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Mean curvature evolution of entire graphs

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We consider immersions

$$\mathbf{F}: M^n \to \mathbf{R}^{n+1}$$

of n-dimensional hypersurfaces in $\mathbf{R}^{n+1}$. We say that M^n moves by mean curvature if there is a one-parameter family $\mathbf{F}_t = \mathbf{F}(\cdot, t)$ of immersions with corresponding images $M_t = \mathbf{F}_t(M)$ such that

$$(1) \qquad \frac{d}{dt}\mathbf{F}(p,t) = \mathbf{H}(p,t), \qquad p \in M,$$
$$\mathbf{F}(p,0) = \mathbf{F}_0(p)$$

is satisfied for some initial data $\mathbf{F}_0$. Here $\mathbf{H}(p,t)$ is the mean curvature vector of the hypersurface M_t at $\mathbf{F}(p,t)$.

Mean curvature flow was first studied by Brakke [1], in the context of geometric measure theory. Smooth compact surfaces moving by their mean curvature were investigated in [5], [6].

Here we shall assume that M can be written as an entire graph; i.e. there exists a vector $\omega \in \mathbf{R}^{n+1}$, $|\omega| = 1$, such that for a choice of unit normal ν for M we have

$$\langle \nu, \omega \rangle > 0$$

everywhere on M. Then the system

$$(2) \qquad \left(\frac{d}{dt}\mathbf{F}(p,t)\right)^{\perp} = \mathbf{H}(p,t)$$

which up to tangential diffeomorphisms is equivalent to (1), corresponds to the quasilinear equation

$$(2') \qquad \frac{d}{dt}u = \sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Here $\perp$ denotes the normal component of a vector, u is the graph representation for M_t with respect to the hyperplane defined by ω and D indicates differentiation in this hyperplane. In the graphical setting it is therefore possible to deal with problem (1) entirely from the viewpoint of partial differential equations. However, due to the geometric nature of equation (1) it turns out to be much more convenient to derive estimates directly from calculations on the hypersurfaces; see Sections 2, 3 and 4 below.

Our first result says that any polynomial growth rate for the height and the gradient of the initial surface M_0 is preserved during the evolution. We then show in Section 4 that in the case of Lipschitz initial data with linear growth, equation (1) has a smooth solution for all times. This result follows from a priori estimates for the curvature and higher derivatives of the curvature on M_t, which are global in space direction and interior in time direction. In Section 5 we study the asymptotic behaviour of these solutions as $t \to \infty$. We prove that our family of surfaces asymptotically approaches a *selfsimilar* solution of (1), provided the initial graph was "straight" at infinity in the sense that for some $\delta > 0$, $C < \infty$,

$$|\langle \mathbf{F}, \boldsymbol{\nu} \rangle| \leq C(1 + |\mathbf{F}|)^{1-\delta}. \tag{3}$$

More precisely, we show that after appropriate rescaling, the surfaces converge to a solution of the equation

$$\mathbf{F}^{\perp} = \mathbf{H}, \tag{4}$$

which characterizes *expanding* selfsimilar solutions of (1). This result should be compared with Theorem 3.5 in [6], where it was shown that singularities of the mean curvature flow behave asymptotically like *contracting* selfsimilar solutions of (1), characterized by the equation

$$\mathbf{F}^{\perp} = -\mathbf{H}. \tag{5}$$

While equation (4) has many non-trivial solutions, we show in the appendix that the only entire graphs satisfying equation (5) are planes.

Finally we prove in Section 6 that condition (3) is indeed necessary for asymptotic convergence. We give an example of a slowly oscillating initial surface violating (3) which does not converge asymptotically.

1. The monotonicity formula

In the following we shall not distinguish between the image $\mathbf{F}(p, t)$ of a point $p \in M$ and its coordinate vector $\mathbf{x} = \mathbf{x}(p, t)$. For a fixed point $(\mathbf{x}_0, t_0) \in \mathbf{R}^{n+1}$ we define the "backward heat kernel" $\rho = \rho(\mathbf{x}, t)$ by

$$\rho(\mathbf{x}, t) = \left(4\pi(t_0 - t)\right)^{-n/2} \exp\left(\frac{-|\mathbf{x}_0 - \mathbf{x}|^2}{4(t_0 - t)}\right), \qquad t_0 > t,$$

such that

$$\frac{d}{dt}\rho = -\Delta\rho + \rho\left(\frac{\langle \mathbf{x}_0 - \mathbf{x}, \mathbf{H}\rangle}{(t_0 - t)} - \frac{1}{4}\frac{\left|(\mathbf{x}_0 - \mathbf{x})^{\perp}\right|^2}{(t_0 - t)^2}\right).$$

Here and in the following, $\Delta = \Delta_t$ denotes the Laplace-Beltrami operator on M_t. It was shown in [6], Theorem 3.1, that this implies the monotonicity formula

$$\text{(6)} \qquad \frac{d}{dt}\int_{M_t}\rho\, d\mu_t = -\int_{M_t}\rho\left|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x} - \mathbf{x}_0)^{\perp}\right|^2 d\mu_t$$

where $d\mu_t$ is the measure on M_t and $\tau = t_0 - t$. Proceeding as in [6] we obtain more generally for a function $f = f(\mathbf{x}, t)$ on M that

$$\text{(7)} \quad \frac{d}{dt}\int_M f\rho\, d\mu_t = \int_M\left(\frac{d}{dt}f - \Delta f\right)\rho\, d\mu_t - \int_M f\rho\left|\mathbf{H} + \frac{1}{2\tau}(\mathbf{x} - \mathbf{x}_0)^{\perp}\right|^2 d\mu_t.$$

All integrals are finite and integration by parts is permitted for the surfaces and functions we are going to consider in the sequel.

1.1 COROLLARY. *Suppose the function $f = f(\mathbf{x}, t)$ satisfies the inequality*

$$\text{(8)} \qquad \left(\frac{d}{dt} - \Delta\right)f \le \mathbf{a}\cdot\nabla f$$

for some vector field $\mathbf{a}$, *where* ∇ *denotes the tangential gradient on* M. *If* $a_0 = \sup_{M\times[0,t_1]}|\mathbf{a}| < \infty$ *for some* $t_1 > 0$, *then*

$$\sup_{M_t} f \le \sup_{M_0} f$$

for all $t \in [0, t_1]$.

Proof. Let $k = \sup_{M_0} f$ and define $f_k = \max(f - k, 0)$. Then we derive from (8)

$$\left(\frac{d}{dt} - \Delta\right)f_k^2 \le 2f_k\mathbf{a}\cdot\nabla f_k - 2|\nabla f_k|^2.$$

Using Young's inequality we obtain, in the weak sense, that

$$\left(\frac{d}{dt} - \Delta\right)f_k^2 \le \frac{1}{2}a_0^2 f_k^2.$$

We may now employ (7) with f_k^2 instead of f and choose $t_0 > t$, $\mathbf{x}_0$ arbitrary in the definition of ρ to conclude

$$\frac{d}{dt}\int f_k^2\rho\, d\mu_t \le \frac{1}{2}a_0^2\int f_k^2\rho\, d\mu_t$$

which yields the desired result. □

2. A priori height estimates

We define the *height* of M with respect to the hyperplane orthogonal to ω by

$$u = \langle \mathbf{x}, \omega \rangle$$

and observe from (1) that

$$\left(\frac{d}{dt} - \Delta\right) u = 0. \tag{9}$$

In terms of a local orthonormal frame $\{\mathbf{e}_i\}_{1 \le i \le n}$ on M we then have the formula

$$\nabla u = \langle \mathbf{e}_i, \omega \rangle \mathbf{e}_i.$$

Notice that the function $\mathscr{u}$ mentioned in the introduction denotes the height of M over a fixed point $\hat{\mathbf{x}}$ in the hyperplane, whereas $u(p, t)$ denotes the height of $\mathbf{F}(p, t)$ for a fixed $p \in M$. It immediately follows from (9) and Corollary 1.1 that u stays uniformly bounded, if it is uniformly bounded at time $t = 0$. To deal with the case of polynomial growth, we need a technical lemma.

2.1 Lemma i). *The function $\eta_1 = \eta_1(\mathbf{x}, t)$ defined by*

$$\eta_1(\mathbf{x}, t) = |\mathbf{x}|^2 + 2nt$$

satisfies

$$\left(\frac{d}{dt} - \Delta\right) \eta_1 = 0.$$

ii) *The function $\eta_2 = \eta_2(\mathbf{x}, t)$ defined by*

$$\eta_2(\mathbf{x}, t) = 1 + |\mathbf{x}|^2 - u^2 + 2nt$$

satisfies for arbitrary p

$$\left(\frac{d}{dt} - \Delta\right) \eta_2^p = -\, p(p-1)|\nabla \eta_2|^2 \eta_2^{p-2} + 2p\eta_2^{p-1}|\nabla u|^2.$$

Proof. We have in view of (1)

$$\frac{d}{dt}\eta_1 = 2\langle \mathbf{x}, \mathbf{H} \rangle + 2n$$

and the first identity then follows from

$$\Delta \eta_1 = \mathbf{e}_i(2\langle \mathbf{x}, \mathbf{e}_i \rangle) = 2\langle \mathbf{x}, \mathbf{H} \rangle + 2n.$$

Since by (9)

$$\left(\frac{d}{dt} - \Delta\right)u^2 = -2|\nabla u|^2,$$

we have, using (i),

$$\left(\frac{d}{dt} - \Delta\right)\eta_2 = 2|\nabla u|^2$$

which implies the second identity. □

Now let M_t be a smooth solution of (1) which grows at most polynomially. We show that $u(\cdot, t)$ satisfies the same polynomial growth estimate as $u(\cdot, 0)$. Note that the nonnegative function $|\mathbf{x}|^2 - u^2$ measures distance in the hyperplane orthogonal to ω.

2.2 Proposition. *If for some $c_0 < \infty$, $p \geq 0$, the inequality*

$$u^2 \leq c_0\left(1 + |\mathbf{x}|^2 - u^2\right)^p$$

is satisfied on M_0, then for all $t > 0$,

$$u^2 \leq c_0\left(1 + |\mathbf{x}|^2 - u^2 + (2n + 4(p-1))t\right)^p.$$

Proof. From Lemma 2.1 we compute for

$$\eta = \eta(\mathbf{x}, t) = 1 + |\mathbf{x}|^2 - u^2 + (2n + 4(p-1))t$$

the evolution equation

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)u^2\eta^{-p} = &-2\eta^{-p}|\nabla u|^2 - p(p+1)\eta^{-p-2}|\nabla\eta|^2u^2\\ &-2p\eta^{-p-1}u^2|\nabla u|^2 - 4(p-1)p\eta^{-p-1}u^2\\ &-4p\eta^{-p-1}u\nabla u\cdot\nabla\eta.\end{aligned}$$

Using Young's inequality we obtain

$$|4p\eta^{-p-1}u\nabla u\cdot\nabla\eta| \leq 2\eta^{-p}|\nabla u|^2 + 2p^2u^2\eta^{-p-2}|\nabla\eta|^2.$$

Now observe that $\nabla_i u = \langle \mathbf{e}_i, \omega\rangle$ implies

$$\nabla_i\eta = 2\langle \mathbf{e}_i, \mathbf{x} - \langle \mathbf{x}, \omega\rangle\omega\rangle$$

which yields

$$|\nabla\eta|^2 \leq 4\eta.$$

Thus we derive

$$\left(\frac{d}{dt} - \Delta\right) u^2 \eta^{-p} \leq 0$$

and the result follows from Corollary 1.1. □

3. A priori gradient estimates

To ensure that M_t stays a graph for all times we have to estimate $\langle \nu, \omega \rangle$ from below or equivalently

$$v = \langle \nu, \omega \rangle^{-1}$$

from above. Let $A = \{h_{ij}\}$ be the second fundamental form.

3.1 LEMMA. *The quantity v satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2v^{-1} |\nabla v|^2.$$

Proof. From [5], Lemma 3.3, we know that $(d/dt)\nu = \nabla H$. Therefore

$$\frac{d}{dt} v = -v^2 \langle \nabla H, \omega \rangle.$$

On the other hand

$$\begin{aligned} \Delta v &= \mathbf{e}_i\left(-v^2 \langle \nabla_{\mathbf{e}_i} \nu, \omega \rangle\right) = \mathbf{e}_i\left(-v^2 \langle h_{il} \mathbf{e}_l, \omega \rangle\right) \\ &= -v^2 \langle \nabla H, \omega \rangle + v|A|^2 + 2v^{-1} |\nabla v|^2 \end{aligned}$$

and the conclusion follows.

3.2 COROLLARY. *If v is bounded at time $t = 0$, it remains bounded by the same constant.*

3.3 *Remark.* For the equation on $\mathbf{R}^n$

$$\frac{d}{dt} u = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right)$$

which describes mean curvature flow in e_{n+1} direction, a corresponding result is obtained in [2] from interior gradient estimates.

We can also derive polynomial estimates for v similar to those derived in Section 2.

3.4 PROPOSITION. *If for some $c_1 < \infty$, $p \geq 0$,*

$$v \leq c_1\left(1 + |\mathbf{x}|^2 - u^2\right)^p$$

at time $t = 0$, then for $t > 0$ the inequality

$$v(\mathbf{x}, t) \leq c_1\left(1 + |\mathbf{x}|^2 - u^2 + 2nt\right)^p$$

holds.

Proof. From Lemma 2.1 and Lemma 3.1 we compute

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) v\eta_2^{-p} = &- |A|^2 v\eta_2^{-p} - 2v^{-1}|\nabla v|^2\eta_2^{-p} \\ &- p(p+1)v|\nabla\eta_2|^2\eta_2^{-p-2} - 2pv\eta_2^{-p-1}|\nabla u|^2 \\ &- 2p\eta_2^{-p-1}\nabla v \cdot \nabla\eta_2.\end{aligned}$$

We estimate

$$\left|2p\eta_2^{-p-1}\nabla v \nabla\eta_2\right| \leq 2v^{-1}|\nabla v|^2\eta_2^{-p} + \tfrac{1}{2}p^2 v\eta_2^{-p-2}|\nabla\eta_2|^2$$

and the conclusion follows since $p \geq 0$. □

4. Curvature estimates and longtime existence

From now on we shall only consider the case of *linear growth*; i.e. we assume that for some fixed constant $c_1 \geq 1$ the inequality

$$v \leq c_1 \tag{10}$$

holds everywhere on M_0. Corollary 3.2 then ensures that (10) remains valid for all $t > 0$.

To guarantee longtime existence of a solution for the mean curvature flow, it is crucial to obtain a priori bounds for the second fundamental form on M_t. In Theorem 4.4 we derive uniform estimates for the curvature and all derivatives of the curvature which are interior in time and allow us to prove existence of a longtime smooth solution to the flow for Lipschitz initial data. Notice that these estimates do not follow from the standard quasilinear theory since u may be unbounded.

4.1 LEMMA. *The curvature satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right)|A|^2v^2 \leq -2v^{-1}\nabla v \cdot \nabla\left(|A|^2v^2\right).$$

Proof. From [5], Corollary 3.5, we have the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)|A|^2 = -2|\nabla A|^2 + 2|A|^4$$
$$\leq -2|\nabla|A||^2 + 2|A|^4.$$

Together with the identity

$$\left(\frac{d}{dt} - \Delta\right)v^2 = -2|A|^2v^2 - 6|\nabla v|^2$$

derived from Lemma 3.1 this yields

$$\left(\frac{d}{dt} - \Delta\right)|A|^2v^2 \leq -2|\nabla|A||^2v^2 - 6|\nabla v|^2|A|^2$$
$$-2\nabla|A|^2\cdot\nabla v^2.$$

Using Young's inequality we then estimate

$$\begin{aligned} -2\nabla|A|^2\cdot\nabla v^2 &\leq -\nabla|A|^2\cdot\nabla v^2 - 4v|A|\nabla|A|\cdot\nabla v \\ &= -v^{-2}\nabla v^2\cdot\nabla\left(|A|^2v^2\right) + v^{-2}|\nabla v^2|^2|A|^2 \\ &\quad - 4v|A|\nabla|A|\cdot\nabla v \\ &\leq -2v^{-1}\nabla v\cdot\nabla\left(|A|^2v^2\right) + 6|\nabla v|^2|A|^2 \\ &\quad + 2|\nabla|A||^2v^2 \end{aligned}$$

which implies the result. □

4.2 Corollary. *If M_t is a smooth solution of* (1) *with bounded gradient and bounded curvature on each M_t, then there is the a priori estimate*

$$\sup_{M_t}|A|^2v^2 \leq \sup_{M_0}|A|^2v^2.$$

Proof. Notice that

$$v^{-1}|\nabla v| \leq |A|v.$$

The result then follows from Lemma 4.1 and Corollary 1.1 with $\mathbf{a} = -2v^{-1}\nabla v$. □

Using the uniform estimate on $|A|^2$ we can now proceed exactly as in [6], Proposition 2.3, to estimate all derivatives of A in terms of their initial data.

4.3 Proposition. *If M_t is a smooth solution of* (1) *such that $v, |A|^2$, $|\nabla A|^2, \ldots, |\nabla^m A|^2$ are bounded on each M_t, then we have for all $t \geq 0$ the* *a priori estimate*

$$\sup_{M_t} |\nabla^m A| \le C(m)$$

where $C(m)$ *only depends on* m, n, c_1 *and* $\sup_{M_0} |\nabla^j A|$ *for* $0 \le j \le m$.

We now derive estimates interior in time for the curvature and all its derivatives.

4.4 PROPOSITION. *Let* M_t *be a smooth solution of* (1) *satisfying* (10). *Then for each* $m \ge 0$ *there is a constant* $C(m)$ *depending only on* c_1, n *and* m *such that*

$$t^{m+1} |\nabla^m A|^2 \le C(m) \tag{11}$$

holds uniformly on M_t.

Proof. To establish the case $m = 0$ we compute from Lemma 3.1 and Lemma 4.1

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)(2t|A|^2 v^2 + v^2) &\le -2v^{-1} \nabla v \cdot \nabla(2t|A|^2 v^2) - 6|\nabla v|^2 \\ &\le -2v^{-1} \nabla v \cdot \nabla(2t|A|^2 v^2 + v^2).\end{aligned}$$

Again Corollary 1.1 yields that the estimate

$$2t|A|^2 v^2 + v^2 \le c_1^2$$

holds uniformly on M_t. We now proceed by induction on m in a way similar to that in [6], Proposition 2.3. From [5], Theorem 7.1 we have for arbitrary $l \ge 0$ the inequality

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)(t^{l+1} |\nabla^l A|^2) \le &-2t^{l+1} |\nabla^{l+1} A|^2 + (l+1) t^l |\nabla^l A|^2 \\ &+ C(l, n) t^{l+1} \sum_{i+j+k=l} |\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^l A|.\end{aligned}$$

Suppose (11) is established up to $(m-1)$. Then we estimate

$$\begin{aligned}&t^{l+1} \sum_{i+j+k=l} |\nabla^i A|\,|\nabla^j A|\,|\nabla^k A|\,|\nabla^l A| \\ &\le C t^{l+1} \sum_{i+j+k=l} t^{-i/2-j/2-1} |\nabla^k A|\,|\nabla^l A| \le C t^{l/2} \sum_{k \le l} t^{k/2} |\nabla^k A|\,|\nabla^l A| \\ &\le C \sum_{k \le l} t^k |\nabla^k A|^2\end{aligned}$$

with constants C depending only on l, n and c_1. Thus we obtain for all $l \le m$

the inequality

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{l+1}|\nabla^l A|^2\right) \le -2t^{l+1}|\nabla^{l+1}A|^2 + C\sum_{k\le l} t^k|\nabla^k A|^2.$$

In a first step we may then choose k_1 so large that

$$\left(\frac{d}{dt} - \Delta\right)\left(t^{m+1}|\nabla^m A|^2 + k_1 t^m|\nabla^{m-1}A|^2\right) \le C\sum_{k\le m-1} t^k|\nabla^k A|^2.$$

We can then successively select $k_2, k_3, \dots, k_{m+1}$ such that finally

$$\left(\frac{d}{dt} - \Delta\right)\Big(t^{m+1}|\nabla^m A|^2 + k_1 t^m|\nabla^{m-1}A|^2 + k_2 t^{m-1}|\nabla^{m-2}A|^2$$
$$+ \cdots + k_m t|A|^2 + k_{m+1}v^2\Big) \le 0.$$

Here we used again the evolution equation for v^2 and the fact that $v \ge 1$. Proposition 4.4 then follows from Corollary 1.1. □

Having obtained decay estimates in t we are now able to show also that initial spatial decay behaviour is maintained during the evolution. We will make use of this fact in Section 6.

4.5 Proposition. *Let M_t be a smooth solution of* (1) *satisfying* (10) *and the additional assumption*

$$|\nabla^m A|^2(\mathbf{x}s) \le c_2(m)\left(1 + |\mathbf{x}|^2\right)^{-m-1}, \qquad m \ge 0 \tag{12}$$

at time $t = 0$. Then for all $t > 0$,

$$|\nabla^m A|^2 \le C_m\left(1 + \left(\sqrt{|\mathbf{x}|^2 + 2nt} - \sqrt{\beta t}\right)^2\right)^{-m-1} \tag{13}$$

where $\beta = \beta(c_1) > 0$ and $C_m = C_m(n, m, c_1, c_2(0), \dots, c_2(m))$.

Proof. Let us prove (13) for $m = 0, 1$ only. From there we can proceed by induction in a way similar to that in the proof of Proposition 4.4.

Let $g = |A|^2v^2\eta + Lv^2$ where η is an arbitrary nonnegative function and $L > 0$ will be determined later. Using Lemma 4.1 we compute as in the proof of Proposition 4.4 for the case $m = 0$:

$$\left(\frac{d}{dt} - \Delta\right)g \le \mathbf{b}\cdot\nabla g + 2L\eta^{-2}|\nabla\eta|^2v^2$$
$$+\left(c_1 t^{-1/2}|\nabla\eta| + 2\eta^{-1}|\nabla\eta|^2 + \left(\frac{d}{dt} - \Delta\right)\eta - 2L\right)|A|^2v^2$$

where $\mathbf{b} = -2(v^{-1}\nabla v + \eta^{-1}\nabla\eta)$. Note that we estimated the vector $\mathbf{a} =$

$-2v^{-1}\nabla v$ from Lemma 4.1 using the inequality $v^{-1}|\nabla v| \le |A|v$, estimate (11) for $m = 0$ and the fact that $C(0) = c_1^2/2$. We now define

$$\eta(\mathbf{x}, t) = 1 + \left(\sqrt{|\mathbf{x}|^2 + 2nt} - \sqrt{\beta t}\right)^2$$

where $\beta > 0$ will be chosen later. We obviously have the inequalities

$$|\nabla \eta|^2 \le 4\eta$$

and

$$|\nabla \eta| \le 2\sqrt{|\mathbf{x}|^2 + 2nt} + 2\sqrt{\beta t}\,.$$

In view of Lemma 2.1.i), η also satisfies

$$\left(\frac{d}{dt} - \Delta\right)\eta \le \beta - \sqrt{\frac{\beta}{t}\left(|\mathbf{x}|^2 + 2nt\right)}\,.$$

Therefore we conclude for large enough $\beta = \beta(c_1)$,

$$\left(\frac{d}{dt} - \Delta\right)g \le \mathbf{b}\cdot\nabla g + 8L\eta^{-1}v^2 + \left(2c_1\beta^{1/2} + \beta + 8 - 2L\right)|A|^2v^2.$$

If we now choose L large depending on β and c_1 and define $k = \sup_{M_0} g + 9Lc_1^2$, we achieve

$$\left(\frac{d}{dt} - \Delta\right)g \le \mathbf{b}\cdot\nabla g - \eta^{-1}(g - k)$$

where we also used the estimate $v(x, t) \le c_1$ again.

Let $g_k = \max(g - k, 0)$. Since $g_k\cdot(g - k) = g_k^2$, we obtain the result using Corollary 1.1 with $f = g_k^2$.

In order to prove (13) for $m = 1$, we compute as in Proposition 4.4:

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)|\nabla A|^2\eta^2 &\le -2|\nabla^2 A|^2\eta^2 + c(n)|A|^2\eta^2|\nabla A|^2 - 2\nabla\eta^2\cdot\nabla|\nabla A|^2\\ &\le c(n)|A|^2\eta^2|\nabla A|^2 + 8|\nabla\eta|^2|\nabla A|^2.\end{aligned}$$

Since $|\nabla\eta|^2 \le 4\eta$ and $|A|^2\eta \le C_0$ (recall that $v \ge 1$) we estimate

$$\left(\frac{d}{dt} - \Delta\right)|\nabla A|^2\eta^2 \le c(n, C_0)|\nabla A|^2\eta.$$

Similarly we derive

$$\left(\frac{d}{dt} - \Delta\right)|A|^2\eta \le -|\nabla A|^2\eta + c(n, C_0)|A|^2.$$

Since by Lemma 3.1 and the fact that $v \geq 1$ one has

$$\left(\frac{d}{dt} - \Delta\right)v^2 \leq -2|A|^2$$

it is now easy to see that

$$\left(\frac{d}{dt} - \Delta\right)\left(|\nabla A|^2\eta^2 + K|A|^2\eta + Lv^2\right) \leq 0$$

for large enough positive constants K and L depending on n and C_0. The proposition then follows from Corollary 1.1. □

Using Proposition 4.4 we can prove existence of a longtime solution for Lipschitz initial data.

4.6 THEOREM. *If the initial hypersurface M_0 is Lipschitz continuous and satisfies*

$$\sup_{M_0} v \leq c_1,$$

then the mean curvature flow problem (1) *has a longtime solution which is smooth for all $t > 0$ and satisfies the a priori estimates in Corollary* 3.2 *and Proposition* 4.4.

Proof. Suppose first that the initial hypersurface is smooth with $\sup_{M_0}|\nabla^m A|$ bounded for all $m \geq 0$. In view of the bound on v the linearization of (2′) is a uniformly parabolic equation and the implicit function theorem guarantees the existence of a unique smooth solution on some short time interval. Our uniform a priori estimates then ensure that this solution extends to all $t > 0$. For Lipschitz initial data the result follows by approximation in view of our interior estimates in Proposition 4.4. □

5. Asymptotic behaviour

In this section we study the behaviour of solutions M_t of the mean curvature flow system (1) for large times t in the case of linear growth. For simplicity we shall additionally assume that the initial surface M_0 has bounded curvature. We saw in Proposition 4.4 that the surfaces M_t "flatten out" as $t \to \infty$, and if they do not diverge to ∞ (e.g. if u is bounded), then they must converge to a plane. However, in general the surfaces will move out to infinity at speed proportional to $t^{-1/2}$ such that Proposition 4.4 does not yield any information about their global shape.

To study the global shape of M_t for $t \to \infty$ we will now rescale the surfaces in such a way that they do not diverge to infinity and at the same time retain a bound on their curvature.

Similarly, as in [6], we define

$$\tilde{\mathbf{F}}(s) = \frac{1}{\sqrt{2t+1}}\mathbf{F}(t)$$

where the new time variable s is given by $s = \frac{1}{2}\log(2t+1)$, $0 \leq s < \infty$. The normalized equation then becomes

$$\frac{d}{ds}\tilde{\mathbf{F}} = \tilde{\mathbf{H}} - \tilde{\mathbf{F}} \tag{14}$$

and the estimates from Proposition 2.2, Corollary 3.2, Corollary 4.2 and Proposition 4.4 translate to

$$\begin{aligned} \tilde{u}^2(\tilde{\mathbf{x}}, s) &\leq \tilde{c}_0\left(1 + |\tilde{\mathbf{x}}|^2 - \tilde{u}^2(\tilde{\mathbf{x}}, s)\right), \\ \tilde{v}(\tilde{\mathbf{x}}, s) &\leq c_1, \\ |\tilde{A}|^2(\tilde{\mathbf{x}}, s) &\leq c_2, \end{aligned}$$

with constants depending only on the initial bounds for the respective quantities on M_0.

For the rescaled surfaces $\tilde{M}_s = \tilde{\mathbf{F}}(\cdot, s)(M)$ we then establish the following result concerning asymptotic convergence.

5.1 Theorem. *Suppose M_0 satisfies the linear growth condition* (10) *and has bounded curvature. If in addition the estimate*

$$\langle \mathbf{F}, \nu\rangle^2 \leq c_3\left(1 + |\mathbf{F}|^2\right)^{1-\delta} \tag{3}$$

is valid on M_0 for some constants $c_3 < \infty$, $\delta > 0$, then the solution $\tilde{M}_s$ of the normalized equation (14) *converges for $s \to \infty$ to a limiting surface $\tilde{M}_\infty$ satisfying the equation*

$$\mathbf{F}^\perp = \mathbf{H}. \tag{4}$$

We will see in the last section that condition (3) is indeed necessary.

5.2 *Remarks.* i) The result follows from the estimate

$$\sup_{\tilde{M}_s} \frac{\left(\tilde{H} + \langle\tilde{\mathbf{x}}, \tilde{\nu}\rangle\right)^2 \tilde{v}^2}{\left(1 + \alpha|\tilde{\mathbf{x}}|^2\right)^{1-\varepsilon}} \leq e^{-\beta s} \sup_{M_0} \frac{\left(H + \langle\mathbf{x}, \nu\rangle\right)^2 v^2}{\left(1 + \alpha|\mathbf{x}|^2\right)^{1-\varepsilon}} \tag{15}$$

which we derive for all $\varepsilon < \delta$ with some constants $\alpha > 0$, $\beta > 0$ depending only on ε, n, c_1 and c_2. This implies, in particular, exponentially fast convergence on compact subsets, a result much stronger than the corresponding result in [6] concerning asymptotic behaviour of singularities.

ii) In view of the interior estimates in Proposition 4.4 the conclusion of Theorem 5.1 remains valid for Lipschitz initial data provided condition (3) is satisfied for some $t_0 > 0$.

iii) Any initial surface M_0 given by $\mathbf{F}_0$: $M^n \to \mathbf{R}^{n+1}$ satisfying (4) gives rise to an *expanding* selfsimilar solution of the mean curvature flow in the sense that

$$\mathbf{F}(t) = \sqrt{2t+1}\,\mathbf{F}_0$$

satisfies

$$\left(\frac{d}{dt}\mathbf{F}\right)^{\perp} = \mathbf{H}.$$

Theorem 5.1 therefore says that M_t becomes asymptotically selfsimilar. Nontrivial examples of surfaces satisfying (4) can be constructed in arbitrary dimensions by choosing for instance a rotationally symmetric cone $\mathscr{C}$ in $\mathbf{R}^{n+1}$ as initial surface for the evolution. From the maximum principle applied to the evolution equation for H we obtain a family of surfaces lying above $\mathscr{C}$ for all times. Thus the limiting hypersurface $M_{\mathscr{C}}$ for the rescaled flow obtained from Theorem 5.1 is nonlinear. Moreover, from Proposition 4.5 one can see that $M_{\mathscr{C}}$ satisfies the estimate

$$|\langle \mathbf{x}, \nu\rangle| = |H| \leq \frac{c}{1+|\mathbf{x}|}$$

and is therefore strongly asymptotic to $\mathscr{C}$.

In the one-dimensional case an example for "curves of constant shape" evolving from a corner was numerically obtained by Brakke in [1], Fig. 3. It is an open problem to understand and possibly classify solutions of equation (4). We show in the appendix that the equation (5) characterizing *contracting* selfsimilar solutions of the mean curvature flow has only trivial solutions in the class of entire graphs.

We begin the proof of the theorem with the following lemma.

5.2 Lemma. *The quantity* $\langle \mathbf{x}, \nu\rangle$ *satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right)\langle \mathbf{x}, \nu\rangle = |A|^2\langle \mathbf{x}, \nu\rangle - 2H.$$

Proof. From the equation $(d/dt)\nu = \nabla H$ we compute

$$\frac{d}{dt}\langle \mathbf{x}, \nu\rangle = -H + \langle \mathbf{x}, \nabla H\rangle,$$

while

$$
\begin{aligned}
\Delta\langle \mathbf{x}, \boldsymbol{\nu}\rangle &= \mathbf{e}_i\langle \mathbf{x}, \nabla_{e_i}\boldsymbol{\nu}\rangle = \mathbf{e}_i\langle \mathbf{x}, h_{ik}\mathbf{e}_k\rangle \\
&= H + \langle \mathbf{x}, \nabla_{\mathbf{e}_i}\mathbf{e}_k\rangle h_{ik} + \langle \mathbf{x}, \mathbf{e}_k\rangle \nabla_k H \\
&= H - |A|^2\langle \mathbf{x}, \boldsymbol{\nu}\rangle + \langle \mathbf{x}, \nabla H\rangle.
\end{aligned}
$$

We can now show that up to a time-dependent factor condition (3) is preserved for all $s > 0$. □

5.4 LEMMA. *Suppose M_0 satisfies the assumptions of Theorem 5.1. Then on $\tilde{M}_s$ we have the estimate*

$$\langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}}\rangle^2 \le C(s)\left(1 + |\tilde{\mathbf{x}}|^2\right)^{1-\delta}$$

with a constant depending on s and c_2.

Proof. Since the constant in the estimate is allowed to depend on time it is sufficient to look at the unnormalized flow. From Lemma 5.3 we infer

$$
\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\langle \mathbf{x}, \boldsymbol{\nu}\rangle^2 &= 2\langle \mathbf{x}, \boldsymbol{\nu}\rangle^2|A|^2 - 4H\langle \mathbf{x}, \boldsymbol{\nu}\rangle - 2|\nabla\langle \mathbf{x}, \boldsymbol{\nu}\rangle|^2 \\
&\le C\left(\langle \mathbf{x}, \boldsymbol{\nu}\rangle^2 + 1\right) - 2|\nabla\langle \mathbf{x}, \boldsymbol{\nu}\rangle|^2
\end{aligned}
$$

where from now on we denote all constants depending only on c_2 and s by C. We now write $f = \langle \mathbf{x}, \boldsymbol{\nu}\rangle$, multiply the above equation by a test function ρ and estimate

$$\left(\frac{d}{dt} - \Delta\right)f^2\rho \le \left(C + 2\rho^{-2}|\nabla\rho|^2 + \rho^{-1}\left(\frac{d}{dt} - \Delta\right)\rho\right)f^2\rho + c\rho.$$

Choosing $\rho = \eta_1^{\delta-1}$ where $\eta_1 = 1 + |\mathbf{x}|^2 + 2nt$ we derive from Lemma 2.1.i), since $0 < \delta \le 1$,

$$\left(\frac{d}{dt} - \Delta\right)\rho = -(\delta - 1)(\delta - 2)\eta_1^{\delta-3}|\nabla\eta_1|^2 \le 0.$$

Furthermore,

$$\rho^{-2}|\nabla\rho|^2 = (1-\delta)^2\eta_1^{-2}|\nabla\eta_1|^2 \le 4(1-\delta)^2\eta_1^{-1} \le 4$$

since $|\nabla\eta_1|^2 \le 4\eta_1$. Altogether we conclude

$$\left(\frac{d}{dt} - \Delta\right)f^2\rho \le C\left(f^2\rho + 1\right)$$

such that by Corollary 1.1, $f^2\rho$ can at most grow exponentially in time. This implies the result. □

5.5 LEMMA. *The normalized quantity* $\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle$ *satisfies the evolution equation*

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\left(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle\right) = \left(|\tilde{A}|^2 - 1\right)\left(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle\right). \tag{16}$$

Proof. From [5], Lemma 9.1, and Lemma 5.3 we compute the identities

$$\left(\frac{d}{ds} - \Delta\right)\tilde{H} = |\tilde{A}|^2\tilde{H} + \tilde{H},$$

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle = |\tilde{A}|^2\langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle - 2\tilde{H} - \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle.$$

This yields (16). □

Proof of Theorem 5.1. From (16) we infer

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\left(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle\right)^2 = 2\left(|\tilde{A}|^2 - 1\right)\left(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle\right)^2 - 2\left|\nabla\left(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle\right)\right|^2.$$

The normalized gradient $\tilde{v}$ satisfies the equation

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{v}^2 = -2|\tilde{A}|^2\tilde{v}^2 - 6|\nabla\tilde{v}|^2.$$

We may then proceed exactly as in the proof of Lemma 4.1 to obtain for $f^2 = (\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle)^2\tilde{v}^2$ the inequality

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)f^2 \leq -2f^2 - 2\tilde{v}^{-1}\nabla\tilde{v} \cdot \nabla f^2.$$

Multiplying by a test function ρ we compute

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right)f^2\rho \leq &-2f^2\rho - 2\rho\tilde{v}^{-1}\nabla\tilde{v} \cdot \nabla f^2 \\ &+f^2\left(\frac{d}{ds} - \tilde{\Delta}\right)\rho - 2\nabla\rho \cdot \nabla f^2.\end{aligned} \tag{17}$$

Now let $0 < \varepsilon < \delta$ and define $\rho(\tilde{\mathbf{x}}, s) = \eta_\alpha^{\varepsilon-1}(\tilde{\mathbf{x}})e^{\beta s}$ with $\eta_\alpha(\tilde{\mathbf{x}}) = 1 + \alpha|\tilde{\mathbf{x}}|^2$ where α, β are small positive constants to be determined later. Then the normalized equation (14) implies

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\eta_\alpha = -2\alpha\left(|\tilde{\mathbf{x}}|^2 + n\right)$$

and therefore

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\rho \le (\beta + 2(1 - \varepsilon)(\alpha n + 1))\rho. \tag{18}$$

Moreover,

$$\begin{aligned} &- 2\rho\tilde{v}^{-1}\nabla\tilde{v} \cdot \nabla f^2 - 2\nabla\rho \cdot \nabla f^2 \\ &= - 2\left(\tilde{v}^{-1}\nabla\tilde{v} + \rho^{-1}\nabla\rho\right) \cdot \nabla\left(f^2\rho\right) \\ &\quad + 2|\nabla\rho|^2 f^2\rho^{-1} + 2f^2\tilde{v}^{-1}b\nabla\tilde{v} \cdot \nabla\rho \end{aligned} \tag{19}$$

and we obtain from $|\nabla \eta_\alpha|^2 \le 4\alpha\eta_\alpha$ the estimate

$$|\nabla\rho| \le 2\alpha^{1/2}\rho.$$

Combining now (17), (18) and (19) and using the fact that $|\nabla\tilde{v}| \le |\tilde{A}|\tilde{v}^2$ we derive for $g = f^2\rho$ the inequality

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)g \le \mathbf{a} \cdot \nabla g + \left(\beta + c\alpha^{1/2} - 2\varepsilon\right)g$$

where $\mathbf{a} = - 2(\tilde{v}^{-1}\nabla\tilde{v} + \tilde{\rho}^{-1}\nabla\tilde{\rho})$ and c depends on c_1, c_2 and n. Choosing then α, β suitably small depending on ε and c we see that

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)g \le \mathbf{a} \cdot \nabla g$$

for all $s \ge 0$. Lemma 5.4 ensures that g vanishes at infinity which enables us to apply the parabolic maximum principle to conclude that g is uniformly bounded by its initial data. This proves estimate (15) and completes the proof of Theorem 5.1.

6. A counterexample

In this section we want to show that the result of Theorem 5.1 is optimal. The main observation underlying the following arguments is that in case the mean curvature decays at infinity the normalized equation (14) primarily describes a "blow-down" of the initial surface. Hence for initial surfaces which slowly oscillate at infinity and violate condition (3), one would not expect asymptotic convergence as described in Section 5. We have the following explicit examples.

6.1 Proposition. *If M_0 is the graph of the function*

$$u_0(\hat{\mathbf{x}}) = u_0(|\hat{\mathbf{x}}|) = \begin{cases} |\hat{\mathbf{x}}| \sin\log|\hat{\mathbf{x}}|, & |\hat{\mathbf{x}}| \ge 1 \\ \text{smooth}, & |\hat{\mathbf{x}}| \le 1, \end{cases}$$

where $\hat{\mathbf{x}}$ denotes coordinates in $\mathbf{R}^n$, then the normalized surfaces $\tilde{M}_s$ do not converge to a solution of the limiting equation (4).

The proposition is a special case of the following lemma from which a large number of further examples can be obtained.

6.2 LEMMA. *Suppose M_0 satisfies the assumptions of Proposition 4.5 for $m = 0, 1$ and suppose there exists a sequence of points $p_k \in M^n$ such that $|\mathbf{x}(p_k, 0)| \to \infty$ and $\langle \mathbf{x}(p_k, 0), \nu\rangle^2 = \gamma|\mathbf{x}(p_k, 0)|^2$ for some $\gamma > 0$. Then there exists a sequence of times s_k for which $(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\nu}\rangle)$ does not converge to zero.*

Proof. Let us assume for simplicity that $\gamma = 1/2$. Rescaling estimate (13) we immediately infer that for all $s > 0$,

$$|\tilde{H}|^2(\tilde{\mathbf{x}}, s) \le \frac{c_4}{|\tilde{\mathbf{x}}|^2},$$

$$|\nabla \tilde{H}|^2(\tilde{\mathbf{x}}, s) \le \frac{c_5}{|\tilde{\mathbf{x}}|^4}$$

if $|\tilde{\mathbf{x}}| \ge C = C(n, \beta)$. We now derive by direct calculation from (14)

$$\left|\frac{d}{ds}\frac{\langle \tilde{\mathbf{x}}, \tilde{\nu}\rangle^2}{|\tilde{\mathbf{x}}|^2}\right| \le \frac{c_6}{|\tilde{\mathbf{x}}|^2} \tag{20}$$

as well as

$$\left||\tilde{\mathbf{x}}(p, s)|^2 - e^{-2s}|\tilde{\mathbf{x}}(p, 0)|^2\right| \le c_7. \tag{21}$$

Now let p_k be the sequence of points which for all $k \ge 1$ satisfy the relation

$$\langle \mathbf{x}(p_k, 0), \nu\rangle^2 = \tfrac{1}{2}|\mathbf{x}(p_k, 0)|^2.$$

Let us choose a sequence of times s_k such that

$$e^{2s_k} = \varepsilon c_7^{-1}|\mathbf{x}(p_k, 0)|^2$$

with some fixed $0 < \varepsilon < \frac{1}{2}$ to be determined. It follows from (20) and (21) that for $s \le s_k$,

$$\left|\frac{d}{ds}\frac{\langle \tilde{\mathbf{x}}(p_k, s), \tilde{\nu}\rangle^2}{|\tilde{\mathbf{x}}(p_k, s)|^2}\right| \le 2\varepsilon c_6 c_7^{-1} e^{2(s - s_k)}$$

and therefore

$$\frac{\langle \tilde{\mathbf{x}}(p_k, s_k), \tilde{\nu}\rangle^2}{|\tilde{\mathbf{x}}(p_k, s_k)|^2} \ge \frac{1}{2} - \varepsilon c_6 c_7^{-1}.$$

Furthermore we see from (21) that

$$c_7(\varepsilon^{-1} - 1) \le |\tilde{\mathbf{x}}(p_k, s_k)|^2 \le c_7(\varepsilon^{-1} + 1).$$

In view of the a priori estimate

$$|\tilde{H}(\tilde{\mathbf{x}}, s)|^2 \le \frac{c_5}{|\tilde{\mathbf{x}}|^2}$$

for $|\tilde{\mathbf{x}}| \ge c(n, \beta)$, we may choose $\varepsilon > 0$ so small that the quantity $(\tilde{H} + \langle \tilde{\mathbf{x}}, \tilde{\boldsymbol{\nu}} \rangle)^2(p_k, s_k)$ is uniformly bounded from below. This completes the proof of Lemma 6.2. □

Appendix

We show that in the case of entire graphs the only *contracting* selfsimilar solutions of the mean curvature flow are hyperplanes.

PROPOSITION. *If M is an entire graph of at most polynomial growth satisfying the equation*

$$H = \langle \mathbf{x}, \boldsymbol{\nu} \rangle, \tag{22}$$

then M is a plane.

Proof. From (22) we easily compute $\nabla_i H = \langle \mathbf{x}, \mathbf{e}_l \rangle h_{il}$ and hence

$$\Delta v = |A|^2 v + 2v^{-1}|\nabla v|^2 + \langle \mathbf{x}, \mathbf{e}_i \rangle \nabla_i v.$$

We multiply this equation by $\rho = \exp(-|\mathbf{x}|^2/2)$ which after integration by parts leads to

$$\int_M |A|^2 v\rho \, d\mu + 2\int_M v^{-1}|\nabla v|^2 \rho \, d\mu = 0,$$

thus implying the result.

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