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Local and Global Light Bending in Einstein's and other Gravitational Theories

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To remedy a certain confusion in the literature, we stress the distinction between local and global light bending. Local bending is a purely kinematic effect between mutually accelerating reference frames tracking the same signal, and applies via Einstein's equivalence principle exactly and equally in Newton's, Einstein's, Nordström's and other gravitational theories, independently of all field equations. Global bending, on the other hand, arises as an integral of local bending and depends critically on the conformal spacetime structure and thus on the specific field equations of a given theory.

KEY WORDS : Einstein's equivalence principle ; Nordström's theory

1. INTRODUCTION

The present paper is written in reaction to a false rumor that has a certain currency in the literature. This asserts that, since Einstein's equivalence principle is somewhat vague and heuristic, none of its conclusions can be fully trusted. In particular, its conclusion about light bending is held to be contradicted by Nordström's second theory [1] (for a modern account see Ref. 2) which contains the equivalence principle and is in effect based on conformally flat spacetime: it is alleged that because of the latter there can

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be *no* light bending in that theory.[3] Even Wolfgang Pauli, in his celebrated 1921 encyclopedia article on Relativity [4] makes statements that could be construed in the above sense.[4]

To clarify the situation, it is essential to make the distinction between *two types of light bending, local and global*. However obvious, this distinction has rarely been sufficiently stressed,[5] and this omission may lead to confusion.

2. LOCAL AND GLOBAL BENDING DISTINGUISHED

To specify local bending, we suppose a freely moving particle (which could be a photon) to be tracked from two relatively accelerated, nearly rigid, non-rotating frames of reference, F and $\tilde{F}$, instantaneously at rest with respect to each other. Then, if the particle moves uniformly and on a straight line in F, its spatial path in $\tilde{F}$ is curved, the curvature being determined by the acceleration of $\tilde{F}$ with respect to F and by the velocity of the particle with respect to $\tilde{F}$. This effect, first pointed out by Einstein in his famous review article on special relativity in 1907 [7], is *local*; it refers to an arbitrarily small neighbourhood of an event. It holds in Newtonian as well as in (special and general) relativistic kinematics and is independent of spacetime curvature. The connection of this kinematical effect with gravity and light is through Einstein's equivalence principle as applied to light: *at all events in spacetime there exist local inertial frames* (freely falling nonrotating "Einstein elevators") *in which light travels uniformly at velocity c*. In any theory accepting this principle there is local light bending in all reference frames that accelerate relative to the elevators, in particular in frames that are fixed in a stationary gravitational field. And this bending is in principle measurable.

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[3] For example, one can hardly interpret N. Straumann's remarks in Ref. 3, Section 1.2.3, p. 86, otherwise.

[4] Actually, Pauli got it right. In his comments on Nordström's theory, Pauli writes (loc. cit., p. 179) that in that theory "eine Strahlenablenkung im Schwerefeld findet nicht statt" — no deflection of light rays in a gravitational field takes place. Later, he writes (p. 180) that the theory contradicts experience since it gives no "Krümmung der Lichtstrahlen" — curvature of light rays. The last expression lends itself to misinterpretation, though from the context ("experience") it seems clear that Pauli all along refers to global bending, and then there is no problem.

[5] One of the few writers who have stressed this point is C. M. Will [5]. He writes (p. 111): "The first [local] contribution to the deflection is universal: it is the same in any theory compatible with the equivalence principle..." Another is one of us, W.R. Cf. penultimate paragraph on p. 21 of Ref. 6.

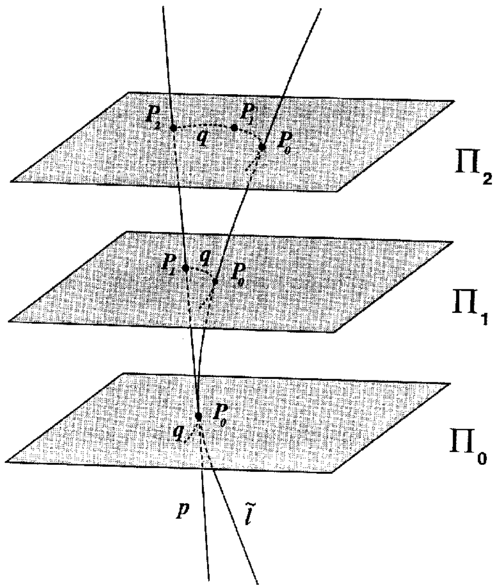

Figure 1(i).

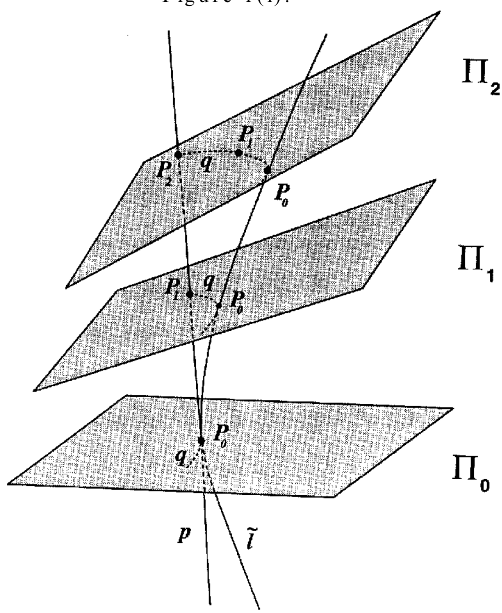

Figure 1(ii).

Figure 1. The geometry of local path bending in (i) Newtonian and (ii) relativistic kinematics.

A four-dimensional view of local bending — in Newton's theory and in general relativity, respectively — is shown in Figure 1. A particle or photon has a (geodesically) straight worldline p, while an accelerated observer has a curved worldline $\tilde{l}$. The hyper-"planes" Π_0, Π_1, Π_2 are orthogonal to $\tilde{l}$ and represent closely successive instants relative to $\tilde{l}$. They are parallel in Newton's theory, but not in relativity. The intersection point P of p with Π (now regarded as a single plane moving in time) traces out a curve q in Π whose curvature relative to Π is the measure of what we call local bending. In relativity the situation is apparently complicated by the non-parallelism of the Π's, which Einstein allowed for in a "tortured, yet sophisticated" (Ref. 8, p. 180) approximate argument without the benefit of 4-geometry in his 1907 paper [7]. But this turns out to be a "third order" correction having no effect on the result.

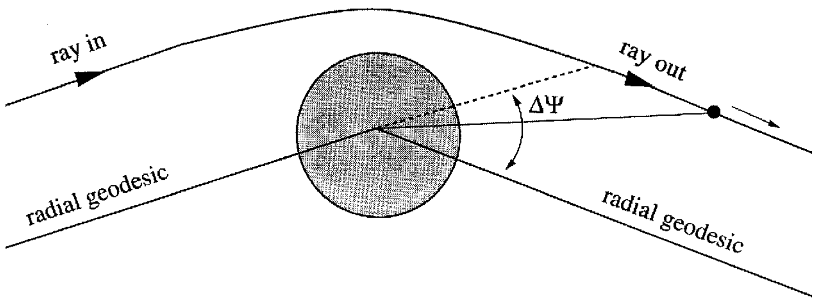

Figure 2. Global light bending in Schwarzschild spacetime

The second type of bending occurs when a light ray from a distant source traverses the gravitational field of a massive body and proceeds to a distant observer. Then, in general, the direction of the outgoing ray will differ from that of the incoming ray by some angle $\Delta\psi$. In order to define this deflection angle invariantly in curved spacetime, we consider first Schwarzschild spacetime. In that case, the spatial path of a light ray is well defined and has asymptotic "in" and "out" directions, defined in terms of the limiting positions of radial geodesics ending at points on the light path as the points are pushed towards infinity; see Figure 2. The deflection angle depends on the mass of the gravitating object and the distance of the light path from that object.

The definition of $\Delta\psi$ can be generalized to those (not necessarily stationary or symmetrical) weakly asymptotically flat spacetimes as defined in Ref. 9, sec. 9.6, which admit a unique, continuous null cone at spatial infinity.

This bending is clearly *global* and depends critically on gravity, namely on the conformal curvature of spacetime implied by the field equation of a given gravitational theory. Nonvanishing conformal curvature is also necessary for null cones to develop caustics and thus for the occurence of gravitational lensing, which is another important manifestation of global bending.

While in static fields global bending can be regarded as resulting from a "patching together" (integration) of all local bendings along the path of a light signal, the patching itself depends on the field equations, i.e. on how the local frames fit together (space curvature!).

In Nordström's or any other conformally-flat-spacetime theory global bending is absent. It is for this reason that Nordström's theory has been recognized (perhaps first by Roman Sexl) as a counterexample to the old and by now well discredited claim that the equivalence principle by itself (without field equations) implies the general-relativistic and empirically confirmed global bending of light.

3. LOCAL BENDING FROM THE ELEVATOR ARGUMENT

To discuss the local bending quantitatively, we introduce some standard geometric machinery. The center O of the freely falling elevator F is represented in spacetime by a geodesic worldline l (not shown in Figure 1), while the origin $\tilde{O}$ of the accelerated frame of reference $\tilde{F}$ is represented by an arbitrary worldline $\tilde{l}$ that is tangent to l at the event E in question. At that event, $\tilde{O}$ shall have proper acceleration g, which can be interpreted in $\tilde{F}$ as a gravitational field $-$g. Both frames are coordinatized by Fermi-transported (Ref. 10, Ch. II, sec. 10) (spatially normal) coordinates, say x, y, z, t and $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}$, centered on l and $\tilde{l}$ respectively. We choose the spatial coordinates so as to coincide in the hyperplane $\tilde{t} = t = 0$ through E, with the y-axes in the direction of g there. Then x, y, z, t are as close to inertial coordinates as one can get in a curved spacetime while $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}$ are nonrotating and nearly rigid; it is for this reason that we shall find it possible to use essentially Newtonian kinematics with small correction terms.

Suppose now that some particle, e.g. a photon, passing through E, has zero acceleration relative to the elevator at E, and a velocity, say of magnitude c, in the plane of x and y. Then we can specify its worldline p

near E by the equations

$$x = ct\cos\vartheta + O(t^3),$$
$$y = ct\sin\vartheta + O(t^3), \tag{1}$$
$$z = O(t^3),$$

ϑ being the inclination of the path to the "horizontal" in $\tilde{F}$.

Now from the standard theory of Fermi coordinates (see, for example, Ref. 10) we have, on l

$$g_{\alpha\beta} = \eta_{\alpha\beta}, \qquad \Gamma^{\alpha}_{\beta\gamma} = 0, \tag{2}$$

and, on $\tilde{l}$,

$$g_{\tilde\alpha\tilde\beta} = \eta_{\tilde\alpha\tilde\beta}, \qquad \Gamma^{\tilde y}_{\tilde t\tilde t} = g$$
$$\Gamma^{\tilde t}_{\tilde y\tilde t} = g \qquad \Gamma^{\cdot}_{\cdot\cdot} = 0 \quad \text{otherwise.} \tag{3}$$

The transformation law of the $\Gamma's$ then shows that at E, where $\partial\tilde{x}^{\alpha}/\partial x^{\beta} = \delta^{\alpha}_{\beta}$, we have

$$\Gamma^{\tilde\alpha}_{\tilde\beta\tilde\gamma} = -\frac{\partial^2 x^{\tilde\alpha}}{\partial x^{\beta}\partial x^{\gamma}}, \tag{4}$$

whence

$$\frac{\partial^2\tilde{x}^{\alpha}}{\partial x^{\beta}\partial x^{\gamma}} = 0 \qquad \text{except} \quad \frac{\partial^2\tilde{y}}{\partial t^2} = -g. \tag{5}$$

And this, in turn, implies the validity of the quasi-Newtonian coordinate transformation at E:

$$\left.\begin{aligned}\tilde{x} &= x\\ \tilde{y} &= y - \tfrac{1}{2}gt^2\\ \tilde{z} &= z\end{aligned}\right\} + \text{terms of third and higher order in } x, y, z, t. \tag{6}$$

The transformation of $\tilde{t}$ will not be needed. Observe that the relativistic c does then not enter the argument. From (1) and (6) we find for the path of the particle in $\tilde{F}$

$$\tilde{y} = \tilde{x}\tan\vartheta - \tfrac{1}{2}c^{-2}g\tilde{x}^2\sec^2\vartheta + O(\tilde{x}^3) \tag{7}$$

and thus for its curvature κ at E ($\tilde{x} = \tilde{y} = \tilde{z} = 0$)

$$\kappa = \frac{d^2\tilde{y}/d\tilde{x}^2}{(1 + (d\tilde{y}/d\tilde{x})^2)^{3/2}} = -\frac{1}{c^2}\,g\cos\vartheta, \tag{8}$$

exactly. So if the proper acceleration of $\tilde{O}$ is interpreted in $\tilde{F}$ as the negative of a gravitational field (as it would be, for example, for a point "at rest" in a stationary field) then eq. (8) tells us that the curvature vector κn (n = unit principal normal) of the spatial path of a free particle or a photon as observed in $\tilde{F}$ equals c^{-2} times the component of the field normal to the path.

This is what Einstein showed (almost rigorously) in 1907, except that we have had the benefit of 4-geometry and Fermi-coordinate theory to estimate the correction terms.

Note, incidentally, that one characterisation of a local inertial frame is now seen to be that light paths in *all* directions have zero curvature. Conversely the curvature of a curved light signal in a frame $\tilde{F}$ serves as a measure of the proper acceleration of $\tilde{F}$.

The formula (8) and its derivation also hold rigorously for local light bending in *Newtonian* static gravitational fields, if the kinematic assumption is made that in the *one* elevator that is momentarily at rest in absolute space "light corpuscles" always travel with speed c. *Dynamically*, this constancy of c would violate Newtonian energy conservation unless we enrich the model with $h\upsilon = mc^2$ and allow m to vary.

But, as we already said, eq. (8) holds independently of gravity and of light. It applies to the motion of all free "particles" as observed in accelerating reference frames, whether it be Newton's theory, Special Relativity, General Relativity, or indeed any metric theory of gravity such as Nordström's.

4. GLOBAL LIGHT BENDING REVISITED

To see intuitively how local bending is related to global bending in what are perhaps the three most interesting cases — the Newtonian, the Einsteinian, and the Nordströmian — we can proceed as follows.

First we note that even in Newton's theory, with "light corpuscles" moving only approximately at speed c, formula (8) will be of sufficient accuracy in "weak" fields like that of the sun. For example, for a corpuscle to get to infinity with speed c from near the sun it must start with velocity $c' = c\sqrt{1 + v_\odot^2/c^2}$, where $v_\odot = \sqrt{2GM_\odot/R_\odot}$ is the escape velocity from the sun. But this makes $c' \approx c$ to rather high accuracy.

Now imagine the following drawing: In the middle there is a circle representing the sun, somewhat as in Figure 2. At the top of this circle we draw a small piece of a tangent line — it will represent a light path grazing the sun. Then we continue this line in both directions by computation: using formula (8) for κ and the fact that κ, by definition, is the arc rate of turning of the tangent, $\kappa = d\psi/dl$, we can compute $\psi(l) = \int_0^l \kappa dl$ for the angle the curve makes with the horizontal at distance l from the center. The resulting path turns out to be essentially made up of two straight-line segments joined near the sun by an arc, somewhat like one branch of a hyperbola. The angle between the asymptotes, when we use the data for the sun, is $0''.87$ (see Appendix for the calculation). This is the "Newtonian" global bending of light: it is simply the integral of the local curvature.

In the case of a static spacetime in general relativity, the curvature of a light path as given by (8) equals the geodesic curvature of that path with respect to the spatial Riemannian metric of a t = const. hypersurface, since in such a spacetime one can choose coordinates $(\tilde{t}, \tilde{x}^a)$ such that at an arbitrary fixed point $\tilde{x}^a = 0$ and $g_{\tilde{t}\tilde{t}} = 1$, and such that $\tilde{x}^a$ are normal coordinates with respect to the spatial metric. Then $(\tilde{t}, \tilde{x}^a)$ define an accelerated frame of reference for that fixed point. Thus, one again obtains the "Newtonian" contribution to the global deflection angle due to the integrated curvature. There is, however, a second contribution:[6] We consider the bent Newtonian light path to be the central line of a narrow strip which we imagine to be cut out of the plane. This strip we now glue onto what is known as *Flamm's Paraboloid* (see, for example, Ref. 6). This is essentially an infinite plane with a circular funnel-shaped hole in the middle, somewhat like the wide end of a trumpet, and it represents the *real* geometry of the central plane of the sun's field in which the ray lies. A little experimenting with such a curved strip will quickly convince the reader that the depression in the middle will impart an extra amount to the total deflection of the path "from infinity to infinity". In fact, the Newtonian deflection is exactly doubled to $1''.74$.

In Nordström's theory, the real geometry of such a central plane can-

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[6] When Einstein in 1911 first recognized the possibility of observing global light bending by the solar gravitational field [11], he was well aware that this effect does *not* follow from his equivalence principle alone, which originally referred to static, homogenous fields only. To obtain the observable deflection angle he used and explicitly stated the additional assumption that the local bending formula applies pointwise also in an inhomogenous field, and he assumed implicitly that the spatial metric is euclidian. This last assumption he "corrected" without comment, almost in passing, in 1915, using his field equation [12].

not be represented by a surface of revolution. (Instead of having too much space near the center, which can funnel out, we now have too little.) Of course, we know the result in advance from conformal flatness: the global bending is now zero. More experimenting with the paper strip (holding its ends to one straight line on a table) will make it plausible that the Nordström 2-geometry of a central "plane" indeed correspons to (part of) an infinite plane far from the sun, but that near the sun there is a deficit rather than an excess of area.

To look at Nordström's theory a little more closely, we recall [2] that its spacetime has a metric of the form

$$\mathrm{d}s^2 = e^{2\Phi/c^2}(c^2 dt^2 - dx^2 - dy^2 - dz^2) =: e^{2\Phi/c^2}\,\mathrm{d}\tilde{s}^2, \tag{9}$$

where Φ is essentially the Newtonian potential. (In fact, in the case of spherical symmetry, $e^{\Phi/c^2} = 1 - Gm/rc^2$.) By a well-known theorem, the null geodesics of conformally equivalent spaces coincide. And since they are straight lines in $\mathrm{d}\tilde{s}^2$, there is no global bending in $\mathrm{d}s^2$, i.e. in Nordström's theory. But local bending there is! Suppose (9) refers to a static field, with Φ independent of t. Since light travels straight in $\mathrm{d}\tilde{s}^2$, its spatial tracks are the straight lines in the metric $d\tilde{l}^2 = dx^2 + dy^2 + dz^2$, and thus satisfy three equations like

$$x = a\tilde{l} + b. \tag{10}$$

But these tracks are *not* geodesics in the spatial lattice of (9) which has metric $dl^2 = e^{2\Phi/c^2}(dx^2 + dy^2 + dz^2)$. Geodesics in *this* lattice must satisfy three Euler–Lagrange equations like

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial x} = 0, \tag{11}$$

where $\mathcal{L} = e^{2\Phi/c^2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ and "$\cdot$" $= d/dl = e^{-\Phi/c^2}d/d\tilde{l}$; and it is easily seen that this will not be the case for (10) unless $\Phi =$ constant, i.e. unless there is no gravity. Hence in general the tracks have nonvanishing geodesic curvature; its exact value is given by our eq. (8).

Finally, a few numerical values may be of interest. As is well known, the value of g on earth in units of years and light-years is ≈ 1. By (8), therefore, the radius of curvature of a horizontal light path at the earth's surface is $\kappa^{-1} \approx 1$ light year; for a ray grazing the sun the value is smaller by a factor of $\approx 1/30$. Measuring such a minute curvature locally is, of course, out of the question. That the integrated effect nevertheless leads to the observable Einstein angle $\Delta\psi = 1''.74 \approx 10^{-5}$ is due to the considerable length d, of a few solar diameters, of that part of the path which contributes effectively to $\int \kappa dl$. For $d \approx 10^7$km we get $\Delta\psi \approx \kappa d \approx 10^{-12} \times 10^7 = 10^{-5}$, the right order of magnitude.

5. CONCLUSION

In conclusion we note that the local bending of light — though it can probably never be observed directly because of its smallness — is nevertheless one of only two *in principle* measurable non-classical gravitational effects that spring directly and rigorously from the equivalence principle, without use of field equations. The other is, of course, the by now well-validated local gravitational frequency-shift (Pound–Rebka–Snider Harvard Tower experiment.) Both effects are free of the frequently discussed difficulties one faces when trying to formulate the equivalence principle generally and rigorously, as should be clear from our use of the elevator argument which refers only to a restricted form of the principle. It is the field equation that determines the spatial geometry of the spatial lattice in static spacetimes and thus, in conjunction with asymptotic flatness and local bending, the global deflection of light.

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APPENDIX

To calculate the global "Newtonian" deflection of light, we first approximate the light path with a straight tangent line to the circle representing the sun. If ϑ is the angle between that line and the direction of the field g at any of its points, then $g = M_\odot \sin^2\vartheta / R_\odot^2$ in units making $c = G = 1$. Also, from (8), $\kappa = g \sin\vartheta$ and the distance along the line from the point of tangency is $l = R_\odot \cot\vartheta$, whence $dl = R_\odot \operatorname{cosec}^2\vartheta\, d\vartheta$ (omitting signs all along). Thus the Newtonian global bending angle is given by

$$\Delta\psi = \int_{-\infty}^{+\infty} \kappa\, dl = \frac{2M_\odot}{R_\odot} \int_0^{\pi/2} \sin\vartheta\, d\vartheta = \frac{2M_\odot}{R_\odot}\,. \tag{A.1}$$

This is just half of the general-relativistic value.

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