Generalized $\mathcal{F}$-statistic: Multiple detectors and multiple gravitational wave pulsars

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The $\mathcal{F}$-statistic, derived by Jaranowski, Krolak and Schutz (1998), is the optimal (frequentist) statistic for the detection of nearly periodic gravitational waves from known neutron stars, in the presence of stationary, Gaussian detector noise. The $\mathcal{F}$-statistic was originally derived for the case of a single detector, whose noise spectral density was assumed constant in time, and for a single known neutron star. Here we show how the $\mathcal{F}$-statistic can be straightforwardly generalized to the cases of (1) a network of detectors with time-varying noise curves, and (2) a collection of known sources (e.g., all known millisecond pulsars within some fixed distance). Fortunately, all the important ingredients that go into our generalized $\mathcal{F}$-statistics are *already* calculated in the single-source/single-detector searches that are currently implemented, e.g., in the Laser Interferometer Gravitational-Wave Observatory software library, so implementation of optimal multidetector, multisource searches should require negligible additional cost in computational power or software development. This paper also includes an analysis of the likely efficacy of a collection-type search, and derives criteria for deciding which candidate sources should be included in a collection, if one is trying to maximize the detectability of the whole. In particular we show that for sources distributed uniformly in a thin disk, the strongest source in the collection should have signal-to-noise-squared ~5 times larger than weakest source, for an optimized collection. We show that gravitational waves from collection of the few brightest (in gravitational waves) neutron stars could perhaps be detected before the single brightest source, but that this is far from guaranteed. Once gravitational waves from the few brightest neutron stars have been discovered, grouping more distant (individually undetectable) pulsars into collections, and then searching for those collections, should be an effective way of measuring the average gravitational-wave strengths of those more distant pulsars.

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I. INTRODUCTION

The $\mathcal{F}$-statistic, as first derived by Jaranowski, Krolak and Schutz [1] (hereinafter referred to as JKS), is the optimal frequentist statistic for the detection of nearly periodic gravitational waves (GWs) from a known neutron star. In the original JKS version, the $\mathcal{F}$-statistic was derived only for the case of a single GW detector (which was assumed to have stationary noise characteristics) and a single known neutron star (assumed to be emitting GWs at the neutron star's rotation frequency and/or at twice its rotation frequency). Here we show how the $\mathcal{F}$-statistic can be generalized in a straightforward manner to the cases of (1) a network of detectors with time-varying noise curves, and (2) an entire collection of known sources. Fortunately, all the important ingredients that go into the generalized $\mathcal{F}$-statistic are *already* calculated in the single-detector/single-source searches that are currently implemented, e.g., in the Laser Interferometer Gravitational-Wave Observatory (LIGO) software library [2], so implementation of optimal multidetector and/or multisource searches should require negligible additional cost in software development and computation.

We note that the problem of optimally combining data from different detectors has already been solved for several types of GW searches. For the case of inspiralling binaries, we refer the reader to Bose, Pai and Dhurandhar [3] and to Finn [4]; for the case of GW bursts, to Sylvestre [5]; and for the case of Laser Interferometer Space Antenna (LISA) observations of galactic, stellar-mass binaries, to both Krolak *et al.* [6] and Rogan and Bose [7]. (LISA can effectively be treated as a network of three independent GW detectors.) Our analysis in Sec. III is especially similar to that of Krolak *et al.* [6] and Rogan and Bose [7], since formally the sources considered there are equivalent to GW pulsars. Like GW pulsars, the stellar-mass binaries visible to LISA are effectively monochromatic sources that can be characterized by four amplitude parameters, in addition to the GW frequency and the two angles specifying the source position on the sky.

The basic idea of somehow combining the signals from many individually undetectable sources or events, in hopes of finding a statistical excess, is also hardly a new one. In GW astronomy, a good example is the suggestion of looking for GW bursts associated with gamma-ray bursts by cross correlating the outputs of LIGO's L1 and H1 detectors over short time windows coincident with hundreds of observed gamma-ray bursts [8]. But our application of this idea to the population of known millisecond pulsars appears to be new. We investigate when this strategy is likely to be effective and derive useful criteria for deciding how many and which sources should be included in the collection, in order to maximize the detectability of that group.

The plan of this paper is as follows. In Sec. II we briefly establish notation; we generally try to align our notation with that of JKS, to ease comparison with their work. In Sec. III we derive the $\mathcal{F}$-statistic for a network of N

detectors and a single source. This multidetector $\mathcal{F}$-statistic follows a χ^2 distribution with 4 degrees of freedom, exactly as with the single-detector version. We consider the general case where the detectors have correlated noises, but of course our expressions simplify in the case where noises from different detectors are uncorrelated. As a bonus, our results immediately show how to appropriately time-weight the data in the (realistic) case that the detector noise spectra are slowly time-varying. (The appropriate time-weighting for a single detector has already been derived by Itoh *et al.* [9] and is implemented in the LIGO software library. We give an independent derivation, since here it follows trivially.)

In Sec. IV we extend the $\mathcal{F}$-statistic to the case of a collection of known sources. If there are M known sources (each emitting at a single, known frequency), then the correct $\mathcal{F}$-statistic for the entire collection follows a χ^2 distribution with $4M$ degrees of freedom. (This is true for both the single-detector case and for an N-detector network.) The most interesting target population is clearly (some subset of) the known millisecond pulsars. We consider two particularly interesting cases: (1) a collection of the few very brightest GW pulsars, and (2) a larger collection of more distant GW pulsars. We investigate the expected gains from both these types of multisource searches, under the reasonable assumption that there exists some population of GW pulsars that is uniformly spread throughout the Galactic disk. As a further illustration of multisource searches, we estimate the sensitivity of the LIGO network to the collection consisting of the five "most promising" millisecond pulsars, assuming they all have the same ellipticity. Our conclusions are summarized in Sec. V.

II. NOTATION

Let us consider an N-detector network, with output $x^\alpha(t)$, $\alpha = 1, \ldots, N$. (In cases where a single instrument outputs k independent data streams—e.g., a spherical bar detector that encodes for two GW polarizations in five data streams—we regard these formally as the outputs of k different detectors.) For simplicity, we begin by assuming that the detector noise is both stationary and Gaussian. We allow, however, for the possibility that the noises are correlated. Then we have

$$\langle \tilde{n}^\alpha(f)\tilde{n}^\beta(f')^*\rangle = \tfrac{1}{2}\delta(f - f')S_h^{\alpha\beta}(f). \tag{2.1}$$

Here tildes denote Fourier transforms, according to the convention that

$$\tilde{x}(f) = \int_{-\infty}^{\infty} e^{2\pi i f t}x(t)dt, \tag{2.2}$$

and "$\langle\cdots\rangle$" denotes "expectation value." We note that $S_h^{\alpha\beta}(f)$ is the *single-sided* noise spectral density, which is also the convention followed in JKS. [If we were using the *double-sided* convention, the factor $\frac{1}{2}$ on the right-hand side in Eq. (2.1) would be replaced by 1.]

The Gaussian random process $\mathbf{n}(t)$ determines a natural inner product $(\ldots|\ldots)$ on the space of functions $\mathbf{x}(t)$ [10]:

$$(\mathbf{x}|\mathbf{y}) \equiv 4\Re\int_0^\infty df\tilde{x}^\alpha(f)^*[S_h^{-1}(f)]_{\alpha\beta}\tilde{y}^\beta(f), \tag{2.3}$$

where $[S_h^{-1}(f)]_{\alpha\beta}S_h^{\beta\gamma}(f) = \delta_\alpha^\gamma$ and where $\Re$ means "the real part of." Here and below, to reduce index clutter, we sometimes represent a signal vector, having one component for each detector, by simply using boldface without an index; e.g., $\mathbf{x}(t)$ instead of $x^\alpha(t)$. The inner product Eq. (2.3) is such that the probability distribution function (pdf) for the noise $\mathbf{n}(t)$ takes the form

$$\mathrm{pdf}[\mathbf{n}] = \mathcal{N}e^{-(\mathbf{n}|\mathbf{n})/2}, \tag{2.4}$$

where $\mathcal{N}$ is a normalization constant. It follows that the expectation value of the product $(\mathbf{x}|\mathbf{n})(\mathbf{y}|\mathbf{n})$, over many realizations of the noise, is simply given by

$$\langle(\mathbf{x}|\mathbf{n})(\mathbf{y}|\mathbf{n})\rangle = (\mathbf{x}|\mathbf{y}). \tag{2.5}$$

III. $\mathcal{F}$-STATISTIC FOR A DETECTOR NETWORK

Given gravitational-wave data from a single detector, the $\mathcal{F}$-statistic developed by JKS is the optimal frequentist statistic for the detection of GWs from a single known neutron star (NS) in that single data stream. This section answers the question: if we have data from a network of detectors (possibly including bars as well as interferometers) how does one combine the different data streams to produce the optimal detection statistic for the entire network?[1]

A. $\mathcal{F}$-statistic for a single source and multiple detectors, all with time-invariant noise curves

Consider the search for nearly periodic GWs from a single source with known position and known (possibly time-varying) frequency, e.g., pulsar (PSR) 1937 + 21. The GW signal is characterized by four unknowns: an overall amplitude A_0 (equivalent to the combination $h_0 \sin\zeta\sin^2\theta$ in the notation of JKS), two angles ι and ψ that characterize the waves' polarization (equivalent to determining the direction of the NS's spin axis), and an overall phase Φ_0. The GW signal $h^\alpha(t)$ depends nonlinearly on ι, ψ, Φ_0, but, crucially, one can make a simple

[1]JKS briefly consider this question and sketch a claimed answer, in Sec. 4 of their paper [1], but their answer is quite wrong. In particular, they claim that the appropriate $\mathcal{F}$-statistic for a network with N detectors follows a χ^2 distribution with $4N$ degrees of freedom, but we shall see below that the right number of degrees of freedom is just 4—the same as for the single-detector case. This is because there are still just four unknowns in the problem: the amplitude and phase of each of the two GW polarizations.

change of variables—to $(\lambda^1, \lambda^2, \lambda^3, \lambda^4)$—such that dependence of $h^\alpha(t)$ is linear in these new variables:

$$h^\alpha(t) = \sum_{a=1}^{4} \lambda^a h_a^\alpha(t) \tag{3.1}$$

where the four basis waveforms $h_a^\alpha(t)$ are defined by

$$h_1^\alpha(t) = F_+^\alpha(t)\cos\Phi^\alpha(t), \qquad h_2^\alpha(t) = F_\times^\alpha(t)\cos\Phi^\alpha(t),$$
$$h_3^\alpha(t) = F_+^\alpha(t)\sin\Phi^\alpha(t), \qquad h_4^\alpha(t) = F_\times^\alpha(t)\sin\Phi^\alpha(t). \tag{3.2}$$

Here $\Phi(t)$ is the waveform phase at the detector:

$$\Phi(t) \approx 2\pi \int^t f_{\rm gw}(t')dt', \tag{3.3}$$

where $f_{\rm gw}(t')$ is the measured GW frequency at the detector at time t'. The measured frequency includes the Doppler effect from the detector's motion relative to the source, as well as Einstein and Shapiro delays associated with the Earth's orbit around the Sun. When the GW pulsar is in a binary, then $f_{\rm gw}(t')$ also includes the Roemer, Einstein, and Shapiro delays associated with that binary orbit. We emphasize that the known-pulsar searches described here do *not* require the GW pulsar be isolated, but just that there exist an accurate timing model for the emitted waves. The $F_+^\alpha(t)$ and $F_\times^\alpha(t)$ terms in Eq. (3.2) are the beam-pattern functions giving the response of the αth detector to the + and × polarizations, respectively. We note that the exact form of $F_+^\alpha(t)$ and $F_\times^\alpha(t)$ depends on one's convention for decomposing the waveform into "plus" and "cross" polarizations; a one-parameter family of choices is possible, corresponding to the freedom to rotate the axes around the line of sight. JKS follow the conventions of Bonazzola and Gourgoulhon [11].

A further word on our index notation: as above, we use Greek indices from the beginning of the alphabet (α, β, γ) to indicate the various detectors in the network; we use Latin letters from the beginning of the alphabet (a, b, c) to indicate the four independent waveform components from a single NS (emitting at a single frequency); and we use Latin letters from the middle of the alphabet (i, j, k) to label different NSs. As above, we sometimes remove the Greek index and instead represent the vector in boldface: $\mathbf{h}_a(t)$ instead of $h_a^\alpha(t)$. Finally, we use the capital Latin letter "J" to label different time intervals [always intervals over which the noise spectral density $S_h^{\alpha\beta}(f)$ can be safely approximated as constant].

Next we define the 4×4 matrix Γ_{ab} by

$$\Gamma_{ab} \equiv \left(\frac{\partial \mathbf{h}}{\partial \lambda^a} | \frac{\partial \mathbf{h}}{\partial \lambda^b}\right) = (\mathbf{h_a}|\mathbf{h_b}). \tag{3.4}$$

Because both the observation time and 1 day [the time scale on which the $F_{+,\times}^\alpha(t)$ vary] are vastly larger than the period of the sought-for GWs (typically 10^{-2}–10^{-3} s), we can replace $\cos^2\Phi(t)$, $\sin^2\Phi(t)$, and $\cos\Phi(t)\sin\Phi(t)$ by their time averages: $\cos^2\Phi(t), \sin^2\Phi(t) \rightarrow \frac{1}{2}$, while $\cos\Phi(t)\sin\Phi(t) \rightarrow 0$. Then we have

$$\begin{aligned}
\Gamma_{11} &\approx \sum_{\alpha,\beta} (S_h^{-1}(f_{\rm gw}))_{\alpha\beta} \int F_+^\alpha(t) F_+^\beta(t) dt \\
\Gamma_{12} &\approx \sum_{\alpha,\beta} (S_h^{-1}(f_{\rm gw}))_{\alpha\beta} \int F_+^\alpha(t) F_\times^\beta(t) dt \\
\Gamma_{22} &\approx \sum_{\alpha,\beta} (S_h^{-1}(f_{\rm gw}))_{\alpha\beta} \int F_\times^\alpha(t) F_\times^\beta(t) dt;
\end{aligned} \tag{3.5}$$

additionally, $\Gamma_{33} \approx \Gamma_{11}$, $\Gamma_{34} \approx \Gamma_{12}$, $\Gamma_{44} \approx \Gamma_{22}$, and $\Gamma_{13} \approx \Gamma_{14} \approx \Gamma_{23} \approx \Gamma_{24} \approx 0$.

The best-fit values of λ^a satisfy

$$\frac{\partial}{\partial \lambda^a}\left(\mathbf{x} - \sum_b \lambda^b \mathbf{h_b} | x - \sum_c \lambda^c \mathbf{h_c}\right) = 0 \tag{3.6}$$

implying

$$\lambda^a = \sum_b (\Gamma^{-1})^{ab} (\mathbf{x}|\mathbf{h_b}), \tag{3.7}$$

and our optimal statistic $2\mathcal{F}$ is then just twice the log of the likelihood ratio:

$$\begin{aligned}
2\mathcal{F} &= (\mathbf{x}|\mathbf{x}) - \left(\mathbf{x} - \sum_b \lambda^b \mathbf{h_b} | \mathbf{x} - \sum_c \lambda^c \mathbf{h_c}\right) \\
&= \sum_{a,d} (\Gamma^{-1})^{ad} (\mathbf{x}|\mathbf{h_a})(\mathbf{x}|\mathbf{h_d}).
\end{aligned} \tag{3.8}$$

Therefore using $2\mathcal{F}$ as one's detection statistic satisfies the Neyman-Pearson criterion for an optimum test: it minimizes the false dismissal (FD) rate for any given false alarm (FA) rate.

Writing $\mathbf{x} = \mathbf{n} + \mathbf{h}$, and plugging into Eq. (3.8), we find

$$\langle 2\mathcal{F} \rangle = 4 + (\mathbf{h}|\mathbf{h}), \tag{3.9}$$

where we have used Eq. (2.5) and the fact that $\langle(\mathbf{h}|\mathbf{n})\rangle = 0$. More generally, it is easy to show that $y \equiv 2\mathcal{F}$ follows a χ^2 distribution with 4 degrees of freedom (d.o.f.) and noncentrality parameter $\rho^2 \equiv (\mathbf{h}|\mathbf{h})$:

$$P(y) = \chi^2(y|4; \rho^2). \tag{3.10}$$

B. $\mathcal{F}$-statistic for a single source and multiple detectors with time-varying noise curves

It is trivial to generalize the above results to a network of detectors with time-varying noise curves. Divide the total observation time into segments that are short enough that all noise correlation functions $S_h^{\alpha\beta}$ can be approximated as constant during each segment. (We assume the segments are still much longer than the GW period.) Let there be p such segments in all. Let the beginning and end points of these time intervals be $(t_0, t_1, \cdots, t_p)$. (In this scheme, we can formally represent gaps in the output of one or more detectors by intervals where some of the components $S_h^{\alpha\beta}$ go to infinity.) While our signals come from N detectors with time-varying noise curves, we can formally regard them as coming from pN detectors, each with stationary noise (but such that only N detectors are turned "on" at any instant; when N more turn on, the previous N turn off). But we know how to construct the $\mathcal{F}$-statistic for pN detectors with stationary noise characteristics, from the previous subsection. (Nothing in that subsection required all the detectors to be on simultaneously.) The noise spectral density coefficients $S_h^{\alpha\beta}(f)$ are now labeled by time interval J: $S_{h,J}^{\alpha\beta}(f)$. Then Γ_{11} becomes

$$\Gamma_{11} \approx \sum_{\alpha,\beta=1}^{N}\sum_{J=1}^{p}(S_{h,J}^{-1}(f_{\rm gw}))_{\alpha\beta}\int_{t_{J-1}}^{t_J} F_+^{\alpha}(t)F_+^{\beta}(t)dt$$
$$\rightarrow \sum_{\alpha,\beta=1}^{N}\int_{t_0}^{t_p}(S_h^{-1}(f_{\rm gw},t))_{\alpha\beta}F_+^{\alpha}(t)F_+^{\beta}(t)dt \qquad (3.11)$$

where we have made the notational shift $S_{h,J}^{\alpha\beta}(f_{\rm gw}(t)) \rightarrow S_h^{\alpha\beta}(f_{\rm gw}(t),t)$; i.e., we have replaced the discrete label "J" by the continuous label "t." (In practice, the noise spectral density at any instant is estimated from the data itself, e.g., by use of a running mean.)

Similarly,

$$\Gamma_{12} \approx \sum_{\alpha,\beta=1}^{N}\int_{t_0}^{t_p}(S_h^{-1}(f_{\rm gw}(t),t))_{\alpha\beta}F_+^{\alpha}(t)F_\times^{\beta}(t)dt$$
$$\Gamma_{22} \approx \sum_{\alpha,\beta=1}^{N}\int_{t_0}^{t_p}(S_h^{-1}(f_{\rm gw}(t),t))_{\alpha\beta}F_\times^{\alpha}(t)F_\times^{\beta}(t)dt, \qquad (3.12)$$

and again $\Gamma_{33} \approx \Gamma_{11}$, $\Gamma_{34} \approx \Gamma_{12}$, $\Gamma_{44} \approx \Gamma_{22}$, while $\Gamma_{13} \approx \Gamma_{14} \approx \Gamma_{23} \approx \Gamma_{24} \approx 0$.

If we define

$$(\mathbf{x}|\mathbf{h_a}) \equiv \sum_{\alpha,\beta}\int_{t_0}^{t_p}(S_h^{-1}(f_{\rm gw}(t),t))_{\alpha\beta}x^{\alpha}(t)h_a^{\beta}(t)dt, \qquad (3.13)$$

then Eq. (3.8) remains the correct expression for $2\mathcal{F}$, and Eq. (3.10) remains its correct distribution function, with $y \equiv 2\mathcal{F}$ and $\rho^2 = (\mathbf{h}|\mathbf{h})$. That is, given our notation, the expression for the multidetector $\mathcal{F}$-statistic is the same as for the single-detector case.

C. $\mathcal{F}$-statistic for a single source and N detectors with uncorrelated noises

The expressions simplify somewhat in the (common) case where the noises from different detectors are uncorrelated: $S_h^{\alpha\beta}(f,t) = S_h^{\alpha}(f,t)\delta^{\alpha\beta}$. Then the inner product $(\mathbf{x}|\mathbf{y})$ is given by

$$(\mathbf{x}|\mathbf{y}) \equiv 2\sum_{\alpha}\int_{t_0}^{t_p}\frac{x^{\alpha}(t)y^{\alpha}(t)dt}{S_h^{\alpha}(f_{\rm gw}(t),t)}. \qquad (3.14)$$

We define A, B, C by

$$\tfrac{1}{2}A \equiv (\mathbf{h_1}|\mathbf{h_1}), \qquad \tfrac{1}{2}B \equiv (\mathbf{h_2}|\mathbf{h_2}), \qquad \tfrac{1}{2}C \equiv (\mathbf{h_1}|\mathbf{h_2}). \qquad (3.15)$$

(Note that the A, B, C terms defined here are, in the single-detector case, larger than the A, B, C terms in JKS by a factor of $T_0/S_h(f_{\rm gw})$, where T_0 is the observation time.)

Then $\Gamma_{11} = \frac{1}{2}A$, $\Gamma_{22} = \frac{1}{2}B$, and $\Gamma_{12} = \frac{1}{2}C$. So Γ^{-1} takes the form

$$\Gamma^{-1} = \frac{2}{D}\begin{pmatrix} B & -C & 0 & 0 \\ -C & A & 0 & 0 \\ 0 & 0 & B & -C \\ 0 & 0 & -C & A \end{pmatrix}. \qquad (3.16)$$

where $D \equiv AB - C^2$. Thus we arrive at

$$2\mathcal{F} = \frac{2}{D}[B\{(\mathbf{x}|\mathbf{h_1})(\mathbf{x}|\mathbf{h_1}) + (\mathbf{x}|\mathbf{h_3})(\mathbf{x}|\mathbf{h_3})\}$$
$$+ A\{(\mathbf{x}|\mathbf{h_2})(\mathbf{x}|\mathbf{h_2}) + (\mathbf{x}|\mathbf{h_4})(\mathbf{x}|\mathbf{h_4})\}$$
$$- 2C\{(\mathbf{x}|\mathbf{h_1})(\mathbf{x}|\mathbf{h_2}) + (\mathbf{x}|\mathbf{h_3})(\mathbf{x}|\mathbf{h_4})\}]. \qquad (3.17)$$

As a check, consider the case of N identical, nearby detectors (assumed to have uncorrelated noises). Then A, B and C all scale like N, while $D \propto N^{-2}$. In the absence of a GW signal, the only terms in the (implied) double sum over α, β in (3.17) that contribute, on average, are those with $\beta = \alpha$. Thus terms like $(\mathbf{x}|\mathbf{h_1})(\mathbf{x}|\mathbf{h_1})$ scale like N in the absence of a true GW, and so $\langle 2\mathcal{F}\rangle$ remains invariant (always equalling 4) under changes of N when there is no true signal. However when there is a true signal, then terms like $(\mathbf{x}|\mathbf{h_1})(\mathbf{x}|\mathbf{h_1})$ scale like N^2, so the noncentrality parameter ρ^2 of the distribution scales like N—just as one would expect.

Equation (3.17) can be rewritten more compactly if we use complexified variables, as done in JKS. Defining

$$2F_a \equiv (\mathbf{x}|\mathbf{h_1} - i\mathbf{h_3}), \qquad 2F_b \equiv (\mathbf{x}|\mathbf{h_2} - i\mathbf{h_4}), \qquad (3.18)$$

Eq. (3.17) becomes

$$2\mathcal{F} = \frac{8}{D}[B|F_a|^2 + A|F_b|^2 - 2C\Re(F_aF_b^*)]. \qquad (3.19)$$

IV. $\mathcal{F}$-STATISTIC FOR MULTIPLE SOURCES

In this section we consider a search for a *collection* of M nearly periodic GW sources, all with known positions and frequencies. In this case, the signal $h^{\alpha}(t)$ depends linearly on $4M$ unknown parameters. Assuming that the M different GW frequencies f_i $(i = 1,\cdots,M)$ are all sufficiently well separated that the detector noises are uncorrelated [i.e.,$\langle\tilde{n}^{\alpha}(f_i)\tilde{n}^{\beta}(f_j)^*\rangle = 0$ for $i \neq j$], then a trivial repetition of the arguments in Sec. III shows that the optimum statistic (for either the single-detector or the multidetector case) $2\mathcal{F}$ is simply the sum of the optimal statistics for the individual sources:

$$2\mathcal{F} \equiv \sum_i 2\mathcal{F}_i. \quad (4.1)$$

It also easy to show that $y \equiv 2\mathcal{F}$ follows a χ^2 distribution with $4M$ degrees of freedom:

$$P(y) = \chi^2(y|4M; \rho^2), \quad (4.2)$$

where the noncentrality parameter $\rho^2 = \sum_i \rho_i^2$.

There are currently $\sim$100 known millisecond (ms) pulsars[2] (defined as pulsars with period $P < 10$ msec), of which $\sim$60 are in binaries. We can of course consider any subset of these as some collection, sum their individual $\mathcal{F}$-statistics (derived from existing GW data) as in Eq. (4.1), and test whether or not the collection has been detected. But when is such a strategy likely to be advantageous, and for which subsets? It seems that there are at least two interesting applications of this idea. First, one might hope that the nearest $\sim$5–50 (say) ms pulsars, searched for as a collection, might be more detectable than any individual member. If this were the case, a multisource search might hasten the first discovery of GWs from rotating neutron stars. We shall see below, however, that it is highly unlikely that a collection of more than a few ($\sim$ 2–5) of the brightest (in GWs) ms pulsars is more detectable than the very brightest source alone. While it is reasonably likely that the brightest few sources, taken together, are more detectable than the single brightest one—and we give a realistic example of this in Sec. IV D—this will certainly not be the case for the brightest 20 or 50 sources. If there are too many sources, the strongest ones are effectively diluted by mixing them with the weaker ones, in the multisource $\mathcal{F}$-statistic.

To understand the second interesting application of multisource searches, imagine a day when GWs have already been detected from the few brightest, closest GW pulsars (all at distances of $\sim$0.1–0.5 kpc, say), but when the GW pulsars in the range $d > 0.5$ kpc are still too faint to be detected. In that situation, it could make sense to take as a collection all (or some promising-looking fraction of) the ms pulsars in some *annulus*—say those in the range $0.5 < d < 1.0$ kpc. This might allow one to measure the *average* GW strength of those more distant sources, even if no single one of them could be positively detected in GWs, and therefore to begin to make interesting statistical statements based on this larger sample.

We investigate the likely advantages of multisource searches in the next five subsections. First, in Sec. IV A, we consider a collection of M sources, for large M, and ask: when does adding one more source to the collection increase that collection's overall detectability? In Sec. IV B we rederive the distribution function of signal-to-noise-squared for any spatially uniform population of sources. These results from IV A and IV B are utilized in IV C, where we show that for a uniform planar distribution of GW pulsars (representing a somewhat idealized version of the population in our neighborhood of the Galactic disk), one might reasonably expect the few brightest sources, taken together, to be more detectable than the very brightest one. However this is hardly guaranteed, and any advantages of a multisource search are likely to be small in this case. This is illustrated in IV D, where we consider a fairly realistic example based on the closest known ms pulsars. In IV E we consider searching collectively for more numerous, more distant GW pulsars, after the nearest, brightest ones have been detected, and the advantages of multisource searching are shown to be much greater in that case.

[2] However $\sim$60% of these are in globular clusters, at distances of several kpc, and so are roughly an order of magnitude further away than the closest known sources.

A. The large-M case

Here we compare the sensitivities of a single-source search and a search for a collection with M members, when M is much larger than 1. For the single-source search, the threshold value of $\mathcal{F}$ that gives a 1% FA rate is given by $2\mathcal{F}_{\rm th}(\equiv y_{\rm th}) = 13.277$ [i.e., $\int_{13.277}^{\infty} \chi^2(y|4)dy = 0.01$]. To be detectable with FD rate $\geq 50\%$, the signal strength must be at least $\rho^2 = 10.234$ [i.e., $\int_{13.277}^{\infty} \chi^2(y|4; 10.234)dy = 0.50$].

By comparison, when M is large, the χ^2 distribution with $4M$ d.o.f. is well approximated by a Gaussian. Let $y \equiv 2\mathcal{F}$, and let $\rho_{\rm tot}^2 \equiv \sum_{i=1}^{M} \rho_i^2$. Then

$$P(y) = \chi^2(y|4M; \rho_{\rm tot}^2) \approx (8\pi M)^{-1/2} e^{-(y-\langle y\rangle)^2/(8M)} \quad (4.3)$$

where $\langle y\rangle = 4M + \rho_{\rm tot}^2$. The threshold value $y_{\rm th}$ such that $\int_{y_{\rm th}}^{\infty} = 0.01$ is then

$$y_{\rm th} \approx 4M + 4.652\sqrt{M} \quad (\text{large } M). \quad (4.4)$$

[Note that the approximate threshold value that one obtains by inserting $M = 1$ into Eq. (4.4) is only 8.652, which is considerably less than the actual threshold $y_{\rm th} = 13.277$ for the $M = 1$ case. Clearly, this is because the χ^2 distribution with only 4 d.o.f. has a substantial tail—i.e., is more skewed to the right than the higher-M distributions.]

When will a collection be more detectable than its single brightest member? To answer this, let us order the pulsars in the sample such that

$$\rho_1 \geq \rho_2 \geq \cdots \geq \rho_M. \quad (4.5)$$

Let T_1 be the integration time necessary to detect the brightest source, and let $T_{\rm coll}$ be the integration time required to detect the M-member collection. For large M, the ratio of these 2 times is

$$T_{\rm coll}/T_1 = 0.455 M^{1/2} \frac{\rho_1^2}{\rho_{\rm tot}^2} = 0.455 M^{-1/2} \frac{\rho_1^2}{\rho_{\rm ave}^2}, \quad (4.6)$$

where $\rho_{\rm ave}^2 \equiv \rho_{\rm tot}^2/M$. As an extreme example, if $M = 25$

and all members have the same strength ($\rho_1 = \rho_2 = \cdots = \rho_{25}$), then $T_{\rm coll}/T_1 = 1/11.0$. (More realistic cases will be considered in the next two subsections.) More generally, we say that a collection is more detectable than its brightest member if $T_{\rm coll}/T_1 < 1$.

How many pulsars should one include in the sample, assuming the goal is to hasten its detection? Imagine that pulsars $1, 2, \cdots, M-1$ are included in the sample, and we want to decide whether to include pulsar M. By Eq. (4.6), the change $\Delta T_{\rm coll}$ in the time required to confidently detect the collection is

$$\frac{\Delta T_{\rm coll}}{T_{\rm coll}} = M^{-1}(0.5 - \rho_M^2/\rho_{\rm ave}^2). \qquad (4.7)$$

Thus it is advantageous to increase the sample size (because $\Delta T_{\rm coll} < 0$) iff $\rho_M^2 > 0.5\rho_{\rm ave}^2$.

Of course, *a priori* both ρ_M^2 and $\rho_{\rm ave}^2$ are unknown; nevertheless one can use both some general statistical arguments and the measured parameters of nearby millisecond pulsars to make a reasonably informed choice. We shall illustrate this in the next two subsections.

B. Distribution of ρ^2 for Galactic GW pulsars

What is the distribution function of ρ^2 for the GW pulsars in the Galactic disk, within a few kpc of us? We can get quite far in answering this question, based on quite general considerations.

Let r represent a pulsar's distance from the Earth. For simplicity, we shall consider two different spatial distributions: a uniform (i.e., homogeneous and isotropic) three-dimensional distribution and a uniform planar distribution. (These roughly represent the pulsar distributions at distances $r \lesssim 300$ pc and 300 pc $\lesssim r \lesssim 5$ kpc, respectively.) Let $\sigma(r, f, A, \alpha_i)$ represent the probability density of GW pulsars in parameter space. Here f is again the NS's gravitational-wave frequency, A represents the signal's source's intrinsic amplitude (proportional to $\sqrt{\dot{E}_{\rm GW}/f^2}$, where $\dot{E}_{\rm GW}$ is the source's GW luminosity), and the α_i are the relevant angles in the problem. For a 3D distribution there are 4 such angles: two for the NS's angular location on the sky and two for the direction of its spin. For the planar (2D) distribution there are only 3 relevant angles, since one angle suffices for the sky location. For either uniform distribution, the r dependence can clearly be factored out:

$$\sigma(r, f, A, \alpha_i) \equiv F(r)\hat{\sigma}(f, A, \alpha_i), \qquad (4.8)$$

where

$$F(r) = \begin{cases} 4\pi r^2 & \text{for 3D,} \\ 2\pi H r & \text{for 2D.} \end{cases} \qquad (4.9)$$

Here $H \approx 600$ pc is the thickness of the Galactic disk.

The source's signal-to-noise-squared, ρ^2, can clearly be written in the following form:

$$\rho^2 = A^2 r^{-2}\lambda(f, \alpha_i), \qquad (4.10)$$

where $\lambda(f, \alpha_i)$ is some function of f and the source's angular parameters. For a single detector with time-invariant noise characteristics, the f dependence can also be factored out of $\lambda(f, \alpha_i)$:

$$\lambda(f, \alpha_i) = \hat{\lambda}(\alpha_i)/S_h(f); \qquad (4.11)$$

however this factorization of the f dependence is not necessary for our argument.

For notational convenience, we again define $y \equiv \rho^2$. We now change variables: $(r, f, A, \alpha_i) \rightarrow (y, f, A, \alpha_i)$. The density function σ on the new variables is

$$\sigma(y, f, A, \alpha_i) = \sigma(r, f, A, \alpha_i)\left|\frac{\partial(r, f, A, \alpha_i)}{\partial(y, f, A, \alpha_i)}\right|, \qquad (4.12)$$

where the second term on the right-hand side of Eq. (4.12) is the Jacobian of the transformation. It is easy to check that this Jacobian factor is just

$$\left|\frac{\partial(r, f, A, \alpha_i)}{\partial(y, f, A, \alpha_i)}\right| = \frac{1}{2}y^{-3/2}A\lambda^{1/2}. \qquad (4.13)$$

Combining Eqs. (4.8), (4.9), (4.12), and (4.13), we therefore have

$$\sigma(y, f, A, \alpha_i) = \hat{\sigma}(f, A, \alpha_i) \times \begin{cases} y^{-5/2}\frac{2\pi}{A\lambda^{1/2}}, & \text{(3D)} \\ y^{-2}\pi H, & \text{(2D).} \end{cases} \qquad (4.14)$$

Integrating Eq. (4.14) over the variables (f, A, α_i), we arrive at the density function for y alone:

$$\sigma(y) = n_3 y^{-5/2} \quad \text{(3D)} \qquad (4.15)$$

$$\sigma(y) = n_2 y^{-2} \quad \text{(2D),} \qquad (4.16)$$

for some constants n_3 and n_2.

We emphasize that *no assumption about the distribution of GW pulsars in f and A went into this result.* All that was required was spatial uniformity—that the nearby pulsars are drawn from the same distribution as the more distant ones, and that the total number within some radius scales as r to some power. Indeed, the same scaling applies to any source-type having a spatially uniform distribution in Euclidean space; e.g., to the extent that one can ignore cosmological effects, the scaling in Eq. (4.15) also applies to detections of binary inspirals. [To appreciate why this is at least a bit remarkable, consider trying to estimate the density function $\sigma(y)$ for all the pulsars in some globular cluster (say, 47 Tuc). In this case, all the pulsars are effectively at the same distance, but we would need to somehow estimate the distribution of GW pulsars in f and A, and then to fold in the detector's noise curve, in order to estimate $\sigma(y)$ for that cluster.] Of course, the scaling law Eqs. (4.15) is well known in other areas of astronomy, where it is the basis for the ubiquitous $\log N$- $\log S$ test of source strength distributions.

C. Implications of $\sigma(y)$ for collection searches

We can now return to the question of when a large sample of the brightest (in GWs) pulsars, taken together, might be more detectable than the single brightest member. From Eq. (4.6), the "figure of merit" that characterizes the detectability of the collection is $M^{-1/2}\sum_{i=1}^{M} y^i$. In the next two subsections we show how this quantity varies with collection size for spherical and planar distributions, respectively.

1. Uniform 3D pulsar distribution

The spherically symmetric case is the less interesting one, from a practical standpoint, since there only a few known ms pulsars with ~300 pc of the Earth. Nevertheless we begin with this case since it is somewhat simpler and illustrates our general line of reasoning.

Imagine that we have included in our collection all the brightest (in GWs) ms pulsars, down to some lower limit y_l. Then $M \approx \int_{y_l}^{\infty} n_3 y^{-5/2}$ and

$$\sum_{i=1}^{M} y^i \approx \int_{y_l}^{\infty} n_3 y^{-3/2} dy, \tag{4.17}$$

so

$$M^{-1/2}\sum_{i=1}^{M} y^i \approx \sqrt{6n_3} y_l^{1/4}. \tag{4.18}$$

This is a strictly increasing function of y_l (albeit that it increases rather slowly). But increasing y_l means shrinking the collection. Thus for a uniform 3D distribution, it would be unlikely that a large collection of the brightest sources would be detectable before the single brightest member was detected.

2. Uniform 2D pulsar distribution

We turn now to the planar case. We begin by considering the detectability of all GW pulsars with ρ^2 in the interval $y_l < \rho^2 < y_u$, (so y_l and y_u are the lower and upper limits of the interval, respectively). Again, assuming the number of sources in the interval is large, the appropriate figure of merit, characterizing the detectability of the whole collection, is $M^{-1/2}\sum_{i=1}^{M} y^i$. The continuous version of this is clearly

$$M^{-1/2}\sum_{i=1}^{M} y^i \approx \left[\int_{y_l}^{y_u} n_2 y^{-2} dy\right]^{-1/2} \int_{y_l}^{y_u} n_2 y^{-1} dy$$

$$= n_2^{1/2} \ln(y_u/y_l)[y_l^{-1} - y_u^{-1}]^{-1/2} \tag{4.19}$$

$$= (n_2 y_u)^{1/2}\left[\frac{-\ln(x)}{(x^{-1}-1)^{1/2}}\right] \tag{4.20}$$

where in the last line we introduced the dimensionless ratio $x \equiv y_l/y_u < 1$. We gain some insight into Eq. (4.20) if we reexpress n_2 in terms of $y_{\rm max}$, which we define to be the y value of the strongest galactic source. Let $\tilde{y}_{\rm max}$ represent the median value of $y_{\rm max}$, for our distribution function Eq. (4.16). Then $\tilde{y}_{\rm max}$ is given implicitly by

$$\int_{\tilde{y}_{\rm max}}^{\infty} n_2 y^{-2} dy = 0.5, \tag{4.21}$$

(since then there is a 50% chance of finding a stronger source than $\tilde{y}_{\rm max}$), so $\tilde{y}_{\rm max} = 2n_2$. The actual value of $y_{\rm max}$ for our Galaxy is therefore $y_{\rm max} = 2\beta n_2$, where we expect β is some number of order one. The right-hand side in Eq. (4.20) can therefore be written as

$$\left(\frac{y_{\rm max} y_u}{2\beta}\right)^{1/2}\left[\frac{-\ln(x)}{(x^{-1}-1)^{1/2}}\right]. \tag{4.22}$$

Next we consider the function $f(x) \equiv -\ln x/(x^{-1}-1)^{1/2}$, which is displayed in Fig. 1.

Note that it has a maximum at $x \approx 0.203$, where $f(x) \approx 0.805$. Thus an optimized source collection has a ratio of weakest-to-brightest source (in terms of their signal-to-noise-squared) of ~1/5. However the maximum in $f(x)$ is rather broad; at a brightness ratio of 20 ($x = 0.05$), $f(x)$ has decreased only ~15%, to 0.687. Assuming the collection includes all the brightest sources down to some limiting brightest y_l, the number of sources in the collection is

$$M \approx n_2/y_l = \tilde{y}_{\rm max}/(2y_l) \tag{4.23}$$

$$= 2.5\beta^{-1}\left(\frac{y_{\rm max}/y_l}{5}\right). \tag{4.24}$$

Thus including all sources down to strength y_l while also optimizing $y_l/y_{\rm max}$ leads to a rather small value of M, which is then somewhat outside the range of validity of

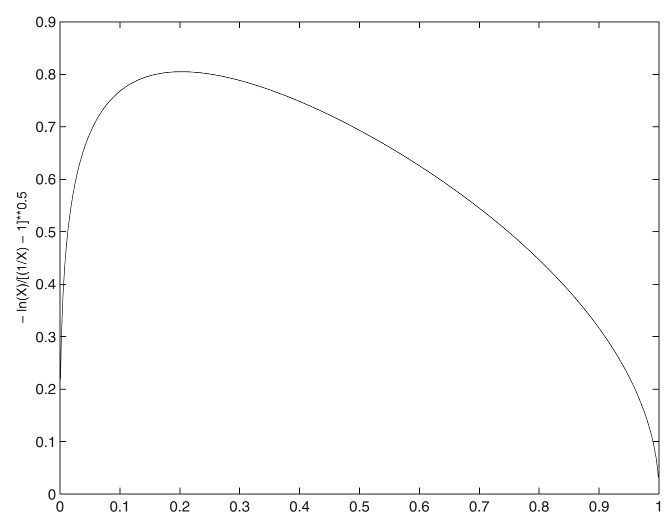

FIG. 1. Plot of function $f(x) \equiv -\ln(x)/(x^{-1}-1)^{1/2}$, which displays a maximum at $x \approx 0.203$.

the Gaussian approximation that led to our figure of merit $M^{-1/2}\sum_{i=1}^{M} y^i$; nevertheless it is clear that the "most-detectable" collection will have at most a few members.

To estimate $T_{\rm coll}/T_1$ (the time to detect this collection divided by the time to detect the single brightest source), it would seem we need to evaluate the integral $\int_{y_1}^{\infty} n_2 y^{-1}$ (i.e., the continuous version of $\sum_{i=1}^{M} y^i$), which is logarithmically divergent. Physically, though, it seems sensible to simply cut off the upper end of the integral at some $y_{\rm cut}$ of order $y_{\rm max}$. I.e., we cut off the integral at the y value of the brightest source.[3]

Thus, setting $y_{\rm u}$ equal to $y_{\rm max}$ in Eq. (4.22) and plugging the result into Eq. (4.6), we estimate

$$\frac{T_{\rm coll}}{T_1} \approx 0.455(2\beta)^{1/2}/f(0.203) \approx 0.80\beta^{1/2}. \tag{4.25}$$

Again, the use of Eq. (4.6) in deriving Eq. (4.25) is strictly valid only for large M; nevertheless the basic moral is clear: $T_{\rm coll}/T_1$ is of order unity, and whether in actual experience it is greater or less than one depends strongly on β, i.e., on whether the strongest source is stronger or weaker than one would expect, based on the source distribution function.

D. Example: The best candidates among known millisecond pulsars

We next consider a potentially realistic example: a collection drawn from the population of known millisecond pulsars. We attempt to construct the most-detectable collection from these. Of course, we do not know their actual GW strengths, so for this exercise we will estimate their strengths by assuming that they all have the same non-axisymmetry $I\epsilon \equiv I_{xx} - I_{yy}$ (where the NS is assumed to be spinning about its z axis). This nonaxisymmetry might be generated, e.g., by lateral variations in the crustal composition or strong toroidal magnetic fields in the NS interiors [12].

For each of the millisecond pulsars, we estimate ρ^2 as follows. First, we estimate h_0 at the Earth from the pulsar's measured spin and the best available estimate of its distance r [13], using

$$h_0 = 4\pi^2(G/c^4)I\epsilon f^2 r^{-1} \tag{4.26}$$

where here we will assume the GW frequency f is exactly twice the pulsar's measured spin frequency, ν. Then we estimate ρ^2 using

$$\rho^2 = 2\left(\frac{h_0^2 T_0}{S_h(f)}\right)K(\alpha_i) \tag{4.27}$$

where T_0 is some fiducial observation time and $K(\alpha_i)$ is factor that depends on the sky location and spin orientation of the source. The spin orientations of the millisecond sources are poorly constrained, so for simplicity, in our estimates, we will simply replace K by its average value (over all angles). For $S_h(f)$, we use the values for the advanced LIGO noise curve, as generated by the BENCH software package [14]. [Equation (4.27) is for a single detector; if one optimally combined the outputs of LIGO's L1, H1 and H2 interferometers, then ρ^2 should be approximately 2.25 times greater than for either L1 or H1 alone.]

Given the above inputs, we find that there are 5 ms pulsars that stand out as the best candidates for detection by the advanced LIGO interferometers. They are PSRs J0437-4715, J0030 + 0451, J2124-3358, J1744-1134, and J1024-0719. These 5 are also the closest known ms pulsars. And pulsar 1 (PSR J0437-4715), which at $d = 0.14$ kpc is the closest of all the known ms pulsars, is estimated to be the strongest GW source. Relative to pulsar 1, the GW strengths of the other four sources are given by: $(\rho_2/\rho_1)^2 = 0.38$, $(\rho_3/\rho_1)^2 = 0.32$, $(\rho_4/\rho_1)^2 = 0.17$, and $(\rho_5/\rho_1)^2 = 0.16$.

Assuming the above estimates of ρ^2 for the five best candidates were correct, what would be $T_{\rm coll}/T_1$ (the ratio of the integration times necessary to detect the 5-member collection and the brightest individual source)? For our 5-member sample, the threshold value for detection with 1% FA rate is $y_{\rm th} = 37.57$ [i.e., $\int_{37.57}^{\infty}\chi^2(y|20)dy = 0.01$]. To be detectable with FD rate $\geq 50\%$, the signal strength must be at least $\rho_{\rm tot}^2 = 18.45$ [i.e., $\int_{37.57}^{\infty}\chi^2(y|4; 18.45)dy = 0.50$]. Thus

$$\frac{T_{\rm coll}}{T_1} = (\rho_1^2/\rho_{\rm tot}^2)\left(\frac{18.45}{10.23}\right) = 0.87. \tag{4.28}$$

E.g., if it took two years to confidently detect the strongest source, the 3-member ensemble would be detectable in about 21 months. [Note that in deriving Eq. (4.28) we have used the actual χ^2 distribution with 20 d.o.f., not the Gaussian approximation to it.]

Should we add a sixth pulsar to the sample? From the analysis in the previous subsection, this would be advantageous if $\rho_6^2/(\sum_{i=1}^{5}\rho_i^2) > 0.10$. But we estimate that the sixth most-detectable pulsar is J1012 + 5307, with $(\rho_6/\rho_1)^2 = 0.07$, so we restrict the sample to the most promising five. [Indeed, if we had restricted the sample to only the most promising 3 pulsars, we would coincidentally have arrived at the same estimate for $T_{\rm coll}/T_1$. For the 3-member case, $(\rho_{\rm tot}^2/\rho_1^2) = 1.70$, while the threshold for detection with 1% FA rate is $y_{\rm th} = 26.22$, and the signal strength must be $\rho_{\rm tot}^2 \geq 15.13$ to be detectable with a FD

[3] Of course, our planar approximation breaks down at $r < H/2$, and this "switchover" from an effective 2D to a 3D distribution at short distances would obviate the need for an artificial cutoff in a more realistic treatment of this problem.

rate $\geq 50\%$. Thus we would estimate $T_{\rm coll}/T_1 = 15.13/(2.03 * 10.23) = 0.87$, the same as for the 5-member collection. However we highlighted the 5-member result since that one is clearly somewhat more robust against deviations of the actual source strengths away from our fiducial estimates.]

Clearly, since the actual orientations of the ms pulsars are unknown and the distances are known only to within a factor ~ 2, the above estimate merely gives a rough indication of the time savings that a multisource search might reasonably lead to.

We also note that if we were to estimate source strengths by assuming that all ms pulsars are spinning down primarily due to GW emission, thus using $h_0 = (5GI\dot{\nu}/2c^3\nu)^{1/2}r^{-1}$ instead of Eq. (4.26), and then repeat the above analysis from that starting point, we would find that *no* subset of the known ms pulsars is more detectable than the single brightest source, PSR J0437-4715. This just highlights the fact that the ratio $T_{\rm coll}/T_1$—and especially whether that ratio is greater or less than one—depends rather sensitively on the relative strengths of the few brightest sources, which of course we will not know in advance.

E. Search for a collection of weaker GW pulsars

The last two subsections showed that a search for a collection of the very brightest GW pulsars may offer some advantages, compared to a search for the very brightest one, but any such advantages are likely to be quite modest. We now turn to a case where the advantages of a whole-collection search are much more impressive.

Consider some time in the future, when the few brightest GW pulsars are presumed to have already been detected. These are presumably among the closest GW pulsars, while the GWs from their more distant cousins are still too weak (at the Earth) to be detected. Now once again consider collecting together all known ms pulsars in the range $y_{\rm l} < y < y_{\rm u}$. (Of course, again, one does not know precisely which these are, but whatever lessons are learned from the brightest GW pulsars, combined with the known distances and spin rates of the remaining millisecond pulsars, will probably allow one to make fairly educated guesses.) If all ms pulsars had the same intrinsic GW strength, the same frequency, and the same angular factor $K(\alpha_i)$ [from Eq. (4.27)], then clearly these pulsars would occupy a circular annulus in the disk. In fact, of course, these GW pulsars will *not* have the same intrinsic amplitude, frequency, or angular factor, but we still find it helpful, conceptually, to imagine the GW pulsars with $y_{\rm l} < y < y_{\rm u}$ as filling a roughly annular region.

For an optimally chosen annular region (one that minimizes the integration time required for positive detection), what is the optimal value of $x \equiv y_{\rm l}/y_{\rm u}$. We worked this out in Sec. IV C 2; the optimum selection has $x \sim 1/5$. How many sources are in this range? For a planar distribution, the answer is clearly

$$M \approx \int_{y_{\rm u}}^{y_{\rm u}/5} n_2 y^{-2} dy = 2\left(\frac{\tilde{y}_{\rm max}}{y_{\rm u}}\right), \tag{4.29}$$

where we have used $n_2 = \tilde{y}_{\rm max}/2$. Similarly the ratio $T_{\rm coll}/T_{\rm u}$ (where $T_{\rm u}$ is the integration time required to detect a GW pulsar whose ρ^2 equals $y_{\rm u}$) is given by

$$T_{\rm coll}/T_{\rm u} \approx 0.455 y_{\rm u}/[(n_2 y_{\rm u})^{1/2} f(x \approx 0.203)] \tag{4.30}$$

$$\approx 0.8\left(\frac{y_{\rm u}}{\tilde{y}_{\rm max}}\right)^{1/2}. \tag{4.31}$$

For example, if $y_{\rm u}/\tilde{y}_{\rm max} = 1/10$, then the optimal number of sources for the "annulus" is $M \approx 20$, and the integration time required to detect that whole collection of 20 GW pulsars is only $0.8/\sqrt{10} \approx 0.25$ as long as the time required to detect the brightest single member in that group. Again, such a detection would provide an estimate of the average ρ^2 for ms pulsars in that collection, even though none could be detected individually in GWs.

V. CONCLUSIONS

The most sensitive GW detectors (currently the LIGO L1, H1, and H2 interferometers) have very similar sensitivities, and this is likely to remain the case for some years. In such a case, one can significantly increase the effective signal-to-noise of any source by optimally combining the data streams. Here we have derived the appropriate formulas for doing so, for GW pulsar searches. For N GW detectors with the same sensitivity, the observation time $T_{\rm det}$ required to detect any particular GW pulsar scales like N^{-1}, and so combining the data streams this way is clearly a useful strategy.

However we remind the reader that our analysis of the multidetector statistics has assumed the noise is Gaussian. In the more realistic case, one would one want to veto candidate detections that had a large $\mathcal{F}$-statistic but that did not sufficiently resemble actual GW pulsar signals, e.g., because the relative sizes of the signal in the various detectors did not conform with expectations for *any* choice of parameters (λ^1, λ^2, λ^3, λ^4). In particular, we imagine that a realistic implementation would incorporate some multidetector version of the chi-square veto developed in Itoh *et al.* [15]; however we have not considered this in any detail.

We have also pointed out that one can search for collections of pulsars, and that the optimal frequentist search for such collections simply adds up the $\mathcal{F}$-statistics of the individual members. We considered two cases in detail. We first asked whether the few brightest GW pulsars might be discovered, collectively, before the very brightest one. The answer turns out to depend rather sensitively on the relative strengths of the few brightest sources, and so we can only equivocate: maybe yes, maybe no. But even if some collection turns out to be more detectable than the single brightest source, it is unlikely to ''win'' by much. However, *after* the few brightest GW pulsars have been discovered, searching for more distant pulsars by summing their $\mathcal{F}$-statistics should prove to be an effective strategy, allowing one to measure the average strength of many sources that are not individually detectable.

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[1] P. Jaranowski, A. Krolak, and B. F. Schutz, Phys. Rev. D **58**, 063001 (1998).
[2] http://www.lsc-group.phys.uwm.edu/daswg/.
[3] S. Bose, A. Pai, and S. V. Dhurandhar, Int. J. Mod. Phys. D **9**, 325 (2000).
[4] L. S. Finn, Phys. Rev. D **63**, 102001 (2001).
[5] J. Sylvestre, Phys. Rev. D **68**, 102005 (2003).
[6] A. Krolak, M. Tinto, and M. Vallisneri, Phys. Rev. D **70**, 022003 (2004).
[7] A. Rogan and S. Bose, Classical Quantum Gravity **21**, S1607 (2004).
[8] L. S. Finn, S. D. Mohanty, and J. D. Romano, Phys. Rev. D **60**, 121101 (1999).
[9] Y. Itoh, G. Mendell, and X. Siemens, LIGO Technical Report No. T050052-00-Z, 2005.
[10] L. S. Finn, Phys. Rev. D **46**, 5236 (1992).
[11] S. Bonazzola and E. Gourgoulhon, Astron. Astrophys. **312**, 675 (1996).
[12] C. Cutler and K. S. Thorne, in *Proceedings of the 16th Conference on General Relativity and Gravitation, Durban, South Africa, 2001*, edited by N. Bishop and S. D. Maharaj (World Scientific, Singapore, 2002), p. 72.
[13] http://www.atnf.csiro.au/research/pulsar/psrcat/.
[14] http://cosmos.nirvana.phys.psu.edu/lsf/Benchmarks/main.html.
[15] Y. Itoh, M. A. Papa, B. Krishnan, and X. Siemens, Classical Quantum Gravity **21**, S1667 (2004).