Compositional Abstractions of Interconnected Discrete-Time Stochastic Control Systems

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Abstract— This paper is concerned with a compositional approach for constructing abstractions of interconnected discrete-time stochastic control systems. The abstraction framework is based on new notions of so-called stochastic simulation functions, using which one can quantify the distance between original interconnected stochastic control systems and their abstractions in the probabilistic setting. Accordingly, one can leverage the proposed results to perform analysis and synthesis over abstract interconnected systems, and then carry the results over concrete ones. In the first part of the paper, we derive sufficient small-gain type conditions for the compositional quantification of the distance in probability between the interconnection of stochastic control subsystems and that of their abstractions. In the second part of the paper, we focus on the class of discrete-time linear stochastic control systems with independent noises in the abstract and concrete subsystems. For this class of systems, we propose a computational scheme to construct abstractions together with their corresponding stochastic simulation functions. We demonstrate the effectiveness of the proposed results by constructing an abstraction (totally 4 dimensions) of the interconnection of four discrete-time linear stochastic control subsystems (together 100 dimensions) in a compositional fashion.

I. INTRODUCTION

Large-scale interconnected systems have received significant attentions in the last few years due to their presence in real life systems including power networks, air traffic control, and so on. Each complex real-world system can be regarded as an interconnected system composed of several subsystems. Since these large-scale network of systems are inherently difficult to analyze and control, one can develop compositional schemes to employ the abstractions of the given systems as a replacement in the controller design process. In other words, in order to overcome the computational complexity in large-scale interconnected systems, one can abstract the original concrete system by a simpler one with lower dimension. Those abstractions allow us to design controllers for them, and then refine the controllers to the ones for the concrete complex systems, while provide us with the quantified errors in this controller synthesis detour.

In the past few years, there have been several results on the construction of (in)finite abstractions for stochastic systems. Existing results include infinite approximations for a class of stochastic hybrid systems [1] and finite approximations for discrete-time stochastic models with continuous state spaces [2], [3], [4]. Construction of finite bisimilar abstractions for stochastic control systems is proposed in [5], [6]. Recent results address stochastic switched systems [7], [8] and

propose compositional construction of infinite abstractions of continuous-time stochastic control systems [9], [10] using small-gain type compositional reasoning.

In this paper, we provide a compositional approach for the construction of infinite abstractions of interconnected discrete-time stochastic control systems. Our abstraction framework is based on a new notion of so-called stochastic simulation functions under which an abstraction, which is itself a discrete-time stochastic control system with lower dimension, performs as a substitute in the controller design process. The stochastic simulation function is leveraged to quantify the error in probability in this controller synthesis scheme. As a consequence, one can use the proposed results here to solve particularly safety/reachability problems over the abstract interconnected systems and then carry the results over the concrete interconnected ones. It should be noted that the existing compositional results in [9], [10] are for *continuous-time* stochastic systems and assume that the noises in the concrete and abstract systems are the same, which means the abstraction has access to the noise of the concrete system, which is a strong assumption. In this paper, we do not have such an assumption meaning that the noises of the abstraction can be completely independent of that of the concrete system.

II. DISCRETE-TIME STOCHASTIC CONTROL SYSTEMS

A. Preliminaries

We consider a probability space $(\Omega, \mathcal{F}_\Omega, \mathbb{P}_\Omega)$, where Ω is the sample space, $\mathcal{F}_\Omega$ is a sigma-algebra on Ω comprising subsets of Ω as events, and $\mathbb{P}_\Omega$ is a probability measure that assigns probabilities to events. We assume that random variables introduced in this article are measurable functions of the form $X : (\Omega, \mathcal{F}_\Omega) \to (S_X, \mathcal{F}_X)$. Any random variable X induces a probability measure on its space $(S_X, \mathcal{F}_X)$ as $Prob\{A\} = \mathbb{P}_\Omega\{X^{-1}(A)\}$ for any $A \in \mathcal{F}_X$. We often directly discuss the probability measure on $(S_X, \mathcal{F}_X)$ without explicitly mentioning the underlying probability space and the function X itself.

A topological space S is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space). Examples of a Borel space are the Euclidean spaces $\mathbb{R}^n$, its Borel subsets endowed with a subspace topology, as well as hybrid spaces. Any Borel space S is assumed to be endowed with a Borel sigma-algebra, which is denoted by $\mathcal{B}(S)$. We say that a map $f : S \to Y$ is measurable whenever it is Borel measurable.

B. Notation

The following notation is used throughout the paper. We denote the set of nonnegative integers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and the set of positive integers by $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$. The symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote the set of real, positive, and

nonnegative real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. The symbols I_n and $\mathbf{1}_n$ denote the identity matrix in $\mathbb{R}^{n \times n}$, and the vector in $\mathbb{R}^n$ with all its elements to be one, respectively. We denote by $\text{diag}(a_1, \ldots, a_N)$ a diagonal matrix in $\mathbb{R}^{N \times N}$ with diagonal matrix entries $a_1, \ldots, a_N$ starting from the upper left corner. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \ldots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \ldots, x_N) = [f_1(x_1); \ldots; f_N(x_N)]$. For any set A we denote by $A^{\mathbb{N}}$ the Cartesian product of a countable number of copies of A, i.e., $A^{\mathbb{N}} = \prod_{k=0}^{\infty} A$. Given a measurable function $f : \mathbb{N} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_{\infty} := (\text{ess})\sup\{\|f(k)\|, k \geq 0\}$. A function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_{\infty}$ if $\gamma(r) \to \infty$ as $r \to \infty$.

C. Discrete-Time Stochastic Control Systems

We consider stochastic control systems in discrete time (dt-SCS) defined over a general state space adopted from [11] and characterized by the tuple

$$\Sigma = (X, W, U, \{U(x, \omega) | x \in X, \omega \in W\}, Y, T_\mathsf{x}, h), \quad (1)$$

where X is a Borel space as the state space of the system. We denote by $(X, \mathcal{B}(X))$ the measurable space with $\mathcal{B}(X)$ being the Borel sigma-algebra on the state space. Sets W and U are Borel spaces as the *internal* and *external* input spaces of the system. The set $\{U(x, \omega) | x \in X, \omega \in W\}$ is a family of non-empty measurable subsets of U with the property that

$$K := \{(x, \omega, \nu) : x \in X, \omega \in W, \nu \in U(x)\}$$

is measurable in $X \times W \times U$. Intuitively, $U(x, \omega)$ is the set of inputs that are feasible at state $x \in X$ with the internal input $\omega \in W$. Set Y is a Borel space as the output space of the system. Map $T_\mathsf{x} : \mathcal{B}(X) \times X \times W \times U \to [0, 1]$, is a conditional stochastic kernel that assigns to any $x \in X$, $\omega \in W$ and $\nu \in U(x, \omega)$ a probability measure $T_\mathsf{x}(\cdot | x, \omega, \nu)$ on the measurable space $(X, \mathcal{B}(X))$ so that for any set $A \in \mathcal{B}(X), \mathbb{P}_{x,\omega,\nu}(A) = \int_A T_\mathsf{x}(d\bar{x} | x, \omega, \nu)$, where $\mathbb{P}_{x,\omega,\nu}$ denotes the conditional probability $\mathbb{P}(\cdot | x, \omega, \nu)$. Finally, $h : X \to Y$ is a measurable function that maps a state $x \in X$ to its output $y = h(x) \in Y$.

Given the dt-SCS in (1), we are interested in *Markov policies* to control the system.

Definition 2.1: A Markov policy for the dt-SCS Σ in (1) is a sequence $\rho = (\rho_0, \rho_1, \rho_2, \ldots)$ of universally measurable stochastic kernels ρ_n [12], each defined on the input space U given $X \times W$ and such that for all $(x_n, \omega_n) \in X \times W$, $\rho_n(U(x_n, \omega_n) | (x_n, \omega_n)) = 1$. The class of all Markov policies is denoted by Π_M.

For given inputs $\omega(\cdot), \nu(\cdot)$, the stochastic kernel T_x captures the evolution of the state of the system. This kernel features an equivalent dynamical representation: there exists a measurable function $f_a : X \times W \times U \times V_\varsigma \to X$ such that the evolution of the state of the system can be written as

$$x(k + 1) = f_a(x(k), \omega(k), \nu(k), \varsigma(k)),$$

where $\{\varsigma(k) : \Omega \to V_\varsigma, \ k \in \mathbb{N}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables on the set

V_ς. In this paper we assume that the state space X is a subset of $\mathbb{R}^n$ and are interested in the specific form of the function

$$f_a(x, \omega, \nu, \varsigma) = f(x, \omega, \nu) + g(x)\varsigma.$$

Therefore, the dt-SCS Σ in (1) can be described as:

$$\Sigma : \begin{cases} x(k + 1) = f(x(k), \omega(k), \nu(k)) + g(x(k))\varsigma(k), \\ y(k) = h(x(k)), \end{cases} \quad (2)$$

for any $x(k) \in X, \omega(k) \in W$, and $\nu(k) \in U(x(k), \omega(k))$. Note that T_x in (1) contains the information of functions f and g and the distribution of noise $\varsigma(\cdot)$ in the dynamical representation (2).

For the sake of simplicity, we also assume that the set of valid inputs is the whole input space: $U(x, \omega) = U$ for all $x \in X$ and $\omega \in W$, but the obtained results are generally applicable. We associate respectively to U and W the sets $\mathcal{U}$ and $\mathcal{W}$ to be collections of sequences $\{\nu(k) : \Omega \to U, \ k \in \mathbb{N}\}$ and $\{\omega(k) : \Omega \to W, \ k \in \mathbb{N}\}$, in which $\nu(k)$ and $\omega(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$.

For any initial state $a \in X, \nu(\cdot) \in \mathcal{U}$, and $\omega(\cdot) \in \mathcal{W}$, the random sequences $x_{a\omega\nu} : \Omega \times \mathbb{N} \to X$ and $y_{a\omega\nu} : \Omega \times \mathbb{N} \to Y$ that satisfy (2) are called respectively the *solution process* and *output trajectory* of Σ under external input ν, internal input ω and initial state a. We here call the tuple $(\omega, \nu, x_{a\omega\nu}, y_{a\omega\nu})$ a *trajectory* of Σ.

III. STOCHASTIC PSEUDO-SIMULATION AND SIMULATION FUNCTIONS

In this section we first introduce a notion of so-called pseudo-simulation functions for the discrete-time stochastic control systems with both internal and external inputs and then define the stochastic simulation functions for systems with only external input. These definitions can be used to quantify closeness of two dt-SCS with the same internal input and output spaces.

Definition 3.1: Consider dt-SCS $\Sigma = (X, W, U, Y, T_\mathsf{x}, h)$ and $\hat{\Sigma} = (\hat{X}, W, \hat{U}, Y, \hat{T}_\mathsf{x}, \hat{h})$ with the same internal input and output spaces. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic pseudo-simulation function (SPSF) from $\hat{\Sigma}$ to Σ if

- $\exists \alpha \in \mathcal{K}_{\infty}$ such that

$$\forall x \in X, \forall \hat{x} \in \hat{X}, \quad \alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad (3)$$

- $\forall x \in X, \hat{x} \in \hat{X}, \hat{\nu} \in \hat{U}$, and $\forall \hat{\omega} \in \hat{W}, \exists \nu \in U$ such that $\forall \omega \in W$

$$\mathbb{E}\Big[V(x(k + 1), \hat{x}(k + 1)) \,\big|\, x(k), \hat{x}(k), \omega(k) = \omega,$$
$$, \hat{\omega}(k) = \hat{\omega}, \nu(k) = \nu, \hat{\nu}(k) = \hat{\nu} \Big] - V(x(k), \hat{x}(k)) \leq$$
$$-\kappa(V(x(k), \hat{x}(k))) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|) + \rho_{\text{ext}}(\|\hat{\nu}\|) + \psi, \quad (4)$$

for some $\kappa \in \mathcal{K}, \rho_{\text{int}}, \rho_{\text{ext}} \in \mathcal{K}_{\infty} \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.

We utilize notation $\hat{\Sigma} \preceq_{\mathcal{PS}} \Sigma$ if there exists a pseudo-simulation function V from $\hat{\Sigma}$ to Σ, in which control system $\hat{\Sigma}$ is considered as an abstraction of concrete (original) system Σ. The second condition above implies implicitly existence of a function $\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{\omega})$ for satisfaction of (4). This function is called the *interface function* and can

be used to refine a synthesized policy $\hat{\nu}$ for $\widehat{\Sigma}$ to a policy ν for Σ.

In this paper we study interconnected discrete-time stochastic control systems without internal inputs, resulting from the interconnection of discrete-time stochastic control subsystems having both internal and external signals. In this case, the interconnected dt-SCS reduces to the tuple (X, U, Y, T_x, h). Thus we modify the above notion for systems without internal inputs.

Definition 3.2: Consider two dt-SCS $\Sigma = (X, U, Y, T_x, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, Y, \hat{T}_x, \hat{h})$ with the same output spaces. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a *stochastic simulation function* (SSF) from $\widehat{\Sigma}$ to Σ if

- $\exists \alpha \in \mathcal{K}_\infty$ such that

$$\forall x \in X, \forall \hat{x} \in \hat{X}, \quad \alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad (5)$$

- $\forall x \in X, \hat{x} \in \hat{X}, \hat{\nu} \in \hat{U}, \exists \nu \in U$ such that

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \big| x(k), \hat{x}(k), \nu(k) = \nu, \hat{\nu}(k) = \hat{\nu}\Big]$$
$$- V(x(k), \hat{x}(k)) \leq -\kappa(V(x(k), \hat{x}(k))) + \rho_{\text{ext}}(\|\nu\|) + \psi, \quad (6)$$

for some $\kappa \in \mathcal{K}$, $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.

The next theorem shows usefulness of SSF in comparing output trajectories of two dt-SCS in a probabilistic sense.

Theorem 3.3: Let Σ and $\widehat{\Sigma}$ be two dt-SCS with the same output spaces. Suppose V is an SSF from $\widehat{\Sigma}$ to Σ, and there exists a constant $0 < \hat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}$ in (6) satisfies $\kappa(r) \geq \hat{\kappa} r \; \forall r \in \mathbb{R}_{\geq 0}$. For any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds provided that there exists a constant $\hat{\psi} \geq 0$ satisfying $\hat{\psi} \geq \rho_{\text{ext}}(\|\hat{\nu}\|_\infty) + \psi$:

$$\mathbb{P}\left\{\sup_{0 \leq k \leq T} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\} \quad (7)$$

$$\leq \begin{cases} 1 - \left(1 - \frac{V(a,\hat{a})}{\alpha(\varepsilon)}\right)\left(1 - \frac{\hat{\psi}}{\alpha(\varepsilon)}\right)^T & \text{if } \alpha(\varepsilon) \geq \frac{\hat{\psi}}{\hat{\kappa}}, \\ \left(\frac{V(a,\hat{a})}{\alpha(\varepsilon)}\right)(1 - \hat{\kappa})^T + \left(\frac{\hat{\psi}}{\hat{\kappa}\alpha(\varepsilon)}\right)(1 - (1 - \hat{\kappa})^T) & \text{if } \alpha(\varepsilon) < \frac{\hat{\psi}}{\hat{\kappa}}. \end{cases}$$

Proof: Since V is an SSF from $\widehat{\Sigma}$ to Σ, we have

$$\mathbb{P}\left\{\sup_{0 \leq k \leq T} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\}$$

$$= \mathbb{P}\left\{\sup_{0 \leq k \leq T} \alpha(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|) \geq \alpha(\varepsilon) \,|\, [a; \hat{a}]\right\}$$

$$\leq \mathbb{P}\left\{\sup_{0 \leq k \leq T} V(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k)) \geq \alpha(\varepsilon) \,|\, [a; \hat{a}]\right\}. \quad (8)$$

The equality holds due to α being a $\mathcal{K}_\infty$ function. The inequality is also true due to condition (5) on the SSF V. The results follows by applying Theorem 3 in [13, pp. 81] to (8) and utilizing inequality (6). ∎

The results shown in Theorem 3.3 provide closeness of output behaviours of two systems in finite-time horizon. We can extend the result to infinite-time horizon provided that constant $\hat{\psi} = 0$ as the following.

Corollary 3.4: Let Σ and $\widehat{\Sigma}$ be two dt-SCS with the same output spaces. Suppose V is an SSF from $\widehat{\Sigma}$ to Σ such that $\rho_{\text{ext}}(\cdot) \equiv 0$ and $\psi = 0$. For any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ preserving Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, there exists $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds:

$$\mathbb{P}\left\{\sup_{0 \leq k < \infty} \|y_{a\nu}(k) - \hat{y}_{\hat{a}0}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon)}.$$

Proof: Since V is an SSF from $\widehat{\Sigma}$ to Σ with $\rho_{\text{ext}}(\cdot) \equiv 0$ and $\psi = 0$, for any $x(k) \in X$ and $\hat{x}(k) \in \hat{X}$ and any $\hat{\nu}(k) \in \hat{U}$, there exists $\nu(k) \in U$ such that

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,|\, x(k), \hat{x}(k), \nu(k), \hat{\nu}(k)\Big]$$
$$- V((x(k), \hat{x}(k))) \leq -\kappa(V(x(k), \hat{x}(k)), \quad (9)$$

showing that $V(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k))$ is a nonnegative supermartingale [14]. Following the same reasoning as in the proof of Theorem 3.3 we have

$$\mathbb{P}\left\{\sup_{0 \leq k < \infty} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\}$$

$$= \mathbb{P}\left\{\sup_{0 \leq k < \infty} \alpha(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\|) \geq \alpha(\varepsilon) \,|\, [a; \hat{a}]\right\}$$

$$\leq \mathbb{P}\left\{\sup_{0 \leq k < \infty} V(x_{a\nu}(k), \hat{x}_{\hat{a}\hat{\nu}}(k)) \geq \alpha(\varepsilon) \,|\, [a; \hat{a}]\right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon)},$$

where the last inequality is due to the nonnegative supermartingale property [13]. ∎

The stochastic simulation function defined before can be used to guarantee an upper bound on the probability of the maximum difference in output trajectories. This idea can be used in conjunction with stochastic safety/reachability analysis of the systems, which is discussed next.

Suppose that V is a stochastic simulation function from $\widehat{\Sigma}$ to Σ. Then for any input strategy $\hat{\nu}$ of the system $\widehat{\Sigma}$ there exists an input strategy ν of Σ, such that the following probability is bounded

$$\mathbb{P}\left\{\sup_{0 \leq k \leq T} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}]\right\} \leq \delta,$$

with δ being defined in Theorem 3.3 based on ε and T. Given the unsafe set A_1 for Σ, we can construct another set A_2, which is the ε neighborhood of A_1, i.e.,

$$A_2 = \{y' | \exists y \in A_1, \|y' - y\| \leq \varepsilon\}.$$

Now, we can provide the following corollary.

Corollary 3.5: Suppose V is an SSF from $\widehat{\Sigma}$ to Σ. For any input $\hat{\nu}(\cdot)$ there exists $\nu(\cdot)$ such that the following inequality holds:

$$\mathbb{P}\{\exists k \leq T, \; y_{a\nu}(k) \in A_1\} \leq \mathbb{P}\{\exists k \leq T, \; \hat{y}_{\hat{a}\hat{\nu}}(k) \in A_2\} + \delta.$$

Proof: Denote the events $\mathcal{E}_1 := \{\exists k \leq T, \; y_{a\nu}(k) \in A_1\}$ and $\mathcal{E}_2 := \{\exists k \leq T, \; \hat{y}_{\hat{a}\hat{\nu}}(k) \in A_2\}$. Then we have

$$\mathbb{P}\{\mathcal{E}_1\} = \mathbb{P}\{\mathcal{E}_1 \cap \mathcal{E}_2\} + \mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\} \leq \mathbb{P}\{\mathcal{E}_2\} + \mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\},$$

where $\bar{\mathcal{E}}_2$ is the complement of $\mathcal{E}_2$. Notice that the term $\mathbb{P}\{\mathcal{E}_1 \cap \bar{\mathcal{E}}_2\}$ is bounded by δ due to the above results, which concludes the proof. ∎

IV. COMPOSITIONAL ABSTRACTIONS FOR INTERCONNECTED SYSTEMS

Here, we first provide a formal definition of interconnection between discrete-time stochastic control systems.

A. Interconnected Stochastic Control Systems

Consider a complex stochastic control system Σ composed of $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i interconnected with each other as follows:

$$\Sigma_i = (X_i, W_i, U_i, Y_i, T_{\mathsf{x}_i}, h_i), \quad i \in [1; N],$$

with partitioned internal inputs and outputs

$$\omega_i = [\omega_{i1}; \ldots; \omega_{i(i-1)}; \omega_{i(i+1)}; \ldots; \omega_{iN}],$$
$$y_i = [y_{i1}; \ldots; y_{iN}], \tag{10}$$

and also output space and function

$$h_i(x_i) = [h_{i1}(x_i); \ldots; h_{iN}(x_i)],$$
$$Y_i = \prod_{j=1}^{N} Y_{ij}. \tag{11}$$

We interpret the outputs y_{ii} as *external* ones, whereas the outputs y_{ij} with $i \neq j$ are *internal* ones which are used to define the interconnected stochastic control systems. In particular, we assume that the dimension of ω_{ij} is equal to the dimension of y_{ji}. If there is no connection from stochastic control subsystem Σ_i to Σ_j, then we assume that the connecting output function is identically zero for all arguments, i.e., $h_{ij} \equiv 0$. Now, we define the *interconnected stochastic control systems* as the following.

Definition 4.1: Consider $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems $\Sigma_i = (X_i, W_i, U_i, Y_i, T_{\mathsf{x}_i}, h_i)$, $i \in [1; N]$, with the input-output configuration as in (10) and (11). The interconnection of Σ_i for any $i \in [1, \ldots, N]$, is the interconnected stochastic control system $\Sigma = (X, U, Y, T_{\mathsf{x}}, h)$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, such that $X := \prod_{i=1}^{N} X_i$, $U := \prod_{i=1}^{N} U_i$, function $f_a := \prod_{i=1}^{N} f_{ai}$, characterizing the stochastic kernel T_{x} based on those of subsystems (i.e. f_{ai}), $Y := \prod_{i=1}^{N} Y_{ii}$, and $h = \prod_{i=1}^{N} h_{ii}$, subjected to the following constraint:

$$\omega_{ij} = y_{ji} \quad \forall i, j \in [1; N], i \neq j. \tag{12}$$

B. Compositional Abstractions of Interconnected Systems

This subsection contains one of the main contributions of the paper. We assume that we are given N stochastic control subsystems

$$\Sigma_i = (X_i, W_i, U_i, Y_i, T_{\mathsf{x}_i}, h_i),$$

together with their corresponding abstractions $\widehat{\Sigma}_i = (\hat{X}_i, W_i, \hat{U}_i, Y_i, \hat{T}_{\mathsf{x}_i}, \hat{h}_i)$ with SPSF V_i from $\widehat{\Sigma}_i$ to Σ_i. For providing the main compositionality result of the paper, we raise the following assumption.

Assumption 1: For any $i, j \in [1; N]$, $i \neq j$, there exist $\mathcal{K}_\infty$ functions γ_i and constants $\lambda_i \in \mathbb{R}_{>0}$ and $\delta_{ij} \in \mathbb{R}_{\geq 0}$ such that for any $s \in \mathbb{R}_{\geq 0}$

$$\kappa_i(s) \geq \lambda_i \gamma_i(s) \tag{13}$$
$$h_{ji} \equiv 0 \implies \delta_{ij} = 0 \text{ and} \tag{14}$$
$$h_{ji} \not\equiv 0 \implies \rho_{i\text{int}}((N-1)\alpha_i^{-1}(s)) \leq \delta_{ij}\gamma_j(s), \tag{15}$$

where κ_i, α_j, and $\rho_{i\text{int}}$ represent the corresponding $\mathcal{K}$ and $\mathcal{K}_\infty$ functions of V_i appearing in Definition 3.1. Prior to presenting the next theorem, we define $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_N)$, $\Delta := \{\delta_{ij}\}$, where $\delta_{ii} = 0 \ \forall i \in [1; N]$, and $\Gamma(\vec{s}) := [\gamma_1(s_1); \ldots; \gamma_N(s_N)]$, where $\vec{s} = [s_1; \ldots; s_N]$. In the next theorem, we leverage a small-gain type condition to quantify the error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional way.

Theorem 4.2: Consider the interconnected stochastic control system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i. Suppose that each stochastic control subsystem Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding SPSF V_i. If Assumption 1 holds and there exists a vector $\mu \in \mathbb{R}_{>0}^N$ such that the inequality

$$\mu^T(-\Lambda + \Delta) < 0 \tag{16}$$

is also met, then

$$V(x, \hat{x}) := \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i)$$

is an SSF function from $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ to $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$.

The proof is similar to that of Theorem 4.5 in [9], and is omitted here due to lack of space.

V. DISCRETE-TIME LINEAR STOCHASTIC CONTROL SYSTEMS

In this section, we focus on a class of discrete-time linear stochastic control systems, defined as follows:

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + B\nu(k) + D\omega(k) + F\varsigma(k), \\ y(k) = Cx(k), \end{cases} \tag{17}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. We use the tuple $\Sigma = (A, B, C, D, F)$ to refer to the class of systems in (17). Here, we provide conditions under which a candidate V is an SPSF function facilitating the construction of an abstraction $\hat{\Sigma}$.

Let us assume that there exist matrix K and positive definite matrix M such that the matrix inequalities

$$C^T C \preceq M, \tag{18}$$
$$\left((1+\pi)(A+BK)^T M(A+BK) - M\right) \preceq -\widehat{\kappa}M, \tag{19}$$

hold for some positive constants π and $0 < \widehat{\kappa} < 1$. We employ the following quadratic SPSF

$$V(x, \hat{x}) = (x - P\hat{x})^T M(x - P\hat{x}), \tag{20}$$

where $P \in \mathbb{R}^{n \times \hat{n}}$ is a matrix of appropriate dimension. Assume that the equalities

$$AP = P\hat{A} - BQ \tag{21}$$
$$D = P\hat{D} - BS \tag{22}$$
$$CP = \hat{C}, \tag{23}$$

hold for some matrices Q and S of appropriate dimensions and possibly with the lowest possible $\hat{n}$. In the next theorem, we show that under the aforementioned conditions V in (20) is an SPSF from $\widehat{\Sigma}$ to Σ.

Theorem 5.1: Let $\Sigma = (A, B, C, D, F)$ and $\widehat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{F})$ be two discrete-time linear stochastic control subsystems with two independent additive noises. Suppose that there exist matrices $M, K, P, Q,$ and S satisfying (18), (19), (21), (22), and (23). Then, V defined in (20) is an SPSF from $\widehat{\Sigma}$ to Σ.

Proof: Here, we show that $\forall x, \forall \hat{x}, \forall \hat{\nu}, \forall \hat{\omega}, \exists \nu, \forall \omega,$ such that V satisfies $\|Cx - \hat{C}\hat{x}\|^2 \leq V(x, \hat{x})$ and

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,|\, x(k), \hat{x}(k), \omega(k) = \omega, \hat{\omega}(k) = \hat{\omega},$$
$$, \hat{\nu}(k) = \hat{\nu}\Big] - V(x(k), \hat{x}(k))$$
$$\leq -\hat{\kappa}(V(x(k), \hat{x}(k))) + (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{M}D\|^2 \|\omega - \hat{\omega}\|^2$$
$$+ (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{M}(B\widetilde{R} - P\hat{B})\|^2 \|\hat{\nu}\|^2$$
$$+ \mathrm{Tr}\Big(F^T M F + \hat{F}^T P^T M P \hat{F}\Big). \tag{24}$$

According to (23), we have $\|Cx - \hat{C}\hat{x}\|^2 = (x - P\hat{x})^T C^T C (x - P\hat{x})$. By applying (18), it can be easily verified that $\|Cx - \hat{C}\hat{x}\|^2 \leq V(x, \hat{x})$ holds $\forall x, \forall \hat{x}$. Now, we show inequality (24). Given any $x, \hat{x}, \hat{\nu},$ and $\hat{\omega}$, we choose ν via the following *linear interface function*:

$$\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{w}) := K(x - P\hat{x}) + Q\hat{x} + \widetilde{R}\hat{\nu} + S\hat{\omega}, \tag{25}$$

for some matrix $\widetilde{R}$ of appropriate dimension. By Employing equations (21), (22), and the definition of the interface function in (25), we simplify

$$Ax + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu}, \hat{\omega}) + D\omega - P(\hat{A}\hat{x} + \hat{B}\hat{\nu} + \hat{D}\hat{\omega})$$
$$+ \Big(F\varsigma(k) - P\hat{F}\hat{\varsigma}(k)\Big)$$

to $(A + BK)(x - P\hat{x}) + D(\omega - \hat{\omega}) + (B\widetilde{R} - P\hat{B})\hat{\nu} + \Big(F\varsigma(k) - P\hat{F}\hat{\varsigma}(k)\Big)$. One obtains:

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,|\, x(k), \hat{x}(k), \omega(k) = \omega, \hat{\omega}(k) = \hat{\omega},$$
$$, \hat{\nu}(k) = \hat{\nu}\Big] - V(x(k), \hat{x}(k))$$
$$= (x - P\hat{x})^T \Big[(A + BK)^T M (A + BK) - M\Big](x - P\hat{x})$$
$$+ \Big[2(x - P\hat{x})^T (A + BK)^T\Big] M \Big[D(\omega - \hat{\omega})\Big]$$
$$+ \Big[2(x - P\hat{x})^T (A + BK)^T\Big] M \Big[(B\widetilde{R} - P\hat{B})\hat{\nu}\Big]$$
$$+ \Big[2(\omega - \hat{\omega})^T D^T\Big] M \Big[(B\widetilde{R} - P\hat{B})\hat{\nu}\Big] + \|\sqrt{M}D(\omega - \hat{\omega})\|^2$$
$$+ \|\sqrt{M}(B\widetilde{R} - P\hat{B})\hat{\nu}\|^2 + \mathrm{Tr}\big(F^T M F + \hat{F}^T P^T M P \hat{F}\big).$$

Using Young's inequality [15] as $ab \leq \frac{\pi}{2}a^2 + \frac{1}{2\pi}b^2$, for any $a, b \geq 0$ and any $\pi > 0$, and by employing Cauchy-Schwarz inequality and (19), one obtains the following upper bound:

$$\mathbb{E}\Big[V(x(k+1), \hat{x}(k+1)) \,|\, x(k), \hat{x}(k), \omega(k) = \omega, \hat{\omega}(k) = \hat{\omega},$$
$$, \hat{\nu}(k) = \hat{\nu}\Big] - V(x(k), \hat{x}(k))$$
$$\leq -\hat{\kappa}(V(x, \hat{x})) + (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{M}D\|^2 \|\omega - \hat{\omega}\|^2$$
$$+ (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{M}(B\widetilde{R} - P\hat{B})\|^2 \|\hat{\nu}\|^2$$
$$+ \mathrm{Tr}\Big(F^T M F + \hat{F}^T P^T M P \hat{F}\Big). \tag{26}$$

Hence, the proposed V in (20) is an SPSF from $\widehat{\Sigma}$ to Σ, which completes the proof. Note that the $\mathcal{K}$ and $\mathcal{K}_\infty$ functions $\kappa, \alpha,$ and ρ_{ext}, in Definition 3.1 associated with the SPSF in (20) are $\alpha(s) := s^2, \kappa(s) := \hat{\kappa}s,$ and $\rho_{\mathrm{int}}(s) := (1 + \frac{2}{\pi} + \frac{\pi}{2})\|\sqrt{M}D\|^2 s^2, \rho_{\mathrm{ext}}(s) := (1 + \frac{2}{\pi} + \frac{2}{\pi})\|\sqrt{M}(B\widetilde{R} - P\hat{B})\|^2 s^2, \forall s \in \mathbb{R}_{\geq 0}$. Moreover, positive constant ψ in (4) is $\psi = \mathrm{Tr}\big(F^T M F + \hat{F}^T P^T M P \hat{F}\big)$. ∎

Remark 5.2: One can readily verify from the result of Theorem 5.1 that choosing $\hat{F}$ equal to zero results in smaller constant ψ and, hence, more closeness of linear subsystems and their abstractions. Observe that this is not the case when one assumes the noise of the concrete subsystem and its abstraction are the same as in [9], [10].

Remark 5.3: Note that the results in Theorem 5.1 do not impose any condition on matrix $\hat{B}$ and, hence, it can be chosen arbitrarily. As an example, one can choose $\hat{B} = I_{\hat{n}}$ which makes the abstract system $\widehat{\Sigma}$ fully actuated and, hence, the synthesis problem over it much easier.

Remark 5.4: Since Theorem 5.1 does not impose any condition on matrix $\widetilde{R}$, we choose $\widetilde{R}$ to minimize function ρ_{ext} for V as suggested in [16]. The following choice for $\widetilde{R}$

$$\widetilde{R} = (B^T M B)^{-1} B^T M P \hat{B}. \tag{27}$$

minimizes ρ_{ext}.

VI. EXAMPLE

Here, we demonstrate the effectiveness of the proposed results for an interconnected system consisting of four discrete-time linear stochastic control subsystems, i.e. $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$. The interconnection scheme of Σ with four inputs and two outputs is illustrated in Figure 1.

As seen, the output of Σ_1 (resp. Σ_2) is connected to the internal input of Σ_4 (resp. Σ_3) and the output of Σ_3 (resp. Σ_4) connects to the internal input of Σ_1 (resp. Σ_2). The system matrices are given by

$$A_i = I_{25}, \; B_i = I_{25}, \; C_i^T = 0.1\mathbf{1}_{25}, \; F_i = 0.01\mathbf{1}_{25},$$

for $i \in \{1, 2, 3, 4\}$. The internal input and output matrices are also given by:

$$C_{14}^T = C_{23}^T = C_{31}^T = C_{42}^T = 0.1\mathbf{1}_{25},$$
$$D_{13} = D_{24} = D_{32} = D_{41} = 0.1\mathbf{1}_{25}.$$

In order to construct an abstraction for $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, we construct an abstraction $\widehat{\Sigma}_i$ of each individual subsystem $\Sigma_i, i \in \{1, 2, 3, 4\}$. We first fix $\hat{\kappa}$ and π for each subsystem,
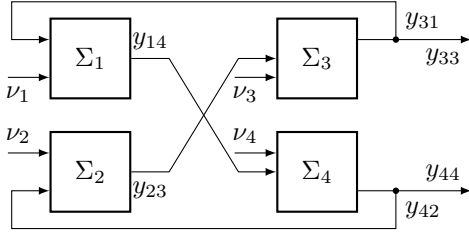
Fig. 1. The interconnected system $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$.

and then determine the matrices M and K such that (18) and (19) hold for $i \in \{1, 2, 3, 4\}$:

$$M_i = I_{25}, \ K_i = -0.95I_{25}, \ \widehat{\kappa}_i = 0.98, \ \pi_i = 0.99.$$

We continue with determining other matrices such that (21), (22), and (23) hold:

$$P_i = \mathbf{1}_{25}, \ Q_i = \mathbf{1}_{25}, \ S_i = -0.0031\mathbf{1}_{25},$$

for $i \in \{1, 2, 3, 4\}$. Accordingly, the matrices of abstract subsystems are computed as:

$$\hat{A}_i = 2, \ \hat{C}_i = 2.5, \ \hat{D}_i = 0.096,$$

for $i \in \{1, 2, 3, 4\}$. Note that here $\hat{F}_i$, $i \in \{1, 2, 3, 4\}$, are considered zero in order to reduce constants ψ_i for each V_i. Moreover, $\hat{B}_i$ are chosen 1 and we compute $\widetilde{R}_i$, $i \in \{1, 2, 3, 4\}$, using (27) as $\widetilde{R}_i = \mathbf{1}_{25}$. The interface function for $i \in \{1, 2, 3, 4\}$ follows by (25) as:

$$\nu_i = -0.95I_{25}(x_i - \mathbf{1}_{25}\hat{x}_i) + \mathbf{1}_{25}\hat{x}_i + \mathbf{1}_{25}\hat{\nu}_i - 0.0031\mathbf{1}_{25}\hat{\omega}_i.$$

Hence, Theorem 5.1 holds and $V_i(x_i, \hat{x}_i) = (x_i - \mathbf{1}_{25}\hat{x}_i)^T M_i(x_i - \mathbf{1}_{25}\hat{x}_i)$ is an SPSF function from $\widehat{\Sigma}_i$ to Σ_i satisfying conditions (3) and (4) with $\alpha_i(s) = s^2$, $\kappa_i(s) = 0.98s$, $\rho_{i\text{ext}}(s) = 0$, $\rho_{i\text{int}}(s) = 0.88s^2$, and $\psi_i = 0.0025$, for $i \in \{1, 2, 3, 4\}$. We now proceed with Theorem 4.2 to construct a stochastic simulation function form $\widehat{\Sigma}$ to Σ. Assumption 1 holds with $\gamma_i(s) = s$ and:

$$\Delta = \begin{bmatrix} 0 & 0 & 0.88 & 0 \\ 0 & 0 & 0 & 0.88 \\ 0 & 0.88 & 0 & 0 \\ 0.88 & 0 & 0 & 0 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 0.98 & 0 & 0 & 0 \\ 0 & 0.98 & 0 & 0 \\ 0 & 0 & 0.98 & 0 \\ 0 & 0 & 0 & 0.98 \end{bmatrix}.$$

Additionally, one can readily verify that a vector $\mu \in \mathbb{R}^4_{>0}$ exists here since the spectral radius of $\Lambda^{-1}\Delta$ is strictly less than one [17]. By choosing vector μ as $\mu^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, the function

$$V(x, \hat{x}) = V_1(x_1, \hat{x}_1) + V_2(x_2, \hat{x}_2) + V_3(x_3, \hat{x}_3) + V_4(x_4, \hat{x}_4)$$

is an SSF from $\mathcal{I}(\widehat{\Sigma}_1, \widehat{\Sigma}_2, \widehat{\Sigma}_3, \widehat{\Sigma}_4)$ to $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ satisfying conditions (5) and (6) with $\alpha(s) = s^2$, $\kappa(s) = 0.1s$, $\rho_{\text{ext}}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi = 0.01$. If the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are started from zero, one can readily verify that the norm of error between outputs of Σ and of $\widehat{\Sigma}$ will not exceed 1 with probability at least 90% computed by the stochastic simulation function V using inequality (7) for $T = 10$.

VII. Discussion

In this paper, we provided a compositional approach for abstractions of interconnected discrete-time stochastic control systems, with independent noises in the abstract and concrete subsystems. First, we introduced new notions of stochastic pseudo-simulation and stochastic simulation functions in order to quantify the distance in a probability setting between original stochastic control subsystems and their abstractions and their interconnections, respectively. Therefore, one can employ the proposed results here to potentially solve safety/reachability problems over the abstract interconnected systems and then refine the results to the concrete interconnected ones. Furthermore, we provided a computational scheme for the class of discrete-time linear stochastic control systems to construct abstractions together with their corresponding stochastic pseudo-simulation functions. Finally, we demonstrated the effectiveness of the results by constructing an abstraction (totally 4 dimensions) of the interconnection of four discrete-time linear stochastic control subsystems (together 100 dimensions) in a compositional fashion.

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