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On tachyon condensation and open-closed duality in the $c = 1$ string theory

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Abstract: We present an exact representation for decaying ZZ-branes within the dual matrix model formulation of $c = 1$ string theory. It is shown how to trade the insertion of decaying ZZ-branes for a shift of the closed string background. Our formalism allows us to demonstrate that the conjectured world-sheet mechanism behind the open-closed dualities (summing over disc insertions) is realized here in a clear way. On the way we need to clarify certain infrared issues — insertion of ZZ-branes creates solitonic superselection sectors.

Keywords: Gauge-gravity correspondence, Tachyon Condensation, D-branes.

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Contents

1. Introduction

The study of noncritical string theories has recently seen a renaissance, initiated by the appearance of [1 – 3]. One of the reasons for the renewed interest in these string theoretical toy models is the observation that the dualities between noncritical string theories and matrix models can be seen as examples for open-closed dualities in string theory. In particular it was proposed in [1, 2] that the free fermionic field theory conjectured to be dual to the $c = 1$ string theory is nothing but the open string theory on a "gas" of unstable D0 branes. This was recently further substantiated in [4]. Another example in which the mechanism of open-closed dualities is exhibited in a particularly explicit way is the duality between the Kontsevich matrix model and pure topological gravity [5].

Another source of recent interest in these toy models was the realization that non-perturbatively stable definitions of the relevant theories exist [4, 7], after all [8]. These developments open the possibility to study certain time-dependent phenomena such as D-brane decay in an exactly soluble framework. One may therefore hope to improve our understanding of certain foundational aspects of string theory for which time-dependent phenomena represent a challenge. One may in particular hope to learn how to describe the final state of a decaying D-brane, and to what extend one may describe the process with the help of the usual perturbative approach to string theory.

Our initial aim was therefore to use the duality between noncritical string theories and matrix models in order to find an exact description for the decay of unstable D0-branes in the two-dimensional string theory. It then becomes possible to learn about scope and limitations of the perturbative approach to the same problem.

A particularly interesting feature of the exact description for the D-brane decay in two-dimensional string theory that we are about to present is the fact that it exhibits an example of open-closed duality in a very explicit way: Insertion of decaying D0-branes can be traded for a shift of the closed string background. It is furthermore possible to show that this shift of the background is perturbatively generated by summing over the disc insertions which represent the emission of closed strings from the decaying brane. This fits well into the picture proposed in [1, 2]: If the $c = 1$ background is generated by the insertion of a gas unstable D0 branes, it should be possible to trade addition of further probe D-branes for a shift of the closed string background.

On the way we will need to clarify certain infrared issues. It will be shown that the insertion of D-branes creates "solitonic" superselection sectors. D-branes are solitons after all. Excitations in these sectors can not be represented by normalizable vectors in the sector which describes pure closed string excitations. Nevertheless there is a clear sense in which these sectors are equivalent to the sector with zero D-branes: These sectors can not be distinguished by measuring any local observable like the expectation values of the tachyon field. They are distinguished by the values of *global* observables, though. This limits the extend to which a narrow-minded version of open-closed duality is true: The trade of D-branes for a shift of the closed string background is not perfect, it works to the extend to which we may regard the different sectors as physically equivalent. For some questions it may nevertheless be important to keep in mind that the insertion of D0-branes does not generate a *normalizable* deformation of the background, similar to the phenomenon emphasized in [9].

Previous work on similar questions is contained in [10, 12, 11]. The present paper will describe a new approach to this problem which allows us to go somewhat further and to clarify a number of aspects which have not yet been discussed in the literature. In order to simplify the presentation we have chosen to focus of the case of the bosonic $c = 1$ string theory. However, a good part of our formalism carries over with only small changes to the case of type 0B $\hat{c} = 1$ string theory.

2. The $c = 1$ string as free fermionic field theory

We are going to revisit some aspects of the conjectured duality between the $c = 1$ string theory and free fermionic field theory, see [13] for a review. One of our main aims is to introduce a formalism which will be particularly well-suited for our later discussions of D-branes within the free fermionic field theory. This will also allow us to present a simplified representation for the S-matrix of bosonic excitations [14, 15] within the free fermionic field theory.

The presentation will be brief, the necessary technical details are contained in the appendix A.

2.1 $c = 1$ string

The $c = 1$ string theory is a two-dimensional string background with coordinates $(X_0, \phi) \in \mathbb{R}^2$, where X_0 represents time. This background is characterized by the following expectation values for the target-space metric $G_{\mu\nu}$, the dilaton Φ and the tachyon field T:

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad \Phi = \phi, \quad T = \mu e^{2\phi}\,. \tag{2.1}$$

The worldsheet-description of this theory is characterized by the world-sheet action

$$S = \int d^2x \left(-\frac{1}{4\pi}\partial_+ X_0 \partial_- X_0 + \frac{1}{\pi}\partial_+\phi\partial_-\phi - \mu(2\phi + \ln \pi\mu)e^{2\phi} \right) + (\text{ghosts})\,. \tag{2.2}$$

The string theory has one propagating space-time field, the tachyon T . The vertex operators which create the modes of this field with definite space-time energy ω will be denoted as $T_\iota(\omega)$, where $\iota = -$ creates the asymptotic in-states, whereas $\iota = +$ corresponds to the asymptotic out-states,

$$T_\pm(\omega) \;=\; e^{-i\omega X_0}\, e^{2(1\mp i\omega)\phi}. \tag{2.3}$$

Standard CFT methods will allow us to define arbitrary string scattering amplitudes in an asymptotic expansion in powers of the string coupling constant g_s,

$$\begin{aligned}\left\langle\, T_{\rm out}(\omega_1)\dots T_{\rm out}(\omega_n)\, T_{\rm in}(\omega_1')\dots T_{\rm in}(\omega_m')\,\right\rangle_{c=1} &\equiv \\ \equiv \sum_{h=0}^{\infty} g_s^{2h-2}\left\langle\, T_{\rm out}(\omega_1)\dots T_{\rm out}(\omega_n)\, T_{\rm in}(\omega_1')\dots T_{\rm in}(\omega_m')\,\right\rangle_{c=1}^{(h)}.\end{aligned} \tag{2.4}$$

The amplitudes $\langle\cdots\rangle_{c=1}^{(h)}$ associated to Riemann surfaces with genus h are defined in the usual way by integrating CFT-correlation functions over the moduli space of Riemann surfaces.

2.2 Free fermionic field theory

Let us consider the quantum field theory of free fermions in the inverted harmonic oscillator potential. The one-particle hamiltonian will be

$$\mathsf{h} = -\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} - \frac{1}{4}\lambda^2. \tag{2.5}$$

There exists a complete set of *real* eigenfunctions

$$\left\{\, \mathrm{F}_p(\omega|\lambda)\,;\, \omega\in\mathbb{R}\,,\, p\in\{+,-\}\,\right\}$$

such that the labels (ω, p) of $\mathrm{F}_p(\omega|.)$ correspond to the eigenvalues of h and the parity operator P respectively.

For each pair (ω, p) of eigenvalues for the hamiltonian h and parity P, we introduce a pair of fermionic creation- and annihilation operators $(\mathrm{c}_p^\dagger(\omega), \mathrm{c}_p(\omega))$ and require that they satisfy the canonical anticommutation relations

$$\left\{\ \mathrm{c}_{p_1}(\omega)\ ,\ \mathrm{c}_{p_2}^\dagger(\omega')\ \right\}\ =\ \delta_{p_1p_2}\delta(\omega-\omega'). \tag{2.6}$$

We shall also use the vector notation

$$\mathbf{c}(\omega) = \begin{pmatrix}\mathrm{c}_+(\omega)\\ \mathrm{c}_-(\omega)\end{pmatrix}, \qquad \mathbf{F}(\omega|\lambda) = \begin{pmatrix}\mathrm{F}_+(\omega|\lambda)\\ \mathrm{F}_-(\omega|\lambda)\end{pmatrix}, \qquad \mathbf{A}\cdot\mathbf{B} \equiv A_+B_+ + A_-B_-.$$

The fermionic Fock-vacuum $|\mu\rangle\!\rangle$ is defined by the conditions

$$\begin{aligned}\mathbf{c}\ (\omega)\,|\mu\rangle\!\rangle\ &=\ 0\quad \text{for}\ \ \omega > -\mu,\\ \mathbf{c}^\dagger(\omega)\,|\mu\rangle\!\rangle\ &=\ 0\quad \text{for}\ \ \omega < -\mu.\end{aligned} \tag{2.7}$$

The Hilbert space $\mathcal{H}$ of the theory is then defined as the completion of the dense subspace spanned by vectors of the form

$$\mathsf{c}[\mathbf{f}_n]\cdots\mathsf{c}[\mathbf{f}_1]\,\mathsf{c}^\dagger[\mathbf{g}_m]\cdots\mathsf{c}^\dagger[\mathbf{g}_1]\,|\,\mu\,\rangle\!\rangle\,,$$

where

$$\mathsf{c}[\mathbf{f}]\ =\ \int_{\mathbb{R}} d\omega\ \mathbf{f}(\omega)\cdot\mathbf{c}(\omega), \quad \mathsf{c}^\dagger[\mathbf{g}]\ =\ \int_{\mathbb{R}} d\omega\ \mathbf{g}(\omega)\cdot\mathbf{c}^\dagger(\omega),$$

with $\mathbf{f}(\omega)$, $\mathbf{g}(\omega)$ smooth and rapidly decaying at $\pm\infty$. The resulting Hilbert space decomposes into sectors with a definite fermion number:

$$\mathcal{H} \;=\; \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n \,. \tag{2.8}$$

The $\mathcal{H}_n$ are eigenspaces of the fermion number operator

$$\mathsf{N} \;\equiv\; \int_{-\mu}^{\infty} d\omega \, \mathbf{c}^\dagger(\omega)\cdot \mathbf{c}\,(\omega) + \int_{-\infty}^{-\mu} d\omega \, \mathbf{c}\,(\omega)\cdot \mathbf{c}^\dagger(\omega)\,. \tag{2.9}$$

The second-quantized fermionic field operators are then defined as

$$\begin{aligned} \Psi(\lambda,t) \;&=\; \int d\omega \;\; e^{+i\omega t}\, \mathbf{F}(\omega|\lambda)\cdot \mathbf{c}\,(\omega) \;, \\ \Psi^\dagger(\lambda,t) \;&=\; \int d\omega \;\; e^{-i\omega t}\, \mathbf{c}^\dagger(\omega)\cdot \mathbf{F}(\omega|\lambda) \;. \end{aligned} \tag{2.10}$$

The dynamics of the theory is generated by the hamiltonian

$$\mathsf{H} \;=\; \int d\lambda \;\Psi^\dagger \left(-\frac{\partial^2}{\partial\lambda^2} - \frac{1}{4}\lambda^2 \right) \Psi \,. \tag{2.11}$$

As usual in fermionic field theories one may construct observables as bilinear expressions in the fermionic fields. One may e.g. consider the collective field

$$\partial_\lambda \chi(\lambda,t) = \Psi^\dagger(\lambda,t)\Psi(\lambda,t) - \langle\!\langle \mu|\Psi^\dagger(\lambda,t)\Psi(\lambda,t)|\mu\rangle\!\rangle \,. \tag{2.12}$$

The dynamics generated by the hamiltonian H becomes rather complicated in terms of the field χ. Consideration of observables like (2.12) will therefore only be useful in certain limiting cases.

2.3 In- and Out-fields

A crucial feature of the inverted harmonic oscillator potential is that the asymptotic behavior of the wave-function $\psi(\lambda,t)$ for late/early times can be represented in a very simple way:

$$\psi(\lambda,t) \underset{t\to\pm\infty}{\sim} (2\pi)^{-\frac{1}{2}} e^{\frac{i}{4}\lambda^2}\, e^{\mp\frac{t}{2}}\, \chi_\pm(u_\pm), \tag{2.13}$$

where $u_\pm \equiv \lambda e^{\mp t}$. A proof of this claim is given in appendix A.3.This means that asymptotically for $t \to \pm\infty$ the time evolution becomes represented by scale transformations. In terms of the coordinate $x = \ln|\lambda|$ one finds free relativistic motion.

The asymptotic wave-functions $\chi_\pm(u_\pm)$ can be calculated from the wave function $\psi(\lambda) \equiv \psi(\lambda, 0)$ by means of the integral transformations

$$\chi_\pm(u_\pm) \;\equiv\; (\mathsf{M}_\pm\psi)(u_\pm) \;\equiv\; \int_{\mathbb{R}} d\lambda \; K_\pm(u_\pm|\lambda)\,\psi(\lambda), \tag{2.14}$$

with kernels $K_\pm(u_\pm|\lambda)$ given by the explicit formulae

$$K_+(u_+|\lambda) \;=\; e^{-i\frac{\pi}{4}} e^{\frac{i}{2}u_+^2 - i\lambda u_+ + \frac{i}{4}\lambda^2}, \quad K_-(u_-|\lambda) \;=\; K_+^*(u_-|\lambda). \tag{2.15}$$

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These observations lead to a natural definition of the in- and out-fields. Let us consider the asymptotics for $t \to \pm\infty$ of the fermionic operators

$$\Psi^\dagger[\,\psi\,|\,t\,) \;\equiv\; \int d\lambda\;\psi(\lambda)\,\Psi^\dagger(\lambda,t) = \int d\lambda\;\psi(\lambda,t)\,\Psi^\dagger(\lambda). \tag{2.16}$$

It is then natural to define the in- and out fields $\Psi^\dagger_\pm(u)$ by the asymptotic relation

$$\Psi^\dagger[\,\psi\,|\,t\,) \underset{t\to\pm\infty}{\sim} \Psi^\dagger_\pm[\,\chi\,|\,t\,) \equiv \int \frac{du}{2\pi}\,\chi_\pm(ue^{\mp t})\,\Psi^\dagger_\pm(u), \tag{2.17}$$

where $\chi_\pm$ is defined in (2.13). It is shown in the appendix A.3 that $\Psi^\dagger[\,\psi\,|\,t\,)$ indeed has asymptotics of the required form (2.17), with $\Psi^\dagger_\pm(u_\pm)$ being related to $\Psi^\dagger(\lambda)$ by the integral transformations

$$\Psi^\dagger_\pm(u_\pm) = \int_{\mathbb{R}} d\lambda\; K^*_\pm(u_\pm|\lambda)\Psi^\dagger(\lambda)\,. \tag{2.18}$$

The transformation between in- and the out-field becomes particularly simple,

$$\Psi_+(u_+) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} du_-\; e^{-iu_+u_-}\;\Psi_-(u_-)\,. \tag{2.19}$$

It is useful to translate this into the energy representation. The observation that the transformation (2.14) maps the single-particle hamiltonian h into the generator for scale transformations of the coordinates $u_\pm$ makes it easy to find the expansion into energy eigenfunctions:

$$\Psi^\dagger_\pm(u_\pm) \;=\; \int_{\mathbb{R}} \frac{d\omega}{\sqrt{2\pi}}\,|u_\pm|^{\pm i\omega-\frac{1}{2}}\left(\Theta(-u_\pm)\mathrm{d}^{\dagger L}_\pm(\omega)+\Theta(u_\pm)\mathrm{d}^{\dagger R}_\pm(\omega)\right) \tag{2.20}$$

where $\Theta(u)$ is the usual step function. The operators $\mathrm{d}^{\dagger L}_\pm(\omega)$ and $\mathrm{d}^{\dagger R}_\pm(\omega)$ create fermions which are asymptotically located either on the right or left of the potential $V = -\frac{1}{4}\lambda^2$, respectively. It will be convenient to regard $\mathrm{d}^{\dagger L}_\pm(\omega)$ and $\mathrm{d}^{\dagger R}_\pm(\omega)$ as the two components of a vector $\mathbf{d}^\dagger_\pm(\omega)$. The relation between the oscillators $\mathbf{d}^\dagger_\pm(\omega)$ and $\mathbf{c}^\dagger(\omega)$ takes the form

$$\mathbf{d}^\dagger_\pm(\omega) = \mathbf{M}^\dagger_\pm(\omega)\cdot\mathbf{c}^\dagger(\omega), \qquad \mathbf{d}^\dagger_+(\omega) = \mathbf{R}^\dagger(\omega)\cdot\mathbf{d}^\dagger_-(\omega)\,, \tag{2.21}$$

where the matrices $\mathbf{M}_\pm$ represent the unitary operators $\mathsf{M}_\pm$ defined in (2.14) in the energy representation, and

$$\mathbf{R}(\omega) \;\equiv (\mathbf{M}_+(\omega))^2 \;=\begin{pmatrix}\rho(\omega) & \theta(\omega)\\ \theta(\omega) & \rho(\omega)\end{pmatrix}, \tag{2.22}$$

with matrix elements $\rho(\omega)$, $\theta(\omega)$ having the following explicit expressions:

$$\rho(\omega) = \frac{1}{\sqrt{2\pi}}e^{-\frac{\pi}{2}\omega}\Gamma\big(\tfrac{1}{2}-i\omega\big), \qquad \theta(\omega) = -ie^{\pi\omega}\rho(\omega). \tag{2.23}$$

Note that $\mathbf{R}$ is the matrix which represents the reflection of a single fermion in the potential $V = -\frac{1}{4}\lambda^2$. The definition of in- and out-fields leads straightforwardly to the definition of the unitary S-operator, which may be represented as

$$\mathsf{S} \;=\; \exp\left(-\int_{\mathbb{R}} d\omega\;\mathbf{d}^\dagger_-(\omega)\cdot\log\mathbf{R}(\omega)\cdot\mathbf{d}_-(\omega)\right). \tag{2.24}$$

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Remark: the formalism presented above is clearly closely related to the light cone formalism of [16]. What seems to be new is our proof (appendix A) of the equivalence between this formalism and the conventional definition of in- and out fields in terms of time-asymptotics. This explains the observation in [16] that the Fourier transformation (2.19) correctly describes the scattering of free fermions in the inverted harmonic oscillator potential.

2.4 Scattering of the bosonic excitations

However, we are also interested in the bosonic fields $\mathrm{S}_\pm(x)$ which are defined by bosonizing the in- and out fields $\Psi_\pm^\dagger$ as

$$\begin{aligned} u_\pm\partial_\pm \mathrm{S}_\pm^s(u_\pm) \;&=\; \int_{\mathbb{R}} \frac{d\omega}{\sqrt{2\pi}}\,|u_\pm|^{\pm i\omega}\left(\Theta(-u_\pm)\mathrm{a}_\pm^{\mathrm{L}}(\omega)+\Theta(u_\pm)\mathrm{a}_\pm^{\mathrm{R}}(\omega)\right), \\ \mathrm{a}_\pm^s(\omega) \;&=\; \int_{\mathbb{R}} d\omega'\;\mathrm{d}_\pm^{s\,\dagger}(\omega')\,\mathrm{d}_\pm^s(\omega+\omega')\,, \qquad s\in\{+,-\}\,\widehat{=}\,\{\mathrm{R},\mathrm{L}\} \end{aligned} \tag{2.25}$$

where $\partial_\pm \equiv \frac{\partial}{\partial u_\pm}$. The operators $\mathrm{a}_\pm^s(\omega)$ satisfy the following commutation relations,

$$[\,\mathrm{a}_\pm^s(\omega)\;,\;\mathrm{a}_\pm^{s'}(\omega')\,] \;=\; \omega\,\delta^{s,s'}\,\delta(\omega+\omega'). \tag{2.26}$$

With the help of the oscillators $\mathrm{a}_\imath^s(\omega)$, $\imath\in\{+,-\}$ we may construct the states

$$\mathrm{a}_\imath^{\mathrm{s}_n}[f_n]\cdots\mathrm{a}_\imath^{\mathrm{s}_1}[f_1]\,\mathrm{a}_\imath^{\mathrm{t}_n}[g_n]\cdots\mathrm{a}_\imath^{\mathrm{t}_1}[g_1]\,|\,\mu\,\rangle\!\rangle\,, \qquad \mathrm{a}_\imath^{\mathrm{s}}[h]\equiv\int d\omega\;h(\omega)\mathrm{a}_\imath^s(\omega) \tag{2.27}$$

which generate subspaces $\mathcal{H}_{\mathsf{o},\mathsf{o}}^\pm$ of $\mathcal{H}$. It is important to note, however, that the vectors (2.27) do not even exhaust the subspace $\mathcal{H}_\mathsf{o}\subset\mathcal{H}$ of fermion number zero. The operator

$$\begin{aligned} \mathsf{K}_\pm \;\equiv\; &\int_{-\mu}^{\infty} d\omega\,\left(\mathrm{d}_\pm^{\mathrm{R}\,\dagger}(\omega)\mathrm{d}_\pm^{\mathrm{R}}(\omega)-\mathrm{d}_\pm^{\mathrm{L}\,\dagger}(\omega)\mathrm{d}_\pm^{\mathrm{L}}(\omega)\right)+ \\ &\qquad +\int_{-\infty}^{-\mu} d\omega\,\left(\mathrm{d}_\pm^{\mathrm{R}}(\omega)\mathrm{d}_\pm^{\mathrm{R}\,\dagger}(\omega)-\mathrm{d}_\pm^{\mathrm{L}}(\omega)\mathrm{d}_\pm^{\mathrm{L}\,\dagger}(\omega)\right). \end{aligned} \tag{2.28}$$

measures the difference between the numbers of fermions which asymptotically end up to the right and left sides of the potential respectively. This means in particular that $\mathcal{H}_\mathsf{o}$ decomposes into "k-sectors" [17] $\mathcal{H}_{\mathsf{o},k}$ as follows:

$$\mathcal{H}_\mathsf{o} \;=\; \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_{\mathsf{o},k}^\pm\,. \tag{2.29}$$

In order to generate all of $\mathcal{H}_\mathsf{o}$ from $|\,\mu\,\rangle\!\rangle$ we also need to consider operators like

$$\begin{aligned} \mathrm{B}_\pm(\omega) \;=\; \int_{\mathbb{R}} d\omega'\,\big(&K^{\mathrm{LR}}(\omega|\omega')\,\mathrm{d}_\pm^{\mathrm{L}\,\dagger}(\omega')\,\mathrm{d}_\pm^{\mathrm{R}}(\omega+\omega')+ \\ &+K^{\mathrm{RL}}(\omega|\omega')\,\mathrm{d}_\pm^{\mathrm{R}\,\dagger}(\omega')\,\mathrm{d}_\pm^{\mathrm{L}}(\omega+\omega')\big)\,. \end{aligned} \tag{2.30}$$

The S-operator does not map the sector $\mathcal{H}_{\mathsf{o},k}^{\mathrm{in}}$ to $\mathcal{H}_{\mathsf{o},k}^{\mathrm{out}}$. In order to see this, let us notice that inserting (2.21) into (2.25) yields a relation of the form

$$\mathrm{a}_{\mathrm{out}}^s(\omega) \;=\; [\,\mathrm{a}_{\mathrm{out}}^s(\omega)\,]_{\mathsf{o},\mathrm{in}} + [\,\mathrm{a}_{\mathrm{out}}^s(\omega)\,]_{\mathsf{o},\mathrm{in}}^{\perp} \tag{2.31}$$

where $[\mathrm{a}^s_{\text{out}}(\omega)]_{\mathfrak{o},\text{in}}$ preserves the sectors $\mathcal{H}^{\text{in}}_{\mathfrak{o},k}$, whereas $[\,\mathrm{a}^s_{\text{out}}(\omega)\,]^{\perp}_{\mathfrak{o},\text{in}}$ is of the form (2.30). The term $[\mathrm{a}^s_{\text{out}}(\omega)]_{\mathfrak{o},\text{in}}$ is dominant for $|\omega| \ll \mu$, in which case either pure reflection or pure transmission dominate the fermionic scattering of particle-hole pairs.

The perturbative (in μ^{-1}) part of the bosonic S-matrix is encoded in

$$\begin{aligned} R^{(m\mapsto n)}(\underline{\omega}_1,\dots,\underline{\omega}_n|\underline{\omega}'_1,\dots,\underline{\omega}'_m) \;&= \\ =\;& \langle\!\langle\, \mu \,|\, \mathrm{a}^{s_1}_{\text{out}}(\omega_1)\dots \mathrm{a}^{s_n}_{\text{out}}(\omega_n)\, \mathrm{a}^{s'_1}_{\text{in}}(-\omega'_1)\dots \mathrm{a}^{s'_m}_{\text{in}}(-\omega'_m)\,|\,\mu\,\rangle\!\rangle\,, \end{aligned} \tag{2.32}$$

where we have abbreviated $\underline{\omega} \equiv (\omega, s)$. These matrix elements are unambiguously defined by (2.33), (2.26) together with $\mathrm{a}^s_{\text{out}}(\omega)|\,\mu\,\rangle\!\rangle = 0$ for $\omega > 0$. A diagrammatic formalism for the explicit evaluation of the S-matrix elements has been developed in [15]. It will be useful for us to observe that the diagonal[1] part of the scattering of the bosonic excitations can alternatively be encoded in the following operator relations:

$$\boxed{\begin{aligned} &[\,\mathrm{a}^s_{\text{out}}(\omega)\,]_{\mathfrak{o},\text{in}} \;= \\ &= \sum_{s'=\mathrm{L},\mathrm{R}} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1} \int_{\omega_1}^{\infty} \frac{d\omega_2}{\omega_2} \cdots \int_{\omega_{n-1}}^{\infty} \frac{d\omega_n}{\omega_n} \times \\ &\qquad\qquad \times\; R^{ss'}_{(n)}(\,\omega\,|\,\omega_1,\dots,\omega_n\,)\, \mathrm{a}^{s'}_{\text{in}}(\omega_1)\cdots \mathrm{a}^{s'}_{\text{in}}(\omega_n)\,. \end{aligned}} \tag{2.33}$$

The proof of formula (2.33) together with the explicit expressions for the coefficient functions $R^{ss'}_{(n)}$ are given in appendix A.4. The amplitude $R^{m\mapsto 1}$ that one can read off directly from (2.33) is very similar to the corresponding result of [20] for type 0A two-dimensional string theory.

It is also shown in appendix A.4 that the leading asymptotics of this relation for $\mu \to \infty$, $\omega \ll \mu$ is given by

$$\begin{aligned} [\,\mathrm{a}^s_{\text{out}}(\omega)\,]_{\mathfrak{o},\text{in}} \;=\; & \sum_{n=1}^{\infty} \mu^{-i\omega+1-n} \int_{-\infty}^{\infty} d\omega_1 \int_{\omega_1}^{\infty} d\omega_2 \cdots \int_{\omega_{n-1}}^{\infty} d\omega_n \times \\ & \times\; \frac{\Gamma(1+i\omega)}{\Gamma(2-n+i\omega)} \delta(\omega - \textstyle\sum_{r=1}^{n} \omega_r)\, \mathrm{a}^s_{\text{in}}(\omega_1)\cdots \mathrm{a}^s_{\text{in}}(\omega_n)\,. \end{aligned} \tag{2.34}$$

This is a trivial generalization of the formula derived in [18, 19].

2.5 Duality conjecture

From now on let us restrict attention to the excitations supported on the right of the maximum of the inverted harmonic oscillator potential. Ignoring the other side will be a good approximation as long as all energies are well below the top of the potential. The conjectured duality between $c = 1$ string theory and free fermionic field theory can be formulated most simply in terms of rescaled bosonic oscillators

$$\mathrm{b}_{\pm}(\omega) \;\equiv\; e^{\pm i\delta(\omega)} \mathrm{a}^{\mathrm{R}}_{\pm}(\omega), \tag{2.35}$$

[1] w.r.t. the decomposition into k-sectors (2.29)

provided the phase δ is chosen as

$$e^{i\delta(\omega)} \equiv \frac{\Gamma(+i\omega)}{\Gamma(-i\omega)}. \tag{2.36}$$

One manifestation of the conjectured duality between the $c = 1$ string theory and the free fermionic field theory may then be formulated as the validity of

$$\begin{aligned} &\langle\!\langle\, \mu \,|\, \mathrm{b}_{\mathrm{out}}(\omega_1) \dots \mathrm{b}_{\mathrm{out}}(\omega_n)\, \mathrm{b}_{\mathrm{in}}(-\omega_1') \dots \mathrm{b}_{\mathrm{in}}(-\omega_m') \,|\, \mu \,\rangle\!\rangle \asymp_{g_s} \\ &\qquad\qquad \asymp_{g_s} \big\langle\, T_{\mathrm{out}}(\omega_1) \dots T_{\mathrm{out}}(\omega_n)\, T_{\mathrm{in}}(\omega_1') \dots T_{\mathrm{in}}(\omega_m') \,\big\rangle_{c=1}, \end{aligned} \tag{2.37}$$

where $\asymp_{g_s}$ means equality of asymptotic expansions in $g_s = \mu^{-1}$. Note that the matrix elements on the left of (2.37) by themselves do not define a unitary S-matrix, but the deviation from unitarity is nonperturbative ($\propto e^{-\frac{1}{g_s}}$), as follows from (2.23).

3. D0-branes versus fermions - leading order

Given the duality between the $c = 1$ noncritical string theory and the free fermionic field theory it is natural to ask how to interpret the fermionic fields within the $c = 1$ noncritical string theory. A proposal for how to answer this question emerged from [1, 2]: The excitations created by the fermionic fields can be interpreted as the unstable D0-branes of the $c = 1$ noncritical string theory. In the following section we will review the existing evidence for this identification.

3.1 D0-branes in type 0B $c = 1$ string theory

The $c = 1$ noncritical string theory contains unstable D0-branes [21]. These D0-branes are localized in the strong coupling region $\phi = \infty$. In order to describe their decay one may consider the boundary interaction

$$S_{\mathrm{int}} = \kappa \int_{\partial\Sigma} d\tau \;\cosh X_0\,. \tag{3.1}$$

A construction for the corresponding boundary states was first proposed in [22]. These boundary states have to be tensored with the boundary states for Liouville theory which describe the D0-branes [21]. In this way one arrives at the following result for the leading order closed string emission amplitudes:

$$\langle\, T_{\mathrm{out}}(\omega) \,|\, B_\kappa \,\rangle_{\mathrm{HH}} = e^{i\delta(\omega)} e^{-i\omega \log\sin\pi\kappa} \mu^{-i\frac{\omega}{2}}, \tag{3.2}$$

The notation $|\, B_\kappa \,\rangle_{\mathrm{HH}}$ reminds of the fact [23] that the definition of the boundary state associated to the boundary interaction (3.1) depends on a choice of integration contour, $|\, B_\kappa \,\rangle_{\mathrm{HH}}$ being the boundary state associated to the so-called Hartle-Hawking contour [23].

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3.2 Evidence for the correspondence between D0-branes and fermions

The authors of [2] propose that the state $|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle$ which describes a fermion with a well-defined initial localization at λ_{o} may - at least to leading order in the semiclassical limit - be represented in the following bosonized form:

$$|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle \;=:\exp\left(i\mathrm{S}_{\mathrm{out}}(\lambda_{\mathsf{o}})\right):|\,\mu\,\rangle\!\rangle\;. \tag{3.3}$$

We will later discuss the applicability of the approximation

$$|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle\;\simeq\;\Psi^{\dagger}_{\mathrm{out}}(\lambda_{\mathsf{o}})\,|\,\mu\,\rangle\!\rangle \tag{3.4}$$

underlying the proposal (3.3). Adopting (3.3) as a working hypothesis for the moment, one seems to find straightforwardly that

$$\langle\!\langle\,\mu\,|\,\mathrm{b}_{\mathrm{out}}(\omega)\,|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle\;=\;e^{i\delta(\omega)}e^{-i\omega\log\lambda_{\mathsf{o}}}. \tag{3.5}$$

This matches the result of the worldsheet-computation provided that the initial location λ_{o} of the fermion is related to the parameter κ of the unstable ZZ-brane via

$$\lambda_{\mathsf{o}}\;=\;\sqrt{\mu}\sin\pi\kappa. \tag{3.6}$$

The precise match of (3.5) with the worldsheet-computation for the closed string emission from a decaying ZZ-brane represents evidence for the identification of the single fermion state with the ZZ-brane.

3.2.1 The UV problem

So far we have been considering the state $|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle$ which corresponds to a definite D0 brane parameter κ via (3.6). However, this state is clearly not normalizable. It was pointed out in [2] that the resulting divergence of energy expecation values accounts for the corresponding singular behavior in the expectation values of the energy emitted from a decaying D-brane as discussed in [23]. The natural way to resolve this problem is to average over the initial localization with a given wave-function $\varphi(\lambda_{\mathsf{o}})$,

$$|\,\varphi\,\rangle\!\rangle\;\equiv\;\int d\lambda_{\mathsf{o}}\,\varphi(\lambda_0)\,|\,\lambda_{\mathsf{o}}\,\rangle\!\rangle \tag{3.7}$$

Indeed, the norm of the resulting state will be bounded by the norm of the wave-function φ, making it obvious that the ultraviolet problem is resolved.

3.2.2 The IR problem

On the other hand one must observe that the overlaps on the left hand sides of (3.5) are identically zero since fermion number is conserved in the free fermionic field theory. To be more explicit, let us note that $\Psi^{\dagger}_{\mathrm{out}}(\lambda_0)\,|\,\mu\,\rangle\!\rangle\in\mathcal{H}_{\mathsf{1}}$, whereas $\mathrm{a}_{\mathrm{out}}(\omega)|\,\mu\,\rangle\!\rangle\in\mathcal{H}_{\mathsf{o}}$. This implies that the overlaps in (3.5) are indeed identically zero. There is no contradiction with (3.3) since the bosonization formula (3.3) has serious infrared problems[2].

The aim of the next section is to discuss how to resolve this puzzle and how to reconcile the essence of the proposal of [2] with the fermion number conservation in the free fermionic field theory. More precisely, we will propose answers to the following two questions:

[2]In Mandelstam's work one was dealing with a *massive* theory!

- ▷ What are reasonable nonvanishing analogs of the amplitudes (3.5)?
- ▷ What is the proper string-theoretic interpretation of these amplitudes? Can they be interpreted in terms of ZZ-brane decay?

4. Fermions vs. solitons

We are now going to explain how to resolve the IR problem that was pointed out at the end of previous section.

4.1 Solitonic sectors

To begin with, let us interpret the sectors $\mathcal{H}_n$ from the bosonic perspective. To simplify the notation let us temporarily restrict attention to the in-fields $\Psi(x) \equiv e^{\frac{x}{2}}\Psi_-(e^x)$ and $S(x) \equiv S_-(e^x)$.

An important point to observe is the fact that the different sectors can only be distinguished with the help of *global* observables. States $|\varphi\rangle\!\rangle_1$ in $\mathcal{H}_1$ can be created from $|\mu\rangle\!\rangle$ via

$$|\varphi\rangle\!\rangle_1 = \Psi^\dagger[\varphi]\,|\mu\rangle\!\rangle, \quad \Psi^\dagger[\varphi] \equiv \int_{\mathbb{R}} dx\,\varphi(x)\Psi^\dagger(x)\,. \tag{4.1}$$

We shall analyze the physical content of the states $|\varphi\rangle\!\rangle_1$ from the bosonic perspective - the observables used to measure properties of the states $|\varphi\rangle\!\rangle_1$ will, as usual, be constructed out of the *bosonic* oscillators $\mathrm{a}(\omega)$. It is not terribly difficult to show that

$$\langle\, \partial_x \mathrm{S}(x)\,\rangle_\varphi \equiv {}_1\langle\!\langle\, \varphi\,|\,\partial_x\mathrm{S}(x)\,|\,\varphi\,\rangle\!\rangle_1 > 0 \quad \forall x\in\mathbb{R}. \tag{4.2}$$

This means that the states $|\varphi\rangle\!\rangle_1$ are solitonic in the classical sense: They describe kinks of the bosonic field S. The states $|\varphi\rangle\!\rangle_1$ differ in their *global* properties from any state $|\varphi\rangle\!\rangle_0 \in \mathcal{H}_0$. The latter satisfy

$$\lim_{\Lambda\to\infty}\int_{-\Lambda}^{+\Lambda} dx\;{}_0\langle\!\langle\,\varphi\,|\,\partial_x\mathrm{S}(x)\,|\,\varphi\,\rangle\!\rangle_0 = 0. \tag{4.3}$$

This should be compared to the expectation value taken in the state $|\varphi_1\rangle\!\rangle$,

$$\lim_{\Lambda\to\infty}\int_{-\Lambda}^{+\Lambda} dx\;{}_1\langle\!\langle\,\varphi\,|\,\partial_x\mathrm{S}(x)\,|\,\varphi\,\rangle\!\rangle_1 = 1. \tag{4.4}$$

The difference between the asymptotic values of S measures the number of solitons $\equiv$ fermions.

Nevertheless, as long as one uses only *local* observables to measure properties of the states $|\varphi\rangle\!\rangle_n$, $n=0,1$ one will not be able to determine which sector $\mathcal{H}_n$ a given state $|\varphi\rangle\!\rangle$ belongs to. It is impossible to measure the soliton charge by using local observables like $\partial\mathrm{S}[f] \equiv \int_{\mathbb{R}} dx\, f(x)\partial_x\mathrm{S}(x)$ for $f(x)$ nonzero only in a compact subset of $\mathbb{R}$. This point can be understood quite clearly by looking back at (4.2). Imagine we are measuring

$$\langle\,\partial\mathrm{S}[f]\,\rangle_\varphi \equiv {}_1\langle\!\langle\,\varphi\,|\,\partial\mathrm{S}[f]\,|\,\varphi\,\rangle\!\rangle_1$$

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for $f > 0$ having support in small intervals. If $|\varphi\rangle\rangle \in \mathcal{H}_1$ we will find a positive result for whatever interval we have chosen. After having performed such measurements for a large number of different intervals one may feel inclined to say that the probability that the state under consideration is solitonic is rather high. Nevertheless one can never be sure that one will *always* find a positive result if one was able to continue the measurements ad infinitum.

One may therefore regard the sectors $\mathcal{H}_0$ and $\mathcal{H}_1$ as physically equivalent as long as only measurements of *local* observables are concerned. However, mathematically the sectors are not equivalent at all. This is illustrated most clearly by the fact that the sector $\mathcal{H}_1$, as opposed to $\mathcal{H}_0$, does not contain a normalizable ground state (state of energy $-\mu$), as proven in appendix B. This implies that the sectors $\mathcal{H}_1$ and $\mathcal{H}_0$ are *not* unitarily equivalent as representations of the algebra generated by the $\partial S[f]$. The mathematical non-equivalence between the sectors becomes physically relevant as soon as a physical meaning is assigned to *global* observables like the difference between the asymptotic values of the scalar field.

4.1.1 Approximate vacua

The discussion of the previous subsection may be reformulated in terms of energies as the the statement that we are unable to distinguish states in $\mathcal{H}_0$ and $\mathcal{H}_1$ as long as our detectors are insensitive to states below a certain minimal energy $\omega_{\min}$.

If however, as is usually the case, one is interested in measuring local observables only, it is perfectly sufficient to have states within $\mathcal{H}_1$ which resemble the ground state $|\,\mu\,\rangle\rangle \in \mathcal{H}_0$ to any given accuracy. A simple example for such states are the vectors

$$\begin{aligned} |\,y\,\rangle\rangle_1 \,&\equiv\, (2y)^{\frac{1}{2}}\, e^{-\mu y}\, \Psi^\dagger(-iy)\,|\,\mu\,\rangle\rangle \\ &\equiv\, (2y)^{\frac{1}{2}} \int_{-\mu}^{\infty} d\omega\; e^{-(\omega+\mu)y}\, \mathrm{d}^\dagger(\omega)\,|\,\mu\,\rangle\rangle, \qquad y>0, \end{aligned} \tag{4.5}$$

for large positive values of y. It is easy to show that

$$\begin{aligned} &\text{(i)} \quad {}_1\langle\langle\, y\,|\,\mathsf{H}\,|\,y\,\rangle\rangle_1 \,=\, (2y)^{-1}, \\ &\text{(ii)} \quad {}_1\langle\langle\, y\,|\,\partial_x \mathrm{S}(x)\,|\,y\,\rangle\rangle_1 \,=\, \frac{2y}{x^2+y^2} \end{aligned} \tag{4.6}$$

For given sensitivity of our detectors we only need to make y large enough to get states which resemble the bosonic vacuum $|\,\mu\,\rangle\rangle$ as much as we want. We will call such states approximate vacua. Equation (4.6), (ii) offers an intuitive picture of these states: The profile of the expectation value of $\partial_x \mathrm{S}(x)$ becomes arbitrarily flat.

One may then consider states like $\mathrm{a}(-\omega)|\,y\,\rangle\rangle_1$. By generalizing the previous discussion slightly one may convince oneself that such a state resembles a single boson state. Overlaps like

$${}_1\langle\langle\, y\,|\,\mathrm{a}(\omega)\,|\,\varphi\,\rangle\rangle_1$$

will be nonvanishing and can be interpreted as the amplitude for transition of a single fermion state into a state which resembles a state with a single boson created from the

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vacuum. For large y one finds

$$ {}_{\mathbf{1}}\langle\!\langle\, y\,|\,\mathrm{a}(\omega)\,\Psi^\dagger(x)\,|\,\mu\,\rangle\!\rangle_{\mathbf{1}} \simeq_y (2y)^{-\frac{1}{2}}e^{-i(\omega-\mu)t}, \tag{4.7} $$

where the notation $\simeq_y$ means equality up to an error controlled by y^{-1}.

4.1.2 Partial bosonization

As a convenient formal device to deal with the infrared problem of the usual bosonization formula we propose to replace it by the following "partial bosonization" formula

$$ \Psi^\dagger(x) \;=\; e^{iT_<(x|y)}\,\Psi^\dagger(y)\,e^{iT_>(x|y)}, \tag{4.8} $$

where

$$ T_>(x|y) \;\equiv\; \int_0^\infty \frac{d\omega}{\omega}\,(e^{i\omega x}-e^{\omega y})\mathrm{a}(\omega)\,, \tag{4.9} $$

and similarly for $T_<(x|y)$. The states

$$ \Psi^\dagger(x)|\,\mu\,\rangle\!\rangle \;\propto\; e^{iT_<(x|y)}|\,y\,\rangle\!\rangle_{\mathbf{1}} $$

are then naturally interpreted as coherent states of bosons created from an approximate vacuum within $\mathcal{H}_{\mathbf{1}}$.

It is quite clear that y plays the role of an IR cutoff in formula (4.8). It may be puzzling that one can not remove the cut-off y from transition amplitudes such as (4.7). The small prefactor $(2y)^{-\frac{1}{2}}$ on the right hand side of (4.7) implies that the probability for finding only a single boson in the state $\Psi^\dagger(x)\,|\,\mu\,\rangle\!\rangle$ would vanish if we would try to remove the IR cutoff y. This is physically perfectly appropriate: The state $|\,y\,\rangle\!\rangle$ represents the unobservable "cloud" of radiation with energy too small to be detected. An increase of y means our detector was replaced by a better one which can detect quanta of lower energy. The probability to observe *only one* closed string will be lowered.

4.2 String-theoretical interpretation

Let us now apply the discussion in the previous subsections to the case of the 2d string theory. It is natural to consider the following transition amplitude as the proper representative of the amplitude for emission of a single tachyon from a decaying ZZ brane within the fermionic field theory:

$$ \langle\!\langle\, y\,|\,\mathrm{b}_{\mathrm{out}}(\omega)\,|\,\lambda_\circ\,\rangle\!\rangle \;=\; e^{i\delta(\omega)}e^{-i\omega\log\lambda_\circ}\,\langle\!\langle\, y\,|\,\lambda_\circ\,\rangle\!\rangle, \tag{4.10} $$

where

$$ \langle\!\langle\, y\,|\,\lambda_\circ\,\rangle\!\rangle \;=\; \sqrt{2y}\frac{e^{i\mu\log\lambda_\circ}}{y+i\log\lambda_\circ} \simeq_y (2y)^{-\frac{1}{2}}\,e^{i\mu\log\lambda_\circ}. \tag{4.11} $$

When trying to interpret this result as the amplitude for emission of a closed string from a decaying D-brane one may be bothered by the additional factor $\langle\!\langle\, y\,|\,\lambda_\circ\,\rangle\!\rangle$ in (4.10). However, we shall imagine performing a gedankenexperiment in which the radiation from a decaying ZZ-brane is measured with the help of a tachyon detector installed in the weak coupling region $\phi \to -\infty$ of our two-dimensional space time. More generally one may consider amplitudes like $\langle\!\langle\, y \,|\, \mathcal{O} \,|\, \lambda_{\mathsf{o}} \,\rangle\!\rangle$ for an arbitrary local bosonic observable $\mathcal{O}$. *Any* such amplitude will be proportional to $\langle\!\langle\, y \,|\, \lambda_{\mathsf{o}} \,\rangle\!\rangle$, leading us to the conclusion that this overall factor is not physically relevant for the gedankenexperiment we are considering.

To exhibit the content of the formulae above from the perspective of 2d string theory let us assume that our tachyon detector measures the expectation value of the tachyon field as a function of the time t,

$$\langle\, \partial_t T_{\rm out}(t) \,\rangle_\chi \equiv {}_{\mathbf{1}}\langle\!\langle \chi_{\rm out} |\, \partial_t T_{\rm out}(t) \,| \chi_{\rm out} \rangle\!\rangle_{\mathbf{1}}. \tag{4.12}$$

The expectation value is evaluated in a state $|\chi_{\rm out}\rangle\!\rangle_{\mathbf{1}}$, which is defined as

$$|\chi_{\rm out}\rangle\!\rangle_{\mathbf{1}} \,=\, \int_{\mathbb{R}} dx\, \chi_{\rm out}(x) \Psi^\dagger_{\rm out}(x) \,|\, \mu \,\rangle\!\rangle. \tag{4.13}$$

Let us assume for simplicity that the wave-function $\chi_{\rm out}$ has gaussian decay away from a narrow interval. The expectation value $\langle\, \partial_t T_{\rm out}(t) \,\rangle_\chi$ can be calculated from the expectation value of $\partial_t S^{\rm R}_{\rm out}(t)$ and by taking into account the so-called leg-pole transformation,

$$\langle\, \partial_t T_{\rm out}(t) \,\rangle_{\lambda_{\mathsf{o}}} = \int\limits_{-\infty}^{\infty} dx\; K(t-x)\, \langle\!\langle \chi_{\rm out} | \partial_x S^{\rm R}_{\rm out}(x) | \chi_{\rm out} \rangle\!\rangle, \tag{4.14}$$

where $K(x)$ is defined by

$$K(x) = \int_{\mathbb{R}} d\omega\; e^{i\omega x}\, e^{i\delta(\omega)} = -\frac{z}{2} J_1(z), \qquad z \equiv 2e^{-\frac{y}{2}}.$$

The expectation value on the right hand side of (4.14) is sharply peaked. Following the discussion in [24] one may then conclude that the resulting profile for $\langle \partial_t T_{\rm out}(t) \rangle_\chi$ will first exhibit an exponential growth, reach a maximum, and then decay to zero faster than exponentially.

On the basis of these observations it seems natural to propose the following physical interpretation in terms of 2d string theory. The closed string observer in the weak coupling region will conclude that he/she has observed the radiation from the decay of a ZZ-brane. After some time, there will be no detectable radiation any more. Most of the energy of the ZZ-brane went into radiation, the missing bit not being detectable. Although fermion number conservation implies that there is a low energy "remnant" hidden behind the Liouville wall, there will always be a time after which the existence of a ZZ-brane becomes unobservable. In this sense the remnant is unphysical, not being distinguishable from the true vacuum by any local measurement.

There is yet another point of view that one may adopt. Given that the D-branes are sources for closed strings one may interpret the presence of D-branes as a deformation of the original $c = 1$ closed string background. The fact that the fermion number distinguishes superselection sectors then translates into the statement that backgrounds containing D-branes are not small deformations of the original $c = 1$-background but rather distinguished from it by boundary conditions related to the asymptotic values of certain fields. It seems interesting to note that - in contrast to previous appearances of topological charges in string theory - here we find that the topological charges are given by asymptotic boundary conditions in time rather than space.

5. Manifestation of open-closed duality

5.1 General features of the worldsheet description

We now want to propose a hopefully suggestive formal line of arguments leading to a proposal which was made in many discussions (see e.g. the discussion in [5] and references therein) of possible world-sheet explanations for open-closed dualities: The insertion of discs into string-worldsheets is equivalent to the insertion of a particular on-shell closed string vertex operator. Summing over disc insertions amounts to exponentiating the vertex operator, which describes a shift of the closed string background. In particular we shall try to identify some issues connected to this line of thought on which we shall gain some insight by the subsequent comparison with the results from the free fermionic field theory.

What sort of amplitudes are we looking for? We want to analyze the particle production by the time-dependent background that is furnished by the "decaying" D0-brane(s). So very schematically we are interested in

$$\left\langle T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\right\rangle_{\rm D0,\kappa}. \tag{5.1}$$

The notation $\langle\cdots\rangle_{\rm D0,\kappa}$ is supposed to indicate that the expectation value is not taken in the usual $c=1$ closed string background, but rather in the modified background obtained by the insertion of a $D0$-brane with parameter κ. The standard world-sheet definition of amplitudes like (5.1) takes the following schematic form:

$$\begin{aligned}\left\langle T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\right\rangle_{\rm D0,\kappa} &\asymp_{g_s}\\ &\asymp_{g_s}\sum_{h=0}^{\infty}\sum_{d=1}^{\infty} g_s^{2h-2+d}\left\langle T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\right\rangle_{c=1.}^{(h,d)}\end{aligned} \tag{5.2}$$

The notation $\asymp_{g_s}$ means equality of asymptotic expansions in powers of g_s. The terms $\langle T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\rangle_{h,d}$ in the expansion (5.2) are associated to Riemann surfaces $\Sigma_{h,n,d}$ with genus h, n punctures and d discs. In general one might try to represent these terms as integrals over the moduli space $\mathcal{M}_{h,n,d}$ of Riemann surfaces $\Sigma_{h,n,d}$,

$$\begin{aligned}\left\langle T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\right\rangle_{c=1}^{h,d} &=\\ &=\int_{\mathcal{M}_{h,n,d}}\Omega_{h,n,d}\left\langle v_{\rm out}(\omega_1)\otimes\cdots\otimes v_{\rm out}(\omega_n)\right\rangle_{\Sigma_{h,n,d}(M)}^{\rm CFT},\end{aligned} \tag{5.3}$$

where we have put the ghost contributions into the definition of the top form $\Omega_{h,n,d}$ and $\langle\cdots\rangle^{\rm CFT}_{\Sigma_{h,n,d}(M)}$ is a correlation function in the conformal field theory

$$\mathrm{CFT} = (\mathrm{Super-Liouville})\otimes(X_0\mathrm{-CFT}).$$

The correlation functions $\langle\cdots\rangle^{\rm CFT}_{\Sigma_{h,n,d}(M)}$ are viewed as machines which for each point $M\in\mathcal{M}_{h,n,d}$ transform vectors $v\in\mathcal{H}_{\rm CFT}^{\otimes n}$ into numbers.

We do not expect unusual problems in the construction of arbitrary correlation functions $\langle \cdots \rangle^{\rm CFT}_{\Sigma_{h,n,d}(M)}$ as long as $d = 0$. The potentially troublesome X_0-CFT is free in the bulk, which allows us to define the contribution from the X_0-CFT by means of analytic continuation w.r.t. the energies ω_k, $k = 1, \dots, n$. In order to construct the amplitudes with disc insertions a standard approach would be to start from correlation functions $\langle \cdots \rangle^{\rm CFT}_{\Sigma_{h,n+d,0}(M)}$, from which one may try to construct $\langle \cdots \rangle^{\rm CFT}_{\Sigma_{h,n,d}(M)}$ by sewing punctured discs to d of the $n + d$ punctures. In the case $d = 1$ this would lead to a representation of the following type

$$\begin{aligned} \big\langle\, v_{\rm out}(\omega_1)\otimes \cdots \otimes v_{\rm out}(\omega_n)\, \big\rangle^{\rm CFT}_{\Sigma_{h,n,1}(M)} = \qquad\qquad\qquad\qquad\qquad\qquad \\ = \big\langle\, v_{\rm out}(\omega_1) \otimes \cdots \otimes v_{\rm out}(\omega_n) \otimes (e^{-\tau(L_0+\bar{L}_0-2)}|B_\kappa\rangle)\, \big\rangle^{\rm CFT}_{\Sigma_{h,n+1,0}(M)}, \end{aligned} \tag{5.4}$$

where $|B_\kappa\rangle$ is the boundary state associated to the boundary interaction (3.1). The gluing parameter $\tau \in \mathbb{R}_+$ represents the deformations of $\Sigma_{h,n,1}$ which change the radius of the disc. In the general case $d \geq 0$ one will have d such gluing parameters $\tau_1, \dots, \tau_d$, and an obvious generalization of formula (5.4).

In formula (5.4) we observe an unusual source of trouble: The spectrum of $L_0 + \bar{L}_0$ is unbounded from below since the X_0-CFT contains eigenstates with arbitrarily negative eigenvalues. It is therefore not clear to me how to make sense out of the right hand side of (5.4) in the present context.

If, however, a good definition for the right hand side of (5.4) is ultimately found, we could proceed with the integration over moduli space as follows: By using coordinates for the moduli space $\mathcal{M}_{h,n,d}$ such as those used in [25] one may realize that

$$\mathcal{M}_{h,n,d} \simeq \mathcal{M}_{h,n+d,0} \times \mathbb{R}^d_+, \tag{5.5}$$

where we may think of $\mathcal{M}_{h,n+d,0}$ as parametrizing the complex structures on the surface that is obtained from $\Sigma_{h,n,d}$ by gluing punctured discs into the d boundaries of $\Sigma_{h,n,d}$. The moduli corresponding to the factor $\mathbb{R}^d_+$ in (5.5) can be identified with the radii of the discs, and therfore with the parameters $\tau_1, \dots, \tau_d$ that were introduced after (5.4). This means that we can factor off the integration over $\mathbb{R}^d_+$ in (5.3) and represent it explicitly by integrating over $\tau_1, \dots, \tau_d$. As a symbolic notation for the result of this procedure we shall propose

$$\left\langle\, v_{\rm out}(\omega_1) \otimes \cdots \otimes v_{\rm out}(\omega_n) \otimes \Big(\frac{1}{L_0 + \bar{L}_0 - 2}\, |B_\kappa\rangle \Big)^{\otimes d} \right\rangle^{\rm CFT}_{\Sigma_{h,n+d,0}(M)}. \tag{5.6}$$

The physical interpretation of the insertions of $(L_0 + \bar{L}_0 - 2)^{-1}|B_\kappa\rangle$ should be clear: They represent the propagation of closed strings from the brane into the region where interactions with other closed strings take place. This leads us to formulate a physically motivated requirement on possible definitions of (5.4),(5.6): They should be such that only on-shell physical states contribute in (5.6). We are thereby lead to the expectation that (5.3) can be replaced by an expression of the following form

$$
\begin{aligned}
&\big\langle\, T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\,\big\rangle_{h,d} = \\
&\quad = \int\limits_{\mathcal{M}_{h,n+d,0}} \Omega_{h,n+d,0}\,\Big\langle\, v_{\rm out}(\omega_1)\otimes\cdots\otimes v_{\rm out}(\omega_n)\otimes\big(w_{\rm in}(\kappa)\big)^{\otimes d}\Big\rangle^{\rm CFT}_{\Sigma_{h,n+d,0}(M)} \\
&\quad = \int d\omega_1'\cdots d\omega_d'\,\prod_{r=1}^{d}\langle\, T_{\rm in}(\omega_r')\,|\,B_\kappa\,\rangle \\
&\qquad\qquad\qquad \times\big\langle\, T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\,T_{\rm in}(\omega_1')\cdots T_{\rm in}(\omega_m')\,\big\rangle^{(h,0)}_{c=1.}
\end{aligned} \tag{5.7}
$$

where we have assumed (with some hindsight) that the state $w_{\rm in}(\kappa)$ may be represented in the form

$$
w_{\rm in}(\kappa) \;=\; \int d\omega\;\langle\, T_{\rm in}(\omega)\,|\,B_\kappa\,\rangle\, v_{\rm in}(\omega)\,. \tag{5.8}
$$

Equation (5.7) is the sought-for representation of disc insertions in terms of certain closed string operators.

We believe that the following point deserves some emphasis: *Despite the fact that we do not know the precise definition for the right hand side of (5.4), we are rather confident that the representation (5.7) for perturbative closed string emission amplitudes in terms of a sum over disc insertions should be valid.*

We will soon see that these expectations are nicely supported by results from the fermionic field theory. This will allow us to demonstrate that the above ideas about the world-sheet mechanism behind open-closed duality are realized in the present context in a rather concrete and well-defined manner.

5.2 Fermionic definition of amplitudes

Our aim is to calculate the amplitudes for emission of closed strings from a decaying ZZ-brane. Identifying the ZZ-branes with the fermions in the free fermionic field theory leads one to consider overlaps of the form

$$
\langle\!\langle\, y\,|\,{\rm b}_{\rm out}(\omega_1)\ldots{\rm b}_{\rm out}(\omega_n)\,|\,\varphi\,\rangle\!\rangle \tag{5.9}
$$

where $|\,\varphi\,\rangle\!\rangle$ represents a single fermion created from the vacuum $|\,\mu\,\rangle\!\rangle$,

$$
|\,\varphi\,\rangle\!\rangle \;=\; \int d\lambda\;\varphi(\lambda)\,|\,\lambda\,\rangle\!\rangle_{\mathbf{1}}\,,\qquad |\,\lambda\,\rangle\!\rangle_{\mathbf{1}}\;\equiv\;\Psi^\dagger(\lambda)\,|\,\mu\,\rangle\!\rangle. \tag{5.10}
$$

To be fully specific let us agree that the pseudo-vacuum $|\,y\,\rangle\!\rangle$ in (5.9) is defined as in (4.5) by using ${\rm d}_-^{\dagger\rm R}(\omega)$ instead of ${\rm d}_+^\dagger(\omega)$.[3]

[3]It may seem unnatural that we define $|\,y\,\rangle\!\rangle$ in terms of the in-fermionic creation operators ${\rm d}_-^{\dagger\rm R}(\omega)$ rather than ${\rm d}_+^{\dagger\rm R}(\omega)$. The difference is in many respects inessential, though. Thanks to the fact that the energy distribution in (4.5) is peaked around $\omega=-\mu$ we may approximate the reflection matrix $\mathbf{R}(\omega)$ in (2.21) by $\mathbf{R}(-\mu)$. This means that replacing ${\rm d}_-^{\dagger\rm R}(\omega)$ by ${\rm d}_+^{\dagger\rm R}(\omega)$ results in an overall factor that depends on μ only, and will therefore be irrelevant for most questions. However, this factor will have to be taken into account when calculating the asymptotic expansions in μ^{-1}. This will be done in the following subsections 5.3 and 5.4, where our present choice will turn out to be the most convenient one.

We are trying to establish a relation of the form

$$\langle\!\langle\, y \,|\, \mathrm{b}_{\rm out}(\omega_1)\dots \mathrm{b}_{\rm out}(\omega_n)\,|\,\varphi\,\rangle\!\rangle \asymp_{g_s} \big\langle\, T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\,\big\rangle_{\rm D0,\kappa}\,. \tag{5.11}$$

This clearly requires choosing a particular wave-function $\varphi \equiv \varphi_\kappa$. We know that $|\,\varphi\,\rangle\!\rangle$ has the following equivalent descriptions:

$$\begin{aligned} |\,\varphi\,\rangle\!\rangle \;&=\; \int_{\mathbb{R}} du_+\, \chi_{\rm out}(u_+)\Psi^\dagger_{\rm out}(u_+)\,|\,\mu\,\rangle\!\rangle \\ &= \int_{\mathbb{R}} d\lambda\; \psi(\lambda)\Psi^\dagger(\lambda)\,|\,\mu\,\rangle\!\rangle \\ &= \int_{\mathbb{R}} du_-\, \chi_{\rm in}(u_-)\Psi^\dagger_{\rm in}(u_-)\,|\,\mu\,\rangle\!\rangle. \end{aligned} \tag{5.12}$$

Let us furthermore recall that given any one of the wave-functions $(\chi_{\rm out}, \phi, \chi_{\rm in})$, we can calculate the two others via the integral transformations (2.14). These relations describe the dispersion that a wave-packet suffers in the time-evolution between any finite time and and the asymptotics $t \to \pm\infty$. It seems natural to suspect that the correct choice of φ_κ must correspond to point-like initial localization, with parameter κ being related to the initial position. Still we have two options to consider: Point-like initial localization at finite time or point-like initial localization for time $t \to -\infty$. These two possibilities are of course inequivalent as the effects of dispersion will be substantial in general. We are going to show that only point-like initial localization for time $t \to -\infty$ has the chance to yield amplitudes that can be identified with the world-sheet description. Let us therefore consider the choice

$$\chi_{\rm in}(u_-) \;\equiv\; \delta(u_- - u_-^\circ) \tag{5.13}$$

which corresponds to choosing

$$|\,\varphi\,\rangle\!\rangle \;\equiv\; |\,u_-^\circ\,\rangle\!\rangle\,, \quad |\,u_-^\circ\,\rangle\!\rangle \equiv \Psi^\dagger_{\rm in}(u_-^\circ)\,|\,\mu\,\rangle\!\rangle\,. \tag{5.14}$$

Our next aim will be to show that this choice indeed leads to a relation of the desired form (5.11).

5.3 Closed string picture

One possible way to expand the amplitude in powers of $g_s = \mu^{-1}$ proceeds by using the expansion (2.33) in order to express the bosonic operators $\mathrm{b}_{\rm out}(\omega)$ in terms of the $\mathrm{b}_{\rm in}(\omega)$. This leads to an expression of the following form:

$$\begin{aligned} &\langle\!\langle\, y \,|\, \mathrm{b}_{\rm out}(\omega_1)\cdots \mathrm{b}_{\rm out}(\omega_n)\,|\,u_-^\circ\,\rangle\!\rangle \;\asymp_{g_s} \\ &\asymp_{g_s}\; \sum_{m=1}^\infty \frac{1}{m!} \int_{\mathbb{R}_+} d\omega_1' \,\dots \int_{\mathbb{R}_+} d\omega_m'\; R^{(m\mapsto n)}(\omega_1,\dots,\omega_n|\omega_1',\dots,\omega_m') \\ &\qquad\qquad \times \langle\!\langle\, y \,|\, \mathrm{b}_{\rm in}(\omega_1')\cdots \mathrm{b}_{\rm in}(\omega_m')\,|\,u_-^\circ\,\rangle\!\rangle \end{aligned} \tag{5.15}$$

We are only claiming equality of asymptotic expansions in $g_s = \mu^{-1}$ since we have been ignoring the non-perturbative correction associated to the second term in the decomposition (2.31). In the case that we have $\mu \to \infty$, $\frac{\omega}{\mu} \ll 1$, where $\omega \equiv \sum_{r=1}^n \omega_r$, we may approximate

$R^{(m\mapsto n)}$ by its asymptotic expansion in powers of $g_s = \mu^{-1}$. It is important to note that up to terms of order y^{-1}, the S-matrix elements $R^{(m\mapsto n)}$ in the one-fermion sector are equal to the their counterparts in the zero fermion sector. The asymptotic expansion of the latter was identified with the correlation functions of the $c = 1$ string theory in (2.37). This allows us to write

$$\begin{aligned} & R^{(m\mapsto n)}(\omega_1,\dots,\omega_n|\omega_1',\dots,\omega_m') \asymp_{g_s} \\ & \asymp_{g_s} \sum_{h=0}^{\infty} g_s^{2h-2+n+m} \left\langle T_{\rm out}(\omega_1)\dots T_{\rm out}(\omega_n)\, T_{\rm in}(\omega_1')\dots T_{\rm in}(\omega_m') \right\rangle_{c=1,}^{(h,0)} \end{aligned} \tag{5.16}$$

Let us also note the following simple relation

$$\langle\!\langle\, y\,|\, \mathrm{b}_{\rm in}(\omega_1')\dots \mathrm{b}_{\rm in}(\omega_m')\,|\, u_-^\circ \,\rangle\!\rangle \;=\; \langle\!\langle\, y\,|\, u_-^\circ\,\rangle\!\rangle \prod_{r=1}^{d} \langle\, T_{\rm in}(\omega_r')\;|\, B_\kappa\,\rangle. \tag{5.17}$$

We thereby arrive at an expansion of the following form

$$\boxed{\begin{aligned} & \langle\!\langle\, y\,|\, u_-^\circ\,\rangle\!\rangle^{-1} \langle\!\langle\, y\,|\, \mathrm{b}_{\rm out}(\omega_1)\cdots \mathrm{b}_{\rm out}(\omega_n)\,|\, u_-^\circ\,\rangle\!\rangle \asymp_{g_s} \\ & \asymp_{g_s} \sum_{d=1}^{\infty}\frac{1}{d!}\sum_{h=0}^{\infty} g_s^{2h-2+n+d}\int_{\mathbb{R}_+} d\omega_1' \,\dots \int_{\mathbb{R}_+} d\omega_d' \prod_{r=1}^{d}\langle\, T_{\rm in}(\omega_r')\;|\, B_\kappa\,\rangle \\ & \qquad\qquad \times \left\langle T_{\rm out}(\omega_1)\dots T_{\rm out}(\omega_n)\, T_{\rm in}(\omega_1')\dots T_{\rm in}(\omega_d') \right\rangle_{c=1.}^{(h,0)} \end{aligned}}$$

This is just what we are expecting on the basis of the discussion in subsection 5.1, cf. in particular with equation (5.7).

Remarks

1. The leading asymptotics $\mu\to\infty$, $\frac{\omega}{\mu}\ll 1$ of the one-point function is in found to be

$$\langle\!\langle\, y\,|\, \mathrm{b}_{\rm out}(\omega)\,|\, F\,\rangle\!\rangle \;\simeq_y\; (2y)^{-\frac{1}{2}}\, e^{i\delta(\omega)}\, e^{-i(\omega-\mu)\log u_-^\circ}\,\mu^{-i\omega} \tag{5.18}$$

Up to inessential factors (cf. our discussion in subsection 4.2) we find a result that matches the world-sheet computation provided that we identify the parameters as

$$\sin\pi\kappa \;=\; \sqrt{\mu}\, u_-^\circ. \tag{5.19}$$

It should be noted that the initial localization u_-° has nothing to do with the turning point of the corresponding classical motion. The latter may be estimated from the average value of the energy when we form wave-packets rather than considering point-like localized "states". This point will be further elaborated upon at the end of the following subsection.

2. One may/should worry about the convergence of the various summations/integrations in (5.18). Let us first note that the integrations do not pose any problem. The UV convergence is ensured by the delta-function in the integrand together with $\langle\!\langle y|\mathrm{b}_{\rm in}(\omega)\sim e^{y\omega}\langle\!\langle y|$ for $\omega<0$. The absence of IR problems can be inferred from formula (A.19), noting that $Q_{(n)}^{ss'}$ vanishes for $\omega_r\to 0$. The summation over h will not be convergent but only asymptotic, but it is interesting to note that the sum over the number of discs d is probably convergent even *after* exchanging the summations over h and d. We have checked this claim explicitly in the case $n = 1$ using the tree approximation (2.34) to the closed string S-matrix.

3. There is another way of writing the expansion (5.15) which appears to be instructive. Introducing the notation

$$\mathrm{B}_{\mathrm{in}}(\kappa) \equiv \int_{\mathbb{R}_+} d\omega \, \langle \, T_{\mathrm{in}}(\omega) \mid B_\kappa \, \rangle \, T_{\mathrm{in}}(\omega)$$

allows us to write (5.15) in the following form:

$$\begin{aligned} \langle\!\langle \, y \, | \, u_-^\circ \, \rangle\!\rangle^{-1} \langle\!\langle \, y \, | \, \mathrm{b}_{\mathrm{out}}(\omega_1) \cdots \mathrm{b}_{\mathrm{out}}(\omega_n) \, | \, u_-^\circ \, \rangle\!\rangle \; &\asymp_{g_s} \\ \asymp_{g_s} \sum_{d=1}^{\infty} \frac{g_s^d}{d!} \, \big\langle \, T_{\mathrm{out}}(\omega_1) \cdots T_{\mathrm{out}}(\omega_n) \, (\mathrm{B}_{\mathrm{in}}(\kappa))^d \, \big\rangle_{c=1} & \\ \asymp_{g_s} \big\langle \, T_{\mathrm{out}}(\omega_1) \cdots T_{\mathrm{out}}(\omega_n) \, \exp\big[\mathrm{B}_{\mathrm{in}}(\kappa)\big] \, \big\rangle_{c=1.} & \end{aligned} \tag{5.20}$$

It seems natural to call the representation (5.20) the closed string representation. This representation suggests the following interpretation in terms of 2d string theory. The initial state of the D-brane is represented as a coherent state of incoming closed strings[4]. The resulting out-state is obtained by applying the closed string S-matrix to the closed string oscillators which generate the initial state.

5.4 Open string picture

There is an alternative way to calculate the amplitude (5.9). Let us first note the simple relation

$$\begin{aligned} &\langle\!\langle \, y \, | \, \mathrm{b}_{\mathrm{out}}(\omega_1) \ldots \mathrm{b}_{\mathrm{out}}(\omega_n) \; | \, F \, \rangle\!\rangle \; = \\ &\quad = \int_0^\infty du_+ \; \chi_{\mathrm{out}}(u_+) \prod_{r=1}^{n} e^{i\delta(\omega_r)} \, e^{-i\omega_r \log u_+} \, \langle\!\langle \, y \, | \, u_+ \, \rangle\!\rangle. \end{aligned} \tag{5.21}$$

When calculating the matrix element $\langle\!\langle \, y \, | \, u_+ \, \rangle\!\rangle$ one should not forget that we had adopted the convention to create the state $| \, y \, \rangle\!\rangle$ with the help of the fermionic in-field, cf. the footnote in subsection 5.2. It follows that

$$\langle\!\langle \, y \, | \, u_+ \, \rangle\!\rangle \; \simeq_y \; (2y)^{-\frac{1}{2}} \, e^{i\mu \log u_+} \, \rho^*(-\mu) \, u_+^{-\frac{1}{2}}, \tag{5.22}$$

where $\rho(\omega)$ is the diagonal element of the single particle reflection matrix defined in (2.23). We thereby arrive at the expression

$$\langle\!\langle \, y \, | \, \mathrm{b}_{\mathrm{out}}(\omega_1) \ldots \mathrm{b}_{\mathrm{out}}(\omega_n) \; | \, F \, \rangle\!\rangle \; \simeq_y \; (2y)^{-\frac{1}{2}} \rho^*(-\mu) \, \tilde{\chi}^{\mathrm{R}}_{\mathrm{out}}(\omega - \mu) \prod_{r=1}^{n} e^{i\delta(\omega_r)} \tag{5.23}$$

where $\omega = \sum_{r=1}^n \omega_r$, and $\tilde{\chi}^{\mathrm{R}}_{\mathrm{out}}(\omega)$ is the Fourier-transformation of $\chi^{\mathrm{R}}_{\mathrm{out}}(u_+)$,

$$\tilde{\chi}^{\mathrm{R}}_{\mathrm{out}}(\omega) = \int_0^\infty du \; u^{-i\omega - \frac{1}{2}} \, \chi^{\mathrm{R}}_{\mathrm{out}}(u).$$

[4]on top of an unobservable D0-remnant, if you wish.

It remains to calculate $\tilde{\chi}_{\rm out}$ for the choice of $\tilde{\chi}_{\rm in}$ which corresponds to the state $|\,u_-^\circ\,\rangle\!\rangle$, cf. (5.13), namely

$$\begin{pmatrix}\chi_{\rm in}^{\rm L}(\omega)\\ \chi_{\rm in}^{\rm R}(\omega)\end{pmatrix} = e^{-i\omega\log u_-^\circ}\begin{pmatrix}0\\1\end{pmatrix}.$$

The relation (2.21) simplifies to $\chi_{\rm out}^{\rm R}(\omega)=\rho(\omega)\chi_{\rm in}^{\rm R}(\omega)$. We finally arrive at the simple result

$$\boxed{\begin{aligned}&\langle\!\langle\, y\,|\,{\rm b}_{\rm out}(\omega_1)\dots{\rm b}_{\rm out}(\omega_n)\,|\,F\,\rangle\!\rangle\simeq_y\\ &\quad\simeq_y\;(2y)^{-\frac{1}{2}}\prod_{r=1}^{n}e^{i\delta(\omega_r)}\,\rho^*(-\mu)\rho(\omega-\mu)e^{-i(\omega-\mu)\log u_-^\circ}.\end{aligned}} \tag{5.24}$$

Remarks

1. Viewing the fermionic field theory as a representation for the open string theory on a gas of $D0$-branes [1, 2] motivates us to call the resulting representation the "open string picture". Quantum corrections to the D-brane dynamics are calculated in the dual open string theory before we analyze the final state in terms of closed string observables.

In the gedankenexperiment proposed in subsection 4.2 we are of course not tracking the evolution of the D-brane state at finite times, we only observe outgoing radiation at late times. It is therefore completely arbitrary[5] if we prefer to interpret the state at finite times as a D-brane or as a coherent state of closed strings. We may in particular imagine that the D-brane is "created" by an incoming coherent state of closed strings, and that it subsequently decays back into an outgoing coherent state of closed string radiation, as suggested by the "full-brane" picture.

2. It is possible to calculate the expansion in powers of g_s by noting that

$$\rho(\omega-\mu)\;=\;\frac{e^{-\frac{\pi}{2}(\omega-\mu)}}{2\cosh\pi(\omega-\mu)}e^{i\xi(\omega-\mu)}\underset{\mu\to\infty}{\asymp}e^{i\xi(\omega-\mu)},\quad \xi(x)\equiv\arg\Gamma\big(\tfrac{1}{2}-ix\big), \tag{5.25}$$

for $\mu\to\infty$ and $\frac{\omega}{\mu}\ll 1$, and using

$$e^{i\xi(\omega-\mu)}\asymp e^{i(\mu\ln\mu-\mu)}\,\mu^{-i\omega}\,\exp\left(i\sum_{n=1}^{\infty}\frac{(-1)^n B_{2n}}{2n(2n-1)}(1-2^{-(2n-1)})(\omega-\mu)^{-(2n-1)}\right).$$

Note that we reproduce our previous result (5.18) for the leading asymptotics $\mu\to\infty$, $\frac{\omega}{\mu}\ll 1$ of the one-point function.

The individual terms in the resulting expansion are naturally interpreted as perturbative contributions to the amplitude for emission of closed strings in the time-dependent background that is furnished by the decaying ZZ-brane, corresponding to the following reorganization of the perturbative expansion (5.2):

$$\big\langle\,T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\,\big\rangle_{{\rm D0},\kappa}=\sum_{r=0}^{\infty}g_s^{r-2}\sum_{\substack{h,d=0\\2h+d=r}}^{\infty}\big\langle\,T_{\rm out}(\omega_1)\cdots T_{\rm out}(\omega_n)\,\big\rangle_{c=1}^{(d,g)}. \tag{5.26}$$

[5] to the extend that we can identify the different solitonic superselection sectors as physically equivalent

One should bear in mind that the perturbative expansion in powers of g_s will give a useful approximation only if $\mu \to \infty$, and if only low energy tachyons with $\frac{\omega}{\mu} \ll 1$ are "measured".

3. It may seem puzzling that in (5.19) we are identifying the D-brane parameter with the initial localization for time $t \to -\infty$, whereas in [2, 7] it is associated with the turning point of the classical motion in the inverted harmonic oscillator potential. By forming wave-packets it is of course possible to get states which are in the classical limit $\mu \to \infty$ well-localized in the sense that the uncertainties $\frac{1}{\mu}(\delta u_\pm)^2$ and $\frac{1}{\mu}(\delta\lambda)^2$ are small. For those wave-packets one recovers (4.10) as the leading approximation to the tachyon emission amplitude.

5.5 More general $c = 1$-backgrounds

More generally we may consider string scattering amplitudes of the form

$$\langle\!\langle\, f_+ \,|\, \mathrm{b}_+(\omega_1) \cdots \mathrm{b}_+(\omega_n)\, \mathrm{b}_-(-\omega_1') \cdots \mathrm{b}_-(-\omega_m') \,|\, f_- \,\rangle\!\rangle, \tag{5.27}$$

where

$$\begin{aligned} |\, f_- \,\rangle\!\rangle \; &= \exp\left(\int_{-\infty}^0 d\omega \; f_-(\omega)\mathrm{b}_-(\omega) \right) |\, \mu \,\rangle\!\rangle \\ \langle\!\langle\, f_+ \,| \; &= \langle\!\langle\, \mu \,|\, \exp\left(\int_0^\infty d\omega \; f_+(\omega)\mathrm{b}_-(\omega) \right). \end{aligned} \tag{5.28}$$

This may be interpreted as a string scattering amplitude in a time-dependent background that is explicitly represented in terms of coherent states $|\, f_+ \,\rangle\!\rangle$, $|\, f_- \,\rangle\!\rangle$ of closed strings. Of course it is sufficient to study

$$\langle\!\langle\, f_+ \,|\, f_- \,\rangle\!\rangle, \tag{5.29}$$

from which (5.27) can be recovered by taking functional derivatives.

We now want to describe the scattering amplitudes in a background that contains a decaying ZZ-brane on top of the closed string background described by $|\, f_+ \,\rangle\!\rangle$ and $|\, f_- \,\rangle\!\rangle$. Our previous discussions suggest that

$$_{\scriptscriptstyle 1}\langle\!\langle\, f_+ \,|\, \Psi_-^\dagger(u_-^\circ) \,|\, f_- \,\rangle\!\rangle \tag{5.30}$$

represents the generating functional for the amplitudes in question. The state $_{\scriptscriptstyle 1}\langle\!\langle\, f_+ \,|$ is defined by replacing the vacuum $\langle\!\langle\, \mu \,|$ in (5.28) by an approximate vacuum in the one fermion sector. By using our partial bosonization formula (4.8) it becomes easy to show that

$$\boxed{\begin{aligned} &_{\scriptscriptstyle 1}\langle\!\langle\, f_+ \,|\, \Psi_-^\dagger(u_-^\circ) \,|\, f_- \,\rangle\!\rangle \; = \\ &\qquad = \exp\left(i \int_{\mathbb{R}_+} d\omega \; (u_-^\circ)^{i\omega} f_-(-\omega) \right) \;\; {}_{\scriptscriptstyle 1}\big\langle\!\big\langle\, f_+ \,|\, \mathsf{S} \,|\, f_- + e_\tau \,\big\rangle\!\big\rangle_{\scriptscriptstyle 1} \end{aligned}}\,, \tag{5.31}$$

where e_τ is explicitly given as

$$e_\tau(\omega) \;=\; i\, \frac{(u_-^\circ)^{i\omega} - e^{\omega y}}{\omega}.$$

Equation (5.31) shows that the insertion of a decaying ZZ-brane generates a shift in the closed string background that is linear in the variables f_+. The open-closed duality expressed by equation (5.31) is not perfect, though. The infrared region near $\omega = 0$ is effectively removed by our cut-off y. We therefore do not generate a shift of the cosmological constant μ, corresponding to the zero energy tachyon. The fact that we can not remove the infrared cut-off y limits the extend to which strict open-closed duality is realized in our context. On the other hand, our previous discussion of this issue shows that what we are missing to strict open-closed duality is associated with low-energy quanta that we are not able to observe anyway. In this sense one may well regard the failure of strict open-closed duality as unphysical.

6. Comparison with the euclidean case

It seems worth pointing out a close analogy between the results of the previous subsection and discussions of the integrable structure of two-dimensional string theory in [26],[16] and [27] respectively. In the following we shall review those features of the formalisms developed in [26, 16, 27] which we need to see the analogy with the results in the previous section. Our discussion will not be self-contained, the reader not sufficiently familiar with the results of [26, 16, 27] may need to consult these references while reading the following section.

6.1 Euclidean generating function

We will now consider the case of euclidean target space for the two-dimensional string theory which is obtained by $X_0 \to -iX_0 \equiv X$. Compactification of euclidean time via $X \equiv X + 2\pi R$ will be introduced to describe finite temperature. One is then in particular interested in deformations of the background induced by changing the world-sheet action as

$$S_{\rm WS} \to S_{\rm WS} + \sum_{k\neq 0} t_k \, T_{\rm E}(p_k), \tag{6.1}$$

where $p_k = k/R$ and $T_{\rm E}(p)$ is the on-shell vertex operator

$$T_{\rm E}(p) \sim \int d^2z \; e^{ipX} e^{2(1-|p|)\phi} \,.$$

The central object to study is the deformed partition function

$$Z(\{t_k\};\mu,R) \;=\; \Big\langle \exp\Big(-\sum_{k\neq 0} t_k \, T_{\rm E}(p_k)\Big)\Big\rangle^{\rm Eucl}_{\rm c=1.} \tag{6.2}$$

Turning to the free fermionic field theory, it is in fact straightforward to introduce a natural euclidean counterpart to (5.29) as follows [15]: Let $u_\pm = e^{t\mp x}$ and continue $t = i\theta$. Periodicity w.r.t. $\theta \to \theta + 2\pi R$ then leads to quantization of the euclidean energies as $\omega = ik/R$. The euclidean counterparts of in- and out bosonic fields,

$$\partial_{u_\pm} S_\pm(u_\pm) = \sum_{k\in\mathbb{Z}} a_k^{(\pm)} u_\pm^{-\frac{1}{R}(k+1)}, \quad \pm = \pm \tag{6.3}$$

will be single-valued. It is then natural to consider

$$\mathcal{Z}(\{t_k\};\mu,R) \;=\; \langle\!\langle\, T_+\,|\,T_-\,\rangle\!\rangle, \tag{6.4}$$

where $|T_-\rangle\!\rangle$, $\langle\!\langle T_+|$ are coherent states of bosonic excitations defined as

$$|\,T_+\,\rangle\!\rangle \;\equiv\; \exp\left(\sum_{k<0} t_k a_k^{\rm in}\right)|\,\mu\,\rangle\!\rangle, \qquad \langle\!\langle\, T_-\,| \;\equiv\; \langle\!\langle\,\mu\,|\exp\left(\sum_{k>0} t_k a_k^{\rm out}\right). \tag{6.5}$$

The generating function (6.4) can be evaluated by first expressing the bosonic oscillators $a_k^{(\pm)}$ in terms of fermions as

$$a_k^{(\pm)} \;=\; \sum_{l\in\mathbb{Z}+\frac12} \mathrm{d}_k^{(\pm)}\mathrm{d}_{k-l,}^{\dagger(\pm)} \qquad \begin{aligned} \psi_\pm(u_\pm) &= \sum_{k\in\mathbb{Z}+\frac12} \mathrm{d}_k^{(\pm)} u_\pm^{-\frac{k}{R}-\frac{1}{2R}},\\ \psi_\pm^\dagger(u_\pm) &= \sum_{k\in\mathbb{Z}+\frac12} \mathrm{d}_k^{\dagger(\pm)} u_\pm^{-\frac{k}{R}-\frac{1}{2R}}, \end{aligned} \tag{6.6}$$

and then using the algebra $\{\mathrm{d}_k^{(\pm)},\mathrm{d}_l^{\dagger(\pm)}\} = \delta_{k+l}$ as well as the following relation between in- and out oscillators:

$$\mathrm{d}_k^{\rm out} \;=\; \rho(p_k)\,\mathrm{d}_k^{\rm in}. \tag{6.7}$$

Equation (6.4) is therefore good enough to define $\mathcal{Z}$ as a formal series in the variables t_k. The conjectured duality between the euclidean versions of $c = 1$ string theory and free fermionic field theory is coincidence of the objects defined in (6.4) and (6.2), respectively.

6.2 Deformations of the Fermi level curve

The starting point of the formalism developed in [16] is the reformulation of the free fermionic field theory in terms of light-cone variables $u_\pm$ for the phase space of the single particle problem,

$$u_\pm \equiv \frac{1}{2}\lambda \pm p. \tag{6.8}$$

The classical single particle hamiltonian is then simply $h = -u_+u_-$, so that the vacuum of the classical limit of free fermionic field theory, the filled Fermi sea, gets represented by the equation

$$u_+u_- = \mu. \tag{6.9}$$

Complexifying $u_\pm$ one may regard (6.9) as the definition for a noncompact Riemann surface which may be covered by two patches $\mathcal{U}_\pm$ with coordinates $u_\pm$ respectively. Eqn. (6.9) defines the transition between the patches $\mathcal{U}_+$ and $\mathcal{U}_-$.

In the quantized theory $u_\pm$ get represented by the operators

$$\mathsf{u}_\pm \;=\; \frac{1}{2}\lambda \mp i\partial_\lambda.$$

One may introduce representations for the single particle Hilbert space in which either u_+ or u_- are diagonal. The representation of eigenfunctions of the hamiltonian $\mathsf{h} = -\mathsf{u}_+\mathsf{u}_- - \mathsf{u}_-\mathsf{u}_+$ becomes very simple,

$$\zeta_\pm^s(\omega|u_\pm) \;=\; \frac{1}{\sqrt{2\pi}}\Theta(su_\pm)|u_\pm|^{\pm i\omega-\frac12}\,. \tag{6.10}$$

Thanks to the fact that $[\mathsf{u}_+, \mathsf{u}_-] = -i$ one may realize the unitary operator between these two representations simply as Fourier transformation,

$$\phi(u_+) = \frac{1}{\sqrt{2\pi}} \int du_- \; e^{-iu_+u_-} \phi(u_-). \tag{6.11}$$

The relation between the formalism introduced in this paper and the light-cone formalism of [16] follows from the fact that the unitary transformation from wave-functions $\psi(\lambda)$ to their time asymptotics $\phi_\pm(u_\pm)$ diagonalizes the operators $\mathsf{u}_\pm$. This is proven in appendix A.2.

It is not hard to set up a formalism for the second quantized theory in terms of the variables $u_\pm$. Key ingredients of this formalism will be the fermionic field operators $\Psi_\pm(u_\pm)$. It is useful to associate the two field operators with the corresponding patches $\mathcal{U}_\pm$. It follows from (6.11) that the operators $\Psi_+(u_+)$ and $\Psi_-(u_-)$ are also related by Fourier transformation,

$$\Psi_+(u_+) = \frac{1}{\sqrt{2\pi}} \int du_- \; e^{iu_+u_-} \Psi_-^s(u_-). \tag{6.12}$$

In order to treat deformed backgrounds of the euclidean along the lines of [16] one may start from the following key idea: *The deformations can be described by a change of the vacuum in which to calculate expectation values* only. The deformation should therefore not change any of the relations which characterize the operator algebra of the theory including the relation between in- and out fields (6.12).

The deformation will induce, however, a deformation of the energy eigenfunctions (6.10) which appear in the expansion of the Fermi-fields into creation- and annihilation operator with a specific energy. The authors of [16] propose that the deformation of the energy eigenfunctions will take the form[6]

$$\zeta_\pm^\theta(\omega|u_\pm) \;=\; \frac{1}{\sqrt{4\pi}} \, e^{i\theta_\pm(\omega|u_\pm)} u_\pm^{\pm i\omega - \frac{1}{2}}\,, \tag{6.13}$$

with phases $\theta_\pm(\omega|u_\pm)$ that are of the form

$$\theta_\pm(\omega|u_\pm) \;=\; \frac{1}{2}\theta_0(\omega) + \sum_{k\geq 1} t_{\pm k}\, u_\pm^{\frac{k}{R}} - \sum_{k\geq 1} \frac{1}{k} v_{\pm k}(\omega) u_\pm^{-\frac{k}{R}}. \tag{6.14}$$

A basic idea behind this proposal is that the "field" $\theta_\pm(\omega|u_\pm)$ should essentially coincide with the expectation value of the bosonic fields $S_\pm(u_\pm)$ obtained by bosonizing the fermionic fields $\Psi_\pm(u_\pm)$. More precisely, the relation with the deformed partition function $\mathcal{Z} = \mathcal{Z}(\{t_k\}; \mu, R)$ is expected to be

$$\theta_\pm(\omega|u_\pm) \;=\; \frac{1}{\mathcal{Z}} \left(\frac{1}{2}\frac{\partial}{\partial\mu} + \sum_{k\geq 1} t_{\pm k}\, u_\pm^{\frac{k}{R}} - \sum_{k\geq 1} u_\pm^{-\frac{k}{R}} \frac{1}{k}\frac{\partial}{\partial t_{\pm k}} \right) \mathcal{Z}. \tag{6.15}$$

In order for (6.12) to remain valid in the deformed theory one then needs that the deformed energy eigenfunctions are related by

$$\zeta_+^\theta(\omega|u_+) \;=\; \frac{1}{\sqrt{2\pi}} \int du_- \; e^{-iu_+u_-} \zeta_-^\theta(\omega|u_-) \tag{6.16}$$

[6] For the ease of notation we restrict ourselves to $u_\pm > 0$ in the following.

This defines an intricate problem. Regarding the coefficients t_k as given input data one finds from (6.16) a non-trivial set of relations between the parameters t_k and the coefficients $v_{\pm k}(\omega)$. One may expect that these relations can generically be solved to uniquely to define $v_{\pm k}(\omega)$ as a function of the t_k. The deformed partition function $\mathcal{Z}$ is then defined via (6.15), where the integrability of these equations follows from the observation that a solution to this problem defines a particular solution of the Toda integrable hierarchy [16].

All this can be understood much more concretely in the classical limit $\mu \to \infty$. In this case one may evaluate (6.16) via the saddle point method [16], leading to the conditions

$$u_+ u_- = \begin{cases} u_+ \partial_+ S_+(u_+) \equiv \mu + \sum_{k\geq 1} k\, t_{+k}\, u_+^{\frac{k}{R}} + \sum_{k\geq 1} v_{+k}\, u_+^{-\frac{k}{R}} \\ u_- \partial_- S_-(u_-) \equiv \mu + \sum_{k\geq 1} k\, t_{-k}\, u_-^{\frac{k}{R}} + \sum_{k\geq 1} v_{-k}\, u_-^{-\frac{k}{R}} \,. \end{cases} \tag{6.17}$$

The coefficients v_k are now defined as functions of the t_k by the mutual consistency of the two equations in (6.17), see [16] for details. Having chosen the v_k in such a way that the equations (6.17) are consistent one may view either of these equations as the defining equation for a Riemann surface that is obtained as a deformation of the surface (6.9).

It seems worth remarking that the corresponding classical free energy $\mathcal{F}_{\rm cl}$, defined by

$$v_k = -\frac{\partial}{\partial t_k} \mathcal{F}_{\rm cl} \,, \tag{6.18}$$

defines a natural Kähler potential, whose associated symplectic form identifies the coefficients v_k as the dual momenta to the coordinates t_k for the space of deformations of the surface (6.9). This line of thought naturally leads to the proposal that the partition function $\mathcal{Z}$ of the quantized theory can be interpreted as the wave-function of a particular state in the quantization of the symplectic space with Kähler potential $\mathcal{F}_{\rm cl}$. The Fourier transformation (6.12) may then be regarded as the natural quantum counterpart of the transition between the patches $\mathcal{U}_+$ and $\mathcal{U}_-$. This point of view is strongly supported by the observation from [16] that the Fourier transformation (6.12) reduces to (6.17) in the classical limit.

A very similar framework was shown in [27] to follow from a general formalism for solving the topological B-model on certain classes of noncompact Calabi-Yau manifolds. The case of the $c = 1$ string corresponds to the hypersurface

$$zw - H(p, \lambda) = 0, \quad H(p, \lambda) = p^2 - \frac{1}{4}\lambda^2 - \mu.$$

In this context one interprets the fermionic fields $\Psi_\pm(u_\pm)$ as representatives for topological D-branes that may be present in the relevant Calabi-Yau geometry. These branes are parameterized by points of the surface $H(p, \lambda) = 0$, or alternatively by the corresponding values of the coordinates $u_\pm$.

6.3 Comparison

Although this has not been shown non-perturbatively yet, it seems very likely that the formalisms outlined in the previous two subsections are ultimately all equivalent. One way to establish this is to observe that all these formalisms produce solutions of the equations of the Toda hierarchy with initial conditions given by the partition function of the undeformed two-dimensional string theory background.

In any of these formalisms an important role is played by the one-point functions of the fermionic fields, which will be denoted as

$$\left\langle \Psi_{\pm}(u_{\pm}) \right\rangle_{\{t_k\};\mu,R.} \tag{6.19}$$

By using standard bosonization formulae it is then straightforward to show that e.g.

$$\left\langle \Psi_{-}(u_{-}) \right\rangle_{\{t_k\};\mu,R} = \exp\left(\sum_{k\geq 1} \frac{1}{k}\, t_k\, u_-^{-\frac{k}{R}} \right) \mathcal{Z}\Big(\Big\{ t_k + \frac{i}{k} u_-^{\frac{k}{R}} \Theta(-k) \Big\} \, ; \, \mu, R \Big). \tag{6.20}$$

Following [27] one may read this as follows: *Insertion of a topological B-brane at position u_- generates the shift*

$$t_k \to +\frac{i}{k}\, u_-^{\frac{k}{R}}\, \Theta(-k)$$

of the closed string background.

The analogy between (6.20) and (5.31) should be clear. But our discussion also shows that the relation between (6.20) and (5.31) is more than just an analogy: Bear in mind that the coordinates $u_{\pm}$ with which we describe the in- and out states are *identical* with the light cone coordinates which play a central role in the euclidean formalisms. It follows that the fermionic fields $\Psi_{\pm}(u_{\pm})$ of these formalisms are nothing but the euclidean counterparts of the fermionic in- and out fields in the minkowskian formalism used in this paper. One of our main results is to show that the insertion of fermionic in-fields describes decaying ZZ-branes. This finally leads us to propose that *the topological B-branes of [27] are the euclidean counterparts of the rolling ZZ-branes.*

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A. Free fermionic field theory revisited

A.1 Single particle quantum mechanics

As emphasized in [16], it is convenient to start by representing the single particle hamiltonian h in terms of the "light-cone variables" $\mathsf{u}_\pm = \frac{1}{2}\lambda \pm i\partial_\lambda$,

$$\mathsf{h} = -\mathsf{u}_+\mathsf{u}_- - \mathsf{u}_-\mathsf{u}_+.$$

There exist representations for the Hilbert space $\mathcal{K}$ of the single particle problem in which either u_+ or u_- are represented as multiplication operators. Observing that h is the generator of dilatations of the coordinates $u_\pm$ it becomes easy to find a complete set of eigenfunctions for h,

$$\zeta^s_\pm(\omega|u_\pm) \;=\; \frac{1}{\sqrt{2\pi}}\Theta(su_\pm)|u_\pm|^{\pm i\omega-\frac{1}{2}}\,, \qquad s\in\{+,-\}, \quad \omega\in\mathbb{R}. \tag{A.1}$$

These representations are related to the usual Schrödinger representation by means of integral transformations of the form

$$\phi_\pm(u_\pm) \;=\; \int_{\mathbb{R}} d\lambda\; \mathrm{M}_\pm(u_\pm|\lambda)\,\psi(\lambda), \tag{A.2}$$

with kernels $\mathrm{M}_\pm(u_\pm|\lambda) \equiv \langle u_\pm|\lambda\rangle$ given by the explicit formulae

$$\mathrm{M}_+(u_+|\lambda) \;=\; e^{-i\frac{\pi}{4}}e^{\frac{i}{2}u_+^2 - i\lambda u_+ + \frac{i}{4}\lambda^2}, \quad \mathrm{M}_-(u_-|\lambda) \;=\; \big(\mathrm{M}_+(u_-|\lambda)\big)^*. \tag{A.3}$$

This claim is easily verified using the fact that the kernels $\mathrm{M}_\pm(u_\pm|\lambda)$ satisfy the differential equations

$$\left(\pm i\partial_\lambda + \frac{1}{2}\lambda\right)\mathrm{M}_\pm(u_\pm|\lambda) \;=\; u_\pm \mathrm{M}_\pm(u_\pm|\lambda).$$

When working in the Schrödinger representation one may construct a convenient set of eigenfunctions for the single particle hamiltonian h by applying the inverse of the transformation (A.2) to the eigenfunctions (A.1). In this way one may construct in particular

$$G(\omega|\lambda) \;=\; \frac{e^{-\frac{\pi}{2}\omega - i\frac{\pi}{4}}}{\Gamma\big(\frac{1}{2}-i\omega\big)}\,e^{i\frac{\lambda^2}{4}} \int_0^{\infty+i\epsilon} d\sigma\;\sigma^{-i\omega-\frac{1}{2}}\,e^{i\lambda\sigma+\frac{i}{2}\sigma^2}. \tag{A.4}$$

$G(\omega|\lambda)$ is an eigenfunction of h with eigenvalue ω which has particularly simple asymptotics for $\lambda\to+\infty$, namely

$$G(\omega|\lambda) \underset{\lambda\to\infty}{\sim} e^{i\frac{\lambda^2}{4}}\lambda^{i\omega+\frac{1}{2}}. \tag{A.5}$$

The functions $G(\omega|\lambda)$ are related to the standard parabolic cylinder functions $U(a,x)$ [28] via

$$G(\omega|\lambda) = e^{-\frac{\pi}{4}\omega - i\frac{\pi}{8}}\,U\big(-i\omega, \lambda e^{-i\frac{\pi}{4}}\big). \tag{A.6}$$

Three further solutions with simple asymptotic behavior can be obtained as $G(\omega|-\lambda)$, $G^*(\omega|\lambda)$, $G^*(\omega|-\lambda)$, where the asterisk denotes complex conjugation. A normalized set of real parity eigenfunctions is finally constructed as

$$\mathrm{F}_p(\omega|\lambda) \equiv \frac{1}{\sqrt{2\pi}}\Big(m_p(\omega)G(\omega|\lambda) + m_p^*(\omega)G^*(\omega|\lambda)\Big)\,, \tag{A.7}$$

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where the coefficients $m_p(\omega)$ are defined as

$$\mathrm{m}_p(\omega) \;\equiv\; \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\frac{k(\omega)-ip}{\sqrt{k^2(\omega)+1}}\left(\frac{\Gamma(\frac{1}{2}-i\omega)}{\Gamma(\frac{1}{2}+i\omega)}\right)^{\frac{1}{4}}, \tag{A.8}$$

with $k(\omega)=\sqrt{1+e^{-2\pi\omega}}-e^{-\pi\omega}$. The label $p=\pm$ is identified with the parity eigenvalue of $\mathrm{F_p}(\omega|\lambda)$. The functions $\mathrm{F_p}(\omega|\lambda)$ have asymptotics

$$\mathrm{F}_p(\omega|\lambda) \underset{|\lambda|\to\infty}{\sim} \frac{1}{\sqrt{2\pi|\lambda|)}}\Big(e^{\frac{i}{4}\lambda^2}\, e^{i\omega\ln|\lambda|}\,\mathrm{M}_p^s(\omega)+e^{-\frac{i}{4}\lambda^2}\, e^{i\omega\ln|\lambda|}\,\overline{\mathrm{M}}{}_p^s(\omega)\Big), \tag{A.9}$$

where $\mathrm{M}_p^s(\omega)=s^{\Theta(-p)}\mathrm{m}_p(\omega)$ with $\Theta(-p)$ being the usual step function. It is known [14] that the functions $\mathrm{F_p}(\omega|\lambda)$ fulfil the following orthogonality and completeness relations :

$$\begin{aligned}\int_{\mathbb{R}} d\lambda\;\mathrm{F}_{p_1}(\omega_1|\lambda)\mathrm{F}_{p_2}(\omega_2|\lambda) \;&=\; \delta_{p_2p_1}\,\delta(\omega_1-\omega_2),\\ \int_{\mathbb{R}} d\omega\;\big(\,\mathrm{F}^+(\omega|\lambda_1)\mathrm{F}^+(\omega|\lambda_2)+\mathrm{F}^-(\omega|\lambda_1)\mathrm{F}^-(\omega|\lambda_2)\,\big) \;&=\; \delta(\lambda_1-\lambda_2).\end{aligned} \tag{A.10}$$

A.2 Asymptotics of wave-packets

Claim: *The asymptotics of a wave-packet $\psi(\lambda,t)$ for $t\to\pm\infty$ is of the form*

$$\psi(\lambda,t) \underset{t\to\pm\infty}{\frown} (2\pi)^{-\frac{1}{2}}e^{\frac{i}{4}\lambda^2}\,e^{\mp\frac{t}{2}}\,\phi_\pm(u_\pm), \tag{A.11}$$

where $u_\pm\equiv\lambda e^{\mp t}$. The asymptotic wave-functions $\phi_\pm(u_\pm)$ can be calculated from the wave function $\psi(\lambda)\equiv\psi(\lambda,0)$ by means of the integral transformations (A.2).

Proof: We may represent $\psi(\lambda,t)$ as

$$\psi(\lambda,t) \;=\; \int d\omega\; e^{-i\omega t}\,\mathbf{F}(\omega|\lambda)\cdot\tilde{\psi}(\omega), \quad \tilde{\psi}(\omega)\equiv\int_{\mathbb{R}}\mathrm{d}\lambda\,\mathbf{F}(\omega|\lambda)\psi(\lambda), \tag{A.12}$$

Standard stationary phase arguments show that $\psi(\lambda,t)$ will vanish rapidly at any fixed λ when $|t|\to\infty$. We should therefore regard the asymptotics where $|\lambda|$ tends to ∞ as well. In this case we may replace the wave-functions $\mathbf{F}(\omega|\lambda)$ by their leading asymptotics for $|\lambda|\to\infty$ as given in equation (A.9).

Only the term containing the factor $e^{-i\omega(t\mp\ln|\lambda|)}$ will contribute in the limit $t\to\pm\infty$. This is enough to establish the first half of our claim, equation (A.11), with the two-component vector $\phi_\pm(u_\pm)$ formed out of the functions $\phi^s_\pm(u_\pm)$, $s=\pm$ given by

$$\phi_\pm(u_\pm) \;=\; \int_{\mathbb{R}} d\omega\; u_\pm^{\pm i\omega}\,\mathbf{M}_\pm(\omega)\cdot\tilde{\psi}(\omega)\,, \qquad \begin{aligned}\mathbf{M}_+(\omega)&\equiv\mathbf{M}(\omega),\\ \mathbf{M}_-(\omega)&\equiv\overline{\mathbf{M}}(\omega),\end{aligned} \tag{A.13}$$

where $\mathbf{M}(\omega)$ is the matrix with matrix elements $\mathrm{M}_p^s(\omega)$, or explicitly

$$\mathbf{M}(\omega) \;\equiv\; \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\mathrm{m}_+(\omega) & 0\\ 0 & \mathrm{m}_-(\omega)\end{pmatrix}. \tag{A.14}$$

It remains to calculate the asymptotic wave-functions $\phi_\pm^s(t)$ more explicitly. To this aim let us consider $\mathbf{M}(\omega)\cdot\mathbf{F}(\omega|\lambda)$. By using (A.7), the expression that follows from (A.7) by $F_p(\omega|-\lambda) = (-)^{\Theta(-p)}F_p(\omega|\lambda)$ as well as $|m_p(\omega)|^2 = \frac{1}{2}$ one arrives at

$$\mathbf{M}(\omega)\cdot\mathbf{F}(\omega|\lambda) \;=\; \frac{1}{2\sqrt{2\pi}}\big(m_+^2(\omega)-m_-^2(\omega)\big)\begin{pmatrix} G(\omega|-\lambda)\\ G(\omega|+\lambda)\end{pmatrix}.$$

The factor $m_+^2(\omega)-m_-^2(\omega)$ equals $2(2\pi)^{-\frac{1}{2}}e^{\frac{\pi\omega}{2}}\Gamma\big(\frac{1}{2}-i\omega\big)$, leading us to

$$\mathbf{M}(\omega)\cdot\mathbf{F}(\omega|\lambda) \;=\; \frac{1}{2\pi}e^{-i\frac{\pi}{4}+\frac{i}{4}\lambda^2}\int_0^{\infty+i\epsilon} d\sigma\;\sigma^{-i\omega-\frac{1}{2}}e^{\frac{i}{2}\sigma^2}\begin{pmatrix} e^{-i\lambda\sigma}\\ e^{+i\lambda\sigma}\end{pmatrix}.$$

This should then be inserted into (A.13). After exchanging the integrations one can easily do the integration over ω, thereby producing a delta-function. This straightforwardly yields our claim that $\phi_\pm$ are given by the integral transformation (A.2). □

A.3 In- and Out-fields

Our next aim is to study the asymptotics for $t\to\pm\infty$ of the fermionic operators

$$\Psi^\dagger[\,\psi\,|\,t\,) \;\equiv\; \int d\lambda\;\psi(\lambda)\,\Psi^\dagger(\lambda,t)\,. \tag{A.15}$$

Introducing the operators $\mathbf{d}_\pm^\dagger(\omega)$ by $\mathbf{d}_\pm^\dagger(\omega)=\mathbf{M}_\pm^\dagger(\omega)\cdot\mathbf{c}^\dagger(\omega)$ allows us to write

$$\begin{aligned}\Psi^\dagger[\,\psi\,|\,t\,) \;&=\; \int_{\mathbb{R}} d\omega\;e^{-i\omega t}\,\mathbf{c}^\dagger(\omega)\cdot\tilde{\psi}(\omega)\\ &=\; \int_{\mathbb{R}} d\omega\;e^{-i\omega t}\,\mathbf{d}_\pm^\dagger(\omega)\cdot\tilde{\phi}_\pm(\omega),\end{aligned} \tag{A.16}$$

where the definition of $\tilde{\phi}_\pm(\omega)$ can be read off from (A.13). Using (2.20) it is then straightforward to rewrite the result in the following form:

$$\Psi^\dagger[\,\psi\,|\,t\,) \;=\; \int\frac{du}{2\pi}\;\phi_\pm(ue^{\mp t})\,\Psi_\pm^\dagger(u)\,. \tag{A.17}$$

This clearly identifies the fermionic fields $\Psi_\pm^\dagger(u)$ as the in- and out-fields.

A.4 Relation between bosonic in- and out-oscillators

We want to demonstrate the validity of the expansion

$$\begin{aligned}[\,\mathrm{a}_{\rm out}^s(\omega)\,]_{\rm o,in} \;=\; \sum_{s'=\mathrm{L,R}}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega_1}{\omega_1}\int_{\omega_1}^{\infty}\frac{d\omega_2}{\omega_2}\cdots\int_{\omega_{n-1}}^{\infty}\frac{d\omega_n}{\omega_n}\times\\ \times\;R_{(n)}^{ss'}(\,\omega\,|\,\omega_1,\dots,\omega_n\,)\,\mathrm{a}_{\rm in}^{s'}(\omega_1)\cdots\mathrm{a}_{\rm in}^{s'}(\omega_n)\,.\end{aligned} \tag{A.18}$$

This formula is a nonperturbative generalization of Polchinski's result [18] for the classical limit $\mu^{-1} \to 0$. The coefficients $R^{ss'}_{(n)}$ are of the form $R^{ss'}_{(n)}(\omega|\omega_1,\dots,\omega_n) = 2\pi\delta(\omega - \sum_{r=1}^{s}\omega_s)Q^{ss'}_{(n)}(\omega_1,\dots,\omega_n)$, where

$$Q^{ss'}_{(n)}(\,\omega_1,\dots,\omega_n\,) \;=\; \int_{-\infty}^{\infty} \frac{d\tau}{i\tau}\; e^{-i\mu\tau} K^{ss'}(\omega|\tau) \prod_{r=1}^{n} 2i\sin\tfrac{\omega_r\tau}{2}. \tag{A.19}$$

The kernels $K^{ss'}(\omega|t)$ are given by the following integrals:

$$K^{ss'}(\,\omega\,|\,t\,) = \int_{-\infty}^{\infty} d\omega'\; e^{i\omega' t}\, \bar{\mathrm{R}}^{ss'}(\tfrac{\omega'-\omega}{2})\mathrm{R}^{ss'}(\tfrac{\omega'+\omega}{2}) = \begin{cases} J_{-i\omega}(+2ie^t) & \text{if} \quad s = s', \\ J_{-i\omega}(-2ie^t) & \text{if} \quad s \neq s'\,. \end{cases} \tag{A.20}$$

Proof of (A.18): We start from the expression

$$[\,\mathrm{a}^s_{\mathrm{out}}(\omega)\,]_{\mathrm{o,in}} \;=\; 2\int_{-\infty}^{\infty} d\omega' \sum_{s'=\mathrm{L,R}} \bar{R}^{ss'}(\tfrac{\omega'-\omega}{2})R^{ss'}(\tfrac{\omega'+\omega}{2})\, \mathrm{d}^{\dagger\, s'}_{\mathrm{in}}(\tfrac{\omega'-\omega}{2})\mathrm{d}^{s'}_{\mathrm{in}}(\tfrac{\omega'+\omega}{2}). \tag{A.21}$$

By inserting

$$\mathrm{d}^{\dagger\, s'}_{\mathrm{in}}(\omega) = \int \frac{dx_1}{2\pi}\, e^{i\omega x_1}\, \bar{\Psi}^{\dagger\, s'}_{\mathrm{in}}(x_1), \quad \mathrm{d}^{s'}_{\mathrm{in}}(\omega) = \int \frac{dx}{2\pi}\, e^{i\omega x_2}\, \bar{\Psi}^{s'}_{\mathrm{in}}(x_2)\,,$$

where $\bar{\Psi}^s_{\mathrm{in}}(x) \equiv e^{\frac{x}{2}}\Psi^s_{\mathrm{in}}(e^x)$, exchanging the order of integrations and changing variables to $x_1 = x + \frac{\tau}{2}$, $x_2 = x + \frac{\tau}{2}$, we arrive at the expression

$$\begin{aligned}[\,\mathrm{a}^s_{\mathrm{out}}(\omega)\,]_{\mathrm{o,in}} &= \\ &= \int \frac{dx}{2\pi}\, e^{-i\omega x} \int \frac{d\tau}{2\pi} \sum_{s'=\mathrm{L,R}} K^{ss'}(\,\omega\,|\,\tau\,)\, \Psi^{\dagger\, s'}_{\mathrm{in}}(x+\tfrac{\tau}{2})\Psi^{s'}_{\mathrm{in}}(x-\tfrac{\tau}{2}),\end{aligned} \tag{A.22}$$

where the kernel $K^{ss'}(\,\omega\,|\,\tau\,)$ is the one defined in (A.19). For the product of fermionic field operators which appears in (A.22) we may use the bosonization formula

$$\begin{aligned}\Psi^{\dagger\, s}_{\mathrm{in}}(x+\tfrac{\tau}{2})\Psi^s_{\mathrm{in}}(x-\tfrac{\tau}{2}) &= \\ &= \frac{1}{i\tau} e^{-i\mu\tau} e^{i(T^s_<(x+\frac{\tau}{2}) - T^s_<(x-\frac{\tau}{2}))} e^{i(T^s_>(x+\frac{\tau}{2}) - T^s_>(x-\frac{\tau}{2}))}\end{aligned} \tag{A.23}$$

Note that this formula is free from the infrared problems of the corresponding bosonization formula for a single fermionic field operator. It may be proven by using the partial bosonization formulae from §4.1.2, noting that the cut-off y may be removed in expressions that are bilinear in fermionic fields.

By using series expansions for the exponentials in (A.23) we then get

$$\begin{aligned}\Psi^{\dagger\, s}_{\mathrm{in}}(x+\tfrac{\tau}{2})\Psi^s_{\mathrm{in}}(x-\tfrac{\tau}{2}) &= \\ = \frac{e^{-i\mu\tau}}{i\tau} \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{(2i)^k}{l!(k-l)!} & \int_{-\infty}^{\infty} \frac{d\omega_1}{\omega_1}\cdots\frac{d\omega_k}{\omega_k}\, e^{i\bar{\omega}x} \prod_{r=1}^{k} \sin\tfrac{\omega_r\tau}{2} \times \\ & \times \mathrm{a}^{s-}_{\mathrm{in}}(\omega_1)\cdots\mathrm{a}^{s-}_{\mathrm{in}}(\omega_l)\mathrm{a}^{s+}_{\mathrm{in}}(\omega_{l+1})\cdots\mathrm{a}^{s+}_{\mathrm{in}}(\omega_k),\end{aligned} \tag{A.24}$$

where we have used the notation $\bar{\omega} = \sum_{r=1}^{k} \omega_r$ as well as $\mathrm{a}_{\mathrm{in}}^{s\pm}(\omega) \equiv \Theta(\pm\omega)\mathrm{a}_{\mathrm{in}}^{s}(\omega)$. Passing to integration over ordered sets of integration variables finally yields the formula

$$
\begin{aligned}
&\Psi_{\mathrm{in}}^{\dagger\, s}(x+\tfrac{\tau}{2})\Psi_{\mathrm{in}}^{s}(x-\tfrac{\tau}{2}) = \\
&= \frac{e^{-i\mu\tau}}{i\tau}\sum_{k=0}^{\infty}(2i)^k \int_{-\infty}^{\infty}\frac{d\omega_1}{\omega_1}\int_{\omega_1}^{\infty}\frac{d\omega_2}{\omega_2}\cdots\int_{\omega_{n-1}}^{\infty}\frac{d\omega_n}{\omega_n}\, e^{i\bar{\omega}x}\prod_{r=1}^{k}\sin\tfrac{\omega_r\tau}{2}\times \\
&\qquad\qquad \times\ \mathrm{a}_{\mathrm{in}}^{s}(\omega_1)\cdots\mathrm{a}_{\mathrm{in}}^{s}(\omega_k),
\end{aligned}
\tag{A.25}
$$

Inserting this into (A.22) and noting that the integration over x produces a delta-function completes our derivation of formula (A.18).

Finally we would like to show how to calculate the leading asymptotics for $\mu \to \infty$, $\frac{\omega_r}{\mu} \ll 1$ from the general formula (A.18). First, it is not hard to see that in this limit the integral which represents $Q_{(n)}^{ss'}$, cf. equation (A.19), is dominated by the contributions from small τ. Approximating $\sin\frac{\omega_r\tau}{2} \simeq \frac{1}{2}\omega_r\tau$ and inserting the explicit expression for the matrix elements $\mathrm{R}^{ss'}$ we arrive at

$$
\begin{aligned}
&Q_{(n)}^{ss'}(\omega_1,\dots,\omega_n) \simeq \\
&\simeq \prod_{r=1}^{n}(i\omega_r)\int_{\mathbb{R}}\frac{d\tau}{i\tau}\, e^{-i\tau\mu}\,\tau^n\int_{\mathbb{R}} d\omega' e^{\sigma\frac{\pi}{2}\omega'}\, e^{i\omega'\tau}\Gamma\big(\tfrac{1}{2}+\tfrac{i}{2}(\omega'-\omega)\big)\Gamma\big(\tfrac{1}{2}-\tfrac{i}{2}(\omega'+\omega)\big),
\end{aligned}
\tag{A.26}
$$

where $\sigma = -$ if $s = s'$, $\sigma = +$ otherwise. The integral over τ can be represented in terms of $\delta(\mu - \omega')$, allowing us to do the integral over ω' as well. This yields

$$
\begin{aligned}
&Q_{(n)}^{ss'}(\omega_1,\dots,\omega_n) \simeq \\
&\qquad \simeq \frac{1}{i}\prod_{r=1}^{n} i\omega_r \left(i\frac{\partial}{\partial\mu}\right)^{n-1} e^{-\sigma\frac{\pi}{2}\mu}\,\Gamma\big(\tfrac{1}{2}-\tfrac{i}{2}(\mu+\omega)\big)\Gamma\big(\tfrac{1}{2}+\tfrac{i}{2}(\mu-\omega)\big).
\end{aligned}
\tag{A.27}
$$

With the help of Stirling's formula it is not hard to show that

$$
e^{-\sigma\frac{\pi}{2}\mu}\Gamma\big(\tfrac{1}{2}-\tfrac{i}{2}(\mu+\omega)\big)\Gamma\big(\tfrac{1}{2}+\tfrac{i}{2}(\mu-\omega)\big) \underset{\mu\to\infty}{\simeq} \mu^{-i\omega}\begin{cases} 1 \text{ if } s = s' \\ 0 \text{ if } s \neq s'. \end{cases}
$$

By inserting this relation into (A.27) it becomes easy to verify (2.34).

B. Solitonic sectors - Non-existence of normalizable ground-states

Our aim is to prove that the sectors with nonzero fermion number do not have normalizable ground states. The basic idea is very simple: We should be able to decompose any state into energy eigenstates. A state $|\Omega\rangle\!\rangle_n$ could only be a ground state in the sector with fermion number $n \neq 0$ if it would get contributions from states with energy $-\mu$ only. Due to the fact that the single particle spectrum is purely continuous, one may suspect that the problem to construct normalizable states with energy $-\mu$ is similar to the problem to construct point-like localized states in a theory with purely continuous spectrum. There do not exist *normalizable* states of this type. However, one may be confused by the fact that we are certainly able to construct sequences of normalized vectors which have energy expectation values that converge to the vacuum expecation value $-\mu$. It may therefore be worth demonstrating in some detail that no such sequence of vectors can be convergent.

For simplicity let us restrict attention to the case in which one has only a single set of fermionic creation- and annihilation operators $\mathrm{c}(\omega)$, $\mathrm{c}^\dagger(\omega)$. As a preliminary remark let us observe that the sector $\mathcal{H}_f$ with fermion number f may be represented as

$$\mathcal{H}_f \simeq \int_{\mathbb{R}_+}^{\oplus} d\omega\, \mathcal{H}_f(\omega), \tag{B.1}$$

where $\mathcal{H}_f(\omega)$ is generated by expressions of the form

$$\begin{aligned}&\int\limits_{-\infty}^{-\mu} d\omega_1 \dots d\omega_m \int\limits_{-\mu}^{\infty} d\omega_1' \dots d\omega_{m+f}'\, \delta\Big(\omega - \sum_{r=1}^{m+f}\omega_r' + \sum_{s=1}^{m}\omega_s\Big)\times \\ &\qquad \times F(\omega_1,\dots,\omega_m;\omega_1',\dots,\omega_{m+f}')\mathrm{c}(\omega_1)\cdots\mathrm{c}(\omega_m)\mathrm{c}^\dagger(\omega_1')\cdots\mathrm{c}(\omega_{m+f}')|\,\mu\,\rangle\!\rangle.\end{aligned} \tag{B.2}$$

In the representation (B.1) one represents vectors $|\,\Psi\,\rangle\!\rangle_f \in \mathcal{H}_f$ by vector-valued functions $\Psi_f(\omega) \in \mathcal{H}_f(\omega)$. Upon choosing suitable normalizations one may assume that the scalar product takes the form

$$\langle\!\langle\,\Psi_f\,|\,\Phi_f\,\rangle\!\rangle = \int_{-\mu}^{\infty} d\omega\, \big(\,\Psi_f(\omega)\,,\,\Phi_f(\omega)\,\big)_f,$$

where $(.,.)_f$ is the scalar product in $\mathcal{H}_f(\omega)$.

We will consider sequences $(\Psi_n)_{n\in\mathbb{N}}$ of vectors in $\mathcal{H}_f$ such that

$$\text{(i)}\qquad \lim_{n\to\infty}\langle\!\langle\,\Psi_n\,|\,\mathsf{H}\,|\,\Psi_n\,\rangle\!\rangle = -\mu. \tag{B.3}$$

$$\text{(ii)}\qquad \langle\!\langle\,\Psi_n\,|\,\Psi_n\,\rangle\!\rangle = 1. \tag{B.4}$$

Our aim is to show that no such sequence converges, which means that there exists an $\epsilon > 0$ such that for any $n \in \mathbb{N}$ one can find $m > n$ for which $\|\Psi_n - \Psi_m\| > \epsilon$. Keeping in mind that $\|\Psi_n - \Psi_m\| = 2 - 2\mathrm{Re}\,\langle\!\langle\Psi_n|\Psi_m\rangle\!\rangle$ it suffices to show that for any fixed n, the sequence $(|\langle\!\langle\Psi_n|\Psi_m\rangle\!\rangle|)_{m\in\mathbb{N}}$ does not converge to 1.

So let us pick any $n \in \mathbb{N}$. Define $\delta > -\mu$ by

$$\int_{-\mu}^{\delta} d\omega\, \|\Psi_n(\omega)\|_f^2 = \frac{1}{2}\,. \tag{B.5}$$

We claim that for all $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $m > M$ we have

$$\int_{\delta}^{\infty} d\omega\, \|\Psi_m(\omega)\|_f^2 < \epsilon\,. \tag{B.6}$$

Indeed, if this was not the case we could find an $\epsilon > 0$ such that for all $M \in \mathbb{N}$ there exists $m > M$ with $\int_\delta^\infty d\omega\ \|\Psi_m\|^2 > \epsilon$, which implies that also

$$
\begin{aligned}
\langle\!\langle\, \Psi_m \,|\, \mathsf{H} \,|\, \Psi_m \,\rangle\!\rangle \geq & (-\mu) \int_{-\mu}^{\delta} d\omega\ \|\Psi_m(\omega)\|_f^2 + \delta \int_\delta^\infty d\omega\ \|\Psi_m(\omega)\|_f^2 \\
\geq & (-\mu) \left(1 - \int_\delta^\infty d\omega\ \|\Psi_m(\omega)\|_f^2 \right) + \delta \int_\delta^\infty d\omega\ \|\Psi_m(\omega)\|_f^2 \\
\geq & -\mu + (\delta + \mu) \int_\delta^\infty d\omega\ \|\Psi_m(\omega)\|_f^2 \\
> & -\mu + (\delta + \mu)\epsilon .
\end{aligned}
$$

Since $\delta + \mu > 0$ we would have a contradiction to the convergence of the energy expecation values, condition (B.3).

So let us now present an estimate for $|\langle\!\langle \Psi_n | \Psi_m \rangle\!\rangle|$ that holds for any m which satisfies (B.6).

$$
\begin{aligned}
|\langle\!\langle \Psi_n | \Psi_m \rangle\!\rangle| \leq & \left| \int_{-\mu}^{\delta} d\omega\ \big(\Psi_n(\omega), \Psi_m(\omega)\big)_f \right| + \left| \int_\delta^\infty d\omega\ \big(\Psi_n(\omega), \Psi_m(\omega)\big)_f \right| \\
\leq & \left(\int_{-\mu}^{\delta} d\omega\ \|\Psi_n(\omega)\|_f^2 \right)^{\frac{1}{2}} \left(\int_{-\mu}^{\delta} d\omega\ \|\Psi_m(\omega)\|_f^2 \right)^{\frac{1}{2}} + \\
& + \left(\int_\delta^\infty d\omega\ \|\Psi_n(\omega)\|_f^2 \right)^{\frac{1}{2}} \left(\int_\delta^\infty d\omega\ \|\Psi_m(\omega)\|_f^2 \right)^{\frac{1}{2}} \\
< & \frac{1}{2} + \frac{1}{4}\epsilon .
\end{aligned} \tag{B.7}
$$

To go from the first to the second line we have used the Cauchy-Schwartz inequality, to arrive at the last inequality we have used (B.5) and (B.6). This estimate will hold for any $m > M$ with $M \in \mathbb{N}$ being such that the validity of (B.6) is guaranteed for $m > M$. Since ϵ is at our disposal we are sure that $|\langle\!\langle \Psi_n | \Psi_m \rangle\!\rangle|$ will stay below $2/3 < 1$, say. This clearly shows that the sequence $(|\langle\!\langle \Psi_n | \Psi_m \rangle\!\rangle|)_{m \in \mathbb{N}}$ can not converge to 1.

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