

Spikes in the Mixmaster regime of G_2 cosmologies

Woei Chet Lim*

*Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany. and
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA.*

Lars Andersson†

*Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany. and
Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA.*

David Garfinkle‡

Department of Physics, Oakland University, Rochester, Michigan 48309, USA.

Frans Pretorius§

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA.

(Dated: April 9, 2009)

We produce numerical evidence that spikes in the Mixmaster regime of G_2 cosmologies are transient and recurring, supporting the conjecture that the generalized Mixmaster behavior is asymptotically non-local where spikes occur.

AEI preprint number: AEI-2009-033

PACS numbers: 98.80.Jk, 04.20.-q, 04.20.Jb, 04.25.D-

I. INTRODUCTION

Belinskii, Khalatnikov and Lifshitz (BKL) [1, 2, 3] conjectured that, according to general relativity, the approach to the generic spacelike singularity is vacuum dominated (assuming $p < \rho$), local, and oscillatory (labeled ‘Mixmaster’). Here local means that the contribution of terms in the evolution equations with spatial derivatives becomes negligible. The Mixmaster dynamics consists of Kasner epochs bridged by transitions represented by the vacuum Bianchi type II solution. Numerical studies of the asymptotics of the Gowdy models, which represent the simplest inhomogeneous vacuum spacetimes, reveal that on approach to the singularity, spiky structures form [4, 5, 6]. These spikes become ever narrower as the singularity is approached. At first, the presence of such spikes might seem inconsistent with the ‘local’ part of the BKL conjecture since the spatial derivative of such a spike grows without bound as the singularity is approached. Remarkably, such spiky behavior in Gowdy spacetimes actually *is* consistent with BKL locality. The reason for this is that in the evolution equations for Gowdy spacetimes, the spatial derivatives are multiplied by a quantity that goes to zero even faster than the spatial derivatives go to infinity. This has been verified in detailed numerical simulations, as well as by the discovery of closed form Gowdy solutions with the spike

property [7, 8].

Nonetheless, the Gowdy models are a very special class of spacetimes, so it remains to be seen whether this property of BKL locality and persistent spikes holds in more general spacetimes. To that end, we will examine the properties of spikes in G_2 models which are a slightly more general class that includes the Gowdy spacetimes.

Studies of G_2 and more general models have produced numerical evidence that the BKL conjecture generally holds except possibly at isolated points where spiky structures form [9, 10, 11, 12]. Here, BKL locality is violated due to large spatial gradients. However, the ability to draw conclusions about spikes from such simulations is severely limited due to the enormous numerical resources needed to resolve the narrowing spikes. In this paper, we will use a different numerical method that does have adequate resolution to provide reliable conclusions about spike behavior in G_2 spacetimes. We present numerical evidence in support of the following conjectures: that recurring “spike transitions” are a general type of oscillation as the singularity is approached, and that higher-order spike transitions split into separate first-order spike transitions. Section 2 presents the equations for the evolution of G_2 spacetimes. Our numerical method is presented in section 3, results in section 4, and conclusions in section 5. Appendix A presents the procedure to match a numerical solution with an explicit spike solution. Appendix B gives the formula for the BKL parameter u in term of the parameter w . Appendix C gives the formulae for the Weyl scalar invariants.

*Electronic address: wclim@aei.mpg.de

†Electronic address: lars.andersson@aei.mpg.de

‡Electronic address: garfinkl@oakland.edu

§Electronic address: fpretori@princeton.edu

II. G_2 SPACETIMES

The metric of the general G_2 class takes the form [10, eq (7)]

$$\begin{aligned} ds^2 = & -e^{(\lambda-3\tau)/2}d\tau^2 + e^{(\lambda+\mu+\tau)/2}dx^2 \\ & + e^{P-\tau}[dy + Qdz + (G_1 + QG_2)dx \\ & \quad + (M_1 + QM_2)(-e^{-\tau}d\tau)]^2 \\ & + e^{-P-\tau}[dz + G_2dx + M_2(-e^{-\tau}d\tau)]^2. \end{aligned} \quad (1)$$

Here all metric quantities depend only on the time coordinate τ and spatial coordinate x , thus there is symmetry in two spatial directions. The singularity is approached as $\tau \rightarrow \infty$. The choice of gauge used here is the same as in [10, 12].

Our choice of variables are the β -normalized variables [13, 14] in the orthonormal frame formalism [15], related to the metric components as follows:

$$\beta = 2e^{\frac{\lambda-3\tau}{4}}, \quad \mathcal{N} = -\frac{1}{2}, \quad (2)$$

$$E_1^1 = 2e^{-\frac{\mu}{4}-\tau}, \quad \Sigma_2 = \frac{K}{\sqrt{3}}e^{\frac{\lambda+2P+3\tau}{4}}, \quad (3)$$

$$\Sigma_- = -\frac{P_\tau}{\sqrt{3}}, \quad N_\times = -\frac{e^{-\frac{\mu}{4}-\tau}P_x}{\sqrt{3}}, \quad (4)$$

$$\Sigma_\times = -\frac{e^P Q_\tau}{\sqrt{3}}, \quad N_- = \frac{e^{P-\frac{\mu}{4}-\tau}Q_x}{\sqrt{3}}, \quad (5)$$

where K is a constant, and the τ and x subscripts denote partial differentiation.

The evolution equations for the β -normalized variables are:

$$\partial_\tau E_1^1 = -\frac{1}{2}(2 - 3\Sigma_2^2)E_1^1 \quad (6)$$

$$\begin{aligned} \partial_\tau \Sigma_- = & -\frac{1}{2}[-3\Sigma_2^2 \Sigma_- + 2\sqrt{3}(\Sigma_\times^2 - N_-^2) - \sqrt{3}\Sigma_2^2 \\ & - E_1^1 \partial_x N_\times] \end{aligned} \quad (7)$$

$$\partial_\tau N_\times = -\frac{1}{2}[(2 - 3\Sigma_2^2)N_\times - E_1^1 \partial_x \Sigma_-] \quad (8)$$

$$\begin{aligned} \partial_\tau \Sigma_\times = & -\frac{1}{2}[(-3\Sigma_2^2 - 2\sqrt{3}\Sigma_-)\Sigma_\times - 2\sqrt{3}N_\times N_- \\ & + E_1^1 \partial_x N_-] \end{aligned} \quad (9)$$

$$\begin{aligned} \partial_\tau N_- = & -\frac{1}{2}[(2 - 3\Sigma_2^2 + 2\sqrt{3}\Sigma_-)N_- + 2\sqrt{3}\Sigma_\times N_\times \\ & + E_1^1 \partial_x \Sigma_\times] \end{aligned} \quad (10)$$

$$\partial_\tau \Sigma_2 = -\frac{1}{2}[-3\Sigma_2^2 - 3\Sigma_+ + \sqrt{3}\Sigma_-]\Sigma_2, \quad (11)$$

where

$$\Sigma_+ = \frac{1}{2}(1 - \Sigma_-^2 - \Sigma_\times^2 - \Sigma_2^2 - N_-^2 - N_\times^2). \quad (12)$$

There is one constraint equation:

$$E_1^1 \partial_x \Sigma_2 = (3N_- \Sigma_\times - 3N_\times \Sigma_- - \sqrt{3}N_\times)\Sigma_2. \quad (13)$$

For state-space presentations, we will use the Hubble-normalized variables [8]:

$$\begin{aligned} & (\Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_-, N_\times)^H \\ & = \frac{1}{1 - \Sigma_+}(\Sigma_+, \Sigma_-, \Sigma_\times, \Sigma_2, N_-, N_\times). \end{aligned} \quad (14)$$

See [12] for the evolution equations for Hubble-normalized variables, and the derivation of the evolution equations.

The Gowdy spacetimes are that class of G_2 spacetimes for which $\Sigma_2 = 0$. Note that it then follows from equation (6) that $E_1^1 = \exp(-\tau)$. An interesting class of solutions of the Gowdy equations are the exact spike solutions of [8]

$$\Sigma_- = \frac{-1}{\sqrt{3}}\left(1 + \frac{f^2 - 1}{f^2 + 1}[w \tanh(w\tau) - 1]\right) \quad (15)$$

$$N_\times = \frac{2f}{f^2 + 1} \frac{w}{\sqrt{3}} \operatorname{sech}(w\tau) \quad (16)$$

$$\Sigma_\times = \frac{f^2 - 1}{f^2 + 1} \frac{w}{\sqrt{3}} \operatorname{sech}(w\tau) \quad (17)$$

$$N_- = \frac{2f}{f^2 + 1} \frac{1}{\sqrt{3}}(1 - w \tanh(w\tau)) \quad (18)$$

where w is a constant and the quantity f is given by

$$f = we^\tau \operatorname{sech}(w\tau)x \quad (19)$$

For $|w| < 1$ this solution describes a spike because $f = 0$ at $x = 0$ but f becomes large as $\tau \rightarrow \infty$ for all $x \neq 0$. Nonetheless, BKL locality is preserved because in the equations of motion all spatial derivatives are multiplied by E_1^1 and $E_1^1 \partial_x f = w \operatorname{sech}(w\tau)$ which goes to zero as $\tau \rightarrow \infty$.

Note however that this conclusion depends on the fact that $E_1^1 = \exp(-\tau)$ which in turn depends on the fact that in equation (6) we could set Σ_2 to zero, something that we can only do in Gowdy spacetimes, not the more general G_2 spacetimes. The dynamics in a G_2 spacetime consists of eras where Σ_2 is very small (and which can thus be well described by the dynamics of Gowdy spacetimes) punctuated by short "frame bounces" where Σ_2 rapidly grows and then rapidly shrinks to become again negligible. During a frame bounce E_1^1 shrinks more slowly than $\exp(-\tau)$ and thus it is not clear whether spatial derivatives continue to remain negligible. To resolve this issue, we will need to perform numerical simulations of the dynamics of G_2 spacetimes. Furthermore, those simulations will need to have enough resolution to accurately model the rapidly shrinking spikes.

III. NUMERICAL METHODS

One numerical method for resolving small scale structure is adaptive mesh refinement (AMR). However, if one knows beforehand the location of the structure, one need not use AMR and can instead use a coordinate system adapted to the structure that one wants to study. In particular, here we are studying spikes that shrink exponentially with time, so we choose a coordinate system that does the same.

We introduce new coordinates (T, X) to zoom in on the worldline $x = x_{\text{zoom}}$.

$$T = \tau, \quad X = e^{A\tau}(x - x_{\text{zoom}}), \quad (20)$$

$\partial_X F$ on grid point i by

$$\frac{4}{3} \frac{F_{i+1} - F_{i-1}}{2\Delta X} - \frac{1}{3} \frac{F_{i+2} - F_{i-2}}{4\Delta X} \quad (34)$$

On the last gridpoint, the excision boundary, we evaluate the spatial derivative using one sided differences. That is we approximate $\partial_X F$ at the final gridpoint N by

$$\frac{25F_N - 48F_{N-1} + 36F_{N-2} - 16F_{N-3} + 3F_{N-4}}{12\Delta X} \quad (35)$$

and at the second last grid point $N - 1$ by

$$\frac{3F_N + 10F_{N-1} - 18F_{N-2} + 6F_{N-3} - F_{N-4}}{12\Delta X} \quad (36)$$

For spikes, non-symmetric data would be problematic for implementing the local perspective as the spike worldline is not stationary in this case. Therefore we shall choose symmetric initial data (around $X = 0$) and simulate only $X \in [0, X_0]$, with enforcement of the symmetry at the left boundary $X = 0$. For comparison, we also simulate along non-spike worldlines, in which case the data are not symmetric and the left boundary at $-X_0$ is an excision boundary.

We choose the first gridpoint to be either an excision boundary at $X = -X_0$ or a point of symmetry at $X = 0$. If it is an excision boundary, then $\partial_X F$ is approximated by the one sided differences

$$\frac{-F_1 + 48F_2 - 36F_3 + 16F_4 - 3F_5}{12\Delta X} \quad (37)$$

at the first grid point, and

$$\frac{-3F_1 - 10F_2 + 18F_3 - 6F_4 + F_5}{12\Delta X} \quad (38)$$

at the second. However, if first grid point is a point of symmetry then we choose all quantities to be either even or odd there. For even functions, $\partial_X F = 0$ at the first grid point, and

$$\frac{4}{3} \frac{F_3 - F_1}{2\Delta X} - \frac{1}{3} \frac{F_4 - F_2}{4\Delta X} \quad (39)$$

and the second, while for odd functions we approximate $\partial_X F$ by

$$\frac{8F_2 - F_3}{6\Delta X} \quad (40)$$

at the first grid point, and

$$\frac{4}{3} \frac{F_3 - F_1}{2\Delta X} - \frac{1}{3} \frac{F_4 + F_2}{4\Delta X} \quad (41)$$

at the second.

The standard double precision real variables (with 16 digits of significance) are normally used in the numerical code. When necessary, quad precision real variables (with 32 digits of significance) are used to lower the numerical roundoff errors by 10^{16} folds, thereby preventing

it from prematurely swamping small values. The variables N_- , Σ_\times and N_\times take small values during Kasner epochs, and can be swamped by the roundoff error in the spatial derivative term of another variable with a larger value. Usually this happens to N_\times first, when the term $\frac{1}{2}e^{AT}E_1^{-1}\partial_X\Sigma_-$ in equation (24) becomes 10^{16} times smaller (if double precision is used) than the value of Σ_- . The usage of quad precision real variables increases the runtime by 4 to 8 folds.

No numerical dissipation is used, as it is unnecessary.

To verify that numerical solutions converge with fourth order accuracy, we compare the constraint (27) in numerical runs with different resolutions (different number of grid points). We observe that doubling the resolution reduces the constraint by a factor of 16 when adequate numerical resolution is used. We also compare the numerical solutions with a matching exact spike solution. The procedure for matching is described in Appendix A. The formula for the BKL parameter u for the Kasner epochs between transitions are given in Appendix B. The Weyl scalar invariants are used to measure the difference between numerical and exact solutions. Appendix C gives their formulae.

In this paper, we shall focus on obtaining numerically accurate results, which require much higher numerical resolution than qualitative results do. This requirement also places severe limit on how far into the asymptotic regime one can simulate, because the numerical error must not be larger than the distance from the solution to the nearest Kasner point in the state space, and this in turn require high numerical resolution. When a numerical simulation takes up to months to run in order to meet the accuracy, it becomes impractical. Despite this difficulty, we want to provide more than just qualitative results, because numerically accurate results can provide evidence supporting convergence to the exact spike solution, while qualitative results cannot. In presentation, we shall round the numbers to 4 decimals, even though the accuracy is higher.

Qualitative numerical results are still valuable in providing evidence supporting the general behavior of the solution. Compared with other aspects of the solution, the timing of a transition is most sensitive to numerical inaccuracy. At lower resolutions, the timing of a transition differs greatly while other aspects of the solution remain robust.

IV. RESULTS

We clarify a few terms we use below. A (true) spike point is where a (true) spike can occur (the spike may be active or smoothed out). In our variables, a spike point is where

$$N_- = 0. \quad (42)$$

We shall hold the spike point fixed (at $x = 0$), so that we can easily locate it and zoom in on it. To do so, we

require N_- and N_\times to be odd functions around the spike point, and Σ_- , Σ_\times and Σ_2 to be even functions.

A false spike point is where $\Sigma_\times = 0$. To hold it fixed, we require Σ_\times and N_\times to be odd functions around the false spike point, and Σ_- , Σ_2 and N_- to be even functions.

We will present three sets of numerical results. The first set chooses a perturbed spike solution as the initial condition, and shows two recurrences of the spike solution, within the same Gowdy era. The purpose is to show that spike recurs within the same Gowdy era. The second set chooses a generic initial condition, and shows two occurrences of the spike solution, one in each Gowdy era. The purpose is to show that spike recurs over different Gowdy eras. The third set consists of two simulations, with a perturbed second and third order spike solution as the initial condition, respectively. The purpose is to show that second and third order spikes break up into separate first order spikes. All three sets demonstrate the attractor nature of the first order spike.

The reader will notice that different numerical resolutions are used for different sets. The length of simulation also differs. Both numerical resolutions and length are not arbitrarily chosen, but are dictated by the cost of computation to maintain accuracy. For example, in the first set, we do not try to extend the simulation to show the third spike recurrence over different Gowdy eras, as it would be too costly.

A. Perturbed spike

The format of the initial data is a perturbed spike solution at $\tau = 0$:

$$E_1^1 = 2, \quad \Sigma_2 = 10^{-5}, \quad (43)$$

$$\Sigma_- = \frac{(wx)^2 - 1}{(wx)^2 + 1} \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \epsilon, \quad N_\times = \frac{2wx}{(wx)^2 + 1} \frac{w}{\sqrt{3}}, \quad (44)$$

$$\Sigma_\times = \frac{(wx)^2 - 1}{(wx)^2 + 1} \frac{w}{\sqrt{3}}, \quad N_- = \frac{2wx}{(wx)^2 + 1} \frac{1}{\sqrt{3}}, \quad (45)$$

where $x = X + x_{\text{zoom}}$, with $\epsilon = 0.001$, $w = 9.5$, $A = 1$, $x_{\text{zoom}} = 0$. Here, small perturbations are applied to the variables Σ_2 and Σ_- . Compare with the exact spike solution at $\tau = 0$ [8, eq (36)]. The value $w = 9.5$ allows for two spike recurrences at roughly $w = 5.5$ and $w = 1.5$ before the next Gowdy era.

Because the data chosen are symmetric about $X = 0$, only $X \geq 0$ needs to be simulated. A resolution of 100001 grid points on the X -interval $[0, 2]$ is sufficient for convergence during the time interval $T \in [0, 16]$. Double precision is used. Beyond $T = 16$ a higher numerical precision is needed to maintain accuracy. To compare the orbit along a different worldline, we also use the same initial data but with $x_{\text{zoom}} = 1$ and 200001 grid points on the X -interval $[-2, 2]$.

The simulation shows a perturbed spike solution recurs twice over the same Gowdy era. For each of the two recurrences, the numerical solution is matched with an exact spike solution. The difference between the numerical and exact spike solution is computed in the four Weyl scalar invariants, and is observed to be smaller in the second recurrence than in the first (see Figures 2 and 3). This suggests that the closer to the singularity, the closer the numerical solution gets to an exact spike solution. This supports the conjecture that the exact spike solutions are attractors.

Figure 4 shows the orbit along the spike point $x = 0$ (from the first simulation) and the worldline $x = 1$ (from the second simulation) projected onto the (Σ_+^H, Σ_-^H) plane in the state space of Hubble-normalized variables. It shows the orbits follow the expected paths as predicted from the spike solution and the Σ_\times and N_- transition sets (see Figures 5 and 6 in [8]). This subsection is similar to the work in [16], which was done in the Gowdy class ($\Sigma_2 = 0$), in the sense that the simulations here focus on what happens within one Gowdy era. The approximate values for the w parameter in [8] and corresponding u parameter when near a Kasner point for the $x = 0$ orbit in Figure 4 are given below (rounded to 4 decimal points). Linking the Kasner epochs are alternating frame and spike transitions.

$$w \approx -7.8330 \xrightarrow{\text{frame}} 7.5401 \xrightarrow{\text{spike}} -3.7592 \xrightarrow{\text{frame}} 3.8264 \xrightarrow{\text{spike}} 0.1674 \quad (46)$$

$$u \approx 3.4165 \xrightarrow{u} 3.2700 \xrightarrow{u-2} 1.3796 \xrightarrow{u} 1.4132 \xrightarrow{\frac{1}{u-1}-1} 1.4021 \quad (47)$$

Note that Kasner epochs linked by frame transitions are not distinct physically. One also observes that the numbers above do not follow the maps very closely, suggesting that the solution is not yet very close to the generalized Mixmaster attractor. The difference is due to perturbation present in the initial data. Over time, the difference gradually decreases. Also note that the map $u \rightarrow u - 2$ has an adjustment algorithm when the new value is less than 1, namely

$$u \rightarrow \begin{cases} u - 2 & \text{if } u \geq 3 \\ \frac{1}{u-2} & \text{if } 2 < u \leq 3 \\ \frac{1}{\frac{1}{u-1}-1} & \text{if } \frac{3}{2} \leq u < 2 \\ \frac{1}{u-1} - 1 & \text{if } 1 < u \leq \frac{3}{2}. \end{cases} \quad (48)$$

B. Generic initial condition

Having seen that perturbed spike initial data lead to recurring spikes within the same Gowdy era, we now go further and ask whether a generic initial data also lead to recurring spikes, and whether the recurrence continues in

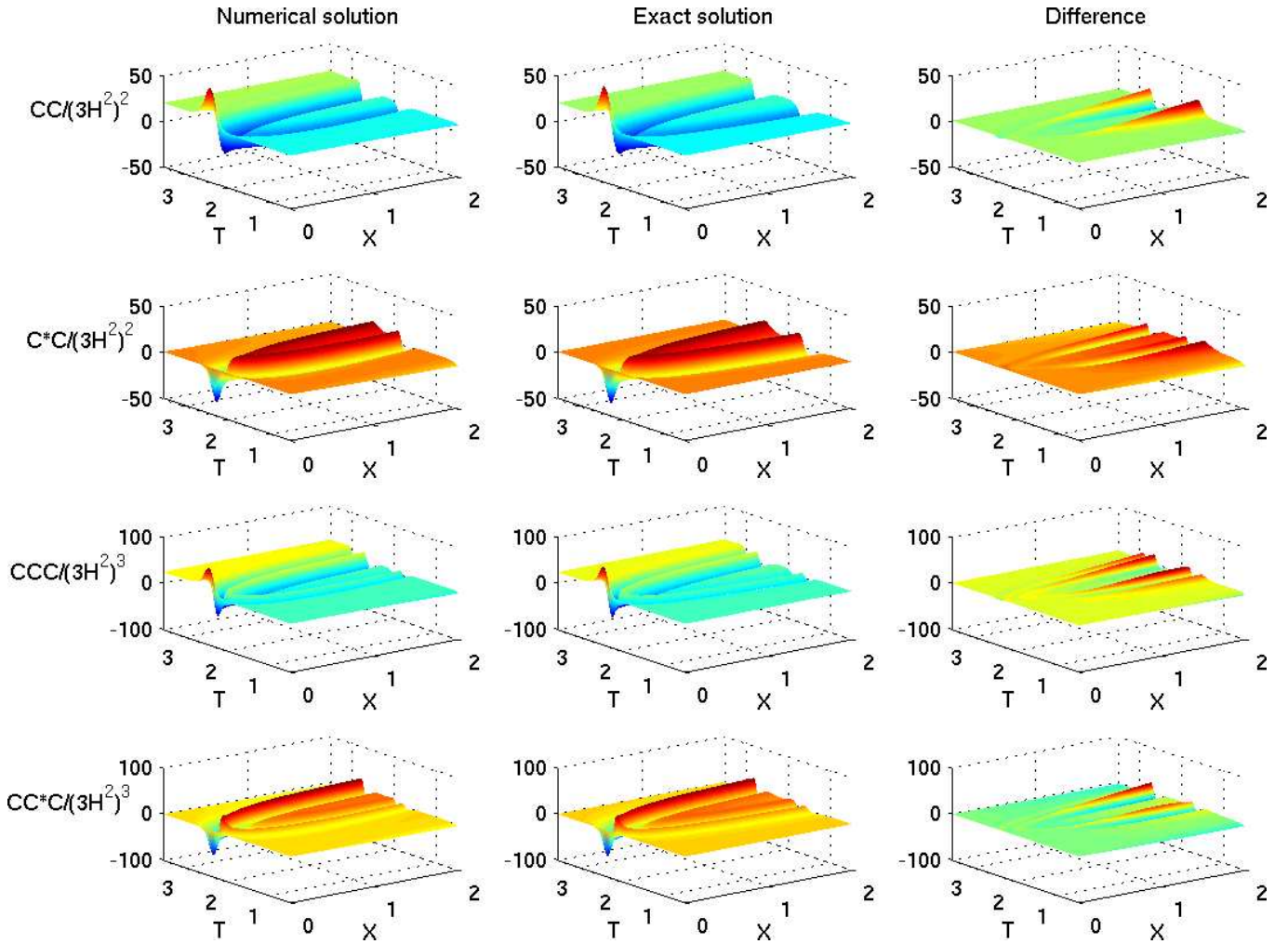


FIG. 2: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact spike solution (with $w = 5.8644$, $\tau_0 = 2.2205$), and their difference, during the first spike recurrence over the time interval $[0.5, 3.5]$.

the next Gowdy era. To answer these questions, we shall start with a generic initial data, and evolve the solution through to the next Gowdy era.

We give an example of a generic initial condition for a true spike below.

$$E_1^1 = 2, \quad \Sigma_2 = (\Sigma_2)_0 e^{\int_0^x C_T/E_1^1 dx}, \quad (49)$$

$$\Sigma_- = a_1 + a_2 x^2, \quad N_x = a_5 x, \quad (50)$$

$$\Sigma_x = a_3 + a_4 x^2, \quad N_- = a_6 x, \quad (51)$$

with

$$\begin{aligned} \int_0^x C_T/E_1^1 dx &= \frac{1}{4}(-3a_1 a_5 - \sqrt{3}a_5 + 3a_3 a_6)x^2 \\ &+ \frac{1}{8}(-3a_2 a_5 + 3a_4 a_6)x^4, \end{aligned} \quad (52)$$

where $x = X + x_{\text{zoom}}$. For example, we choose

$$a_1 = 3.25, \quad a_2 = 0.002, \quad a_3 = 0.3, \quad (53)$$

$$a_4 = -0.001, \quad a_5 = 0.04, \quad a_6 = -0.05, \quad (54)$$

$$(\Sigma_2)_0 = 0.2, \quad (55)$$

with $A = 1$, $x_{\text{zoom}} = 0$, 6401 grid points over the X -interval $[0, 10]$, and time interval $[0, 40]$. Quadruple precision is used. For comparison, another simulation with $x_{\text{zoom}} = 1$, 12801 grid points over the X -interval $[-10, 10]$ is used. Beyond $T = 40$, the solution gets too close to a Kasner point, and a higher numerical resolution is needed to maintain accuracy.

Two recurrences of spike are observed, one in the same Gowdy era, and the other in the next (after a Σ_2 transition). Figure 5 shows the orbits along $x = 0, 1$ passing close to various identical points during two Gowdy eras. A difference in position of the final points is observed, and is attributed to the lag between the two worldlines that becomes more pronounced over time. The approximate values for the w and u parameters when near a Kasner point (except the initial point) for the $x = 0$ orbit in Figure 5 are given below (rounded to 4 decimal

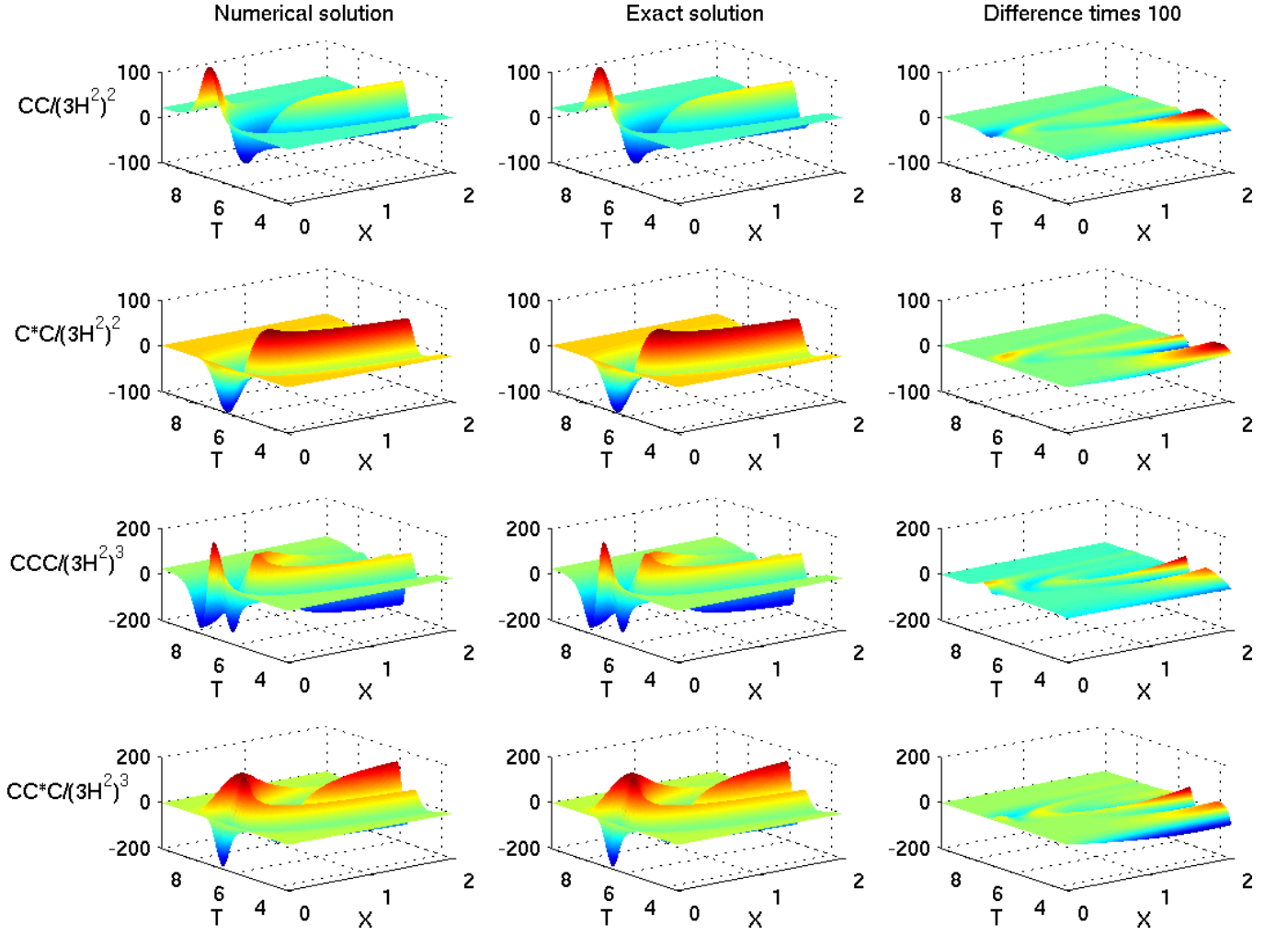


FIG. 3: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact spike solution (with $w = 1.8329$, $\tau_0 = 6.6029$), and their difference, during the second spike recurrence over the time interval $[3.5, 9.5]$.

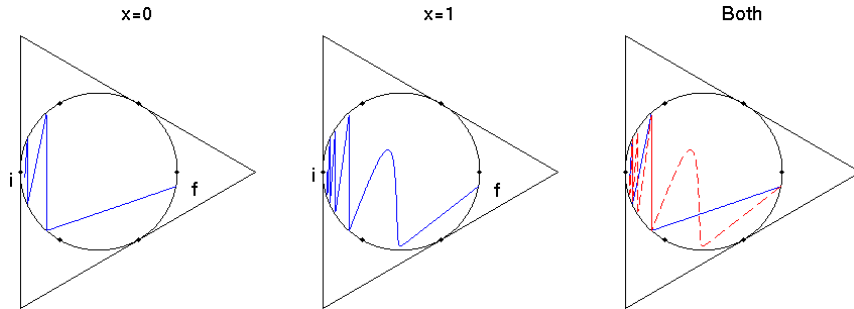


FIG. 4: The orbit along the spike point $x = 0$ and a nearby non-spike point $x = 1$ projected onto the (Σ_+^H, Σ_-^H) plane for the perturbed spike simulation. The initial (at $T = 0$) and final (at $T = 16$) points are marked by the letters i and f respectively.

points).

$$w \approx 5.7070 \xrightarrow{\text{spike}} -1.7114 \xrightarrow{\text{frame}} 1.7114 \xrightarrow{\Sigma_2 \text{ frame}} 6.6228$$

$$\xrightarrow{\text{spike}} -2.6228 \quad (56)$$

$$u \approx 2.3535 \xrightarrow{u-2} 2.8114 \xrightarrow{u} 2.8114 \xrightarrow{u} 2.8114$$

$$\xrightarrow{\frac{1}{u-2}} 1.2324 \quad (57)$$

Observe that the numbers here follow the map much more closely in the later stage.

The numerical solution is matched with an exact spike solution and the Weyl scalars are plotted in Figures 6 and 7. As in the previous subsection, matching improves with time (towards the singularity). The remarkable im-

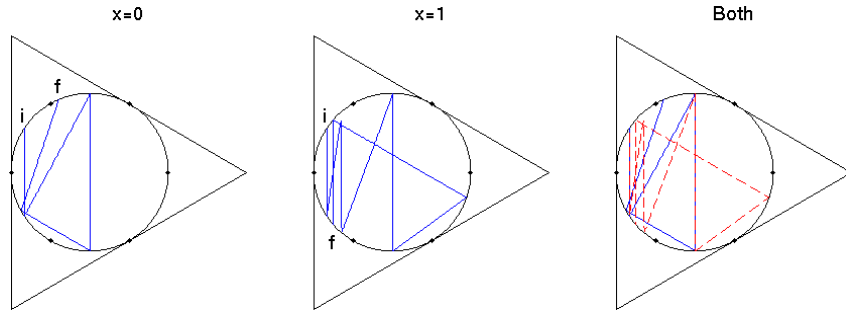


FIG. 5: The orbit along the spike point $x = 0$ and a nearby non-spike point $x = 1$ projected onto the (Σ^H_+, Σ^H_-) plane for the generic initial data simulation. The initial (at $T = 0$) and final (at $T = 40$) points are marked by the letters i and f respectively.

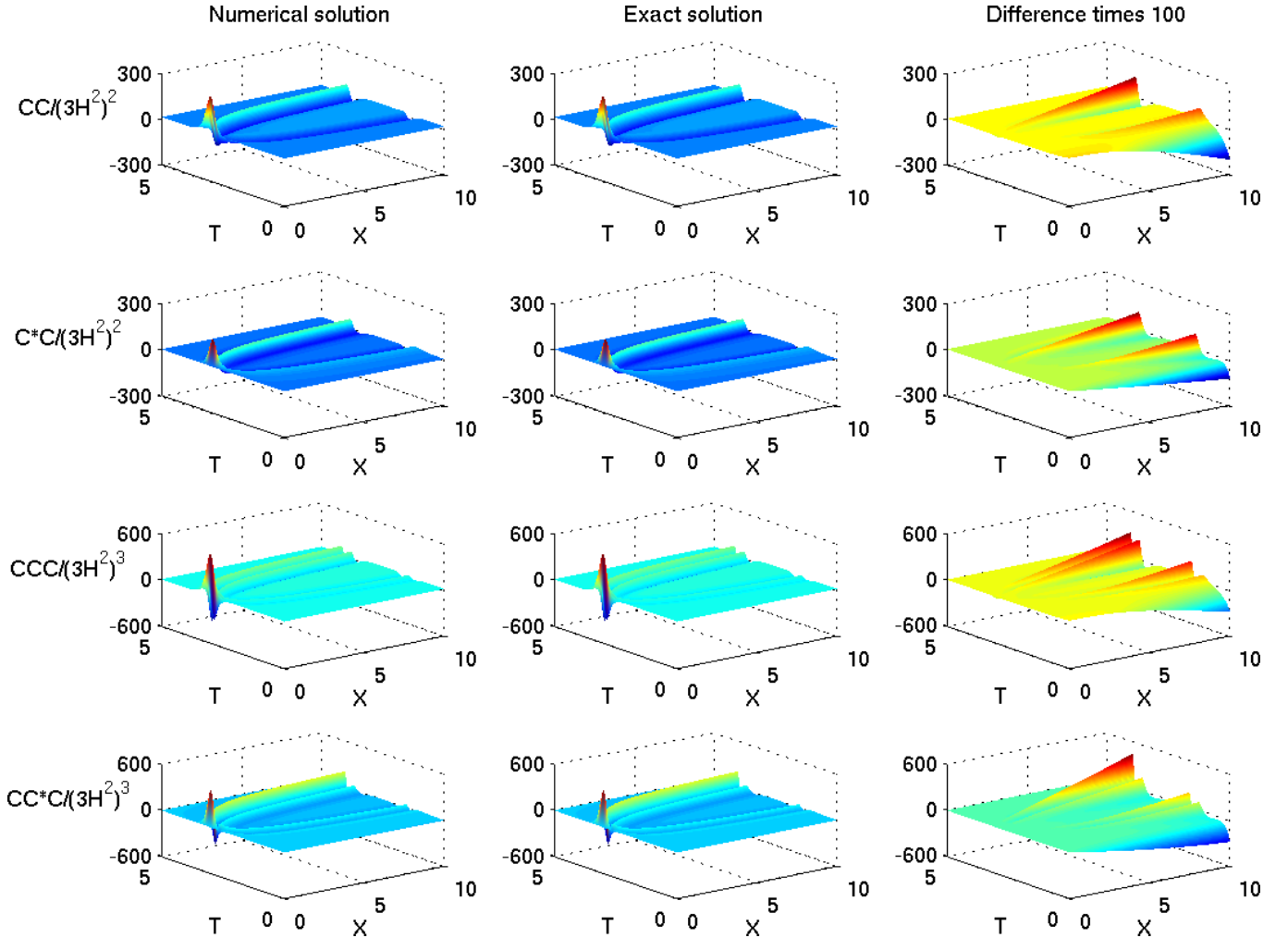


FIG. 6: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact spike solution (with $w = -3.7114$, $\tau_0 = 2.6461$), and their difference, during the first spike recurrence over the time interval $[0, 5]$.

provement from the first to the second recurrence also suggest an exponential rate of convergence to the exact spike solution. This provides a very strong evidence that the spike solution is an attractor not only for perturbed spike initial data, but also for generic ones. The exponential rate of convergence is also a curse for accurate

numerical simulations, as the need for numerical resolution also increases exponentially with time.

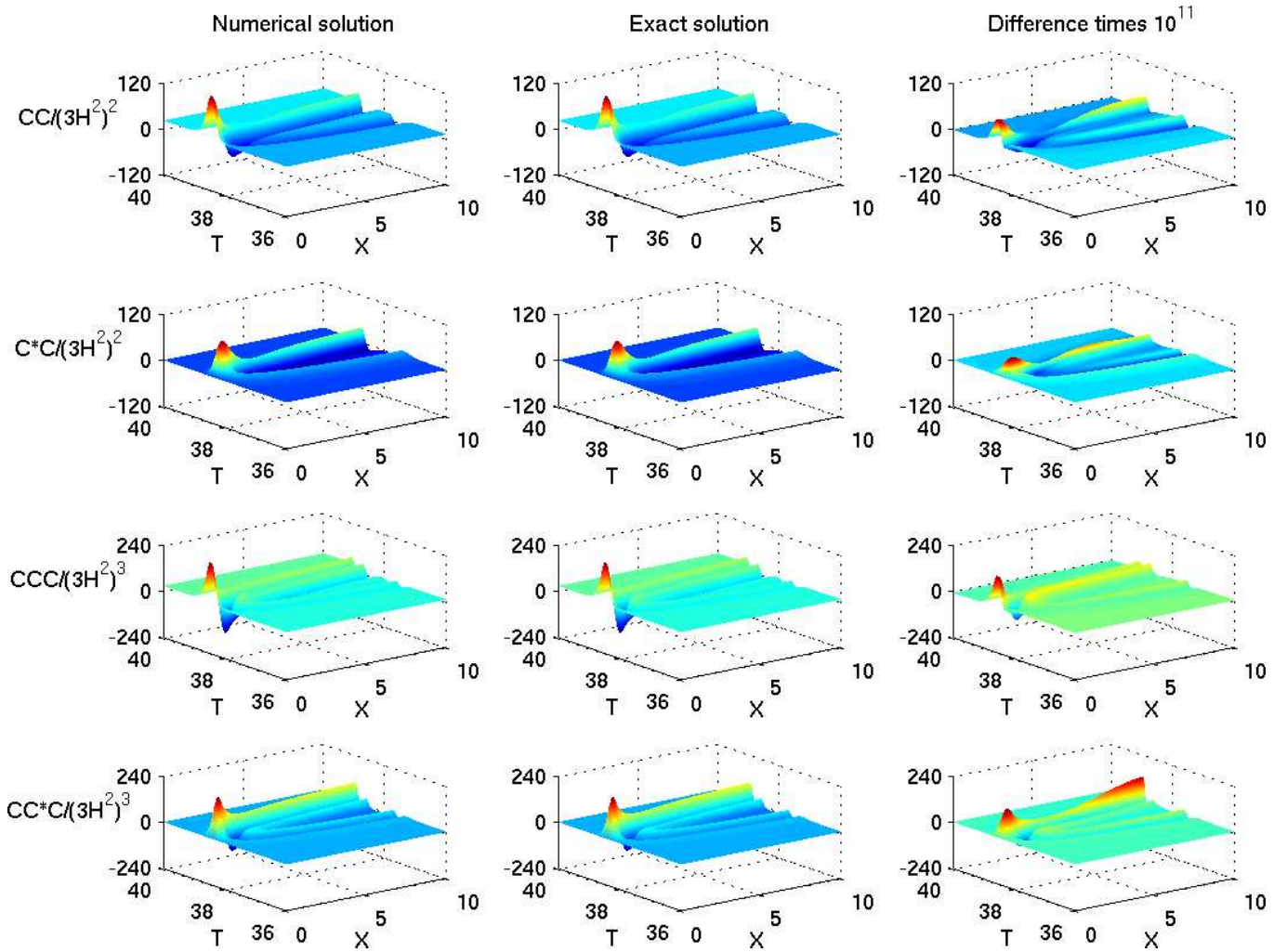


FIG. 7: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact spike solution (with $w = -4.6228$, $\tau_0 = 38.1923$), and their difference, during the second spike recurrence over the time interval $[35, 40]$.

C. Perturbed higher order spikes

Having seen that the spike solution is an attractor, we now investigate whether higher order spikes [8, Section 5.5] are also attractors. In this subsection we shall use perturbed second and third order spike solutions as initial data, and see whether they recur as first, second or third order spikes.

The initial data for a perturbed second order spike solution at $\tau = 0$ is given recursively in terms of the first order spike solution:

$$E_1^1 = 2, \quad (58)$$

$$\Sigma_- = -(c\Sigma_{-1} + s\Sigma_{\times 1}) - \frac{1}{\sqrt{3}}, \quad N_{\times} = sN_{-1} - cN_{\times 1}, \quad (59)$$

$$\Sigma_{\times} = cN_{-1} + sN_{\times 1}, \quad N_- = -s\Sigma_{-1} + c\Sigma_{\times 1}, \quad (60)$$

where $(\Sigma_-, N_{\times}, \Sigma_{\times}, N_-)_1$ are the perturbed first order

spike solution in (44)–(45). c and s are given by

$$c = \frac{f_1^2 - 1}{f_1^2 + 1}, \quad s = \frac{2f_1}{f_1^2 + 1}, \quad (61)$$

$$f_1 = \frac{1}{(wx)^2 + 1} \left[-2w(1 - 2x^2) + \frac{Q_2}{2Q_0} \right], \quad (62)$$

where $x = X + x_{\text{zoom}}$. Σ_2 is specified by numerically evaluating the constraint

$$\Sigma_2 = (\Sigma_2)_0 e^{\int_0^x C_T/E_1^1 dx}. \quad (63)$$

We perform a simulation with the parameters

$$\epsilon = 10^{-3}, \quad w = 9.5, \quad A = 1, \quad (64)$$

$$(\Sigma_2)_0 = 10^{-3}, \quad Q_2 = 0, \quad (65)$$

and $x_{\text{zoom}} = 0$, $X \in [0, 2]$ with 10001 grid points, and $T \in [0, 12]$.

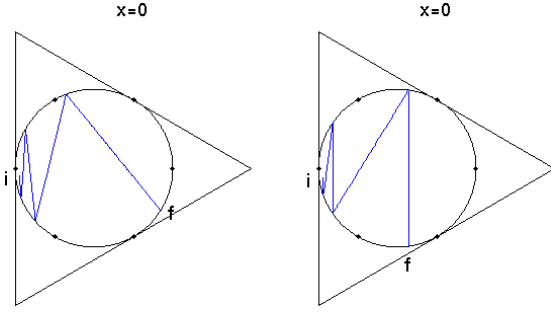


FIG. 8: The orbit along the spike point $x = 0$ projected onto the (Σ_+, Σ_-) plane for the perturbed second (left) and third (right) order spike simulation. The orbits follow the false and the true spike orbits respectively. The initial (at $T = 0$) and final (at $T = 12$) points are marked by the letters i and f respectively.

The initial data for a perturbed third order spike solution at $\tau = 0$ is also given recursively:

$$E_1^1 = 2, \quad (66)$$

$$\Sigma_- = -(c\Sigma_{-2} + s\Sigma_{\times 2}) - \frac{1}{\sqrt{3}}, \quad N_{\times} = sN_{-2} - cN_{\times 2}, \quad (67)$$

$$\Sigma_{\times} = cN_{-2} + sN_{\times 2}, \quad N_- = -s\Sigma_{-2} + c\Sigma_{\times 2}, \quad (68)$$

where $(\Sigma_-, N_{\times}, \Sigma_{\times}, N_-)_2$ are the perturbed second order spike solution in (59)–(60) above. c and s are given by

$$c = \frac{f_2^2 - 1}{f_2^2 + 1}, \quad s = \frac{2f_2}{f_2^2 + 1}, \quad (69)$$

$$f_2 = \frac{1}{[(wx)^2 + 1][f_1^2 + 1]} \times \left[-\frac{1}{3}x \left(3[(wx)^2 + 1][(f_1^2 - 1)w - 4f_1] + 4w(w^2 + 2)x^2 \right) + \frac{Q_3}{2Q_0} \right], \quad (70)$$

where f_1 is given in (62), and $x = X + x_{\text{zoom}}$. Σ_2 is specified by numerically evaluating the constraint

$$\Sigma_2 = (\Sigma_2)_0 e^{\int_0^x C_T / E_1^1 dx}. \quad (71)$$

We perform a simulation with the parameters

$$\epsilon = 10^{-3}, \quad w = 9.5, \quad A = 1, \quad (72)$$

$$(\Sigma_2)_0 = 10^{-3}, \quad Q_2 = 0, \quad Q_3 = 0, \quad (73)$$

and $x_{\text{zoom}} = 0$, $X \in [0, 2]$ with 10001 grid points, and $T \in [0, 12]$.

The center or inner part of a perturbed second order spike evolves into a false first order spike, as suggested by Figures 9 and 10. False spikes are merely a spiky representation of the vacuum Bianchi type II solution. The center or inner part of a perturbed third order spike

evolves into a true first order spike, as suggested by Figure 11. The outer parts of the perturbed spikes beyond the domain of simulation and are suspected to evolve into first order spikes, with a moving spike point. A global numerical scheme might be needed to follow their evolution, but one with enough numerical resolution would take months to run, which is impractical. Figure 8 shows that the orbit for a perturbed second (third) order spike later follows the predicted orbit for the false (true) first order spike. This suggests that higher order spikes break into separate first order spikes, and therefore are not attractor themselves.

The approximate values for the w and u parameters when near a Kasner point (except the first point) for the $x = 0$ orbits in Figure 8 are given below (rounded to 4 decimal points). For the left figure, two false spike transitions and (Σ_{\times}) curvature transition link the Kasner epochs.

$$w \approx -6.5005 \xrightarrow{\text{false spike}} 4.4161 \xrightarrow{\text{curvature}} 2.4980 \xrightarrow{\text{false spike}} 0.5003 \quad (74)$$

$$u \approx 2.7502 \xrightarrow{u-1} 1.7081 \xrightarrow{\frac{1}{u-1}} 1.3351 \xrightarrow{\frac{1}{u-1}} 3.0024 \quad (75)$$

Recall that false spike transitions and curvature transition are physically the same.

For the right figure, two frame transitions and a spike transition link the Kasner epochs.

$$w \approx -5.5004 \xrightarrow{\text{frame}} 5.4999 \xrightarrow{\text{spike}} -1.4979 \xrightarrow{\text{frame}} 1.5002 \quad (76)$$

$$u \approx 2.2502 \xrightarrow{u} 2.2500 \xrightarrow{\frac{1}{u-2}} 4.0169 \xrightarrow{u} 3.9984 \quad (77)$$

V. CONCLUSION

We have found numerical evidence (from both perturbed solutions and generic initial data) that the spike solution is part of the generalized Mixmaster attractor. We have found that the second and third order spikes are not part of the attractor, and conjecture that all higher order spikes are not part of the attractor.

We summarize the above conjectures as follows:

1. Spike transitions are a new type of oscillation on approach to the singularity, with each transition approximated by a spike solution. A spike transition has a map of $u \rightarrow u - 2$ and is different from the previously known Mixmaster oscillation, which has a map of $u \rightarrow u - 1$. It occurs in a causal neighborhood of special 2D surfaces of worldlines in generic spacetimes.
2. Higher order spike transitions (with maps $u \rightarrow u - 3$, etc) split into first order spike transitions and so are not general. i.e. the generic behavior towards singularity is either $u \rightarrow u - 1$ or $u \rightarrow u - 2$.

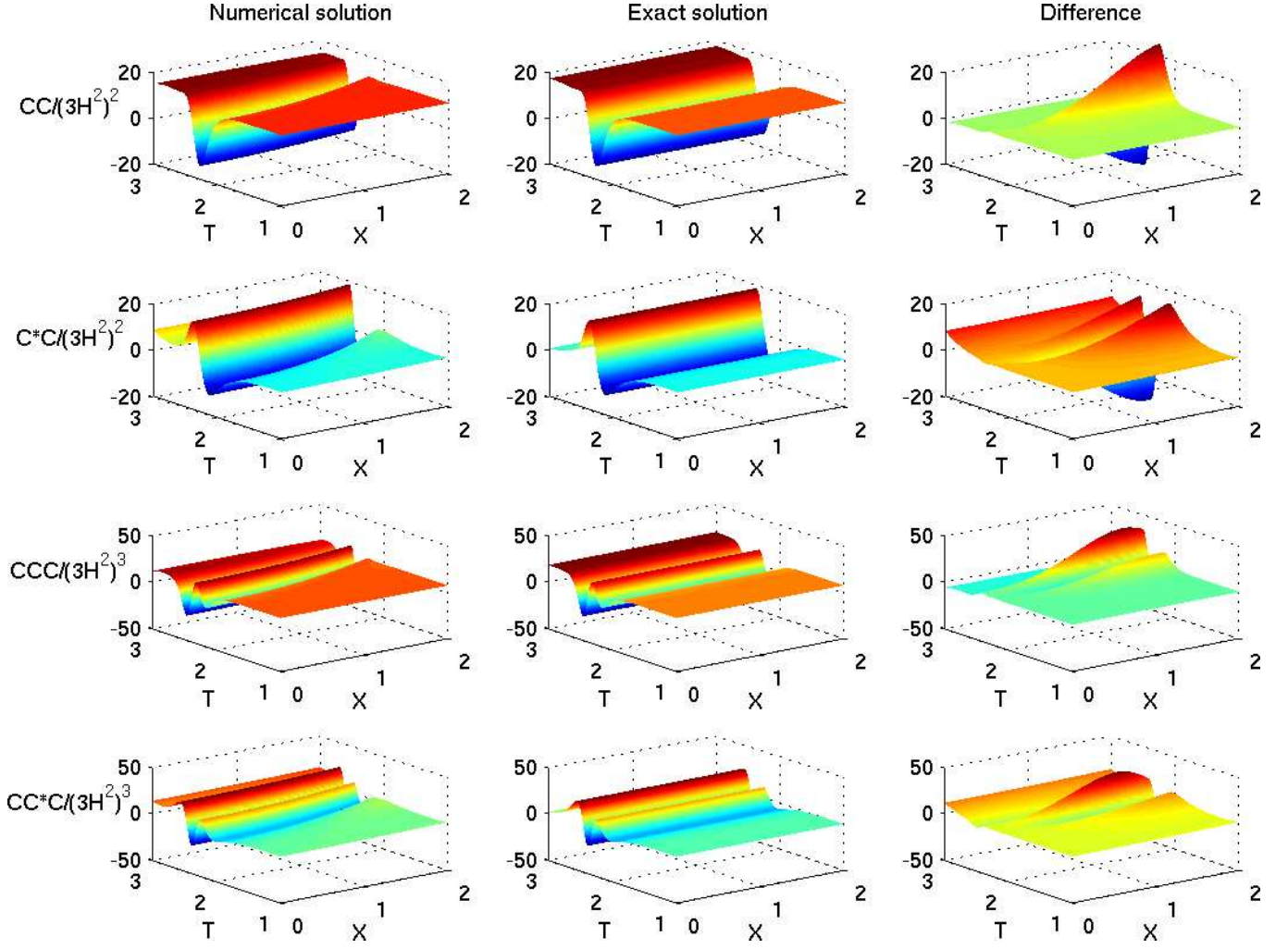


FIG. 9: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact false spike solution (with $w = 5.5005$, $\tau_0 = 2.3329$), and their difference, during the first false spike recurrence over the time interval $[1, 3]$.

We have used symmetric data in order to hold the spike point fixed, so that we can zoom in on it. We believe that for non-symmetric data, in which the spike point can move (by a little when the spike is active, and sometimes by a lot when the spike is smoothed out), the above conclusion should also hold. This remains to be confirmed numerically. At present we do not know how to zoom in on a moving spike point.

What remains unanswered is the following. Because we simulate only the neighborhood of a spike, we do not know what happens outside this domain. We also do not know what happens to new spike points that are created and move out of the domain, how they interact with other spike points or false spike points. Existing numerical simulations from the global view suffers from expensive resources needed to resolve spikes, which severely limit the length of simulation. We envision a new way to simulate spikes, by combining the zoom-in view with the global view. The biggest benefit of such a combination is much longer simulations. The zoom-in view can also provide

boundary conditions, so that the assumption of spatial periodicity can be dropped. Implementing the combination will be challenging.

APPENDIX A: MATCHING WITH EXPLICIT SOLUTIONS

For the purpose of matching the numerical solutions with explicit solutions, we will need the explicit spike solutions with generic time and space constants. To restore these constant, perform the transformation

$$\tau \rightarrow \tau - \tau_0, \quad x \rightarrow \frac{2}{(E_1^1)_0}(x - x_0). \quad (\text{A1})$$

The expression of the metric, the governing equations, and the solutions will change accordingly. In particular, E_1^1 is now given by

$$E_1^1 = (E_1^1)_0 e^{\tau - \tau_0}. \quad (\text{A2})$$

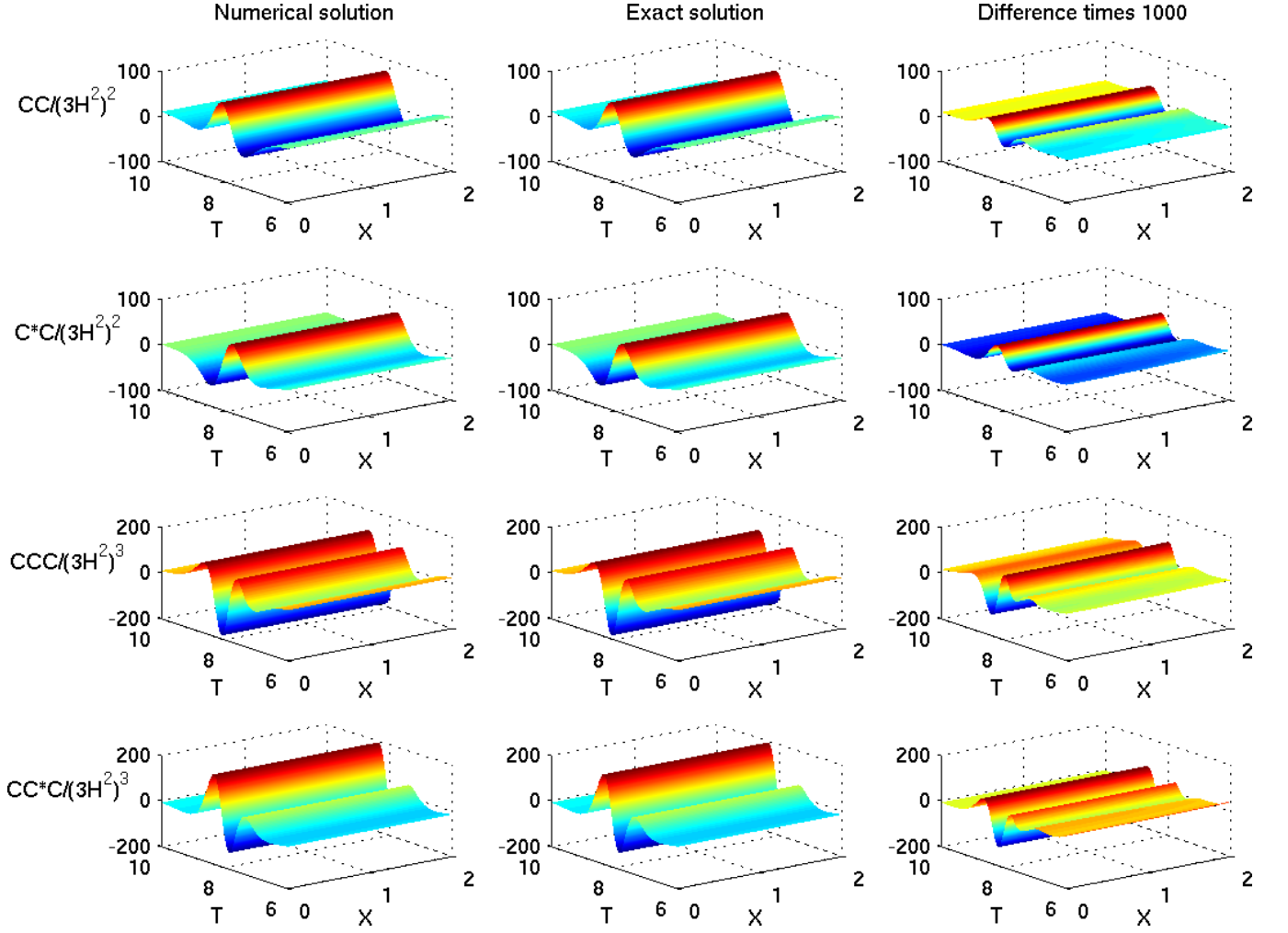


FIG. 10: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact false spike solution (with $w = 1.5006$, $\tau_0 = 7.6904$), and their difference, during the second false spike recurrence over the time interval $[6, 10]$.

The spike solution is now given by

$$P = 2(\tau - \tau_0) + \ln[\operatorname{sech}(w(\tau - \tau_0))] - \ln[f^2 + 1] - \ln(2Q_0) \quad (\text{A3})$$

$$Q = -Q_0 w [2(w \tanh(w(\tau - \tau_0)) - 1) \left(\frac{2}{(E_1^1)_0} (x - x_0)\right)^2 + e^{-2(\tau - \tau_0)}] + Q_2 \quad (\text{A4})$$

$$\lambda = -4 \ln[\operatorname{sech}(w(\tau - \tau_0))] + 2 \ln[f^2 + 1] - (w^2 + 4)(\tau - \tau_0) + \lambda_2, \quad (\text{A5})$$

where

$$f = w e^{\tau - \tau_0} \operatorname{sech}(w(\tau - \tau_0)) \frac{2}{(E_1^1)_0} (x - x_0) \quad (\text{A6})$$

is the the factor Qe^P for the vacuum Bianchi type II solution. Correspondingly, the β -normalized variables be-

come

$$\Sigma_- = \frac{-1}{\sqrt{3}} \left(1 + \frac{f^2 - 1}{f^2 + 1} [w \tanh(w(\tau - \tau_0)) - 1]\right) \quad (\text{A7})$$

$$N_\times = \frac{2f}{f^2 + 1} \frac{w}{\sqrt{3}} \operatorname{sech}(w(\tau - \tau_0)) \quad (\text{A8})$$

$$\Sigma_\times = \frac{f^2 - 1}{f^2 + 1} \frac{w}{\sqrt{3}} \operatorname{sech}(w(\tau - \tau_0)) \quad (\text{A9})$$

$$N_- = \frac{2f}{f^2 + 1} \frac{1}{\sqrt{3}} [1 - w \tanh(w(\tau - \tau_0))]. \quad (\text{A10})$$

The other solutions are similarly restored.

We take this opportunity to correct errors in [8]: the third minus sign in Equation (28) should be a plus sign, and the factor 4 in Equation (34) should not be there.

In order to match with an explicit spike solution, we will need to guess the value of the parameter w . This can be done in two ways. The first way is to choose a predetermined value, the second is to obtain a guess

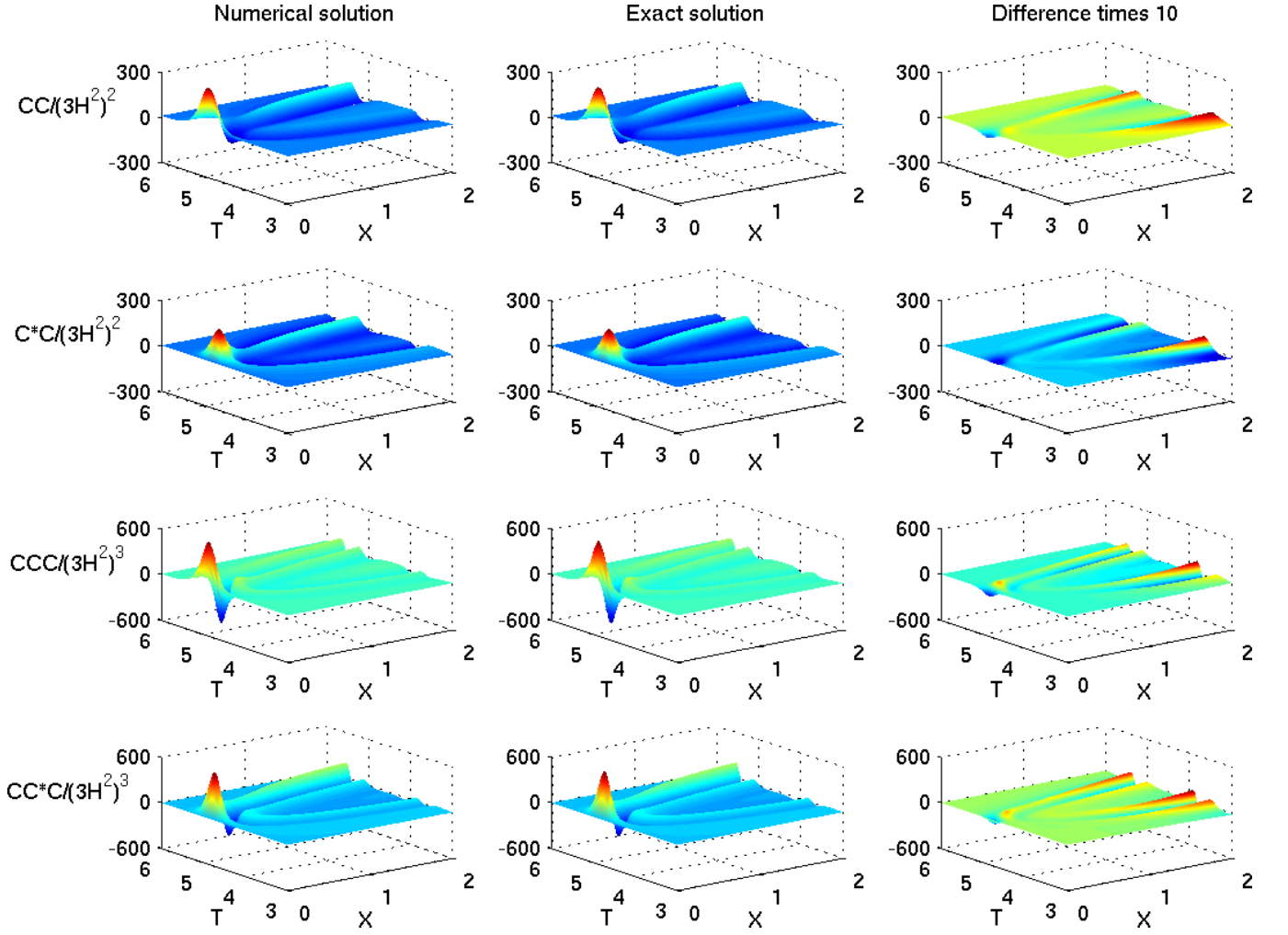


FIG. 11: The Weyl scalar invariants (normalized by Hubble scalar) for the numerical solution, the matching exact spike solution (with $w = -3.5002$, $\tau_0 = 4.4985$), and their difference, during the first spike recurrence over the time interval $[3, 6]$.

from the numerical solution. To do so we compute the expression

$$\operatorname{arcsinh}\left(\frac{\Sigma_- + \frac{2}{\sqrt{3}}}{\Sigma_\times}\right), \quad (\text{A11})$$

along $X = 0$, which equals

$$w(\tau - \tau_0) \quad (\text{A12})$$

for the spike solution. The value τ_0 is then obtained through interpolation.

We then compute the expression

$$\left(\Sigma_- + \frac{2}{\sqrt{3}}\right)^2 + \Sigma_\times^2 \quad (\text{A13})$$

along $X = 0$. For the spike solution this expression equals $w^2/3$. In practise the numerical solution will give an close-to-constant time function of w , from which we choose one value. For example we can take the maximum value of this time function.

APPENDIX B: OBTAINING THE BKL PARAMETER u FOR THE KASNER EPOCHS

Matching the Kasner epochs with Kasner solutions is straightforward. Recall from equation (18) of [8] that for a Kasner solution

$$\Sigma_- = -\frac{w}{\sqrt{3}}, \quad (\text{B1})$$

where w is a constant. One then obtains the local maximum and minimum values for Σ_- along a worldline and convert them to w . Then one computes the BKL parameter u from w using the following formula.

$$u = \begin{cases} \frac{|w|-1}{2} & |w| \geq 3, \\ \frac{|w|-1}{|w|} & 1 < |w| \leq 3, \\ \frac{1+|w|}{1-|w|} & 0 \leq w < 1. \end{cases} \quad (\text{B2})$$

APPENDIX C: THE WEYL SCALAR INVARIANTS

The orthonormal frame components C_{abcd} of the Weyl tensor can be conveniently expressed in terms of the electric and magnetic components $E_{\alpha\beta}$ and $H_{\alpha\beta}$ [15]:

$$C_{\alpha 0\beta 0} = E_{\alpha\beta}, \quad C_{\alpha\beta\gamma\delta} = -\epsilon^\mu{}_\alpha \epsilon^\nu{}_\delta E_{\mu\nu}, \quad (\text{C1})$$

$$C_{\alpha\beta\gamma 0} = \epsilon^\mu{}_{\alpha\beta} H_{\gamma\mu}, \quad (\text{C2})$$

which are then normalized by $3\beta^2$:

$$\mathcal{E}_{\alpha\beta} = \frac{1}{3\beta^2} E_{\alpha\beta}, \quad \mathcal{H}_{\alpha\beta} = \frac{1}{3\beta^2} H_{\alpha\beta}, \quad (\text{C3})$$

and further decomposed as follows:

$$\mathcal{E}_{\alpha\beta} = \begin{pmatrix} -2\mathcal{E}_+ & \sqrt{3}\mathcal{E}_3 & \sqrt{3}\mathcal{E}_2 \\ \sqrt{3}\mathcal{E}_3 & \mathcal{E}_+ + \sqrt{3}\mathcal{E}_- & \sqrt{3}\mathcal{E}_\times \\ \sqrt{3}\mathcal{E}_2 & \sqrt{3}\mathcal{E}_\times & \mathcal{E}_+ - \sqrt{3}\mathcal{E}_- \end{pmatrix} \quad (\text{C4})$$

and similarly for $\mathcal{H}_{\alpha\beta}$. The components are given by

$$\mathcal{E}_+ = \frac{1}{3}\Sigma_+ - \frac{1}{3}(\Sigma_-^2 + \Sigma_\times^2) + \frac{2}{3}(N_-^2 + N_\times^2) + \frac{1}{6}\Sigma_2^2 \quad (\text{C5})$$

$$\mathcal{E}_- = \frac{1}{3}(1 - 3\Sigma_+)\Sigma_- + \frac{2}{3}N_+N_- + \frac{1}{3}(e^{AT}E_1^1\partial_X - r)N_\times + \frac{1}{2\sqrt{3}}\Sigma_2^2 \quad (\text{C6})$$

$$\mathcal{E}_\times = \frac{1}{3}(1 - 3\Sigma_+)\Sigma_\times + \frac{2}{3}N_+N_\times - \frac{1}{3}(e^{AT}E_1^1\partial_X - r)N_- \quad (\text{C7})$$

$$\mathcal{E}_3 = -\frac{1}{\sqrt{3}}\Sigma_\times\Sigma_2 \quad (\text{C8})$$

$$\mathcal{E}_2 = \frac{1}{3}(1 + \sqrt{3})\Sigma_-\Sigma_2 \quad (\text{C9})$$

$$\mathcal{H}_+ = -N_-\Sigma_- - N_\times\Sigma_- \quad (\text{C10})$$

$$\mathcal{H}_- = -\Sigma_+N_- - \frac{2}{3}N_+\Sigma_- - \frac{1}{3}(e^{AT}E_1^1\partial_X - r)\Sigma_\times \quad (\text{C11})$$

$$\mathcal{H}_\times = -\Sigma_+N_\times - \frac{2}{3}N_+\Sigma_\times + \frac{1}{3}(e^{AT}E_1^1\partial_X - r)\Sigma_- \quad (\text{C12})$$

$$\mathcal{H}_3 = -\frac{1}{\sqrt{3}}N_\times\Sigma_2 \quad (\text{C13})$$

$$\mathcal{H}_2 = \frac{1}{\sqrt{3}}N_-\Sigma_2 \quad (\text{C14})$$

where $N_+ = \sqrt{3}N_-$, $r = -3(N_\times\Sigma_- - N_-\Sigma_\times)$. The four Weyl scalar invariants are computed as follows:

$$C_{abcd}C^{abcd} = 8(E_{\alpha\beta}E^{\alpha\beta} - H_{\alpha\beta}H^{\alpha\beta}) \quad (\text{C15})$$

$$C_{abcd}{}^*C^{abcd} = 16E_{\alpha\beta}H^{\alpha\beta} \quad (\text{C16})$$

$$C_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab} = -16(E_\alpha{}^\beta E_\beta{}^\gamma E_\gamma{}^\alpha - 3E_\alpha{}^\beta H_\beta{}^\gamma H_\gamma{}^\alpha) \quad (\text{C17})$$

$$C_{ab}{}^{cd}C_{cd}{}^{ef}{}^*C_{ef}{}^{ab} = 16(H_\alpha{}^\beta H_\beta{}^\gamma H_\gamma{}^\alpha - 3E_\alpha{}^\beta E_\beta{}^\gamma H_\gamma{}^\alpha), \quad (\text{C18})$$

where ${}^*C_{abcd} = \frac{1}{2}\eta_{ab}{}^{ef}C_{efcd}$, and η^{abcd} is the totally antisymmetric permutation tensor, with $\eta^{0123} = 1$.

The drawback of plotting the Weyl scalars for spikes is that the blow-up of the Weyl scalars towards the singularity makes the spiky structures invisible. For example, in Figure 8 of [8], level curves have to be plotted to make the structure visible. In this paper we plot Hubble-normalized Weyl scalars so that spiky structures are clearly visible. The Weyl scalars are normalized as follows:

$$\frac{C_{abcd}C^{abcd}}{(3H^2)^2}, \quad \frac{C_{abcd}{}^*C^{abcd}}{(3H^2)^2}, \quad (\text{C19})$$

$$\frac{C_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab}}{(3H^2)^3}, \quad \frac{C_{ab}{}^{cd}C_{cd}{}^{ef}{}^*C_{ef}{}^{ab}}{(3H^2)^3}. \quad (\text{C20})$$

ACKNOWLEDGMENTS

LA is partially supported by the NSF, grant no. DMS 0707306. DG is partially supported by the NSF, grant no. PHY 0456655. FP is partially supported by the NSF, grant no. PHY 0745779, and the Alfred P. Sloan Foundation. WCL and LA thank the Mittag-Leffler Institute for hospitality during part of the work on this paper.

[1] E. M. Lifshitz and I. M. Khalatnikov, *Adv. Phys.* **12**, 185 (1963).
 [2] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Adv. Phys.* **19**, 525 (1970).
 [3] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, *Adv. Phys.* **31**, 639 (1982).
 [4] B. K. Berger and V. Moncrief, *Phys. Rev. D* **48**, 4676 (1993).
 [5] B. K. Berger and D. Garfinkle, *Phys. Rev. D* **57**, 4767 (1998).
 [6] S. D. Hern and J. M. Stewart, *Class. Quantum Grav.* **15**, 1581 (1998).
 [7] A. Rendall and M. Weaver, *Class. Quantum Grav.* **18**,

2959 (2001).
 [8] W. C. Lim, *Class. Quantum Grav.* **25**, 045014 (2008).
 [9] S. D. Hern, Ph.D. thesis, University of Cambridge (1999), gr-qc/0004036.
 [10] B. K. Berger, J. Isenberg, and M. Weaver, *Phys. Rev. D* **64**, 084006 (2001).
 [11] D. Garfinkle, *Phys. Rev. Lett.* **93**, 161101 (2004).
 [12] L. Andersson, H. van Elst, W. C. Lim, and C. Uggla, *Phys. Rev. Lett.* **94**, 051101 (2005).
 [13] H. van Elst, C. Uggla, and J. Wainwright, *Class. Quantum Grav.* **19**, 51 (2002).
 [14] W. C. Lim, Ph.D. thesis, University of Waterloo (2004), gr-qc/0410126.

- [15] H. van Elst and C. Uggla, *Class. Quantum Grav.* **14**, 2673 (1997). (2003).
- [16] D. Garfinkle and M. Weaver, *Phys. Rev. D* **67**, 124009