Lett Math Phys (2010) 91:265–287
DOI 10.1007/s11005-010-0374-8

Exact Spectrum of Anomalous Dimensions of Planar $N=4$ Supersymmetric Yang–Mills Theory: TBA and excited states

NIKOLAY GROMOV[1,2,3], VLADIMIR KAZAKOV[4], ANDRII KOZAK[5], and PEDRO VIEIRA[6]
[1] *DESY Theory, Hamburg, Germany. e-mail: nikgromov@gmail.com*
[2] *II. Institut für Theoretische Physik Universität, Hamburg, Germany.*
[3] *St.Petersburg INP, St.Petersburg, Russia.*
[4] *Ecole Normale Superieure, LPT, 75231 Paris Cedex-5, France.*
[5] *l'Université Paris-VI, Paris, France.*
[6] *Max-Planck-Institut für Gravitationphysik, Albert-Einstein-Institut, 14476 Potsdam, Germany.*

Received: 31 May 2009 / Revised: 21 December 2009 / Accepted: 6 January 2010
Published online: 23 February 2010 –

Abstract. Using the thermodynamic Bethe ansatz method we derive an infinite set of integral non-linear equations for the spectrum of states/operators in AdS/CFT. The Y-system conjectured in Gromov et al. (Integrability for the Full Spectrum of Planar AdS/CFT. arXiv:0901.3753 [hep-th]) for the spectrum of all operators in planar $N=4$ SYM theory follows from these equations. In particular, we present the integral TBA type equations for the spectrum of all operators within the $sl(2)$ sector. We prove that all the kernels and free terms entering these TBA equations are real and have nice fusion properties in the relevant mirror kinematics. We find the analog of DHM formula for the dressing kernel in the mirror kinematics.

Mathematics Subject Classification (2000). 81T13, 81T20, 81T30, 81T60.

Keywords. gauge/string duality, integrability, finite volume spectrum.

1. Introduction

Recently, a set of functional equations, the so-called Y-system, defining the spectrum of all local operators in planar AdS/CFT correspondence, was proposed by three of the current authors [1]. The Y-system has the form of functional equations

$$\frac{Y_{a,s}^{+}Y_{a,s}^{-}}{Y_{a+1,s}Y_{a-1,s}}=\frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}, \tag{1}$$

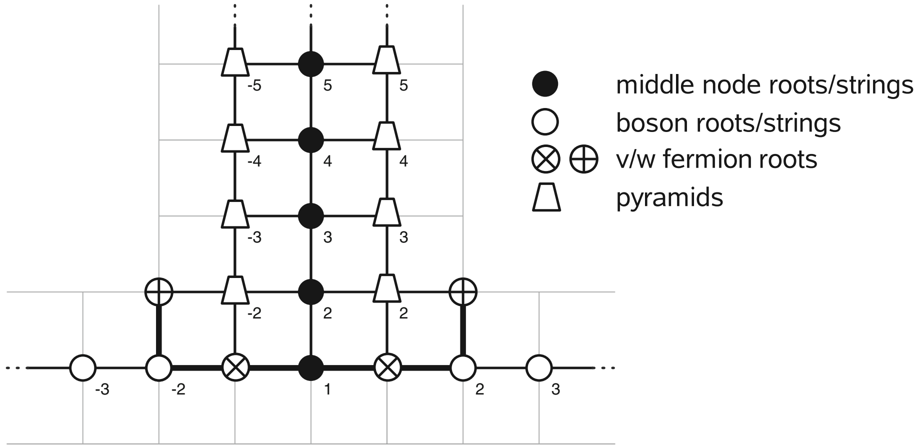

Figure 1. **T**-shaped “fat hook” (**T**-hook) uniting two $SU(2|2)$ fat hooks, see [2] for details on fat hooks and super algebras.

where $f^{\pm} \equiv f(u \pm i/2)$ are simple shifts in the imaginary direction. The functions $Y_{a,s}(u)$ are defined only on the nodes marked by ○, ⊕, ⊗, ◇, ● on Figure 1. Its solutions with appropriate analytical properties define the energy of a state (anomalous dimension of an operator in $N=4$ SYM) through the formula[1]

$$E = \sum_j \epsilon_1(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a^*}{\partial u} \log\left(1 + Y_{a,0}^*(u)\right). \tag{2}$$

where ϵ_n^* is the mirror “momentum” defined in the text below and the rapidities $u_{4,j}$ are fixed by the exact Bethe ansatz equations

$$Y_{1,0}(u_{4,j}) = -1. \tag{3}$$

The Y-system is equivalent to the Hirota bilinear equation

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}, \tag{4}$$

where the functions $T_{a,s}(u)$ are non-zero only on the visible part of the 2D lattice drawn on Figure 1 and

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}. \tag{5}$$

It was shown that the Y-system passes a few non-trivial tests, and in particular it is completely consistent with the asymptotic Bethe ansatz (ABA) [4–7], is compatible with the crossing relation [3] and reproduces the first wrapping corrections at weak coupling for Konishi and other twist two operators [8–11].

[1] In some cases the integration contour could encircle singularities of the integrand situated away from the real axe. In the large L asymptotics these singularities can be responsible for the Lüscher μ-terms. See also discussion in Section 7.

In this paper, we will provide a derivation of the Y-system similar in spirit to that employed in the derivation of the TBA-type non-linear integral equations for the finite volume spectra of relativistic 2D models. It is based on the Matsubara trick relating the ground state of a euclidean QFT on a cylinder to the free energy of the same theory in finite temperature. If we take instead of the cylinder a torus with a small circumference L and a large circumference R we can represent the partition function in two different channels as a sum over energy levels. In the large R limit, we can identify the free energy $\mathcal{F}(L)$ per unit length of a "mirror" QFT living in the space section along the infinite direction of the torus and having a temperature $T=1/L$, with the ground state energy $E_0(L)$ of the original QFT living on a space circle of the radius L

$$Z(L,R)=\sum_k \mathrm{e}^{-L\tilde{E}_k(R)}=\sum_j \mathrm{e}^{-RE_j(L)} \underset{R\to\infty}{\to} \mathrm{e}^{-R\mathcal{F}(L)}=\mathrm{e}^{-RE_0(L)}.$$

In the relativistic QFTs the original theory and the mirror theory are essentially equivalent and differ only in the boundary conditions [12]. An example of such a TBA calculation, useful for our further purposes, for the $SU(2)$ principle chiral field (PCF), can be seen in the Appendix A of [13]. In the superstring sigma model on $AdS_5\times S^5$ background in the light cone gauge relevant to our problem, we have to deal with the non-relativistic original and mirror sigma models (see [14,15]).

Particularly important for our discussion is the form of the energy and momentum of the elementary excitations for both the physical and mirror theories in infinite volume. They are conveniently parameterized in terms of the Zhukowsky variables,

$$x(u)+\frac{1}{x(u)}=\frac{u}{g} \tag{6}$$

which admits two solutions, one of them outside the unit circle $|x(u)|>1$ and another inside the unit circle, $|x(u)|<1$. The energy $\epsilon_a(u)$ and momentum $p_a(u)$ of the physical bound states are then given by [16]

$$\epsilon_a(u)=a+\frac{2ig}{x^{[+a]}}-\frac{2ig}{x^{[-a]}},\quad p_a(u)=\frac{1}{i}\log\frac{x^{[+a]}}{x^{[-a]}} \tag{7}$$

where $x^{[\pm a]}\equiv x(u\pm ia/2)$ are evaluated in the physical kinematics where $|x^{[\pm a]}|>1$.

The mirror energy and momentum are obtained by the usual Wick rotation $(E,p)\to(ip,iE)$. To stress this we denote the mirror energy by ip_a^* and the mirror momentum by $i\epsilon_a^*$. The quantities ϵ_a^* and p_a^* are defined precisely as in (7) where $x^{[a]}$ are now evaluated in the mirror kinematics where $|x^{[a]}|>1$ but $|x^{[-a]}|<1$, for $a>0$.

Let us now return to our general review of the TBA method. This method is based on the so-called string hypothesis: all the eigenstates of an integrable model in the infinite volume are represented by bound states (the simplest ones are called "strings") described by some density ρ_A. In terms of these densities the asymptotic Bethe equations simply read

$$\bar{\rho}_A(u)+\rho_A(u)=\frac{i}{2\pi}\frac{\mathrm{d}\epsilon_A^*(u)}{\mathrm{d}u}-K_{BA}(v,u)*\rho_B(v). \tag{8}$$

Here, $K_{BA}(v,u)=\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}u}\log S_{AB}(u,v)$ is the kernel describing the interaction between the bound states A and B which scatter via an S-matrix S_{AB}. $i\epsilon_A^*$ is the momentum of a magnon labeled by A. For the same aforementioned reasons in the discussion of the AdS/CFT dispersion relations we use this notation to emphasize that the momenta of these mirror particles are obtained from the energy of the physical particles $\epsilon_A(u)$ by the Wick rotation. Finally, $\bar{\rho}_A$ is the density of holes associated with the bound state A.

To compute the free energy we must minimize the functional

$$\mathcal{F}=\sum_A\int_{-\infty}^{\infty}\mathrm{d}u\left(\left(Lip_A^*+h_A\right)\rho_A-\left[\rho_A\log\left(1+\frac{\bar{\rho}_A}{\rho_A}\right)+\bar{\rho}_A\log\left(1+\frac{\rho_A}{\bar{\rho}_A}\right)\right]\right) \tag{9}$$

with respect to $\rho_A(u)$, $\bar{\rho}_A(u)$ and exclude $\delta\bar{\rho}_A$ by the use of the constraint imposed by the BAEs (8). The physical origin of each term in the expression for the free energy is as follows: The first term accounts for the energy (times inverse "temperature" L); the term in the square brackets represent the entropy contribution; we added a generic chemical potential h_A for each kind of bound states. This chemical potential is needed if the theory contains fermionic excitations, as is the case for the AdS/CFT system, since we want to compute the Witten index rather than the thermal partition function where the physical fermions are periodic. This amounts to choosing $h_A=i\pi=\log(-1)$ for the fermionic states and $h_A=0$ for the bosonic states.

The minimization of the free energy yields the TBA equations

$$\log\mathcal{Y}_A(u)=K_{AB}(u,v)*\log[1+1/\mathcal{Y}_B(v)]+iLp_A^*(u)+h_A \tag{10}$$

for the quantities $\mathcal{Y}_A=\frac{\bar{\rho}_A}{\rho_A}$. Finally, at this saddle point, the free energy can be simply written as

$$\mathcal{F}=\sum_A\int\frac{\mathrm{d}u}{2\pi i}\frac{\mathrm{d}\epsilon_A^*}{\mathrm{d}u}\log\left(1+1/\mathcal{Y}_A(u)\right). \tag{11}$$

In this way one obtains the finite volume ground state energy for a generic integrable field theory. The excited physical states are recovered by the usual procedure of analytic continuation [17–20] and will be also discussed in this paper.

In what follows, we will apply the TBA method to the "mirror" superstring sigma model and derive this AdS/CFT Y-system conjectured in [1]. The actual TBA equations arising as an intermediate step towards the Y-system may be very useful for the numerical calculations of the energies of low-lying states.

2. The Starting Point: Beisert–Staudacher Equations

The basis of our derivation of TBA for AdS/CFT are the Beisert–Staudacher (BS) ABA equations of [4,5,7] in their mirror form [15,21]. We write them in our compact notations, introducing three types of Baxter functions

$$
\begin{aligned}
&R_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{x(u)-x_{l,j}^{\mp}}{\sqrt{x_{l,j}^{\mp}}}, \quad B_l^{(\pm)}(u) \equiv \prod_{j=1}^{K_l} \frac{\frac{1}{x(u)}-x_{l,j}^{\mp}}{\sqrt{x_{l,j}^{\mp}}}, \\
&Q_l(u) = \prod_{j=1}^{K_l} (u-u_{l,j}) = (-g)^{K_l} R_l(u) B_l(u).
\end{aligned} \tag{12}
$$

The index l takes the values $l=1L,2L,3L$ or $l=1R,2R,3R$ parametrizing the rapidities of the left and right $SU(2|2)$ wings of the model, correspondingly. $R^{(\pm)}$ and $B^{(\pm)}$ with no subscript l correspond to the roots $x_{4,j}$ of the middle node and R_l, B_l without superscript $(+)$ or $(-)$ are defined as in (12) with $x_j^{\pm}$ replaced by x_j. In these notations the left wing ABAs read:

$$
1 = \left.\frac{Q_{2L}^{+} B^{(-)}}{Q_{2L}^{-} B^{(+)}}\right|_{u_{1L,k}}, \quad -1 = \left.\frac{Q_{2L}^{--} Q_{1L}^{+} Q_{3L}^{+}}{Q_{2L}^{++} Q_{1L}^{-} Q_{3L}^{-}}\right|_{u_{2L,k}}, \quad 1 = \left.\frac{Q_{2L}^{+} R^{(-)}}{Q_{2L}^{-} R^{(+)}}\right|_{u_{3L,k}} \tag{13}
$$

with a similar set of equations for the right wing replacing $L \to R$. The Bethe equation for the middle node for the full AdS/CFT ABA of [5] fixes the positions of the $u_{4,j}$ roots from[2]

$$
\begin{aligned}
-1 = &\left[e^{R\epsilon_1^*} \left(\frac{Q_4^{--}}{Q_4^{++}} \frac{B_{1L}^{+} R_{3L}^{+}}{B_{1L}^{-} R_{3L}^{-}} \frac{B_{1R}^{+} R_{3R}^{+}}{B_{1R}^{-} R_{3R}^{-}} \right) \left(\frac{B^{+(+)}}{B^{-(-)}} \right)^2 S^2 \right] \times \\
&\times \left. \prod_{j=1}^{K_4} \frac{x_j^{+}}{x_j^{-}} \left(\frac{x_k^{+}}{x_k^{-}} \right)^{\frac{K_{1R}-K_{3R}+K_{1L}-K_{3L}}{2}} \right|_{u=u_{4,k}}
\end{aligned} \tag{14}
$$

for the $sl(2)$ favored grading. The dressing factor is $S(u) = \prod_j \sigma(x(u), x_{4,j})$ where σ is the BES dressing kernel [7] (see [22] for a nice integral representation of the dressing kernel).

3. Bound States and TBA Equations for the Mirror "Free Energy"

To write the TBA for the full AdS/CFT, we have to find the BAEs for the densities of all complexes of Bethe roots in the infinite volume $R=\infty$. The string hypothesis

[2]This equation is identical to the Eq. (6.6) from [15]. The factors of x^+/x^- outside of the square brackets were not present in our earlier preprints. However, they are present inside the kernels [the last column of (19)] starting from the preprint arXiv:0902.4458v2 of our paper. We thank the referee for pointing us out this misprint which fortunately does not affect any of our results.

implies the full description of the infinite volume solutions. They are easy to classify: there is only one type of momentum carrying complexes, strings in the middle nodes, similar to standard $SU(2)$ strings [16]; the rest are the same complexes as found by Takahashi in the Hubbard model [23,24] (see also [25]).

As a result, we find that in the large R limit of BAEs the roots regroup into the following bound states:

$u_4 = u + ij,$	$j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}:$	middle node bound states	$:\bullet_n$
$u_2^{L,R} = u + ij,$	$j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}:$	L, R string bound states	$:\bigcirc_{\pm n}$
$u_3^{L,R} = u + ij,$ $u_2^{L,R} = u + ij,$ $u_1^{L,R} = u + ij,$	$j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$ $j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}:$ $j = -\frac{n-3}{2}, \dots, \frac{n-3}{2}$	L, R trapezia	$:\triangle_{\pm n}$
$u_1^{L,R} = u:$		L, R single fermion	$:\oplus_\pm$
$u_3^{L,R} = u:$		L, R single fermion	$:\otimes_\pm$

where by u we denote the real center of a complex. Thus, the index A in formulae (8–11) takes the values

$$A = \{\bigcirc_{\pm n}, \oplus_\pm, \otimes_\pm, \triangle_{\pm n}, \bullet_n\} \tag{15}$$

or, in the notation used in [1] for the points on the T-hook

$$A = \{(1, \pm n), (2, \pm 2), (1, \pm 1), (n, \pm 1), (n, 0)\}. \tag{16}$$

Multiplying the Bethe equations along each complex we obtain the fused equations (8) for the densities (of particles and holes, $\rho_A(u)$ and $\bar{\rho}_A(u)$) of the centers of complexes (10). It is useful to introduce the following notation for $\mathcal{Y}_A$:

$$\left\{\mathcal{Y}_{\bigcirc_{\pm n}}, \mathcal{Y}_{\oplus_\pm}, \mathcal{Y}_{\otimes_\pm}, \mathcal{Y}_{\triangle_{\pm n}}, \mathcal{Y}_{\bullet_{\pm n}}\right\} = \left\{Y_{\bigcirc_{\pm n}}, Y_{\oplus_\pm}, \frac{1}{Y_{\otimes_\pm}}, \frac{1}{Y_{\triangle_{\pm n}}}, \frac{1}{Y_{\bullet_{\pm n}}}\right\} \tag{17}$$

In particular, notice that the Y functions $Y_{a,s}$ arrange nicely into a T-shaped form as depicted in Figure 1. As shown below, these functions are precisely those appearing in the Y-system (1).

The only complexes which carry energy and momentum are those made out of middle node roots $u_{4,j}$,

$$\epsilon_A^* = \delta_{A,\bullet_n} \epsilon_n^*, \quad p_A^* = \delta_{A,\bullet_n} p_n^* \tag{18}$$

where ϵ_n^* and p_n^* are explained after (7). The fused kernels K_{AB} are given by

$K_{AB} =$ (19)

$A\backslash B$	$\bigcirc_m$	$\oplus_+$	$\otimes_+$	$\square_m$	$\bullet_m$
$\bigcirc_n$	$+K_{n-1,m-1}$	$-K_{n-1}$	$+K_{n-1}$	0	0
$\oplus_+$	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{B}_{1m}^{(01)}$
$\otimes_+$	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{R}_{1m}^{(01)}$
$\square_n$	0	$-K_{n-1}$	$+K_{n-1}$	$+K_{n-1,m-1}$	$-\mathcal{R}_{nm}^{(01)} - \mathcal{B}_{n-2,m}^{(01)}$
$\bullet_n$	0	$\mathcal{B}_{n1}^{(10)}$	$-\mathcal{R}_{n1}^{(10)}$	$-\mathcal{R}_{nm}^{(10)} - \mathcal{B}_{n,m-2}^{(10)}$	$-2\mathcal{S}_{nm} - \mathcal{B}_{nm}^{(11)} + \mathcal{R}_{nm}^{(11)}$

where the block entrees of this infinite matrix are defined as

$$K_n \equiv \frac{1}{2\pi i}\frac{d}{dv}\log\frac{u-v+in/2}{u-v-in/2}, \qquad K_{nm} \equiv \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} K_{2j+2k+2} \tag{20}$$

$$\mathcal{S}_{nm}(u,v) \equiv \frac{1}{2\pi i}\frac{d}{dv}\log\sigma(x^{\pm n}(u), x^{\pm m}(v)) \tag{21}$$

$$\mathcal{B}_{nm}^{(ab)}(u,v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i}\frac{d}{dv}\log\frac{b(u+ia/2+ij, v-ib/2+ik)}{b(u-ia/2+ij, v+ib/2+ik)} \tag{22}$$

$$\mathcal{R}_{nm}^{(ab)}(u,v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i}\frac{d}{dv}\log\frac{r(u+ia/2+ij, v-ib/2+ik)}{r(u-ia/2+ij, v+ib/2+ik)} \tag{23}$$

where

$$r(u,v) = \frac{x(u)-x(v)}{\sqrt{x(v)}}, \qquad b(u,v) = \frac{1/x(u)-x(v)}{\sqrt{x(v)}}. \tag{24}$$

In the table above, we only wrote the interaction between the complexes of the left $SU(2|2)$ wing, between those complexes and the middle node bound states, as well as between the middle node bound states themselves. The right wing interaction is of course absolutely identical and the complexes of different wings do not interact. Equation (10) in the notation of (17) then read

$$\log Y_{\otimes_\pm} = +K_{m-1} * \log\frac{1+1/Y_{\bigcirc_{\pm m}}}{1+Y_{\square_{\pm m}}} + \mathcal{R}_{1m}^{(01)} * \log(1+Y_{\bullet_m}) + \log(-1) \tag{25}$$

$$\log Y_{\oplus_\pm} = -K_{m-1} * \log\frac{1+1/Y_{\bigcirc_{\pm m}}}{1+Y_{\square_{\pm m}}} - \mathcal{B}_{1m}^{(01)} * \log(1+Y_{\bullet_m}) - \log(-1) \tag{26}$$

$$\log Y_{\square_{\pm n}} = -K_{n-1,m-1} * \log(1+Y_{\square_{\pm m}}) - K_{n-1} * \log\frac{1+Y_{\otimes_\pm}}{1+1/Y_{\oplus_\pm}} + \\ + \left(\mathcal{R}_{nm}^{(01)} + \mathcal{B}_{n-2,m}^{(01)}\right) * \log(1+Y_{\bullet_m}) \tag{27}$$

$$\log Y_{\bigcirc_{\pm n}} = K_{n-1,m-1} * \log(1+1/Y_{\bigcirc_{\pm m}}) + K_{n-1} * \log \frac{1+Y_{\otimes_\pm}}{1+1/Y_{\oplus_\pm}} \tag{28}$$

$$\begin{aligned}\log Y_{\bullet_n} = {} & L \log \frac{x^{[-n]}}{x^{[+n]}} + \left(2\mathcal{S}_{nm} - \mathcal{R}^{(11)}_{nm} + \mathcal{B}^{(11)}_{nm}\right) * \log(1+Y_{\bullet_m}) - \\ & -\mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus_+}) + \mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes_+}) + \\ & + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m}) - \\ & -\mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus_-}) + \mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes_-}) + \\ & + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_{-m}}) \end{aligned} \tag{29}$$

All convolutions are to be understood in the usual sense with the second variable being integrated over so that $K * f = \int \mathrm{d}v K(u,v) f(v)$. Summation over the repeated index m is assumed ($m = 2, \ldots, \infty$ for the convolutions involving pyramids $\triangle_{\pm m}$ and strings $\bigcirc_{\pm m}$ and $m = 1, \ldots, \infty$ for the convolutions with the middle node bound states $\bullet_m$). There are still some ambiguities involved in these integral equations concerning the choice of the integration contours. We will discuss this, still not completely elucidated, point when we will consider equations for the excited states where some of the ambiguities will be lifted.

4. Derivation of the AdS/CFT Y-system

We will now derive, from the TBA equations, the Y-system (1) and (4) for the AdS/CFT spectrum conjectured in [1]. We shall do it separately for each type of excitations.

The key idea in the derivation is to use the discrete Laplace operator acting on the free variable u and free index n in the TBA equations. We notice that

$$\Delta K_n(u) \equiv K_n(u+i/2-i0) + K_n(u-i/2+i0) - K_{n+1}(u) - K_{n-1}(u) = \delta_{n,1}\delta(u)$$

As a simple consequence of this identity we find

$$\begin{aligned} \Delta K_{nm}(v-u) = \Delta\mathcal{R}^{(11)}_{nm}(v,u) = \delta_{n,m+1}\delta(v-u) + \delta_{n,m-1}\delta(v-u) \\ \Delta\mathcal{R}^{(01)}_{nm}(v,u) = \Delta\mathcal{R}^{(10)}_{nm}(v,u) = \delta_{n,m}\delta(v-u) \end{aligned} \tag{30}$$

whereas the Laplacian kills all other kernels, $\Delta\mathcal{S}_{nm} = 0$, etc. For example, the fact that the dressing factor is killed by the Laplacian follows from its harmonic form

$$\sigma_{nm}(u,v) = e^{\chi(u+in/2,v+in/2)+\chi(u-in/2,v-in/2)-\chi(u-in/2,v+in/2)-\chi(u+in/2,v-in/2)} \tag{31}$$

without any singularities in the physical kinematics (this fact was already used in [1] when constructing the large L solutions of the Y-system). By virtue of these identities we can easily compute the combinations $\log \frac{Y^+_{\bigcirc_n} Y^-_{\bigcirc_n}}{Y_{\bigcirc_{n+1}} Y_{\bigcirc_{n-1}}}$, $\log \frac{Y^+_{\triangle_n} Y^-_{\triangle_n}}{Y_{\triangle_{n+1}} Y_{\triangle_{n-1}}}$ and $\log \frac{Y^+_{\bullet_n} Y^-_{\bullet_n}}{Y_{\bullet_{n+1}} Y_{\bullet_{n-1}}}$, where $f^\pm \equiv f(u \pm i/2 \mp i0)$, using, respectively, (28), (27) and (29). We find

$$\log \frac{Y^+_{\bigcirc_n} Y^-_{\bigcirc_n}}{Y_{\bigcirc_{n+1}} Y_{\bigcirc_{n-1}}} = \log(1+1/Y_{\bigcirc_{n+1}})(1+1/Y_{\bigcirc_{n-1}}), \quad n>2 \tag{32}$$

and

$$\log \frac{Y^+_{\bigcirc_2} Y^-_{\bigcirc_2}}{Y_{\bigcirc_3}} = \log \frac{(1+Y_{\otimes_+})(1+1/Y_{\bigcirc_3})}{1+1/Y_{\oplus_+}} \tag{33}$$

for the string bound states. The equations for $Y_{1,n}$ at $n \leq -2$, as well as their derivation, are similar. For the pyramid complexes we obtain

$$\log \frac{Y^+_{\triangle_n} Y^-_{\triangle_n}}{Y_{\triangle_{n+1}} Y_{\triangle_{n-1}}} = \log \frac{1+Y_{\bullet_n}}{(1+Y_{\triangle_{n+1}})(1+Y_{\triangle_{n-1}})}, \quad n>2 \tag{34}$$

and

$$\log \frac{Y^+_{\triangle_2} Y^-_{\triangle_2}}{Y_{\triangle_3}} = \log \frac{(1+Y_{\oplus})(1+Y_{\bullet_2})Y_{\otimes_+}}{(1+Y_{\triangle_3})(1+Y_{\otimes_+})} - \\ - \log Y_{\otimes_+} Y_{\oplus_+} + \sum_n \left(\mathcal{R}^{(01)}_{n1} - \mathcal{B}^{(01)}_{n1}\right) * \log(1+Y_{\bullet_n}) .$$

The first term in the r.h.s. of this equation reproduces again the correct structure of the Y-system (1). In fact, we will see below that the last two terms cancel each other and hence this equation perfectly fits the Y-system (1). Finally, for the middle node bound states, we kill again the kernels when applying the discrete Laplace operator and obtain

$$\log \frac{Y^+_{\bullet_n} Y^-_{\bullet_n}}{Y_{\bullet_{n+1}} Y_{\bullet_{n-1}}} = \log \frac{(1+Y_{\triangle_n})(1+Y_{\triangle_{-n}})}{(1+Y_{\bullet_{n+1}})(1+Y_{\bullet_{n-1}})}, \quad n>1 \tag{35}$$

and

$$\log \frac{Y^+_{\bullet_1} Y^-_{\bullet_1}}{Y_{\bullet_2}} = \log \frac{1+Y_{\otimes_+}}{1+Y_{\bullet_2}} . \tag{36}$$

We are left with the equations for the two fermionic nodes $Y_{1,1} = Y_{\otimes_+}$ and $Y_{2,2} = Y_{\oplus_+}$ (for $Y_{1,-1}$ and $Y_{2,-2}$ it will be similar). We consider first the node $Y_{1,1}$. Combining Equation. (25) for $u \to u \pm i/2 \mp i0$ with Eqs. (27) and (28) for real u and $n=2$ we obtain (again using the fusion properties of several kernels),

$$\log \frac{Y^+_{\otimes_+} Y^-_{\otimes_+}}{Y_{\triangle_2} Y_{\bigcirc_2}} = \log \frac{(1+1/Y_{\bigcirc_2})(1+Y_{\bullet_1})}{1+Y_{\triangle_2}} \tag{37}$$

perfectly reproducing the equation for $Y_{1,1}$ from the Y-system (1). Finally, to find the equation for the last fermion node $Y_{2,2}$ we simply add up Eqs. (26) and (25) to get

$$\log Y_{\otimes_+} Y_{\oplus_+} = \sum_m \left(\mathcal{R}^{(01)}_{1m} - \mathcal{B}^{(01)}_{1m}\right) * \log(1+Y_{\bullet_m}) \tag{38}$$

This shows indeed that the two last terms in (35) cancel. The equation for $Y_{22} = Y_{\oplus_+}$ is not a part of Y-system (1) since in the standard form it would contain the ratio $\frac{1+Y_{23}}{1+1/Y_{32}} = \frac{0}{0}$. It is thus natural that one cannot render this equation local if we only use the finite Y functions (see also [26]). However, in terms of the T-functions appearing in 5 we believe, and partially checked, that Hirota equation 4 is well defined on the full T-shaped fat-hook of Figure 1.

All these equations precisely reproduce the Y-system (1) under the identification

$$\left\{Y_{\bigcirc_{\pm n}}, Y_{\oplus_\pm}, Y_{\otimes_\pm}, Y_{\triangle_{\pm n}}, Y_{\bullet_{\pm n}}\right\} = \left\{Y_{1,\pm n}, Y_{2,\pm 2}, Y_{1,\pm 1}, Y_{n,\pm 1}, Y_{n,0}\right\} \tag{39}$$

mentioned in the previous section!

5. Integral Equations for Excited States

In this section we will consider the non-linear integral TBA-type equations for excited states. For the sake of simplicity, we shall consider only the states in the $SL(2)$ sector, corresponding to operators of the form tr $(D^S Z^J)$ + permutations. Notice that since none of the wings are excited the Y-functions will have the symmetry $Y_{a,s} = Y_{a,-s}$ which also means that $Y_{\otimes_+} = Y_{\otimes_-} \equiv Y_\otimes, \ldots$. To consider such excited states we employ the standard analytic continuation trick [17,18,20] where we pick extra singularities in the convolutions with $Y_{\bullet_1}$ at the points where $Y_{\bullet_1}(u_{4,j}) = -1$. This procedure contains some ambiguities and the result should be considered as a conjecture. In this way, the free energy (11) becomes (2) while the non-linear integral equations of Section 3 are modified by the terms in the square brackets

$$\log Y_\otimes = +K_{m-1} * \log \frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} + \mathcal{R}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) + \\ + \left[\log \frac{R^{(+)}}{R^{(-)}}\right] + \log(-1) \tag{40}$$

$$\log Y_\oplus = -K_{m-1} * \log \frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} - \mathcal{B}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) - \\ - \left[\log \frac{B^{(+)}}{B^{(-)}}\right] - \log(-1) \tag{41}$$

$$\log Y_{\triangle_n} = -K_{n-1,m-1} * \log(1+Y_{\triangle_m}) - K_{n-1} * \log \frac{1+Y_\otimes}{1+1/Y_\oplus} + \\ + \left(\mathcal{R}^{(01)}_{nm} + \mathcal{B}^{(01)}_{n-2,m}\right) * \log(1+Y_{\bullet_m}) + \\ + \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \frac{R^{(+)}(u+ik)}{R^{(-)}(u+ik)} + \sum_{k=-\frac{n-3}{2}}^{\frac{n-3}{2}} \log \frac{B^{(+)}(u+ik)}{B^{(-)}(u+ik)}\right] \tag{42}$$

$$\log Y_{\bigcirc_n} = K_{n-1,m-1} * \log(1+1/Y_{\bigcirc_m}) + K_{n-1} * \log \frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} \tag{43}$$

$$\begin{aligned}\log Y_{\bullet_n} = & L \log \frac{x^{[-n]}}{x^{[+n]}} + \left(2\mathcal{S}_{nm} - \mathcal{R}^{(11)}_{nm} + \mathcal{B}^{(11)}_{nm}\right) * \log(1+Y_{\bullet_m}) + \\ & + \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} i\Phi(u+ik)\right] + 2\left(\mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes}) - \right. \\ & \left. - \mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus}) + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m})\right)\end{aligned} \tag{44}$$

where

$$\Phi(u) = \frac{1}{i} \log\left(S^2 \frac{B^{(+)+} R^{(-)-}}{B^{(-)-} R^{(+)+}} \right). \tag{45}$$

and B and R and S containing the positions of rapidities of the excited states are defined in section 2. These rapidities are constrained by the *exact* Bethe equations

$$Y_{\bullet_1}(u_{4,j}) = -1, \quad j = 1, \ldots, M. \tag{46}$$

In the convolutions involving the fermionic Y-functions $Y_{\oplus}$ and $Y_{\otimes}$ we integrate over $v \in [-2g, 2g]$.[3] We found that prescription to be consistent with the asymptotical large L solution of the Y-system derived in [1]. In fact, as one can see from these integral equations we can think of the two functions $Y_{\oplus}$ and $1/Y_{\otimes}$ as two branches of the same function. In this language the convolutions can be recasted into some nice B-cycle contour integrals in the $x(u)$ Riemann sheet. This is reminiscent of the inversion symmetry in the BS equations which allows one to reduce the seven Bethe equations to a smaller set of five equations [5].

An important check of these equations is the limit where $L \to \infty$. The solution of the Y-system in this limit was constructed explicitly in [1]. We checked numerically that for large L our integral equations are consistent with the large volume solution.

6. Physical and Mirror Choices of Branches

The above system of TBA equations should be valid for any value of the spectral parameter u, and it should be possible to analytically continue it to any point of the Riemann surface of the multi-valued Y-functions. But the choice of branches to formulate the TBA equations can be very important for its good definition and in particular for the future numerical applications. In this section we will fix a particular choice of branches in the kernels involved in the integral equations. This

[3] Another possibility, consistent with the infinite length solution of [1], is to choose $v \in]-\infty, -2g] \cup [2g, \infty[$. We will examine that possibility in detail in the next section. We thank G. Arutyunov and S. Frolov for the correspondence on this issue.

choice will be quite unique, with the following nice properties for the Y-functions and the integration kernels:

- They have only a minimal number of cuts, in general only a pair of cuts, which means that they obey an ordinary fusion procedure where all the intermediate constituents of a bound state but the first and the last cancel.
- They are real functions of the spectral parameter u on the real axis. It fits well their physical meaning in TBA as of the ratios of densities of physical particles and holes.

These properties will stem of course from the similar properties of integration kernels and free terms (with no convolutions) in the TBA equations (40)–(44).

There are two natural possibilities to define $x(u)$ compatible with (6). We define two functions

$$x^{\rm ph}(u)=\frac{1}{2}\left(\frac{u}{g}+\sqrt{\frac{u}{g}-2}\sqrt{\frac{u}{g}+2}\right), \quad x^{\rm mir}(u)=\frac{1}{2}\left(\frac{u}{g}+i\sqrt{4-\frac{u^2}{g^2}}\right). \tag{47}$$

They both solve (6). It is easy to check that with this choice of branches (7) reproduces the physical and mirror dispersion relations, correspondingly [15]. They coincide above the real axes and have the following properties under complex conjugation:

$$\overline{x^{\rm ph}}=x^{\rm ph}, \quad \overline{x^{\rm mir}}=1/x^{\rm mir}. \tag{48}$$

Basically both representations (47) describe the same function, with the same Riemann surface but extended from the upper half plane to the plane with the cut $(-2g,2g)$ for $x^{\rm ph}$, and to the plane with the infinite cut $(-\infty,-2g)\cup(2g,\infty)$ for the function $x^{\rm mir}$. One can say that they are two sections of the same Riemann surface.

We can plot them in Mathematica by running, e.g.,

```
z=a+b I;xmr=1/2(z+I Sqrt[4-z^2]);xph=1/2(z+Sqrt[z-2]Sqrt[z+2]);
Plot3D[{Im[xph],Im[xmr]+0.1},{a,-3,3},{b,-1,1},PlotStyle->{Red,Yellow}]
```

Notice that in the mirror ABA [15] (13) and (14) which we started from, the choice $x^{\rm mir}$ is employed [21,15]. However, for the physical ABA of Beisert and Staudacher [5] we only use the physical choice $x^{\rm ph}$. Thus, to have a link with the ABA in the physical channel one should use the same definition (12) with

$$x_j\equiv x^{\rm ph}(u_j), \quad x_j^{\pm}\equiv x^{\rm ph}(u_j\pm i/2), \tag{49}$$

in various free terms (with no convolutions) in the TBA equations.

On the other hand, since all the kernels in the TBA equations are coming from the mirror theory, both arguments should be in mirror kinematics. Hence we

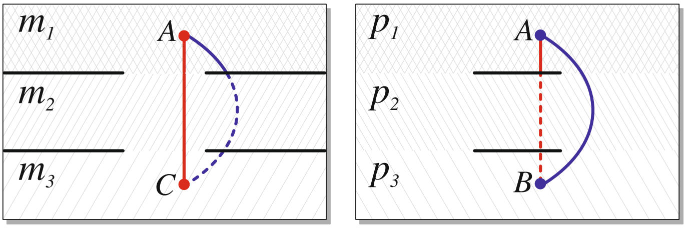

Figure 2. Structure of the cuts and conjugation paths on the mirror and physical sheets.

specify in definitions (24) for the integration kernels the following branches:[4]

$$r(u,v)=\frac{x^{\rm mir}(u)-x^{\rm mir}(v)}{\sqrt{x^{\rm mir}(v)}},\qquad b(u,v)=\frac{1/x^{\rm mir}(u)-x^{\rm mir}(v)}{\sqrt{x^{\rm mir}(v)}}. \tag{50}$$

With this choice of branches, it is easy to check that the kernels $\mathcal{R}^{(ab)}_{nm}$, $\mathcal{B}^{(ab)}_{nm}$ entering our TBA integral equations (40)–(44) are all real! In the next section we show that the kernel involving the dressing factor, $2\mathcal{S}_{nm}$, is real as a consequence of crossing, up to a simple square root factor which we identify there. Moreover, together with $\mathcal{R}^{11}_{nm}-\mathcal{B}^{11}_{nm}$ appearing in (44), it has very simple analytic properties. Namely, it has only four branch points for each of two variables, confirming the nice fusion property announced earlier. We will also present a simple integral representation for this combination.

6.1. REALITY AND CROSSING

One of the important consequences of the crossing for the $SU(2)\times SU(2)$ principal chiral model considered in [13] was the reality of the function Y_0 corresponding to the single middle node in the finite size TBA equations. Here, we show that exactly the same phenomenon is taking place in the present AdS/CFT TBA equations.

Similarly to the Beisert–Eden–Staudacher physical dressing factor, the mirror S-matrix ought to be a pure phase. Let us here explain why this follows indeed in a simple way from the crossing relation for the dressing factor. The same argument can be easily adapted to prove that the leading large volume Y-functions found in [1] are indeed real.

We present schematically the mirror and physical sheets on the Figure 2. They are naturally divided by cuts into three regions denoted by m_1, m_2, m_3 and p_1, p_2, p_3, correspondingly. Since $x^{\rm ph}(u)$ coincides with $x^{\rm mir}(u)$ in the upper half-plane the regions p_1 and m_1 are equivalent, $p_1=m_1$.

Let us consider two points u_A and v_A above the upper cut, i.e., in the region $p_1=m_1$. Conjugation in the mirror sheet sends these points to $u_B\equiv\tilde u_A$ and $v_B\equiv\tilde v_A$

[4] The same branches are used in [31,32].

(belonging to the physical sheet) while conjugation in the physical sheet maps them to $u_C \equiv \bar{u}_A$ and $v_C \equiv \bar{v}_A$ (belonging to the mirror sheet).

Notice that crossing condition relates the dressing factor with argument u_B with the dressing factor at the point u_C. More precisely, we have [3]

$$\sigma(u_B, v_B)\sigma(u_C, v_B) = \frac{y^-}{y^+}\frac{x^- - y^-}{x^+ - y^-}\frac{1/x^- - y^+}{1/x^+ - y^+}, \quad x = x^{\text{ph}}(u_B), \quad y = x^{\text{ph}}(v_B) \tag{51}$$

Notice also that we can now analytically continue both sides of this equality with respect to the v_B root; in particular, we can generate the crossing relation where v_B is replaced by v_C. Using again the (analytically continued) crossing relation to transform v_B into v_C, we get

$$\sigma(u_C, v_C) = \frac{x^- y^+}{x^+ y^-}\sigma(u_B, v_B), \quad x = x^{\text{ph}}(u_B), \quad y = x^{\text{ph}}(v_B) \tag{52}$$

Taking the complex conjugate of this expression and using the fact that the dressing factor is a pure phase on the physical sheet, we get [15]

$$(\sigma(\bar{u}_A, \bar{v}_A))^* = \frac{x^+ y^-}{x^- y^+}\frac{1}{\sigma(u_A, v_A)}, \quad x = x^{\text{mir}}(u_A), \quad y = x^{\text{mir}}(v_A) \tag{53}$$

Notice that we replaced $x^{\text{ph}}(u_A)$ and $x^{\text{ph}}(v_A)$ by their mirror counterparts because A is in the region $p_1 = m_1$. Furthermore, in the left-hand side, we explicitly wrote $u_C = \bar{u}_A$ and $v_C = \bar{v}_A$ to recognize the explicit definition of the conjugated function on the mirror sheet. It is now clear that up to a simple factor of $\sqrt{\frac{x^+ y^-}{x^- y^+}}$ the dressing factor in the mirror theory is indeed a pure phase function. More precisely, the combination $\sqrt{\frac{x^- y^+}{x^+ y^-}}\sigma(u, v)$ is a pure phase in the real axis of the mirror sheet. The same kind of arguments can be used to prove the reality of the large L Y-functions of [1].

6.2. INTEGRAL REPRESENTATION

We will show that the dressing phase on the mirror sheet admits some concise integral representation. Based on that representation we can explicitly see that it has very simple analytical properties. In particular, up to a simple multiplier, namely the simple square root factor identified in the previous section, we can clearly see that this dressing phase is indeed a pure phase function.

6.2.1. *A New Representation of the Dressing Kernel in (mir,mir) Kinematics*

We will start form the DHM integral representation [22] for $\sigma(x^{\text{ph}}(u \pm i/2), x^{\text{ph}}(v \pm i/2))$,

$$\sigma \equiv \exp\left[i\chi^{++} + i\chi^{--} - i\chi^{+-} - i\chi^{-+}\right] \tag{54}$$

where $\chi^{\pm\pm} = \chi(u \pm i/2, v \pm i/2)$,

$$\chi(u,v) = \frac{1}{i} \oint_{|z_1|=1} \frac{dz_1}{2\pi} \oint_{|z_2|=1} \frac{dz_2}{2\pi} \frac{1}{z_1 - x^{\text{ph}}(u)} \frac{1}{z_2 - x^{\text{ph}}(v)} \log \frac{\Gamma(iw_1 - iw_2 + 1)}{\Gamma(iw_2 - iw_1 + 1)} \tag{55}$$

and $w_{1,2} = g(z_{1,2} + 1/z_{1,2})$. This representation is valid for the physical kinematics and in particular for u, v in the region p_1. Since $p_1 = m_1$ we can start with the same expression with $x^{\text{ph}}(u)$ and $x^{\text{ph}}(v)$ replaced by $x^{\text{mir}}(u)$ and $x^{\text{mir}}(v)$ for u and v in the region m_1, above the upper cut.

For the kernel $\mathcal{S}(u,v) \equiv \frac{1}{2\pi i} \partial_v \log \sigma(u,v)$ appearing in our TBA equations we have

$$\mathcal{S}(u,v) = -\int_{-2g}^{2g} \int_{-2g}^{2g} \left(\mathcal{R}^{(10)}(u, w_1 - i0) - \mathcal{B}^{(10)}(u, w_1 + i0) \right) \mathcal{G}(w_1 - w_2) \times \\ \times \left(\mathcal{R}^{(01)}(w_2 - i0, v) - \mathcal{B}^{(01)}(w_2 + i0, v) \right) dw_1 dw_2 \tag{56}$$

where

$$\mathcal{G}(u) \equiv \frac{\partial_u}{2\pi i} \log \frac{\Gamma(1 - iu)}{\Gamma(1 + iu)} = \sum_{a=1}^{\infty} \left(K_{2a} - \frac{1}{a\pi} \right) + \frac{\gamma}{\pi}. \tag{57}$$

Let us briefly recall how to derive this representation for the dressing kernel from the integral representation (55). First, the pole terms $1/(z_1 - x(u))$ and $1/(z_2 - x(v))$ are written as derivatives of log which will give rise to the $\mathcal{B}$s and $\mathcal{R}$s in this expression (in this section we often omit the lower indices of $\mathcal{B}$s and $\mathcal{R}$s in which case they are equal to $\ldots_{11}$). The extra derivative to make a kernel out of the phase can also be transported to the log of gamma function by integration by parts and this generates the function $\mathcal{G}$. The integration contour around the unit circle in the $z_{1,2}$ variable is mapped to an integral from $2g$ to $-2g$ slightly above the real axis and then back from $-2g$ to $2g$ slightly below the real axis for the variable $w_{1,2}$. When w_1 is above the real axis we have $z_1 = x^{\text{ph}}(w_1) = x^{\text{mir}}(w_1)$, but when we are below we have $z_1 = x^{\text{ph}}(w_1) = 1/x^{\text{mir}}(w_1)$. This explains why we get that combination of $\mathcal{R}$s and $\mathcal{B}$s in the last formula.

Now that we have transformed the original contour integrals into usual integrals in the real axis, we can further replace the integration limits in this expression by $\mp\infty$ because for $|w_1| > 2g$ we have $\mathcal{R}^{(10)}(u, w_1 - i0) - \mathcal{B}^{(10)}(u, w_1 + i0) = 0$ and similarly for w_2. Hence, we arrive at the following integral representation for the dressing kernel in the mirror kinematics:

$$\mathcal{S}(u,v) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\mathcal{R}^{(10)}(u, w_1 - i0) - \mathcal{B}^{(10)}(u, w_1 + i0) \right) \mathcal{G}(w_1 - w_2) \times \\ \times \left(\mathcal{R}^{(01)}(w_2 - i0, v) - \mathcal{B}^{(01)}(w_2 + i0, v) \right) dw_1 dw_2. \tag{58}$$

Recall that this expression is derived for u and v in the region m_1. The reason for which we cannot use this integral representation everywhere on the mirror sheet is the presence of poles of $\mathcal{R}$ under the integral at $u=\pm w_1 \pm i/2 - i0$. To get rid of them we use the relations[5] $\mathcal{R}^{(10)} = K_1 - \mathcal{B}^{(10)}$, $\mathcal{R}^{(01)} = K_1 - \mathcal{B}^{(01)}$ and then evaluate the integrals with K_1 by poles using[6]

$$\int K_1(u-w_1)\mathcal{G}(w_1-w_2)\mathcal{B}^{(01)}(w_2+i0,v)\,\mathrm{d}w_1\mathrm{d}w_2 = -\mathcal{B}^{(01)}(u+i/2,v) - \frac{i}{2}\mathcal{P}^{(1)}(v),$$

$$\int \mathcal{B}^{(10)}(u,w_1+i0)\,\mathcal{G}(w_1-w_2)K_1(w_2-v)\mathrm{d}w_1\mathrm{d}w_2 = -\mathcal{B}^{(10)}(u,v+i/2),$$

where

$$\mathcal{P}^{(a)}(v) = -\frac{1}{2\pi}\partial_v \log \frac{x^{\mathrm{mir}}(v+ia/2)}{x^{\mathrm{mir}}(v-ia/2)}. \tag{59}$$

In this way, we get the following representation valid everywhere in m_1, m_2, m_3 for both variables

$$\mathcal{S}(u,v) = -\mathcal{B}^{(11)}(u,v) - \frac{i}{2}\mathcal{P}^{(1)}(v) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\Big(\mathcal{B}^{(10)}(u,w_1+i0) - \mathcal{B}^{(10)}(u,w_1-i0)\Big)\times$$
$$\times\,\mathcal{G}(w_1-w_2)\Big(\mathcal{B}^{(01)}(w_2+i0,v) - \mathcal{B}^{(01)}(w_2-i0,v)\Big)\Big]\mathrm{d}w_1\mathrm{d}w_2. \tag{60}$$

We see that the integrals can be combined to a contour integral around the cuts $(-\infty,-2g)\cup(2g,\infty)$! This implies that we can write the result in a fashion similar to (55). Introducing

$$\hat{\chi}(u,v) \equiv \frac{1}{i}\int_{|z_1|>1}\frac{\mathrm{d}z_1}{2\pi}\int_{|z_2|>1}\frac{\mathrm{d}z_2}{2\pi}\left[\frac{1}{(z_1 - x^{\mathrm{mir}}(u))} - \frac{1}{(z_1 - \overline{x^{\mathrm{mir}}}(u))}\right]\times$$
$$\times\left[\frac{1}{(z_2 - x^{\mathrm{mir}}(v))} - \frac{1}{(z_2 - \overline{x^{\mathrm{mir}}}(v))}\right]\log\frac{\Gamma(iu_1 - iu_2 + 1)}{\Gamma(iu_2 - iu_1 + 1)} \tag{61}$$

with the integration going along the part of the real axes over $(-\infty,-1)\cup(1,\infty)$. Then for the physical dressing factor, analytically continued to the mirror in both variables, we get the following representation

$$\sigma^{m,m}(u,v) = \frac{1-1/(x^-y^+)}{1-1/(x^+y^-)}\,\hat{\sigma}(u,v), \quad \hat{\sigma} \equiv \exp\left[i\hat{\chi}^{++} + i\hat{\chi}^{--} - i\hat{\chi}^{+-} - i\hat{\chi}^{-+}\right], \tag{62}$$

where $x = x^{\mathrm{mir}}(u)$, $y = x^{\mathrm{mir}}(v)$ and $\hat{\chi}^{\pm\pm} = \hat{\chi}(u \pm i/2, v \pm i/2)$. We see that the second factor $\hat{\sigma}$ has the same properties under the fusion procedure on the mirror sheet as

[5] It is often useful to change from $\mathcal{R}$s to $\mathcal{B}$s because the latter are much more regular than the former. In particular, since $\mathrm{Im}\,x^{\mathrm{mir}}(u), \mathrm{Im}\,x^{\mathrm{mir}}(v) > 0$ we can never have $1/x^{\mathrm{mir}}(u) = x^{\mathrm{mir}}(v)$ and thus $\mathcal{B}$ will be pole free when both variables are taken in the mirror sheet. Similarly, in the physical sheet, $|x^{\mathrm{ph}}(u)|, |x^{\mathrm{ph}}(v)| > 1$ and again $\mathcal{B}$ is regular. Only when u and v are in different kinematics we should worry about regularity of the $\mathcal{B}$ functions.

[6] In these formulae, w is the variable being integrated over in the last convolution.

the physical dressing phase σ had on the physical sheet – one simply replaces shifts by $\pm i/2$ by $\pm in/2$ for u and by $\pm im/2$ for v in $\hat{\chi}$. Note that $\hat{\chi}$ is a real function and thus $\hat{\sigma}$ is a pure phase. Thus, $\hat{\sigma}(u,v)$ is nothing but the dressing phase of the mirror theory!

Finally, let us present yet another interesting representation of the dressing phase in the mirror kinematics. It is easy to see that $\mathcal{R}^{(10)}(u,w)$ and $\mathcal{R}^{(01)}(w,v)$, as functions of w, are regular below the real axis. Moreover, $\mathcal{B}^{(10)}(u,w)$ and $\mathcal{B}^{(01)}(w,v)$ are regular on the whole complex plane except for the Zhukoswky cuts (see previous footnote). That implies that the terms with $\mathcal{BB}$ and $\mathcal{RR}$ in (58) vanish because for those terms we can deform the integration contour to $+i\infty$ and $-i\infty$, correspondingly. For the remaining terms, the integration with $\mathcal{G}$ can be done explicitly to yield

$$2\mathcal{S}_{nm}(u,v)-\mathcal{R}^{(11)}_{nm}(u,v)+\mathcal{B}^{(11)}_{nm}(u,v)=-\mathcal{K}_{n,m}(u-v)-\frac{i}{2}\mathcal{P}^{(m)}(v)-$$
$$-2\sum_{a=1}\int\Big[\mathcal{B}^{(10)}_{n1}(u,w+ia/2)\,\mathcal{B}^{(01)}_{1m}(w-ia/2,v)+$$
$$+\mathcal{B}^{(10)}_{n1}(u,w-ia/2)\,\mathcal{B}^{(01)}_{1m}(w+ia/2,v)\Big]\mathrm{d}w \tag{63}$$

where we wrote the result already after fusion, i.e., for the dressing factor between magnon bound states n and m. Quite remarkably this combination of kernels, which is precisely the one appearing in the TBA equations contains no cuts apart from those at $\mathrm{Im}(u)=\pm n/2$ in the u plane and $\mathrm{Im}(v)=\pm m/2$ for the v variable, precisely as expected. This property was also noticed independently in [33].

6.2.2. *A New Representation of the Dressing Kernel in the (mir,ph) Kinematics*

In this section, we analyze the dressing kernel when $\sigma(u,v)$ when the first variable u takes values in the mirror sheet while the second variable v lives in the physical sheet. This is precisely the case for the free terms (without convolutions) in the TBA equations. For example, in (43) the term Φ contains $S(u)=\prod_j\sigma(u,u_{4,j})$ where $u_{4,j}$ are the Bethe roots of the physical theory while u is in the mirror kinematics. The derivation of a nice integral representation for this dressing factor goes along the same lines as in the previous section. We find

$$\log S(u)=-\Big[\mathcal{B}^{(10)}(u,w+i0)-\mathcal{R}^{(10)}(u,w-i0)\Big]*\mathcal{G}*$$
$$*\left[\log\frac{B^{(+)}(u+i0)}{B^{(-)}(u+i0)}-\log\frac{R^{(+)}(u-i0)}{R^{(-)}(u-i0)}\right] \tag{64}$$

where R and B are defined like in (12) with $x(u)=x^{\mathrm{mir}}(u)$ and $x_j^{\pm}=x^{\mathrm{ph}}(u_j\pm i/1)$. As in the previous section, this relation is derived in the region $m_1=p_1$, and the next step is to transform this expression in such a way that it allows for a trivial analytical continuation to the full mirror sheet for the u variable. Actually the r.h.s.

is not singular in m_1, m_2 (but not in m_3) and thus should coincide with analytical continuation of the dressing factor. Next we recall that

$$B^{(+)}(u_j - i/2) = 0, \quad R^{(-)}(u_j + i/2) = 0 \tag{65}$$

while $\log B^{(-)}$ and $\log R^{(+)}$ are regular in m_1, m_2, m_3. Assuming u to be real we can again drop BB and RR terms and convert $\mathcal{R}$ to $\mathcal{B}$ as in the previous section to obtain

$$\log S(u) = \mathcal{B}^{(10)}(u, w + i0) * \mathcal{G} * \log \frac{R^{(+)}(u - i0)}{R^{(-)}(u - i0)} - \\ -\mathcal{B}^{(10)}(u, w - i0) * \mathcal{G} * \log \frac{B^{(+)}(u + i0)}{B^{(-)}(u + i0)} + K_1 * \mathcal{G} * \log \frac{B^{(+)}(u + i0)}{B^{(-)}(u + i0)} \tag{66}$$

The last term can be computed explicitly, $K_1 * \mathcal{G} * \log \frac{B^{(+)}(u+i0)}{B^{(-)}(u+i0)} = -\log \frac{B^{(+)}(u+i/2)}{B^{(-)}(u+i/2)} - \sum_j \frac{1}{2} \log \frac{x_j^+}{x_j^-}$, and in this way we obtain the following integral representation valid in the full mirror sheet m_1, m_2, and m_3

$$\log S = \log \frac{B^{(-)+}}{B^{(+)+}} + \sum_j \frac{1}{2} \log \frac{x_j^+}{x_j^-} + \\ + \left(\mathcal{B}^{(10)}(u, w + i0) * \mathcal{G} * \log \frac{R^{(+)}(u - i0)}{R^{(-)}(u - i0)} + \right. \\ \left. + \mathcal{B}^{(10)}(u, w - i0) * \mathcal{G} * \log \frac{B^{(-)}(u + i0)}{B^{(+)}(u + i0)} \right) \tag{67}$$

Fusion is again trivial and yields

$$\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \log S(u + ik) = \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \log \frac{B^{(-)}(u + i/2 + ik)}{B^{(+)}(u + i/2 + ik)} + \sum_j \frac{1}{2} \log \frac{x_j^{[+m]}}{x_j^{[-m]}} + \\ + \left(\mathcal{B}_{m1}^{(10)}(u, w + i0) * \mathcal{G} * \log \frac{R^{(+)}(u - i0)}{R^{(-)}(u - i0)} + \right. \\ \left. + \mathcal{B}_{m1}^{(10)}(u, w - i0) * \mathcal{G} * \log \frac{B^{(-)}(u + i0)}{B^{(+)}(u + i0)} \right). \tag{68}$$

Using the same arguments as in the previous section we could explicitly eliminate $\mathcal{G}$ and one of the convolutions in this representation at the expense of introducing an extra infinite sum over a. In this way we could derive an alternative integral representation very similar to that in (63).

In the next section, the reality property is discussed in further detail and in particular we explain why the Y-functions which solve our integral equations are indeed real.

6.3. REALITY AND ANALYTICITY PROPERTIES OF *Y*S

We can easily check using the explicit large L solution for the Y-functions presented in [1] together with the explicit representation of the dressing kernel derived in the previous section that *all* Y-functions are real when u is in the real axis.[7] To understand that this property actually holds for the Y-functions even at finite L we should study the reality of several kernels in the TBA equations and also the reality of the free term (without convolutions). If both are real then the exact finite L solution for Y-functions will be also real. The reality property is of a particular interest for the future numerical applications of our equations which can be now done by iterations starting from the known large L solution.

The most complicated equation to analyze is the one for the middle node, Equation (44), which contains the dressing factor in the (fused) kernel and in the free term. We will focus now only on this equation since the reality of all other equations can be checked trivially. Let us explicit in (44) only the "*dangerous*" terms:

$$\log Y_{\bullet_n} = 2\mathcal{S}_{nm} * \log(1+Y_{\bullet_m}) + \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} i\Phi(u+ik) + \\ + 2\mathcal{R}_{n2}^{(10)} * \log(1+Y_{\square_2}) + \dots$$

where the … stand for the rest of the terms, which are explicitly real. The reason why we also kept the last term as *dangerous* (i.e. potentially not real) will become clear in the following:

Inside the kernel $\mathcal{S}_{nm}$ the only non-real contribution comes from the square root of $-\frac{i}{2}\mathcal{P}^{(m)}$ in (63) and the dangerous terms coming from the fusion of Φ are those in the first line of (68) so that we can re-write the *dangerous* terms in the r.h.s of (44) as[8]

$$\log Y_{\bullet_n} = -i\mathcal{P}^{(m)} * \log(1+Y_{\bullet_m}) + \sum_j \log\frac{x_j^+}{x_j^-} + \log\left[\frac{B^{(-)}(u+\frac{in}{2})}{B^{(-)}(u-\frac{in}{2})}\right]^2 \times \\ \times \frac{Q(u-i\frac{n+1}{2})Q(u-i\frac{n-1}{2})}{Q(u+i\frac{n+1}{2})Q(u+i\frac{n-1}{2})} + 2\mathcal{R}_{n2}^{(10)} * \log(1+Y_{\square_2}) + \cdots \quad (69)$$

---

[7] For the Y-functions Y_{11} and Y_{22} associated to the fermionic roots this property is true for $|u| < 2g$.

[8] To simplify the first line in (68) we use the identity

$$\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log\left[\frac{B^{(-)}(u+i/2+ik)}{B^{(+)}(u+i/2+ik)}\right]^2 \frac{B^{(+)}(u+i/2+ik)R^{(-)}(u-i/2+ik)}{B^{(-)}(u-i/2+ik)R^{(+)}(u+i/2+ik)} = \\ = \log\left[\frac{B^{(-)}(u+\frac{in}{2})}{B^{(-)}(u-\frac{in}{2})}\right]^2 \frac{Q(u-i\frac{n+1}{2})Q(u-i\frac{n-1}{2})}{Q(u+i\frac{n+1}{2})Q(u+i\frac{n-1}{2})}$$

Now we notice that the first line is nothing but the total corrected momentum (compare with the expression (2) for the corrected energy) which should vanish due to the string theory level matching constraint![9] Thus, the only danger stems from both terms in the second line: in fact, they are not real (even though the kernel $\mathcal{R}_{n2}^{(10)}$ is real), but their combination will be shown to be real.

The reason for the second term to be not real is that the function $Y_{\square_2}$ contains pole divergencies on the real axis located precisely at the positions of the Bethe roots u_j. This can be seen from the free terms (containing no convolutions) in the TBA equation (42). We see that iff $n=2$, then we do get singularities in the real axis coming from the zeros of $R^{(-)}(u+i/2)=0$ which are precisely the Bethe roots u_j. The zeros of this function induce, via this integral equation, the poles in $Y_{\square_2}$.[10] Analyzing all other TBA equations in a similar way we can easily see that no other free terms give rise to poles in the real axis for any other Y-function.

Now let us explain why the second line in (69) is explicitly real. The convolution in the presence of these poles should be understood as $2\mathcal{R}_{n2}^{(10)} * \log(1+Y_{\square_2}(u-i0))$ which we can rewrite as a principal value integral (which will be of course explicitly real) plus half of each residue of the singularities at the Bethe roots, i.e.,[11]

$$2\mathcal{R}_{n2}^{(10)} * \log(1+Y_{\square_2}(u-i0)) = 2\mathcal{R}_{n2}^{(10)} *_{\text{p.v.}} \log(1+Y_{\square_2}) + \\ + \log \frac{R^{(-)}(u+\frac{in}{2})B^{(+)}(u+\frac{in}{2})}{R^{(-)}(u-\frac{in}{2})B^{(+)}(u-\frac{in}{2})}$$

The last term in this expression is not real. However, it can be easily seen that it combines with the first term in the second line of (69) to give a real contribution!

This concludes our check of reality of all the kernels and free terms in all TBA equations. The reality means that the Y-functions solving these equations will be real, at least on some stretch of the real u-axis.

7. Discussion and Conclusions

The integral equations we present are suitable for the numerical study. In the large L limit we can drop all convolutions containing the black nodes $Y_{\bullet_n}$ and recover in this way the large L solutions of [1] (we also checked this statement numerically). However, compared with the Y-system equation in functional form these equations are of easy numerical implementation, and the iteration from the large L solution to the finite L case is now accessible. This numerical approach is currently under investigation.

---

[9]The gauge theory analog of level matching is the cyclicity of the trace in the definition of local gauge invariant operators.

[10]For $n>2$ we also have poles for the corresponding Y-functions but they will lie away from the real axis.

[11]Notice that the residues at these singularities depend only on the prefactor of the log since when integrating by parts we get a log derivative which has always unit residue.

In conclusion, we derived in this paper the system of non-linear integral equations of the TBA type, describing, in principle, the spectrum of the states/operators in the full planar AdS/CFT system, including the low lying ones, such as Konishi operator. Not only do these equations confirm our Y-system conjectured in [1], but they also give a practical way to the numerical calculation of the anomalous dimensions as functions of the coupling λ. An alternative, usually numerically quite efficient, would be the derivation of the Destri–DeVega type equations along the guidelines presented in [13] for the $SU(2)$ principal chiral field. In any case, a better understanding of the analytical structure of these equations is needed for the efficient numerics.

A point which we do not completely understand in detail concerns the role of the so-called μ-term contributions in the TBA equations. In particular, we might need to pick extra contributions in (2) coming from further singularities which might arise in the $Y_{n,0}$ functions. In the large L limit such extra terms could probably be identified with the Lüscher's μ term contributions. The role of these extra contributions, if they are present at all, needs to be further elucidated.

One more unclear point concerns the underlying $PSU(2,2|4)$ symmetry of the problem. In our approach, the starting point is the string theory in the light cone gauge where this symmetry is broken to $SU(2|2)^2$. It would be extremely interesting to understand how the full superconformal symmetry emerges in the TBA equations.

Interesting questions yet to be considered concern the derivation of a full set of finite size Bethe equations for *any* type of excitations of the theory, again along the lines of [13] as well as the generalization of these TBA equations to another integrable example of the AdS/CFT correspondence, the ABJM duality [27] (see [28–30] and references therein for the integrability related works on this theory).

The set of TBA equations derived here should give us access to the full spectrum of AdS/CFT for any coupling. Hopefully, it will help to understand deep physical reasons of the integrability of $N=4$ SYM theory. Knowing the exact results always helps understanding physics.

Note Added

After the work on this project was already finished the paper [31] appeared where essentially the similar equations for the vacuum have been derived except the corner, fermionic nodes $Y_{2,\pm 2}$. The corresponding equation 5.71 proposed in [31] appears to be incorrect. We derive here the correct equation and also propose the TBA equations for the excited states.[12]

[12] In the preprint arXiv:0902.3930v2 of [31], which appeared after our preprint arXiv:0902.4458v1 of the current paper, the Eq. 5.16 (Equation 5.71 in arXiv:0902.3930v1), associated with $Y_{2,2}$, was indeed recognized to be incorrect.

Note Added for preprint arXiv:0902.4458v3

When we were preparing a paper with the results of the new section 6, the second version of the paper arXiv:0904.4575 [33] appeared where a part of our new results, concerning the fusion properties for the mirror dressing factor, was established. We decided to update our old paper with these new results. We also stated more explicitly our preferable chose of the contours and branches in the integral equations. It is consistent with [21,15] and agrees with that of [31,32]. We also show that for that choice the Y-functions for excited states have particulary nice analytic properties and are real.

Acknowledgements

The work of NG was partly supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 and RFFI project grant 06-02-16786 and the grant RFFI 08-02-00287. This research was supported in part by the National Science Foundation under Grand No. NSF PHY05-51164. The work of VK was partly supported by the ANR grants INT-AdS/CFT (ANR36ADS-CSTZ) and GranMA (BLAN-08-1-313695) and the grant RFFI 08-02-00287. We thank N.Beisert, M.Staudacher and Z.Tsuboi for discussions. We acknowledge discussions with G. Arutyunov and S.Frolov about the integrations contours in the integral equations. VK and PV thank the Kavli Institute for Theoretical Physics of Santa Barbara University, where a part of the work was done, for the kind hospitality.

References

1. Gromov, N., Kazakov, V., Vieira, P.: Integrability for the Full Spectrum of Planar AdS/CFT. arXiv:0901.3753 [hep-th]
2. Kazakov, V., Sorin, A., Zabrodin, A.: Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics. Nucl. Phys. B **790**, 345 (2008). arXiv:hep-th/0703147
3. Janik, R.A.: The $AdS(5) \times S^5$ superstring worldsheet S-matrix and crossing symmetry. Phys. Rev. D **73**, 086006 (2006). hep-th/0603038
4. Staudacher, M.: The factorized S-matrix of CFT/AdS. JHEP **0505**, 054 (2005). arXiv:hep-th/0412188
5. Beisert, N., Staudacher, M.: Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings. Nucl. Phys. B **727**, 1–62 (2005) hep-th/0504190
6. Beisert, N.: The $SU(2|2)$ dynamic S-matrix. Adv. Theor. Math. Phys. **12**, 945 (2008). arXiv:hep-th/0511082
7. Beisert, N., Eden, B., Staudacher, M.: Transcendentality and crossing. J. Stat. Mech. **0701**, P021 (2007). hep-th/0610251
8. Bajnok, Z., Janik, R.A.: Four-loop perturbative Konishi from strings and finite size effects for multiparticle states. Nucl. Phys. B **807**, 625 (2009)
9. Bajnok, Z., Janik, R.A., Lukowski, T.: Four loop twist two, BFKL, wrapping and strings. arXiv:0811.4448 [hep-th]

10. Fiamberti, F., Santambrogio, A., Sieg, C., Zanon, D.: Anomalous dimension with wrapping at four loops in $N=4$ SYM. Nucl. Phys. B **805**, 231 (2008)
11. Velizhanin, V.N.: Leading transcedentality contributions to the four-loop universal anomalous dimension in $N=4$ SYM. arXiv:0811.0607 [hep-th]
12. Zamolodchikov, Al.B.: On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories. Phys. Lett. B **253**, 391 (1991)
13. Gromov, N., Kazakov, V., Vieira, P.: Finite Volume Spectrum of 2D Field Theories from Hirota Dynamics. arXiv:0812.5091 [hep-th]
14. Arutyunov, G., Frolov, S.: Foundations of the AdS_5xS^5 Superstring. Part I. arXiv:0901.4937 [hep-th]
15. Arutyunov, G., Frolov, S.: On String S-matrix, Bound States and TBA. JHEP **0712**, 024 (2007). arXiv:0710.1568 [hep-th]
16. Dorey, N.: Magnon bound states and the AdS/CFT correspondence. J. Phys. A **39**, 13119 (2006). arXiv:hep-th/0604175
17. Dorey, P., Tateo, R.: Excited states by analytic continuation of TBA equations. Nucl. Phys. B **482**, 639 (1996). arXiv:hep-th/9607167
18. Bazhanov, V.V., Lukyanov, S.L., Zamolodchikov, A.B.: Quantum field theories in finite volume: Excited state energies. Nucl. Phys. B **489**, 487 (1997). arXiv:hep-th/9607099
19. Fioravanti D., Mariottini A., Quattrini E., Ravanini F. (1997) Phys. Lett. B **390**, 243 (1997). arXiv:hep-th/9608091
20. Teschner, J.: On the spectrum of the Sinh-Gordon model in finite volume. Nucl. Phys. B **799**, 403 (2008). arXiv:hep-th/0702214
21. Ambjorn, J., Janik, R.A., Kristjansen, C.: Wrapping interactions and a new source of corrections to the spin-chain/string duality. Nucl. Phys. B **736**, 288–301 (2006). hep-th/0510171
22. Dorey, N., Hofman, D.M., Maldacena, J.M.: Phys. Rev. D **76**, 025011 (2007). arXiv: hep-th/0703104
23. Takahashi: Thermodynamics of One-Dimensional Solvable Models. Cambridge University Press, Cambridge (1999)
24. Essler, F.H.L., Frahm, H., Göhmann, F., Klümper, A., Korepin, V.: The One-Dimensional Hubbard Model. Cambridge University Press, Cambridge (2005)
25. Arutyunov, G., Frolov, S.: String hypothesis for the AdS_5xS^5 mirror. arXiv:0901.1417 [hep-th]
26. Juttner, G., Klumper, A., Suzuki, J.: From fusion hierarchy to excited state TBA. Nucl. Phys. B **512**, 581 (1998). arXiv:hep-th/9707074
27. Aharony, O., Bergman, O., Jafferis, D.L., Maldacena, J.: $N=6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals. JHEP **0810**, 091 (2008). [arXiv:0806.1218 [hep-th]]
28. Minahan, J.A., Zarembo, K.: The Bethe ansatz for superconformal Chern–Simons. JHEP **0809**, 040 (2008)
29. Gromov, N., Vieira, P.: The AdS4/CFT3 algebraic curve. arXiv:0807.0437 [hep-th]
30. Gromov, N., Vieira, P.: The all loop AdS4/CFT3 Bethe ansatz. arXiv:0807.0777 [hep-th]
31. Bombardelli, D., Fioravanti, D., Tateo, R.: Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal. arXiv:0902.3930 [hep-th]
32. Arutyunov, G., Frolov, S.: Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ mirror model. JHEP **0905**, 068 (2009). arXiv:0903.0141 [hep-th]
33. Arutyunov, G., Frolov, S.: The Dressing Factor and Crossing Equations. arXiv:0904.4575 [hep-th]