Weakly relativistic dielectric tensor for arbitrary wavenumbers

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Abstract. A semi-relativistic approximation of Trubnikov's dielectric tensor is derived without making assumptions on the refractive index along and across the ambient magnetic field, $N_{||}$ and $N_\perp$. The results extend the validity of Shkarofsky's treatment, previously restricted to quasi-perpendicular incidence, and permit to handle cases in which Doppler and relativistic widths of electron cyclotron resonances are comparable. Also, as $N_\perp$ is arbitrary, the proposed approximation is adequate for Bernstein waves, characterized by large $N_\perp$. For ease of calculation and to emphasize the link with previous results, the new tensor is expressed in terms of Shkarofsky functions of shifted argument and modified width.

INTRODUCTION

Weakly relativistic effects in the propagation and absorption of waves in a magnetized plasma are typically studied for quasi-perpendicular incidence ($N_{||} \ll \beta_T$), when they dominate over Doppler broadening, and are described by power series of the finite Larmor radius parameter $\lambda = (N_\perp \beta_T/Y)^2/2$ [1, 2, 3, 4, 5, 6, 7, 8] or of $1/\lambda$ [9, 10, 11], or expanded in Bessel functions of argument λ. Here $\beta_T = \sqrt{2k_BT/mc^2}$ is the thermal velocity in c units and $Y = \omega_c/\omega$ the magnetic field in dimensionless units. All these approximations, some of which are truncated at the leading order in λ or $1/\lambda$, are summarized in table 1. We also note that not the whole ε but only an electrostatic approximation is considered in [9, 10, 11]. Ref.[12] contains the most general result, although of little practical use, as it is a series in a 4D space multiplied by a sum in 2D.

TABLE 1. Summary of approximations on $N_{||}$ and finite Larmor radius parameter λ in available weakly relativistic dielectric tensors

	$\lambda \ll 1$ trunc.	$\lambda \ll 1$	$\lambda \gg 1$	$\lambda \gg 1$ trunc.
$N_{||} = 0$	[4, 5]			
$N_{||} \ll \beta_T$	[1, 2]	[3, 6, 7, 8]	[10]	[9]
arbitr. $N_{||}$		[12]	[11]	

In the case of electron Bernstein waves (EBWs) the small $N_\perp$ limit is incompatible with the weakly relativistic limit $\beta_T \ll 1$, because for EBWs $N_\perp \approx 1/\beta_T$. These waves also violate the $N_{||} \ll 1$ assumption when they are generated via ordinary-extraordinary-Bernstein (O-X-B) mode conversion of an *obliquely* injected O-mode [13, 14] or detected by *oblique* observation of the reverse, B-X-O, process [15, 16]. Besides, even for other excitation or detection schemes characterized by small $N_{||}$ at the antenna, still

EBWs tend to develop large $N_{||}$ as a consequence of the large $N_\perp$ and of the evolution of the ray in a bent magnetic field [14, 17].

Finite $N_{||}$ are also important for ion Bernstein waves, and were previously treated numerically with the aid of root finders [18].

Finally, the (mild) relativistic mass gain is obviously important for electromagnetic electron cyclotron waves, as it resolves the emission/absorption line. In the $N_{||} \ll \beta_T$ limit, this mechanism dominates over Doppler broadening and is satisfactorily described by Shkarofsky functions [1]. In the opposite limit, relativistic effects are neglected and the warm non-relativistic dielectric tensor can be utilized. This includes Doppler broadening only. The fully relativistic tensor due to Trubnikov [19] embodies both the broadening mechanisms, thus in principle it suits intermediate angles such that $N_{||} \approx \beta_T$, however it is very complicated and not in a closed form.

Motivated by these considerations, a purely semi-relativistic and relatively easy-to-compute approximation of Trubnikov's tensor is derived in the present work without making any specific assumption on $N_{||}$ and $N_\perp$.

WEAKLY RELATIVISTIC APPROXIMATION

The steady-state solution of the linearized Vlasov-Maxwell problem for a relativistic, uniformly magnetized thermal plasma [19] is, in notations similar to [7],

$$\varepsilon_{ij} = \delta_{ij} + iX \frac{1}{\beta_T^4 K_2(2\beta_T^{-2})} \pi^{-1} \int_0^\infty d\tau T_{jk}^{(1)} \int \frac{d^3\mathbf{u}}{\gamma} u_i u_k \exp\left[-\left(\frac{2}{\beta_T^2} - i\tau \right) \gamma - i\mathscr{N} \cdot \mathbf{u} \right] \tag{1}$$

where K_2 is the modified Bessel function of the second kind of order 2, also known as MacDonald function [20], $\gamma = \sqrt{1 + \mathbf{u}^2}$ is the Lorentz factor and $\mathbf{u} = \mathbf{p}/mc$ the normalized momentum. $X = \omega_p^2/\omega^2$ and $Y = \omega_c/\omega$ are the adimensional density and magnetic field. The convention of implicitly summing on repeated indices is adopted. The tensor

$$T^{(1)} = \begin{pmatrix} \cos Y\tau & -\sin Y\tau & 0 \\ \sin Y\tau & \cos Y\tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

defines a rotation of the reference frame at angular frequency equal to the gyrofrequency ω_c around the field-aligned axis z. The remaining axes are chosen to yield $\mathbf{N} = (N_\perp, 0, N_{||})$. The time τ is renormalized to the wave period ω^{-1}. Finally,

$$\mathscr{N}_x = \frac{1}{Y} N_\perp \sin Y\tau \qquad \mathscr{N}_y = \frac{1}{Y} N_\perp (\cos Y\tau - 1) \qquad \mathscr{N}_z = N_{||}\tau \tag{3}$$

All parameters and independent variables in integral eq.1 are real, apart from the refractive index components. These are complex, with $\Re(N_\perp) \geq 0$, $\Im(N_\perp) \leq 0$ and $\Im(N_{||}) \leq 0$, i.e. waves' amplitude cannot grow.

Let us Taylor-expand γ up to the second order in u. Apart from a factor, the integral in eq.1 rewrites:

$$\int_0^\infty d\tau e^{i\tau} T_{jk}^{(1)} \int d^3\mathbf{u} \, u_i u_k (2 - u_x^2 - u_y^2 - u_z^2) \mathscr{E}_x \mathscr{E}_y \mathscr{E}_z \tag{4}$$

where

$$\mathscr{E}_x = \exp\left[-\left(\frac{1}{\beta_T^2} - i\frac{\tau}{2}\right)u_x^2 - i\mathscr{N}_x u_x\right] \tag{5}$$

and equivalent definitions apply to $\mathscr{E}_y$ and $\mathscr{E}_z$.

The main advantage of eq.4 over eq.1 is that the parallel and perpendicular degree of freedom, which were previously coupled by γ, are now decoupled. The approximation of γ^{-1} with a sum and of the exponent with a product of functions of u_x, u_y or u_z only, ease the integration over $\mathbf{u}$. In fact, this reduces to a sum of products of integrals of type:

$$\int_{-\infty}^{\infty} du_x e^{-i\mathscr{N}_x u_x - bu_x^2} = \sqrt{\frac{\pi}{b}} e^{-\mathscr{N}_x^2/4b} \tag{6}$$

and its derivatives up to the 4th order in $\mathscr{N}_x$. Similar integrals in u_y and u_z, as well as their derivatives, are also involved. In the integrals above, the real part of

$$b = \frac{1}{\beta_T^2} - i\frac{\tau}{2} \tag{7}$$

is positive, of course. After some algebra,

$$\varepsilon_{ij} = \delta_{ij} + iX \frac{1}{\beta_T^4 K_2(2\beta_T^{-2})} \frac{\pi^{1/2}}{2} \int_0^{\infty} d\tau \frac{e^{-2b}e^{-\mathscr{N}^2/4b}}{16b^{11/2}} Q_{ij} \tag{8}$$

where

$$Q_{ij} = T_{ji}^{(1)} 2b(8b^2 - 10b + \mathscr{N}^2) - T_{ij}^{(2)}(8b^2 - 14b + \mathscr{N}^2) \tag{9}$$

and $T_{ij}^{(2)} = \mathscr{N}_i T_{jk}^{(1)} \mathscr{N}_k$ is the tensor

$$T^{(2)} = \begin{pmatrix} \mathscr{N}_x^2 & \mathscr{N}_x\mathscr{N}_y & \mathscr{N}_x\mathscr{N}_z \\ -\mathscr{N}_x\mathscr{N}_y & -\mathscr{N}_y^2 & -\mathscr{N}_y\mathscr{N}_z \\ \mathscr{N}_x\mathscr{N}_z & \mathscr{N}_y\mathscr{N}_z & \mathscr{N}_z^2 \end{pmatrix} \tag{10}$$

For very small β_T, the Lorentz factor can be further approximated in the $2 - u_x^2 - u_y^2 - u_z^2 \simeq 2$ factor in eq.4, which is a weak function of $\mathbf{u}$, compared to the exponent, where the second order approximation will be kept. Then eq.9 becomes:

$$Q_{ij} = (T_{ji}^{(1)} 2b - T_{ij}^{(2)}) 8b^2 \tag{11}$$

An alternative way of deriving eq.8 consists in taking the mildly relativistic limit of the relativistic dielectric tensor (eq.1) integrated over momenta [19]:

$$\varepsilon_{ij} = \delta_{ij} + 4iX \frac{1}{\beta_T^4 K_2(2\beta_T^{-2})} \int_0^{\infty} d\tau \left[T_{ij}^{(1)} \frac{K_2(R^{1/2})}{R} - T_{ij}^{(2)} \frac{K_3(R^{1/2})}{R^{3/2}} \right] \tag{12}$$

where asymptotic limits can be used for the modified Bessel functions as their argument

$$R = 4b^2 + \mathscr{N}^2 = \frac{4}{\beta_T^4} - 4i\frac{\tau}{\beta_T^2} + 2\frac{N_\perp^2}{Y^2}(1 - \cos Y\tau) + (N_\parallel^2 - 1)\tau^2 \tag{13}$$

diverges. Finally, truncating at the 2nd order in $\mathcal{N}$ and at the two lowest orders in $1/b$ returns eq.8.

After the change of variables $\beta_T^2 \tau/2 \to t$ and $Y = \beta_T^2 y/2$ and after expanding

$$\exp\left[\frac{\lambda \cos yt}{1-it}\right] = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left[\frac{\lambda}{2(1-it)}\right]^{p+q}\frac{e^{i(p-q)yt}}{p!q!} = \sum_{n=-\infty}^{\infty}\sum_{m=|n|}^{\infty}\frac{(\lambda/2)^m}{\left(\frac{m+n}{2}\right)!\left(\frac{m-n}{2}\right)!}\frac{e^{inyt}}{(1-it)^m},$$
(14)

with the double sum restricted to even values of $m+n$, all integrals in eq.8 can be brought into the form

$$-i\int_0^{\infty}\frac{(it)^r}{(1-it)^q}\exp\left[izt - \frac{\lambda + at^2}{1-it}\right]dt = e^{\lambda}\mathscr{F}_{q,r}(z-\lambda, a-\lambda)$$
(15)

where the definition of generalized Shkarofsky functions [21, 22, 23]

$$\mathscr{F}_{q,r}(z,a) = -i\int_0^{\infty}\frac{(it)^r}{(1-it)^q}\exp\left[izt - \frac{at^2}{1-it}\right]dt,$$
(16)

was invoked and dispersion was found to depend on Larmor radius, in agreement with relativistic eq.1. A correction to generalized Shkarofsky functions arises in the form of a shift of arguments z and a by an amount λ. Such a linear perturbation can be interpreted as a finite-Larmor-radius correction to the resonance condition (through $z = 2(nY+1)/\beta_T^2$) and to its width in inhomogenoeus plasmas, through the ratio of Doppler to relativistic width, $a = N_{||}/\beta_T$. The physical meaning of $z \to z - \lambda$ is the well-known relativistic downshift.

Putting together eqs.8-14 and, for consistency with earlier weakly relativistic approximations, the asymptotic limit

$$K_2(R^{1/2}) \simeq e^{-R^{1/2}}\sqrt{\frac{\pi}{2}}\left[\frac{1}{R^{1/4}} + \frac{15}{8}\frac{1}{R^{3/4}}\right]$$
(17)

yields the following expression for the slightly relativistic dielectric tensor:

$$\varepsilon_{ij} = \delta_{ij} + iX\frac{e^{\lambda}}{16+15\beta_T^2}\sum_{n=-\infty}^{\infty}\sum_{m=|n|}^{\infty}\frac{(\lambda/2)^m \mathscr{Q}_{mn,ij}}{\left(\frac{m+n}{2}\right)!\left(\frac{m-n}{2}\right)!}$$
(18)

where, as usual, the summation is restricted to even values of $m+n$ and

$$\mathscr{Q}_{mn,11} = 8i\mathscr{F}^+_{m+\frac{5}{2},0,n\pm 1} + 2i\beta_T^2\frac{N_\perp^2}{Y^2}\left(\mathscr{F}^+_{m+\frac{7}{2},0,n\pm 2} - 2\mathscr{F}_{m+\frac{7}{2},0,n}\right)$$
(19)

$$\mathscr{Q}_{mn,22} = 8i\mathscr{F}^+_{m+\frac{5}{2},0,n\pm 1} + 2i\beta_T^2\frac{N_\perp^2}{Y^2}\left(\mathscr{F}^+_{m+\frac{7}{2},0,n\pm 2} - 4\mathscr{F}^+_{m+\frac{7}{2},0,n\pm 1} + 6\mathscr{F}_{m+\frac{7}{2},0,n}\right)$$
(20)

$$\mathscr{Q}_{mn,33} = 16i\mathscr{F}_{m+\frac{5}{2},0,n} + \frac{32i}{\beta_T^2}N_{||}^2\mathscr{F}_{m+\frac{7}{2},2,n}$$
(21)

$$\mathscr{Q}_{mn,12} = 8\mathscr{F}^-_{m+\frac{5}{2},0,n\pm 1} + 2\beta_T^2\frac{N_\perp^2}{Y^2}\left(\mathscr{F}^-_{m+\frac{7}{2},0,n\pm 2} - 2\mathscr{F}^-_{m+\frac{7}{2},0,n\pm 1}\right)$$
(22)

$$\mathscr{Q}_{mn,13} = 8i\frac{N_\perp}{Y}N_{||}\mathscr{F}^-_{m+\frac{7}{2},0,n\pm1} \tag{23}$$

$$\mathscr{Q}_{mn,23} = 8\frac{N_\perp}{Y}N_{||}\left(\mathscr{F}^+_{m+\frac{7}{2},1,n\pm1} - 2\mathscr{F}_{m+\frac{7}{2},1,n}\right) \tag{24}$$

Moreover $\mathscr{Q}_{mn,21} = -\mathscr{Q}_{mn,12}$, $\mathscr{Q}_{mn,31} = \mathscr{Q}_{mn,13}$ and $\mathscr{Q}_{mn,32} = -\mathscr{Q}_{mn,23}$.
This was for Q_{ij} as in eq.11. However, if eq.9 is used instead, one has to add:

$$\delta\mathscr{Q}_{mn,11} = -4iN_{||}^2\mathscr{F}^+_{m+\frac{9}{2},2,n\pm1} - 10i\beta_T^2\mathscr{F}^+_{m+\frac{7}{2},0,n\pm1} + i\beta_T^2\frac{N_\perp^2}{Y^2}N_{||}^2\left(\mathscr{F}^+_{m+\frac{11}{2},2,n\pm2} - 2\mathscr{F}_{m+\frac{11}{2},2,n}\right) \tag{25}$$

$$\delta\mathscr{Q}_{mn,22} = -4iN_{||}^2\mathscr{F}^+_{m+\frac{9}{2},2,n\pm1} \tag{26}$$

$$-10i\beta_T^2\mathscr{F}^+_{m+\frac{7}{2},0,n\pm1} - i\beta_T^2\frac{N_\perp^2}{Y^2}N_{||}^2\left(\mathscr{F}^+_{m+\frac{11}{2},2,n\pm2} - 4\mathscr{F}^+_{m+\frac{11}{2},2,n\pm1} + 6\mathscr{F}_{m+\frac{11}{2},2,n}\right) \tag{27}$$

$$\delta\mathscr{Q}_{mn,33} = -\frac{16i}{\beta_T^2}N_{||}^4\mathscr{F}_{m+\frac{11}{2},4,n} - 64iN_{||}^2\mathscr{F}_{m+\frac{9}{2},2,n} \tag{28}$$

$$-20i\beta_T^2\mathscr{F}_{m+\frac{7}{2},0,n} - 4i\beta_T^2\frac{N_\perp^2}{Y^2}N_{||}^2\left(\mathscr{F}^+_{m+\frac{11}{2},2,n\pm1} - 2\mathscr{F}_{m+\frac{11}{2},2,n}\right) \tag{29}$$

$$\delta\mathscr{Q}_{mn,12} = -4N_{||}^2\mathscr{F}^-_{m+\frac{9}{2},2,n\pm1} - 10\beta_T^2\mathscr{F}^-_{m+\frac{7}{2},0,n\pm1} - \beta_T^2\frac{N_\perp^2}{Y^2}N_{||}^2\left(\mathscr{F}^-_{m+\frac{11}{2},2,n\pm2} - 2\mathscr{F}^-_{m+\frac{11}{2},2,n\pm1}\right) \tag{30}$$

$$\delta\mathscr{Q}_{mn,13} = -4i\frac{N_\perp}{Y}N_{||}^3\mathscr{F}^-_{m+\frac{11}{2},3,n\pm1} - 14i\beta_T^2\frac{N_\perp}{Y}N_{||}\mathscr{F}^-_{m+\frac{9}{2},0,n\pm1} \tag{31}$$

$$\delta\mathscr{Q}_{mn,23} = -4\frac{N_\perp}{Y}N_{||}^3\left(\mathscr{F}^-_{m+\frac{11}{2},3,n\pm1} - 2\mathscr{F}_{m+\frac{11}{2},3,n}\right) - 14\beta_T^2\frac{N_\perp}{Y}N_{||}\left(\mathscr{F}^-_{m+\frac{9}{2},1,n\pm1} - 2\mathscr{F}_{m+\frac{9}{2},1,n}\right) \tag{32}$$

Powers of β_T^2 up to the 1st order were retained here for consistency with eq.19-24. Arguments were omitted for brevity from generalized Shkarofsky functions and summarized by an index for the harmonic number:

$$\mathscr{F}_{q,r,n} = \mathscr{F}_{q,r}\left(\frac{2}{\beta_T^2}(nY+1)-\lambda,\ \frac{N_{||}^2}{\beta_T^2}-\lambda\right) \tag{33}$$

Additionally, the compact notations

$$\mathscr{F}^+_{q,r,n\pm p} = \mathscr{F}_{q,r,n+p} + \mathscr{F}_{q,r,n-p} \qquad \mathscr{F}^-_{q,r,n\pm p} = \mathscr{F}_{q,r,n+p} - \mathscr{F}_{q,r,n-p} \tag{34}$$

were employed for some frequently occurring sums and differences.

At this point it should be remembered that it is customary to treat as a constant the Lorentz factor γ at denominator in the integrand of eq.1 and to Taylor-expand only the exponent, on the ground that relativistic corrections at denominator have a comparatively small effect on the integral. This is equivalent to use eq.11 instead of eq.9 and, ultimately, to neglect corrections 25-32 to eqs.19-24, which is not always legitimate. It is only partly legitimate for those elements which become $\mathscr{O}(\beta_T^4)$ under the assumption $N_{||} \ll \beta_T$ which, by the way, was not adopted in the present work.

Note also that simple Shkarofsky functions of $r = 0$ appear in most functions $\mathscr{Q}_{mn,ij}$.

The double sum in eq.18 might look computationally expensive, but functions $\mathscr{F}$ of different q (thus, $\mathscr{Q}$ of different m) can recursively be related to one another [2, 3, 8, 22,

24]. Besides, for relativistically broadened but well-resolved lines, the double sum can be restricted to diagonal terms ($m = n$).

An alternative form of eq.18 can be obtained by recognizing, in the sum over m in eq.14, the generating function for the modified Bessel function of the first kind:

$$\exp\left[\frac{\lambda \cos yt}{1-it}\right] = \sum_{n=-\infty}^{\infty} I_n\left(\frac{\lambda}{1-it}\right) e^{inyt} \tag{35}$$

This generalizes an identity utilized in deriving the warm non-relativistic dielectric tensor [25]. However, it does so by replacing λ with $\lambda/(1-it)$, i.e. by introducing a time dependence for I_n that cannot be factored out of the time integral anymore. As a consequence, integrals in

$$\varepsilon_{ij} = \delta_{ij} + 2iX\frac{\beta_T^4}{16+15\beta_T^2}\sum_{n=-\infty}^{\infty}\int_0^{\infty} dt \frac{Q_{ij}}{(1-it)^{11/2}} \exp\left[\frac{2}{\beta_T^2}(nY+1) - \frac{\lambda + N_\parallel^2 t^2/\beta_T^2}{1-it}\right] I_n\left(\frac{\lambda}{1-it}\right) \tag{36}$$

are more complicated than eq.15. On the other hand, they avoid the nuisance of the double sum and can be useful when a high m is required for convergence.

In summary a novel formulation of the semi-relativistic dielectric tensor valid for arbitrary wavenumbers was derived starting from Trubnikov's fully relativistic tensor. The new tensor describes the propagation (including electrostatic propagation based on finite Larmor radius effects) and damping (corrected for relativistic effects and Doppler broadening) of modes of arbitrary orientation relative to the magnetic field, including the Bernstein mode.

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