IIA and IIB spinors from $K(E_{10})$

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Received 28 March 2006; accepted 7 April 2006

Available online 21 April 2006

Editor: L. Alvarez-Gaumé

Abstract

We analyze the decomposition of recently constructed unfaithful spinor representations of $K(E_{10})$ under its $SO(9) \times SO(9)$, and $SO(9) \times SO(2)$ subgroups, respectively, where $K(E_{10})$ is the 'maximal compact' subgroup of the hyperbolic Kac–Moody group E_{10}. We show that under these decompositions, respectively, *one and the same* $K(E_{10})$ spinor gives rise to both the fermionic fields of IIA supergravity, *and* to the (chiral) fermionic fields of IIB supergravity. This result is thus the fermionic analogue of the decomposition of E_{10} under its $SO(9, 9)$ and $SL(9) \times SL(2)$ subgroups, respectively, which yield the correct bosonic multiplets of (massive) IIA and IIB supergravity. The essentially unique Lagrangian for the supersymmetric $E_{10}/K(E_{10})$ σ-model therefore can also capture the dynamics of IIA and IIB including bosons and fermions in the known truncations.

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1. Introduction

Recent work has established the existence of two unfaithful spinorial representations of the infinite-dimensional, 'maximal compact' subgroup $K(E_{10})$ of the hyperbolic Kac–Moody group E_{10}, namely a 'Dirac-spinor-type' representation with 32 real components [1], and a 'gravitino-type' representation with 320 real components [2–4].[1] In this Letter, we analyze the decomposition of these two representations under the maximal finite-dimensional subgroups $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$ of $K(E_{10})$, respectively, and show that the '$K(E_{10})$-gravitino' **320** and the '$K(E_{10})$-Dirac-spinor' **32** decompose and transform correctly as required by the fermionic multiplets of IIA and IIB supergravity, respectively. These consist, respectively, of two gravitini and dilatini, each pair with both chiralities, for the IIA theory; and two gravitini of one chirality, and two dilatini, of the opposite chirality, for IIB. The decomposition of the 'Dirac-spinor' representation **32** similarly yields the correct supersymmetry parameters of these theories.

We thus find that *one and the same* 320 component (and 32 component) $K(E_{10})$ spinor gives rise to the fermions of both IIA *and* IIB supergravity, depending on how one 'slices' the infinite-dimensional group $K(E_{10})$ under its finite-dimensional subgroups. This is our main result: it extends previous ones on the emergence of the bosonic multiplets of these theories from E_{10} under appropriate 'level decompositions' of E_{10} under its A_9, D_9 and $A_8 \times A_1$ subgroups, respectively, [6–8], and earlier results on the embedding of these theories into E_{11} [9–13]. The present results thus strengthen the case for the (essentially unique) supersymmetric $E_{10}/K(E_{10})$ σ-model proposed in [2] as a candidate for a unification of the maximally extended supergravity theories in ten and eleven space–time dimensions into a single theory.

This Letter has the following structure. In Section 2, we review, following [2], the definition and 'low level' commutation relations of $K(E_{10})$ viewed from its $SO(10)$ subgroup, and we identify the $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$ subgroups of $K(E_{10})$, relevant for the IIA and IIB theories. In Section 3, we decompose the unfaithful 'Dirac-spinor' and 'gravitino' representations of $K(E_{10})$ under these subgroups. We also briefly discuss the relation to type I supergravity and DE_{10}. We end with some concluding remarks in Section 4.

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[1] The corresponding unfaithful representations for $K(E_9)$ had already been identified and studied in [5].

2. $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$ subgroups

The maximal compact group $K(E_{10})$ is defined as the subgroup of E_{10} whose Lie algebra is invariant under the Chevalley involution θ (see [14] for an introduction to the theory of Kac–Moody algebras). In this section, we will study its distinguished subgroups $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$, starting from the $SO(10)$ decomposition of $K(E_{10})$ at low levels presented in [2]. These two subgroups together generate all of $K(E_{10})$; there is no finite-dimensional R symmetry in the present scheme that would accommodate the fermions of both IIA and IIB supergravity.

2.1. $K(E_{10})$ in terms of $SO(10)$

In the approach of [2], the $K(E_{10})$ algebra was written in terms of generators derived from the $SL(10)$ decomposition of E_{10} [6]. Up to $SL(10)$ level $\ell = 3$, the generators of the associated Lie algebra $\mathfrak{k}_{10} \equiv \mathrm{Lie}(K(E_{10}))$ are defined by

$$J^{ab} = K^a{}_b - K^b{}_a,$$
$$J^{a_1 a_2 a_3} = E^{a_1 a_2 a_3} - F_{a_1 a_2 a_3},$$
$$J^{a_1 \dots a_6} = E^{a_1 \dots a_6} - F_{a_1 \dots a_6},$$
$$J^{a_0 | a_1 \dots a_8} = E^{a_0 | a_1 \dots a_8} - F_{a_0 | a_1 \dots a_8}, \tag{1}$$

with the $GL(10)$ generators $K^a{}_b$ and the basic (level ± 1) E_{10} generators E^{abc} and $F_{abc} = -\theta(E^{abc}) \equiv (E^{abc})^T$; thus, in terms of those generators, the compact generators are generically 'anti-symmetric', i.e., of type $J = E - E^T \equiv E - F$. Henceforth, we shall refer to J^{ab}, $J^{a_1 a_2 a_3}$, $J^{a_1 \dots a_6}$, and $J^{a_0 | a_1 \dots a_8}$ as being of 'levels' $\ell = 0, 1, 2, 3$, respectively, although this 'level' is not a grading of $\mathfrak{k}_{10}$;[2] in fact, $\mathfrak{k}_{10}$ is not even a Kac–Moody algebra [15]. Up to $\ell = 3$ (and neglecting higher level contributions), the $\mathfrak{k}_{10}$ commutation relations are given by [2,3][3]

$$\big[J^{ab}, J^{cd} \big] = \delta^{bc} J^{ad} + \delta^{ad} J^{bc} - \delta^{ac} J^{bd} - \delta^{bd} J^{ac}$$
$$\equiv 4 \delta^{bc} J^{ad},$$

$$\big[J^{a_1 a_2 a_3}, J^{b_1 b_2 b_3} \big] = J^{a_1 a_2 a_3 b_1 b_2 b_3} - 18 \delta^{a_1 b_1} \delta^{a_2 b_2} J^{a_3 b_3},$$

$$\big[J^{a_1 a_2 a_3}, J^{b_1 \dots b_6} \big]$$
$$= J^{[a_1 | a_2 a_3] b_1 \dots b_6} - 5! \delta^{a_1 b_1} \delta^{a_2 b_2} \delta^{a_3 b_3} J^{b_4 b_5 b_6},$$

$$\big[J^{a_1 \dots a_6}, J^{b_1 \dots b_6} \big] = -6 \cdot 6! \delta^{a_1 b_1} \cdots \delta^{a_5 b_5} J^{a_6 b_6} + \cdots,$$

$$\big[J^{a_1 a_2 a_3}, J^{b_0 | b_1 \dots b_8} \big]$$
$$= -336 \big(\delta^{b_0 b_1 b_2}_{a_1 a_2 a_3} J^{b_3 \dots b_8} - \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} J^{b_4 \dots b_8 b_0} \big) + \cdots,$$

$$\big[J^{a_1 \dots a_6}, J^{b_0 | b_1 \dots b_8} \big]$$
$$= -8! \big(\delta^{b_0 b_1 \dots b_5}_{a_1 \dots a_6} J^{b_6 b_7 b_8} - \delta^{b_1 \dots b_6}_{a_1 \dots a_6} J^{b_7 b_8 b_0} \big) + \cdots,$$

$$\big[J^{a_0 | a_1 \dots a_8}, J^{b_0 | b_1 \dots b_8} \big]$$
$$= -8 \cdot 8! \big(\delta^{a_1 \dots a_8}_{b_1 \dots b_8} J^{a_0 b_0} - \delta^{a_1 \dots a_8}_{b_0 b_1 \dots b_7} J^{a_0 b_8} - \delta^{a_0 a_1 \dots a_7}_{b_1 \dots b_8} J^{a_8 b_0}$$
$$+ 8 \delta^{a_0}_{b_0} \delta^{a_1 \dots a_7}_{b_1 \dots b_7} J^{a_8 b_8} + 7 \delta^{a_1}_{b_0} \delta^{a_0 a_2 \dots a_7}_{b_1 \dots b_7} J^{a_8 b_8} \big) + \cdots. \tag{2}$$

Here, all indices $a, b, \dots = 1, \dots, 10$ are to be regarded as ('flat') $SO(10)$ indices. As in [2] we make use of a shorthand notation in (2), where the terms on the r.h.s. are to be anti-symmetrized (with weight one) according to the anti-symmetries on the l.h.s., as explicitly written out for the $SO(10)$ generators J^{ab} in the first line. As also explained there, the generator $J^{a_0 | a_1 \dots a_8}$ decomposes into two irreducible components under $SO(10)$, the trace and a non-trivial mixed (Young tableau) representation.[4] The $SO(10)$ generators rotate the higher level generators in the standard way as tensor representations.

2.2. $SO(9) \times SO(9)$ subgroup

In Ref. [7], the E_{10} group was analyzed under its regular $SO(9,9)$ subgroup. Similarly, one can study $K(E_{10})$ under its $K(SO(9,9)) = SO(9) \times SO(9)$ subgroup. The corresponding generators of $SO(9) \times SO(9)$ can be written in terms of the $SO(10)$ generators of (1) as

$$X^{ij} := \frac{1}{2} \big(J^{(0)ij} + J^{(1)ij\,10} \big),$$
$$X^{\bar{i}\bar{j}} := \frac{1}{2} \big(J^{(0)ij} - J^{(1)ij\,10} \big) \quad \text{for } i, j = 1, \dots, 9. \tag{3}$$

We use the notation of [7] with barred indices for the second $SO(9)$ in order to distinguish the two factors. It is worthwhile to note that this definition uses both 'level zero' and 'level one' generators in terms of (1). We have indicated the A_9 'level' explicitly in parentheses in (3). The form of these generators can be found by tracing back the definitions of X^{ij} and $X^{\bar{i}\bar{j}}$ in terms of the Chevalley generators of E_{10} given in [7] and then re-expressing them in terms of the $SO(10)$ tensors (1).

From (2) it is straightforward to check that the generators (3) satisfy the $SO(9) \times SO(9)$ commutation relations

$$\big[X^{ij}, X^{kl} \big] = \delta^{jk} X^{il} + \delta^{il} X^{jk} - \delta^{jl} X^{ik} - \delta^{ik} X^{jl},$$
$$\big[X^{\bar{i}\bar{j}}, X^{\bar{k}\bar{l}} \big] = \delta^{\bar{j}\bar{k}} X^{\bar{i}\bar{l}} + \delta^{\bar{i}\bar{l}} X^{\bar{j}\bar{k}} - \delta^{\bar{j}\bar{l}} X^{\bar{i}\bar{k}} - \delta^{\bar{i}\bar{k}} X^{\bar{j}\bar{l}},$$
$$\big[X^{ij}, X^{\bar{k}\bar{l}} \big] = 0. \tag{4}$$

Evidently, no 'higher level' ($\ell \geqslant 2$) generators of $K(E_{10})$ are excited in these commutators.

2.3. $SO(9) \times SO(2)$ subgroup

The analysis of [8] started from an $SL(9) \times SL(2)$ decomposition of E_{10} where the $SL(2)$ was identified with the $SL(2)$ symmetry of type IIB supergravity after establishing a dynamical correspondence. At the level of compact subgroups the decomposition is $SO(9) \times SO(2) \equiv K(SL(9) \times SL(2)) \subset K(E_{10})$

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[2] Instead, we have a so-called 'filtration' $[\mathfrak{k}^{(\ell)}, \mathfrak{k}^{(\ell')}] \subset \mathfrak{k}^{(\ell+\ell')} \oplus \mathfrak{k}^{(|\ell-\ell'|)}$.

[3] Neglecting the non-trace part of $J^{a_0 | a_1 \dots a_8}$, the corresponding commutators for $K(E_{11})$ were already computed in [16].

[4] Whereas $J^{a_0 | a_1 \dots a_8}$ is irreducible as a representation of $SL(10)$.

and the generators are defined as

$$R^{rs} := J^{(0)rs}, \qquad R^{r9} := -R^{9r} := J^{(1)r9\,10},$$
$$\text{for } r, s = 1, \ldots, 8,$$
$$R := J^{(0)9\,10}. \tag{5}$$

The nine $SO(9)$ indices had to be split into $(8+1)$ since eight directions are in common with an $SO(8) \subset SO(10)$ but one direction is different. This is in line with standard views on T-duality [17–20]. The expressions (5) can again be deduced by going via the explicit relation to the standard Chevalley basis of E_{10}.

From (2) one can show that the generators (5) satisfy the $SO(9) \times SO(2)$ relations

$$\left[R^{ij}, R^{kl}\right] = 4\delta^{jk} R^{il}, \qquad \left[R, R^{ij}\right] = 0 \tag{6}$$

for $i, j, k, l = 1, \ldots, 9$ (again with anti-symmetrizations understood). In particular, R commutes with all $SO(9)$ generators, as required. In the supergravity context the compact $SO(2)$ is usually referred to as $U(1)$ and we will use both terms interchangeably.

3. Unfaithful $K(E_{10})$ spinor representations

In [1–3] two unfaithful representations of $K(E_{10})$ were defined. These consist of 32 and 320 real components, respectively. In terms of the $SO(10)$ generators (1), the 'Dirac spinor' representation **32**, denoted by ϵ, transforms as [1]

$$J^{(0)ab} \cdot \epsilon = \frac{1}{2}\Gamma^{ab}\epsilon, \qquad J^{(1)abc} \cdot \epsilon = \frac{1}{2}\Gamma^{abc}\epsilon. \tag{7}$$

The 'vector-spinor' representation **320**, denoted by ψ^a, transforms as [2,3]

$$J^{(0)ab} \cdot \psi^c = \frac{1}{2}\Gamma^{ab}\psi^c + 2\delta^{c[a}\psi^{b]},$$
$$J^{(1)abc} \cdot \psi^d = \frac{1}{2}\Gamma^{abc}\psi^d + 4\delta^{d[a}\Gamma^b\psi^{c]} - \Gamma^{d[ab}\psi^{c]}. \tag{8}$$

Here, Γ^a ($a = 1, \ldots, 10$) are the real (32×32) spatial $SO(1, 10)$ Γ-matrices of eleven-dimensional supergravity [21], which we use in the basis defined in [7]. In both cases we have given the transformations only up to $SO(10)$ 'level one' since this suffices to characterize the consistent unfaithful $K(E_{10})$ representation [4].[5] Furthermore, the $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$ subgroup generators (3) and (5) only require the $SO(10)$ levels zero and one. The transformation rules (7) and (8) were derived in [2,3] by demanding a dynamical correspondence between a $K(E_{10})$ covariant spinor equation and the gravitino equation of motion of $D = 11$ supergravity.

In the following we will decompose the Dirac- and vector-spinor under the $SO(9) \times SO(9)$ and $SO(9) \times SO(2)$ subgroups of $K(E_{10})$. For this it will be important that the irreducible Dirac-spinor of $SO(9)$ has 16 real components and that in our

basis the (32×32) matrix Γ^{10} is of the block diagonal form

$$\Gamma^{10} = \begin{pmatrix} \mathbf{1}_{16} & 0 \\ 0 & -\mathbf{1}_{16} \end{pmatrix}. \tag{9}$$

The combinations

$$P_\pm = \frac{1}{2}\left[1 \pm \Gamma^{10}\right] \tag{10}$$

are the standard orthogonal projectors onto two 16-component subspinors of the 32 component Majorana spinor of $SO(1, 10)$; viewed from $D = 10$, they become the two spinors of IIA supergravity of opposite chirality.

3.1. $SO(9) \times SO(9)$ decomposition

3.1.1. Dirac-spinor
We diagonalize the action of (3) on the $K(E_{10})$ Dirac-spinor ϵ by defining

$$\varepsilon_\pm = P_\pm \epsilon \tag{11}$$

using the projectors (10). We immediately check the action of the first $SO(9)$ factor $SO(9) \times SO(9)$ from (7) as

$$X^{ij} \cdot \varepsilon_\pm = \frac{1}{4}P_\pm\Gamma^{ij}\left(1 + \Gamma^{10}\right)\epsilon = \frac{1}{2}\Gamma^{ij}P_\pm P_+\epsilon, \tag{12}$$

such that ε_+ transforms as a Majorana spinor under this $SO(9)$, and ε_- transforms trivially.[6] Under the second $SO(9)$ these properties are obviously interchanged, and therefore we deduce the following decomposition of the $K(E_{10})$ Dirac-spinor under $SO(9) \times SO(9)$

$$\mathbf{32} \rightarrow (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{1}). \tag{13}$$

Under the diagonal $SO(9)_{\text{diag}}$ these two spinors become two spinors of opposite handedness.[7]

3.1.2. Vector-spinor
In order to find the irreducible components of the $K(E_{10})$ vector-spinor (or gravitino) ψ_a in terms of $SO(9) \times SO(9)$ representations we have to redefine the fermionic components according to [7] as

$$\tilde{\psi}_k = \psi_k + \frac{1}{2}\Gamma_k\Gamma^{10}\psi_{10},$$
$$\tilde{\psi}_{10} = -\frac{3}{2}\psi_{10} - \Gamma_{10}\Gamma^k\psi_k. \tag{14}$$

For example, acting with X^{ij} on $P_-\tilde{\psi}_{10}$, using the formulas (8) leads to (after some computation)[8]

$$X^{ij} \cdot P_-\tilde{\psi}_{10} = \frac{1}{2}\Gamma^{ij}P_-\tilde{\psi}_{10}, \tag{15}$$

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[5] The transformation rules up to 'level three' are known and can be found in [1–3].

[6] The matrices $\Gamma^{ij}P_\pm$ can be identified with the (16×16) gamma matrices of $SO(9)$.

[7] This terminology obviously refers to the chirality w.r.t. the space–time Lorentz group $SO(1, 9)$, from which these spinors originate. Although $SO(9)$ by itself has no chiral representations, we will nevertheless make occasional use of this terminology.

[8] Of course, (7) does not fix the normalization of $\tilde{\psi}_{10}$ but only the relative coefficients. The normalization was fixed in [7] by demanding a canonical form of the resulting kinetic term.

so that the $K(E_{10})$ action (8) expressed in terms of $SO(9) \times SO(9)$ via (3) is equivalent to the projection on one chiral component which transforms in the usual $SO(9)$ spinor representation. In other words, $P_-\tilde{\psi}_{10}$ transforms in the **16** representation under the first $SO(9)$ of $SO(9) \times SO(9)$ and trivially under the second $SO(9)$ (the projectors are orthogonal). The opposite behaviour is deduced for $P_+\tilde{\psi}_{10}$, so that the $\tilde{\psi}_{10}$ part of the **320** of $K(E_{10})$ gives rise to the $(\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16})$ under $SO(9) \times SO(9)$. A similar calculation for the $\tilde{\psi}_k$ component gives

$$X^{ij} \cdot \tilde{\psi}^k = \frac{1}{2}\Gamma^{ij}P_+\tilde{\psi}^k + 2\delta^{k[i}P_-\tilde{\psi}^{j]},$$
$$X^{\bar{i}\bar{j}} \cdot \tilde{\psi}^k = \frac{1}{2}\Gamma^{ij}P_-\tilde{\psi}^k + 2\delta^{k[i}P_+\tilde{\psi}^{j]}. \tag{16}$$

The two different projectors appearing in this computation imply that the $P_+\tilde{\psi}^k$ part transforms in the spinor **16** of the first $SO(9)$, and the vector **9** of the second $SO(9)$, and oppositely for the other chiral projection $P_-\tilde{\psi}^k$.

Together with (7) we therefore deduce the expected total result

$$\mathbf{320} \to (\mathbf{9}, \mathbf{16}) \oplus (\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{9}) \oplus (\mathbf{16}, \mathbf{1}). \tag{17}$$

This is precisely the representation used in [7] for which also a partial dynamical check was carried out there, analogous to the one performed in [6]. The representations on the r.h.s. are to be interpreted as the two gravitini and the two dilatini of (massive) IIA supergravity. We emphasize again the chirality-symmetric nature of these spinors, which here manifests itself in the symmetry under interchange of the two $SO(9)$ groups.

3.2. $SO(9) \times SO(2)$ decomposition

For the analysis of the subgroup $SO(9) \times SO(2)$, we first define the matrix

$$\Gamma^* := \Gamma^9\Gamma^{10}. \tag{18}$$

Obviously, $(\Gamma^*)^2 = -\mathbf{1}$, whence Γ^* can be regarded as an imaginary unit. This will turn out to be a main difference with the $SO(9) \times SO(9)$ decomposition of the previous section: unlike Γ^{10} (which squares to $+\mathbf{1}$), we cannot use Γ^* to define projectors unless we complexify the representation. Two useful relations are

$$P_\pm\Gamma^* = \Gamma^*P_\mp, \qquad P_\pm\Gamma^{10} = \pm P_\pm. $$

3.2.1. Dirac-spinor
For the $K(E_{10})$ Dirac-spinor we define new components via

$$\varepsilon_1 := P_-\epsilon, \qquad \varepsilon_2 := P_-\Gamma^*\epsilon = \Gamma^*P_+\epsilon. \tag{19}$$

The use of the projector P_- here makes explicit that we are really dealing with two 16-component objects, from which the original spinor can be reconstructed via

$$\epsilon = \varepsilon_1 - \Gamma^*\varepsilon_2. \tag{20}$$

Equivalently, we could work with the complex 16-component spinor $\varepsilon_1 - i\varepsilon_2$, replacing the matrix Γ^* by the imaginary unit.

Acting with an $SO(2)$ rotation R from (5) on this spinor we obtain

$$R \cdot \varepsilon_1 = +\frac{1}{2}\varepsilon_2, \qquad R \cdot \varepsilon_2 = -\frac{1}{2}\varepsilon_1. \tag{21}$$

The pair $(\varepsilon_1, \varepsilon_2)$ therefore transforms as a doublet under $SO(2)$: the complex spinor $\varepsilon_1 \pm i\varepsilon_2$ carries $U(1)$ charge $\mp\frac{1}{2}$.

Under $SO(9)$ the components ε_1 and ε_2 both transform as (suppressing the $SO(2)$ indices)

$$R^{rs} \cdot \varepsilon = \frac{1}{2}\Gamma^{rs}\varepsilon, \qquad R^{r9} \cdot \varepsilon = +\frac{1}{2}\Gamma^r\Gamma^*\varepsilon. \tag{22}$$

This is the correct transformation of an $SO(9)$ Dirac-spinor, as it does not mix ε_1 and ε_2 (both Γ^{rs} and $\Gamma^r\Gamma^*$ commute with $P_\pm$ and Γ^*). Due to the presence of the projector P_- in the definition of ε_1 and ε_2 the action of the (32×32) Γ-matrices of $SO(1, 10)$ can be seen as the action of (16×16) γ-matrices of $SO(9)$. The present realization of the $SO(9)$ Clifford algebra is in terms of (32×32) matrices $\{\Gamma^{rs}, \Gamma^r\Gamma^*\}$. We conclude that the **32** of $K(E_{10})$ decomposes under the $SO(9) \times SO(2)$ subgroup into an $SO(2)$ doublet of $SO(9)$ spinors:

$$\mathbf{32} \to (\mathbf{16}, \mathbf{2}). \tag{23}$$

3.2.2. Vector-spinor
The $K(E_{10})$ vector-spinor ψ_a gives rise to two kinds of spinors, namely the dilatini

$$\lambda_1 := P_-\left(\psi_9 - \Gamma^*\psi_{10}\right),$$
$$\lambda_2 := P_-\left(\Gamma^*\psi_9 + \psi_{10}\right), \tag{24}$$

and the gravitini

$$\chi_1^9 := P_-\left(\psi_9 + \Gamma^*\psi_{10}\right),$$
$$\chi_2^9 := P_-\left(-\Gamma^*\psi_9 + \psi_{10}\right) = -P_-\Gamma^*\left(\psi_9 + \Gamma^*\psi_{10}\right),$$
$$\chi_1^r := P_-\left(-\frac{4}{3}\psi^r - \frac{1}{3}\Gamma^r\Gamma^9\psi_9 - \frac{1}{3}\Gamma^r\Gamma^{10}\psi_{10}\right),$$
$$\chi_2^r := P_-\Gamma^*\left(-\frac{4}{3}\psi^r - \frac{1}{3}\Gamma^r\Gamma^9\psi_9 - \frac{1}{3}\Gamma^r\Gamma^{10}\psi_{10}\right). \tag{25}$$

Again, the lower index $(1, 2)$ is the $SO(2)$ index, while the upper index (with $r = 1, \ldots, 8$ as in (5)) is an $SO(9)$ vector index. The linear combinations appearing in the above equation were arrived at by demanding proper behavior under $SO(9)$ transformations, to wit, by requiring that χ^9 and χ^r combine into a vector-spinor χ^i ($i = 1, \ldots, 9$) of $SO(9)$, starting from (7) and (8), see below. As before, one can also combine these spinors into a complex dilatino $\lambda \equiv \lambda_1 \pm i\lambda_2$ and a complex gravitino $\chi^i \equiv \chi_1^i \pm i\chi_2^i$.

First we determine the $SO(2)$ properties of the new fermions (24) and (25). A straightforward calculation using (5) and (8) together with (24) shows that

$$R \cdot \lambda_1 = +\frac{3}{2}\lambda_2, \qquad R \cdot \lambda_2 = -\frac{3}{2}\lambda_1, \tag{26}$$

whence the complex dilatino $\lambda_1 \pm i\lambda_2$ carries $U(1)$ charge $\mp\frac{3}{2}$. The pair of gravitini turns out to have the same $SO(2)$ charge as the Dirac-spinor representation (in agreement with the fact that

the gravitino and the supersymmetry transformation parameter should transform in the same $SO(2)$ representation). Using (5) and (8) again, we derive

$$R \cdot \chi_1^i = +\frac{1}{2}\chi_2^i, \qquad R \cdot \chi_2^i = -\frac{1}{2}\chi_1^i. \tag{27}$$

Under the $SO(9)$ part $SO(9) \times SO(2)$ one finds after some computation that

$$R^{rs} \cdot \lambda = \frac{1}{2}\Gamma^{rs}\lambda, \qquad R^{r9} \cdot \lambda = -\frac{1}{2}\Gamma^r\Gamma^*\lambda \tag{28}$$

for both λ_1 and λ_2. This is the transformation of an $SO(9)$ spinor where now the representation is in terms of (32×32) matrices $\{\Gamma^{rs}, -\Gamma^r\Gamma^*\}$, projected onto (16×16) matrices by the projector P_- in the redefinition of the fermions. Notice that this Clifford algebra now differs from formula (22) by the sign in front of $\Gamma^r\Gamma^*$. Although the representations are equivalent over $SO(9)$ (for which there is no chirality), we take this difference of sign as a manifestation of the opposite chiralities of these two spinors in IIB supergravity w.r.t. the Lorentz group $SO(1, 9)$ in ten dimensions [22,23]. For the gravitino components (25) one finds similarly, using (8) and the redefinitions (25),

$$R^{rs} \cdot \chi^k = \frac{1}{2}\Gamma^{rs}\chi^k + 2\delta^{k[r}\chi^{s]},$$
$$R^{r9} \cdot \chi^k = +\frac{1}{2}\Gamma^r\Gamma^*\chi^k + 2\delta^{k[r}\chi^{9]}, \tag{29}$$

where $k = 1, \dots, 9$, and where we have suppressed the $SO(2)$ indices. This is the correct transformation of a vector-spinor (with Γ-trace) under $SO(9)$. Therefore, we conclude that the $K(E_{10})$ vector-spinor decomposes as

$$\mathbf{320} \to (\overline{\mathbf{16}}, \mathbf{2}) \oplus (\mathbf{144}, \mathbf{2}) \tag{30}$$

under $SO(9) \times SO(2)$, giving the dilatino and gravitino doublets (where the bar over the first $\mathbf{16}$ is meant to indicate opposite chirality w.r.t. $SO(1, 9)$).[9] In summary, we have obtained perfect agreement between the IIB supergravity assignments of the fermionic fields and the $SO(2)$ (or $U(1)$) charges and $SO(9)$ assignments as they emerge from $K(E_{10})$!

3.3. Embedding $K(DE_{10}) \subset K(E_{10})$

The truncation to pure type I theory in ten space–time dimensions is conjectured to be associated with the algebra DE_{10} in the hyperbolic KMA setting [24] (for very-extended algebras one expects DE_{11} [25]). It was shown in [7] that DE_{10} is a proper subgroup of E_{10}. Moreover, it was shown there that in the $SO(9, 9)$ decomposition of DE_{10} only *tensorial* representations appear, whereas for the decomposition of E_{10} one also finds $SO(9, 9)$ spinor representations (for example associated with the RR fields). Under the compact $SO(9) \times SO(9)$ of $SO(9, 9)$, an $SO(9, 9)$ Dirac-spinor decomposes into the tensor

product of two $SO(9)$ spinors associated with the two $SO(9)$ factors of $SO(9) \times SO(9)$. Such a generator was denoted $E_{\alpha\bar\alpha}$ in [7] and evidently can be used to transform an $SO(9) \times SO(9)$ representation of the type $(\mathbf{1}, \mathbf{16})$ into $(\mathbf{16}, \mathbf{1})$, i.e. changing the 'chirality' under $SO(9) \times SO(9)$. Since generators like $E_{\alpha\bar\alpha}$ do not exist in DE_{10} this can never happen and we are led to conclude that there are two inequivalent 16-dimensional Dirac-spinor representations of $K(DE_{10})$ which we denote by $\mathbf{16}$ and $\overline{\mathbf{16}}$. They are distinguished by their decomposition under $SO(9) \times SO(9)$ via

$$\mathbf{16} \to (\mathbf{16}, \mathbf{1}), \qquad \overline{\mathbf{16}} \to (\mathbf{1}, \mathbf{16}). \tag{31}$$

We also find that under $K(DE_{10}) \subset K(E_{10})$ the $K(E_{10})$ Dirac-spinor decomposes as

$$\mathbf{32} \to \mathbf{16} \oplus \overline{\mathbf{16}}. \tag{32}$$

Similarly, the decomposition of the $K(E_{10})$ vector-spinor is expected to be

$$\mathbf{320} \to \mathbf{160} \oplus \overline{\mathbf{160}}. \tag{33}$$

In this sense, and because it gives rise to $D = 10$ spinors of a given chirality, $K(DE_{10})$ can be viewed as a 'chiral half' of $K(E_{10})$.

3.4. Unfaithful spinor representations of $K(E_n)$

Our results can be extended to unfaithful spinor representations of $K(E_n)$ for any $n \geqslant 9$, most notably $K(E_9)$ [5] and $K(E_{11})$ [16]. The form of the transformation rules (7) and (8) imply that they define consistent unfaithful representation for the maximal compact subgroup $K(E_n)$ of E_n for $n \geqslant 9$.[10] These representations are written in terms of the $SO(n)$ subgroups of $K(E_n)$, but one could also introduce the flat metric of $SO(n - p, p)$ (with corresponding real gamma matrices) in (8), in particular, $SO(1, 10)$ for $K(E_{11})$ by using the so-called temporal involution [26,27].[11] The reason that (7) and (8) define unfaithful representations for any $K(E_n)$ ($n \geqslant 9$) is that the necessary consistency conditions do not involve traces (and so are independent of n) and take the same form for all $n \geqslant 9$.

The dimension of the Dirac-spinor representation of $K(E_n)$ is given by the dimension of the real spinor representation of $SO(n - p, p)$, i.e., 16 for $(n, p) = (9, 0)$ and 32 for $(n, p) = (11, 1)$. The naïve dimension of the vector-spinor is then $= 144$ for $K(E_9)$ and $= 352$ for $K(E_{11})$ (with temporal involution). However, the resulting vector-spinor of $K(E_9)$ is reducible since one can construct the gamma trace of the vector-spinor in a $K(E_9)$ invariant fashion. Therefore the irreducible vector-spinor of $K(E_9)$ has dimension 128, as already established in [5]. For all other $n > 9$, the gamma matrices are not invariant objects under $K(E_n)$. For $K(E_{11})$, the vector-spinor has

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[9] In the $\mathbf{144}$ of $SO(9)$ a γ-trace can be separated (whence the representation is reducible), but we prefer to write the $\mathbf{144}$ as one representation since it corresponds to the physical gravitino.

[10] For $n < 9$, it is straightforward to check that the relevant maximal compact subgroups $K(E_8) \equiv \mathrm{Spin}(16)/Z_2$, etc., are faithfully generated by the $\ell \leqslant 3$ elements.

[11] See [28] for an analysis of the orbits of non-Euclidean signatures under the E_{11} Weyl group.

dimension 352, and we have checked that under the IIA and IIB decompositions of $K(E_{11})$ this **352** reduces to the correct $SO(1, 9)$ covariant vector-like and chiral spinors of the IIA and IIB theories, respectively.[12] However, these considerations concern the representations, and are therefore purely kinematical. We have not investigated the form of the $K(E_{11})$ covariant spinor equations corresponding to the $D = 11$ gravitino variation and equation of motion.

4. Outlook

In this Letter, we have demonstrated that, at the *kinematical* level, the unfaithful **32** and **320** spinor representations of $K(E_{10})$ decompose into the correct fermionic representations under the subalgebras relevant for the IIA and IIB analysis of E_{10}. At the *dynamical* level, partial checks concerning the $SO(9) \times SO(9)$ decomposed fermionic sector were carried out already in [7]. Using the formulation of [2], the dynamical system with manifest full local $K(E_{10})$ invariance (and global E_{10} invariance) is correctly described by the gauge-fixed Lagrangian

$$\mathcal{L} = \frac{1}{2n}\langle\mathcal{P}|\mathcal{P}\rangle - i(\chi|\mathcal{D}\chi)_{\text{vs}}. \tag{34}$$

This $E_{10}/K(E_{10})$ σ-model describes the motion of a massless spinning particle on the $E_{10}/K(E_{10})$ coset space, where we take the 'spin' in the **320** unfaithful vector-spinor representation of $K(E_{10})$. Here, $\mathcal{P}$ is the $K(E_{10})$ covariant bosonic velocity and χ the matter fermion with $\mathcal{D}$ being the $K(E_{10})$ covariant derivative. As shown in [2] (see also [3]), the Lagrangian (34) correctly reproduces both the fermionic and bosonic equations of motion of eleven-dimensional supergravity at linearized fermion order and with the appropriate truncations on the supergravity side, i.e., neglecting second and higher order spatial gradients for the bosonic fields, and all spatial gradients of the fermionic fields. As shown in [7], the action of IIA supergravity reduces to a $K(E_{10})$-invariant action of the type (34). Given the kinematical 'versatility' of $K(E_{10})$ and these results for $D = 11$ (massive) IIA supergravity we expect that (34) will also describe the fermionic equations of motion of IIB supergravity (in the same truncation). Moreover, the $K(E_{10})$-invariant model (34) would exhibit also a dynamical versatility!

In light of the gradient hypothesis of [6], it is also interesting to note that the *space–time dimension* of the corresponding theory depends upon which subgroup of E_{10} (or $K(E_{10})$) is selected to perform the level decomposition: $D = 11$ for $A_9 \equiv SL(10)$, and $D = 10$ for $D_9 \equiv SO(9, 9)$ and $A_8 \times A_1 \equiv SL(9) \times SL(2)$. In this sense, the dimension of space–time is no longer a fundamental *datum* of this theory, but an emergent phenomenon. Let us also note that a different proposal for the emergence of space–time and its dimension within the framework of E_{11} was already made in [16,27] where, however,

space–time is implemented in terms of an E_{11} representation rather than within the algebra itself.

To be sure, we envisage that ultimately the unfaithful, finite-dimensional representations of $K(E_{10})$ studied here will be replaced by faithful representations (probably by taking tensor products with the coset or the compact subgroup itself), replacing the 320 time-dependent gravitino components ψ_a by a infinite tower of components characterizing the spatial dependence of the fermions in line with the gradient conjecture of [6].

Acknowledgement

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[12] The inner products on these vector-spinor representations are, however, not given by the same form as for $K(E_{10})$, as the invariance of the latter requires ten dimensions [2].