|u^n(x,t)+u^n(x,s)|\,dx \leq 6L/a. \tag{2.34}$$

From the proof of Lemma 2.2.5**(i)** we have that

$$\int_0^\infty xu^n(x,t)\,dx \leq L \qquad \text{and} \qquad \int_0^\infty x^{-2\sigma}u^n(x,t)\,dx \leq L.$$

Splitting the integral domain leads us to

$$\int_0^\infty x^{-\sigma}u^n(x,t)\,dx \leq L. \tag{2.35}$$

By multiplying (2.10) by ϕ and integrating w.r.t. x from 0 to a as well as from a to ∞, w.r.t. τ form s to t and using (2.33), (2.34) and $t\geq s$ we get

$$\begin{aligned} &\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \\ &\quad\leq \left|\int_0^a \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| + \left|\int_a^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \\ &\quad\leq \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t \left[\frac{1}{2}\int_0^a\int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)\,dy\,dx\right. \\ &\qquad\qquad \left. + \int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx\right] d\tau + \epsilon/2. \end{aligned}$$

Now, changing the order of integration, then making a change of variable $x-y=x'$ and replacing x' by x gives

$$\begin{aligned}&\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right|\\ &\quad=\|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_0^a\int_0^{a-x}K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx\right.\\ &\qquad\left.+\int_0^a\int_0^{n-x}K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx\right]d\tau+\epsilon/2.\end{aligned}$$

Using the estimation of $K(x,y)$

$$\begin{aligned}&\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right|\\ &\quad\le\|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_{1/n}^{a-1/n}\int_{1/n}^{a-x}(1+x+y)^\lambda(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\right.\\ &\qquad\left.+\int_{1/n}^{a}\int_{1/n}^{n-x}(1+x+y)^\lambda(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\right]d\tau+\epsilon/2\\ &\quad\le\frac{3}{2}\int_s^t\int_0^\infty\int_0^\infty(1+x+y)^\lambda(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau+\epsilon/2.\end{aligned}\tag{2.36}$$

Making use of the inequalities (2.4) and (2.5) in the following way

$$(1+x+y)^p\le\nu(1+x^p+y^p)\quad\text{where}\quad\nu=\begin{cases}1 & \text{if}\quad 0\le p\le 1\\ 2^{2p-2} & \text{if}\quad p>1,\end{cases}\tag{2.37}$$

for $p=\lambda$ we find that

$$\begin{aligned}&\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right|\\ &\quad\le 3\nu\|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\int_0^\infty\int_0^\infty(1+x^\lambda+y^\lambda)(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau+\epsilon/2\\ &\quad=3\nu\|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\int_0^\infty\int_0^\infty(x^{-\sigma}y^{-\sigma}+x^{\lambda-\sigma}y^{-\sigma}+y^{\lambda-\sigma}x^{-\sigma})u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau+\epsilon/2.\end{aligned}$$

By symmetry in the last two integral terms we have

$$\begin{aligned}\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \\ = 3\nu\|\phi\|_{L^\infty(]0,\infty[)}\Bigg[\int_s^t\int_0^\infty\int_0^\infty x^{-\sigma}y^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau \\ +2\int_s^t\int_0^\infty\int_0^\infty x^{\lambda-\sigma}y^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau\Bigg]+\epsilon/2.\end{aligned}$$

From where, by using Lemma 2.2.5(i) and (2.35) we get

$$\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \le 21\nu\|\phi\|_{L^\infty(]0,\infty[)}(t-s)L^2+\epsilon/2<\epsilon, \tag{2.38}$$

whenever $(t-s)<\delta$ for some $\delta>0$ sufficiently small. The argument given above similarly holds for $s<t$. Hence (2.38) holds for all n and $|t-s|<\delta$. Then the sequence $\left(u^n(t)\right)_{n\in\mathbb{N}}$ is time equicontinuous in $L^1\left(]0,\infty[\right)$. Thus, $\left(u^n(t)\right)_{n\in\mathbb{N}}$ lies in a relatively compact subset of a gauge space Ω_1. The gauge space Ω_1 is $L^1\left(]0,\infty[\right)$ equipped with the weak topology. For details about gauge spaces, see Appendix B. Then, we may apply a version of the Arzela-Ascoli Theorem, see Appendix A [Theorem A.0.5], to conclude that there exists a subsequence $\left(u^{n_k}\right)_{k\in\mathbb{N}}$ such that

$$u^{n_k}(t)\to u(t)\quad\text{in}\quad\Omega_1\quad\text{as}\quad n_k\to\infty,$$

uniformly for $t\in[0,T]$ for some $u\in C\left([0,T];\Omega_1\right)$.

Now let us consider $v^n(x,t)=x^{-\sigma}u^n(x,t)$ where we have to deal with a stronger singularity at 0.

We take $\epsilon>0$, ϕ, s and t as they were defined before. Using Lemma 2.2.5, for each n, we get using $a>1$ chosen to satisfy (2.33)

$$\begin{aligned}\int_a^\infty |v^n(x,t)-v^n(x,s)|\,dx &= \int_a^\infty \left|x^{-\sigma}u^n(x,t)-x^{-\sigma}u^n(x,s)\right|dx \\ &\le \frac{1}{a}\int_a^\infty x^{1-\sigma}\,|u^n(x,t)+u^n(x,s)|\,dx \\ &\le \frac{1}{a}\int_a^\infty x\,|u^n(x,t)+u^n(x,s)|\,dx \le 6L/a. \end{aligned}\tag{2.39}$$

By using (2.10), (2.33), (2.39), for $t \geq s$ and the definition of $v^n(x)$ we obtain

$$
\begin{aligned}
&\left|\int_0^\infty \phi(x)\left[v^n(x,t)-v^n(x,s)\right]dx\right| \\
&\quad\leq \int_0^a \phi(x)\left[v^n(x,t)+v^n(x,s)\right]dx+\epsilon/2 \\
&\quad\leq \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_0^a\int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right. \\
&\qquad\qquad\left.+\int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right]d\tau+\epsilon/2 \\
&\quad= \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_0^a\int_0^{a-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x+y)^{-\sigma}dy\,dx\right. \\
&\qquad\qquad\left.+\int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right]d\tau+\epsilon/2.
\end{aligned}
$$

Taking $y=0$ in the term $(x+y)^{-\sigma}$

$$
\begin{aligned}
&\left|\int_0^\infty \phi(x)\left[v^n(x,t)-v^n(x,s)\right]dx\right| \\
&\quad\leq \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_0^a\int_0^{a-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right. \\
&\qquad\qquad\left.+\int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right]d\tau+\epsilon/2 \\
&\quad= \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\left[\frac{1}{2}\int_{1/n}^{a-1/n}\int_{1/n}^{a-x} K(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right. \\
&\qquad\qquad\left.+\int_{1/n}^a\int_{1/n}^{n-x} K(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx\right]d\tau+\epsilon/2 \\
&\quad\leq \frac{3}{2}\|\phi\|_{L^\infty(]0,\infty[)}\int_s^t\int_0^\infty\int_0^\infty K(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma}dy\,dx+\epsilon/2.
\end{aligned}
$$

Using the estimation of $K(x,y)$ and the inequality (2.4) together (2.37) for $p=\lambda$ we have

$$\left| \int_0^\infty \phi(x)\left[v^n(x,t)-v^n(x,s)\right]dx \right|$$
$$\leq \frac{3}{2}\|\phi\|_{L^\infty(]0,\infty[)} \int_s^t \int_0^\infty \int_0^\infty (1+x+y)^\lambda (xy)^{-\sigma} u^n(x,\tau)u^n(y,\tau)x^{-\sigma}\,dy\,dx\,d\tau + \epsilon/2$$
$$\leq \frac{3}{2}\nu\|\phi\|_{L^\infty(]0,\infty[)} \int_s^t \int_0^\infty \int_0^\infty (x^{-2\sigma}y^{-\sigma} + x^{\lambda-2\sigma}y^{-\sigma} + y^{\lambda-\sigma}x^{-2\sigma})u^n(x,\tau)u^n(y,\tau)\,dy\,dx\,d\tau + \epsilon/2.$$

By using Lemma 2.2.5**(i)** and (2.35) we obtain

$$\left| \int_0^\infty \phi(x)\left[v^n(x,t)-v^n(x,s)\right]dx \right| \leq \frac{27}{2}\nu\|\phi\|_{L^\infty(]0,\infty[)}(t-s)L^2 + \epsilon/2.$$

We can use now the same argument used for u^n to conclude that there exists a subsequence $\left(v^{n_k}\right)_{k\in\mathbb{N}}$ such that

$$v^{n_k}(t) \to v(t) \quad \text{in} \quad \Omega_1 \quad \text{as} \quad n_k \to \infty,$$

uniformly for $t \in [0,T]$ for some $v \in C\left([0,T];\Omega_1\right)$.

Since $T>0$ is arbitrary we obtain $u, v \in C_B\left([0,\infty[;\Omega_1\right)$. □

Lemma 2.2.7. *For $v^n(\cdot,t)$ defined as before, we have*

$$v^n(\cdot,t) \rightharpoonup v(\cdot,t) \quad \textit{where} \quad v(x,t)=x^{-\sigma}u(x,t) \quad \textit{for all} \quad t\in[0,T] \quad \textit{in} \quad L^1\big(]0,a]\big).$$

Proof. By Lemma 2.2.6, we know that $v^n(t) \rightharpoonup v(t)$ in $L^1\big(]0,\infty[\big)$ as $n\to\infty$ uniformly for $t\in[0,T]$. Then, we just need to prove that $v(x,t)=x^{-\sigma}u(x,t)$.

By definition of weak convergence we have

$$\int_0^a \varphi(x)\left[v^n(x,t)-v(x,t)\right]dx \to 0 \quad \text{for all} \quad \varphi \in L^\infty\big(]0,a]\big).$$

As $x^\sigma \in L^\infty\big(]0,a]\big)$ for all $\varphi \in L^\infty\big(]0,a]\big)$

$$\int_0^a \varphi(x)\left[x^\sigma v^n(x,t) - x^\sigma v(x,t)\right]dx = \int_0^a \varphi(x)\left[u^n(x,t)-x^\sigma v(x,t)\right]dx \to 0.$$

Since $u^n \rightharpoonup u$ we have due to the uniqueness of the weak limit of weak convergence, $v(x,t) = x^{-\sigma}u(x,t)$. □

2.3 The existence theorem

2.3.1 Convergence of the integrals

In order to show that the limit function which we obtained above is indeed a solution to (2.1)-(2.2), we define the operators M_i^n, M_i, $i = 1, 2$

$$M_1^n(u^n)(x) = \tfrac{1}{2}\int\limits_0^x K_n(x-y,y)u^n(x-y)u^n(y)dy \quad M_1(u)(x) = \tfrac{1}{2}\int\limits_0^x K(x-y,y)u(x-y)u(y)dy$$
$$M_2^n(u^n)(x) = \int\limits_0^{n-x} K_n(x,y)u^n(x)u^n(y)dy \qquad M_2(u)(x) = \int\limits_0^{\infty} K(x,y)u(x)u(y)dy,$$

where $u \in L^1\big(]0,\infty[\big)$, $x \in [0,\infty[$ and $n = 1, 2, \ldots$. Set $M^n = M_1^n - M_2^n$ and $M = M_1 - M_2$.

Lemma 2.3.1. *Suppose that* $\big(u^n\big)_{n\in\mathbb{N}} \subset Y^+$, $u \in Y^+$ *where* $\|u^n\|_Y \le L$, $\|u\|_Y \le Q$, $u^n \rightharpoonup u$ *and* $v^n \rightharpoonup v$ *in* $L^1\big(]0,\infty[\big)$ *as* $n \to \infty$. *Then for each* $a > 0$

$$M^n(u^n) \rightharpoonup M(u) \quad in \quad L^1\big(]0,a[\big) \quad as \quad n \to \infty.$$

Proof: Choose $a > 0$ and let $\phi \in L^\infty\big(]0,\infty[\big)$. We show that $M_i^n(u^n) \rightharpoonup M_i(u)$ in $L^1\big(]0,a[\big)$ as $n \to \infty$ for $i = 1, 2$.

Case i=1: For $u \in Y^+$ and $x \in [0,a]$ we define the operator g by

$$g(v)(x) = \frac{1}{2}\int\limits_0^{a-x} \phi(x+y)K(x,y)(xy)^\sigma v(y)\,dy \quad \text{where} \quad v = x^{-\sigma}u.$$

We consider

$$\begin{aligned}
\int\limits_0^a \phi(x)M_1^n(u^n)(x)\,dx &= \int\limits_0^a \phi(x)\frac{1}{2}\int\limits_0^x K_n(x-y,y)u^n(x-y)u^n(y)\,dy\,dx \\
&= \frac{1}{2}\int\limits_0^a \int\limits_0^{a-x} \phi(x+y)K_n(x,y)u^n(x)u^n(y)\,dy\,dx \\
&= \frac{1}{2}\int\limits_{1/n}^{a-1/n} \int\limits_{1/n}^{a-x} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx.
\end{aligned}$$

Splitting the inner integral in the right hand side we obtain

$$\begin{aligned}
&\int_0^a \phi(x)M_1^n(u^n)(x)\,dx \\
&= \frac{1}{2}\int_{1/n}^{a-1/n}\int_0^{a-x} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \\
&\quad -\frac{1}{2}\int_{1/n}^{a-1/n}\int_0^{1/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \\
&= \int_{1/n}^{a-1/n} g(v^n)(x)v^n(x)\,dx - \frac{1}{2}\int_{1/n}^{a-1/n}\int_0^{1/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx. \qquad (2.40)
\end{aligned}$$

In a similar way we also find that

$$\int_0^a \phi(x)M_1(u)(x)\,dx = \int_0^a g(v)(x)x^{-\sigma}u(x)\,dx = \int_0^a g(v)(x)v(x)\,dx. \qquad (2.41)$$

For *a.e.* $x \in [0,a]$ the function defined by

$$\varphi_x(y) := \frac{1}{2}\chi_{[0,a-x]}(y)\phi(x+y)K(x,y)(xy)^\sigma \le \frac{1}{2}\chi_{[0,a-x]}(y)\phi(x+y)(1+x+y)^\lambda$$

where χ denotes the characteristic function, is in $L^\infty\big(]0,\infty[\big)$. Since $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$, it follows that

$$g(v^n)(x) \to g(v)(x) \quad \text{for a.e.} \quad x \in [0,a]. \qquad (2.42)$$

Also, by (2.35) we have

$$\begin{aligned}
|g(v^n)(x)| &= \frac{1}{2}\int_0^{a-x} \phi(x+y)K(x,y)(xy)^\sigma v^n(y)\,dy \\
&\le \frac{1}{2}\int_0^{a-x} \phi(x+y)(1+x+y)^\lambda v^n(y)\,dy \\
&\le \frac{1}{2}(1+2a)^\lambda \|\phi\|_{L^\infty([0,a])}L \quad \text{for a.e.} \quad x \in [0,a]. \qquad (2.43)
\end{aligned}$$

This holds similarly for $g(v)$. Thus, both, $g(v^n)$ and $g(v)$ are in $L^\infty\big([0,a]\big)$ with bound

$$\|g(v^n)\|_{L^\infty([0,a])} + \|g(v)\|_{L^\infty([0,a])} \le \frac{1}{2}(1+2a)^\lambda\|\phi\|_{L^\infty([0,a])}(L+Q). \qquad (2.44)$$

It follows by (2.42) and Egorov's Theorem, see Appendix A [Theorem A.0.7], that

$$g(v^n) \to g(v) \quad \text{as} \quad n \to \infty \quad \text{almost uniformly in} \quad [0,a]. \qquad (2.45)$$

Remember that almost uniformly means that for any given δ there exists a set $E \subseteq [0,a]$ such that $\mu(E) < \delta$ and $g(v^n) \to g(v)$ uniformly on $[0,a] \setminus E$ as $n \to \infty$.

By Lemma 2.2.5**(iii)**, since $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$ there is a $\delta > 0$ such that for all n

$$\int\limits_A v^n(x)\,dx < \epsilon / \Big[(1+2a)^\lambda \|\phi\|_{L^\infty([0,a])}(L+Q)\Big] \quad \text{whenever} \quad \mu(A) < \delta. \tag{2.46}$$

Taking $A = E$ we obtain using (2.44) and (2.46)

$$\begin{aligned}
&\left|\int\limits_0^a [g(v^n)(x) - g(v)(x)]\, v^n(x)\,dx\right| \\
&\quad \le \left|\int_{[0,a]\setminus E} [g(v^n)(x) - g(v)(x)]\, v^n(x)\,dx\right| + \left|\int_E [g(v^n)(x) - g(v)(x)]\, v^n(x)\,dx\right| \\
&\quad \le \|g(v^n) - g(v)\|_{L^\infty([0,a]\setminus E)} \int_{[0,a]\setminus E} v^n(x)\,dx + \big(\|g(v^n)\|_{L^\infty(E)} + \|g(v)\|_{L^\infty(E)}\big) \int_E v^n(x)\,dx \\
&\quad \le \|g(v^n) - g(v)\|_{L^\infty([0,a]\setminus E)} \int_{[0,a]\setminus E} v^n(x)\,dx + \frac{\epsilon}{2} \le \epsilon \quad \text{for} \quad n \ge n_0.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrarily chosen

$$\left|\int\limits_0^a [g(v^n)(x) - g(v)(x)]\, v^n(x)\,dx\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.47}$$

Also, since $g(v) \in L^\infty\big([0,a]\big)$ is bounded independently of n by (2.43) and $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$ as $n \to \infty$

$$\left|\int\limits_0^a g(v)(x)\,[v^n(x) - v(x)]\,dx\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.48}$$

Now, since $g(v^n) \in L^\infty\big([0,a]\big)$ and $v^n \in L^1\big([0,\infty[\big)$ and uniformly bounded in the respective norms, by the absolute continuity of the Lebesgue integral, we have

$$\left|\int\limits_{a-1/n}^a g(v^n)(x) v^n(x)\,dx\right| \le \|g(v^n)\|_{L^\infty([0,a])} \left|\int\limits_{a-1/n}^a v^n(x)\,dx\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.49}$$

In the same way we get

$$\left|\int\limits_0^{1/n} g(v^n)(x) v^n(x)\,dx\right| \to 0 \quad \text{as} \quad n \to \infty, \tag{2.50}$$

and

$$
\begin{aligned}
&\left| \int_{1/n}^{a-1/n} \int_{0}^{1/n} \phi(x+y)K(x,y)u(x)u(y)\,dy\,dx \right| \\
&\quad \leq \left| \int_{1/n}^{a-1/n} \int_{0}^{1/n} \phi(x+y)(1+x+y)^{\lambda} v(x)v(y)\,dy\,dx \right| \to 0 \quad \text{as} \quad n \to \infty. \qquad (2.51)
\end{aligned}
$$

Now, it follows from (2.40) and (2.41) that

$$
\begin{aligned}
&\left| \int_{0}^{a} \phi(x)\left[M_1^n(u^n)(x) - M_1(u)(x)\right] dx \right| \\
&= \left| \int_{1/n}^{a-1/n} g(v^n)(x)v^n(x)\,dx - \int_{1/n}^{a-1/n} \int_{0}^{1/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx - \int_{0}^{a} g(v)(x)v(x)\,dx \right| \\
&= \Bigg| \int_{0}^{a} g(v^n)(x)v^n(x)\,dx - \int_{0}^{a} g(v)(x)v(x)\,dx - \int_{a-1/n}^{a} g(v^n)(x)v^n(x)\,dx \\
&\quad - \int_{0}^{1/n} g(v^n)(x)v^n(x)\,dx - \int_{1/a}^{a-1/n} \int_{0}^{1/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \Bigg| \\
&\leq \left| \int_{0}^{a} g(v^n)(x)v^n(x)\,dx - \int_{0}^{a} g(v)(x)v(x)\,dx \right| + \left| \int_{a-1/n}^{a} g(v^n)(x)v^n(x)\,dx \right| \\
&\quad + \left| \int_{0}^{1/n} g(v^n)(x)v^n(x)\,dx \right| + \left| \int_{1/n}^{a-1/n} \int_{0}^{1/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \right|.
\end{aligned}
$$

By addition and subtraction of the term $\int_{0}^{a} g(v)(x)v^n(x)\,dx$ in the first term of the above inequality, it results that

$$\begin{aligned}
&\left|\int_0^a \phi(x)\left[M_1^n(u^n)(x) - M_1(u)(x)\right] dx\right| \\
&\quad \leq \left|\int_0^a g(v)(x)\left[v^n(x) - v(x)\right] dx\right| + \left|\int_0^a \left[g(v^n)(x) - g(v)(x)\right] v^n(x)\, dx\right| \\
&\qquad + \left|\int_{a-1/n}^{a} g(v^n)(x) v^n(x)\, dx\right| + \left|\int_0^{1/n} g(v^n)(x) v^n(x)\, dx\right| \\
&\qquad + \left|\int_{1/n}^{a-1/n} \int_0^{1/n} \phi(x+y) K(x,y) u^n(x) u^n(y)\, dy\, dx\right|.
\end{aligned}$$

Now, by (2.47)-(2.51) and taking $n \to \infty$ we have

$$\left|\int_0^a \phi(x)\left[M_1^n(u^n)(x) - M_1(u)(x)\right] dx\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.52}$$

It follows, since ϕ is arbitrary, that

$$M_1^n(u^n)(x) \rightharpoonup M_1(u)(x) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{2.53}$$

Case $i = 2$: For every $\epsilon > 0$ and ν defined by (2.37) we can choose b such that

$$\nu \|\phi\|_{L^\infty([0,a])} \left[\left(2b^{-(1+\sigma)} + b^{\lambda-\sigma-1}\right)(L^2 + Q^2)\right] < \frac{\epsilon}{3}. \tag{2.54}$$

Redefining the operator g for $u \in Y^+$ and $x \in [0,a]$ by

$$g(v)(x) = \int_0^b \phi(x) K(x,y)(xy)^\sigma v(y)\, dy.$$

For $a.e.$ $x \in [0,a]$ the function defined by

$$\varphi_x(y) := \frac{1}{2}\chi_{[0,b]}(y)\phi(x+y)K(x,y)(xy)^\sigma$$

where, as before, χ denotes the characteristic function, is in $L^\infty\big(]0,\infty[\big)$. Using a similar argument as the one that was used in (2.42)-(2.47) it can be shown that also for the above redefined

g (2.47) and (2.48) hold. By **(H3)** and ν as in (2.37) we have

$$\begin{aligned}
&\left|\int_0^a\int_b^\infty \phi(x)K(x,y)\left[u^n(x)u^n(y)-u(x)u(y)\right]dy\,dx\right| \\
&\quad\le \nu\int_0^a\int_b^\infty |\phi(x)|\left(1+x^\lambda+y^\lambda\right)(xy)^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx \\
&\quad\le \nu\|\phi\|_{L^\infty([0,a])}\left[\int_0^a\int_b^\infty x^{-\sigma}y^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\right. \\
&\qquad\qquad+\int_0^a\int_b^\infty x^{\lambda-\sigma}y^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx \\
&\qquad\qquad\left.+\int_0^a\int_b^\infty x^{-\sigma}y^{\lambda-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\right].
\end{aligned} \tag{2.55}$$

We can estimate the first integral term of (2.55) as follows

$$\begin{aligned}
&\int_0^a\int_b^\infty x^{-\sigma}y^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx \\
&\le \int_b^\infty y^{-\sigma}\left[\int_0^1 x^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx+\int_1^a x^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy \\
&\le \int_b^\infty y^{-\sigma}\left[\int_0^1 x^{-1}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx+\int_1^a x\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy \\
&\le \int_b^\infty y^{-\sigma}\left[\int_0^a (x^{-1}+x)\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy \\
&\le b^{-(1+\sigma)}\int_b^\infty y\left[Lu^n(y)+Qu(y)\right]dy\le b^{-(1+\sigma)}(L^2+Q^2).
\end{aligned} \tag{2.56}$$

In the similar way we have

$$\int_0^a\int_b^\infty x^{\lambda-\sigma}y^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\le b^{-(1+\sigma)}(L^2+Q^2), \tag{2.57}$$

and

$$\int_0^a\int_b^\infty x^{-\sigma}y^{\lambda-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\le b^{\lambda-\sigma-1}(L^2+Q^2). \tag{2.58}$$

By (2.56)-(2.58) and (2.54), (2.55) becomes

$$\begin{aligned}&\left|\int_0^a\int_b^\infty \phi(x)K(x,y)\left[u^n(x)u^n(y)-u(x)u(y)\right]dy\,dx\right|\\&\quad\le \nu\|\phi\|_{L^\infty([0,a])}\left[(2b^{-(1+\sigma)}+b^{\lambda-\sigma-1})(L^2+Q^2)\right]<\frac{\epsilon}{3}.\end{aligned} \tag{2.59}$$

Now, using Lemma 2.2.5**(i)** and (2.35) together with the absolute continuity of the Lebesgue integral, we have

$$\left|\int_0^a\int_0^{1/n}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx\right|\le\frac{\epsilon}{3}\quad\text{for } n \text{ larger than some } n_0, \tag{2.60}$$

and

$$\left|\int_0^{1/n}\int_{1/n}^{n-x}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx\right|\le\frac{\epsilon}{3}\quad\text{for } n\ge n_0. \tag{2.61}$$

Also, proceeding as before, for $n>a$ we have

$$\begin{aligned}&\left|\int_0^a\int_{n-x}^\infty \phi(x)K(x,y)\left[u^n(x)u^n(y)\right]dy\,dx\right|\\&\quad\le \nu\|\phi\|_{L^\infty([0,a])}\left[(2(n-a)^{-(1+\sigma)}+(n-a)^{\lambda-\sigma-1})(L^2+Q^2)\right].\end{aligned} \tag{2.62}$$

Now we have

$$\begin{aligned}&\left|\int_0^a\phi(x)\left[M_2^n(u^n)(x)-M_2(u)(x)\right]dx\right|\\&=\left|\int_0^a\int_0^{n-x}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx-\int_0^a\int_0^\infty\phi(x)K(x,y)u(x)u(y)\,dy\,dx\right.\\&\quad\left.-\int_0^a\int_0^{1/n}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx-\int_0^{1/n}\int_{1/n}^{n-x}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx\right|.\end{aligned}$$

From (2.59)-(2.62) together with the analogues of (2.47) and (2.48), for $n > a$

$$\begin{aligned}
&\left| \int_0^a \phi(x) \left[M_2^n(u^n)(x) - M_2(u)(x) \right] dx \right| \\
&= \left| \int_0^a \int_0^b \phi(x) K(x,y) \left[u^n(x) u^n(y) - u(x) u(y) \right] dy\, dx \right. \\
&\quad + \int_0^a \int_b^\infty \phi(x) K(x,y) \left[u^n(x) u^n(y) - u(x) u(y) \right] dy\, dx \\
&\quad - \int_0^a \int_0^{1/n} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx - \int_0^{1/n} \int_{1/n}^{n-x} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \\
&\quad \left. - \int_0^a \int_{n-x}^\infty \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| \\
&\le \left| \int_0^a \left[g(v^n)(x) - g(v)(x) \right] v^n(x)\, dx \right| + \left| \int_0^a g(v)(x) \left[v^n(x) - v(x) \right] dx \right| + \epsilon \\
&\quad + \nu \|\phi\|_{L^\infty([0,a])} \left[\left((n-a)^{-(1+\sigma)} + (n-a)^{\lambda - \sigma - 1} \right) (L^2 + Q^2) \right] \to \epsilon \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, since ϕ and ϵ are arbitrary, we conclude that

$$M_2^n(u^n)(x) \rightharpoonup M_2(u)(x) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{2.63}$$

Lemma 2.3.1 follows from (2.53) and (2.63). $\square$

2.3.2 The existence result

Theorem 2.3.2. *Suppose that **(H1)**, **(H2)**, and **(H3)** hold and assume that $u_0 \in Y^+$. Then (2.1) has a solution $u \in C_B\left([0,\infty[; L^1\big(]0,\infty[\big)\right)$.*

Proof. Choose $T, m > 0$, and let $\left(u^n\right)_{n\in\mathbb{N}}$ be the weakly convergent subsequence of approximating solutions obtained above, in the proof of Lemma 2.2.6. From Lemma 2.2.6 we have $u \in C_B\left([0,\infty[; \Omega_1\right)$. For $t \in [0,T]$ we obtain due to weak convergence

$$\int_0^m x u(x,t)\, dx = \lim_{n\to\infty} \int_0^m x u^n(x,t)\, dx \quad \text{and} \quad \int_{1/m}^m x^{-1} u(x,t)\, dx = \lim_{n\to\infty} \int_{1/m}^m x^{-1} u^n(x,t)\, dx.$$

Using the mass conservation property (2.24) and (2.25), this gives the uniform estimate

$$\int_0^m x u(x,t)\, dx + \int_{1/m}^m x^{-1} u(x,t)\, dx \le 2L \quad \text{for any } n \in \mathbb{N}.$$

Then taking $m \to \infty$ the uniqueness of weak limits implies that $u \in Y^+$ with $\|u\|_Y \leq 2L$. Let $\phi \in L^\infty(]0,a[)$. From Lemma 2.2.6 we have for each $s \in [0,t]$

$$u^n(t) \rightharpoonup u(t) \quad \text{in} \quad L^1(]0,a[) \quad \text{as} \quad n \to \infty. \tag{2.64}$$

For Lemma 2.2.6 and Lemma 2.3.1 for each $s \in [0,t]$ we have for $M^n = M_1^n - M_2^n$ and $M = M_1 - M_2$

$$\int_0^a \phi(x)\left[M^n(u^n(s))(x) - M(u(s))(x)\right] dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.65}$$

Also, for $s \in [0,t]$, using Lemma 2.2.5**(i)**, (2.35), $\|u\|_Y \leq 2L$, and ν as in (2.37) we find that

$$\begin{aligned}
&\int_0^a |\phi(x)|\,|M^n(u^n(s))(x) - M(u(s))(x)|\,dx \\
&\leq \|\phi\|_{L^\infty(]0,a[)}\left[\frac{1}{2}\int_0^a\int_0^x K(x-y,y)\left[u^n(x-y,s)u^n(y,s) + u(x-y,s)u(y,s)\right] dy\,dx\right. \\
&\qquad\left. + \int_0^a\int_0^{n-x} K(x,y)u^n(x,s)u^n(y,s)\,dy\,dx + \int_0^a\int_0^\infty K(x,y)u(x,s)u(y,s)\,dy\,dx\right] \\
&\leq \|\phi\|_{L^\infty(]0,a[)}\left[\frac{5}{2}(1+2a)^\lambda + 19\nu\right] L^2.
\end{aligned} \tag{2.66}$$

Since the left hand side of (2.66) is in $L^1(]0,t[)$ we have by (2.65), (2.66) and the dominated convergence theorem

$$\left|\int_0^t\int_0^a \phi(x)\left[M^n(u^n(s))(x) - M(u(s))(x)\right] dx\,ds\right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.67}$$

Since ϕ was chosen arbitrarily the limit (2.67) holds for all $\phi \in L^\infty(]0,a[)$. By Fubini's Theorem we get

$$\int_0^t M^n(u^n(s))(x)\,ds \rightharpoonup \int_0^t M(u(s))(x)\,ds \quad \text{in} \quad L^1(]0,a[) \quad \text{as} \quad n \to \infty. \tag{2.68}$$

From the definition of M^n for $t \in [0,T]$

$$u^n(t) = \int_0^t M^n(u^n(s))\,ds + u^n(0).$$

Thus it follows by (2.68), (2.64) and the uniqueness of weak limits that

$$u(x,t) = \int_0^t M(u(s))(x)\,ds + u(x,0) \quad \text{for a.e.} \quad x \in [0,a]. \tag{2.69}$$

It follows from the fact that T and a are arbitrary that u is a solution to (2.1) on $C_B\big([0,\infty[;\Omega_1\big)$. In order to show that $u \in C_B\left([0,\infty[; L^1\big(]0,\infty[\big)\right)$ we consider $t_n > t$ and by using (2.69) we have that

$$\begin{aligned}\int_0^\infty |u(x,t_n)-u(x,t)|dx &= \int_0^\infty \Bigg|\frac{1}{2}\int_t^{t_n}\int_0^x K(x-y,y)u(x-y,\tau)u(y,\tau)dy\,d\tau \\ &\qquad - \int_t^{t_n}\int_0^\infty K(x,y)u(x,\tau)u(y,\tau)dy\,d\tau\Bigg|dx \\ &\le \frac{3}{2}\int_t^{t_n}\int_0^\infty\int_0^\infty K(x,y)u(x,\tau)u(y,\tau)dy\,dx\,d\tau\end{aligned}$$

By using the definition (2.37) of C, **(H3)** and Lemma 2.2.5**(i)** we find that

$$\begin{aligned}\int_0^\infty |u(x,t_n)-u(x,t)|dx &\le \frac{3}{2}\int_t^{t_n}\int_0^\infty\int_0^\infty (1+x+y)^\lambda (xy)^{-\sigma}u(x,\tau)u(y,\tau)\,dy\,dx \\ &\le \frac{3}{2}C\int_t^{t_n}\int_0^\infty\int_0^\infty (1+x^\lambda+y^\lambda)(xy)^{-\sigma}u(x,\tau)u(y,\tau)\,dy\,dx \\ &= \frac{3}{2}C\int_t^{t_n}\int_0^\infty\int_0^\infty \left[(xy)^{-\sigma}+x^{\lambda-\sigma}y^{-\sigma}+y^{\lambda-\sigma}x^{-\sigma}\right]u(x,\tau)u(y,\tau)\,dy\,dx \\ &\le \frac{45}{2}CL^2(t_n-t). \end{aligned} \tag{2.70}$$

Then from (2.70) we obtain that

$$\int_0^\infty |u(x,t_n)-u(x,t)|dx \to 0 \quad \text{as} \quad t_n \to t. \tag{2.71}$$

The same argument holds when $t_n < t$. Hence (2.71) holds for $|t_n - t| \to 0$ and we can conclude that $u \in C_B\left([0,\infty[; L^1\big(]0,\infty[\big)\right)$. This completes the proof of Theorem 2.3.2. $\square$

2.4 Uniqueness of solutions

In this section we study the uniqueness of the solution to (2.1)-(2.2) under the following further restriction on the kernels.

(H3') $K(x,y) \le \kappa_1(x^{-\sigma}+x^{\lambda-\sigma})(y^{-\sigma}+y^{\lambda-\sigma})$ *such that* $\sigma,\ \lambda-\sigma \in [0,1/2]$ *and* $\kappa_1 > 0$.

The restriction $\lambda-\sigma \in [0,1/2]$ in **(H3')** limits our uniqueness result to a subset of the kernels of the class defined in **(H3)**, namely to the ones for which $\lambda-\sigma \in [0,1/2]$ holds. But the class of kernels defined in **(H3')** is also wider than the one defined in **(H3)** for $\lambda - \sigma \in [0, 1/2]$. In this way we are also giving a uniqueness result for kernels which are not included in the class defined in **(H3)**, see Appendix D.

Then, in order to prove the uniqueness of solutions to (2.1)-(2.2) we set the following hypotheses

Hypotheses 2.4.1.

(H1) $K(x,y)$ *is a continuous non-negative function on* $]0,\infty[\times]0,\infty[$,

(H2) $K(x,y)$ *is a symmetric function, i.e.* $K(x,y) = K(y,x)$ *for all* $x, y \in]0,\infty[$,

(H3') $K(x,y) \le \kappa_1 (x^{-\sigma} + x^{\lambda-\sigma})(y^{-\sigma} + y^{\lambda-\sigma})$ *such that* $\sigma,\ \lambda - \sigma \in [0, 1/2]$ *and* $\kappa_1 > 0$.

2.4.1 The uniqueness theorem

Theorem 2.4.2. *Assume that* ***(H1)***, ***(H2)***, *and* ***(H3')*** *hold then the problem (2.1)-(2.2) has a unique solution* $u \in C_B\left([0,\infty[; L^1\left(]0,\infty[\right)\right)$.

Proof: Let us consider u_1 and u_2 to be solutions to (2.1)-(2.2) on $[0, T[$ for $T > 0$, with $u_1(x, 0) = u_2(x, 0)$ and set $U = u_1 - u_2$. We define for $n = 1, 2, 3, \ldots$

$$m^n(t) = \int_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \, |U(x,t)| \, dx.$$

Now, for $\delta \in \mathbb{R}$, we define $\operatorname{sgn}\left(\delta\right)$ as follows

$$\operatorname{sgn}\left(\delta\right) = \begin{cases} 1 & \delta > 0 \\ 0 & \delta = 0 \\ -1 & \delta < 0. \end{cases}$$

Note that if $\zeta(\cdot)$ is an absolutely continuous function of t, then so is $t \mapsto |\zeta(t)|$, and

$$\frac{d}{dt}|\zeta(t)| = \operatorname{sgn}\left(\zeta(t)\right) \frac{d}{dt}\zeta(t) \quad \text{a.e.} \tag{2.72}$$

Also note that by Definition 2.1.2**(iv)**, $u(x,\cdot)$ is absolutely continuous on $[0, T]$ for a.e. $x \in [0,\infty[$ and therefore $u(x,t)$ satisfies (2.1) for a.e. $t \in [0, T]$, see Appendix A. Then, taking the difference of the derivative of the solutions u_1 and u_2 in (2.1) we have

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} = & \frac{1}{2} \int_0^x K(x-y, y) \left[u_1(x-y,t) u_1(y,t) - u_2(x-y,t) u_2(y,t) \right] dy \\ & - \int_0^\infty K(x,y) \left[u_1(x,t) u_1(y,t) - u_2(x,t) u_2(y,t) \, dy \right]. \end{aligned}$$

Applying (2.72) we find that

$$\frac{\partial |U(x,t)|}{\partial t} = \operatorname{sgn}\big(U(x,t)\big)\left[\frac{1}{2}\int_0^x K(x-y,y)\,[u_1(x-y,t)u_1(y,t)-u_2(x-y,t)u_2(y,t)]\,dy \right. \\ \left. -\int_0^\infty K(x,y)\,[u_1(x,t)u_1(y,t)-u_2(x,t)u_2(y,t)]\,dy\right].$$

Multiplying by $(x^{-\sigma}+x^{\lambda-\sigma})$ and integrating from 0 to t and from 0 to n w.r.t. τ and x respectively, and applying Fubini's Theorem, for each n and $0<t<T$ we obtain

$$m^n(t) = \int_0^t \int_0^n (x^{-\sigma}+x^{\lambda-\sigma})\operatorname{sgn}\big(U(x,\tau)\big) \\ \cdot\left[\frac{1}{2}\int_0^x K(x-y,y)\,[u_1(x-y,\tau)u_1(y,\tau)-u_2(x-y,\tau)u_2(y,\tau)]\,dy \right. \\ \left. -\int_0^\infty K(x,y)\,[u_1(x,\tau)u_1(y,\tau)-u_2(x,\tau)u_2(y,\tau)]\,dy\right] dx\,d\tau. \tag{2.73}$$

Using the substitution $y-x=x'$ in the first inner integrals w.r.t. x and y on the right hand side of (2.73) we find that it becomes

$$\int_0^n (x^{-\sigma}+x^{\lambda-\sigma})\operatorname{sgn}\big(U(x,\tau)\big)\frac{1}{2}\int_0^x K(x-y,y) \\ \cdot[u_1(x-y,\tau)u_1(y,\tau)-u_2(x-y,\tau)u_2(y,\tau)]\,dy\,dx \\ = \int_0^n \int_0^{n-x} \frac{1}{2}\left[(x+y)^{-\sigma}+(x+y)^{\lambda-\sigma}\right]\operatorname{sgn}\big(U(x+y,\tau)\big)K(x,y) \\ \cdot[u_1(x,\tau)u_1(y,\tau)-u_2(x,\tau)u_2(y,\tau)]\,dy\,dx.$$

Inserting this into (2.73) gives

$$m^n(t) = \int_0^t\int_0^n\int_0^{n-x}\left[\frac{1}{2}\left[(x+y)^{-\sigma}+(x+y)^{\lambda-\sigma}\right]\operatorname{sgn}\big(U(x+y,\tau)\big)-(x^{-\sigma}+x^{\lambda-\sigma})\operatorname{sgn}\big(U(x,\tau)\big)\right] \\ \cdot K(x,y)\,[u_1(x,\tau)u_1(y,\tau)-u_2(x,\tau)u_2(y,\tau)]\,dy\,dx\,d\tau \\ -\int_0^t\int_0^n\int_{n-x}^\infty (x^{-\sigma}+x^{\lambda-\sigma})\operatorname{sgn}\big(U(x,\tau)\big)K(x,y) \\ \cdot[u_1(x,\tau)u_1(y,\tau)-u_2(x,\tau)u_2(y,\tau)]\,dy\,dx\,d\tau. \tag{2.74}$$

We note now, using the symmetry of K, by interchanging the order of integration and interchanging the roles of x and y yields the identity

$$\begin{aligned}&\int_0^n \int_0^{n-x} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) K(x,y)\, [u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)]\, dy\, dx \\ &= \int_0^n \int_0^{n-x} (y^{-\sigma} + y^{\lambda-\sigma}) \operatorname{sgn}\big(U(y,\tau)\big) K(x,y)\, [u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)]\, dy\, dx.\end{aligned} \tag{2.75}$$

For $x, y \geq 0$ and $t \in [0, T[$ we define the function w by

$$\begin{aligned} w(x,y,t) = &\Big[(x+y)^{-\sigma} + (x+y)^{\lambda-\sigma}\Big] \operatorname{sgn}\big(U(x+y,t)\big) \\ &-(x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,t)\big) - (y^{-\sigma} + y^{\lambda-\sigma}) \operatorname{sgn}\big(U(y,t)\big). \end{aligned} \tag{2.76}$$

Using (2.75) and this definition, we can rewrite (2.74) as

$$\begin{aligned} m^n(t) = &\frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau) K(x,y)\, [u_1(x,\tau)U(y,\tau) + u_2(y,\tau)U(x,\tau)]\, dy\, dx\, d\tau \\ &- \int_0^t \int_0^n \int_{n-x}^{\infty} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) K(x,y) \\ &\qquad \cdot [u_1(x,\tau)U(y,\tau) + u_2(y,\tau)U(x,\tau)]\, dy\, dx\, d\tau. \end{aligned} \tag{2.77}$$

Since the second term in the third integral of (2.77) is positive, we can eliminate and get the following estimate

$$\begin{aligned} m^n(t) \leq &\frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau) K(x,y) u_1(x,\tau) U(y,\tau)\, dy\, dx\, d\tau \\ &+ \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau) K(x,y) u_2(y,\tau) U(x,\tau)\, dy\, dx\, d\tau \\ &- \int_0^t \int_0^n \int_{n-x}^{\infty} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) K(x,y) u_1(x,\tau) U(y,\tau)\, dy\, dx\, d\tau \\ = &\int_0^t [I_{31}(\tau) + I_{32}(\tau) + I_{33}(\tau)]\, d\tau. \end{aligned} \tag{2.78}$$

Taking in account that for all $p_1, p_2 \in \mathbb{R}$ that $\operatorname{sgn}(p_1)\operatorname{sgn}(p_2) = \operatorname{sgn}(p_1 p_2)$ and $|p_1| = p_1 \operatorname{sgn}(p_1)$ hold, we can estimate

$$w(x,y,t)U(y,t) \leq \Big[\Big[(x+y)^{-\sigma} + (x+y)^{\lambda-\sigma}\Big] + (x^{-\sigma} + x^{\lambda-\sigma}) - (y^{-\sigma} + y^{\lambda-\sigma})\Big] |U(y,t)|\,.$$

Using the inequalities (2.4) and (2.7) we find that

$$\begin{aligned}
&w(x,y,t)U(y,t)\\
&\leq \left[\left[(x^{-\sigma}+y^{-\sigma})+(x^{\lambda-\sigma}+y^{\lambda-\sigma})\right]+(x^{-\sigma}+x^{\lambda-\sigma})-(y^{-\sigma}+y^{\lambda-\sigma})\right]|U(y,t)|\\
&\leq 2(x^{-\sigma}+x^{\lambda-\sigma})\,|U(y,t)|\,. \qquad (2.79)
\end{aligned}$$

Now, we use (2.79) to work on each term of the right hand side of (2.78)

$$\int_0^t I_{31}(\tau)\,d\tau \leq \int_0^t\int_0^n\int_0^{n-x}(x^{-\sigma}+x^{\lambda-\sigma})K(x,y)u_1(x,\tau)\,|U(y,\tau)|\,dy\,dx\,d\tau. \qquad (2.80)$$

Using the estimate **(H3')** of $K(x,y)$ and (2.5) we get

$$\begin{aligned}
\int_0^t I_{31}(\tau)\,d\tau &\leq \kappa_1\int_0^t\int_0^n\int_0^{n-x}(x^{-\sigma}+x^{\lambda-\sigma})(x^{-\sigma}+x^{\lambda-\sigma})(y^{-\sigma}+y^{\lambda-\sigma})u_1(x,\tau)\,|U(y,\tau)|\,dy\,dx\,d\tau\\
&=\kappa_1\int_0^t\int_0^n\int_0^{n-x}(x^{-\sigma}+x^{\lambda-\sigma})^2u_1(x,\tau)(y^{-\sigma}+y^{\lambda-\sigma})\,|U(y,\tau)|\,dy\,dx\,d\tau\\
&\leq 2\kappa_1\int_0^t\int_0^n\int_0^{n-x}\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)(y^{-\sigma}+y^{\lambda-\sigma})\,|U(y,\tau)|\,dy\,dx\,d\tau.
\end{aligned}$$

Due to $\lambda-\sigma,\sigma\in[0,1/2]$ and the definition of $m^n(t)$ from the inequality above follows

$$\begin{aligned}
&\int_0^t I_{31}(\tau)\,d\tau\\
&\leq 2\kappa_1\int_0^t m^n(\tau)\left[\int_0^1\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)\,dx+\int_1^n\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)dx\right]d\tau\\
&\leq 2\kappa_1\int_0^t m^n(\tau)\left[\int_0^1(x^{-1}+1)u_1(x,\tau)\,dx+\int_1^n(1+x)u_1(x,\tau)\,dx\right]d\tau\\
&\leq 2\kappa_1\int_0^t m^n(\tau)\left[2\int_0^n x^{-1}u_1(x,\tau)\,dx+2\int_0^n xu_1(x,\tau)\,dx\right]d\tau\\
&\leq \Lambda_1\int_0^t m^n(\tau)\,d\tau, \quad \text{where} \quad \Lambda_1=4\kappa_1\sup_{s\in[0,t]}\|u_1(s)\|_Y. \qquad (2.81)
\end{aligned}$$

In the same way, there is a constant Λ_2 such that

$$\int_0^t I_{32}(\tau)\,d\tau \leq \Lambda_2\int_0^t m^n(\tau)\,d\tau. \qquad (2.82)$$

To consider I_{33} we first see that

$$\begin{aligned} &\left| \int_0^\infty \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,t)\big) K(x,y) u_1(x,t) U(y,t)\, dy\, dx \right| \\ &\qquad \le 2\kappa_1 \int_0^\infty \int_0^\infty \left(x^{-2\sigma} + x^{2(\lambda-\sigma)}\right) (y^{-\sigma} + y^{\lambda-\sigma}) u_1(x,t) \left|U(y,t)\right| dy\, dx < \infty. \end{aligned}$$

Thus, the dominated convergence theorem leads to

$$\int_0^t I_{33}(\tau)\, d\tau \to 0 \quad \text{as} \quad n \to \infty. \tag{2.83}$$

Therefore, due to (2.78), (2.81), (2.82), (2.83) and taking $\Lambda = \Lambda_1 + \Lambda_2$ we obtain

$$\begin{aligned} m(t) := \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \left|U(x,t)\right| dx &= \lim_{n\to\infty} m^n(t) \\ &\le \lim_{n\to\infty} \int_0^t \left[I_{31}(\tau) + I_{32}(\tau) + I_{33}(\tau)\right] d\tau \\ &\le \lim_{n\to\infty} \Lambda \int_0^t m^n(\tau)\, d\tau + \lim_{n\to\infty} \int_0^t I_{33}(\tau)\, d\tau \\ &= \Lambda \int_0^t \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \left|U(x,t)\right| dx\, d\tau. \end{aligned}$$

From where we have the inequality

$$m(t) \le \Lambda \int_0^t m(\tau)\, d\tau. \tag{2.84}$$

Applying Gronwall's inequality, see e.g. Appendix A[Theorem A.0.6] to (2.84), we obtain

$$m(t) = \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \left|U(x,t)\right| dx = 0 \quad \text{for all} \quad t \in [0, T[.$$

Thus,

$$u_1(x,t) = u_2(x,t) \quad \text{for a.e.} \quad x \in]0, \infty[.$$

$\square$

Chapter 3

The coagulation equations with multifragmentation

In this chapter we deal with our result on existence and uniqueness of solutions to the singular coagulation equations with multifragmentation. As a base of our proof we applied a weak L^1 compactness method to a suitably chosen approximating equations. Our result is obtained again in a suitable weighted Banach space of L^1 functions

$$Y^+ = \left\{ u \in L^1 : \int_0^\infty \left(x + x^{-2\sigma} \right) |u| dx < \infty,\, u > 0 \,\text{a.e.} \right\}$$

for non-negative initial data $u_0 \in Y^+$. The result we give here is an extension of the previous result in Chapert 2. This result includes coagulation kernels with singularities on the axes and multifragmentation kernels which can also present singularities on the axes.

The chapter is organized as follows. In Section 3.1 we present the hypotheses of our problem and introduce some necessary definitions. In Section 3.2 we define the a sequence of truncated problems and prove in Theorem 3.2.4 the existence and uniqueness of solutions to them. We extract a weakly convergent subsequence in L^1 from a sequence of unique solutions for truncated equations to (3.1)-(3.2). In Section 3.3 we show that the solution of (3.1) is actually the limit function obtained from the weakly convergent subsequence of solutions of the truncated problem. In Section 3.4 we prove the uniqueness, based on the method of Stewart [37], of the solutions to (3.1)-(3.2) for a modification of the classes of coagulation and fragmentation kernels. We obtain uniqueness for some kernels which are not covered by the existence result.

3.1 Introduction

Let us represent, as in Chapter 2, by the non-negative variables x and t the size of a particle and time respectively. By $u(x,t)$ we denote the number density of particles with size x at time t. The rate at which particles of size x coalesce with particles of size y is represented by the coagulation kernel $K(x,y)$. Now, let us denote by $S(x)$ the rate at which particles of size x are selected to break. The breakage function $b(x,y)$ gives the number of particles of size x produced when a particle of size y breaks up. Then we recall the coagulation equation with multifragmentation (1.5) from Chapter 1

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)u(x-y,t)u(y,t)\,dy - \int_0^\infty K(x,y)u(x,t)u(y,t)\,dy + \int_x^\infty b(x,y)S(y)u(y,t)\,dy - S(x)u(x,t), \tag{3.1}$$

with initial condition

$$u(x,0) = u_0(x) \geq 0 \quad \text{a.e.} \tag{3.2}$$

In order to study the existence of solutions of (3.1)-(3.2), we define for some given $\sigma \in [0,1/2]$ the space Y to be the following space with norm $\|\cdot\|_Y$

$$Y = \left\{u \in L^1\big(]0,\infty[\big) : \|u\|_Y < \infty\right\} \quad \text{where} \quad \|u\|_Y = \int_0^\infty (x + x^{-2\sigma})|u(x,t)|dx.$$

By taking the function $x\exp\big(-x\big)dx$ we find that

$$\int_0^\infty x\exp(-x)dx = 1 \quad \text{and} \quad \int_0^\infty (x+x^{-2\sigma})x\exp(-x)dx \leq 3,$$

from where we have again that the above defined space Y is not empty.

Lemma 3.1.1. *The space Y is a Banach space*

Lemma 3.1.1 can be proven analogously as Lemma 2.1.1, see Appendix C. We also write

$$\|u\|_x = \int_0^\infty xu(x,t)dx \quad \text{and} \quad \|u\|_{x^{-2\sigma}} = \int_0^\infty x^{-2\sigma}u(x,t)dx,$$

and set

$$Y^+ = \{u \in Y : u \geq 0 \quad a.e.\}\,.$$

Now we define a weak solution to problem (3.1)-(3.2) in the same way as Stewart [36]:

Definition 3.1.2. *Let $T \in]0,\infty]$. A solution $u(x,t)$ of (3.1)-(3.2) is a function $u : [0,T[\longrightarrow Y^+$ such that for a.e. $x \in [0,\infty[$ and $t \in [0,T[$ the following properties hold*

(i) *$u(x,t) \geq 0$ for all $t \in [0,\infty[$,*

(ii) *$u(x,\cdot)$ is continuous on $[0,T[$,*

(iii) *for all* $t \in [0, T[$ *the following integral is bounded*

$$\int_0^t \int_0^\infty K(x,y)u(y,\tau)\,dy\,d\tau < \infty \quad \textit{and} \quad \int_0^t \int_x^\infty b(x,y)S(y)u(y,\tau)\,dy\,d\tau < \infty,$$

(iv) *for all* $t \in [0, T[$, u *satisfies the following weak formulation of (3.1)*

$$u(x,t) = u(x,0) + \int_0^t \left[\frac{1}{2} \int_0^x K(x-y,y)u(x-y,\tau)u(y,\tau)\,dy - \int_0^\infty K(x,y)u(x,\tau)u(y,\tau)\,dy \right.$$
$$\left. + \int_x^\infty b(x,y)S(y)u(y,t)\,dy - S(x)u(x,t) \right] d\tau.$$

In the next sections we make use of the following hypotheses

Hypotheses 3.1.3.

(H1) $K(x,y)$ *is a continuous non-negative function on* $]0,\infty[\times]0,\infty[$,

(H2) $K(x,y)$ *is a symmetric function, i.e.* $K(x,y) = K(y,x)$ *for all* $x, y \in]0,\infty[$,

(H4) $K(x,y) \le \kappa(1+x)^\lambda(1+y)^\lambda(xy)^{-\sigma}$ *for* $\lambda - \sigma \in [0,1[, \sigma \in [0,1/2]$, *and constant* κ,

(H5) $S\ :]0,\infty[\to[0,\infty[$ *is continuous and satisfies the bound* $0 \le S(x) \le x^\theta$ *for* $\theta \in [0,1[$,

(H6) $b(x,y) \ge 0$ *is such that* $\int_0^y b(x,y)x^{-2\sigma}dx \le Cy^{-2\sigma}$,

(H7) *There exist* $q > 1$ *and* $\tau_1, \tau_2 \in [-2\sigma - \theta, 1-\theta]$ *such that*

$$\int_0^y b^q(x,y) \le B_1 y^{q\tau_1}, \ \textit{and} \int_0^y x^{-q\sigma} b^q(x,y) \le B_2 y^{q\tau_2} \quad \textit{for constant } B_1, B_2 > 0$$

In the rest of the chapter we consider $\kappa = 1$ for the simplicity.

We recall the mass conservation property (1.7)

$$\int_0^y xb(x,y)dx = y \quad \text{for all} \quad y > 0, \tag{3.3}$$

and the property (1.8)

$$\int_0^y b(x,y)dx = N < \infty \quad \text{for all} \quad y > 0, \quad \text{and} \quad b(x,y) = 0 \quad \text{for} \quad x > y, \tag{3.4}$$

of the breakage function from Chapter 1.

We also recall the inequalities (2.4)-(2.7) from Chapter 2 Section 2.1.

For any $x, y \geq 0$

$$
\begin{aligned}
2^{p-1}(x^p + y^p) \leq (x+y)^p \leq x^p + y^p \qquad &\text{if} \qquad 0 \leq p \leq 1, &(3.5)\\
2^{p-1}(x^p + y^p) \geq (x+y)^p \geq x^p + y^p \qquad &\text{if} \qquad p \geq 1, &(3.6)\\
(x^p + y^p) \geq (x+y)^p \qquad &\text{if} \qquad p < 0, &(3.7)
\end{aligned}
$$

and for $x, y > 0$

$$
2^{p-1}(x^p + y^p) \geq (x+y)^p \qquad \text{if} \qquad p < 0. \tag{3.8}
$$

3.2 The truncated problem

We prove the existence of a solution to the problem (3.1)-(3.2) by taking the limit of the sequence of solutions of the equations given by replacing the kernel $K(x, y)$ and the selection function $S(x)$ by their respective 'cut-off' kernel $K_n(x, y)$ and $S_n(x)$ for any given $n \in \mathbb{N}$

$$
K_n(x,y) = \begin{cases} K(x,y) & \text{if} \quad x+y \leq n \ \text{ and } \ x,y \geq \sigma/n \\ 0 & \text{otherwise,} \end{cases} \qquad S_n(x) = \begin{cases} S(x) & \text{if} \quad x \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}
$$

Note that if we take $\sigma = 0$ our truncated problem will be defined as in Giri et al.[14]. For the defined kernels the resulting equations are written as

$$
\begin{aligned}
\frac{\partial u^n(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K_n(x-y,y)u^n(x-y,t)u^n(y,t)\,dy - \int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)\,dy \\
&\quad + \int_x^n S_n(y)b(x,y)u^n(y,t)\,dy - S_n(x)u^n(x,t),
\end{aligned} \tag{3.10}
$$

with the truncated initial data

$$
u_0^n(x) = \begin{cases} u_0(x) & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}
$$

where u^n denotes the solution of the problem (3.10)-(3.11) for $x \in [0, n]$. Next, we rewrite our truncated problem (3.10)-(3.11) in an equivalent form. We prove some lemmas, which are used to show the existence and uniqueness of the solution to the truncated problem.

3.2.1 Existence and uniqueness of solutions of the truncated problem

Let us define the operator P as

$$P(x,t,u^n) = \int_0^t \left[\int_0^{n-x} K_n(x,y)u^n(y,\tau)\,dy + S_n(x) \right] d\tau,$$

which allows us to rewrite the truncated problem (3.10)-(3.11) in the equivalent form

$$\frac{\partial}{\partial t}\left[u^n(x,t)\exp\big(P(x,t,u^n)\big)\right] = \frac{1}{2}\exp\big(P(x,t,u^n)\big)\left[\int_0^x K_n(x-y)u^n(x-y,t)u^n(y,t)\,dy + \int_x^n S_n(y)b(x,y)u^n(y,t)dy\right] \tag{3.12}$$

with

$$u_0^n(x) = \begin{cases} u_0(x) & \text{if} \quad 0 \le x \le n \\ 0 & \text{otherwise,} \end{cases} \tag{3.13}$$

Now, we define the operator G as

$$G(c)(x,t) = \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c)-P(x,\tau,c)]\big)\left[\int_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy + \int_x^n S_n(y)b(x,y)c(y,\tau)\,dy\right] d\tau + u_0^n(x)\exp\big(-P(x,t,c)\big), \tag{3.14}$$

for $c \in C\left([0,T];L^1\left(]0,n[\right)\right)$. By using (3.12) and (3.14), it can be easily checked that a solution u^n to (3.10)-(3.11) satisfies

$$u^n(x,t) = G(u^n)(x,t). \tag{3.15}$$

The problems (3.10)-(3.11) and (3.13)-(3.15) are equivalent. As a consequence, we prove the existence of solutions of the problem (3.13)-(3.15). With this aim, we use the contraction mapping principle in some interval $[0,T]$. But first, we introduce some necessary definitions.

Let us set

$$\begin{aligned} M &= \max\left\{\sup\left\{K_n(x,y) : x,y \in [0,n]\right\}, \sup\left\{S_n(x)b(x,y) : x,y \in [0,n]\right\}\right\}, \\ L &= (n^\sigma \sigma^{-\sigma} MT + 1)\|u_0^n\|_Y, \end{aligned} \tag{3.16}$$

and choose $t_1, t_2 \ge 0$ such that

$$\exp\left(2n^{2\sigma}MLt_1\right)\left(2^{1-2\sigma}n^{6\sigma}\sigma^{-2\sigma}MLt_1 + 2Cn^\theta t_1 + 1\right) \le 2, \tag{3.17}$$

$$t_2\exp\left(2n^{2\sigma}MLt_2\right)\left[2^{1-2\sigma}n^{6\sigma}\sigma^{-2\sigma}ML(Lt_2+1) + Cn^\theta(2Lt_2+1) + n^{2\sigma}ML\right] < 1. \tag{3.18}$$

We set

$$t_0 = \min(t_1, t_2, T).$$

For those $c \in C\left([0,T]; L^1\left(]0,n[\right)\right)$ for which $\int\limits_0^1 x^{-2\sigma}|c(x,t)|dx$ is finite for all $t \in [0,t_0]$, we define the norm $\|\cdot\|_D$ by

$$\|c\|_D = \sup_{t\in[0,t_0]} \int\limits_0^n x^{-2\sigma}|c(x,t)|\,dx.$$

Now we set

$$D = \left\{c \in C\left([0,t_0]; L^1\left(]0,n[\right)\right) : \|c\|_D \le 2L\right\}.$$

Then, by definition of P and the non-negativity of $S(x)$, for $c \in D$ we have

$$\begin{aligned}
\exp\big(-[P(x,t,c) - P(x,\tau,c)]\big) &= \exp\left(-\int\limits_\tau^t \left[\int\limits_0^{n-x} K_n(x,y)c(y,s)\,dy + S(x)\right] ds\right)\\
&\le \exp\left(-\int\limits_\tau^t \int\limits_0^{n-x} K_n(x,y)c(y,s)\,dy\,ds\right)\\
&\le \exp\big(n^{2\sigma}M\|c\|_D t\big) \le \exp\big(2n^{2\sigma}MLt\big). \qquad (3.19)
\end{aligned}$$

For the proof of Theorem 3.2.4 some lemmas are necessary which we present now.

Lemma 3.2.1. *The functional G maps the set D into itself.*

Proof. Choose c such that $\|c\|_D \le 2L$. For $t \in [0,t_0]$, using (3.14), (3.19), the definition of M, and Fubini's Theorem, we get

$$
\begin{aligned}
&\int_0^n |G(c)(x,t)| x^{-2\sigma}\, dx \\
&= \int_0^n \left| \frac{1}{2} \int_0^t \exp\big(- [P(x,t,c) - P(x,\tau,c)]\big) \left[\int_0^x K_n(x-y,y) c(x-y,\tau) c(y,\tau)\, dy \right.\right. \\
&\qquad \left.\left. + \int_x^n S_n(y) b(x,y) c(y,\tau)\, dy \right] d\tau + u_0^n(x) \exp\big(- P(x,t,c)\big) \right| x^{-2\sigma} dx \\
&\le \frac{1}{2} \int_0^t \int_0^n \exp\left(\int_\tau^t \int_0^{n-x} K_n(x,y) |c(y,s)| dy\, ds \right) \\
&\qquad \cdot \left[\int_0^x K_n(x-y,y) |c(x-y,\tau)||c(y,\tau)|\, dy + \int_x^n S_n(y) b(x,y) |c(y,\tau)|\, dy \right] x^{-2\sigma} dx\, d\tau \\
&\qquad + \int_0^n |u_0^n(x)| \exp\left(\int_0^t \int_0^{n-x} K_n(x,y) |c(y,s)| dy\, ds \right) x^{-2\sigma} dx \\
&\le \frac{1}{2} \exp\big(2n^{2\sigma} MLt\big) \int_0^t \left[\int_0^n \int_0^x K_n(x-y,y) |c(x-y,\tau)||c(y,\tau)| x^{-2\sigma} dy\, dx \right. \\
&\qquad \left. + \int_0^n \int_x^n S_n(y) b(x,y) |c(y,\tau)| x^{-2\sigma}\, dy\, dx \right] d\tau + \|u_0^n\|_Y \exp\big(2n^{2\sigma} MLt\big).
\end{aligned}
$$

Changing the order of integration, then a change of variable $x - y = z$, and again re-changing the order of integration while replacing z by x gives

$$
\begin{aligned}
&\int_0^n |G(c)(x,t)| x^{-2\sigma} dx \\
&\le \exp\big(2n^{2\sigma} MLt\big) \left[\frac{1}{2} \int_0^t \left[\int_0^n \int_0^{n-x} K_n(x,y) |c(x,\tau)||c(y,\tau)| (x+y)^{-2\sigma} dy\, dx \right.\right. \\
&\qquad \left.\left. + \int_0^n \int_0^y S_n(y) b(x,y) |c(y,\tau)| x^{-2\sigma}\, dx\, dy \right] d\tau + \|u_0^n\|_Y \right]. \quad (3.20)
\end{aligned}
$$

By using the definition of K_n and S_n we have

$$\int_0^n |G(c)(x,t)|x^{-2\sigma}\,dx$$
$$\leq \exp\left(2n^{2\sigma}MLt\right)\left[\frac{1}{2}\int_0^t\left[\int_{\sigma/n}^{n-\sigma/n}\int_{\sigma/n}^{n-x}K(x,y)|c(x,\tau)||c(y,\tau)|(x+y)^{-2\sigma}dy\,dx\,d\tau\right.\right.$$
$$\left.\left.+\int_0^n\int_0^y S(y)b(x,y)|c(y,\tau)|x^{-2\sigma}dx\,dy\right]d\tau+\|u_0^n\|_Y\right]. \quad (3.21)$$

Now, we multiply and divide by $(xy)^{2\sigma}$ inside the first integral term on right hand side of (3.21). Taking in account the definition of M (3.16) and **(H6)** in Hypotheses 3.1.3 we get

$$\int_0^n |G(c)(x,t)|x^{-2\sigma}\,dx$$
$$\leq \exp\left(2n^{2\sigma}MLt\right)\left[2^{-(1+2\sigma)}n^{6\sigma}\sigma^{-2\sigma}M\int_0^t\left[\int_{\sigma/n}^{n-\sigma/n}\int_{\sigma/n}^{n-x}|c(x,\tau)||c(y,\tau)|(xy)^{-2\sigma}dy\,dx\,d\tau\right.\right.$$
$$\left.\left.+Cn^{\theta}\int_0^n|c(y,\tau)|y^{-2\sigma}dy\right]d\tau+\|u_0^n\|_Y\right].$$

Now, using that $c\in D$ we obtain

$$\int_0^n |G(c)(x,t)|x^{-2\sigma}\,dx \leq \exp\left(2n^{2\sigma}MLt\right)\left(2^{-(1+2\sigma)}n^{6\sigma}\sigma^{-2\sigma}Mt\|c\|_D^2+Cn^{\theta}t\|c\|_D+\|u_0^n\|_Y\right)$$
$$\leq \exp\left(2n^{2\sigma}MLt\right)\left(2^{1-2\sigma}n^{6\sigma}\sigma^{-2\sigma}MLt+2Cn^{\theta}t+1\right)L. \quad (3.22)$$

Then, by using (3.17) we find that

$$\int_0^n |G(c)(x,t)|\,x^{-2\sigma}dx \leq 2L.$$

Hence, by definition of $\|\cdot\|_D$ we have $\|G\|_D\leq 2L$ and this completes the proof of the lemma.□

Lemma 3.2.2. *Consider* $c_1,c_2\in C\left([0,t_0];L^1\left(]0,n[\right)\right)$ *and let* $B=\max\{\|c_1\|_D,\|c_2\|_D\}$*, i.e.* $B\leq 2L$ *as well as*

$$H(x,\tau,t)=\exp\big(-[P(x,t,c_1)-P(x,\tau,c_1)]\big)-\exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big).$$

Then, for $0\leq\tau\leq t\leq t_0$ *and* $0\leq x\leq n$ *we have*

$$|H(x,\tau,t)|\leq (t-\tau)n^{2\sigma}M\exp\left((t-\tau)n^{2\sigma}BM\right)\|c_1-c_2\|_D. \quad (3.23)$$

Lemma 3.2.2 can be proven analogously as Lemma 2.2.2, see Appendix C

Lemma 3.2.3. *For $c_1, c_2 \in D$ and t_0 as above there exists $\gamma \in [0,1[$ such that*

$$\|G(c_1) - G(c_2)\|_D \leq \gamma \|c_1 - c_2\|_D,$$

i.e. the operator G is a contraction.

Proof. Choose $c_1, c_2 \in D$. Using the defintion of G we find that

$$\begin{aligned}
&G(c_1) - G(c_2) \\
&\quad = u_0^n(x)\left[\exp\big(-P(x,t,c_1)\big) - \exp\big(-P(x,t,c_2)\big)\right] \\
&\qquad + \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_1) - P(x,\tau,c_1)]\big)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau \\
&\qquad - \frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_0^x K_n(x-y,y)c_2(x-y,\tau)c_2(y,\tau)\,dy\,d\tau \\
&\qquad + \int_0^t \exp\big(-[P(x,t,c_1) - P(x,\tau,c_1)]\big)\int_x^n S_n(y)b(x,y)c_1(y,\tau)\,dy\,d\tau \\
&\qquad - \int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_x^n S_n(y)b(x,y)c_2(y,\tau)\,dy\,d\tau. \qquad (3.24)
\end{aligned}$$

By addition and subtraction of the terms

$$\frac{1}{2}\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy\,d\tau$$

and

$$\int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big)\int_x^n S_n(y)b(x,y)c_1(y,\tau)\,dy\,d\tau,$$

together with the defintion of H in Lemma 3.2.2 we rewrite (3.24) as

$$\begin{aligned}
G(c_1) - G(c_2) = u_0^n(x)H(x,t,0) + \int_0^t H(x,t,\tau)\Bigg[\frac{1}{2}\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy \\
+ \int_x^n S_n(y)b(x,y)c_1(y,\tau)\,dy\Bigg]\,d\tau - \int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big) \\
\cdot\Bigg[\frac{1}{2}\int_0^x K_n(x-y,y)c_1(y,\tau)\,[c_2(x-y,\tau) - c_1(x-y,\tau)]\,dy \\
+\frac{1}{2}\int_0^x K_n(x-y,y)c_2(x-y,\tau)\,[c_2(y,\tau) - c_1(y,\tau)]\,dy \\
+ \int_x^n S_n(y)b(x,y)\,[c_2(y,\tau) - c_1(y,\tau)]\,dy\Bigg]\,d\tau.
\end{aligned}$$

Now, making use of the definition of $\|\cdot\|_{x^{-2\sigma}}$ it follows that

$$\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-2\sigma}} \\
&= \int_0^n \Bigg| u_0^n(x)H(x,0,t) + \int_0^t H(x,\tau,t)\Bigg[\frac{1}{2}\int_0^x K_n(x-y,y)c_1(x-y,\tau)c_1(y,\tau)\,dy \\
&\quad + \int_x^n S_n(y)b(x,y)c_1(y,\tau)\,dy\Bigg]\,d\tau - \int_0^t \exp\big(-[P(x,t,c_2) - P(x,\tau,c_2)]\big) \\
&\quad \cdot\Bigg[\frac{1}{2}\int_0^x K_n(x-y,y)c_1(y,\tau)[c_2(x-y,\tau) - c_1(x-y,\tau)]dy \\
&\quad +\frac{1}{2}\int_0^x K_n(x-y,y)c_2(x-y,\tau)[c_2(y,\tau) - c_1(y,\tau)]dy \\
&\quad + \int_x^n S_n(y)b(x,y)[c_2(y,\tau) - c_1(y,\tau)]dy\Bigg]\,d\tau\Bigg|\, x^{-2\sigma}dx.
\end{aligned}$$

Applying the triangule inequality we obtain

$$
\begin{aligned}
&\|G(c_1)-G(c_2)\|_{x^{-2\sigma}}\\
&\leq \int_0^t |H(x,\tau,t)| \left[\frac{1}{2}\int_0^n\int_0^x K_n(x-y,y)|c_1(x-y,\tau)||c_1(y,\tau)|x^{-2\sigma}dy\,dx\right.\\
&\qquad \left.+\int_0^n\int_x^n S_n(y)b(x,y)|c_1(y,\tau)|x^{-2\sigma}dy\,dx\right]d\tau-\int_0^t \exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big)\\
&\qquad \cdot\left[\frac{1}{2}\int_0^n\int_0^x K_n(x-y,y)|c_1(y,\tau)||c_2(x-y,\tau)-c_1(x-y,\tau)|x^{-2\sigma}dy\,dx\right.\\
&\qquad +\frac{1}{2}\int_0^n\int_0^x K_n(x-y,y)|c_2(x-y,\tau)||c_2(y,\tau)-c_1(y,\tau)|x^{-2\sigma}dy\,dx\\
&\qquad \left.+\int_0^n\int_x^n S_n(y)b(x,y)|c_2(y,\tau)-c_1(y,\tau)|x^{-2\sigma}dy\,dx\right]d\tau+\int_0^n u_0^n(x)|H(x,0,t)|x^{-2\sigma}dx.
\end{aligned}
$$

Now, changing the order of integration, then a change of variable $x-y=z$, then re-changing the order of integration while replacing z by x we have

$$
\begin{aligned}
&\|G(c_1)-G(c_2)\|_{x^{-2\sigma}}\\
&\leq \int_0^t |H(x,\tau,t)| \left[\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)|c_1(x,\tau)||c_1(y,\tau)|(x+y)^{-2\sigma}dy\,dx\right.\\
&\qquad \left.+\int_0^n\int_x^n S_n(y)b(x,y)|c_1(y,\tau)|x^{-2\sigma}dy\,dx\right]d\tau-\int_0^t \exp\big(-[P(x,t,c_2)-P(x,\tau,c_2)]\big)\\
&\qquad \cdot\left[\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)|c_1(y,\tau)||c_2(x,\tau)-c_1(x,\tau)|(x+y)^{-2\sigma}dy\,dx\right.\\
&\qquad +\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)|c_2(x,\tau)||c_2(y,\tau)-c_1(y,\tau)|(x+y)^{-2\sigma}dy\,dx\\
&\qquad \left.+\int_0^n\int_x^n S_n(y)b(x,y)|c_2(y,\tau)-c_1(y,\tau)|x^{-2\sigma}dy\,dx\right]d\tau+\int_0^n u_0^n(x)|H(x,0,t)|x^{-2\sigma}dx.
\end{aligned}
$$

By using Lemma 3.2.2 it gives

$$
\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-2\sigma}} \\
&\quad \leq (t-\tau)n^{2\sigma} M \exp\left(2(t-\tau)n^{2\sigma} ML\right)\|c_1 - c_2\|_D \\
&\quad\quad \cdot \int_0^t \left[\frac{1}{2}\int_0^n \int_0^{n-x} K_n(x,y)|c_1(x,\tau)||c_1(y,\tau)|(x+y)^{-2\sigma} dy \right.\\
&\quad\quad\quad \left. + \int_0^n \int_0^y S_n(y) b(x,y)|c_1(y,\tau)| x^{-2\sigma} dx\, dy \right] d\tau \\
&\quad\quad + \exp\left(2tn^{2\sigma} ML\right) \int_0^t \left[\frac{1}{2}\int_0^n \int_0^{n-x} K_n(x,y)|c_2(x,\tau)||c_1(y,\tau) - c_2(y,\tau)|(x+y)^{-2\sigma} dy\, dx \right.\\
&\quad\quad\quad + \frac{1}{2}\int_0^n \int_0^{n-x} K_n(x,y)|c_1(y,\tau)||c_1(x,\tau) - c_2(x,\tau)|(x+y)^{-2\sigma} dy\, dx \\
&\quad\quad\quad \left. + \int_0^n \int_0^y S_n(y) b(x,y)|c_1(y,\tau) - c_2(y,\tau)| x^{-2\sigma} dx\, dy \right] d\tau \\
&\quad\quad + \|u_0^n\|_Y t n^{2\sigma} M \exp\left(2tn^{2\sigma} ML\right)\|c_1 - c_2\|_D.
\end{aligned}
$$

Since $K_n(x,y) = 0$ for $x, y < \sigma/n$, the maximum value that the term $(x+y)^{-2\sigma}$ can have is $(n/2\sigma)^{2\sigma}$. Using this fact, **(H6)**, the definition of M, and $c_1, c_2 \in D$ we arrive at

$$
\begin{aligned}
&\|G(c_1) - G(c_2)\|_{x^{-2\sigma}} \\
&\quad \leq n^{2\sigma} Mt \exp\left(2n^{2\sigma} MLt\right)\|c_1 - c_2\|_D \left(2^{-(1+2\sigma)} n^{6\sigma}\sigma^{-2\sigma} M\|c_1\|_D^2 + Cn^\theta \|c_1\|_D\right) t \\
&\quad\quad + \exp\left(2n^{2\sigma} MLt\right) \left(2^{-(1+2\sigma)} n^{6\sigma}\sigma^{-2\sigma} M(\|c_1\|_D + \|c_2\|_D)\|c_1 - c_2\|_D + Cn^\theta \|c_1 - c_2\|_D\right) t \\
&\quad\quad + \|u_0^n(x)\|_Y n^{2\sigma} Mt \exp\left(2n^{2\sigma} MLt\right)\|c_1 - c_2\|_D \\
&\quad \leq t \exp\left(2n^{2\sigma} MLt\right) \left[2^{1-2\sigma} n^{6\sigma}\sigma^{-2\sigma} ML^2 t + 2Cn^\theta Lt + 2^{1-2\sigma} n^{6\sigma}\sigma^{-2\sigma} ML \right.\\
&\quad\quad\quad \left. + Cn^\theta + n^{2\sigma} ML\right] \|c_1 - c_2\|_D \\
&\quad = t \exp\left(2n^{2\sigma} MLt\right) \left[2^{1-2\sigma} n^{6\sigma}\sigma^{-2\sigma} ML(Lt+1) + Cn^\theta (2Lt+1) + n^{2\sigma} ML\right] \|c_1 - c_2\|_D,
\end{aligned}
$$

from where we can conclude that

$$\|G(c_1) - G(c_2)\|_D \leq \gamma \|c_1 - c_2\|_D,$$

where $\gamma = t\exp\left(2n^{2\sigma} MLt\right)\left[2^{1-2\sigma} n^{6\sigma}\sigma^{-2\sigma} ML(Lt+1) + Cn^\theta(2Lt+1) + n^{2\sigma} ML\right] < 1$, which completes the proof of the lemma. □

Theorem 3.2.4. *Suppose that **(H1)**, **(H2)**, **(H4)**, **(H5)**, **(H6)** hold and $u_0 \in Y^+$. Then for each $n = 2, 3, 4, \ldots$ the problem (3.13)-(3.15) has a unique solution u^n with $u^n(x,t) \geq 0$ for a.e. $x \in [0, n]$ and $t \in [0, \infty[$. Moreover, for all $t \in [0, \infty[$*

$$\int_0^n x u^n(x,t)\, dx = \int_0^n x u^n(x,0)\, dx. \tag{3.25}$$

Proof. From Lemmas 3.2.1, 3.2.3 and the contraction mapping principle, it follows that there exists a unique solution $u^n(x,t)$ to (3.13)-(3.15) in $[0,t_0]$. We proceed now to check that those solutions are non-negative. If we set

$$c_0 = u_0^n \quad \text{and} \quad c_i = G(c_{i-1}),$$

for $i = 1, 2, 3, \ldots$, we find that fixed point iteration gives

$$c_i \to u^n \quad \text{in} \quad L^1\big(]0,\infty[\big) \quad \text{as} \quad i \to \infty,$$

and u^n is constructed by positivity preserving iterations, using G given in (3.14).

Let us check now that the mass conservation property (3.25) holds. Multiplying (3.10) by x and integrating with respect to x on $[0,n]$ we have by (1.7) and changes of variables and order of integration as in (3.20)

$$\begin{aligned}
&\frac{d}{dt}\int_0^n xu^n(x,t)\,dx \\
&= \frac{1}{2}\int_0^n\int_0^x xK_n(x-y,y)u^n(x-y,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx \\
&\quad + \int_0^n\int_x^n xb(x,y)S_n(y)u^n(y,t)\,dy\,dx - \int_0^n xS_n(x)u^n(x,t)\,dx \\
&= \frac{1}{2}\int_0^n\int_0^{n-x} (x+y)K_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx \\
&\quad + \int_0^n\int_0^y xb(x,y)S_n(y)u^n(y,t)\,dx\,dy - \int_0^n xS_n(x)u^n(x,t)\,dx \\
&= \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx - \int_0^n\int_0^{n-x} xK_n(x,y)u^n(x,t)u^n(y,t)\,dy\,dx \\
&\quad + \int_0^n yS_n(y)u^n(y,t)\,dy - \int_0^n xS_n(x)u^n(x,t)\,dx = 0,
\end{aligned}$$

from where we have

$$\frac{d}{dt}\int_0^n xu^n(x,t)\,dx = 0 \qquad \Longrightarrow \qquad \int_0^n xu^n(x,t)dx = \int_0^n xu_0^n(x)\,dx.$$

Now we show that our solution for $t \in [0, t_0]$ extends to arbitrarily large times, changing variable as we did in (3.20) we proceed to obtain a uniform bound

$$\begin{aligned}
&\int_0^n u^n(x,t)x^{-\sigma}\,dx \\
&= \int_0^t \left[\frac{1}{2} \int_0^n \int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)x^{-\sigma} dy\,dx \right. \\
&\qquad - \int_0^n \int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma} dy\,dx + \int_0^n \int_x^n S_n(y)b(x,y)u^n(y,\tau)x^{-\sigma} dy\,dx \\
&\qquad \left. - \int_0^n S_n(x)u^n(x,\tau)x^{-\sigma} dx \right] d\tau + \int_0^n u_0^n(x)x^{-\sigma} dx \\
&= \int_0^t \left[\frac{1}{2} \int_0^n \int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x+y)^{-\sigma} dy\,dx \right. \\
&\qquad - \int_0^n \int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma} dy\,dx + \int_0^n \int_0^y S_n(y)b(x,y)u^n(y,\tau)x^{-\sigma} dx\,dy \\
&\qquad \left. - \int_0^n S_n(x)u^n(x,\tau)x^{-\sigma} dx \right] d\tau + \int_0^n u_0^n(x)x^{-\sigma} dx.
\end{aligned}$$

Making use of the inequality (3.8) and the symmetry of $K(x,y)$ results in

$$\begin{aligned}
\int_0^n u^n(x,t)x^{-\sigma} dx \le \int_0^t & \left[\frac{1}{2^{2+\sigma}} \int_{\sigma/n}^{n-\sigma/n} \int_{\sigma/n}^{n-x} K(x,y)u^n(x,\tau)u^n(y,\tau)(x^{-\sigma}+y^{-\sigma})\,dy\,dx \right. \\
& - \frac{1}{2} \int_{\sigma/n}^{n-\sigma/n} \int_{\sigma/n}^{n-x} K(x,y)u^n(x,\tau)u^n(y,\tau)(x^{-\sigma}+y^{-\sigma})\,dy\,dx \\
& \left. + \int_{\sigma/n}^{n} \int_{\sigma/n}^{y} S_n(y)b(x,y)u^n(y,\tau)x^{-\sigma} dx\,dy \right] d\tau + \int_0^n u_0^n(x)x^{-\sigma} dx.
\end{aligned}$$

Since $u^n(x,t)$ is positive in $[0,t_0]$ we can eliminate the coagulation terms and obtain

$$\int_0^n u^n(x,t)x^{-\sigma} dx \le \int_0^t \int_{\sigma/n}^{n} \int_{\sigma/n}^{y} S_n(y)b(x,y)u^n(y,\tau)x^{-\sigma} dx\,dy\,d\tau + \int_0^n u_0^n(x)x^{-\sigma} dx.$$

By definition of M and taking the maximum value of $x^{-\sigma}$ in $[\sigma/n, n]$ and then extending the integral intervals to $[0, y]$ and $[0, n]$ in the first integral term, we have

$$\begin{aligned}\int_0^n u^n(x,t)x^{-\sigma}dx &\leq Mn^\sigma\sigma^{-\sigma}\int_0^t\int_0^n\int_0^y u^n(y,\tau)\,dx\,dy\,d\tau + \|u_0^n\|_Y \\ &= Mn^\sigma\sigma^{-\sigma}\int_0^t\int_0^n yu^n(y,\tau)\,dy\,d\tau + \|u_0^n\|_Y \\ &\leq (n^\sigma\sigma^{-\sigma}MT+1)\|u_0^n\|_Y = L. \end{aligned} \tag{3.26}$$

Now we can extend the interval $[0, t_0]$ to $[0, \infty[$ to complete the proof of Theorem 3.2.4. By considering the operator

$$\begin{aligned}G_1(c)(x,t) = \frac{1}{2}\int_{t_0}^t \exp\big(-[P_1(x,t,c) - P_1(x,\tau,c)]\big)\Bigg[\int_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy \\ + \int_x^n S_n(y)b(x,y)u^n(y,t)\,dy\Bigg]\,d\tau + u^n(x,t_0)\exp\big(P_1(x,t,c)\big),\end{aligned}$$

with

$$P_1(x,t,c) = \int_{t_0}^t\left[\int_0^{n-x} K_n(x,y)c(x,\tau)c(y,\tau)\,dx + S_n(x)\right]d\tau,$$

we can repeat the above argument to show, that there is a unique non-negative solution u^n on $[t_0, t_1]$ where $t_1 = 2t_0$. We can extend the unique solution to $[0, t_j]$ $\quad j = 1, 2, 3, \ldots$, repeating this process by considering the operators

$$\begin{aligned}G_{j+1}(c)(x,t) = \frac{1}{2}\int_{t_j}^t \exp\big(-[P_{j+1}(x,t,c) - P_{j+1}(x,\tau,c)]\big)\Bigg[\int_0^x K_n(x-y,y)c(x-y,\tau)c(y,\tau)\,dy \\ + \int_x^n S_n(y)b(x,y)u^n(y,t)\,dy\Bigg]\,d\tau + u^n(x,t_j)\exp\big(P_{j+1}(x,t,c)\big),\end{aligned}$$

with

$$P_{j+1}(x,t,c) = \int_{t_j}^t\left[\int_0^{n-x} K_n(x,y)c(x,\tau)c(y,\tau)\,dx + S_n(x)\right]d\tau.$$

In that way we extend the solution to all of $[0, \infty[$. The argument used to get (3.25) for $[0, t_0]$ shows that (3.25) holds for $[0, \infty[$ and thus we have completed the proof of Theorem 3.2.4 by the arbitrariness of n. $\square$

3.2.2 Properties of the solutions of the truncated problem

Lemma 3.2.5. *Let u^n a solution of the truncated problem (3.10)-(3.11). Then for $0 \leq \alpha \leq 1$ and $n = 1, 2, \ldots$ we obtain the inequality*

$$\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx \leq \int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\, dx.$$

Proof. Multiplying equation (3.10) by $x^{-\alpha}$ and integrating w.r.t x from 0 to n we have

$$\begin{aligned}\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx &= \frac{1}{2}\int_0^n\int_0^x K_n(x-y,y)u^n(x-y,t)u^n(y,t)x^{-\alpha}dy\, dx \\ &-\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)x^{-\alpha}dy\, dx \\ &+\int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\, dx - \int_0^n S_n(x)u^n(x,t)x^{-\alpha}dx.\end{aligned}$$

Changing variables as we did in (3.20) we get

$$\begin{aligned}\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx &= \frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)(x+y)^{-\alpha}dy\, dx \\ &-\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)x^{-\alpha}dy\, dx \\ &+\int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\, dx - \int_0^n S_n(x)u^n(x,t)x^{-\alpha}\, dx.\end{aligned}$$

Now by using inequality (3.8) together with the definition and symmetry of $K_n(x,y)$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx &\leq \frac{1}{2}2^{-(\alpha+1)}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)(x^{-\alpha}+y^{-\alpha})dy\, dx \\ &-\frac{1}{2}\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)(x^{-\alpha}+y^{-\alpha})dy\, dx \\ &+\int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\, dx - \int_0^n S_n(x)u^n(x,t)x^{-\alpha}dx.\end{aligned}$$

Taking the difference of the first two terms on the right hand side we have

$$\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx = \frac{1}{2}\left(2^{-(\alpha+1)}-1\right)\int_0^n\int_0^{n-x} K_n(x,y)u^n(x,t)u^n(y,t)(x^{-\alpha}+y^{-\alpha})dy\,dx$$
$$+\int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\,dx - \int_0^n S_n(x)u^n(x,t)x^{-\alpha}dx.$$

Now we can eliminate the negative terms and obtain

$$\frac{d}{dt}\int_0^n u^n(x,t)x^{-\alpha}dx \leq \int_0^n\int_x^n S_n(y)b(x,y)u^n(y,t)x^{-\alpha}dy\,dx,$$

which complete the proof of the theorem. □

In the rest of the chapter we consider for each u^n their zero extension on $\mathbb{R}$, i.e.

$$\hat{u}^n(x,t) = \begin{cases} u^n(x,t) & 0 \leq x \leq n, \quad t \in [0,T], \\ 0 & x < 0 \;\text{ or }\; x > n. \end{cases}$$

For clarity we drop the notation $\hat{\cdot}$ for the remainder of the chapter.

Lemma 3.2.6. *Assume that **(H1)**, **(H2)**, **(H4)**, **(H5)**, **(H6)**, and **(H7)** hold. We take u^n to be the non-negative zero extension of the solution to the truncated problem found in Theorem 3.2.4. Fix $T > 0$ and let us define*

$$L(T) = \left(e^{NT}(N+1) + e^{CT}(C+1) + 1\right)\|u_0\|_Y.$$

Then the following are true:

(i) *We have the bound*

$$\int_0^\infty (1+x+x^{-2\sigma})u^n(x,t)\,dx \leq L(T) \quad \textit{for all} \quad t \in [0,T].$$

(ii) *Given $\epsilon > 0$ there exists an $R > 1$ such that for all $t \in [0,T]$*

$$\sup_n \left\{ \int_R^\infty (1+x^{-\sigma})u^n(x,t)\,dx \right\} \leq \epsilon.$$

(iii) *Given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n = 2, 3, \ldots$ and $t \in [0,T]$*

$$\int_A (1+x^{-\sigma})u^n(x,t)\,dx < \epsilon \qquad \textit{whenever} \qquad \mu(A) < \delta.$$

Proof. Property (i) By Lemma 3.2.5 for $\alpha = 0$ we have

$$\frac{d}{dt}\int_0^n u^n(x,t)\,dx \le \int_0^n \int_x^n S_n(y)b(x,y)u^n(y,t)\,dy\,dx.$$

Changing the order of integration on the right hand side and making use of (1.8) and **(H5)**, we get

$$\begin{aligned}\frac{d}{dt}\int_0^n u^n(x,t)\,dx &\le \int_0^n S_n(y)u^n(y,t)\int_0^y b(x,y)\,dx\,dy \\ &= N\int_0^1 y^\theta u^n(y,t)\,dy + N\int_1^n y^\theta u^n(y,t)\,dy.\end{aligned}$$

As θ is considered to be in $[0,1[$, by using the mass conservation property (3.25) we have the estimate

$$\frac{d}{dt}\int_0^n u^n(x,t)\,dx \le N\int_0^n u^n(y,t)\,dy + N\|u_0\|_Y.$$

Integrating respect to time it becomes

$$\begin{aligned}\int_0^n u^n(x,t)\,dx &\le N\int_0^t\int_0^n u^n(y,\tau)\,dy\,d\tau + N\|u_0\|_Y - \int_0^n u_0^n(x)\,dx \\ &\le N\int_0^t\int_0^n u^n(y,\tau)\,dy\,d\tau + (N+1)\|u_0\|_Y,\end{aligned}$$

from which we obtain by the Gronwall's inequality, see e.g. Appendix A[Theorem A.0.6], for $A(t) = \int_0^n u^n(x,t)dx$

$$\int_0^n u^n(x)dx \le e^{Nt}(N+1)\|u_0\|_Y, \qquad t \in [0,T]. \tag{3.27}$$

Computing now the term with the weight $x^{-2\sigma}$ using Lemma 3.2.5 for $\alpha = 2\sigma$ we have

$$\frac{d}{dt}\int_0^n u^n(x,t)x^{-2\sigma}dx \le \int_0^n\int_x^n S(y)b(x,y)u^n(y,t)x^{-2\sigma}dy\,dx.$$

Changing the order of integration and using **(H6)** we get

$$
\begin{aligned}
\frac{d}{dt}\int_0^n u^n(x,t)x^{-2\sigma}dx &= \int_0^n\int_0^y S(y)b(x,y)u^n(y,t)x^{-2\sigma}dx\,dy\\
&\le C\int_0^n y^{\theta-2\sigma}u^n(y,t)\,dy\\
&\le C\int_0^1 y^{-2\sigma}u^n(y,t)\,dy + C\int_1^n yu^n(y,t)\,dy\\
&\le C\int_0^n x^{-2\sigma}u^n(x,t)\,dx + C\|u_0\|_Y.
\end{aligned}
$$

From this inequality we find as above that

$$
\int_0^n u^n(x,t)x^{-2\sigma}dx \le e^{Ct}(C+1)\|u_0\|_Y, \qquad t\in[0,T]. \tag{3.28}
$$

Now, by the mass conservation property (3.25), by (3.27), and (3.28) we obtain

$$
\begin{aligned}
\int_0^\infty (1+x+x^{-2\sigma})u^n(x,t)\;dx &= \int_0^n u^n(x,t)\;dx + \int_0^n xu^n(x,t)\;dx + \int_0^n x^{-2\sigma}u^n(x,t)\;dx\\
&\le e^{Nt}(N+1)\|u_0\|_Y + \|u_0\|_Y + e^{Ct}(C+1)\|u_0\|_Y\\
&\le \left(e^{NT}(N+1)+e^{CT}(C+1)+1\right)\|u_0\|_Y =: L(T).
\end{aligned}
$$

Property (ii) Choose $\epsilon>0$ and let $R>1$ be such that $R>\frac{2\|u_0\|_Y}{\epsilon}$. Then using (3.25) we get

$$
\begin{aligned}
\int_R^\infty (1+x^{-\sigma})u^n(x,t)\,dx &= \int_R^\infty \frac{x}{x}u^n(x,t)\,dx + \int_R^\infty x^{-\sigma}\frac{x^{1+\sigma}}{x^{1+\sigma}}u^n(x,t)\,dx\\
&\le \frac{1}{R}\int_R^\infty xu^n(x,t)\,dx + \frac{1}{R^{1+\sigma}}\int_R^\infty xu^n(x,t)\,dx\\
&\le \left(\frac{1}{R}+\frac{1}{R^{1+\sigma}}\right)\int_R^\infty xu^n(x,t)\,dx\\
&\le \frac{2}{R}\int_0^n xu^n(x,t)\,dx\\
&\le \frac{2}{R}\|u_0^n\|_Y \le \frac{2}{R}\|u_0\|_Y < \epsilon.
\end{aligned}
$$

Property (iii) Let χ_A denote the characteristic function of a set A and set

$$
\kappa'(r) := \frac{1}{2}(1+r^\sigma)(1+r)^{2\lambda}. \tag{3.29}
$$

Let us define for all $n = 1, 2, 3, \dots$ and $t \in [0, T]$ using property **(i)**

$$f^n(\delta, t) = \sup \left\{ \int_0^r \chi_A(x)(1 + x^{-\sigma})u^n(x, t)\, dx : \ A \subset]0, r[\text{ and } \mu(A) < \delta \right\} \le L(T). \qquad (3.30)$$

We take $t = 0$ in the definition of f^n and observe that $u^n(x, 0) \le u_0(x)$ pointwise almost everywhere. Then by the absolute continuity of the Lebesgue integral, we have that

$$f^n(\delta, 0) = \sup \left\{ \int_0^r \chi_A(x)(1 + x^{-\sigma})u_0^n(x)\, dx : \ A \subset]0, r[\text{ and } \mu(A) < \delta \right\} \to 0 \text{ as } \delta \to 0 \quad (3.31)$$

Now we multiply (3.10) by $(1+x^{-\sigma})\chi_A(x)$. This we integrate from 0 to t w.r.t. s and over $[0, r[$ w.r.t. x. Using the non-negativity of each u^n we obtain

$$\begin{aligned}
&\int_0^r \chi_A(x)(1 + x^{-\sigma})u^n(x, t)\, dx \\
&\le \frac{1}{2} \int_0^t \int_0^r \int_0^x \chi_A(x)(1 + x^{-\sigma})K_n(x - y, y)u^n(x - y, s)u^n(y, s)\, dy\, dx\, ds \qquad (3.32)\\
&+ \int_0^t \int_0^r \chi_A(x) \int_x^n S_n(y)b(x, y)(1 + x^{-\sigma})u^n(y, s)\, dy\, dx\, ds + \int_0^r \chi_A(x)(1 + x^{-\sigma})u_0^n(x)\, dx.
\end{aligned}$$

Let us denote I_{21} and I_{22} the first and the second integral terms on the right hand side of (3.32) respectively. By changing variables as we did in (3.20) in I_{21} we get

$$I_{21}(t) = \frac{1}{2} \int_0^t \int_0^r \int_0^{r-y} \chi_A(x + y)[1 + (x + y)^{-\sigma}]K_n(x, y)u^n(x, s)u^n(y, s)\, dx\, dy\, ds.$$

By using **(H4)** for $K(x, y)$, then taking $1 + (x + y)^{-\sigma} \le 1 + y^{-\sigma}$ and $x^{-\sigma} \le 1 + x^{-\sigma}$ we have

$$\begin{aligned}
I_{21}(t) &\le \frac{1}{2} \int_0^t \int_0^r \int_0^{r-y} \chi_A(x + y) \left[1 + (x + y)^{-\sigma}\right] (1 + x)^{\lambda}(1 + y)^{\lambda}(xy)^{-\sigma}u^n(x, s)u^n(y, s)\, dx\, dy\, ds \\
&\le \frac{1}{2} \int_0^t \int_0^r \int_0^{r-y} \chi_A(x + y)(1 + y^{-\sigma})(1 + x)^{\lambda}(1 + y)^{\lambda}(1 + x^{-\sigma})y^{-\sigma}u^n(x, s)u^n(y, s)\, dx\, dy\, ds \\
&= \frac{1}{2} \int_0^t \int_0^r \int_0^{r-y} \chi_A(x + y)(1 + y^{\sigma})(1 + x)^{\lambda}(1 + y)^{\lambda}(1 + x^{-\sigma})y^{-2\sigma}u^n(x, s)u^n(y, s)\, dx\, dy\, ds.
\end{aligned}$$

By using the definition (3.29) of $\kappa(r)$ we obtain the following estimates for I_{21}

$$I_{21}(t) \le \kappa(r) \int_0^t \int_0^r u^n(y, s)y^{-2\sigma} \int_0^{\infty} \chi_{A-y\cap[0,r-y]}(x)(1 + x^{-\sigma})u^n(x, s)\, dx\, dy\, ds$$

where $A - y := \{\omega > 0 : \omega = x - y \,\text{for some}\, x \in A\}$. Since $A - y \cap [0, r - y] \subset [0, r]$ and $\mu\left(A - y \cap [0, r - y]\right) \leq \mu(A - y) \leq \mu(A) < \delta$, by using the definition of f^n and property **(i)** we have

$$I_{21}(t) \leq \kappa(r)L(T)\int_0^t f^n(\delta, s)\, ds. \tag{3.33}$$

Working now with the integral term I_{22} we have using **(H5)** that

$$\begin{aligned}
I_{22}(t) &\leq \int_0^t\int_0^r \chi_A(x)\int_x^\infty S_n(y)b(x,y)(1+x^{-\sigma})u^n(y,s)\, dy\, dx\, ds \\
&\leq \int_0^t\int_0^\infty\int_0^y \chi_A(x)S_n(y)b(x,y)(1+x^{-\sigma})u^n(y,s)\, dx\, dy\, ds \\
&\leq \int_0^t\int_0^\infty y^\theta u^n(y,s)\left[\int_0^y \chi_A(x)b(x,y)\, dx + \int_0^y \chi_A(x)b(x,y)x^{-\sigma} dx\right] dy\, ds.
\end{aligned}$$

Then by hypotheses **(H7)** we find

$$\begin{aligned}
I_{22}(t) &\leq \mu(A)^{\frac{p-1}{p}}\int_0^t\int_0^\infty y^\theta u^n(y,s)\left[\left(\int_0^y b^p(x,y)dx\right)^{1/p} + \left(\int_0^y b^p(x,y)x^{-p\sigma}dx\right)^{1/p}\right] dy\, ds \\
&\leq \mu(A)^{\frac{p-1}{p}}\int_0^t\int_0^\infty y^\theta u^n(y,s)\left(B_1y^{\tau_1} + B_2y^{\tau_2}\right) dy\, ds \leq (B_1+B_2)\mu(A)^{\frac{p-1}{p}}L(T)T.
\end{aligned}$$

Using the estimates of $I_{21}(t)$ and $I_{22}(t)$ in (3.32) we have by taking the supremum over all A such that $A \subset]0, r[$ with $\mu(A) \leq \delta$

$$f^n(\delta, t) \leq \kappa(r)L(T)\int_0^t f^n(\delta, s)ds + (B_1+B_2)L(T)T\delta^{\frac{p-1}{p}} + f^n(\delta, 0), \quad t \in [0, T].$$

By using Gronwall's inequality, see e.g. Walter [40, page 361], we get

$$f^n(\delta, t) \leq \left[(B_1+B_2)L(T)T\delta^{\frac{p-1}{p}} + f^n(\delta, 0)\right]\exp\left(\kappa(r)L(T)T\right), \quad t \in [0, T]. \tag{3.34}$$

Since $f^n(\delta, 0) \to 0$ as $\delta \to 0$ (3.34) implies that

$$\lim_{\delta\to 0}\sup_{n\geq 1, t\in[0,T]}\{f^n(\delta, t)\} = 0. \tag{3.35}$$

Lemma 3.2.6(iii) is then a consequence of (3.35) and Lemma 3.2.6(i). $\square$

Let us define $v^n(x,t) = x^{-\sigma}u^n(x,t)$. Due to the Lemma 3.2.6 above and the Dunford-Pettis Theorem, see e.g. Appendix A[Theorem A.0.4], we can conclude that for each $t \in [0, T]$ the sequences $\left(u^n(t)\right)_{n\in\mathbb{N}}$ and $\left(v^n(t)\right)_{n\in\mathbb{N}}$ are weakly relatively compact in $L^1\left(]0, \infty[\right)$.

3.2.3 Equicontinuity in time

Lemma 3.2.7. *Assume that* ***H1)****,* ***H2)****,* ***H4)****,* ***H5)****, and* ***H6)*** *hold. Take* $\left(u^n\right)$ *now to be the sequence of extended solutions to the truncated problems (3.10)-(3.11) found in Theorem 3.2.4 and* $v^n(x,t) = x^{-\sigma}u^n(x,t)$*. Then there exists a subsequences* $\left(u^{n_k}(t)\right)$ *and* $\left(v^{n_l}(t)\right)$ *of* $\left(u^n(t)\right)_{n\in\mathbb{N}}$ *and* $\left(v^n(t)\right)_{n\in\mathbb{N}}$ *respectively such that*

$$
\begin{aligned}
u^{n_k}(t) &\rightharpoonup u(t) \quad in \quad L^1\left(]0,\infty[\right) \quad as \quad n_k \to \infty \\
v^{n_l}(t) &\rightharpoonup v(t) \quad in \quad L^1\left(]0,\infty[\right) \quad as \quad n_l \to \infty
\end{aligned}
$$

uniformly for $t \in [0,T]$*. Giving* $u, v \in C_B\left([0,T];\Omega_1\right) = \{\eta : [0,\infty[\to \Omega_1, \eta \text{ continuous and } \eta(t)$ *bounded for all* $t \geq 0\}$*, where* Ω_1 *is* $L^1\left(]0,\infty[\right)$ *equipped with the weak topology. This convergence is uniform for all* $t \in [0;T]$*.*

Proof: Choose $\epsilon > 0$ and $\phi \in L^\infty\left(]0,\infty[\right)$. Let $s,t \in [0,T]$ and assume that $t \geq s$. Choose $a > 1$ such that

$$
\frac{2L(T)}{a}\|\phi\|_{L^\infty(]0,\infty[)} \leq \epsilon/2. \tag{3.36}
$$

Using Lemma 3.2.6(i), for each n, we have

$$
\int_a^\infty |u^n(x,t) - u^n(x,s)|\,dx \leq \frac{1}{a}\int_a^\infty x\,|u^n(x,t) + u^n(x,s)|\,dx \leq 2L(T)/a. \tag{3.37}
$$

By multiplying (3.10) by ϕ and integrating w.r.t. x from 0 to a as well as from a to ∞, w.r.t. τ form s to t and using (3.36), (3.37) and $t \geq s$ we get

$$
\begin{aligned}
&\left|\int_0^\infty \phi(x)\left[u^n(x,t) - u^n(x,s)\right]dx\right| \\
&\leq \left|\int_0^a \phi(x)\left[u^n(x,t) - u^n(x,s)\right]dx\right| + \left|\int_a^\infty \phi(x)\left[u^n(x,t) - u^n(x,s)\right]dx\right| \\
&\leq \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t \Bigg[\frac{1}{2}\int_0^a\int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_0^a\int_x^n b(x,y)S_n(y)u^n(y,\tau)\,dy\,dx + \int_0^a S_n(x)u^n(x,\tau)\,dx\Bigg]d\tau + \epsilon/2.
\end{aligned}
$$

Changing variables as we did in (3.20) we have

$$\begin{aligned}
&\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \\
&\quad = \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t \Bigg[\frac{1}{2}\int_0^a\int_0^{a-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_0^a\int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_0^a\int_x^n b(x,y)S_n(y)u^n(y,\tau)\,dy\,dx + \int_0^a S_n(x)u^n(x,\tau)\,dx\Bigg]d\tau + \epsilon/2.
\end{aligned}$$

Using the definition (3.9) of $K_n(x,y)$ and $S_n(y)$ and the estimation of $K(x,y)$ we obtain

$$\begin{aligned}
&\left|\int_0^\infty \phi(x)\left[u^n(x,t)-u^n(x,s)\right]dx\right| \\
&\quad \le \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t \Bigg[\frac{1}{2}\int_{\sigma/n}^{a-\sigma/n}\int_{\sigma/n}^{a-x} (1+x)^\lambda(1+y)^\lambda(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_{\sigma/n}^{a}\int_{\sigma/n}^{n-x} (1+x)^\lambda(1+y)^\lambda(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx \\
&\qquad + \int_{\sigma/n}^{a}\int_x^n b(x,y)S(y)u^n(y,\tau)\,dy\,dx + \int_{\sigma/n}^{a} x^\theta u^n(x,\tau)\,dx\Bigg]d\tau + \epsilon/2 \\
&\quad = \|\phi\|_{L^\infty(]0,\infty[)}\int_s^t \left(I_{31}(\tau)+I_{32}(\tau)+I_{33}(\tau)+I_{34}(\tau)\right)d\tau + \epsilon/2.
\end{aligned} \tag{3.38}$$

Now, we estimate the terms $I_{31}(\tau)$, $I_{32}(\tau)$, $I_{33}(\tau)$, and $I_{34}(\tau)$ in (3.38). By using Lemma 3.2.6(i) the first term can be estimated by

$$I_{31}(\tau) \le \frac{1}{2}(1+a)^{2\lambda}\int_{\sigma/n}^{a-\sigma/n}\int_{\sigma/n}^{a-x}(xy)^{-\sigma}u^n(x,\tau)u^n(y,\tau)\,dy\,dx \le \frac{1}{2}(1+a)^{2\lambda}L(T)^2. \tag{3.39}$$

In order to estimate the second term, we define

$$C_1 = \begin{cases} 1 & \text{if } \ 0\le\lambda\le 1 \\ 2^{\lambda-1} & \text{if } \ \lambda\ge 1. \end{cases} \tag{3.40}$$

Then, by using inequalities (3.5) and (3.6) for $p = \lambda$ and Lemma 3.2.6 (i) we find that

$$\begin{aligned} I_{32}(\tau) &\leq C_1 \int_{\sigma/n}^{a} \int_{\sigma/n}^{n-x} (1+x^{\lambda})(1+y^{\lambda})(xy)^{-\sigma} u^n(x,\tau) u^n(y,\tau)\, dy\, dx \\ &\leq C_1(1+a^{\lambda}) \int_{\sigma/n}^{a} \int_{\sigma/n}^{n-x} (1+y^{\lambda})(xy)^{-\sigma} u^n(x,\tau) u^n(y,\tau)\, dy\, dx \\ &\leq C_1(1+a^{\lambda}) \int_{\sigma/n}^{a} \int_{\sigma/n}^{n-x} (y^{-\sigma}+y^{\lambda-\sigma}) x^{-\sigma} u^n(x,\tau) u^n(y,\tau)\, dy\, dx \\ &\leq 2C_1(1+a^{\lambda}) L(T)^2. \end{aligned} \tag{3.41}$$

We also find by using **H5)**, Lemma 3.2.6 (i), and (1.8) that

$$\begin{aligned} I_{33}(\tau) &= \int_{\sigma/n}^{a} \int_{x}^{n} b(x,y) S(y) u^n(y,\tau)\, dy\, dx \\ &= \int_{\sigma/n}^{a} \int_{\sigma/n}^{y} b(x,y) y^{\theta} u^n(y,\tau)\, dx\, dy + \int_{a}^{n} \int_{\sigma/n}^{a} b(x,y) y^{\theta} u^n(y,\tau)\, dx\, dy \\ &\leq N \int_{\sigma/n}^{n} y^{\theta} u^n(y,\tau)\, dy \leq N L(T). \end{aligned} \tag{3.42}$$

Now, by using Lemma 3.2.6(i) we have

$$I_{34}(\tau) = \int_{\sigma/n}^{a} x^{\theta} u^n(x,\tau)\, dx \leq L(T), \tag{3.43}$$

which together with (3.39)-(3.42) brings (3.38) to

$$\begin{aligned} &\left| \int_0^{\infty} \phi(x) \left[u^n(x,t) - u^n(x,s) \right] dx \right| \\ &\leq \left[\left(\frac{1}{2}(1+a)^{2\lambda} + 2C_1 \left(1+a^{\lambda}\right) \right) L(T)^2 + (N+1)L(T) \right] (t-s) \|\phi\|_{L^{\infty}} + \epsilon/2 < \epsilon, \end{aligned} \tag{3.44}$$

whenever $(t-s) < \delta$ for some $\delta > 0$ sufficiently small. The argument given above similarly holds for $s < t$. Hence (3.44) holds for all n and $|t-s| < \delta$. Then the sequence $\left(u^n(t)\right)_{n\in\mathbb{N}}$ is time equicontinuous in $L^1\left(]0,\infty[\right)$. Thus, $\left(u^n(t)\right)$ lies in a relatively compact subset of a gauge space Ω_1. The gauge space Ω_1 is $L^1\left(]0,\infty[\right)$ equipped with the weak topology. For details about the gauge space, see Appendix B. Then, we may apply a version of the Arzela-Ascoli Theorem, see Appendix A [Theorem A.0.5], to conclude that there exists a subsequence $\left(u^{n_k}\right)_{k\in\mathbb{N}}$ such that

$$u^{n_k}(t) \to u(t) \quad \text{in} \quad \Omega_1 \quad \text{as} \quad n_k \to \infty,$$

uniformly for $t \in [0,T]$ for some $u \in C\left([0,T];\Omega_1\right)$.

Now let us consider $v^n(x,t) = x^{-\sigma}u^n(x,t)$ where we have to deal with a stronger singularity at 0.

We take $\epsilon > 0$, ϕ, s and t as they were defined before. Using Lemma 3.2.6, for each n, we get using $a > 1$ chosen to satisfy (3.36)

$$\begin{aligned}\int_a^\infty |v^n(x,t) - v^n(x,s)|\, dx &= \int_a^\infty \left|x^{-\sigma}u^n(x,t) - x^{-\sigma}u^n(x,s)\right| dx \\ &\le \frac{1}{a}\int_a^\infty x^{1-\sigma}\,|u^n(x,t) + u^n(x,s)|\, dx \\ &\le \frac{1}{a}\int_a^\infty x\,|u^n(x,t) + u^n(x,s)|\, dx \le 2L(T)/a. \qquad (3.45)\end{aligned}$$

By using (3.10), (3.36), (3.45), for $t \ge s$ and the definition of $v^n(x)$ we obtain

$$\begin{aligned}&\left|\int_0^\infty \phi(x)\left[v^n(x,t) - v^n(x,s)\right] dx\right| \\ &\le \int_0^a |\phi(x)|\left[v^n(x,t) + v^n(x,s)\right] dx + \epsilon/2 \\ &\le \|\phi\|_{L^\infty(]0,\infty[)} \int_s^t \left[\frac{1}{2}\int_0^a \int_0^x K_n(x-y,y)u^n(x-y,\tau)u^n(y,\tau)x^{-\sigma} dy\, dx\right. \\ &\qquad + \int_0^a \int_0^{n-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma} dy\, dx \\ &\qquad \left. + \int_0^a \int_x^n b(x,y)S_n(y)u^n(y,\tau)\, dy\, dx + \int_0^a S_n(x)u^n(x,\tau)\, dx\right] d\tau + \epsilon/2 \\ &= \|\phi\|_{L^\infty(]0,\infty[)} \int_s^t \left(I_{41}(\tau) + I_{42}(\tau) + I_{43}(\tau) + I_{44}(\tau)\right) d\tau + \epsilon/2.\end{aligned}$$

A change of variables in the first integral gives

$$I_{41}(\tau) = \frac{1}{2}\int_0^a \int_0^{a-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)(x+y)^{-\sigma} dy\, dx.$$

Taking $y = 0$ in the term $(x+y)^{-\sigma}$ we find that

$$I_{41}(\tau) \leq \frac{1}{2}\int_0^a \int_0^{a-x} K_n(x,y)u^n(x,\tau)u^n(y,\tau)x^{-\sigma} dy\, dx. \tag{3.46}$$

Working in a similar way as we did in (3.39) and (3.41) we find the estimations

$$\begin{aligned} I_{41}(\tau) &= \frac{1}{2}\int_{\sigma/n}^{a-\sigma/n} \int_{\sigma/n}^{a-x} (1+x)^{\lambda}(1+y)^{\lambda}x^{-2\sigma}y^{-\sigma}u^n(x,\tau)u^n(x,\tau)\, dy\, dx \\ &\leq \frac{1}{2}(1+a)^{2\lambda}\int_{\sigma/n}^{a-\sigma/n} \int_{\sigma/n}^{a-x} x^{-2\sigma}y^{-\sigma}u^n(x,\tau)u^n(x,\tau)\, dy\, dx \leq \frac{1}{2}(1+a)^{2\lambda}L(T)^2 \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} I_{42}(\tau) &\leq C_1 \int_{\sigma/n}^{a} \int_{\sigma/n}^{n-x} (1+x^{\lambda})(1+y^{\lambda})x^{-2\sigma}y^{-\sigma}u^n(x,\tau)u^n(y,\tau)\, dy\, dx \\ &\leq C_1\left(1+a^{\lambda}\right)\int_{\sigma/n}^{a} \int_{\sigma/n}^{n-x} (y^{-\sigma}+y^{\lambda-\sigma})x^{-2\sigma}u^n(x,\tau)u^n(y,\tau)\, dy\, dx \\ &\leq 2C_1\left(1+a^{\lambda}\right)L(T)^2. \end{aligned} \tag{3.48}$$

Now changing the order of integration in $I_{43}(\tau)$ we have

$$\begin{aligned} I_{43}(\tau) &= \int_{\sigma/n}^{a}\int_{x}^{n} b(x,y)S(y)u^n(y,\tau)x^{-\sigma} dy\, dx \\ &= \int_{\sigma/n}^{a}\int_{\sigma/n}^{y} b(x,y)y^{\theta}u^n(y,\tau)x^{-\sigma} dx\, dy + \int_{a}^{n}\int_{\sigma/n}^{a} b(x,y)y^{\theta}u^n(y,\tau)x^{-\sigma} dx\, dy. \end{aligned} \tag{3.49}$$

By using **H6)** and (1.8) we find that

$$\begin{aligned} \int_0^y b(x,y)x^{-\sigma}dx &= \int_0^1 b(x,y)x^{-\sigma}dx + \int_1^y b(x,y)x^{-\sigma}dx \\ &\leq \int_0^1 b(x,y)x^{-2\sigma}dx + \int_1^y b(x,y)dx \\ &\leq \int_0^y b(x,y)x^{-2\sigma}dx + \int_0^y b(x,y)dx \leq Cy^{-2\sigma} + N. \end{aligned} \tag{3.50}$$

Then by using (3.50) and **H5)** in (3.49) we get

$$\begin{aligned}
I_{43}(\tau) &\le \int_{\sigma/n}^{a} \left(Cy^{-2\sigma}+N\right) y^{\theta}u^{n}(y,\tau)\,dy + \int_{a}^{n} \left(Cy^{-2\sigma}+N\right) y^{\theta}u^{n}(y,\tau)\,dy \\
&= \int_{\sigma/n}^{n} \left(Cy^{-2\sigma}+N\right) y^{\theta}u^{n}(y,\tau)\,dy \\
&= C\int_{\sigma/n}^{n} y^{\theta-2\sigma}u^{n}(y,\tau)\,dy + N\int_{\sigma/n}^{n} y^{\theta}u^{n}(y,\tau)\,dy \le (C+N)L(T). \qquad (3.51)
\end{aligned}$$

Using **H5)** and Lemma 3.2.6 (i) we obtain

$$I_{44}(\tau) = \int_{\sigma/n}^{a} x^{\theta-\sigma}u^{n}(x,\tau)dx \le L(T) \qquad (3.52)$$

which together with (3.46)-(3.51) gives the estimation

$$\begin{aligned}
&\left|\int_{0}^{\infty} \phi(x)\left[v^{n}(x,t)-v^{n}(x,s)\right]dx\right| \\
&\le \left[\left(\frac{1}{2}(1+a)^{2\lambda}+2C_{1}\left(1+a^{\lambda}\right)\right)L(T)^{2}+(C+N+1)L(T)\right](t-s)\|\phi\|_{L^{\infty}(]0,\infty[)}+\epsilon/2.
\end{aligned}$$

We can use now the same argument used for u^n to conclude that there exists a subsequence $\left(v^{n_k}\right)_{k\in\mathbb{N}}$ such that

$$v^{n_k}(t) \to v(t) \quad \text{in} \quad \Omega_1 \quad \text{as} \quad n_k\to\infty,$$

uniformly for $t\in[0,T]$ for some $v\in C\left([0,T];\Omega_1\right)$.

Since $T>0$ is arbitrary we obtain $u,v\in C_B\left([0,\infty[;\Omega_1\right)$. □

Lemma 3.2.8. *For $v^n(\cdot,t)$ defined as before, we have*

$$v^{n}(\cdot,t) \rightharpoonup v(\cdot,t) \quad \textit{where} \quad v(x,t)=x^{-\sigma}u(x,t) \quad \textit{for all} \quad t\in[0,T] \quad \textit{in} \quad L^{1}\left(]0,a]\right).$$

Proof. By Lemma 3.2.7, we know that $v^n(t)\rightharpoonup v(t)$ in $L^1\left(]0,\infty[\right)$ as $n\to\infty$ uniformly for $t\in[0,T]$. Then, we just need to prove that $v(x,t)=x^{-\sigma}u(x,t)$.

By definition of weak convergence we have

$$\int_{0}^{a} \varphi(x)\left[v^{n}(x,t)-v(x,t)\right]dx \to 0 \quad \text{for all} \quad \varphi\in L^{\infty}\left(]0,a]\right).$$

As $x^\sigma \in L^\infty\big(]0,a]\big)$ we obtain for all $\varphi \in L^\infty\big(]0,a]\big)$

$$\int_0^a \varphi(x)\,[x^\sigma v^n(x,t) - x^\sigma v(x,t)]\,dx = \int_0^a \varphi(x)\,[u^n(x,t) - x^\sigma v(x,t)]\,dx \to 0.$$

Since $u^n \rightharpoonup u$ we have due to the uniqueness of the limit of weak convergence, $v(x,t) = x^{-\sigma}u(x,t)$. □

3.3 The existence theorem

3.3.1 Convergence of the integrals

In order to show that the limit function which we obtained above is indeed a solution to (3.1)-(3.2), we define the operators M_i^n, M_i, $i = 1,2,3,4$

$$\begin{aligned}
M_1^n(u^n)(x) &= \tfrac{1}{2}\int_0^x K_n(x-y,y)u^n(x-y)u^n(y)dy & M_1(u)(x) &= \tfrac{1}{2}\int_0^x K(x-y,y)u(x-y)u(y)dy\\
M_2^n(u^n)(x) &= \int_0^{n-x} K_n(x,y)u^n(x)u^n(y)dy & M_2(u)(x) &= \int_0^\infty K(x,y)u(x)u(y)dy\\
M_3^n(u^n)(x) &= \int_x^n b(x,y)S_n(y)u^n(y)dy & M_3(u)(x) &= \int_x^\infty b(x,y)S(y)u(y)dy\\
M_4^n(u^n)(x) &= S_n(x)u^n(x) & M_4(u)(x) &= S(x)u(x),
\end{aligned}$$

where $u \in L^1\big(]0,\infty[\big)$, $x \in]0,\infty[$ and $n = 1,2,\ldots$. Set $M^n = M_1^n - M_2^n + M_3^n - M_4^n$ and $M = M_1 - M_2 + M_3 - M_4$.

Lemma 3.3.1. *Suppose that* $\big(u^n\big)_{n\in\mathbb{N}} \subset Y^+$, $u \in Y^+$ *where* $\|u^n\|_Y \le L$, $\|u\|_Y \le Q$, $u^n \rightharpoonup u$ *and* $v^n \rightharpoonup v$ *in* $L^1\big(]0,\infty[\big)$ *as* $n \to \infty$. *Then for each* $a > 0$

$$M^n(u^n) \rightharpoonup M(u) \quad \textit{in} \quad L^1\big(]0,a[\big) \quad \textit{as} \quad n \to \infty.$$

Proof: Choose $a > 0$ and let $\phi \in L^\infty\big(]0,\infty[\big)$. We show that $M_i^n(u^n) \rightharpoonup M_i(u)$ in $L^1\big(]0,a[\big)$ as $n \to \infty$ for $i = 1,2,3,4$.

Case $i = 1$: For $u \in Y^+$ and $x \in [0,a]$ we define the operator g by

$$g(v)(x) = \frac{1}{2}\int_0^{a-x} \phi(x+y)K(x,y)(xy)^\sigma v(y)\,dy \quad \text{where} \quad v = x^{-\sigma}u.$$

For $a.e.$ $x \in [0,a]$ the function defined by

$$\varphi_x(y) := \frac{1}{2}\chi_{[0,a-x]}(y)\phi(x+y)K(x,y)(xy)^\sigma \le \frac{1}{2}\chi_{[0,a-x]}(y)\phi(x+y)(1+x)^\lambda(1+y)^\lambda,$$

where χ denotes the characteristic function and the estimate is due to **H4)**, is in $L^\infty\big(]0,\infty[\big)$. Since $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$, it follows that

$$g(v^n)(x) \to g(v)(x) \quad \text{for a.e.} \quad x \in [0,a]. \tag{3.53}$$

Also, as $\|u^n\|_Y \leq L$ implies $\int\limits_0^\infty v^n(x)\,dx < L$ we have

$$
\begin{aligned}
|g(v^n)(x)| &= \frac{1}{2}\int\limits_0^{a-x} \phi(x+y)K(x,y)(xy)^\sigma v^n(y)\,dy \\
&\leq \frac{1}{2}\int\limits_0^{a-x} \phi(x+y)(1+x)^\lambda(1+y)^\lambda v^n(y)\,dy \\
&\leq \frac{1}{2}(1+a)^{2\lambda}\|\phi\|_{L^\infty([0,a])}L \quad \text{for a.e.} \quad x \in [0,a]. \qquad (3.54)
\end{aligned}
$$

This holds analogously for $g(v)$. Thus, both, $g(v^n)$ and $g(v)$ are in $L^\infty\big([0,a]\big)$ with bound

$$
\|g(v^n)\|_{L^\infty([0,a])} + \|g(v)\|_{L^\infty([0,a])} \leq \frac{1}{2}(1+a)^{2\alpha}\|\phi\|_{L^\infty([0,a])}(L+Q). \qquad (3.55)
$$

It follows by (3.53) and Egorov's Theorem, see Appendix A [Theorem A.0.7], that

$$
g(v^n) \to g(v) \quad \text{as} \quad n \to \infty \quad \text{almost uniformly in} \quad [0,a]. \qquad (3.56)
$$

Remember that almost uniformly means that for any given δ there exists a set $E \subseteq [0,a]$ such that $\mu(E) < \delta$ and $g(v^n) \to g(v)$ uniformly on $[0,a] \setminus E$ as $n \to \infty$.

By Lemma 3.2.6(iii), since $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$ there is a $\delta > 0$ such that for all n

$$
\int\limits_E v^n(x)\,dx < \epsilon/\left[(1+a)^{2\alpha}\|\phi\|_{L^\infty([0,a])}(L+Q)\right] \quad \text{whenever} \quad \mu(E) < \delta. \qquad (3.57)
$$

We obtain using (3.55) and (3.57)

$$
\begin{aligned}
&\left|\int\limits_0^a [g(v^n)(x) - g(v)(x)]\,v^n(x)\,dx\right| \\
&\quad \leq \left|\int_{[0,a]\setminus E} [g(v^n)(x) - g(v)(x)]\,v^n(x)\,dx\right| + \left|\int_E [g(v^n)(x) - g(v)(x)]\,v^n(x)\,dx\right| \\
&\quad \leq \|g(v^n) - g(v)\|_{L^\infty([0,a]\setminus E)} \int_{[0,a]\setminus E} v^n(x)\,dx + \left[\|g(v^n)\|_{L^\infty(E)} + \|g(v)\|_{L^\infty(E)}\right] \int_E v^n(x)\,dx \\
&\quad \leq \|g(v^n) - g(v)\|_{L^\infty([0,a]\setminus E)} \int_{[0,a]\setminus E} v^n(x)\,dx + \frac{\epsilon}{2} \leq \epsilon \quad \text{for} \quad n \geq n_0.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrarily chosen the almost uniform convergence of $g(v^n)$ leads to

$$
\left|\int\limits_0^a [g(v^n)(x) - g(v)(x)]\,v^n(x)\,dx\right| \to 0 \quad \text{as} \quad n \to \infty. \qquad (3.58)
$$

Also, since $g(v) \in L^\infty\big([0,a]\big)$ is bounded independently of n by (3.54) and $v^n \rightharpoonup v$ in $L^1\big(]0,\infty[\big)$ as $n \to \infty$

$$\left| \int_0^a g(v)(x)\,[v^n(x) - v(x)]\,dx \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.59}$$

Now, since $g(v^n) \in L^\infty\big([0,a]\big)$ and $v^n \in L^1\big([0,\infty[\big)$, by the absolute continuity of the Lebesgue integral, we have

$$\left| \int_{a-\sigma/n}^{a} g(v^n)(x)v^n(x)\,dx \right| \le \|g(v^n)\|_{L^\infty([0,a])} \left| \int_{a-\sigma/n}^{a} v^n(x)\,dx \right| \to 0 \quad \text{as} \quad n \to \infty.$$

In the same way we get

$$\left| \int_0^{\sigma/n} g(v^n)(x)v^n(x)\,dx \right| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.60}$$

and

$$\begin{aligned} &\left| \int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y)K(x,y)u(x)u(y)\,dy\,dx \right| \\ &\le \left| \int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y)(1+x)^\lambda(1+y)^\lambda v(x)v(y)\,dy\,dx \right| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.61}$$

Now, we show that

$$\begin{aligned}
\int_0^a \phi(x) M_1^n(u^n)(x)\,dx &= \int_0^a \phi(x)\frac{1}{2}\int_0^x K_n(x-y,y)u^n(x-y)u^n(y)\,dy\,dx \\
&= \frac{1}{2}\int_0^a \int_0^{a-x} \phi(x+y)K_n(x,y)u^n(x)u^n(y)\,dy\,dx \\
&= \frac{1}{2}\int_{\sigma/n}^{a-\sigma/n} \int_{\sigma/n}^{a-x} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \\
&= \frac{1}{2}\int_{\sigma/n}^{a-\sigma/n} \int_0^{a-x} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \\
&\qquad -\frac{1}{2}\int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx \\
&= \int_{\sigma/n}^{a-\sigma/n} g(v^n)(x)v^n(x)\,dx \\
&\qquad -\frac{1}{2}\int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y)K(x,y)u^n(x)u^n(y)\,dy\,dx. \qquad (3.62)
\end{aligned}$$

In a similar way we also find that

$$\int_0^a \phi(x) M_1(u)(x)\,dx = \int_0^a g(v)(x)x^{-\sigma}u(x)\,dx = \int_0^a g(v)(x)v(x)\,dx. \qquad (3.63)$$

Now, it follows from (3.62) and (3.63) that

$$
\begin{aligned}
&\left| \int_0^a \phi(x) \left[M_1^n(u^n)(x) - M_1(u)(x) \right] dx \right| \\
&= \left| \int_{\sigma/n}^{a-\sigma/n} g(v^n)(x) v^n(x)\, dx - \int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y) K(x,y) u^n(x) u^n(y)\, dy\, dx - \int_0^a g(v)(x) v(x)\, dx \right| \\
&= \left| \int_0^a g(v^n)(x) v^n(x)\, dx - \int_0^a g(v)(x) v(x)\, dx - \int_{a-\sigma/n}^{a} g(v^n)(x) v^n(x)\, dx \right. \\
&\quad \left. - \int_0^{\sigma/n} g(v^n)(x) v^n(x)\, dx - \int_{\sigma/a}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| \\
&\leq \left| \int_0^a g(v^n)(x) v^n(x)\, dx - \int_0^a g(v)(x) v(x)\, dx \right| + \left| \int_{a-\sigma/n}^{a} g(v^n)(x) v^n(x)\, dx \right| \\
&\quad + \left| \int_0^{\sigma/n} g(v^n)(x) v^n(x)\, dx \right| + \left| \int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y) K(x,y) u^n(x) u^n(y)\, dy\, dx \right|.
\end{aligned}
$$

By addition and subtraction of the term $\int_0^a g(v)(x) v^n(x)\, dx$ in the first term of the above inequality, it results that

$$
\begin{aligned}
&\left| \int_0^a \phi(x) \left[M_1^n(u^n)(x) - M_1(u)(x) \right] dx \right| \\
&\quad \leq \left| \int_0^a g(v)(x) \left[v^n(x) - v(x) \right] dx \right| + \left| \int_0^a \left[g(v^n)(x) - g(v)(x) \right] v^n(x)\, dx \right| \\
&\qquad + \left| \int_{a-\sigma/n}^{a} g(v^n)(x) v^n(x)\, dx \right| + \left| \int_0^{\sigma/n} g(v^n)(x) v^n(x)\, dx \right| \\
&\qquad + \left| \int_{\sigma/n}^{a-\sigma/n} \int_0^{\sigma/n} \phi(x+y) K(x,y) u^n(x) u^n(y)\, dy\, dx \right|.
\end{aligned}
$$

Now, by (3.58)-(3.61) and taking $n \to \infty$ we have

$$
\left| \int_0^a \phi(x) \left[M_1^n(u^n)(x) - M_1(u)(x) \right] dx \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.64}
$$

It follows, since ϕ is arbitrary, that

$$M_1^n(u^n)(x) \rightharpoonup M_1(u)(x) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{3.65}$$

Case $i = 2$: For every $\epsilon > 0$ and C_1 defined by (3.40) we can choose η large enough, due to the negative exponents, such that for L, Q from our assumptions

$$2C_1^2 \|\phi\|_{L^\infty([0,a])} \Big[(\eta^{-(1+\sigma)} + \eta^{\lambda-\sigma-1})(L^2 + Q^2)\Big] < \frac{\epsilon}{3}. \tag{3.66}$$

Redefining the operator g for $u \in Y^+$ and $x \in [0,a]$ by

$$g(v)(x) = \int_0^\eta \phi(x) K(x,y)(xy)^\sigma v(y)\, dy.$$

For $a.e.$ $x \in [0,a]$ the function defined by

$$\varphi_x(y) := \frac{1}{2}\chi_{[0,\eta]}(y)\phi(x+y)K(x,y)(xy)^\sigma$$

where, as before, χ denotes the characteristic function, is in $L^\infty\big(]0,\infty[\big)$. Using a similar argument as the one was used in (3.53)-(3.58) it can be shown that also for the above redefined g (3.58) and (3.59) hold. By **H4)** and (3.40) we have

$$\begin{aligned}
&\left| \int_0^a \int_\eta^\infty \phi(x)K(x,y)\left[u^n(x)u^n(y) - u(x)u(y)\right] dy\, dx \right| \\
&\quad \le C_1^2 \int_0^a \int_\eta^\infty |\phi(x)| \Big[(xy)^{-\sigma} + x^{\lambda-\sigma}y^{-\sigma} + x^{-\sigma}y^{\lambda-\sigma} + (xy)^{\lambda-\sigma}\Big] \left[u^n(x)u^n(y) + u(x)u(y)\right] dy\, dx \\
&\quad \le C_1^2 \|\phi\|_{L^\infty([0,a])} \Bigg[\int_0^a \int_\eta^\infty (xy)^{-\sigma}\left[u^n(x)u^n(y) + u(x)u(y)\right] dy\, dx \\
&\qquad + \int_0^a \int_\eta^\infty x^{\lambda-\sigma}y^{-\sigma}\left[u^n(x)u^n(y) + u(x)u(y)\right] dy\, dx \\
&\qquad + \int_0^a \int_\eta^\infty x^{-\sigma}y^{\lambda-\sigma}\left[u^n(x)u^n(y) + u(x)u(y)\right] dy\, dx \\
&\qquad + \int_0^a \int_\eta^\infty (xy)^{\lambda-\sigma}\left[u^n(x)u^n(y) + u(x)u(y)\right] dy\, dx \Bigg].
\end{aligned} \tag{3.67}$$

We can estimate the integral term of (3.67) as follows

$$\begin{aligned}
&\int_0^a\int_\eta^\infty (xy)^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\\
&\le \int_\eta^\infty y^{-\sigma}\left[\int_0^1 x^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx+\int_1^a x^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy\\
&\le \int_\eta^\infty y^{-\sigma}\left[\int_0^1 x^{-1}\left[u^n(x)u^n(y)+u(x)u(y)\right]dx+\int_1^a x\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy\\
&\le \int_\eta^\infty y^{-\sigma}\left[\int_0^a (x^{-1}+x)\left[u^n(x)u^n(y)+u(x)u(y)\right]dx\right]dy\\
&\le \eta^{-(1+\sigma)}\int_\eta^\infty y\left[Lu^n(y)+Qu(y)\right]dy\le \eta^{-(1+\sigma)}(L^2+Q^2).
\end{aligned}\tag{3.68}$$

In the similar way we have

$$\int_0^a\int_\eta^\infty x^{\lambda-\sigma}y^{-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\le \eta^{-(1+\sigma)}(L^2+Q^2),\tag{3.69}$$

$$\int_0^a\int_\eta^\infty x^{-\sigma}y^{\lambda-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\le \eta^{-(1+\sigma-\lambda)}(L^2+Q^2),\tag{3.70}$$

and

$$\int_0^a\int_\eta^\infty (xy)^{\lambda-\sigma}\left[u^n(x)u^n(y)+u(x)u(y)\right]dy\,dx\le \eta^{-(1+\sigma-\lambda)}(L^2+Q^2).\tag{3.71}$$

By (3.68)-(3.71) and (3.66), (3.67) becomes

$$\begin{aligned}
&\left|\int_0^a\int_\eta^\infty \phi(x)K(x,y)\left[u^n(x)u^n(y)-u(x)u(y)\right]dy\,dx\right|\\
&\le 2C_1^2\left(\eta^{-(1+\sigma)}+\eta^{-(1+\sigma-\lambda)}\right)(L^2+Q^2)\|\phi\|_{L^\infty([0,a])}<\frac{\epsilon}{3}.
\end{aligned}\tag{3.72}$$

Now, using Lemma 3.2.6(i) and the absolute continuity of the Lebesgue integral, we have

$$\left|\int_0^a\int_0^{\sigma/n}\phi(x)K(x,y)u^n(x)u^n(y)\,dy\,dx\right|<\frac{\epsilon}{3}\quad\text{for } n \text{ larger than some } n_0,\tag{3.73}$$

and

$$\left| \int_0^{\sigma/n} \int_{\sigma/n}^{n-x} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| < \frac{\epsilon}{3} \quad \text{for } n \geq n_0, \tag{3.74}$$

Also, proceeding as before, for $n > a$ we have

$$\begin{aligned} &\left| \int_0^{a} \int_{n-x}^{\infty} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| \\ &\quad \leq 2 \left[(n-a)^{-(1+\sigma)} + (n-a)^{-(1+\sigma-\lambda)} \right] (L^2 + Q^2) \|\phi\|_{L^\infty([0,a])}. \end{aligned} \tag{3.75}$$

From (3.72)-(3.75) together with the analogues of (3.58) and (3.59), for $n > a$

$$\begin{aligned} &\left| \int_0^a \phi(x) \left[M_2^n(u^n)(x) - M_2(u)(x) \right] dx \right| \\ &= \left| \int_0^a \int_0^{n-x} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx - \int_0^a \int_0^{\infty} \phi(x) K(x,y) u(x) u(y)\, dy\, dx \right. \\ &\qquad \left. - \int_0^a \int_0^{\sigma/n} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx - \int_0^{\sigma/n} \int_{\sigma/n}^{n-x} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| \\ &= \left| \int_0^a \int_0^b \phi(x) K(x,y) \left[u^n(x) u^n(y) - u(x) u(y) \right] dy\, dx \right. \\ &\qquad + \int_0^a \int_b^{\infty} \phi(x) K(x,y) \left[u^n(x) u^n(y) - u(x) u(y) \right] dy\, dx \\ &\qquad - \int_0^a \int_0^{\sigma/n} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx - \int_0^{\sigma/n} \int_{\sigma/n}^{n-x} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \\ &\qquad \left. - \int_0^a \int_{n-x}^{\infty} \phi(x) K(x,y) u^n(x) u^n(y)\, dy\, dx \right| \\ &\leq \left| \int_0^a \left[g(v^n)(x) - g(v)(x) \right] v^n(x)\, dx \right| + \left| \int_0^a g(v)(x) \left[v^n(x) - v(x) \right] dx \right| + \epsilon \\ &\qquad + 2 \left[(n-a)^{-(1+\sigma)} + (n-a)^{-(1+\sigma-\lambda)} \right] (L^2 + Q^2) \|\phi\|_{L^\infty([0,a])} \to \epsilon \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore, since ϕ and ϵ are arbitrary, we conclude that

$$M_2^n(u^n)(x) \rightharpoonup M_2(u)(x) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{3.76}$$

Case $i = 3$: As $\theta \in [0, 1[$ in **H5)** we can choose $r > a$ sucht that for N from (3.4) and L, Q from our assumptions

$$\|\phi\|_{L^\infty([0,a])} N(L+Q) r^{\theta-1} < \epsilon. \tag{3.77}$$

By a change of the order of integration and (3.4) we have

$$\begin{aligned}\left| \int_{\sigma/n}^{a} \int_{r}^{\infty} \phi(x) b(x,y) S(y) \left[u^n(y) - u(y)\right] dy\, dx \right| &= \left| \int_{r}^{\infty} \int_{\sigma/n}^{a} \phi(x) b(x,y) S(y) \left[u^n(y) - u(y)\right] dx\, dy \right| \\ &\leq \|\phi\|_{L^\infty([0,a])} N \int_{r}^{\infty} y^{\theta-1} y \left[u^n(y) + u(y)\right] dy \\ &\leq \|\phi\|_{L^\infty([0,a])} N r^{\theta-1} \int_{r}^{\infty} y \left[u^n(y) + u(y)\right] dy.\end{aligned}$$

As $\|u^n\|_Y \leq L$ and $\|u\|_Y \leq Q$, using (3.77) we find that

$$\left| \int_{\sigma/n}^{a} \int_{r}^{\infty} \phi(x) b(x,y) S(y) \left[u^n(y) - u(y)\right] dy\, dx \right| \leq \|\phi\|_{L^\infty([0,a])} N(L+Q) r^{\theta-1} \leq \epsilon. \tag{3.78}$$

Now, by changing the order of integration and using (3.4) and **H6)** we get

$$\begin{aligned}&\left| \int_{0}^{\sigma/n} \int_{x}^{\infty} \phi(x) b(x,y) S(y) u(y)\, dy\, dx \right| \\ &= \left| \int_{0}^{\sigma/n} \int_{0}^{y} \phi(x) b(x,y) S(y) u(y)\, dx\, dy \right| + \left| \int_{\sigma/n}^{\infty} \int_{0}^{\sigma/n} \phi(x) b(x,y) S(y) u(y)\, dx\, dy \right| \\ &\leq \|\phi\|_{L^\infty([0,a])} N \int_{0}^{\sigma/n} y^{\theta} u(y)\, dy + \|\phi\|_{L^\infty([0,a])} \int_{\sigma/n}^{\infty} \int_{0}^{\sigma/n} y^{\theta} x^{2\sigma} x^{-2\sigma} b(x,y) u(y)\, dx\, dy \\ &\leq \|\phi\|_{L^\infty([0,a])} N n^{-2\sigma} \int_{0}^{\sigma/n} y^{\theta-2\sigma} u(y)\, dy + \|\phi\|_{L^\infty([0,a])} C n^{-2\sigma} \int_{\sigma/n}^{\infty} y^{\theta-2\sigma} u(y)\, dy \\ &= \|\phi\|_{L^\infty([0,a])} (N+C) n^{-2\sigma} \int_{0}^{\sigma/n} y^{\theta-2\sigma} u(y)\, dy \leq \|\phi\|_{L^\infty([0,a])} (N+C) n^{-2\sigma} Q.\end{aligned} \tag{3.79}$$

In a similar way we also have

$$\begin{aligned}&\left|\int_{\sigma/n}^{a}\int_{x}^{r}\phi(x)b(x,y)S(y)\left[u^n(y)-u(y)\right]dy\,dx\right|\\&\leq \|\phi\|_{L^\infty([0,a])}Nr^\theta\int_{\sigma/n}^{r}|u^n(y)-u(y)|\,dy\to 0\quad\text{as}\quad n\to\infty.\end{aligned}\tag{3.80}$$

Now, from (3.78)-(3.80) we find that

$$\begin{aligned}&\left|\int_0^a\phi(x)\left[M_3^n(u^n)(x)-M_3(u)\right]dx\right|\\&\leq\left|\int_{\sigma/n}^{a}\phi(x)\left[M_3^n(u^n)(x)-M_3(u)\right]dx\right|+\left|\int_0^{\sigma/n}\phi(x)M_3(u)dx\right|\\&=\left|\int_{\sigma/n}^{a}\int_{x}^{r}\phi(x)b(x,y)S(y)\left[u^n(x)-u(x)\right]dy\,dx+\int_{\sigma/n}^{a}\int_{r}^{\infty}\phi(x)b(x,y)S(y)\left[u^n(x)-u(x)\right]dy\,dx\right.\\&\quad\left.-\int_{\sigma/n}^{a}\int_{n}^{\infty}\phi(x)b(x,y)S(y)u^n(x)dy\,dx\right|+\left|\int_0^{\sigma/n}\int_x^\infty\phi(x)b(x,y)S(y)u(y)\,dy\,dx\right|\\&\leq\left|\int_{\sigma/n}^{a}\int_{x}^{r}\phi(x)b(x,y)S(y)\left[u^n(x)-u(x)\right]dy\,dx\right|+\epsilon\\&\quad+\|\phi\|_{L^\infty([0,a])}N(L+Q)n^{\theta-1}+\|\phi\|_{L^\infty([0,a])}(N+C)n^{-2\sigma}Q\to\epsilon\quad\text{as}\quad n\to\infty.\end{aligned}$$

Since ϕ and ϵ are arbitrarily chosen, it follows that

$$M_3^n(u^n)(x)\rightharpoonup M_3(u)(x)\quad\text{in}\quad L^1\big(]0,a[\big)\quad\text{as}\quad n\to\infty.\tag{3.81}$$

Case $i=4$: By using **H5)** we have

$$|\phi(x)S(x)|\leq\|\phi\|_{L^\infty([0,a])}a^\theta\quad\text{for}\quad a.e.\ x\in[0,a].$$

Then, as $\phi(x)S(x) \in L^{\infty}\big([0,\infty[\big)$ for $x \in [0,a]$ we find that

$$\begin{aligned}
&\left|\int_0^a \phi(x)\left[M_4^n(u^n)(x) - M_4(u)(x)\right] dx\right| \\
&\leq \left|\int_0^{\sigma/n} \phi(x)\left[S_n(x)u^n(x) - S(x)u(x)\right] dx\right| + \left|\int_{\sigma/n}^a \phi(x)\left[S_n(x)u^n(x) - S(x)u(x)\right] dx\right| \\
&= \left|\int_0^{\sigma/n} \phi(x)S(x)u(x)\, dx\right| + \left|\int_{\sigma/n}^a \phi(x)S(x)\left[u^n(x) - u(x)\right] dx\right| \\
&\leq \|\phi\|_{L^{\infty}([0,a])} n^{-\theta} \int_0^{\sigma/n} u(x)\, dx + \left|\int_{\sigma/n}^a \phi(x)S(x)\left[u^n(x) - u(x)\right] dx\right| \longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Since ϕ is arbitrarily chosen, it follows that

$$M_4^n(u^n)(x) \rightharpoonup M_4(u)(x) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{3.82}$$

Lemma 3.3.1 follows from (3.65), (3.76), (3.81) and(3.82). □

3.3.2 The existence result

Theorem 3.3.2. *Suppose that* ***H1)**, **H2)**, **H4)**, **H5)**, **H6)*** *and* ***H7)*** *hold and assume that $u_0 \in Y^+$. Then (3.1) has a solution $u \in C_B\big([0,\infty[; L^1\big(]0,\infty[\big)\big)$.*

Proof. Choose $T, m > 0$, and let $\big(u^n\big)_{n\in\mathbb{N}}$ be the weakly convergent subsequence of approximating solutions obtained in Lemma 3.2.7. For $t \in [0,T]$ we obtain by weak convergence and Lemma 3.2.6(i)

$$\int_0^m xu(x,t)\, dx = \lim_{n\to\infty} \int_0^m xu^n(x,t)\, dx \leq L(T) < \infty,$$

and

$$\int_{1/m}^m x^{-\sigma}u(x,t)\, dx = \lim_{n\to\infty} \int_{1/m}^m x^{-\sigma}u^n(x,t)\, dx \leq L(T) < \infty.$$

Then taking $m \to \infty$ implies that $u \in Y^+$ with $\|u\|_Y \leq 2L(T)$. Let $\phi \in L^{\infty}\big(]0,a[\big)$. From Lemma 3.2.7 we have for each $s \in [0,t]$

$$u^n(t) \rightharpoonup u(t) \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{3.83}$$

For Lemma 3.2.7 and Lemma 3.3.1 for each $s \in [0,t]$ we have

$$\int_0^a \phi(x)\left[M^n(u^n(s))(x) - M(u(s))(x)\right] dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.84}$$

Also, for $s \in [0,t]$, using Lemma 3.2.6(i) and $\|u\|_Y \leq 2L(T)$ we find that

$$
\begin{aligned}
&\int_0^a |\phi(x)|\,|M^n(u^n(s))(x) - M(u(s))(x)|\,dx \\
&\leq \|\phi\|_{L^\infty(]0,a[)} \Bigg[\frac{1}{2}\int_0^a \int_0^x K(x-y,y)\,[u^n(x-y,s)u^n(y,s) + u(x-y,s)u(y,s)]\,dy\,dx \\
&\qquad + \int_0^a \int_0^{n-x} K(x,y)u^n(x,s)u^n(y,s)\,dy\,dx + \int_0^a \int_0^\infty K(x,y)u(x,s)u(y,s)\,dy\,dx \\
&\qquad + \int_0^a \int_x^n b(x,y)S(y)u^n(y,s)\,dy\,dx + \int_0^a \int_x^\infty b(x,y)S(y)u(y,s)\,dy\,dx \\
&\qquad + \int_0^a S(x)\,[u^n(x,s) + u(x,s)]\,dx \Bigg] \\
&\leq 3\|\phi\|_{L^\infty(]0,a[)}\,[10L(T) + (N+1)]\,L(T).
\end{aligned} \tag{3.85}
$$

Since the left hand side of (3.85) is in $L^1\big(]0,t[\big)$ we have by (3.84), (3.85) and the dominated convergence theorem

$$
\left| \int_0^t \int_0^a \phi(x)\,[M^n(u^n(s))(x) - M(u(s))(x)]\,dx\,ds \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.86}
$$

Since ϕ was arbitrarily chosen the limit (3.86) holds for all $\phi \in L^\infty\big(]0,a[\big)$. By Fubini's Theorem we get

$$
\int_0^t M^n(u^n(s))(x)\,ds \rightharpoonup \int_0^t M(u(s))(x)\,ds \quad \text{in} \quad L^1\big(]0,a[\big) \quad \text{as} \quad n \to \infty. \tag{3.87}
$$

From the definition of M^n for $t \in [0,T]$

$$
u^n(t) = \int_0^t M^n(u^n(s))\,ds + u^n(0),
$$

and thus it follows by (3.87), (3.83) and the uniqueness of weak limits that

$$
u(t) = \int_0^t M(u(s))\,ds + u(0). \tag{3.88}
$$

It follows from the fact that T and a are arbitrary that u is a solution to (3.1) on $C_B\left([0,\infty[;\Omega_1\right)$.

Now we show that $u \in C_B\left([0,\infty[; L^1(]0,\infty[)\right)$. Considering $t_n > t$ and by using (3.88) we have that

$$\begin{aligned}
\int_0^\infty |u(x,t_n) - u(x,t)|dx &= \int_0^\infty \Bigg| \frac{1}{2}\int_t^{t_n}\int_0^x K(x-y,y)u(x-y,\tau)u(y,\tau)dy\,d\tau \\
&\quad - \int_t^{t_n}\int_0^\infty K(x,y)u(x,\tau)u(y,\tau)dy\,d\tau \\
&\quad + \int_t^{t_n}\int_x^\infty b(x,y)S(y)u(y,\tau)dy\,d\tau - \int_t^{t_n} S(x)u(x,\tau)d\tau\Bigg|\,dx \\
&\leq \int_t^{t_n}\Bigg[\frac{3}{2}\int_0^\infty\int_0^\infty K(x,y)u(x,\tau)u(y,\tau)dy\,dx \\
&\quad + \int_0^\infty\int_0^y b(x,y)S(y)u(y,\tau)dx\,dy + \int_0^\infty S(x)u(x,\tau)dx\Bigg]\,d\tau.
\end{aligned}$$

By using the definition (3.40) of C_1, Lemma 3.2.6 (i), **H4)**, **H5)** and (1.8) we find that

$$\begin{aligned}
\int_0^\infty |u(x,t_n) - u(x,t)|dx &\leq \int_t^{t_n}\Bigg[\frac{3}{2}\int_0^\infty\int_0^\infty (1+x)^\lambda(1+y)^\lambda(xy)^{-\sigma}u(x,\tau)u(y,\tau)\,dy\,dx \\
&\quad + N\int_0^\infty yu(y,\tau)dy + \int_0^\infty xu(x,\tau)dx\Bigg]\,d\tau \\
&\leq \int_t^{t_n}\Bigg[\frac{3}{2}C_1^2\int_0^\infty\int_0^\infty \left((xy)^{-\sigma} + x^{\lambda-\sigma}y^{-\sigma} + y^{\lambda-\sigma}x^{-\sigma}\right)u(x,\tau)u(y,\tau)\,dy\,dx \\
&\quad + N\int_0^\infty yu(y,\tau)dy + \int_0^\infty xu(x,\tau)dx\Bigg]\,d\tau \\
&\leq \left[\frac{45}{2}C_1^2L^2(T) + (N+1)L(T)\right](t_n - t). \qquad (3.89)
\end{aligned}$$

Then from (3.89) we obtain that

$$\int_0^\infty |u(x,t_n) - u(x,t)|dx \to 0 \quad \text{as} \quad t_n \to t. \qquad (3.90)$$

The same argument holds when $t_n < t$. Hence (3.90) holds for $|t_n - t| \to 0$ and we can conclude that $u \in C_B\left([0,\infty[; L^1(]0,\infty[)\right)$. This completes the proof of Theorem 3.3.2. □

3.4 Uniqueness of solutions

In this section we study the uniqueness of the solutions to (3.1)-(3.2) under the following further hypotheses

H4') $K(x,y) \leq \kappa_1(x^{-\sigma} + x^{\lambda-\sigma})(y^{-\sigma} + y^{\lambda-\sigma})$ *such that* σ, $\lambda - \sigma \in [0, 1/2]$ *and* $k_1 > 0$,

H5') $S(x) = x^{\theta}$ *for* $\theta \leq \lambda - \sigma$.

The restriction $\lambda - \sigma \in [0, 1/2]$ in **H4')** limits our uniqueness result to a subset of the kernels of the class defined in **H4)**, namely to the ones for which $\lambda - \sigma \in [0, 1/2]$ holds. But the class of kernels defined in **H4')** is also wider than the defined in **H4)** for $\lambda - \sigma \in [0, 1/2]$. In this way we are also giving uniqueness result for kernels which are not included in the class defined in **H4)**. On the other hand, the restriction $\theta \leq \lambda - \sigma$ in **H5')** limits our uniqueness result to a more restricted class of fragmentation kernels, see Appendix D.

In order to prove the uniqueness of solutions to (3.1)-(3.2) we set the following hypotheses

Hypotheses 3.4.1.

H1) $K(x,y)$ *is a continuous non-negative function on* $]0,\infty[\times]0,\infty[$,

H2) $K(x,y)$ *is a symmetric function, i.e.* $K(x,y) = K(y,x)$ *for all* $x, y \in]0,\infty[$,

H4') $K(x,y) \leq \kappa_1(x^{-\sigma} + x^{\lambda-\sigma})(y^{-\sigma} + y^{\lambda-\sigma})$ *such that* $\sigma, \lambda - \sigma \in [0, 1/2]$, *and constant* $k_1 > 0$,

H5') $S :]0,\infty[\to [0,\infty[$ *is continuous and satisfies the bound* $0 \leq S(x) \leq x^{\theta}$ *for* $\theta \leq \lambda - \sigma$,

H6) $b(x,y)$ *is such that* $\int\limits_0^y b(x,y) x^{-2\sigma} dx \leq C y^{-2\sigma}$.

3.4.1 The uniqueness theorem

Theorem 3.4.2. *If **H1)**, **H2)**, **H4')**, **H5')**, and **H6)** hold then the problem (3.1)-(3.2) has a unique solution* $u \in C_B\left([0,\infty[; L^1\left(]0,\infty[\right)\right)$.

Proof: Let us consider u_1 and u_2 to be solutions to (3.1)-(3.2) on $[0,T]$ for $T > 0$ arbitrarily chosen, with $u_1(x,0) = u_2(x,0)$ and set $U = u_1 - u_2$. We recall the definition of m^n in Section 2.4.1 for $n = 1, 2, 3, \ldots$

$$m^n(t) = \int\limits_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \left|U(x,t)\right| dx.$$

Then, using Definition 3.1.2 and working analogously as we did in Section 2.4.1 we find that

$$\begin{aligned} m^n(t) = &\int_0^t \int_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) \\ &\cdot \Bigg[\frac{1}{2} \int_0^x K(x-y,y) \left[u_1(x-y,\tau)u_1(y,\tau) - u_2(x-y,\tau)u_2(y,\tau)\right] dy \\ &\quad - \int_0^\infty K(x,y) \left[u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)\right] dy \\ &\quad + \int_x^\infty b(x,y)S(y) \left[u_1(y,t) - u_2(y,t)\right] dy - S(x) \left[u_1(x,t) - u_2(x,t)\right] \Bigg] \, dx \, d\tau. \end{aligned} \tag{3.91}$$

Using the substitution $y - x = x'$ in the first of the inner integrals w.r.t. x and y on the right hand side of (3.91), as in Section 2.4.1, we find that it becomes

$$\begin{aligned} &\int_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) \frac{1}{2} \int_0^x K(x-y,y) \\ &\qquad\qquad \cdot \left[u_1(x-y,\tau)u_1(y,\tau) - u_2(x-y,\tau)u_2(y,\tau)\right] dy \, dx \\ &= \int_0^n \int_0^{n-x} \frac{1}{2} \left[(x+y)^{-\sigma} + (x+y)^{\lambda-\sigma}\right] \operatorname{sgn}\big(U(x+y,\tau)\big) K(x,y) \\ &\qquad\qquad \cdot \left[u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)\right] dy \, dx. \end{aligned}$$

Inserting this into (3.91) gives

$$\begin{aligned} m^n(t) = &\int_0^t \int_0^n \int_0^{n-x} \left[\frac{1}{2} \left[(x+y)^{-\sigma} + (x+y)^{\lambda-\sigma}\right] \operatorname{sgn}\big(U(x+y,\tau)\big) - (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) \right] \\ &\qquad \cdot K(x,y) \left[u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)\right] dy \, dx \, d\tau \\ &- \int_0^t \int_0^n \int_{n-x}^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) \\ &\qquad \cdot K(x,y) \left[u_1(x,\tau)u_1(y,\tau) - u_2(x,\tau)u_2(y,\tau)\right] dy \, dx \, d\tau \\ &+ \int_0^t \int_0^n \int_x^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) b(x,y)S(y) \left[u_1(y,\tau) - u_2(y,\tau)\right] dy \, dx \, d\tau \\ &- \int_0^t \int_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) S(x) \left[u_1(x,\tau) - u_2(x,\tau)\right] dx \, d\tau. \end{aligned} \tag{3.92}$$

For $x, y \geq 0$ and $t \in [0,T]$ we recall the function w defined in (2.76)

$$w(x,y,t) = \Big[(x+y)^{-\sigma} + (x+y)^{\lambda-\sigma}\Big] \operatorname{sgn}\big(U(x+y,t)\big) \\ -(x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,t)\big) - (y^{-\sigma} + y^{\lambda-\sigma}) \operatorname{sgn}\big(U(y,t)\big).$$

Using (2.75) and this definition, we can rewrite (3.92) as

$$\begin{aligned} m^n(t) =& \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau)K(x,y)\left[u_1(x,\tau)U(y,\tau) + u_2(y,\tau)U(x,\tau)\right] dy\, dx\, d\tau \\ &+ \int_0^t \int_0^n \int_x^{\infty} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) b(x,y)S(y)U(y,\tau) dy\, dx\, d\tau \\ &- \int_0^t \int_0^n (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) S(x)U(x,\tau) dx\, d\tau \\ &- \int_0^t \int_0^n \int_{n-x}^{\infty} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) \\ &\qquad \cdot K(x,y)\left[u_1(x,\tau)U(y,\tau) + u_2(y,\tau)U(x,\tau)\right] dy\, dx\, d\tau. \end{aligned} \tag{3.93}$$

Since the third integral and the second term in the fourth integral of (3.93) are positive, we can delete it and get the following estimate

$$\begin{aligned} m^n(t) \leq& \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau)K(x,y)u_1(x,\tau)U(y,\tau)\, dy\, dx\, d\tau \\ &+ \frac{1}{2}\int_0^t \int_0^n \int_0^{n-x} w(x,y,\tau)K(x,y)u_2(y,\tau)U(x,\tau)\, dy\, dx\, d\tau \\ &+ \int_0^t \int_0^n \int_x^{\infty} \operatorname{sgn}\big(U(x,\tau)\big) b(x,y)S(y)U(y,\tau) dy\, dx\, d\tau \\ &- \int_0^t \int_0^n \int_{n-x}^{\infty} (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,\tau)\big) K(x,y)u_1(x,\tau)U(y,\tau)\, dy\, dx\, d\tau \\ =& \int_0^t \left[I_{31}(\tau) + I_{32}(\tau) + I_{33}(\tau) + I_{34}(\tau)\right] d\tau. \end{aligned} \tag{3.94}$$

Since w and U are defined as in Section 2.4, from (2.79) we have

$$w(x,y,t)U(y,t) \leq 2(x^{-\sigma} + x^{\lambda-\sigma}) \left|U(y,t)\right|. \tag{3.95}$$

Now, we use (3.95) to work on each term of the right hand side of (3.94)

$$\int_0^t I_{31}(\tau)\,d\tau \le \int_0^t\int_0^n\int_0^{n-x} (x^{-\sigma}+x^{\lambda-\sigma})K(x,y)u_1(x,\tau)\,|U(y,\tau)|\,dy\,dx\,d\tau.$$

Using the estimate **H4')** for $K(x,y)$ and (3.6) we get

$$\begin{aligned}\int_0^t I_{31}(\tau)\,d\tau &\le \kappa_1\int_0^t\int_0^n\int_0^{n-x}(x^{-\sigma}+x^{\lambda-\sigma})(x^{-\sigma}+x^{\lambda-\sigma})(y^{-\sigma}+y^{\lambda-\sigma})u_1(x,\tau)\,|U(y,\tau)|\,dy\,dx\,d\tau\\ &= \kappa_1\int_0^t\int_0^n\int_0^{n-x}(x^{-\sigma}+x^{\lambda-\sigma})^2u_1(x,\tau)(y^{-\sigma}+y^{\lambda-\sigma})\,|U(y,\tau)|\,dy\,dx\,d\tau\\ &\le 2\kappa_1\int_0^t\int_0^n\int_0^{n-x}\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)(y^{-\sigma}+y^{\lambda-\sigma})\,|U(y,\tau)|\,dy\,dx\,d\tau.\end{aligned}$$

Due to $\lambda-\sigma,\sigma\in[0,1/2]$ and the definition of $m^n(t)$ from the inequality above it follows that

$$\begin{aligned}\int_0^t I_{31}(\tau)\,d\tau &\le 2\kappa_1\int_0^t m^n(\tau)\left[\int_0^1\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)\,dx\right.\\ &\qquad\left.+\int_1^n\left(x^{-2\sigma}+x^{2(\lambda-\sigma)}\right)u_1(x,\tau)dx\right]d\tau\\ &\le 2\kappa_1\int_0^t m^n(\tau)\left[\int_0^1(x^{-2\sigma}+1)u_1(x,\tau)\,dx+\int_1^n(1+x)u_1(x,\tau)\,dx\right]d\tau\\ &\le 2\kappa_1\int_0^t m^n(\tau)\left[2\int_0^n x^{-2\sigma}u_1(x,\tau)\,dx+2\int_0^n xu_1(x,\tau)\,dx\right]d\tau\\ &\le \Lambda_1\int_0^t m^n(\tau)\,d\tau,\end{aligned}\tag{3.96}$$

where $\Lambda_1 = 4\kappa_1\sup_{s\in[0,t]}\|u_1(s)\|_Y$.

In the same way, there is a constant Λ_2 such that

$$\int_0^t I_{32}(\tau)\,d\tau \le \Lambda_2\int_0^t m^n(\tau)\,d\tau.\tag{3.97}$$

Now, changing the order of integration in I_{33} we have

$$\begin{aligned}
\int_0^t I_{33}(\tau)\,d\tau &= \int_0^t \int_0^n \int_x^\infty \operatorname{sgn}\big(U(x,\tau)\big) b(x,y)S(y)U(y,\tau)dy\,dx\,d\tau \\
&\leq \int_0^t \left[\int_0^n \int_0^y b(x,y)S(y)|U(y,\tau)|dx\,dy + \int_n^\infty \int_0^n b(x,y)S(y)|U(y,\tau)|dx\,dy \right] d\tau \\
&\leq \int_0^t \left[\int_0^n \int_0^y b(x,y)S(y)|U(y,\tau)|dx\,dy + \int_n^\infty \int_0^y b(x,y)S(y)|U(y,\tau)|dx\,dy \right] d\tau.
\end{aligned}$$

By using (3.4) and **H5')** we find that

$$\begin{aligned}
\int_0^t I_{33}(\tau)\,d\tau &\leq \int_0^t \left[N \int_0^n y^{\lambda-\sigma}|U(y,\tau)|dy + N \int_n^\infty y y^{\theta-1}|U(y,\tau)|dx \right] d\tau \\
&\leq \int_0^t \left[N m^n(\tau) + N n^{\theta-1} \int_n^\infty y[u_1(y,\tau) + u_2(y,\tau)]dy \right] d\tau \\
&\leq \int_0^t \Big[N m^n(\tau) + N n^{\theta-1} \left(\|u_1\|_Y + \|u_2\|_Y\right) \Big] d\tau \\
&= N \int_0^t m^n(\tau)\,d\tau + N n^{\theta-1} \left(\|u_1\|_Y + \|u_2\|_Y\right) t.
\end{aligned} \tag{3.98}$$

To consider I_{34} we first see that

$$\begin{aligned}
&\left| \int_0^\infty \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \operatorname{sgn}\big(U(x,t)\big) K(x,y) u_1(x,t) U(y,t)\,dy\,dx \right| \\
&\qquad \leq 2\kappa_1 \int_0^\infty \int_0^\infty \left(x^{-2\sigma} + x^{2(\lambda-\sigma)} \right) (y^{-\sigma} + y^{\lambda-\sigma}) u_1(x,t)\,|U(y,t)|\,dy\,dx < \infty.
\end{aligned}$$

Thus, the dominated convergence theorem leads to

$$\int_0^t I_{34}(\tau)\,d\tau \to 0 \quad \text{as} \quad n \to \infty. \tag{3.99}$$

Therefore, due to (3.94), (3.96)-(3.99) and taking $\Lambda = \Lambda_1 + \Lambda_2 + N$ we obtain

$$\begin{aligned} m(t) := \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \, |U(x,t)| \, dx = \lim_{n\to\infty} m^n(t) \\ &\leq \lim_{n\to\infty} \int_0^t [I_{31}(\tau) + I_{32}(\tau) + I_{33}(\tau) + I_{34}(\tau)] \, d\tau \\ &\leq \lim_{n\to\infty} \Lambda \int_0^t m^n(\tau) \, d\tau + \lim_{n\to\infty} \int_0^t I_{34}(\tau) \, d\tau \\ &= \Lambda \int_0^t \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \, |U(x,t)| \, dx \, d\tau. \end{aligned}$$

From where we have the inequality

$$m(t) \leq \Lambda \int_0^t m(\tau) \, d\tau. \tag{3.100}$$

Applying Gronwall's inequality, see e.g. Appendix A [Theorem A.0.6], we obtain

$$m(t) = \int_0^\infty (x^{-\sigma} + x^{\lambda-\sigma}) \, |U(x,t)| \, dx = 0 \quad \text{for all} \quad t \in [0,T].$$

Thus, we have that

$$u_1(x,t) = u_2(x,t) \quad \text{for a.e.} \quad x \in]0,\infty[.$$

$\square$

Chapter 4

Conclusions

This work presented new results concerning the existence and uniqueness of solutions to the coagulation equation and the coagulation equation with multifragmentation. Next we make some concluding remarks on these results.

In Chapter 2 we presented a proof of an existence theorem of solutions to the Smoluchowski coagulation equation for a very general class of kernels, giving a more general result than Fournier and Laurençot [10]. This class of kernels includes singular kernels. The important Smoluchowski coagulation kernel for Brownian motion, see Smoluchowski [34], the equi-partition of kinetic energy (EKE) kernel, see Hounslow [16] and Tan et al. [38], and the granulation kernel, see [17] are covered by our analysis. Our result is obtained in a suitable weighted Banach space of L^1 functions. We define a sequence of truncated problems from our original problem in order to eliminate the singularities of the kernels. Using the contraction mapping principle, we proved the existence and uniqueness of solutions to them. Using weak compactness theory, we prove that this sequence of solutions converges to a certain function. Then it was shown that the limiting function solves the original problem. The uniqueness result was obtained by taking the difference of two solutions and showing that this difference is equal to zero by appliying Gronwall's inequality.

In Chapter 3 we proved the existence and uniqueness of solutions to the singular coagulation equation with multifragmentation extending our result from Chapter 2. As in Chapter 2, we defined a sequence of truncated problems to eliminate the singularities of the kernels and prove the existence and uniqueness of solutions to them. We extracted a weakly convergent subsequence in L^1 from a sequence of unique solutions from the truncated equations. Next, we showed that the solution to our original problem is actually the limit function obtained from the weakly convergent subsequence of solutions to the truncated problem. We proved the uniqueness of the solutions to singular coagulation equations with multifragmentation for a modifed class of coagulation and fragmentation kernels by taking the difference of two solutions and showing that this difference is equal to zero by appliying Gronwall's inequality.

Currently, we are not aware of any kernels, different than the non-random coalescence kernel - see [31], used in fields of application of these equations that are not covered by our existence result. Unfortunately the equi-partition of kinetic energy kernel is not covered by our uniqueness result. In some sense this thesis fills partially a gap that had remained in the analytical theory.

In the following we would also like to mention some open questions in relation to our work.

- It would be interesting to find a new approach to enlarge our class of coagulation kernels in order to cover the non-random coalescence kernel. According to our knowledge, the present approach is not sufficient for the extension.

- To study the existence and uniqueness of L^1 solutions to coagulation and caogulation-fragmentation equations when the kernels are time-dependent.

- To extend the existence and uniqueness results to multidimensional cases.

Appendix A

Basic definitions and theorems

Definition A.0.3 (Equicontinuity). *[15, page 3]*
A family $\mathcal{F}$ of functions $f(y)$ defined on some y-set $E \subset \mathbb{R}^d$ is said to be equicontinuous if, for every $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ such that $|f(y_1) - f(y_2)| \leq \epsilon$ whenever $y_1, y_2 \in E$, $|y_1 - y_2| \leq \delta$ and all $f \in \mathcal{F}$.

Theorem A.0.4 (Dunford-Pettis). *[8, page 274]*
In order that a subset $\mathcal{P}$ of $\mathcal{L}^1$ be weakly relatively compact, it is necessary and sufficient that the following three conditions be fulfilled:

(1) $\sup \left\{ \int |f| d\mu : f \in \mathcal{P} \right\} < +\infty$.

(2) *Given $\epsilon > 0$, there exists a number $\delta > 0$ such that*

$$\sup \left\{ \int_A |f| d\mu : f \in \mathcal{P} \right\} \leq \epsilon$$

provided $A \subset T$ is integrable and $\mu(A) \leq \delta$.

(3) *Given any $\epsilon > 0$, there exists a compact set $K \subset T$ such that*

$$\sup \left\{ \int_{T \setminus K} |f| d\mu : f \in \mathcal{P} \right\} \leq \epsilon.$$

Theorem A.0.5 (Arzela-Ascoli). *[1, page 228]*
Let Ω be a compact topological space, Ω_1 a Hausdorff gauge space, and $G \subset C(\Omega, \Omega_1)$, with the uniform topology. Then G is compact iff the following three conditions are satisfied:

(a) *G is closed,*

(b) *$\{g(x) : g \in G\}$ is a relatively compact subset of Ω_1 for each $x \in \Omega$, and*

(c) *G is equicontinuous at each point of Ω; that is, if $\epsilon > 0$, $d \in \mathcal{D}(\Omega_1)$, $x_0 \in \Omega$, there is a neighborhood U of x_0 such that if $x \in U$, then*

$$d(g(x), g(x_0)) < \epsilon$$

for all $g \in G$

Theorem A.0.6 (Gronwall's Inequality)**.** *[40, page 361]*
Let $u(t)$ be a continuous function in $J = [0, T]$ which satisfies the inequality

$$u(t) \leq a + b \int_0^t u(s)\, ds \quad in \quad J \quad with \quad b > 0,$$

the it holds

$$u(t) \leq ae^{bt} \quad in \quad J.$$

Where a is an arbitrary constant.

Theorem A.0.7 (Egorov)**.** *[35, page 33]*
Suppose $\big(f_k\big)_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) < \epsilon$ and $f_k \to F$ uniformly on A_ϵ.

Theorem A.0.8 (Absolutely continuous function)**.** *[27, page 354]*
If λ denotes Lebesgue measure on the σ-algebra $\mathcal{M}$ of Lebesgue-measurable subsets of some interval $[a, b]$ and if u is a real- or complex-valued function on $[a, b]$, the following conditions are equivalents:

(a) u is absolutely continuous.

(b) There is a function f in $L^1\big([a, b], \mathcal{M}, \lambda\big)$ such that

$$u(x) = u(a) + \int_{[a,x]} f\, d\lambda$$

for all $x \in [a, b]$.

(c) u is differentiable at λ-a.e. point of $(a.b)$, $u' \in L^1\big([a, b], \mathcal{M}, \lambda\big)$, and

$$u(x) = u(a) + \int_{[a,x]} u'\, d\lambda$$

for all $x \in [a, b]$.

Theorem A.0.9 (Lebesgue dominated convergence)**.** *[4][page 44]*
Let $\big(f_n\big)$ be a sequence of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and

$$\int f\, d\mu = lim \int f_n\, d\mu.$$

Appendix B

Gauge space

Definition B.0.10 (Gauge Space)**.** *[1, page 226]*
A gauge space is a space Ω_1 whose topology is determined by a family $\mathcal{D} = \mathcal{D}(\Omega_1)$ of pseudo-metrics; thus a subbase for the topology is formed by the sets $B_d(x, \delta) = \{y \in \Omega_1 : d(x, y) < \delta\}$, $x \in \Omega_1$, $\delta > 0$, $d \in \mathcal{D}$.

Now, for $\phi \in L^\infty\big(]0, \infty[\big)$ and $u_1, u_2 \in L^1\big(]0, \infty[\big)$ we define by

$$d_\phi(u_1, u_2) = \left| \int_0^\infty \phi(x)[u_1(x) - u_2(x)]dx \right|$$

the distance between two elements $u_1, u_2 \in L^1\big(]0, \infty[\big)$ equipped with the weak topology.

Theorem B.0.11. *Set Ω_1 to be $L^1\big(]0, \infty[\big)$ equipped with the weak topology. Then Ω_1 is a Hausdorff gauge spaces*

Proof: That Ω_1 is a gauge space can be seen from the definition of gauge space. To prove that it is Hausdorff we need to show that for each pair $u_1, u_2 \in L^1\big(]0, \infty[\big)$ there is a $\phi(x) \in L^\infty\big(]0, \infty[\big)$ such that

$$\left| \int_0^\infty \phi(x)[u_1(x) - u_2(x)]dx \right| \neq 0 \quad \text{for } u_1(x) \neq u_2(x). \tag{B.1}$$

For $x \in [0, \infty[$ set $\phi(x)$ as follows

$$\phi(x) = \int_0^x [u_1(y) - u_2(y)]dy.$$

As $u_1, u_2 \in L^1\big(]0, \infty[\big)$ the defined function $\phi \in L^\infty\big(]0, \infty[\big)$. Since the sign of the integral value in (B.1) is not important, we work without the absolute value as follows. Integration by parts for the x-integral gives

$$\begin{aligned}\int_0^\infty \phi(x)[u_1(x)-u_2(x)]dx &= \int_0^\infty \int_0^x [u_1(y)-u_2(y)][u_1(x)-u_2(x)]dy\,dx\\ &= \left[\int_0^x [u_1(y)-u_2(y)]dy\right]^2\Bigg|_0^\infty\\ &\quad - \int_0^\infty [u_1(x)-u_2(x)]\int_0^x [u_1(y)-u_2(y)]dy\,dx.\end{aligned}$$

From there we have

$$\int_0^\infty \phi(x)[u_1(x)-u_2(x)]dx = \frac{1}{2}\left[\int_0^\infty [u_1(y)-u_2(y)]dy\right]^2.$$

Then, since $u_1(x) \neq u_2(x)$ we have that (B.1) holds. □

Appendix C

Proofs for some theorems

Proof of Lemma 3.1.1

In order to prove that Y is a Banach space, we need to show that every Cauchy sequence in Y converges to an element in Y.

Let u_n be a Cauchy sequence in Y. By defintion of the space Y we have that $(x^{-2\sigma}+x)u_n =: w_n$ is a Cauchy sequence in $L^1\big(]0,\infty[\big)$, but $L^1\big(]0,\infty[\big)$ is a Banach space and we get

$$w_n \to w \quad \text{in} \quad L^1\big(]0,\infty[\big).$$

Now, we define $u := (x^{-2\sigma}+x)^{-2\sigma}w$ and we show that u is in Y. As $w \in L^1\big(]0,\infty[\big)$ we find that

$$\|u\|_Y = \int_0^\infty (x^{-2\sigma}+x)|u|dx = \int_0^\infty (x^{-2\sigma}+x)(x^{-2\sigma}+x)^{-2\sigma}|w|dx < \infty.$$

By using $\int |fg| \leq \|f\|_\infty \|g\|_{L^1}$, $v \in L^1\big(]0,\infty[\big)$, and $(x^{-2\sigma}+x)^{-2\sigma} \in L^\infty\big(]0,\infty[\big)$ we show that $u \in L^1\big(]0,\infty[\big)$. Taking the L^1-norm of u we have

$$\int_0^\infty |u|\,dx = \int_0^\infty \big|(x^{-2\sigma}+x)^{-2\sigma}w\big|\,dx \leq \|(x^{-2\sigma}+x)^{-2\sigma}\|_\infty \|w\|_{L^1} < \infty.$$

Then, we have $u \in Y$.

Now, we prove that the sequence u_n converges to u in Y. Taking the norm of the difference between u_n and u we find that

$$\begin{aligned}
\|u_n - u\|_Y &= \int_0^\infty (x^{-2\sigma}+x)|u_n - u|dx \\
&= \int_0^\infty \big|(x^{-2\sigma}+x)u_n - (x^{-2\sigma}+x)(x^{-2\sigma}+x)^{-2\sigma}w\big|\,dx \\
&= \int_0^\infty |w_n - w|dx \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, we have that every Cauchy sequence in Y converge to an element in Y. □

Proof of Lemma 3.2.2

For any fixed $x \in [0, n]$, $t, \tau \in [0, t_0]$ we may assume c_1 to satisfy

$$P(x, t, c_1) - P(x, \tau, c_1) \geq P(x, t, c_2) - P(x, \tau, c_2).$$

Then

$$\begin{aligned} |H(x, \tau, t)| &= -H(x, \tau, t) \\ &= \exp\big(- [P(x, t, c_2) - P(x, \tau, c_2)] \big) \\ &\quad \cdot \big[1 - \exp\big(- [P(x, t, c_1) - P(x, \tau, c_1) - [P(x, t, c_2) - P(x, \tau, c_2)]]\big)\big]. \end{aligned} \tag{C.1}$$

Since $1 - \exp\big(- x\big) \leq x$ for $x \geq 0$, (C.1), together with the definitions of B and M, and the non-negativity of $S_n(x)$ leads to

$$\begin{aligned} &|H(x, \tau, t)| \\ &\leq \exp\big(- [P(x, t, c_2) - P(x, \tau, c_2)]\big) [P(x, t, c_1) - P(x, \tau, c_1) - [P(x, t, c_2) - P(x, \tau, c_2)]] \\ &= \exp\big(- [P(x, t, c_2) - P(x, \tau, c_2)]\big) \int_\tau^t \int_0^{n-x} K_n(x, y)[c_1(y, s) - c_2(y, s)] dy\, ds \\ &\leq \exp\left(- \int_\tau^t \int_0^{n-x} K_n(x, y) c_2(y, s)\, dy\, ds \right) (t - \tau) n^{2\sigma} M \|c_1 - c_1\|_D \\ &\leq (t - \tau) n^{2\sigma} M \exp\big((t - \tau) n^{2\sigma} BM\big) \|c_1 - c_2\|_D. \end{aligned}$$

If $P(x, t, c_1) - P(x, \tau, c_1) \leq P(x, t, c_2) - P(x, \tau, c_2)$ then inequality (2.22) can be derived analogously. □

Appendix D

Kernels

D.1 Coagulation kernels

In this section we show how the Smoluchowski kernel is covered by our analysis as well as the equi-partition of kinetic energy and the granulation kernels. In order to do that we introduces some necesary inequalities. The proof of these inequalities can be found in an appendix of Giri [12]. For any $x, y > 0$

$$(1 + x + y)^p \leq 2(1 + x)^p(1 + y)^p \quad \text{if} \quad 0 < p \leq 1, \tag{D.1}$$

$$(1 + x + y)^p \leq 2^p(1 + x)^p(1 + y)^p \quad \text{if} \quad p \geq 1. \tag{D.2}$$

We recall the class of coagulation kernels studied in Chapter 2

$$K(x, y) \leq k(1 + x + y)^\lambda (xy)^{-\sigma} \quad \text{with} \quad \lambda - \sigma \in [0, 1[,\ \sigma \in [0, 1/2] \tag{D.3}$$

and the one studied in Chapter 3

$$K(x, y) \leq k'(1 + x)^\lambda (1 + y)^\lambda (xy)^{-\sigma} \quad \text{with} \quad \lambda - \sigma \in [0, 1[,\ \sigma \in [0, 1/2]. \tag{D.4}$$

Using the inequalities (D.1) and (D.2) we can find that

$$k(1 + x + y)^\lambda (xy)^{-\sigma} \leq k''(1 + x)^\lambda (1 + y)^\lambda (xy)^{-\sigma}$$

where

$$k'' = \begin{cases} 2k' & if \quad 0 < \lambda \leq 1 \\ 2^\lambda k' & if \quad \lambda \geq 1. \end{cases}$$

Then, since the class of kernels (D.3) is included in the class of kernels (D.4) we just need to show that the Smoluchowski kernel

$$K_1(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3}), \tag{D.5}$$

the equi-partition of kinetic energy kernel

$$K_2(x, y) = (x^{1/3} + y^{1/3})^2 \sqrt{\frac{1}{x} + \frac{1}{y}}, \tag{D.6}$$

and the granulation kernel

$$K_3(x, y) = \frac{(x + y)^a}{(xy)^b}$$

are included in class of kernels (D.3).

Working with the kernel K_1 (D.5) we have

$$K_1(x, y) = (x^{1/3} + y^{1/3})(x^{-1/3} + y^{-1/3}) = (x^{1/3} + y^{1/3})^2 (xy)^{-1/3}.$$

Using (2.4) we find that

$$\begin{aligned} K_1(x, y) &\le 2^{4/3}(x + y)^{2/3}(xy)^{-1/3} \\ &\le 2^{4/3}(1 + x + y)^{2/3}(xy)^{-1/3}, \end{aligned} \tag{D.7}$$

from where we have for $k = 2^{4/3}$, $\lambda = 2/3$, and $\sigma = 1/3$ that the Smoluchowski kernel (D.5) is included in the class of kernels (D.3).

Working now with the kernel K_2 (D.6) we get

$$K_2(x, y) = (x^{1/3} + y^{1/3})^2 \sqrt{\frac{1}{x} + \frac{1}{y}} = (x^{1/3} + y^{1/3})^2 (x + y)^{1/2}(xy)^{-1/2}.$$

Applying the inequality (2.4) we obtain

$$\begin{aligned} K_2(x, y) &\le 2^{4/3}(x + y)^{7/6}(xy)^{-1/2} \\ &\le 2^{4/3}(1 + x + y)^{7/6}(xy)^{-1/2}, \end{aligned} \tag{D.8}$$

from where we have for $k = 2^{4/3}$, $\lambda = 7/6$, and $\sigma = 1/2$ that the equi-partition of kinetic energy kernel (D.5) is included in the class of kernels (D.3).

Taking now $k = 1$, $\lambda = a$, and $\sigma = b$ we find that the granulation kernel is also in the class of kernels (D.3).

$$K_3(x, y) = \frac{(x + y)^a}{(xy)^b} \le \frac{(1 + x + y)^a}{(xy)^b}. \tag{D.9}$$

Now, by using 2.4 on the class of coagulation kernels (D.4) we find that

$$\begin{aligned} K(x, y) = k'(1 + x)^\lambda (1 + y)^\lambda (xy)^{-\sigma} &\le k''' \frac{(1 + x^\lambda)}{x^\sigma} \frac{(1 + y^\lambda)}{y^\sigma} \\ &= k'''(x^{-\sigma} + x^{\lambda-\sigma})(y^{-\sigma} + y^{\lambda-\sigma}) \end{aligned} \tag{D.10}$$

with

$$k''' = \begin{cases} k' & \text{if } \ 0 \le \lambda \le 1 \\ 2^{\lambda-1}k' & \text{if } \ \lambda \ge 1. \end{cases}$$

From (D.10) we have that the class of coagulation kernels used to prove our uniqueness result is wider than the one used for our existence result, but the class of kernels (D.10) has an stronger restriction on the parameters ($\lambda - \sigma \in [0, 1/2]$). Therefore, from (D.7)-(D.9) we can see that the Smoluchowski kernel K_1 and the granulation kernel K_3 are covered by our uniqueness result, but not the equi-partition of kinetic energy kernel K_2 because $\lambda - \sigma = 7/6 - 1/2 > 1/2$.

D.2 Fragmentation kernels

Now, we show how the fragmentation kernel

$$\Gamma(y,x) = (\alpha+2)x^{\alpha}y^{\gamma-(\alpha+1)} \tag{D.11}$$

is included in our existence result for $\alpha > 2\sigma+\epsilon-1$ and $\gamma=\theta$ with $\sigma \in [0,1/2]$ and $\theta \in]0,1[$.

In order to do that, we show that (D.11) holds **H5)**, **H6)**, and **H7)**, see Hypotheses 3.1.3. Taking in (D.11) $S(x)=x^{\gamma}$ we have that **H5)** holds for $\gamma=\theta$ and $b(x,y)$ is defined by

$$b(x,y) = (\alpha+2)\frac{1}{y}\left(\frac{x}{y}\right)^{\alpha}. \tag{D.12}$$

We have to show that $b(x,y)$ as defined above holds **H6)** and **H7)**.

First we proof that (D.12) holds **H7)**. We have to show that for $\alpha > 2\sigma+\epsilon-1$ with $0<\epsilon\le\theta$, $\sigma\in[0,1/2]$, $\theta\in]0,1[$ and $\tau_1,\tau_2\in[-2\sigma-\theta,1-\theta]$ there exist $p>1$ and such that

$$\int_0^y b^q(x,y) \le B_1 y^{q\tau_1} \text{ and } \int_0^y x^{-q\sigma}b^q(x,y) \le B_2 y^{q\tau_2} \quad \text{for constant } B_1,B_2>0. \tag{D.13}$$

Working with the second integral term we find that

$$\begin{aligned}
\int_0^y x^{-q\sigma}b^q(x,y) &= (\alpha+2)^q \int_0^y x^{-q\sigma}\frac{1}{y^q}\left(\frac{x}{y}\right)^{q\alpha} dx \\
&= \frac{(\alpha+2)^q}{y^{(\alpha+1)q}} y^{(\alpha-\sigma)q+1} \quad \text{for } (\alpha-\sigma)q+1>0 \\
&= (\alpha+2)^q y^{-(1+\sigma)q+1} = B_2 y^{q\tau_2}
\end{aligned}$$

where $B_2=(\alpha+2)^q$ and $\tau_2 = -(1+\sigma)+1/q$. We need to show that there exist $q>1$ such that $\tau_2\in[-2\sigma-\theta,1-\theta]$. As $1/q<1$, $\sigma\ge0$, and $\theta\in]0,1[$ we can see that $\tau_2<1-\theta$ for every $q>1$.

Now looking for a $q>1$ such that $\tau_2 \ge -2\sigma-\theta$ we find that

$$\begin{aligned}
-2\sigma-\theta &\le -1-\sigma+\frac{1}{q} \\
1-(\sigma+\theta) &\le \frac{1}{q}.
\end{aligned}$$

We have now three cases. If $1-(\sigma+\theta)<0$

$$q \ge \frac{1}{1-(\sigma+\theta)} < 0, \quad \text{and } q>1 \text{ can be arbitrarily chosen.}$$

If $1-(\sigma+\theta)=0$, can also be $q>1$ arbitrarily chosen. And finally, if $1-(\sigma+\theta)>0$ we have

$$q \le \frac{1}{1-(\sigma+\theta)} > 0. \tag{D.14}$$

So, in order to cover the three cases, we can choose

$$q = \frac{1}{1-\sigma-\epsilon}, \quad \text{with } 0 < \epsilon \leq \theta. \tag{D.15}$$

Now we check that $(\alpha - \sigma)q + 1 > 0$ holds for q chosen in (D.15). We have

$$\begin{aligned} (\alpha-\sigma)q+1 &> 0 \\ \alpha &> \sigma - \frac{1}{q} \\ \alpha &> 2\sigma + \epsilon - 1. \end{aligned} \tag{D.16}$$

Then we find that for $\alpha > 2\sigma + \epsilon - 1$ there exist $q = \frac{1}{1-\sigma-\epsilon} > 1$ such that, for $0 < \epsilon \leq \theta$, $B_2 = (\alpha+2)^q$, and $\tau_2 \in [-2\sigma - \theta, 1-\theta]$, the estimate

$$\int_0^y x^{-q\sigma} b^q(x,y) \leq B_2 y^{q\tau_2}$$

holds.

We now show that for the same chosen q the first estimate in (D.13) holds with $\tau_1 \in [-2\sigma - \theta, 1-\theta]$.

$$\begin{aligned} \int_0^y b^q(x,y)dx &= (\alpha+2)^q \int_0^y \frac{1}{y}\left(\frac{x}{y}\right)^{q\alpha} dx \\ &= \frac{(\alpha+2)^q}{y^{(\alpha+1)q}} y^{q\alpha+1} \quad \text{for } \alpha q + 1 > 0 \\ &= (\alpha+2)^q y^{1-q} = B_2 y^{q\tau} \quad \text{with } B_1 = (\alpha+2)^q. \end{aligned}$$

Since $(\alpha - \sigma)q + 1 > 0$ for q chosen as in (D.15) and $\alpha > 2\sigma + \epsilon - 1$, see (D.16), we find that $\alpha q + 1 > 0$ and $\tau_1 = \frac{1}{q} - 1 < 0 < 1 - \theta$. Then we just have to check that $\tau_1 = \frac{1}{q} - 1 \geq -2\sigma - \theta$. From (D.14) we have

$$\begin{aligned} q &\leq \frac{1}{1-\theta} \\ 1-\theta &\leq \frac{1}{q} \Longrightarrow 1 - \theta \leq \frac{1}{q} + 2\sigma \Longleftrightarrow \tau_1 = \frac{1}{q} - 1 \geq -2\sigma - \theta. \end{aligned}$$

Then we can choose $p = \frac{1}{1-\sigma-\epsilon}$ such that for $\alpha > 2\sigma + \epsilon - 1$ with $0 < \epsilon \leq \theta$ and $\tau \in [-\sigma-\theta, 1-\theta]$ (D.13) holds.

Now, we show that $b(x,y)$ defined in (D.12) satisfice **H6)** for $\alpha > 2\sigma - 1$ and $\gamma = \theta$ with $\sigma \in [0, 1/2]$ and $\theta \in [0, 1[$.

We compute now, for which values of α the condition

$$\int_0^y b(x,y)x^{-2\sigma}dx \leq Cy^{-2\sigma}dx \tag{D.17}$$

holds. Multiplying $b(x,y)$ by $x^{-2\sigma}$ and integrating from 0 to y w.r.t. x we find, for $\alpha > 2\sigma+\epsilon-1$, that

$$\begin{aligned}\int_0^y (\alpha+2)x^{\alpha}y^{-(\alpha+1)}x^{-2\sigma}dx &= (\alpha+2)y^{-(\alpha+1)}\int_0^y x^{\alpha-2\sigma}dx \\ &= \frac{(\alpha+2)}{(1+\alpha-2\sigma)}y^{-2\sigma}\end{aligned}$$

from where we have that (D.17) holds for $C = \frac{(\alpha+2)}{(1+\alpha-2\sigma)}$ and $\alpha > 2\sigma - 1$.

From where we finally have that for $\alpha > 2\sigma + \epsilon - 1$ **H6)** and **H7)** hold.

Appendix E

Computations

In this section we show the change of variable used in equation (2.20) which is equivalent to the one used in equation (3.20). Changing the order of integration in the integral term

$$\int_0^n \int_0^x K(x-y,y)|c(x-y,\tau)||c(y,\tau)|x^{-1}dy\,dx$$

we find that

$$\int_0^n \int_0^x K(x-y,y)|c(x-y,\tau)||c(y,\tau)|x^{-1}dy\,dx$$
$$= \int_0^n \int_y^n K(x-y,y)|c(x-y,\tau)||c(y,\tau)|x^{-1}dx\,dy.$$

Now, we make the change of variables $x-y=z$

$$\int_0^n \int_0^x K(x-y,y)|c(x-y,\tau)||c(y,\tau)|x^{-1}dy\,dx$$
$$= \int_0^n \int_0^{n-y} K(z,y)|c(z,\tau)||c(y,\tau)|(z+y)^{-1}dz\,dy.$$

Rechanging the order of integration and replacing z by x we find that

$$\int_0^n \int_0^x K(x-y,y)|c(x-y,\tau)||c(y,\tau)|x^{-1}dy\,dx.$$
$$= \int_0^n \int_0^{n-x} K(x,y)|c(x,\tau)||c(y,\tau)|(x+y)^{-1}dx\,dy.$$

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