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Chord-arc constants for submanifolds of arbitrary codimension

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Abstract. In this article we show that for k-dimensional submanifolds of $\mathbb{R}^n$ which go through infinity in a smooth way, smallness of the Gromov distortion and some Ahlfors regularity is equivalent to smallness of the BMO norm of the unit normal and globally δ-Reifenberg flatness with small δ. This generalizes a result due to Semmes for hypersurfaces to surfaces of arbitrary codimension.

Keywords. Submanifolds, chord-arc constant, BMO norm.

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1 Introduction

In 1991 Stephen Semmes published three articles [31, 32, 33] in which he extended the well-known chord-arc condition for curves to hypersurfaces of the Euclidean space. These articles had a deep impact in various fields of mathematics like the study of harmonic measures and the regularity of free boundaries (cf. [21, 22, 23, 24, 6, 20]) or in the search for a sufficient criterion for the existence of bi-Lipschitz parametrizations of two-dimensional manifolds (cf. [35, 13, 4]).

In the present work, we extend the definitions of Semmes' constants to submanifolds of arbitrary codimension and prove that the statement of the main theorem in [31] still holds, i.e. that all of the these constants are small if only one of them is sufficiently small.

Semmes considered complete, connected, and embedded C^2 hypersurfaces $\Gamma \subset \mathbb{R}^n$ without boundary. Furthermore, he assumed that $\Gamma \cup \{\infty\}$ is a C^2 hypersurface of $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^n$. Among other things, this guarantees that Γ goes through infinity and that Γ is an orientable manifold that divides the ambient space $\mathbb{R}^n$ into two connected components Ω_+ and Ω_- . Semmes extended the definition of the chord-arc constant of curves to hypersurfaces by setting

$$\tilde{\eta}(\Gamma) := \max \left\{ \sup_{x \neq y \in \Gamma} \left| \frac{d_\Gamma(x,y)}{|x-y|} - 1 \right|, \sup_{x \in \Gamma, R>0} \left| \frac{\mathcal{H}^{n-1}(\Gamma \cap B_R(x))}{\omega_{n-1} R^{n-1}} - 1 \right| \right\},$$

where d_Γ is the geodesic distance on Γ, $\mathcal{H}^k$ the k-dimensional Hausdorff measure, and ω_k denotes the volume of a k-dimensional ball with radius one. Furthermore, he defined

$$\tilde{\gamma}(\Gamma) := \max\left\{ \sup_{x\in\Gamma, R>0} \frac{1}{\mathcal{H}^{n-1}(\Gamma\cap B_R(x))} \int_{\Gamma\cap B_R(x)} |\nu(z) - \nu_{B_R(x)}|\, d\mathcal{H}^{n-1}(z), \right.$$
$$\left. \sup_{x\in\Gamma, R>0} \left(\sup_{y\in\Gamma\cap B_R(x)} \left| \frac{\langle x-y, \nu_{B_R(x)}\rangle}{R} \right| \right) \right\},$$

where ν denotes the unit normal and

$$\nu_{B_R(x)} := \frac{1}{\mathcal{H}^{n-1}(\Gamma\cap B_R(x))} \int_{\Gamma\cap B_R(x)} \nu(z)\, d\mathcal{H}^{n-1}(z).$$

So $\tilde{\gamma}$ controls the BMO norm of the unit normal and contains some flatness condition. Finally, Semmes introduced two other constants $\alpha(\Gamma)$ and $\beta(\Gamma)$ that reflect the boundary behavior of Clifford holomorphic functions on Ω_+ and Ω_- (cf. [31, p. 200] for more details). His main theorem in this context is that all four constants $\alpha(\Gamma)$, $\beta(\Gamma)$, $\tilde{\gamma}(\Gamma)$, and $\tilde{\eta}(\Gamma)$ are small if any of them is sufficiently small. Thus, he proved analogs to some of the well-known relations between the chord-arc constant for curves, the geometry of and the operator theory on such curves, and function theory on the corresponding chord-arc domains (cf. [27, 36, 7, 10, 19, 29, 30]).

For curves, the constant $\eta_1(\Gamma) + 1$ is known as *Gromov distortion* and the quantity $\eta_1(\Gamma)$ is referred to as *chord-arc constant* or *Lavrent'ev constant*. It plays a major role in the context of boundary regularity of of minimal surfaces [17, 12, Kapitel 7.5], minima of Cartan functionals [18], and geometric knot theory [14, 15, 16, 26, 11, 1].

In the present work, we consider k-dimensional complete, connected, and embedded C^1 submanifolds $\Gamma \subset \mathbb{R}^n$ without boundary such that $\Gamma \cup \{\infty\}$ is a k-dimensional C^1 submanifold of $\mathbb{R}^n \cup \{\infty\} \cong \mathbb{S}^n$. Let us call such objects *k-dimensional chord-arc submanifolds* or *k-dimensional knots with ends at infinity*. More precisely, we will assume that $P_N(\Gamma) \cup \{e_{n+1}\}$ is a k-dimensional, compact, and connected submanifold of $\mathbb{S}^n$ without boundary. Here,

$$P_N : \mathbb{R}^n \to \mathbb{S}^n - \{e_{n+1}\}, \quad x \mapsto \frac{4}{|x|^2+4} \cdot (x, -2) + e_{n+1} \tag{1.1}$$

is the inverse of the stereographic projection, and $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. Note, that we do *not* assume a priori that these submanifolds are orientable or that anything else is known about the topology of these objects.

We do not have a chance to generalize the definition of α and β to submanifolds of codimension greater than one since such submanifolds do not partition $\mathbb{R}^n$ into two connected components Ω_+ and Ω_-. So we concentrate our effort on generalizing the constants $\tilde{\eta}$ and $\tilde{\gamma}$ to quantities defined on chord-arc submanifolds of arbitrary codimension. The straightforward generalization of $\tilde{\eta}$ is given by

$$\eta_1(\Gamma) := \sup\left\{\frac{d_\Gamma(x,y)}{|x-y|} - 1 : x, y \in \Gamma, x \neq y\right\}, \tag{1.2}$$

$$\eta_2(\Gamma) := \sup\left\{\left|\frac{\mathcal{H}^k(\Gamma \cap K_R(x))}{\omega_k R^k} - 1\right| : x \in \Gamma, R > 0\right\}, \tag{1.3}$$

and

$$\eta(\Gamma) := \max\{\eta_1(\Gamma), \eta_2(\Gamma)\}. \tag{1.4}$$

Here, $K_R(x)$ is the closed ball around x with radius r.

For the generalization of $\tilde{\gamma}$, let $G_{i,j}$ be the set of all orthogonal projections of $\mathbb{R}^i$ onto j-dimensional subspaces of $\mathbb{R}^i$ and let

$$N : \Gamma \to G_{n,k}$$

map points $x \in \Gamma$ to the orthogonal projection of $\mathbb{R}^n$ onto the normal space at x and $T(x) := \mathrm{id}_{\mathbb{R}^n} - N(x)$ be the projection onto the tangent space. By $\mathfrak{N}_{x,R} \subset G_{n,n-k}$ we denote the set of all $N_{x,R} \in G_{n,n-k}$ which satisfy

$$\int_{\Gamma \cap K_R(x)} \|N(y) - N_{x,R}\| \, d\mathcal{H}^k(y) = \inf_{S \in G_{n,n-k}} \left\{\int_{\Gamma \cap K_R(x)} \|N(y) - S\| \, d\mathcal{H}^k(y)\right\}$$

and $\mathfrak{T}_{x,R} := \{\mathrm{id}_{\mathbb{R}^n} - N_{x,R} : N_{x,R} \in \mathfrak{N}_{x,R}\}$. Then we set

$$\gamma_1(\Gamma) := \sup_{\substack{x\in\Gamma\\ R>0}} \left\{\sup_{N_{x,R}\in\mathfrak{N}_{x,R}} \frac{\int_{\Gamma\cap K_R(x)} \|N(y) - N_{x,R}\| \, d\mathcal{H}^k(y)}{\mathcal{H}^k(\Gamma \cap K_R(x))}\right\}, \tag{1.5}$$

$$\gamma_2(\Gamma) := \sup_{x\in\Gamma,\ R>0} \left\{\sup_{y\in K_R(x)\cap\Gamma, N_{x,R}\in\mathfrak{N}_{x,R}} \frac{|N_{x,R}(x-y)|}{R}\right\}, \tag{1.6}$$

and

$$\gamma(\Gamma) := \max(\gamma_1(\Gamma), \gamma_2(\Gamma)). \tag{1.7}$$

Since an integral mean of the function N does not necessarily correspond to a k-dimensional subspace of $\mathbb{R}^n$ as the Grassmannian $G_{n,k}$ is not convex, we exchanged it by an element of $\mathfrak{N}_{x,r}$ in the definition of γ. Nevertheless, we will see in the next section that γ_1 can be estimated from above and below by the BMO norm of the unit normal.

The main result of this article is the following generalization of Semmes' result for hypersurfaces in [31]:

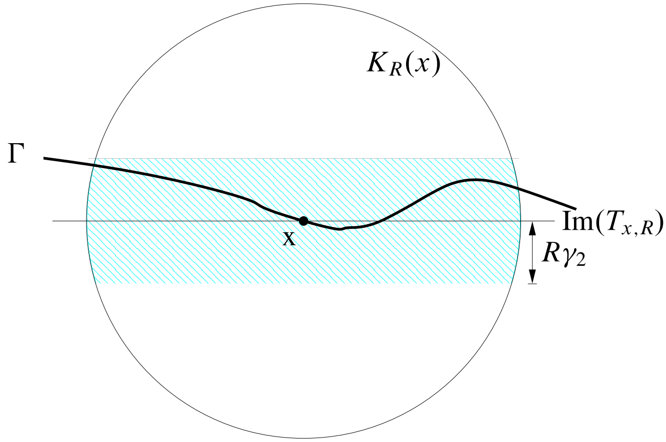

Figure 1. The constant $\gamma_2(\Gamma)$ guarantees that for every $x \in \Gamma$ and every $R > 0$ the distance between a point in $\Gamma \cap K_R(x)$ and the affine space $x + \mathrm{Im}(T_{x,R})$ is less or equal to $R\gamma_2(\Gamma)$.

Theorem 1.1. *(i) There are constants $\varepsilon = \varepsilon(n,k) > 0$ and $C = C(n,k) < \infty$ such that every k-dimensional chord-arc submanifold $\Gamma \subset \mathbb{R}^n$ with $\gamma(\Gamma) \leq \varepsilon$ satisfies*

$$\eta(\Gamma) \leq C\gamma(\Gamma)\log\left(\frac{1}{\gamma(\Gamma)}\right).$$

(ii) There are constants $\varepsilon = \varepsilon(n,k) > 0$ and $C = C(n,k) < \infty$ such that for every k-dimensional chord-arc submanifold $\Gamma \subset \mathbb{R}^n$ the inequality $\eta(\Gamma) \leq \varepsilon$ implies

$$\gamma(\Gamma) \leq C\eta(\Gamma)^{\frac{1}{2}}.$$

Semmes has shown in [33] that for submanifolds of codimension one the smallness of $\gamma_1(\Gamma)$ implies $\gamma_2(\Gamma) \leq C\gamma_1(\Gamma)$ where the constant C only depends on the dimension of the manifold. So for submanifolds of codimension one, one can exchange $\gamma(\Gamma)$ by $\gamma_1(\Gamma)$ in the first part of Theorem 1.1. Except for curves in Euclidean space, it is not known whether this is true for submanifolds of higher codimension. Whether smallness of $\eta_1(\Gamma)$ might imply smallness of $\eta_2(\Gamma)$ for manifolds of dimension greater or equal to two is completely open as well, while the converse is certainly not true.

The main tool in the proof of the first part of Theorem 1.1 is that chord-arc submanifolds with small constants $\gamma(\Gamma)$ contain big portions of C^1 graphs with explicit control over their Lipschitz constant (cf. Theorem 3.1). We show that – except for a small bad set – the part of such a k-dimensional submanifold inside of a ball is contained in the graph of a C^1 function. Semmes only obtains Lipschitz graphs for $k = n - 1$. Using Lusin's Theorem and some extension theorem one can then move on to get C^1 graphs.

But doing so, one would lose the precise characterization of the part of the manifold on the graph of the C^1 function given in the statement of Theorem 3.1.

A set $A \subset \mathbb{R}^n$ is called *globally δ-Reifenberg flat* if and only if for every $x \in A$ and every $R > 0$ there is a k-dimensional linear subspace $L_{x,R} \subset \mathbb{R}^n$ such that

$$d_{\mathcal{H}}(A \cap B_R(x), (L_{x,R} + x) \cap B_R(x)) \le R\delta.$$

Here, $d_{\mathcal{H}}$ denotes the Hausdorff distance between sets. After the proof of Theorem 3.1, we will see that smallness of γ implies global Reifenberg flatness with small δ (cf. Corollary 3.4). Thus we derive the following corollary from Theorem 1.1:

Corollary 1.2. *For every $\delta > 0$ there is a constant $\varepsilon = \varepsilon(n, k, \delta) > 0$ such that the following holds:*

If $\Gamma \subset \mathbb{R}^n$ is a k-dimensional knot with ends at infinity and $\eta(\Gamma) < \varepsilon$, then Γ is globally δ-Reifenberg flat.

In [3], Corollary 1.2 is used to show that k-dimensional knots with ends at infinity are diffeomorphic to spheres and unknotted if the constant η is small. This extends a corresponding results in [11] and [1] for curves in $\mathbb{R}^3$ to submanifolds of arbitrary dimension and codimension.

Comparing γ with $\tilde{\gamma}$ in the case of hypersurfaces Γ, one trivially has $\gamma \le 2\tilde{\gamma}$, while it is not obvious that constant $\tilde{\gamma}$ is small if γ is small, since the new constant γ does not take the orientation of the normal into account. For instance, let $\Gamma \cap K_1(0)$ consist of two parallel hyperplanes near to the origin but such that the unit normal ν on these planes point in opposite directions. Then we get

$$\frac{1}{\mathcal{H}^{n-1}(\Gamma \cap B_1(0))} \int_{\Gamma \cap B_1(0)} |\nu - \nu_{B_1(0)}| \, d\mathcal{H}^{n-1} \cong 1$$

which enters the definition of Semmes' constant $\tilde{\gamma}$ while

$$\frac{1}{\mathcal{H}^{n-1}(\Gamma \cap K_1(0))} \int_{\Gamma \cap K_1(0)} \|N - N_{0,1}\| \, d\mathcal{H}^{n-1} \cong 0.$$

Hence, our generalization of Semmes' main result in [31] is even new in the hypersurface case.

In Section 2 we provide variants of the Hardy–Littlewood maximal theorem and the inequality of John and Nirenberg for spaces with a local doubling property. Later on we apply these results to the intersection of a ball with a chord-arc submanifold Γ with small constant $\gamma(\Gamma)$ to prove that Γ contains big portions of C^1 graphs. Although these intersections are spaces of homogeneous type for which corresponding results are available in the literature (cf. [8, 9]), we cannot use those since in our context it is not at all obvious how to control the defining constants of the spaces of homogeneous type. Furthermore, we gather some elementary facts about the constant $\gamma(\Gamma)$ and cite a very useful characterization of chord-arc submanifolds which tells us that a C^1 submanifold is a chord-arc submanifold if near infinity it is equal to the graph of a C^1 function whose differential vanishes at ∞. For proofs of these statements we refer to [2].

After that we prove in Section 3 that chord-arc submanifolds with a small constant $\gamma(\Gamma)$ contain big portions of C^1 graphs. As an application of this result, we show in Section 4 that η is small if γ is sufficiently small.

To show that the inverse of this statement is true as well, i.e. that γ is small if η is sufficiently small, we carefully carry over an iteration technique due to Semmes from the hypersurface case to our situation of chord-arc submanifolds of arbitrary codimensions in Section 5. Here, the difficulty is to find the corresponding inequalities for the case of codimension greater than one where we cannot work with the unit normal as Semmes does. Instead, we will work with the projection of the ambient space onto the normal spaces.

2 Some preparations

Let (X,d) be a metric space. We denote by $B_r(x) := \{y \in X : d(y,x) < r\}$ the open ball of radius $r > 0$ around $x \in X$ and by $K_r(x) := \{y \in X : d(y,x) \leq r\}$ the closed ball of radius $r \geq 0$ around $x \in X$. We call such a ball non-degenerate if $r > 0$. For a closed ball K with center x and radius r in a metric space (X,d) and $\alpha > 0$ let $\alpha K := K_{\alpha r}(x)$. For a measure μ on some set X, a μ-measurable subset A of X with $0 < \mu(A) < \infty$, and a μ-integrable function $f : X \to \mathbb{R}^n$ we set

$$f_A := \fint_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu.$$

Furthermore, we denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^k$ and for a linear mapping $A : \mathbb{R}^n \to \mathbb{R}^k$ we define

$$\|A\| := \sup_{v \in \mathbb{R}^n - \{0\}} \frac{|A(v)|}{|v|}.$$

2.1 Local doubling spaces

Let us gather some facts about spaces which satisfy a local doubling constant. We will use these facts to show that chord-arc submanifolds contain big portions of C^1-graphs. For detailed proofs we refer to [2, Section 2.2].

Definition 2.1 (Local doubling property). We say that a metric space (X,d) with measure μ has the *local doubling property on scale R with doubling constant* $1 \leq C_d = C_d(R) < \infty$ if and only if

$$\mu(K_{2\rho}(x)) \leq C_d \cdot \mu(K_\rho(x)) < \infty \tag{2.1}$$

for all $0 < \rho \leq \frac{R}{2}$, $x \in \operatorname{spt}(\mu)$.

Definition 2.2 (Variant of the Hardy–Littlewood maximal function). Let $R > 0$ and μ be a measure on some metric space (X, d) with $\mu(K_r(x)) < \infty$ for all $x \in X$ and $0 < r \leq R$. Then we set for a μ-measurable function $f : X \to \mathbb{R}$

$$(\mathfrak{M}_R f)(x) := \begin{cases} \sup_{0<r\leq R} \fint_{K_r(x)} |f| \, d\mu & \text{if } x \in \operatorname{spt}(\mu), \\ 0 & \text{if } x \in X - \operatorname{spt}(\mu). \end{cases}$$

Following the lines of the proof of the classical Hardy–Littlewood maximal theorem one gets

Lemma 2.3 (Hardy–Littlewood maximal theorem for local doubling spaces). *Let μ be a measure on a separable metric space (X, d) such that (X, d, μ) possesses the local doubling property on scale $5R > 0$ with doubling constant $C_d < \infty$. Then*

$$\|\mathfrak{M}_R(f)\|_{L^p((X,\mu),\mathbb{R})} \leq 2 \left(C_d^3 \frac{p}{p-1} \right)^{1/p} \|f\|_{L^p((X,\mu),\mathbb{R})}$$

for all $f \in L^p((X, \mu), \mathbb{R})$, $1 < p < \infty$.

Definition 2.4 (BMO norm). Let μ be a measure on the metric space (X, d) with $\mu(K_r(x)) < \infty$ for all $x \in X$, $r > 0$, and let $f : X \to \mathbb{R}^n$ be a μ-measurable function. We set

$$\|f\|_{\mathrm{BMO}((X,\mu),\mathbb{R}^n)} := \sup_{x\in\operatorname{spt}(\mu), r>0} \fint_{K_r(x)} |f - f_{K_r(x)}| \, d\mu \tag{2.2}$$

and let $\mathrm{BMO}((X, \mu), \mathbb{R}^n)$ be the set of all μ-measurable functions $f : X \to \mathbb{R}^n$ for which $\|f\|_{\mathrm{BMO}((X,\mu),\mathbb{R}^n)} < \infty$.

Observing that actually only the local doubling constant is needed in the proof of the inequality of John and Nirenberg as it can be found for example in [5], we are let to

Lemma 2.5 (Inequality of John and Nirenberg on local doubling spaces). *Let (X, d) be a separable metric space and μ be a Radon measure on X such that the triple (X, d, μ) has the local doubling property up to scale $4R > 0$ with doubling constant $C_d < \infty$. Then there is constant $b = b(n, C_d)$ depending only on n and C_d such that*

$$\fint_{K_R(x)} \exp\left(b \frac{|f(y) - f_{K_R(x)}|}{\|f\|_{\mathrm{BMO}((X,\mu),\mathbb{R}^n)}} \right) < 3e$$

for all $x \in \operatorname{spt}(\mu)$, and $f \in \mathrm{BMO}((X, \mu), \mathbb{R}^n)$.

For subsets of a Euclidean space a local Ahlfors regularity condition implies that the set satisfies a local doubling condition on any scale. Later on, this fact will allow us to use the Hardy–Littlewood maximal theorem and the inequality of John and Nirenberg for chord-arc submanifolds.

Lemma 2.6. *Let μ be a measure on the Euclidean n-space and let $R_0 > 0$, $k \in \mathbb{N}$ be such that there are $M < \infty$, $m > 0$ with*

$$m\rho^k \leq \mu(K_\rho(x)) \leq M\rho^k \quad \forall x \in \operatorname{spt}(\mu), 0 < \rho \leq R_0.$$

Then $(\operatorname{spt}(\mu), |\cdot|, \mu)$ *has the doubling property on any scale $R > 0$ with doubling constant*

$$C_d(R) := 2^k \cdot \begin{cases} \frac{M}{m} & \text{if } R \leq R_0, \\ \frac{M}{m} 4^n \left(\frac{R}{R_0}\right)^n & \text{if } R > R_0. \end{cases}$$

2.2 Chord-arc submanifolds and constants

When dealing with chord-arc submanifolds we do not want to work with the image of Γ under the stereographic projection. The next proposition tells us that a complete, connected, and embedded C^1 submanifold without boundary is a chord-arc submanifold if and only if outside of a large ball around the origin it is the graph of a C^1 function over a k-dimensional subspace of $\mathbb{R}^n$ whose differential vanishes at ∞.

Proposition 2.7 (Proposition 4.2 in [2]). *A set $\Gamma \subset \mathbb{R}^n$ is a k-dimensional chord-arc submanifold if and only if the following two conditions are satisfied:*

- *Γ is an embedded, complete, connected, k-dimensional C^1 submanifold of $\mathbb{R}^n$ that has no boundary.*
- *There are $A \in SO(n)$, $R < \infty$, $\phi \in C^1(\mathbb{R}^k, \mathbb{R}^{n-k})$, such that $A(\Gamma) - K_R(0) = \operatorname{graph}(\phi) - K_R(0)$ and $\lim_{|x|\to\infty} D\phi(x) = 0$.*

The next lemma tells how γ_1 is related to the BMO norm of the normal spaces.

Lemma 2.8. *For k-dimensional chord-arc submanifolds $\Gamma \subset \mathbb{R}^n$ we have*

$$\frac{1}{2}\gamma_1(\Gamma) \leq \|N\|_{\mathrm{BMO}(\mathcal{H}^k \llcorner \Gamma)} \leq 2\gamma_1(\Gamma).$$

Proof. For $x \in \Gamma$, $R > 0$, and $N_{x,R} \in \mathfrak{N}_{x,R}$ one estimates

$$\begin{aligned} \fint_{\Gamma\cap K_R(x)} \|N - N_{K_R(x)}\| \mathcal{H}^k &\leq \fint_{\Gamma\cap K_R(x)} \|N - N_{R,x}\| \mathcal{H}^k + \|N_{R,x} - N_{K_R(x)}\| \\ &\leq 2 \fint_{\Gamma\cap K_R(x)} \|N - N_{R,x}\| \mathcal{H}^k \leq 2\gamma_1(\Gamma). \end{aligned}$$

On the other hand

$$\begin{aligned}\fint_{\Gamma\cap K_R(x)} \|N - N_{R,x}\|\mathcal{H}^k &= \inf_{S\in G_{n,n-k}} \left(\fint_{\Gamma\cap K_R(x)} \|N-S\|\mathcal{H}^k\right)\\ &\le \fint_{\Gamma\cap K_R(x)} \|N - N_{K_R(x)}\|\mathcal{H}^k + \inf_{S\in G_{n,n-k}} \|N_{K_R(x)} - S\|\\ &\le 2\fint_{\Gamma\cap K_R(x)} \|N - N_{K_R(x)}\|\mathcal{H}^k. \end{aligned}$$ □

3 Big portions of graphs

Let us set $K_R^{(k)}(x) := \{y\in\mathbb{R}^k : |y-x|\le R\}$, $B_R^{(k)}(x) := \{y\in\mathbb{R}^k : |y-x| < R\}$, $\omega_k := \mathcal{H}^k(K_1^{(k)}(0))$, and $\mathcal{C}_R := K_R^{(k)}(0)\times K_R^{(n-k)}(0)$. For $T\in G_{n,k}$ we say that a function $g : \mathrm{Im}(T)\to \mathrm{Im}(T)^\perp$ is a *function over* T. In this case we define the graph of g by $\mathrm{graph}(g) := \{v + g(v) : v\in\mathrm{Im}(T)\}$.

Theorem 3.1 (Decomposition Theorem). *There are constants $\varepsilon = \varepsilon(n,k) > 0$, $C = C(n,k) < \infty$, $0 < a = a(n,k)$ such that the following holds:*

If $\Gamma\subset\mathbb{R}^n$ is a k-dimensional chord-arc submanifold with $\gamma := \gamma(\Gamma)\le\varepsilon$, then Γ has the following properties:

(i) The space $(\Gamma, |\cdot|, \mathcal{H}^k\lfloor\Gamma)$ is Ahlfors regular. More precisely, for every $z\in\Gamma$ and every $R > 0$ we have the estimates

$$(1 - C\gamma)\omega_k R^k \le \mathcal{H}^k(\Gamma\cap K_R^{(n)}(z)) \le (1 + C\gamma\log(1/\gamma))\,\omega_k R^k. \tag{3.1}$$

(ii) Let $z\in\Gamma$, $R > 0$, $T_{z,4R}\in\mathfrak{T}_{z,4R}$, and $\mu\in[10\gamma, 1/3]$. After some translation and rotation we can assume that $z = 0$ and $\mathrm{Im}(T_{0,4R}) = \mathbb{R}^k\times\{0\}$. We set

$$\begin{aligned} F &:= \{x\in\mathcal{C}_R\cap\Gamma : \mathfrak{M}_{4R}(T - T_{0,4R})(x)\le\mu\},\\ B &:= (\mathcal{C}_R\cap\Gamma) - F. \end{aligned}$$

Then

$$|N_{0,4R}(y-x)| \le 3\mu|T_{0,4R}(y-x)| \quad \text{for all } x\in F, y\in\mathcal{C}_R\cap\Gamma, \tag{3.2}$$

$$\mathcal{H}^k(B) \le C\exp\left(-a\frac{\mu}{\gamma}\right)R^k, \tag{3.3}$$

and

$$T_{0,4R}(\mathcal{C}_R\cap\Gamma) = K_R^{(k)}(0)\times\{0\}. \tag{3.4}$$

Furthermore, there is a function $g\in C^1(\mathbb{R}^k, \mathbb{R}^{n-k})$ with $\|\nabla g\|_{L^\infty}\le C\mu$ such that the graph G of g satisfies $F\subset G$ and $T_xG = T_x\Gamma$ for all $x\in F$. Here T_xG and $T_x\Gamma$ denote the tangential spaces in x of G and Γ respectively.

The proof relies on an iteration technique. Due to our a priori assumptions, a $\rho_0 := \rho_0(\Gamma) > 0$ exists such that

$$\frac{1}{2}\omega_k R^k \leq \mathcal{H}^k(\Gamma \cap K_R^{(n)}(z)) \leq 2\omega_k R^k \quad \text{for all } 0 < R \leq \rho_0 \text{ and all } x \in \Gamma.$$

This follows from the fact that Γ is an embedded C^1 submanifold that is – outside of a large ball around the origin – the graph of a C^1 function over some k-dimensional subspace whose gradient has a limit at ∞ (cf. Proposition 2.7).

Then the following lemma shows that the conclusions of Theorem 3.1 hold for all $0 < R \leq 2\rho_0$. Since under these conclusions there is an Ahlfors regularity condition, we can iterate this argument to prove that the conclusion of Theorem 3.1 holds in fact for all $R > 0$.

Lemma 3.2. *There is an* $\varepsilon_0 = \varepsilon_0(n,k) > 0$ *and a constant* $C = C(n,k) < \infty$ *such that the following is true:*

If $\Gamma \subset \mathbb{R}^n$ *is a chord-arc submanifold of dimension* k*,* $\gamma(\Gamma) < \varepsilon_0$*, and if there is a* $\rho > 0$ *with*

$$\frac{1}{2}\omega_k R^k \leq \mathcal{H}^k(\Gamma \cap K_R^{(n)}(z)) \leq 2\omega_k R^k \quad \text{for all } 0 < R \leq \rho, z \in \Gamma \tag{3.5}$$

then all the conclusions of Theorem 3.1 hold for $0 < R \leq 2\rho$*.*

Proof. Let $z \in \Gamma$, $0 < R < 2\rho$, and $T_{z,4R} \in \mathfrak{T}_{z,4R}$. After applying a suitable rotation and translation, we can assume that $z = 0$ and $\operatorname{Im}(T_{0,4R}) = \mathbb{R}^k \times \{0\}$. Then the definition of $\gamma_2(\Gamma)$ (cf. (1.6)) leads to

$$\Gamma \cap \mathcal{C}_R \subset \Gamma \cap K_{4R}^{(n)}(0) \subset K_{4R}^{(k)}(0) \times K_{4R\gamma}^{(n-k)}(0). \tag{3.6}$$

Let us furthermore note that F is closed since the Hardy–Littlewood maximal function as the supremum of continuous functions is lower semicontinuous.

Step 1:

$$\boxed{\text{There are constants } 0 < a = a(n,k) \text{ and } C = C(n,k) < \infty \text{ such that } \mathcal{H}^k(B) \leq C \exp(-a\mu\gamma^{-1}) R^k.}$$

Proof. This estimate will be proved using the inequality of John and Nirenberg on balls of radius $8R$ and the Hardy–Littlewood maximal theorem for $\mathfrak{M}_{4R}$ on the metric space $\mathbb{R}^n$ equipped with the measure $\mathcal{H}^k \lfloor \Gamma$ (cf. Lemma 2.3 and Lemma 2.5). Lemma 2.6 and (3.5) tell us that $\mathcal{H}^k \lfloor \Gamma$ has the local doubling property on scale $32R$ with doubling constant $C_d = C_d(n,k) = 2^{k+2} 256^n$. That is all we need to apply Lemma 2.5 and Lemma 2.3 as we do below.

From (2.8) we get $\|T\|_{\mathrm{BMO}(\mathcal{H}^k\lfloor\Gamma)} = \|N\|_{\mathrm{BMO}(\mathcal{H}^k\lfloor\Gamma)} \le 2\gamma(\Gamma)$. Using the inequality of John and Nirenberg in the form of Lemma 2.5, we get a constant $0 < b = b(n,k) < \infty$ such that

$$\fint_{\Gamma\cap K_{8R}^{(n)}(0)} \exp\left(\frac{b}{\gamma}\|T(x) - T_{K_{8R}^{(n)}(0)}\|\right) d\mathcal{H}^k(x) \le C \tag{3.7}$$

where $T_{K_{8R}^{(n)}(0)} := \fint_{\Gamma\cap K_{8R}^{(n)}(0)} T\, d\mathcal{H}^k$. Let $T_{0,8R} \in \mathfrak{T}_{0,8R}$. Since

$$\begin{aligned}
\|T_{0,4R} - T_{K_{8R}^{(n)}(0)}\| \le &\fint_{\Gamma\cap K_{4R}^{(n)}(0)} \|T_{0,4R} - T(x)\|\, d\mathcal{H}^k(x) \\
&+ \fint_{\Gamma\cap K_{4R}^{(n)}(0)} \|T(x) - T_{0,8R}\|\, d\mathcal{H}^k(x) \\
&+ \fint_{\Gamma\cap K_{8R}^{(n)}(0)} \|T_{0,8R} - T(x)\|\, d\mathcal{H}^k(x) \\
\le\ & 2\gamma + \frac{\mathcal{H}^k(\Gamma\cap K_{8R}^{(n)}(0))}{\mathcal{H}^k(\Gamma\cap K_{4R}^{(n)}(0))} \fint_{\Gamma\cap K_{8R}^{(n)}(0)} \|T(x) - T_{0,8R}\|\, d\mathcal{H}^k(x) \\
\overset{\text{doubling}}{\le}\ & C\gamma,
\end{aligned}$$

we get from (3.7)

$$\fint_{\Gamma\cap K_{8R}^{(n)}(0)} \exp\left(\frac{b}{\gamma}\|T(x) - T_{0,4R}\|\right) d\mathcal{H}^k(x) \le C. \tag{3.8}$$

Let $\chi_{K_{8R}^{(n)}(0)}$ be the characteristic function of the set $K_{8R}^{(n)}(0)$. We now apply the Hardy–Littlewood maximal theorem (Lemma 2.3) to $\|T - T_{0,4R}\|\chi_{K_{8R}^{(n)}(0)}$ and use the fact that for all $x \in K_{4R}^{(n)}(0)$

$$\mathfrak{M}_{4R}(\|T - T_{0,4R}\|\chi_{K_{8R}^{(n)}(0)})(x) = \mathfrak{M}_{4R}(\|T - T_{0,4R}\|)(x)$$

to get

$$\begin{aligned}
&\int_{\Gamma\cap K_{4R}^{(n)}(0)} \Big(\mathfrak{M}_{4R}(\|T - T_{0,4R}\|)(x)\Big)^p\, d\mathcal{H}^k(x) \\
&\qquad \le 2^p C_d^3 \frac{p}{p-1} \int_{\Gamma\cap K_{8R}^{(n)}(0)} \|T(x) - T_{0,4R}\|^p\, d\mathcal{H}^k(x)
\end{aligned} \tag{3.9}$$

for all $p > 1$. Since for a measure ν on Ω, a ν-measurable function $f : \Omega \to \mathbb{R}^n$, and a ν-measurable set $A \subset \Omega$ we have

$$\int_A |f|\, d\nu = \int_{A\cap[|f|>1]} |f|\, d\nu + \int_{A\cap[|f|\le 1]} |f|\, d\nu \le \int_A |f|^2\, d\nu + \nu(A), \tag{3.10}$$

we get for $a := b/2$

$$
\begin{aligned}
&\fint_{\Gamma\cap K^{(n)}_{4R}(0)} \exp\left(a\,\frac{\mathfrak{M}_{4R}(\|T-T_{0,4R}\|)(x)}{\gamma}\right)\,d\mathcal{H}^k(x)\\
&\quad= \sum_{l=0}^{\infty} \fint_{\Gamma\cap K^{(n)}_{4R}(0)} \frac{(a\gamma^{-1})^l}{l!}\Big(\mathfrak{M}_{4R}(\|T-T_{0,4R}\|)(x)\Big)^l\,d\mathcal{H}^k(x)\\
&\quad\overset{(3.10)}{\leq} 2\left\{1+\sum_{l=2}^{\infty} \fint_{\Gamma\cap K^{(n)}_{4R}(0)} \frac{(a\gamma^{-1})^l}{l!}\Big(\mathfrak{M}_{4R}(\|T-T_{0,4R}\|)(x)\Big)^l\,d\mathcal{H}^k(x)\right\}\\
&\quad\overset{(3.9)}{\leq} 2\left\{1+C_d\sum_{l=2}^{\infty} C_d^3 2^{l+1} \fint_{\Gamma\cap K^{(n)}_{8R}(0)} \frac{(a\gamma^{-1})^l}{l!}\|T(x)-T_{0,4R}\|^l\,d\mathcal{H}^k(x)\right\}\\
&\quad\leq 4C_d^4 \fint_{\Gamma\cap K^{(n)}_{8R}(0)} \exp\left(b\,\frac{\|T(x)-T_{0,4R}\|)}{\gamma}\right)\,d\mathcal{H}^k(x) \overset{(3.8)}{\leq} C.
\end{aligned}
$$

Since $\mathcal{C}_R \subset K^{(n)}_{4R}(0)$, we finally get by repeated use of the doubling property

$$
\begin{aligned}
\mathcal{H}^k(B) &\leq \int_{\Gamma\cap K^{(n)}_{4R}(0)} \frac{\exp(a\gamma^{-1}\mathfrak{M}_{4R}(\|T-T_{0,4R}\|)(x))}{\exp(a\gamma^{-1}\mu)}\,d\mathcal{H}^k(x)\\
&\leq C\exp(-a\gamma^{-1}\mu)\mathcal{H}^k(\Gamma\cap K^{(n)}_{4R}(0))\\
&\overset{(3.5)\text{ \& doubling}}{\leq} C\exp(-a\gamma^{-1}\mu)R^k.
\end{aligned}
$$

□

Step 2:

For every $x \in F$ and $y \in \Gamma\cap\mathcal{C}_R$ we have $|N_{0,4R}(x-y)| \leq 3\mu|T_{0,4R}(x-y)|$ (cf. Figure 2).

Proof. Let $x \neq y \in \Gamma\cap\mathcal{C}_R$ and $x \in F$. We choose an $N_{x,|x-y|} \in \mathfrak{N}_{x,|x-y|}$. Then

$$
\begin{aligned}
|N_{0,4R}(x-y)| &\leq |N_{x,|x-y|}(x-y)| + |N_{x,|x-y|}(x-y) - N_{0,4R}(x-y)|\\
&\overset{\text{def. of }\gamma_2(\Gamma)}{\leq} \gamma|x-y| + \|N_{x,|x-y|} - N_{0,4R}\|\cdot|x-y|.
\end{aligned}
$$

Using

$$
\begin{aligned}
\|N_{x,|x-y|} - N_{0,4R}\| \leq &\fint_{\Gamma\cap K^{(n)}_{|x-y|}(x)} \|N_{x,|x-y|} - N(\xi)\|\,d\mathcal{H}^k(\xi)\\
&+ \fint_{\Gamma\cap K^{(n)}_{|x-y|}(x)} \|N(\xi) - N_{0,4R}\|\,d\mathcal{H}^k(\xi)\\
&\overset{x\in F}{\leq} \gamma+\mu,
\end{aligned}
$$

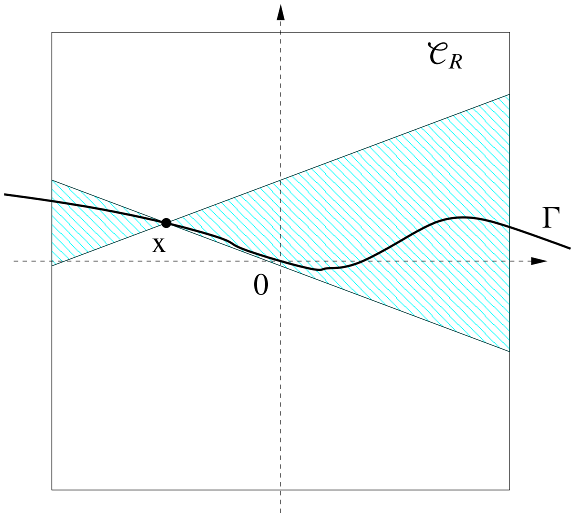

Figure 2. This picture illustrates the statement proven in Step 2. For every point x belonging to the good set $F \subset \Gamma$, we show that $\Gamma \cap \mathscr{C}_R$ is contained in the cone $\{y \in \mathbb{R}^n : |N_{0,4R}(y-x)| \leq 3\mu |T_{0,4R}(y-x)|\}$.

we get

$$|N_{0,4R}(x-y)| \leq (2\gamma + \mu)|x-y|.$$

With $|x-y| \leq |N_{0,4R}(x-y)| + |T_{0,4R}(x-y)|$, we get

$$|N_{0,4R}(x-y)| \leq \frac{2\gamma+\mu}{1-2\gamma-\mu}|T_{0,4R}(x-y)| \leq 3\mu |T_{0,4R}(x-y)|$$

if $\gamma \leq 4/30$ and $\mu \in [10\gamma, 1/3]$. □

Step 3:

$$\boxed{T_{0,4R}(\Gamma \cap \mathscr{C}_R) = K_R^{(k)}(0) \times \{0\}.}$$

Proof. We will use the modulo 2 degree deg [2, Section 3.2] to show that the function

$$f : \Gamma \cap \mathscr{C}_R \to K_R^{(k)}(0) \times \{0\}, \quad x \mapsto T_{0,4R}(x)$$

is surjective. From (3.6) we get $\Gamma \cap \mathscr{C}_R \subset K_R^{(k)}(0) \times K_{4\gamma R}^{(n-k)}(0)$. If $\gamma < 1/4$, we thus have

$$f\,(\partial_\Gamma(\Gamma \cap \mathscr{C}_R)) \subset \Big(\partial_{\mathbb{R}^k}(K_R^{(k)}(0))\Big) \times \{0\}. \tag{3.11}$$

We will now show that there is a $y_0 \in B_R^{(k)}(0) \times \{0\}$ such that

$$\deg(f, \Gamma \cap \mathscr{C}_R, y_0) \equiv 1 \mod 2.$$

It then follows from property the properties of the degree and (3.11) that $\deg(f, \Gamma \cap \mathscr{C}_R, y) \equiv 1 \mod 2$ for all $y \in B_R^{(k)}(0)$. From this and known properties of the degree our assertion follows.

Let us fix $\mu = 1/3$ in Steps 1 and 2 until the end of the current step. Using (3.5) and Step 1 we get

$$\mathscr{H}^k(F) = \mathscr{H}^k(\Gamma \cap \mathscr{C}_R) - \mathscr{H}^k(B) \geq \frac{1}{2}\omega_k \left(\frac{R}{2}\right)^k - C \exp\left(-\frac{a}{3\gamma}\right) R^k > 0$$

if γ is sufficiently small. So there is an $x_0 \in F$ and we set $y_0 := T_{0,4R}(x_0)$. We have

$$\mathfrak{M}_{4R}(\|T - T_{0,4R}\|)(x_0) = \sup_{0 \leq r \leq 4R} ⨍_{\Gamma \cap K_r^{(n)}(x_0)} \|T - T_{0,4R}\| \, d\mathscr{H}^k \leq \mu \leq 1/3.$$

Sending $r \to 0$ we get from the C^1 smoothness of Γ

$$\|T(x_0) - T_{0,4R}\| \leq \frac{1}{3}.$$

We know from Step 2 that $f^{-1}(y_0) = \{x_0\}$ since $x_0 \in F$. Thus y_0 is a regular value of f and we have

$$\deg(f, \Gamma \cap \mathscr{C}_R, y_0) \equiv 1 \mod 2.$$

□

Step 4:

$$\boxed{\text{Construction of } g.}$$

Let $E := \{x \in \mathbb{R}^k : (x, 0) \in T_{0,4R}(F)\}$. Step 2 shows us that for every $x \in E$ there is a unique point $y \in F$ such that $T_{0,4R}(y) = (x, 0)$. We set

$$\tilde{g}(x) := (y_{k+1}, \dots, y_n).$$

From Step 2 we get $|\tilde{g}(x) - \tilde{g}(y)| \leq 3\mu|x - y|$ and $\|T(x, \tilde{g}(x)) - T_{0,4R}\| \leq \mu$ for all $x, y \in E$.

Using that $\tilde{g}$ is a Lipschitz function whose graph is contained in the C^1 submanifold Γ and the last two estimates, it can be shown that there is an open set $\tilde{E} \supset E$ and $h \in C^1(\tilde{E}, \mathbb{R}^{n-k})$ with Lipschitz constant $\leq C\mu$, $g = h|_E$, and $\operatorname{graph} h \subset \Gamma$. Using Kirszbraun's theorem (cf. [25, Hauptsatz $\mathfrak{A}$ 1]), we get a Lipschitz continuous extension $\tilde{h} : \mathbb{R}^k \to \mathbb{R}^{n-k}$ of h with $|\nabla \tilde{h}| \leq C\mu$ almost everywhere. Convolving this function with a smooth kernel we get smooth functions $h_m : \mathbb{R}^k \to \mathbb{R}^{n-k}$

with $|\nabla h_m| \leq C\mu$ and $h_m \to \tilde{h}$ in $L^\infty(\mathbb{R}^k, \mathbb{R}^{n-k})$. Now let $\tilde{\tilde{E}}$ be an open subset with $E \subset\subset \tilde{\tilde{E}} \subset\subset \tilde{E}$ and $\psi \in C^\infty(\mathbb{R}^k, [0,1])$ be a cutoff function satisfying $\chi_{\tilde{\tilde{E}}} \leq \psi \leq \chi_{\tilde{E}}$. For m large enough we set $g := \psi h + (1-\psi)h_m$. Then $g \in C^1(\mathbb{R}^k, \mathbb{R}^{n-k})$, $g|_E \equiv \tilde{g}$, and for almost all $x \in \mathbb{R}^k$

$$|\nabla g(x)| \leq |\nabla \psi||\tilde{h}(x) - h_m(x)| + |\nabla \tilde{h}(x)| + |\nabla h_m(x)| \leq C\mu$$

if m is big enough. Let $G = \operatorname{graph}(g)$. Then $F \subset G$ and since $g(\tilde{\tilde{E}}) = h(\tilde{\tilde{E}}) \subset \Gamma$ we furthermore obtain

$$T_x G = T_x \Gamma \quad \forall x \in F.$$

Step 5:

$$\boxed{\begin{gathered}(1 - C\gamma)\omega_k R^k \leq \mathcal{H}^k(\Gamma \cap K_R^{(n)}(z)) \leq \\ (1 + C\gamma \log(1/\gamma))\, \omega_k R^k.\end{gathered}}$$

For the upper bound we set $\mu = a^{-1}\gamma \log(1/\gamma)$ in the estimates we have derived so far. Since $\gamma \log(1/\gamma) \to 0$ and $\log(1/\gamma) \to \infty$ as $\gamma \to 0$, we get $a^{-1}\gamma\log(1/\gamma) \in [10\gamma, 1/3]$ if γ is small enough. Therefore,

$$\mathcal{H}^k(B) \overset{\text{Step 1}}{\leq} C \exp(-\log(1/\gamma))\, R^k = C\gamma R^k < C\gamma \log(1/\gamma)\, R^k$$

if $\gamma < 1$. Since F is part of the graph of a Lipschitz function on $B_R^{(k)}(0)$ with Lipschitz constant smaller than $C\gamma\log(1/\gamma)$, we get

$$\mathcal{H}^k(F) \leq (1 + C\gamma\log(1/\gamma))\omega_k R^k.$$

This yields

$$\mathcal{H}^k(\Gamma \cap K_R^{(n)}(0)) \leq \mathcal{H}^k(B) + \mathcal{H}^k(F) \leq (1 + C\gamma\log(1/\gamma))\omega_k R^k.$$

For the lower bound we first observe that

$$K_R^{(n)}(0) \cap \Gamma \quad \subset \quad \mathcal{C}_R \cap \Gamma \quad \overset{(3.6)}{\subset} \quad K_R^{(k)}(0) \times K_{4\gamma R}^{(n-k)}(0).$$

Let $x \in K^{(k)}_{R\sqrt{1-16\gamma^2}}(0)$. From Step 3 we know that $T_{0,4R}(\Gamma \cap \mathcal{C}_R) = K_R^{(k)}(0) \times \{0\}$. Thus, there is a $y \in K_{4\gamma R}^{(n-k)}(0)$ such that $(x,y) \in \Gamma \cap \mathcal{C}_R$. We calculate

$$|(x,y)|^2 \leq (1 - 16\gamma^2)R^2 + 16\gamma^2 R^2 = R^2$$

and see that $(x,y) \in K_R^{(n)}(0) \cap \Gamma$ and $T_{0,4R}((x,y)) = (x,0)$. So we have shown that

$$K^{(k)}_{R\sqrt{1-16\gamma^2}}(0) \times \{0\} \subset T_{0,4R}(K_R^{(n)}(0) \cap \Gamma).$$

Hence,

$$\mathcal{H}^k(\Gamma \cap K_R^{(n)}(0)) \geq \mathcal{H}^k(T_{0,4R}(K_R^{(n)}(0) \cap \Gamma)) \geq \mathcal{H}^k(K^{(k)}_{R\sqrt{1-16\gamma^2}}(0) \times \{0\})$$
$$= (1-16\gamma^2)^{k/2}\omega_k R^k \geq (1 - C(k)\gamma)\omega_k R^k$$

for γ sufficiently small. □

Proof of Theorem 3.1. Let $C(n,k)$, $a(n,k)$, and $\varepsilon_0(n,k)$ be the constants from the last lemma. We choose $\varepsilon = \varepsilon(n,k)$ such that $\gamma \leq \varepsilon$ implies $\gamma \leq \varepsilon_0$, $C(n,k)\gamma \leq \frac{1}{2}$, and $C(n,k)\gamma \log(1/\gamma) \leq 1$. Due to our a priori assumptions, there is a $\rho_0 = \rho_0(\Gamma) > 0$ such that

$$\frac{1}{2}\omega_k R^k \leq \mathcal{H}^k(\Gamma \cap K_R(z)) \leq 2\omega_k R^k$$

for all $0 < R < \rho_0$. Using induction and Lemma 3.2, the conclusion of the theorem follows. □

Corollary 3.3. *In the situation of Part 2 of Theorem 3.1 we furthermore have the following estimates:*

(i) $\mathcal{H}^k(\mathcal{C}_R \cap \{(\Gamma - G) \cup (G - \Gamma)\}) \leq C \exp(-a\mu/\gamma) R^k$.

(ii) For all $y \in \Gamma \cap \mathcal{C}_R$ *we have*

$$|y - (y_1, \ldots, y_k, g(y_1, \ldots, y_k))| \leq C\mu \operatorname{dist}(T_{0,4R}(y), T_{0,4R}(F)).$$

Proof. Since $\mathcal{C}_R \cap (\Gamma - G) \subset B$, we get

$$\mathcal{H}^k(\mathcal{C}_R \cap (\Gamma - G)) \leq \mathcal{H}^k(B). \tag{3.12}$$

Using the fact that G is the graph of a Lipschitz function with Lipschitz constant smaller than $C\mu \leq C$, we get

$$\mathcal{H}^k(\mathcal{C}_R \cap (G - \Gamma)) \leq C\mathcal{H}^k(T_{0,4R}(\mathcal{C}_R \cap (G - \Gamma))).$$

Since $T_{0,4R}(F \cup B) = T_{0,4R}(\mathcal{C}_R \cap \Gamma) \overset{(3.4)}{=} K_R^{(k)}(0) \times \{0\}$ and $F \subset G \cap \Gamma$ we conclude that $T_{0,4R}(\mathcal{C}_R \cap (G - \Gamma)) \subset T_{0,4R}(B)$ and thus

$$\mathcal{H}^k(T_{0,4R}(\mathcal{C}_R \cap (G - \Gamma))) \leq C\mathcal{H}^k(T_{0,4R}(B)) \leq C\mathcal{H}^k(B).$$

Together with (3.12) this leads to

$$\mathcal{H}^k(\mathcal{C}_R \cap \{(\Gamma - G) \cup (G - \Gamma)\}) \leq C\mathcal{H}^k(B) \leq C \cdot \exp(-a\mu/\gamma) R^k$$

and the first estimate is shown.

Let $y \in \Gamma$. As $T_{0,4R}(F)$ is a closed set, there is a $z \in F$ with

$$|T_{0,4R}(y) - T_{0,4R}(z)| = \operatorname{dist}(T_{0,4R}(y), T_{0,4R}(F)).$$

We set $\tilde{y} := (y_1, \ldots, y_k)$ and $\tilde{z} := (z_1, \ldots, z_k)$. Since $z \in F$, we know $z = (\tilde{z}, g(\tilde{z}))$ and hence

$$\begin{aligned} |y - (\tilde{y}, g(\tilde{y}))| &= |N_{0,4R}(y - (\tilde{y}, g(\tilde{y})))| \\ &\le |N_{0,4R}(y - z)| + |N_{0,4R}(z - (\tilde{y}, g(\tilde{y})))| \\ &= |N_{0,4R}(y - z)| + |g(\tilde{z}) - g(\tilde{y})| \overset{(3.2)}{\le} C\mu |T_{0,4R}(y - z)| \\ &= C\mu \operatorname{dist}(T_{0,4R}(y), T_{0,4R}(F)). \end{aligned} \qquad \square$$

Furthermore, we get the following relation between the constant $\gamma_2(\Gamma)$ and the constant

$$\tilde{\delta}(\Gamma) := \inf\{\delta \in [0, \infty) : \Gamma \text{ is globally } \delta\text{-Reifenberg flat}\}.$$

Corollary 3.4. *There is an $\varepsilon(n, k) > 0$ such that for every k-dimensional chord-arc submanifold with $\gamma(\Gamma) \le \varepsilon$ we have*

$$\tilde{\delta}(\Gamma) \le 8\gamma_2(\Gamma). \tag{3.13}$$

Proof. Let $x \in \Gamma$ and $R > 0$. After some rotation and translation we can assume that $x = 0$ and $\operatorname{Im}(T_{x,4R}) = \mathbb{R}^k \times \{0\}$. From the definition of $\gamma_2(\Gamma)$ one gets

$$\sup_{y \in \Gamma \cap B_R^{(n)}(x)} d(y, \operatorname{Im}(T_{x,4R}) \cap B_R^{(n)}(x)) \le 4\gamma_2(\Gamma).$$

Applying Proposition 3.1 we get that $T_{0,4R}(\mathscr{C}_R \cap \Gamma) = K_R^k(0) \times \{0\}$ if $\gamma(\Gamma)$ is small enough.

Let $y \in \operatorname{Im}(T_{0,4R}) \cap (B_R^k(0) \times \{0\})$, If $\gamma_2 < \frac{1}{4}$ there is an $\tilde{y} \in B_{R-4R\gamma_2}^{(k)}(0)$ with $|y - \tilde{y}| \le \gamma_2$. Then we get an $z \in \Gamma \cap \mathscr{C}_R$ with $T_{x,4R}(z) = \tilde{y}$ and using the definition of $\gamma_2(\Gamma)$ and $\mathscr{C}_R \subset K_{4R}^{(n)}(0)$ one gets $|z - \tilde{y}| \le 4R\gamma_2$ and hence $z \in \Gamma \cap B_R^{(n)}(0)$. From $|y - z| \le |y - \tilde{y}| + |\tilde{y} - z| 8R\gamma_2$ we finally derive

$$\sup_{y \in \operatorname{Im}(T_{x,4R}) \cap B_R^{(n)}(x)} d(y, \Gamma \cap B_R^{(n)}(0)) \le 8\gamma_2. \qquad \square$$

4 Proof of the first part of Theorem 1.1

Let us briefly sketch the idea of the proof. For $u, v \in \Gamma$ we have to construct a short curve on Γ joining u and v. If Γ were the graph of a Lipschitz function with small constant, this would be easy. Theorem 3.1 implies that $\Gamma \cap K_{2|u-v|}^{(n)}(u)$ looks like the graph G of such a function, except on a small bad set. The idea is, to start with a curve on this graph and then manipulate it on the bad set to get a curve on Γ. Using that the bad set is small, we can control the growth of length in this last step.

Proof of the first part of Theorem 1.1. Let us set $\gamma := \gamma(\Gamma)$, $\eta := \eta(\Gamma)$, $\eta_1 := \eta_1(\Gamma)$, and $\eta_2 := \eta_2(\Gamma)$. From Theorem 3.1, inequality (3.1), and $\lim_{\gamma \searrow 0} \gamma \log(1/\gamma) = 0$ we get $\eta_2 \leq C\gamma \log(1/\gamma)$ if γ is small enough.

Let us set

$$\tilde{\eta}_1 := \sup_{x \neq y \in \Gamma} \frac{d_\Gamma(x,y)}{|x-y|} = \eta_1 + 1$$

and let $u, v \in \Gamma$, $u \neq v$, $R := 2|u-v| > 0$, and $T_{u,4R} \in \mathfrak{T}_{u,4R}$. After a suitable translation and rotation we can assume that $u = 0$, $\operatorname{Im}(T_{0,4R}) = \mathbb{R}^k \times \{0\}$, and $\tilde{v} := T_{0,4R}(v) = \lambda e_k$ for a $\lambda \in \mathbb{R}^+$.

Let $F := \{x \in \Gamma \cap \mathscr{C}_R : \mathfrak{M}_{4R}(T - T_{0,4R})(x) \leq \mu\}$ and $B := (\Gamma \cap \mathscr{C}_R) - F$. Theorem 3.1 tells us that

$$T_{0,4R}(\Gamma \cap \mathscr{C}_R) = K_R^{(k)} \times \{0\} \tag{4.1}$$

and that the set F is contained in the graph of a function $g \in C^1(\mathbb{R}^k, \mathbb{R}^{n-k})$ with $\|\nabla g\|_{L^\infty} \leq C\mu$ and

$$\mathscr{H}^k(B) \leq C \exp\left(-a\frac{\mu}{\gamma}\right) R^k.$$

Using $(K_R^k(0) \times \{0\}) - T_{0,4R}(F) \subset T_{0,4R}((\Gamma \cap \mathscr{C}_R) - F) = T_{0,4R}(B)$ we get

$$\mathscr{H}^k((K_R^k(0) \times \{0\}) - T_{0,4R}(F)) \leq \mathscr{H}^k(B) \leq C \exp\left(-a\frac{\mu}{\gamma}\right) R^k. \tag{4.2}$$

Because of (4.1), for every $\zeta \in K_R^{(k)}(0) \times \{0\} \subset \mathbb{R}^n$ there is an $x_\zeta \in \Gamma \cap \mathscr{C}_R$ such that

$$T_{0,4R}(x_\zeta) = \zeta.$$

Let $0 < e \leq \frac{1}{2}$. We then get for $\theta \in B_{eR}^{(k)}(0) \times \{0\} \subset \mathbb{R}^n$

$$\begin{aligned} d_\Gamma(u,v) = d_\Gamma(0,v) &\leq d_\Gamma(0,x_\theta) + d_\Gamma(x_\theta, x_{\tilde{v}+\theta}) + d_\Gamma(x_{\tilde{v}+\theta}, v) \\ &\leq \tilde{\eta}_1\big(|x_\theta| + |x_{\tilde{v}+\theta} - v|\big) + d_\Gamma(x_\theta, x_{\tilde{v}+\theta}). \end{aligned}$$

Since $\Gamma \cap \mathscr{C}_R \subset K_R^{(k)}(0) \times K_{4\gamma R}^{(n-k)}(0)$ and $\operatorname{Im}(T_{0,4R}) = \mathbb{R}^k \times \{0\}$, we get using the definition of γ (cf. (1.6), (1.7))

$$\begin{aligned} |x_{\tilde{v}+\theta} - v| &\leq |T_{0,4R}(x_{\tilde{v}+\theta} - v)| + |N_{0,4R}(x_{\tilde{v}+\theta} - v)| \\ &\leq |\theta| + |N_{0,4R}(x_{\tilde{v}+\theta})| + |N_{0,4R}(v)| \leq eR + 8\gamma R \end{aligned}$$

and

$$|x_\theta| \le |T_{0,4R}(x_\theta)| + |N_{0,4R}(x_\theta)| \le eR + 4\gamma R.$$

Consequently,

$$\begin{aligned} d_\Gamma(u,v) &\le \tilde{\eta}_1(12\gamma + 2e)R + d_\Gamma(x_\theta, x_{\tilde{v}+\theta}) \\ &\overset{R=2|u-v|}{=} \tilde{\eta}_1(24\gamma + 4e)\cdot|u-v| + d_\Gamma(x_\theta, x_{\tilde{v}+\theta}). \end{aligned} \tag{4.3}$$

To estimate the last term, we need to find a curve $c_\theta : [0,\lambda] \to \Gamma$ on Γ from x_θ to $x_{\tilde{v}+\theta}$ using the graph of g whose length we can estimate. To construct this curve, we set $E := T_{0,4R}(F)$, $E_\theta := \{t \in [0,\lambda] : \theta + te_k \in E\}$, and $E_\theta^C := (0,\lambda) - E_\theta$. We know from (4.2) that

$$\mathcal{H}^k((K_R^{(k)}(0) \times \{0\}) - E) \le \exp\left(-a\frac{\mu}{\gamma}\right) R^k. \tag{4.4}$$

Since E is a closed set and the function $t \mapsto \theta + te_k$ is continuous, the set E_θ^C is open and thus the union of countably many disjoint open intervals $I_j = (a_j, b_j)$, $j \in J \subset \mathbb{N}$. Now let us define c_θ in the following way:

(i) If $t \in E_\theta$, then $c_\theta(t)$ is the unique point in $\Gamma \cap \mathcal{C}_R$ with $T_{0,4R}(c(t)) = \theta + te_1$.

(ii) For $j \in J$ let $c_j : [a_j, b_j] \to \Gamma$ be one of the shortest Lipschitz curves of constant velocity joining the points

- $c_\theta(a_j)$ and $c_\theta(b_j)$ if $0 < a_j$ and $b_j < 1$,
- $c_\theta(a_j)$ and $x_{\tilde{v}+\theta}$ if $0 < a_j$ and $b_j = 1$,
- x_θ and $c_\theta(b_j)$ if $0 = a_j$ and $b_j < 1$,
- x_θ and $x_{\tilde{v}+\theta}$ if $a_j = 0$, $b_j = 1$.

We set $c_\theta(t) := c_j(t)$ if $t \in [a_j, b_j]$.

From the construction of the curve, we get that $c(0) = x_\theta$ and $c(\lambda) = x_{\tilde{v}+\theta}$.

For $t_1, t_2 \in E_\theta$ we get from Step 3 in the proof of Theorem 3.1

$$\begin{aligned} |c_\theta(t_1) - c_\theta(t_2)| &\le |T_{0,4R}(c_\theta(t_1) - c_\theta(t_2))| + |N_{0,4R}(c_\theta(t_1) - c_\theta(t_2))| \\ &\le (1+3\mu)\cdot|T_{0,4R}(c_\theta(t_1) - c_\theta(t_2))| = (1+3\mu)\cdot|t_1 - t_2|. \end{aligned} \tag{4.5}$$

So c_θ is Lipschitz continuous on E_θ. Next we want to derive a Lipschitz estimate for c_θ on one of the components $[a_j, b_j]$.

Let $j \in J$. If $a_j, b_j \in E_\theta$, inequality (4.5) proves

$$|c_\theta(a_j) - c_\theta(b_j)| \le (1+3\mu)\cdot|t_1 - t_2|.$$

In the case that $a_j = 0$ and $b_j \in E_\theta$, or $a_j \in E_\theta$ and $b_j = 1$ we get using $|T_{0,4R}(c_\theta(a_j) - c_\theta(b_j))| = |a_j - b_j|$ and Step 3 in the proof of Theorem (3.1)

$$\begin{aligned}|c_\theta(a_j) - c_\theta(b_j)| &\le |T_{0,4R}(c_\theta(a_j) - c_\theta(b_j))| + |N_{0,4R}(c_\theta(a_j) - c_\theta(b_j))| \\ &\le (1+3\mu) \cdot |T_{0,4R}(c_\theta(a_j) - c_\theta(b_j))| \le (1+3\mu) \cdot |a_j - b_j|.\end{aligned}$$

In the case that $a_j = 0$ and $b_j = 1$ we get using

$$|\tilde{v}| \ge |v| - |N_{0,4R}(v)| = \frac{R}{2} - 8\gamma \frac{R}{2} = (1 - 8\gamma)\frac{R}{2}$$

that

$$\begin{aligned}|c_\theta(a_j) - c_\theta(b_j)| = |u - v| = \frac{R}{2} &\le \frac{1}{1-8\gamma}|\tilde{v}| \le (1+16\gamma)|\tilde{v}| \\ &= (1+16\gamma)|a_j - b_j|\end{aligned}$$

if γ is small enough.

Since $\mu \ge 10\gamma$, we have in either case

$$\begin{aligned}H^1(c_\theta([a_j, b_j])) &= \text{length}(c_\theta|_{[a_j,b_j]}) \le \tilde{\eta}_1 |c_\theta(a_j) - c_\theta(b_j)| \\ &\le \tilde{\eta}_1(1+3\mu)|a_j - b_j|.\end{aligned} \tag{4.6}$$

As $c_\theta|_{[a_j,b_j]}$ has constant velocity, we get

$$|c_\theta(t_1) - c_\theta(t_2)| \le \tilde{\eta}_1(1+3\mu)|t_1 - t_2| \quad \text{for all } t_1, t_2 \in [a_j, b_j]. \tag{4.7}$$

The estimates (4.5) and (4.7) show that c_θ is Lipschitz continuous on the whole interval $[0, \lambda]$. Inequality (4.5) implies

$$\mathcal{H}^1(c_\theta(E_\theta)) \le (1+3\mu)\mathcal{H}^1(E_\theta) \le (1+3\mu)|T_{0,4R}(u-v)| \le (1+3\mu)|u-v|.$$

Combining this with (4.7), we get

$$\begin{aligned}d_\Gamma(x_\theta, x_{\tilde{v}+\theta}) &\le \text{length}(c_\theta) = \mathcal{H}^1(c_\theta(E_\theta)) + \mathcal{H}^1(c_\theta(E_\theta^C)) \\ &= \mathcal{H}^1(c_\theta(E_\theta)) + \sum_{j \in \mathbb{J}} \mathcal{H}^1(c_\theta([a_j, b_j])) \\ &\le (1+3\mu)|u-v| + (1+\mu)\tilde{\eta}_1 \sum_{j \in J} |a_j - b_j| \\ &= (1+3\mu)|u-v| + (1+\mu)\tilde{\eta}_1 \mathcal{H}^1\big(E_\theta^C\big).\end{aligned} \tag{4.8}$$

Then (4.3) and (4.8) yield

$$d_\Gamma(u,v) \le |u-v| \cdot \big(1 + 3\mu + \tilde{\eta}_1(24\gamma + 4e)\big) + (1+\mu)\tilde{\eta}_1 \mathcal{H}^1\big(E_\theta^C\big)\big)$$

for all $\theta \in K_{eR}^{(k)}(0) \times \{0\} \subset \mathbb{R}^n$. Taking the integral mean over all $\theta \in B_{eR}^{(k-1)}(0) \times \{0\} \subset B_{eR}^{(k)}(0) \times \{0\} \subset \mathbb{R}^n$ and using $B_{eR}^{(k-1)}(0) \times [0,\lambda] \subset K_R^{(k)}(0)$ and $\mu \le 1/3$, we get

$$\begin{aligned}
d_\Gamma(u,v) &\le |u-v| \cdot \big(1 + 3\mu + \tilde\eta_1(24\gamma + 4e)\big) \\
&\qquad + 2\tilde\eta_1 \frac{1}{\omega_{k-1} e^{k-1} R^{k-1}} \int_{B_{eR}^{(k-1)}(0)\times\{0\}} \mathcal{H}^1\big(E_\theta^C\big)\, d\mathcal{H}^{k-1}(\theta) \\
&= |u-v| \cdot \big(1 + 3\mu + \tilde\eta_1(24\gamma + 4e)\big) \\
&\qquad + \frac{2\tilde\eta_1}{\omega_{k-1} e^{k-1} R^{k-1}} \int_{B_{eR}^{(k-1)}(0)} \mathcal{H}^1\big((\{\tilde\theta\} \times [0,\lambda] \times \{0\}) - E\big)\, d\mathcal{H}^{k-1}(\tilde\theta) \\
&\le |u-v| \cdot \big(1 + 3\mu + \tilde\eta_1(24\gamma + 4e)\big) \\
&\qquad + 2\tilde\eta_1 \frac{1}{\omega_{k-1} e^{k-1} R^{k-1}} \mathcal{H}^k((K_R^{(k)}(0) \times \{0\}) - E) \\
&\overset{(4.4)}{\le} |u-v| \cdot \big(1 + C\mu + \tilde\eta_1(24\gamma + 4e)\big) + C\tilde\eta_1 e^{1-k} R \exp\left(-a\frac{\mu}{\gamma}\right).
\end{aligned}$$

If we divide through $|u-v|$, take the supremum, and set $\mu = \frac{k}{a}\gamma \log(\frac{1}{\gamma})$ and $e = \gamma$, we derive

$$\tilde\eta_1 \le 1 + C\gamma \log\left(\frac{1}{\gamma}\right) + \tilde\eta_1 \left(28\gamma + C\gamma^{1-k}\gamma^k\right) = 1 + C\gamma \log\left(\frac{1}{\gamma}\right) + C\tilde\eta_1\gamma.$$

The C^1 smoothness of Γ and Proposition 2.7 imply $\tilde\eta_1 < \infty$. Hence,

$$\tilde\eta_1 \le \frac{1 + C\gamma \log(\frac{1}{\gamma})}{1 - C\gamma} \le 1 + C\gamma \log\left(\frac{1}{\gamma}\right)$$

if γ is small enough and thus $\eta_1 = \tilde\eta_1 - 1 \le C\gamma \log\left(\frac{1}{\gamma}\right)$.

□

5 Proof of the second part of Theorem 1.1

As the first part, also the second part will be proved using an iteration argument that starts using the C^1 smoothness of the manifold Γ. Let us introduce some notation and then sketch the structure of the lengthy proof.

For a k-dimensional chord-arc submanifold $\Gamma \subset \mathbb{R}^n$ we set

$$\delta := \sup_{\substack{x\in\Gamma \\ R>0}} \left\{ \inf_{N_0 \in G_{n,n-k}} \max\left(\sup_{y \in \Gamma \cap K_R^{(n)}(x)} \frac{|N_0(y-x)|}{R}, \fint_{\Gamma \cap K_R^{(n)}(x)} \|N - N_0\|\, d\mathcal{H}^k \right) \right\} \tag{5.1}$$

and

$$\delta(R) := \sup_{\substack{x\in\Gamma\\ R\geq r>0}} \left\{ \inf_{N_0\in G_{n,n-k}} \max\left(\sup_{y\in\Gamma\cap K_r^{(n)}(x)} \frac{|N_0(y-x)|}{r}, \fint_{\Gamma\cap K_r^{(n)}(x)} \|N-N_0\|\, d\mathcal{H}^k \right)\right\} \tag{5.2}$$

for $R > 0$. Thus, $\delta = \sup_{R>0}\delta(R)$. We will show below that it is enough to control δ since in fact

$$\gamma \leq 5\delta. \tag{5.3}$$

For $x \in \Gamma$ and $R > 0$ let $\tilde{\mathfrak{N}}_{x,R}$ be the set of all projections $\tilde{N}_{x,R} \in G_{n,n-k}$ satisfying

$$\begin{aligned} &\max\left(\sup_{y\in\Gamma\cap K_R^{(n)}(x)} \frac{|\tilde{N}_{x,R}(y-x)|}{R}, \fint_{\Gamma\cap K_R^{(n)}(x)} \|N-\tilde{N}_{x,R}\|\, d\mathcal{H}^k \right) \\ &\quad = \inf_{N_0\in G_{n,n-k}} \max\left(\sup_{y\in\Gamma\cap K_R^{(n)}(x)} \frac{|N_0(y-x)|}{R}, \fint_{\Gamma\cap K_R^{(n)}(x)} \|N-N_0\|\, d\mathcal{H}^k \right). \end{aligned} \tag{5.4}$$

We set

$$\tilde{\mathfrak{T}}_{x,R} := \{\mathrm{id}_{\mathbb{R}^n} - \tilde{N}_{x,R} : \tilde{N}_{x,R} \in \tilde{\mathfrak{N}}_{x,R}\}. \tag{5.5}$$

Hence to prove the second part of Theorem 1.1 it is enough to show

$$\delta = \sup_{R>0}\delta(R) < C\eta(\Gamma)^{\frac{1}{2}}$$

if η is sufficiently small.

In the proof, we will use the C^1 smoothness of Γ and Proposition 2.7 to get a $\rho_0 := \rho_0(\Gamma)$ such that $\delta(\rho_0)$ is arbitrarily small. Lemma 5.7 then shows that there is a constant $a = a(n,k) > 1$ such that $\delta(a\rho_0)$ can still be estimated. But of course this is not enough to prove the theorem using iteration since the estimate of $\delta(a\rho_0)$ is not as good as the estimate of $\delta(\rho_0)$.

To bridge this gap, we will spend almost all of this section to show that the smallness of η and $\delta(R)$ for some $R > 0$ even implies $\delta(R) \leq C\eta^{\frac{1}{2}}$. This statement is the content of Lemma 5.6. Using this, the theorem follows immediately by iteration.

The keys to the proof of Lemma 5.6 are the Proposition 5.4 and Lemma 5.5. Proposition 5.4 tells us that if there are points $x_0, x_1, \dots, x_k \in \Gamma$ such that the vectors $v_i := \frac{x_i - x_0}{R}$, $i = 1, \dots, k$, are almost orthogonal in the sense that the quantities

$$|\langle v_i, v_j\rangle - \delta_{ij}|$$

are small for all $i, j = 1, \dots, k$, then there is an $N_0 \in G_{n,n-k}$ such that

$$|N_0(y - x_0)| \leq C(n,k)\eta^{\frac{1}{2}} R$$

for all $y \in K_R^{(n)}(x_0) \cap \Gamma$.

We will then use Lemma 5.5 to find such points $x_0, x_1, \ldots, x_k$ under the assumption that $\delta(R)$ and η are small.

The next lemma is the basic step that will finally lead to the proof of Proposition 5.4.

Lemma 5.1 (cf. Lemma 8.5 in [31]). *For $l > 0$ let $c : [0,l] \to \mathbb{R}^n$ be a curve parametrized by arc-length and let $P := c(0)$ and $Q := c(l)$. Then we obtain for all $t \in [0,l]$*

$$\left| c(t) - \left(P + \frac{t}{l}(Q-P) \right) \right| \le 3l \left(\frac{l - |P-Q|}{l} \right)^{\frac{1}{2}}.$$

Proof. Applying a rotation and a translation, we may assume $P = 0$, $Q = |P-Q|e_n$. For $t \in [0,l]$ we estimate vector $\hat{c}(t) := (c_1(t), \ldots, c_{n-1}(t)) \in \mathbb{R}^{n-1}$ by

$$\begin{aligned}
|\hat{c}(t)| &\le \int_0^l |(\dot{c}_1(t), \ldots, \dot{c}_{n-1}(t))| dt \le \sqrt{l} \Big(\int_0^l |(\dot{c}_1(t), \ldots, \dot{c}_{n-1}(t)|^2 dt \Big)^{\frac{1}{2}} \\
&\overset{|\dot{c}|=1}{=} \sqrt{l} \Big(\int_0^l (1 - \dot{c}_n^2) dt \Big)^{\frac{1}{2}} \le \sqrt{l} \Big(2 \int_0^l (1 - \dot{c}_n(t)) dt \Big)^{\frac{1}{2}} \\
&= \sqrt{2l} (l - |P-Q|)^{\frac{1}{2}} \le \sqrt{2} \cdot l \left(\frac{l - |P-Q|}{l} \right)^{\frac{1}{2}}.
\end{aligned}$$

Now $c_n(l) - c_n(t) \le |c_n(l) - c_n(t)| \le l - t$ yields $c_n(t) \ge |P-Q| - (l-t)$ and

$$c_n(t) - \frac{t}{l}|P-Q| \ge (l - |P-Q|) \left(\frac{t}{l} - 1 \right) \ge -(l - |P-Q|).$$

On the other hand, $c_n(t) \le |c(t)| \le t$ implies

$$c_n(t) - \frac{t}{l}|P-Q|e_n \le t - \frac{t}{l}|P-Q| = \frac{t}{l}(l - |P-Q|) \le l - |P-Q|.$$

Hence, $\left| c_n(t) - \frac{t}{l}|P-Q|e_n \right| \le l \big(\frac{l-|P-Q|}{l} \big)$. Using the estimate for $\hat{c}(t)$, we conclude

$$\begin{aligned}
\left| c(t) - \left(P + \frac{t}{l}(Q-P) \right) \right| &\le |\hat{c}(t)| + \left| c_n(t) - \frac{t}{l}|P-Q| \right| \\
&\le l \left(\frac{l - |P-Q|}{l} \right) + \sqrt{2} \cdot l \left(\frac{l - |P-Q|}{l} \right)^{\frac{1}{2}}.
\end{aligned}$$

Since $x \le \sqrt{x}$ for $x \in [0,1]$, we obtain the desired estimate. □

For $A \subset \mathbb{R}^n$ let $\operatorname{conv}(A)$ denote the convex hull of A. Iterating the above lemma we now prove

Lemma 5.2 (Analog to Lemma 8.4 in [31]). *Let* $\Gamma \subset \mathbb{R}^n$ *be a* k*-dimensional chord-arc submanifold with* $18n\eta^{\frac{1}{2}} \leq 1$. *Then for all* $x \in \Gamma$ *and* $R > 0$ *we have*

$$\operatorname{conv}(\Gamma \cap K_R^{(n)}(x)) \subset \left\{z \in \mathbb{R}^n : \operatorname{dist}(z, \Gamma) \leq 18n\eta^{\frac{1}{2}} R\right\}.$$

Proof. Let $y \in \operatorname{conv}(\Gamma \cap K_R^{(n)}(x))$. From Carathéodory's theorem (cf. Theorem 17.1 in [28]) we get that there are $a_1, \ldots, a_\nu \in \Gamma \cap K_R^{(n)}(x)$ and $0 < \lambda_1, \ldots, \lambda_\nu \leq 1$, $\nu \leq n+1$, with $\sum_{i=1}^{\nu} \lambda_i = 1$ such that $y = \sum_{i=1}^{\nu} \lambda_i a_i$. We show now inductively that for $j = 1, \ldots, \nu$ we have

$$\operatorname{dist}\left(\frac{\sum_{i=1}^{j} \lambda_i a_i}{\sum_{i=1}^{j} \lambda_i}, \Gamma\right) \leq 18(j-1)\eta^{\frac{1}{2}} R$$

and thus prove the lemma. The estimate is trivial for $j = 1$. So let the estimate be true for $1 \leq j < \nu$, i.e. let us assume that there is a point $P \in \Gamma$ with

$$\left|\frac{\sum_{i=1}^{j} \lambda_i a_i}{\sum_{i=1}^{j} \lambda_i} - P\right| \leq 18(j-1)\eta^{\frac{1}{2}} R.$$

Let us put $\tilde{P} := \frac{\sum_{i=1}^{j} \lambda_i a_i}{\sum_{i=1}^{j} \lambda_i}$. Then the above estimate reads

$$|\tilde{P} - P| \leq 18(j-1)\eta^{\frac{1}{2}} R. \tag{5.6}$$

Furthermore we set $Q := \tilde{Q} := a_{j+1}$ and thus get

$$\tilde{P} + \frac{\lambda_{j+1}}{\sum_{i=1}^{j+1} \lambda_i} \left(\tilde{Q} - \tilde{P}\right) = \frac{\sum_{i=1}^{j+1} \lambda_i a_i}{\sum_{i=1}^{j+1} \lambda_i} \tag{5.7}$$

and $|P - Q| \leq |P - \tilde{P}| + |\tilde{P} - Q| \leq 3R$. Since $P, Q \in \Gamma$, there is a Lipschitz curve $c : [0, l] \to \Gamma$ parametrized by arc-length joining P and Q with $l \leq (1+\eta)|P - Q|$. If we now apply Lemma 5.1 with $t_0 = \frac{\lambda_{j+1}}{\sum_{i=1}^{j+1} \lambda_i} l$ to this curve we get

$$\begin{aligned} \left|c(t_0) - \left(P + \frac{\lambda_{j+1}}{\sum_{i=1}^{j+1} \lambda_i}(Q - P)\right)\right| &\leq 3l \left(\frac{l - |P - Q|}{l}\right)^{\frac{1}{2}} \\ &\overset{|P-Q| \leq l \leq (1+\eta)|P-Q|)}{\leq} 3(1+\eta)|P - Q|\eta^{\frac{1}{2}} \\ &\overset{\eta \leq \frac{1}{2}, |P-Q| \leq 3R}{\leq} 18\eta^{\frac{1}{2}} R. \end{aligned} \tag{5.8}$$

Hence,

$$\begin{aligned}\operatorname{dist}\left(\frac{\sum_{i=1}^{j}\lambda_i a_i}{\sum_{i=1}^{j}\lambda_i},\Gamma\right) &\overset{c(t_0)\in\Gamma}{\leq} \left|c(t_0)-\frac{\sum_{i=1}^{j}\lambda_i\cdot a}{\sum_{i=1}^{j}\lambda_i}\right|\\ &\overset{(5.7)}{=} \left|c(t_0)-\left(\tilde P+\frac{t_0}{l}(\tilde Q-\tilde P)\right)\right|\\ &\overset{Q=\tilde Q}{\leq} \left|c(t_0)-\left(P+\frac{t_0}{l}(Q-P)\right)\right|+|\tilde P-P|\\ &\overset{(5.8)\&(5.6)}{\leq} 18R\eta^{\frac12}+18R(j-1)\eta^{\frac12}=18Rj\eta^{\frac12}. \qquad \square\end{aligned}$$

A consequence of the last lemma is the following estimate for the volume of the convex hull of $\Gamma\cap K_R^{(n)}(x)$.

Lemma 5.3 (Analog to Lemma 8.7 in [31]). *Let $\Gamma\subset\mathbb{R}^n$ be a k-dimensional chord-arc submanifold, $18n\eta^{\frac12}\leq 1$, and let V be a $(k+1)$-dimensional affine subspace. Then we have*

$$\mathcal{H}^{k+1}(\operatorname{conv}(\Gamma\cap K_R^{(n)}(x))\cap V)\leq C(n,k)\eta^{\frac12}R$$

where $C(n,k):=3\cdot 36\cdot\omega_{k+1}\cdot 8^k\cdot n$.

Proof. From Lemma 5.2 we get

$$\operatorname{conv}(\Gamma\cap K_R^{(n)}(x))\subset\bigcup_{z\in\Gamma}K^{(n)}_{18n\eta^{\frac12}R}(z).$$

Since $\operatorname{conv}(\Gamma\cap K_R^{(n)}(x))\subset K_R^{(n)}(x)$ and $18n\eta^{\frac12}\leq 1$ we obtain

$$\operatorname{conv}(\Gamma\cap K_R^{(n)}(x))\subset\bigcup_{z\in\Gamma\cap K_{2R}^{(n)}(x)}K^{(n)}_{18n\eta^{\frac12}R}(z). \tag{5.9}$$

Using Zorn's lemma if you wish, we can find a maximal subset $L\subset\Gamma\cap K_{2R}^{(n)}(x)$ with respect to the order "$\subset$" with the property that $u\neq v\in L$ implies $|u-v|\geq 18n\eta^{\frac12}R$. From the maximality of the set we deduce that

$$\Gamma\cap K_{2R}^{(n)}(x)\subset\bigcup_{z\in L}K^{(n)}_{18n\eta^{\frac12}R}(z)$$

and hence

$$\operatorname{conv}(\Gamma\cap K_R^{(n)}(x))\subset\bigcup_{z\in L}K^{(n)}_{36n\eta^{\frac12}R}(z). \tag{5.10}$$

Since $18n\eta^{\frac{1}{2}} \leq 1$, we get $R + 9n\eta^{\frac{1}{2}}R \leq 2R$ and thus $B^{(n)}_{9n\eta^{\frac{1}{2}}}(z) \subset B^{(n)}_{2R}(x)$ for all $z \in L$. Using the definition of η_2 (cf. (1.3)) and the fact that the balls $B^{(n)}_{9n\eta^{\frac{1}{2}}R}(z)$, $z \in L$ are pairwise disjoint, we get

$$\#L = \sum_{z\in L} \frac{\mathcal{H}^k(K^{(n)}_{9n\eta^{\frac{1}{2}}R}(z) \cap \Gamma)}{\mathcal{H}^k(K^{(n)}_{9n\eta^{\frac{1}{2}}R}(z) \cap \Gamma)} \leq \sum_{z\in L} \frac{\mathcal{H}^k(K^{(n)}_{9n\eta^{\frac{1}{2}}R}(z) \cap \Gamma)}{\frac{1}{2}\omega_k(9n\eta^{\frac{1}{2}}R)^k} \leq \frac{\mathcal{H}^k(K^{(n)}_{2R}(x) \cap \Gamma)}{\frac{1}{2}\omega_k(9n\eta^{\frac{1}{2}}R)^k}$$
$$\leq 3\left(\frac{2}{9n\eta^{\frac{1}{2}}}\right)^k.$$

Combining this with (5.9) and (5.10), we finally get

$$\begin{aligned}\mathcal{H}^{k+1}(\operatorname{conv}(\Gamma \cap K^{(n)}_R(x)) \cap V) &\leq \mathcal{H}^{k+1}\left(\bigcup_{z\in L} K^{(n)}_{36n\eta^{\frac{1}{2}}R}(z) \cap V\right)\\ &\leq \sum_{z\in L} \mathcal{H}^{k+1}(K^{(n)}_{36n\eta^{\frac{1}{2}}R}(z) \cap V)\\ &\leq (\#L)\omega_{k+1}(36n\eta^{\frac{1}{2}}R)^{k+1}\\ &\leq 3\left(\frac{2}{9n\eta^{\frac{1}{2}}}\right)^k \omega_{k+1}(36n\eta^{\frac{1}{2}}R)^{k+1}\\ &= C(n,k)\eta^{\frac{1}{2}}R^{k+1}\end{aligned}$$

where $C(n,k) := 3 \cdot 36 \cdot \omega_{k+1} \cdot 8^k \cdot n$. □

Proposition 5.4 (Analog to Lemma 8.7 in [31]). *Let $x_0, x_1, \ldots, x_k \subset \Gamma$ be such that the vectors $v_i := \frac{x_i - x_0}{R}$, $i = 1, \ldots, k$ are almost orthogonal, i.e. that*

$$|\langle v_i, v_j\rangle - \delta_{ij}| \leq \varepsilon_k$$

for all $i, j = 1, \ldots, k$, where $\varepsilon_k := \min\{k^{-1/2}(2^{\frac{1}{k-1}} - 1), k^{-\frac{3}{2}}/4\}$. Furthermore, let $18n\eta^{\frac{1}{2}} \leq 1$ and N_0 denote the orthogonal projection of $\mathbb{R}^n$ onto the vector space spanned by $v_1, \ldots, v_k$. Then

$$|N_0(y - x_0)| \leq C(n,k)\eta^{\frac{1}{2}}R$$

for all $y \in \Gamma \cap K^{(n)}_R(x_0)$ with $C(n,k) := 12 \cdot 36 \cdot \omega_{k+1} \cdot 32^k \cdot n$.

Proof. Let us translate the whole setting so that $x_0 = 0$. Let $y \in \Gamma \cap K^{(n)}_R(x_0)$ with $\mu := N_0(y) \neq 0$ and V be the vector space spanned by y and the vectors

$v_1, \dots, v_k$. Then there is a unit vector $v^\perp$ with $\langle v^\perp, v_i \rangle = 0$ for all $i = 1, \dots, k$ and $\nu_1, \dots, \nu_k \in \mathbb{R}$ such that $y = \sum_{i=1}^k \nu_i x_i + \mu v^\perp$. Let us consider the map

$$g : \Delta_{k+1} := \{(\lambda_1, \dots, \lambda_{k+1}) \in (\mathbb{R}_+)^{k+1} : \sum_{i=1}^{k+1} \lambda_i \le 1\} \to \operatorname{conv}(\Gamma \cap K_{2R}^{(n)}(0)) \cap V,$$

$$(\lambda_1, \dots, \lambda_{k+1}) \mapsto \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} y.$$

Using $y = \sum_{i=1}^k \nu_i x_i + \mu v^\perp$ the Jacobian determinant of the function g can be shown to satisfy

$$\begin{aligned} \det\left((Dg)^* \circ Dg)\right) &= \mu^2 \cdot \det\left(\langle x_i, x_j \rangle_{i,j=1,\dots,k}\right) \\ &= \mu^2 R^{2k} \cdot \det\left(\langle v_i, v_j \rangle_{i,j=1,\dots,k}\right). \end{aligned}$$

We set $w_i := (\langle v_1, v_i \rangle, \dots, \langle v_k, v_i \rangle)^T$ and let $e_1, \dots, e_k$ denote the standard basis of $\mathbb{R}^k$. Using the inequality of Hadamard and the multilinearity of the determinant, we obtain

$$\begin{aligned} \det\left(\langle v_i, v_j \rangle_{i,j=1,\dots,k}\right) &= \det(w_1, \dots, w_k) \\ &\ge \det(e_1, \dots, e_k) - |\det(w_1, \dots, w_k) - \det(e_1, \dots, e_k)| \\ &= 1 - |\sum_{i=1}^k \det(e_1, \dots, e_{i-1}, w_i - e_i, w_{i+1}, \dots, w_k)| \\ &\ge 1 - (\sup\{1, |w_1|, \dots, |w_k|\})^{k-1} \sum_{i=0}^k |w_i - e_i|. \end{aligned}$$

Combining this with $|w_i - e_i| \le \sqrt{k}\, \varepsilon_k$ and $|w_i| \le 1 + \sqrt{k}\varepsilon_k$, we get

$$\det\left(\langle v_i, v_j \rangle_{i,j=1,\dots,k}\right) \ge 1 - (1 + \sqrt{k}\varepsilon_k)^{k-1} k^{\frac{3}{2}} \varepsilon_k \ge 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

Thus,

$$\det(Dg^* \circ Dg) \ge \frac{1}{2} \mu^2 R^{2k}.$$

This implies that the function g is a diffeomorphism onto its image. Using Lemma 5.3

and the area formula, we hence get

$$\begin{aligned}\tilde{C}(n,k)\eta^{\frac{1}{2}}(2R)^{k+1} &\geq \mathcal{H}^{k+1}(\operatorname{conv}(\Gamma\cap K^{(n)}_{2R}(x_0))\cap V) \geq \mathcal{H}^{k+1}(\operatorname{Im}(g))\\ &= \int_{\Delta_{k+1}} \sqrt{\det(Dg^*\circ Dg)}\, d\mathcal{H}^{k+1} \geq \frac{1}{2}\mu R^k \mathcal{H}^{k+1}(\Delta_{k+1})\\ &= \left(\frac{1}{2}\right)^{k+1} \mu R^k.\end{aligned}$$

with $\tilde{C}(n,k) := 3\cdot 36\,\omega_{k+1}8^k n$ and thus $\mu \leq 12\cdot 36\,\omega_{k+1}32^k n\eta^{\frac{1}{2}}R$. □

The next lemma will be used to prove the existence of points $x_0,\dots,x_k$ satisfying the assumptions of Proposition 5.4. Let

$$\begin{aligned}&\delta(x,R)\\ &:= \inf_{N_0\in G_{n,n-k}}\left(\max\left(\sup_{y\in\Gamma\cap K^{(n)}_R(x)} \frac{|N_0(y-x)|}{R}, \fint_{\Gamma\cap K^{(n)}_R(x)} |N-N_0|\, d\mathcal{H}^k\right)\right).\end{aligned} \tag{5.11}$$

Note that (5.2) and (5.11) imply $\delta(R) = \sup_{x\in\Gamma}\delta(x,R)$ and $\delta(x,R)\leq\delta(R)\leq\delta$.

Lemma 5.5. *Let $\Gamma\subset\mathbb{R}^n$ be a k-dimensional chord-arc submanifold with $\eta(\Gamma)\leq\frac{1}{2}$, $x\in\Gamma$, and $R>0$.*

(i) If $\delta(R) < \frac{1}{10^5\cdot 176^k}$, then

$$\tilde{T}_{x,R}(\Gamma\cap K^{(n)}_R(x)) \supset \tilde{T}_{x,R}(K^{(n)}_{(1-\delta(x,R))R}(x))$$

for all $\tilde{T}_{x,R}\in\tilde{\mathfrak{T}}_{x,R}$.

(ii) If $\delta(R)\leq\frac{1}{10^5\cdot 176^k}$ and $N_0\in G_{n,k}$ with

$$\frac{|N_0(y-x)|}{R} \leq \mu < \frac{1}{8} \qquad \forall y\in\Gamma\cap K^{(n)}_R(x),$$

and $(\delta(R)+\mu)\leq\frac{1}{12\cdot 8\cdot 10\cdot k}$, then

$$T_0(\Gamma\cap K^{(n)}_R(x)) \supset T_0(K^{(n)}_{(1-\mu)R}(x))$$

where $T_0 := \mathrm{id}_{\mathbb{R}^n} - N_0$.

Proof. The proof relies on degree theory combined with calculations that are similar to those used in the proof of Theorem 3.1.

We consider the map $f_1 := \tilde{T}_{x,R}|_{\Gamma\cap K^{(n)}_R(x)}$. From (5.4), (5.5), and (5.11) we get

$$\left(\tilde{T}_{x,R}(\partial_\Gamma(\Gamma\cap K^{(n)}_R(x)))\right)\cap\left(\tilde{T}_{x,R}(B^{(n)}_{(1-\delta(x,R))R}(x))\right) = \emptyset. \tag{5.12}$$

We will show, that there is a point $w_0 \in \tilde{T}_{x,R}(B^{(n)}_{(1-\delta(x,R))R}(x))$ with

$$\deg(f, \Gamma \cap K^{(n)}_R(x), w_0) = 1 + 2\mathbb{Z}.$$

From the properties of the degree and (5.12) we then get the conclusion of this lemma.

Let $y \neq z \in \Gamma \cap K^{(n)}_R(x)$ and $\tilde{N}_{y,|z-y|} \in \tilde{\mathfrak{N}}_{y,|z-y|}$. We see that

$$\begin{aligned} \left|\tilde{N}_{x,R}(z-y)\right| &\le \left|\tilde{N}_{y,|z-y|}(z-y)\right| + \left|\left(\tilde{N}_{y,|z-y|} - \tilde{N}_{x,R}\right)(z-y)\right| \\ &\le \left(\delta(R) + \left\|\tilde{N}_{y,|z-y|} - \tilde{N}_{x,R}\right\|\right)|z-y| \end{aligned}$$

and

$$\begin{aligned} \left\|\tilde{N}_{y,|z-y|} - \tilde{N}_{x,R}\right\| \le &\fint_{\Gamma\cap K^{(n)}_{|z-y|}(y)} \|\tilde{N}_{y,|z-y|} - N\| \, d\mathcal{H}^k \\ &+ \fint_{\Gamma\cap K^{(n)}_{|z-y|}(y)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \\ &\overset{|x-y|\le 2R}{\le} \delta(R) + \mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y). \end{aligned}$$

We are looking for a $y_0 \in \Gamma \cap K^{(n)}_{\frac{R}{2}}(x)$ with

$$\mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y_0) \le \frac{1}{4}$$

since for such a point we would get

$$\left|\tilde{N}_{x,R}(z-y_0)\right| \le \frac{1}{2}|z-y_0| \quad \forall z \in \Gamma \cap K^{(n)}_R(x) \tag{5.13}$$

if we combine the last two inequalities.

Using the Hardy–Littlewood maximal theorem (cf. Lemma 2.3) and the fact that $\mathcal{H}^k \lfloor \Gamma$ has the doubling property by definition of η, we see that

$$\begin{aligned} &\mathcal{H}^k\left(\left\{y \in \Gamma \cap K^{(n)}_{\frac{R}{2}}(x) : \mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y) > \frac{1}{4}\right\}\right) \\ &\le \mathcal{H}^k\left(\left\{y \in \Gamma : \mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\chi_{K^{(n)}_{\frac{5}{2}R}(x)}\right)(y) > \frac{1}{4}\right\}\right) \\ &\le 4 \cdot 27 \cdot 2^{3k} \int_{\Gamma\cap K^{(n)}_{\frac{5}{2}R}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k. \end{aligned} \tag{5.14}$$

Here $\chi_{K^{(n)}_{\frac{5}{2}R}(x)}$ denotes the characteristic function of the set $K^{(n)}_{\frac{5}{2}R}(x)$. To estimate the last integral, let us choose a maximal subset $L \subset \Gamma \cap K^{(n)}_{\frac{5}{2}R}(x)$ with the property that $u \neq v \in L$ implies $|u-v| \geq \frac{1}{2}R$. From the maximality of the set we get $\bigcup_{z\in L} K^{(n)}_{\frac{R}{2}}(z) \supset K^{(n)}_{\frac{5}{2}R}(x) \cap \Gamma$. Since the balls $B^{(n)}_{\frac{1}{4}R}(z)$, $z \in L$ are pairwise disjoint and $\eta \leq \frac{1}{2}$, we get

$$\begin{aligned}\#L = \sum_{z\in L} \frac{\mathcal{H}^k(\Gamma \cap B^{(n)}_{\frac{1}{4}R}(z))}{\mathcal{H}^k(\Gamma \cap B^{(n)}_{\frac{1}{4}R}(z))} &\overset{(1.3)}{\leq} \frac{2}{\omega_k \left(\frac{1}{4}R\right)^k} \sum_{z\in Z} \mathcal{H}^k(\Gamma \cap B^{(n)}_{\frac{1}{4}R}(z)) \\ &\leq \frac{2}{\omega_k \left(\frac{1}{4}R\right)^k} H^k(\Gamma \cap B^{(n)}_{\frac{11}{4}R}(x)) \overset{(1.3)}{\leq} 3 \cdot 11^k\end{aligned} \tag{5.15}$$

and we see that

$$\int_{\Gamma\cap K^{(n)}_{\frac{5}{2}R}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \leq \sum_{z\in L} \int_{\Gamma\cap K^{(n)}_{\frac{1}{2}R}(z)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k. \tag{5.16}$$

For $z \in \Gamma \cap K^{(n)}_{\frac{5}{2}R}(x)$ there is a curve $c : [0,l] \to \Gamma$ parametrized by the arc-length joining x and z, i.e. with $c(0) = x$ and $c(l) = z$, and with $l \leq (1+\eta) \cdot \frac{5}{2}R \leq 4R$. We set $\tau_i := \frac{l}{8} \cdot i$ for $i = 0, \dots, 8$. For $\tilde{N}_{c(\tau_i),\frac{R}{2}} \in \tilde{\mathfrak{N}}_{c(\tau_i),\frac{R}{2}}$ we get

$$\begin{aligned}\|N - \tilde{N}_{x,R}\| &\leq \|N - \tilde{N}_{z,\frac{R}{2}}\| + \sum_{i=1}^{8} \|\tilde{N}_{c(\tau_i),\frac{R}{2}} - \tilde{N}_{c(\tau_{i-1}),\frac{R}{2}}\| + \|\tilde{N}_{x,\frac{R}{2}} - \tilde{N}_{x,R}\| \\ &\leq \|N - \tilde{N}_{z,\frac{R}{2}}\| \\ &\quad + \sum_{i=1}^{8} \left(\|\tilde{N}_{c(\tau_i),\frac{R}{2}} - \tilde{N}_{c(\tau_{i-1}),R}\| + \|\tilde{N}_{c(\tau_{i-1}),R} - \tilde{N}_{c(\tau_{i-1}),\frac{R}{2}}\| \right) \\ &\quad + \|\tilde{N}_{x,\frac{R}{2}} - \tilde{N}_{x,R}\|.\end{aligned}$$

For $v, u \in \Gamma$ with $K^{(n)}_{\frac{R}{2}}(v) \subset K^{(n)}_{R}(u)$ we have

$$\begin{aligned}\|\tilde{N}_{v,\frac{R}{2}} - \tilde{N}_{u,R}\| &\leq \fint_{\Gamma\cap K^{(n)}_{\frac{R}{2}}(v)} \|\tilde{N}_{v,\frac{R}{2}} - N\| \mathcal{H}^k + \fint_{\Gamma\cap K^{(n)}_{\frac{R}{2}}(v)} \|N - \tilde{N}_{u,R}\| \mathcal{H}^k \\ &\overset{K_{\frac{R}{2}}(v)\subset K_R(u)}{\leq} \delta(R) + \frac{\mathcal{H}^k(\Gamma \cap K^{(n)}_{R}(u))}{\mathcal{H}^k(\Gamma \cap K^{(n)}_{\frac{R}{2}}(v))} \fint_{\Gamma\cap K^{(n)}_{R}(u)} \|N - \tilde{N}_{u,R}\| \mathcal{H}^k \\ &\leq \delta(R) + \frac{1+\eta}{1-\eta} 2^k \delta(R) \leq (1 + 3 \cdot 2^k) \cdot \delta(R),\end{aligned}$$

and we obtain, since $|c(\tau(i)) - c(\tau(i-1))| \leq \frac{1}{2}R$,

$$\|N - \tilde{N}_{x,R}\| \leq \|N - \tilde{N}_{z,\frac{R}{2}}\| + 17 \cdot (1 + 3 \cdot 2^k) \cdot \delta(R). \tag{5.17}$$

Combining the inequalities (5.14)–(5.17) one gets

$$\frac{\mathcal{H}^k\left(\left\{y \in \Gamma \cap K^{(n)}_{\frac{R}{2}}(x) : \mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y) > \frac{1}{4}\right\}\right)}{\mathcal{H}^k(\Gamma \cap K^{(n)}_{\frac{R}{2}}(x))} \leq 10^5 \cdot 176^k \delta(R) < 1.$$

So we can find a $y_0 \in \Gamma \cap K^{(n)}_{\frac{R}{2}}(x)$ such that

$$\left|\mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y_0)\right| \leq \frac{1}{4},$$

and we have by (5.13)

$$|N(z - y_0)| \leq \frac{1}{2}|z - y_0| \quad \forall z \in \Gamma \cap K^{(n)}_R(x) \tag{5.18}$$

and

$$\begin{aligned} \left\|N(y_0) - \tilde{N}_{x,R}\right\| &= \lim_{r \to 0} \fint_{K_r(y_0) \cap \Gamma} \left\|N - \tilde{N}_{x,R}\right\| \, d\mathcal{H}^k \\ &\leq \left|\mathfrak{M}_{2R}\left((N - \tilde{N}_{x,R})\right)(y_0)\right| \leq \frac{1}{4}. \end{aligned} \tag{5.19}$$

From (5.18) and (5.19) one can now deduce that $\deg(f_1, \Gamma \cap K^{(n)}_R(x), w_0) = 1 + 2\mathbb{Z}$ for $w_0 := f_1(y_0)$ and so we the first part of the lemma is shown.

To prove the second part, we set $f_2 := T_0|_{\Gamma \cap K^{(n)}_R(x)}$ and translate $\mathbb{R}^n$ so that we can assume $x = 0$. Arguing as above, it is enough to find a point $w_0 \in T_0(B^{(n)}_{(1-\mu)R}(0))$ with $\deg(f_2, \Gamma \cap K^{(n)}_R(0), w_0) = 1 + 2\mathbb{Z}$ since

$$T_0(\partial_\Gamma(\Gamma \cap K^{(n)}_R(0))) \cap T_0(B^{(n)}_{(1-\mu)R}(0)) = \emptyset.$$

First we estimate $\|N_0 - \tilde{N}_{0,R}\|$. Let $\tilde{e}_1, \ldots, \tilde{e}_k$ be an orthonormal basis of $\mathrm{Im}(\tilde{T}_{0,R})$. Using the first part, we can find $v_1, \ldots, v_k \in \Gamma \cap K^{(n)}_R(0)$ with $\tilde{T}_{0,R}(v_i) = (1 - \delta(R))R\tilde{e}_i$. If we fix $w_i := \frac{1}{(1-\delta(R))R}T_0(v_i)$, we get

$$\begin{aligned} |w_i - \tilde{e}_i| &= \frac{1}{(1 - \delta(R))R}\left|T_0(v_i) - \tilde{T}_{0,R}(v_i)\right| \\ &\overset{\delta(R) \leq \frac{1}{2}}{\leq} \frac{2}{R}\left|N_0(v_i) - \tilde{N}_{0,R}(v_i)\right| \\ &\leq \frac{2}{R}\left(\left|\tilde{N}_{0,R}(v_i)\right| + |N_0(v_i)|\right) \leq 2(\delta(R) + \mu) \end{aligned}$$

for $i = 1, \dots, k$. Let $A, B : \mathbb{R}^k \to \mathbb{R}^n$ be the linear mappings represented by the matrices $(w_1, \dots, w_k)$ and $(e_1, \dots, e_k)$. Then we get

$$\|A - B\| \leq 2k(\delta(R) + \mu) \leq \frac{1}{12 \cdot 8} < 1.$$

Hence, the vectors $w_1, \dots, w_k$ are linearly independent since otherwise there would be a vector $u \in \mathbb{S}^{k-1}$ with

$$A(u) = 0$$

and thus

$$\|A - B\| \geq |(A - B)(u)| \geq |B(u)| - |A(u)| = 1.$$

Hence, we can apply the normal equations (cf. [34, pp. 235–237])

$$T_0 = A \circ \left(A^* \circ A\right)^{-1} \circ A^*$$

and

$$\tilde{T}_{0,R} = B \circ \left(B^* \circ B\right)^{-1} \circ B^*$$

and we can estimate

$$\begin{aligned} \left\|T_0 - \tilde{T}_{0,R}\right\| \leq \|A - B\| & \left\|\left(A^* \circ A\right)^{-1}\right\| \|A^*\| \\ & + \|B\| \left\|\left(A^* \circ A\right)^{-1} - \left(B^* \circ B\right)^{-1}\right\| \|A^*\| \\ & + \|B\| \left\|\left(B^* \circ B\right)^{-1}\right\| \|A^* - B^*\|. \end{aligned}$$

Combining this with

$$\begin{aligned} \|B\| &= 1, \\ \|A^*\| &= \|A\| \leq \|B\| + \|A - B\| < 2, \\ \|\mathrm{id}_k - A^* A\| &\leq 5k(\delta(R) + \mu) \leq \frac{1}{12 \cdot 8}, \\ \left\|\left(A^* \circ A\right)^{-1}\right\| &\leq \frac{1}{1 - \|\mathrm{id}_{\mathbb{R}^k} - A^* \circ A\|} \leq 2, \\ \left\|\left(A^* \circ A\right)^{-1} - \left(B^* \circ B\right)^{-1}\right\| &= \left\|\left(A^* \circ A\right)^{-1} - \mathrm{id}_{\mathbb{R}^k}\right\| \\ &\leq \left\|\left(A^* \circ A\right)^{-1}\right\| \cdot \left\|\mathrm{id}_{\mathbb{R}^k} - A^* \circ A\right\| \\ &\leq 10k \cdot (\delta(R) + \mu) < \frac{1}{12 \cdot 8}, \end{aligned}$$

we get

$$\left\| T_0 - \tilde{T}_{0,R} \right\| \leq \frac{1}{8}. \tag{5.20}$$

In the proof of the first part we have shown that there is a $y_0 \in \Gamma \cap K^{(n)}_{R/2}(0) \subset K^{(n)}_{(1-\mu)R}(0)$ with $\left|\mathfrak{M}_{2R}\left((N - \tilde{N}_{0,R})\right)(y_0)\right| \leq \frac{1}{4}$ and that this implies

$$\left|\tilde{N}_{0,R}(z - y_0)\right| \leq \frac{1}{2}\,|z - y_0| \quad \forall z \in \Gamma \cap K^{(n)}_R(0)$$

and $\left\| N(y_0) - N_{0,R} \right\| \leq \frac{1}{4}$. Combined with (5.20) this leads to

$$|N_0(z - y_0)| \leq \left|\left(N_0 - \tilde{N}_{0,R}\right)(z - y_0)\right| + \left|\tilde{N}_{0,R}(z - y_0)\right| \leq \frac{7}{8}\,|z - y_0|$$

for all $z \in K^{(n)}_R(0)$ and

$$\|N(y_0) - N_0\| \leq \|N(y_0) - \tilde{N}_{0,R}\| + \left\|N_0 - \tilde{N}_{0,R}\right\| \leq \frac{3}{8}.$$

From these estimates and setting $w_0 := T_0(y_0)$ we get $\deg(f_2, \Gamma \cap K^{(n)}_R(0), w_0) = 1 + 2\mathbb{Z}$. □

Let us now show that in fact

$$\delta(R) \leq C\eta^{\frac{1}{2}}$$

if $\delta(R)$ and η are small enough.

Lemma 5.6. *There is an $\varepsilon = \varepsilon(n,k) > 0$ and a constant $C = C(n,k) < \infty$ such that for every k-dimensional chord-arc submanifold $\Gamma \subset \mathbb{R}^n$ of dimension k, then $\eta, \delta(R) \leq \varepsilon$ implies $\delta(R) \leq C(n,k)\eta^{\frac{1}{2}}$.*

Proof. Let $x \in \Gamma$, $R > 0$, $\tilde{T}_{x,R} \in \tilde{\mathfrak{T}}_{x,R}$, and let $e_1, \ldots, e_k$ be an orthonormal basis of $\mathrm{Im}(\tilde{T}_{x,R})$. Lemma 5.5 shows that there are $x_1, \ldots, x_k \in \Gamma \cap K^{(n)}_R(x)$ such that $\tilde{T}_{x,R}(x_i - x) = (1 - \delta(R))Re_i$. We get

$$\begin{aligned}\left|\left\langle \frac{x_i - x}{R}, \frac{x_j - x}{R} \right\rangle - \delta_{ij}\right| &\leq \left| \frac{1}{R^2}\Big(\left\langle \tilde{T}_{x,R}(x_i - x), \tilde{T}_{x,R}(x_j - x) \right\rangle \right.\\ &\qquad \left. + \left\langle \tilde{N}_{x,R}(x_i - x), \tilde{N}_{x,R}(x_j - x) \right\rangle \Big) - \delta_{ij} \right| \\ &\leq 2\delta(R)^2 \leq \varepsilon_k\end{aligned}$$

if $\delta(R)$ is small enough and $\varepsilon_k := \min\left\{\frac{\sqrt[k-1]{2}-1}{k^{\frac{1}{2}}}, \frac{1}{4k^{\frac{3}{2}}}\right\}$ is as in Proposition 5.4. By Proposition 5.4 there is an $N_0 \in G_{n,n-k}$ such that $|N_0(y-x)| \leq C\eta^{\frac{1}{2}}R$ for all $y \in \Gamma \cap K_R^{(n)}(x)$. So it remains to prove that

$$\fint_{\Gamma \cap K_R^{(n)}(x)} \|N - N_0\| \, d\mathcal{H}^k \leq C\eta^{\frac{1}{2}}.$$

Let us translate and rotate the whole picture in such a way that we get $x = 0$ and $\operatorname{Im}(T_0) = \mathbb{R}^k \times \{0\}$. By Lemma 5.5

$$T_0(\Gamma \cap K_R^{(n)}(0)) \supset K^{(k)}_{(1-C\eta^{\frac{1}{2}})R}(0) \times \{0\}.$$

Defining

$$X := \left(\Gamma \cap K_R^{(n)}(0)\right) \cap \left(K^{(k)}_{(1-C\eta^{\frac{1}{2}})R}(0) \times \mathbb{R}^{n-k}\right) \supset \Gamma \cap K^{(n)}_{(1-C\eta^{\frac{1}{2}})R}(0)$$

we get

$$\begin{aligned}
\mathcal{H}^k\left(\left(\Gamma \cap K_R^{(n)}(0)\right) - X\right) &= \mathcal{H}^k\left(\Gamma \cap K_R^{(n)}(0)\right) - \mathcal{H}^k(X) \\
&\overset{(1.3)}{\leq} (1+\eta)\omega_k R^k - (1-\eta)\omega_k\left((1 - C\eta^{\frac{1}{2}})R\right)^k \qquad (5.21) \\
&\leq C\eta^{\frac{1}{2}}R^k
\end{aligned}$$

if η is small enough since the function $\xi \to 1 + \xi^2 - (1-\xi^2)(1-C\xi)^k$ is 0 at $\xi = 0$ and differentiable at this point.

Let $J(y)$ be the Jacobian determinant of $F := T_0|_\Gamma$, i.e.

$$J(y) := \sqrt{\det(DF^*(y) \circ DF(y))}.$$

Using the area formula and the fact that by Lemma 5.5

$$T_0^{-1}(y) \cap X \neq \emptyset$$

for all $y \in K^{(k)}_{(1-C\eta^{\frac{1}{2}})R}(0) \times \{0\}$ we get

$$\begin{aligned}
\int_X J(y)\, d\mathcal{H}^k(y) &= \int_{K^{(k)}_{(1-C\sqrt{\eta})R}(0)\times\{0\}} \mathcal{H}^0(T_0^{-1}(y) \cap X)\, d\mathcal{H}^k(y) \\
&\geq \omega_k((1 - C\eta^{\frac{1}{2}})R)^k.
\end{aligned} \qquad (5.22)$$

Now, we show that

$$J(y) \leq 1 - \frac{\|T(y) - T_0\|^2}{4n}. \qquad (5.23)$$

In order to prove (5.23) we first deduce

$$\det(DF^*(y) \circ DF(y)) = \det(\mathrm{id}_{\mathbb{R}^n} - T_0 \circ N(y) \circ T_0).$$

This is true because $DF(y) = T_0|_{T_y\Gamma}$, $DF^*(y) = T(y) \circ T_0$ and thus

$$\det(DF^*(y) \circ DF(y)) = \det(T(y) \circ T_0|_{T_y\Gamma}) = \det(T(y) \circ T_0 \circ T(y) + N(y)).$$

Furthermore, we have used

$$\begin{aligned} T(y) \circ T_0 \circ T(y) + N(y) &= T(y) \circ (\mathrm{id}_{\mathbb{R}^n} - N_0) \circ T(y) + N(y) \\ &= T(y) + N(y) - T(y) \circ N_0 \circ T(y) \\ &= \mathrm{id}_{\mathbb{R}^n} - T(y) \circ N_0 \circ T(y). \end{aligned}$$

Since $\mathrm{id}_{\mathbb{R}^n} - T(y) \circ N_0 \circ T(y)$ is a symmetric matrix, the inequality between arithmetic and geometric mean leads to

$$J^2(y) = \det(\mathrm{id}_{\mathbb{R}^n} - T(y) \circ N_0 \circ T(y)) \leq \left(\frac{\mathrm{trace}(\mathrm{id}_{\mathbb{R}^n} - T(y) \circ N_0 \circ T(y))}{n} \right)^n.$$

Now,

$$\begin{aligned} \mathrm{trace}(T(y) \circ N_0 \circ T(y)) &= \mathrm{trace}(T(y) - T(y) \circ T_0) = k - \mathrm{trace}(T(y)T_0) \\ &= \frac{1}{2} \mathrm{trace}\left((T(y) - T_0)^2\right) \geq \frac{1}{2} \|T(y) - T_0\|^2 \end{aligned}$$

yields

$$J(y) \leq \left(1 - \frac{\|T(y) - T_0\|^2}{2n}\right)^{\frac{n}{2}} \leq \left(1 - \frac{\|T(y) - T_0\|^2}{2n}\right)^{\frac{1}{2}} \leq 1 - \frac{\|T(y) - T_0\|^2}{4n}.$$

Thus (5.23) is proven. Combining (5.23) with (5.22), we get

$$\begin{aligned} \int_X \|T(y) - T_0\|^2 \, d\mathcal{H}^k(y) &\leq 4n \int_X 1 - J(y) \, d\mathcal{H}^k(y) \\ &\leq 4n \left(\mathcal{H}^k(X) - \omega_k((1 - C\eta^{\frac{1}{2}})R)^k \right) \\ &\leq 4n \left((1 + \eta)\omega_k R^k - \omega_k((1 - C\eta^{\frac{1}{2}})R)^k \right) \leq C\eta^{\frac{1}{2}} R^k, \end{aligned}$$

and thus $\fint_X \|N(y) - N_0\|^2 \, d\mathcal{H}^k(y) \le C\eta^{\frac{1}{2}} R^k$. Using (5.21) we finally get

$$\begin{aligned}
&\fint_{\Gamma\cap K_R^{(n)}(0)} \|N - N_0\| \, d\mathcal{H}^k \\
&\quad \le \fint_{\Gamma\cap K_R^{(n)}(0)} \|N - N_0\|^2 \, d\mathcal{H}^k \\
&\quad = \frac{1}{\mathcal{H}^k(\Gamma \cap K_R^{(n)}(0))} \left(4\mathcal{H}^k(\Gamma \cap K_R^{(n)}(0)) - X) + \int_X \|N - N_0\|^2 \, d\mathcal{H}^k \right) \\
&\quad \overset{(5.21)}{\le} C\eta^{\frac{1}{2}}.
\end{aligned}$$

□

Lemma 5.7. *Let* $0 < \varepsilon \le \frac{1}{2}$, $a := \sqrt[k]{1+\varepsilon} < 2$ *and assume that* $\delta(R), \eta \le \varepsilon$. *Then* $\delta(aR) \le 17\varepsilon$.

Proof. For $R \le r \le aR$ and $x \in \Gamma$ we calculate

$$\begin{aligned}
&\fint_{\Gamma\cap K_r^{(n)}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \\
&\qquad = \frac{1}{\mathcal{H}^k(\Gamma \cap K_r^{(n)}(x))} \left(\int_{(\Gamma\cap K_r^{(n)}(x)) - K_R^{(n)}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \right. \\
&\qquad\qquad \left. + \int_{\Gamma\cap K_R^{(n)}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \right) \\
&\qquad \le 2 \frac{\mathcal{H}^k((\Gamma \cap K_r^{(n)}(x)) - K_R^{(n)}(x))}{\mathcal{H}^k(\Gamma \cap K_r^{(n)}(x))} \\
&\qquad\qquad + \frac{\mathcal{H}^k(\Gamma \cap K_R^{(n)}(x))}{\mathcal{H}^k(\Gamma \cap K_r^{(n)}(x))} \fint_{\Gamma\cap K_R^{(n)}(x)} \|N - \tilde{N}_{x,R}\| \, d\mathcal{H}^k \\
&\qquad \le 2 \frac{(1+\eta)(aR)^k - (1-\eta)R^k}{(1-\eta)R^k} + \delta(R) \\
&\qquad \le 4\left(a^k - 1 + (a^k + 1)\eta \right) + \delta(R) \le 17\varepsilon.
\end{aligned}$$

Now let $y \in K_r^{(n)}(x) \cap \Gamma$. If $y \in K_R^{(n)}(x)$, then we get

$$\left| \tilde{N}_{x,R}(y - x) \right| \le \delta(R) R.$$

If $y \notin K_R^{(n)}(x)$, there is a curve $c : [0, l] \to \Gamma$ parametrized by arc-length, with $c(0) = x$, $c(l) = y$ and $l \le (1+\eta)r$ and there is a $t_0 \in [R, l]$ with $c(t_0) \in \partial K_R^{(n)}(x)$.

We get

$$\begin{aligned}
|\tilde{N}_{x,R}(y-x)| &\le |\tilde{N}_{x,R}(c(l)-c(t_0))| + |\tilde{N}_{x,R}(c(t_0)-c(0))| \\
&\le |c(l)-c(t_0)| + \delta(R)R \le (l-t_0) + \delta(R)R \\
&\le (1+\eta)r - R + \delta(R)R = r - R + \delta(R)R + \eta r \\
&\overset{r\ge R\ge \frac{r}{a}}{\le} \left(\frac{a-1}{a} + \delta(R) + \eta\right) r \overset{a\ge 1}{\le} (a-1+\delta(R)+\eta)\, r \\
&\le 3\varepsilon r. \qquad \square
\end{aligned}$$

Proof of the second part of Theorem 1.1. Let $0 < \varepsilon := \varepsilon(n,k) \le \frac{1}{2}$ be so small that the conclusions of Lemma 5.6 and Lemma 5.7 hold and let $C = C(n,k)$ be the constant from Lemma 5.6. Let us now consider a k-dimensional chord-arc submanifold with $C\eta^{\frac{1}{2}} \le \frac{\varepsilon}{17}$.

Since chord-arc submanifolds are C^1 and since Lemma 2.7 holds, there is an $R_0 > 0$ such that $\delta(R_0) \le \frac{\varepsilon}{17}$. Applying Lemma 5.7, we get $\delta(aR_0) \le \varepsilon$ for $a := \sqrt[k]{1+\frac{\varepsilon}{17}}$ and hence Lemma 5.6 implies

$$\delta(aR_0) \le C\eta^{\frac{1}{2}} \le \frac{\varepsilon}{17}.$$

Repeating this procedure, we get inductively $\delta(a^l R_0) \le C\eta^{\frac{1}{2}}$ for all $l \in \mathbb{N}$ and hence $\delta \le C\eta^{\frac{1}{2}}$. By (5.3) we finally get $\gamma \le 5C\eta^{\frac{1}{2}}$. $\square$

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