On the Integrability of large N
Plane-Wave Matrix Theory

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ABSTRACT

We show the three-loop integrability of large N plane-wave matrix theory in a subsector of states comprised of two complex light scalar fields. This is done by diagonalizing the theory's Hamiltonian in perturbation theory and taking the large N limit. At one-loop level the result is known to be equal to the Heisenberg spin-1/2 chain, which is a well-known integrable system. Here, integrability implies the existence of hidden conserved charges and results in a degeneracy of parity pairs in the spectrum. In order to confirm integrability at higher loops, we show that this degeneracy is not lifted and that (corrected) conserved charges exist. Plane-wave matrix theory is intricately connected to $\mathcal{N} = 4$ Super Yang-Mills, as it arises as a consistent reduction of the gauge theory on a three-sphere. We find that after appropriately renormalizing the mass parameter of the plane-wave matrix theory the effective Hamiltonian is identical to the dilatation operator of $\mathcal{N} = 4$ Super Yang-Mills theory in the considered subsector. Our results therefore represent a strong support for the conjectured three-loop integrability of planar $\mathcal{N} = 4$ SYM and are in disagreement with a recent dual string theory finding. Finally, we study the stability of the large N integrability against nonsupersymmetric deformations of the model.

1 Introduction and Conclusions

Recently a number of fascinating developments have pointed to the existence of integrable structures in the large N limit of four-dimensional $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory [1, 2, 3, 4]. These developments have been paralleled by investigations on the integrability of the dual $AdS_5 \times S^5$ string [5]. In this paper we shall turn to another setting of string/M-theory in which integrable structures emerge: The large N limit of plane-wave matrix theory [6]. This SU(N) matrix quantum mechanics represents a mass deformed version of the "BFSS" matrix theory [7], thereby preserving all of its supersymmetries. In its large N limit it is believed to describe the microscopic degrees of freedom of M-theory in the maximally supersymmetric plane-wave background [8]. Opposed to its flat space cousin the model enjoys a number of pleasant features: It possesses a discrete spectrum, the model may be treated perturbatively in the large mass limit [9], and its spectrum contains an infinite set of exactly protected states due to its underlying SU(2|4) symmetry structure [10, 11, 12]. The present work, however, is mainly motivated by the direct connection of plane-wave matrix theory to $\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R} \times S^3$. The matrix quantum mechanics arises in a consistent reduction of the four-dimensional superconformal gauge theory on the S^3 [13]. As the scaling dimensions of operators in a conformal theory on $\mathbb{R}^4$ are related to the energy spectrum on $\mathbb{R} \times S^3$ in radial quantization, the dilatation operator of the four-dimensional Super Yang-Mills theory should bear a relation to the Hamiltonian of the plane-wave matrix model. And indeed in [13] it was shown that at the one-loop level the spectrum of anomalous dimensions of gauge theory operators agrees with the leading perturbative corrections to the spectrum of associated states in the plane-wave matrix quantum mechanics. An immediate consequence of this observation is, that the one-loop integrability of the Super Yang-Mills model, to be discussed below, carries over to plane-wave matrix theory.

Integrability in $\mathcal{N} = 4$ Super Yang-Mills shows up in the perturbative study of the dilatation operator, whose eigenvalues yield the spectrum of scaling dimensions of composite operators. In principle the scaling dimensions are extracted from diagonal two-point functions, however, in [14, 1] a new efficient technique was introduced to directly extract the dilatation operator order by order in a loop expansion from the gauge theory. This led to the remarkable discovery of Minahan and Zarembo [1] of an exact equivalence between the planar, one-loop piece of the dilatation operator (in the pure scalar excitation sector) with the Hamiltonian of an integrable $SO(6)$ spin chain. In this picture the spins on the chain are given by the scalar fields ϕ_i ($i = 1, \ldots, 6$) of $\mathcal{N} = 4$ Super Yang-Mills thread together in a single trace structure, $\mathrm{tr}(\phi_{i_1} \ldots \phi_{i_n})$. This finding was subsequently generalized to the

full Super Yang-Mills field content [3] establishing an equivalence of the planar one-loop dilatation operator with the Hamiltonian of an integrable $SU(2,2|4)$ spin chain.

Obviously one would like to know whether this integrability is a mere artifact of the one-loop approximation, or whether it represents a genuine hidden symmetry of planar Super Yang-Mills. To answer this question higher loop considerations suggest themselves. For this it is advantageous to restrict one's attention to a minimal subsector of field excitations, which does not mix in perturbation theory with other excitations. Such a sector is given by the $SU(2)$ excitations of two complex scalars, which we denote Z and ϕ. The corresponding one-loop planar dilatation operator is then identical to the famous $XXX_{1/2}$ Heisenberg quantum spin chain model, $H = \sum_{j=1}^{L} \vec{s}_j \cdot \vec{s}_{j+1}$, where $\vec{s}_j$ is the spin $1/2$ operator acting on the jth site of the chain, being in an up (Z) or down (ϕ) state. This model of a one dimensional magnet is known to be integrable, i.e. one may construct a set of $L-1$ mutually commuting charges which contain H [15]. In [2] a two-loop computation in this $SU(2)$ sector of $\mathcal{N} = 4$ Super Yang-Mills was performed showing that integrability is preserved. The higher loop contributions to the planar dilatation operator result in deformations of the spin chain Hamiltonian incorporating spin interactions beyond nearest neighbours, i.e. at l loop level one is dealing with the interactions of $l+1$ spins. The integrability of the higher loop deformation manifests itself in the preservation of degeneracies in the spectrum, which may be attributed at one-loop level to the hidden symmetry. However, a generalization of the powerful methods known from integrable spin chains for the construction of commuting charges at higher loop orders remains unknown, nevertheless the first few charges beyond the deformed Hamiltonian were constructed explicitly. It is important to note that while the one-loop dilatation operator is exactly integrable, the combined one- and two-loop dilatation operator is only integrable up to terms of three-loop order. Hence only the full, planar, all-loop dilatation operator can display exact integrability – a fascinating perspective on the large N spectrum of $\mathcal{N} = 4$ Super Yang-Mills.

Based on the assumption of *three*-loop integrability the authors of [2] where actually able to determine the form of the planar *three*-loop dilatation operator. However, this conjecture was recently challenged by the work of Callan, Lee, McLoughlin, Schwarz, Swanson and Wu [16] from the dual string theory side. These authors computed the corrections to the spectrum of IIB plane-wave superstring theory due to the presence of first curvature corrections that arise in the expansion of $AdS_5 \times S^5$ about the plane-wave limit. These corrections translate into certain $1/J$ corrections of the scaling dimensions of BMN operators in the BMN limit [6] and can be compared to the two-loop result and three-loop conjecture of Beisert, Kristjansen and Staudacher [2]. Indeed the two-loop corrections of the gauge theory match with the string result, however, at three-loop order the conjectured gauge

theory result *disagrees* with the string based findings. Hence, if the involved analysis of [16] is indeed correct, then string theory predicts the breakdown of integrability in $\mathcal{N} = 4$ Super Yang-Mills at the three-loop level! To clarify this issue from a gauge theory perspective a direct three-loop analysis would be very desirable. Fortunately the aforementioned relation of $\mathcal{N} = 4$ Super Yang-Mills to plane-wave matrix theory enables one to perform such a three-loop computation in a simpler quantum mechanical setting. This is the goal of our paper.

Our analysis consists of an essentially straightforward application of degenerate quantum mechanical perturbation theory to the third non-vanishing order, which is equivalent to a field theoretic three-loop computation. Needless to say that this has been only possible by making massive use of symbolic computer algebra systems. The outcome of our analysis is positive for integrability: Plane-wave matrix theory remains integrable up to the three-loop level in the minimal SU(2) sector mentioned above. Technically this means that the degeneracy of "planar parity pairs" noted in [2] extends to the three-loop order. Also the first charge beyond the Hamiltonian can be explicitly constructed. Clearly this does not prove the integrability of the full four-dimensional theory, as we are dealing with a reduced model here, but it does make it very likely. This is furthermore supported by the following remarkable observation. As found in [13] the agreement of the Super Yang-Mills dilatation operator with the perturbative effective plane-wave matrix theory Hamiltonian ceases to exist at two-loop order. However, if one allows for a renormalization of the relation between the (dimensionless) mass parameter M of matrix theory and the Yang-Mills coupling constant g_{YM} of the form[1]

$$\frac{1}{M^3} = \frac{g_{\text{YM}}^2}{32\pi^2} \left(1 + \frac{7}{32\,\pi^2}\, g_{\text{YM}}^2\, N - \frac{11}{252\,\pi^4}\, g_{\text{YM}}^4\, N^2 + \mathcal{O}(g_{\text{YM}}^6) \right) , \tag{1}$$

then this agreement extends to three-loop order! That is our three-loop matrix theory computation reproduces the conjectured three-loop dilatation operator of Beisert, Kristjansen and Staudacher [2] with the above choice of renormalized mass parameter. Indeed such a renormalization is to be expected if one were to truly path integrate out the higher Kaluza-Klein modes of the gauge theory on S^3 to obtain an effective one dimensional theory of the zero-modes. The resulting effective theory would then be given by the plane-wave matrix model plus higher loop corrections in $\lambda = g_{\text{YM}}^2\, N$ and $1/N$. These quantum loop corrections should manifest themselves in a renormalization of the mass parameter M of the plane-wave matrix model and – but our findings indicate that this is not the case – the addition

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[1]This observation grew out of discussions with N. Beisert and M. Staudacher.

of higher order interaction terms. It would be very interesting to study this in more detail, in particular the rôle of (maximal) supersymmetry in this non-emergence of higher order interaction terms.

In addition to that, we have explored how stable the integrability is to deformations of the theory. It turns out that supersymmetry is absolutely *unessential* for integrability in the matrix model up to three-loops. Switching off all fermions in the plane-wave matrix theory does not destroy integrability. Furthermore deformations of the bosonic field content are also possible without loosing integrability, as long as the structure of Yang-Mills type interactions is maintained. Our findings suggest that the hidden integrability, if it exists to all loops, is a genuine property of large N gauge quantum mechanics. It would be very nice to prove integrability in these models. To this end a formalism enabling one to directly work in the planar limit of the theory is needed. Maybe the structures introduced in [17] could be of use here.

Naturally a further important question is to understand how these (potential) integrable structures are to be interpreted in M-theory.

Finally, let us mention that an alternative method to address the question of three-loop integrability of the full four-dimensional $\mathcal{N} = 4$ gauge theory lies in the study of how strongly the underlying $SU(2,2|4)$ superalgebra – or a suitable subgroup thereof – restricts the three-loop structure of the dilatation operator. This will be addressed in [18].

2 Plane-wave Matrix Theory

The Hamiltonian of plane-wave matrix theory can be written as $H = H_0 + V_1 + V_2$ with

$$H_0 = \text{tr} \left[\tfrac{1}{2} P_I P_I + \tfrac{1}{2} \left(\tfrac{M}{2} \right)^2 X_i X_i + \tfrac{1}{2} M^2 X_{i'} X_{i'} - \tfrac{3M}{4} i\theta \gamma_{123} \theta \right] , \tag{2a}$$

$$V_1 = -Mi\varepsilon_{i'j'k'} \, \text{tr} \, X_{i'} X_{j'} X_{k'} - \text{tr} \, \theta \gamma_I [X_I, \theta] , \tag{2b}$$

$$V_2 = -\frac{1}{4} \, \text{tr}[X_I, X_J][X_I, X_J] . \tag{2c}$$

The degrees of freedom are the light scalars X_i ($i = 1, \ldots, 6$) of mass $M/2$, the heavy scalars $X_{i'}$ ($i' = 1, \ldots, 3$) of mass M and a SO(9) Majorana spinor θ_α ($\alpha = 1, \ldots, 16$) of mass $3M/4$ [2]. All fields are $N \times N$ traceless Hermitian matrices by virtue of the gauge group $SU(N)$. The index $I = (i, i')$ embraces all scalars. In the fermionic sector we work

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[2]The mass parameter here is a dimensionless quantity given by $M = \frac{\mu \, l_P^2}{6 \, R}$ in terms of the M-theoretic quantities μ (mass parameter of the plane-wave background, see eq. (1) of [13]), l_P (eleven dimensional Planck length) and R (radius of compact eleventh dimension).

in a representation with charge conjugation matrix equal to unity and with symmetric euclidean Dirac matrices $(\gamma_I)_{\alpha\beta}$. Note that it is the bosonic X_i sector which is intimately related to the scalar field sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.

We introduce the following creation and annihilation operators

$$P_i = \sqrt{\tfrac{M}{4}}\left(a_i + a_i^\dagger\right) \qquad P_{i'} = \sqrt{\tfrac{M}{2}}\left(b_{i'} + b_{i'}^\dagger\right) \qquad \theta_\alpha = \theta_\alpha^+ + \theta_\alpha^-$$

$$X_i = \tfrac{i}{\sqrt{M}}\left(a_i - a_i^\dagger\right) \qquad X_{i'} = \tfrac{i}{\sqrt{2M}}\left(b_{i'} - b_{i'}^\dagger\right) \qquad \text{with } \theta^\pm := \Pi^\pm\theta\,, \qquad (3)$$

where $\Pi^\pm := \tfrac{1}{2}(\mathbf{1} \pm i\gamma_{123})$. They satisfy the canonical (anti-)commutation relations

$$[(a_i)_{rs}, (a_j)_{tu}] = \delta_{ij}\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right)\,,$$
$$[(b_{i'})_{rs}, (b_{j'})_{tu}] = \delta_{i'j'}\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right)\,,$$
$$\{(\theta_\alpha^-)_{rs}, (\theta_\beta^+)_{tu}\} = \tfrac{1}{2}\Pi_{\alpha\beta}^-\left(\delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu}\right) \qquad (4)$$

and the free Hamiltonian now reads

$$H_0 = \text{tr}\left[\frac{M}{2}a_i^\dagger a_i + Mb_{i'}^\dagger b_{i'} + \frac{3M}{2}\theta_\alpha^+\theta_\alpha^-\right]\,. \qquad (5)$$

Physical states are constraint to be gauge invariant and are given by traces over words in the creation operators.

3 Perturbation Theory

We are faced with an eigenvalue problem of the form

$$(H_0 + \kappa V_1 + \kappa^2 V_2)\big|\phi(\kappa)\big\rangle = E(\kappa)\big|\phi(\kappa)\big\rangle, \qquad (6)$$

where the spectrum of the free ($\kappa = 0$) Hamiltonian H_0 is known. Let us sketch how (6) is solved perturbatively in κ and how we deal with potential degeneracies. Say we are interested in the perturbative shift of a specific L-fold degenerate free energy value E_0. Then we choose an arbitrary basis $\{\big|\phi_i\big\rangle, i = 1, \ldots, L\}$ for all states with this free energy

$$H_0\big|\phi_i\big\rangle = E_0\big|\phi_i\big\rangle \qquad (i = 1, \ldots, L)\,. \qquad (7)$$

For later convenience we do not assume this basis to be orthonormal. All other states are labeled by $\big|n\big\rangle$ and their free energies by E_n. These energies may, of course, be degenerate

as well. Here, however, we define the states in such a way that

$$\langle n|m\rangle = \delta_{nm} \quad , \quad \langle n|\phi_i\rangle = 0 \ , \tag{8}$$

which is always possible.

When the interaction is turned on ($\kappa \neq 0$), the states $|\phi_i\rangle$ get corrected to $|\phi_i(\kappa)\rangle$. Clearly, the corrected states $|\phi_i(\kappa)\rangle$ are generically not eigenstates of the full Hamiltonian $H = H_0 + \kappa V_1 + \kappa^2 V_2$ since there may occur mixing among the states $|\phi_i\rangle$. However, we do not want to deal with mixing problems and the possible lift of the original degeneracy at this stage, and therefore introduce an energy matrix $E_{ij}(\kappa)$:

$$(H_0 + \kappa V_1 + \kappa^2 V_2)|\phi_i(\kappa)\rangle = E_{ij}(\kappa)|\phi_j(\kappa)\rangle \ . \tag{9}$$

Later, this matrix can be diagonalized and its eigenvalues are the final corrected energies. In this way we have decoupled the mixing of states in the degenerate subspace from the admixture of states from outside this subspace, which turns out to be extremely convenient for practical computations.

Now we wish to determine an energy operator $T(\kappa)$ whose matrix elements give exactly the energy matrix:

$$E_{ij}(\kappa) = \langle\tilde{\phi}_j|T(\kappa)|\phi_i\rangle \ . \tag{10}$$

In (10) we have introduced the dual basis $\{|\tilde{\phi}_i\rangle, \ i = 1, \ldots, L\}$ defined by

$$\langle\tilde{\phi}_i|\phi_j\rangle = \delta_{ij} \ . \tag{11}$$

This is necessary because we did not demand the orthonormality of $\{|\phi_i\rangle\}$. On the other hand it is also very convenient for the following reason. We can start from any arbitrary basis $\{|\phi_i\rangle\}$ and do not need to worry about orthogonalizing it but may immediately apply $T(\kappa)$ with a result of the form

$$T(\kappa)|\phi_i\rangle = \sum_j t_{ij}(\kappa)|\phi_j\rangle + \sum_n t_{in}(\kappa)|n\rangle \ . \tag{12}$$

From the expansion coefficients we then simply read off the energy matrix as

$$E_{ij}(\kappa) = t_{ij}(\kappa) \ . \tag{13}$$

In order to find $T(\kappa)$ we follow a standard procedure. States and energy matrix are

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expanded in κ:

$$\left|\phi_i(\kappa)\right\rangle = |\phi_i\rangle + \kappa\left(\varphi_{ij}^{(1)}|\phi_j\rangle + \psi_{in}^{(1)}|n\rangle\right) + \kappa^2\left(\varphi_{ij}^{(2)}|\phi_j\rangle + \psi_{in}^{(2)}|n\rangle\right) + \dots \tag{14}$$

$$E_{ij}(\kappa) = E_{ij}^{(0)} + \kappa E_{ij}^{(1)} + \kappa^2 E_{ij}^{(2)} + \dots \qquad \text{with } E_{ij}^{(0)} = E_0\delta_{ij} \tag{15}$$

Plugging these expansions into (9) yields one equation for each power of κ. When the resulting equations are projected onto $\langle\tilde{\phi}_i|$ and $\langle n|$ one obtains expressions for $E_{ij}^{(k)}$ and $\psi_{in}^{(k)}$. The coefficients $\varphi_{ij}^{(k)}$ remain undetermined and may be set to zero. Iteratively we then find the energy operator

$$T(\kappa) = \sum_{k=0}^{\infty} \kappa^k T_k \tag{16}$$

up to third loop order

$$T_0 = H_0 \tag{17a}$$

$$T_2 = V_1\Delta V_1 + V_2 \tag{17b}$$

$$T_4 = V_1\Delta V_1\Delta V_1\Delta V_1 + V_1\Delta V_1\Delta V_2 + V_1\Delta V_2\Delta V_1 + V_2\Delta V_1\Delta V_1 + V_2\Delta V_2 \tag{17c}$$
$$- V_1\Delta^2 V_1 PT_2$$

$$\begin{aligned}
T_6 = {} & V_1\Delta V_1\Delta V_1\Delta V_1\Delta V_1\Delta V_1 \\
& + V_1\Delta V_1\Delta V_1\Delta V_1\Delta V_2 + V_1\Delta V_1\Delta V_1\Delta V_2\Delta V_1 + V_1\Delta V_1\Delta V_2\Delta V_1\Delta V_1 \\
& + V_1\Delta V_2\Delta V_1\Delta V_1\Delta V_1 + V_2\Delta V_1\Delta V_1\Delta V_1\Delta V_1 \\
& + V_1\Delta V_1\Delta V_2\Delta V_2 + V_1\Delta V_2\Delta V_1\Delta V_2 + V_1\Delta V_2\Delta V_2\Delta V_1 \\
& + V_2\Delta V_1\Delta V_1\Delta V_2 + V_2\Delta V_1\Delta V_2\Delta V_1 + V_2\Delta V_2\Delta V_1\Delta V_1 \\
& + V_2\Delta V_2\Delta V_2 \\
& - \big(V_1\Delta^2 V_1\Delta V_1\Delta V_1 + V_1\Delta V_1\Delta^2 V_1\Delta V_1 + V_1\Delta V_1\Delta V_1\Delta^2 V_1 \\
& \quad + V_1\Delta^2 V_1\Delta V_2 + V_1\Delta^2 V_2\Delta V_1 + V_2\Delta^2 V_1\Delta V_1 \\
& \quad + V_1\Delta V_1\Delta^2 V_2 + V_1\Delta V_2\Delta^2 V_1 + V_2\Delta V_1\Delta^2 V_1 \\
& \quad + V_2\Delta^2 V_2\big)PT_2 \\
& - V_1\Delta^2 V_1 PT_4 + V_1\Delta^3 V_1 PT_2 PT_2 \tag{17d}
\end{aligned}$$

giving rise to the expansion of the energy matrix (15). Moreover T_k vanishes for odd k.

Here we have introduced the "propagator" and the projector

$$\Delta = \frac{1-P}{E_0 - H_0} \qquad \text{and} \qquad P = \sum_{i=1}^{L} |\phi_i\rangle\langle\tilde{\phi}_i| \,, \tag{18}$$

respectively, where P projects onto the subspace of states with free energy E_0.

4 Energy Operator in SU(2) subsector

In this section we compute the energy operator T up to third loop order for the SU(2) subsector of the matrix model which is spanned by gauge invariant states composed of the two complex fields

$$Z = \frac{1}{\sqrt{2}}(a_1^\dagger + ia_2^\dagger) \qquad \text{and} \qquad \phi = \frac{1}{\sqrt{2}}(a_3^\dagger + ia_4^\dagger) \,. \tag{19}$$

The arguments given in section 3.1. of [2] for the group theoretical constraints on mixing of the analogue SU(2) sector in $\mathcal{N}=4$ supersymmetric Yang-Mills field theory are inherited by the plane-wave matrix theory. This is so as these arguments are merely based on the representation content of the excitations with respect to the SO(6) R-symmetry group and the classical scaling dimensions of the involved fields, which directly carry over to the matrix model. In terms of the SO(6) Dynkin labels the states we consider transform in the representations $[p, q, p]$ and have free energy $E_0 = \frac{M}{2}(2p+q)$.

We apply a procedure similar to our previous two-loop computation [13]. We normal order the terms given in (17) and discard all pieces which annihilate multi-trace states of Z and ϕ. Furthermore we can neglect all terms which are not relevant for the expansion coefficients t_{ij} but only for t_{in} in (12). The relevant terms are exactly those which conserve the free energy, i.e. the number of fields of a state.

In addition to the above fields, the energy operator consists also of the canonically conjugated fields

$$\bar{Z} = \frac{1}{\sqrt{2}}(a_1 - ia_2) \qquad \text{and} \qquad \bar{\phi} = \frac{1}{\sqrt{2}}(a_3 - ia_4) \,, \tag{20}$$

which satisfy

$$[\bar{Z}_{rs}, Z_{tu}] = [\bar{\phi}_{rs}, \phi_{tu}] = \delta_{st}\delta_{ru} - \tfrac{1}{N}\delta_{rs}\delta_{tu} \,. \tag{21}$$

Since our intention is to compare the energy operator of the matrix theory to the dilatation generator of the field theory, we make the following definitions. Perturbation theory is done

for large masses in an expansion in $1/M^3$. As shown in [13], this mass parameter should be related to the SYM coupling constant g_{YM} by

$$\frac{1}{M^3} = \frac{g_{\text{YM}}^2}{32\pi^2} \quad \left(= \frac{1}{2}G^2 = \frac{\Lambda}{2N} \right) \tag{22}$$

in order to match all one-loop results in the scalar SO(6) sector of both theories. In (22) we have also given the relation to two further shorthands $G := \dfrac{g_{\text{YM}}}{4\pi}$ and $\Lambda := G^2 N$. Now we define the counterpart of the dilatation generator from the energy operator T by factoring out the free energy $M/2$ of an SO(6) scalar, i.e. the mass of Z and ϕ:

$$T(\Lambda) =: \frac{M}{2}D(\Lambda) =: \frac{M}{2}\sum_{k=0}^{\infty} \Lambda^k D_{2k} \ , \tag{23}$$

which explicitly means

$$D_0 = \frac{2}{M}T_0 \ , \qquad D_2 = \frac{M^2}{N}T_2 \ , \qquad D_4 = \frac{M^5}{2N^2}T_4 \ , \qquad D_6 = \frac{M^8}{4N^3}T_6 \ . \tag{24}$$

We will refer to $D(\Lambda)$ as the dilatation operator and to its matrix elements $\langle \tilde{\phi}_j | D(\Lambda) | \phi_i \rangle$ as the dilatation matrix. The eigenvalues of $D(\Lambda)$ will be denoted as $\Delta(\Lambda)$.

Performing the computation using `Form` [19] as well as `Mathematica`, we find the following result:

$$D_0 = \ : \text{tr}(Z\bar{Z} + \phi\bar{\phi}) : \ , \tag{25a}$$

$$D_2 = \ -\frac{2}{N} : \text{tr}[Z,\phi][\bar{Z},\bar{\phi}] : \ , \tag{25b}$$

$$D_4 = \frac{11}{N} : \text{tr}[Z,\phi][\bar{Z},\bar{\phi}] :$$
$$+ \frac{2}{N^2} : \left(\text{tr}[Z,\phi][\bar{Z},[Z,[\bar{Z},\bar{\phi}]]] + \text{tr}[Z,\phi][\bar{\phi},[\phi,[\bar{Z},\bar{\phi}]]] \right) : \ , \tag{25c}$$

$$D_6 = -\frac{115}{N} : \mathrm{tr}[Z,\phi][\bar{Z},\bar{\phi}] :$$

$$+ \frac{61}{8N^3} : \big(\mathrm{tr}[[[\phi,Z],T^a],Z][\bar{Z},[T^a,[\bar{\phi},\bar{Z}]]] + \mathrm{tr}[[[\phi,Z],T^a],\phi][\bar{\phi},[T^a,[\bar{\phi},\bar{Z}]]] \big) :$$

$$- \frac{9}{16N^3} \Big[: \big(\mathrm{tr}\, Z[T^a,[\bar{Z},[\bar{Z},[T^a,[\bar{\phi},[\phi,Z]]]]]] + \mathrm{tr}\, \phi[T^a,[\bar{\phi},[\bar{\phi},[T^a,[\bar{Z},[Z,\phi]]]]]] \big) :$$

$$- \tfrac{111}{31} : \big(\mathrm{tr}\, Z[T^a,[\bar{Z},[\bar{\phi},[T^a,[\bar{Z},[\phi,Z]]]]]] + \mathrm{tr}\, \phi[T^a,[\bar{\phi},[\bar{Z},[T^a,[\bar{\phi},[Z,\phi]]]]]] \big) :$$

$$+ \tfrac{80}{31} : \big(\mathrm{tr}\, Z[T^a,[\bar{\phi},[\bar{Z},[T^a,[\bar{Z},[\phi,Z]]]]]] + \mathrm{tr}\, \phi[T^a,[\bar{Z},[\bar{\phi},[T^a,[\bar{\phi},[Z,\phi]]]]]] \big) : \Big]$$

$$+ \frac{195}{16N^3} \Big[: \big(\mathrm{tr}\, \bar{Z}[T^a,[Z,[Z,[T^a,[\phi,[\bar{\phi},\bar{Z}]]]]]] + \mathrm{tr}\, \bar{\phi}[T^a,[\phi,[\phi,[T^a,[Z,[\bar{Z},\bar{\phi}]]]]]] \big) :$$

$$- \tfrac{111}{31} : \big(\mathrm{tr}\, \bar{Z}[T^a,[Z,[\phi,[T^a,[Z,[\bar{\phi},\bar{Z}]]]]]] + \mathrm{tr}\, \bar{\phi}[T^a,[\phi,[Z,[T^a,[\phi,[\bar{Z},\bar{\phi}]]]]]] \big) :$$

$$+ \tfrac{80}{31} : \big(\mathrm{tr}\, \bar{Z}[T^a,[\phi,[Z,[T^a,[Z,[\bar{\phi},\bar{Z}]]]]]] + \mathrm{tr}\, \bar{\phi}[T^a,[Z,[\phi,[T^a,[\phi,[\bar{Z},\bar{\phi}]]]]]] \big) : \Big]$$

$$- \frac{2}{N^3} : \mathrm{tr}[\phi,Z][[[\bar{Z},\bar{\phi}],\phi],[Z,[\bar{Z},\bar{\phi}]]] :$$

$$+ \frac{4}{N^3} : \big(\mathrm{tr}[[[\bar{\phi},\bar{Z}],Z],\bar{Z}][Z,[\bar{Z},[Z,\phi]]] + \mathrm{tr}[[[\bar{\phi},\bar{Z}],\phi],\bar{\phi}][\phi,[\bar{\phi},[Z,\phi]]]$$

$$+ \mathrm{tr}[[[\bar{\phi},\bar{Z}],\phi],\bar{\phi}][Z,[\bar{Z},[Z,\phi]]] + \mathrm{tr}[[[\bar{\phi},\bar{Z}],Z],\bar{Z}][\phi,[\bar{\phi},[Z,\phi]]] \big) :$$

$$- \frac{2}{N^3} : \big(\mathrm{tr}[[[\phi,Z],\bar{Z}],\bar{\phi}][\bar{\phi},[\bar{Z},[Z,\phi]]] + \mathrm{tr}[[[\phi,Z],\bar{\phi}],\bar{Z}][\bar{Z},[\bar{\phi},[Z,\phi]]]$$

$$- \mathrm{tr}[[[\phi,Z],\bar{Z}],\bar{Z}][\bar{\phi},[\bar{\phi},[Z,\phi]]] - \mathrm{tr}[[[\phi,Z],\bar{Z}],\bar{\phi}][\bar{Z},[\bar{\phi},[Z,\phi]]] \big) : \ . \tag{25d}$$

5 Planar limit

We now take the planar limit of the above SU(2) dilatation operator D, i.e. we derive a new operator (again called D), which yields the dilatation matrix in the limit $N \to \infty$ with $\Lambda =$ fixed. Since in the planar limit single and multi-trace states do not mix, we may and we do restrict the action of D to the closed subset of single trace states.

Hence, the dilatation operator in the planar limit may be written as a sum of permutations of the fields in the trace. We use the notation of [2]: Let P_{k_1,k_2} exchange the fields at sites k_1 and k_2 in the trace (where the sites are periodically identified) and define

$$\{n_1, n_2, \ldots\} := \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1} \cdots \ , \tag{26}$$

where L is the number of fields the operator is applied to, i.e. the length of the trace respectively chain. The identity operator is denoted by $\{\}$. There are many trivial relations

such as

$$\{\dots, n, n, \dots\} = \{\dots, \dots\} \tag{27}$$

$$\{\dots, n, m, \dots\} = \{\dots, m, n, \dots\} \qquad \text{for } |n - m| \geq 2 \tag{28}$$

$$\{n_1, n_2, \dots\} = \{n_1 + m, n_2 + m, \dots\} \tag{29}$$

as well as the identity

$$\{\dots, \dots\} + \{\dots, n \pm 1, n, \dots\} + \{\dots, n, n \pm 1, \dots\}$$
$$- \{\dots, n, \dots\} - \{\dots, n \pm 1, \dots\} - \{\dots, n, n \pm 1, n, \dots\} = 0 \tag{30}$$

which only holds in the SU(2) subsector and expresses the fact that two objects cannot be placed completely antisymmetric onto three sites.

In order to find the planar version of the dilatation operator, we have made an ansatz for all independent permutations and determined their coefficients such that the $N \to \infty$ limit of (25) is correctly captured. We find the rather compact result:

$$D_0 = \{\} \,, \tag{31a}$$

$$D_2 = 2\{\} - 2\{0\} \,, \tag{31b}$$

$$D_4 = -15\{\} + 19\{0\} - 2(\{0, 1\} + \{1, 0\}) \,, \tag{31c}$$

$$D_6 = 187\{\} - 259\{0\} + 38(\{0, 1\} + \{1, 0\})$$
$$+ 4\{0, 2\} - 4(\{0, 1, 2\} + \{2, 1, 0\}) - 2(\{0, 2, 1\} - \{1, 0, 2\}) \,. \tag{31d}$$

The presence of the antisymmetric term $(\{0, 2, 1\} - \{1, 0, 2\})$ in the three-loop part might appear unexpected. However, it is just a consequence of the fact that the energy operator (17) is not Hermitian[3]. By a change of basis one can remove this term: after the following similarity transformation

$$D'(\Lambda) := e^{-\frac{\Lambda}{2}\{0\}} D(\Lambda) e^{\frac{\Lambda}{2}\{0\}} \tag{32}$$

we obtain $D_0' = D_0'$, $D_2' = D_2$, $D_4' = D_4$ and

$$D_6' = 187\{\} - 259\{0\} + 38(\{0, 1\} + \{1, 0\}) + 4\{0, 2\} - 4(\{0, 1, 2\} + \{2, 1, 0\}) \,. \tag{33}$$

In [13] we observed that the two-loop dilatation operator (31c) failed to agree with the two-loop dilatation operator of SYM if one relates M to g_{YM} according to (22). However,

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[3]As a matter of fact T is quasi-Hermitian, i.e. there exists an operator S such that $T^\dagger = S^{-1} T S$.

if one allows a renormalization of this relation, then agreement with the full (planar and non-planar) two-loop dilatation operator of SYM can be achieved. Moreover, we discover that this is true for the planar three-loop operator as well. Let thus G_r be the renormalized coupling related to G by

$$G^2 = G_r^2 + \frac{7N}{2}G_r^4 - 11N^2G_r^6 + \mathcal{O}(G_r^8) \,. \tag{34}$$

Then the dilatation operator D' written in terms of $\Lambda_r := G_r^2 N$ becomes

$$D'_{r,0} = \{\} \,, \tag{35a}$$

$$D'_{r,2} = 2\{\} - 2\{0\} \,, \tag{35b}$$

$$D'_{r,4} = -8\{\} + 12\{0\} - 2(\{0,1\} + \{1,0\}) \,, \tag{35c}$$

$$D'_{r,6} = 60\{\} - 104\{0\} + 24(\{0,1\} + \{1,0\}) + 4\{0,2\} - 4(\{0,1,2\} + \{2,1,0\}) \,, \tag{35d}$$

which is exactly the proposed field theory result, cf. equation (F.3) of [2]. There the planar three-loop dilatation operator of SYM was determined using integrability as an input. Here we find it without any assumption.

6   Integrability

Since the planar dilatation operator (31) of plane-wave matrix theory in the SU(2) subsector can be transformed to the corresponding dilatation operator of SYM, we know from the investigations of [2] that it is integrable. I.e. there exists a charge

$$U(\Lambda) = \sum_{k=1}^{\infty} \Lambda^k U_{2k} \tag{36}$$

which commutes with the dilatation operator. This charge pairs up states of different parity and is therefore responsible for their degenerate energies. Here parity refers to the operation of inverting the order of fields in a trace, e.g. $P\,\mathrm{tr}(Z^3\phi^2Z\phi) = \mathrm{tr}(\phi Z\phi^2Z^3)$. The dilatation operator commutes with parity, which is the reason for states of opposite parity not mixing. A priori there is no reason for a relation between the spectra of positive and negative parity states. In the matrix model, however, we discover that there is a maximal degeneracy of these spectra which we have explicitly evaluated up to dimension 10: Each representation that admits both parities has maximal degeneracy, see figure 1.

It is instructive to see this explicitly in the two lowest dimensional representations with
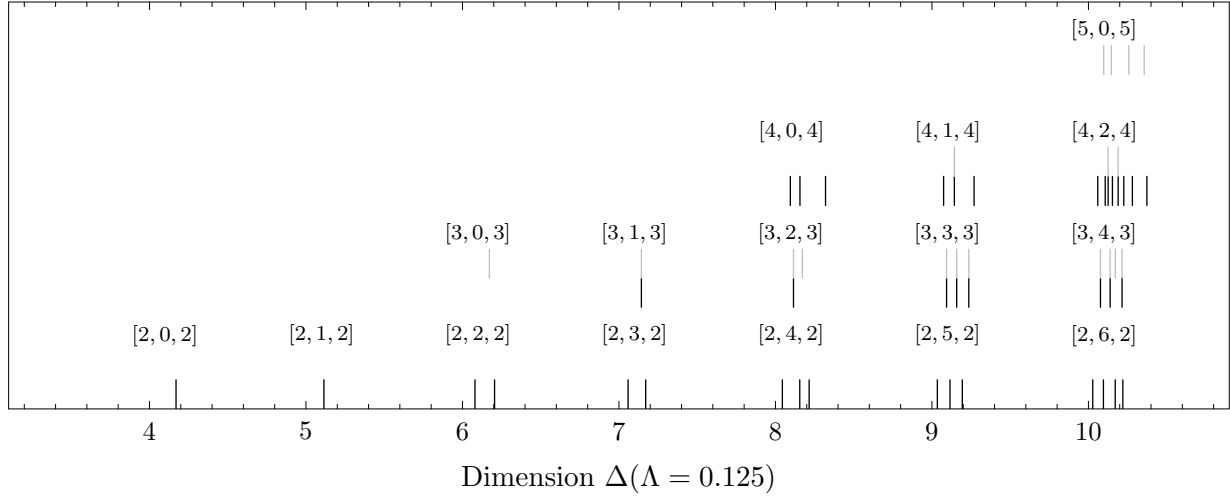
Figure 1: The complete spectrum of unprotected states in the SU(2) subsector up to bare dimension 10 at three-loop order for $\Lambda = 0.125$. States of positive (negative) parity are depicted as dark (light) lines. The states are grouped according to the Dynkin labels of their representation. Note that whenever states of positive and negative parity exist within a representation, then they are degenerate. The explicit results for the energy shifts are spelled out in the apppendix.

SO(6) Dynkin labels $[3, 1, 3]$ and $[3, 2, 3]$. In the $[3, 1, 3]$ representation of $\Delta_0 = 7$ one has two single-trace operators of opposite parity

$$\mathcal{O}_+^{[3,1,3]} = 2\,\text{tr}[\phi\phi\phi ZZZZ] - 3\,\text{tr}[\phi\phi Z\phi ZZZ] + 2\,\text{tr}[\phi\phi ZZ\phi ZZ]$$
$$- 3\,\text{tr}[\phi\phi ZZZ\phi Z] + 2\,\text{tr}[\phi Z\phi Z\phi ZZ]\,,$$
$$\mathcal{O}_-^{[3,1,3]} = -\,\text{tr}[\phi\phi Z\phi ZZZ] + \text{tr}[\phi\phi ZZZ\phi Z]\,.$$

The associated energy shifts up to order Λ^6 read

$$\delta\Delta_+ = \delta\Delta_- = 10\Lambda^2 - 65\Lambda^4 + 765\Lambda^6 \tag{37}$$

and are degenerate.

Similarly for the representation $[3, 2, 3]$ of $\Delta_0 = 8$ one finds one single-trace operator of positive parity

$$\mathcal{O}_+^{[3,2,3]} = \text{tr}[\phi\phi\phi ZZZZZ] - \text{tr}[\phi\phi Z\phi ZZZZ] - \text{tr}[\phi\phi ZZZZ\phi Z] + \text{tr}[\phi Z\phi ZZ\phi ZZ]$$

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and two operators of negative parity

$$\mathcal{O}^{[3,2,3]}_{-,1} = -\operatorname{tr}[\phi\phi Z\phi ZZZ] + \operatorname{tr}[\phi\phi ZZ\phi ZZ] - \operatorname{tr}[\phi\phi ZZZ\phi ZZ] + \operatorname{tr}[\phi\phi ZZZZ\phi Z]\,,$$
$$\mathcal{O}^{[3,2,3]}_{-,2} = -\operatorname{tr}[\phi\phi Z\phi ZZZ] - \operatorname{tr}[\phi\phi ZZ\phi ZZ] + \operatorname{tr}[\phi\phi ZZZ\phi ZZ] + \operatorname{tr}[\phi\phi ZZZZ\phi Z]\,.$$

In order to compute the energy shifts in the negative parity sector one has to diagonalize a 2×2 matrix, $\mathcal{O}_- := (\mathcal{O}^{[3,2,3]}_{-,1}, \mathcal{O}^{[3,2,3]}_{-,2})^\top$

$$D_2\mathcal{O}_- = \begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix}\mathcal{O}_- \quad D_4\mathcal{O}_- = \begin{pmatrix} -78 & -4 \\ -4 & -48 \end{pmatrix}\mathcal{O}_- \quad D_6\mathcal{O}_- = \begin{pmatrix} 938 & 60 \\ 68 & 540 \end{pmatrix}\mathcal{O}_-\,. \tag{38}$$

The outcome of this straightforward exercise are the energy shifts

$$\delta\Delta_+ = 8\Lambda^2 - 48\Lambda^4 + 536\Lambda^6 \tag{39}$$

$$\delta\Delta_- = \begin{cases} 8\Lambda^2 - 48\Lambda^4 + 536\Lambda^6 \\ 12\Lambda^2 - 78\Lambda^4 + 942\Lambda^6 \end{cases} \tag{40}$$

which again display degeneracy of opposite parity states.

Responsible for this degeneracy is the charge U of (36), which commutes with H. Up to three-loop orders it is given by

$$U_2 = 4(\{1,0\} - \{0,1\})\,, \tag{41a}$$
$$U_4 = 8(\{2,1,0\} - \{0,1,2\})\,, \tag{41b}$$
$$\begin{aligned} U_6 = {}& 8(\{1,0,3\} - \{0,1,3\} + \{0,3,2\} - \{0,2,3\}) + 18(\{3,2,1,0\} - \{0,1,2,3\}) \\ & + 38\{0,1,3,2\} - 30\{2,1,0,3\} - 38\{0,3,2,1\} + 30\{1,0,2,3\} \\ & - 34(\{1,0,3,2\} - \{0,2,1,3\})\,. \end{aligned} \tag{41c}$$

7   Nonsupersymmetric Deformations

In this final section we turn to the question of how strongly the observed three-loop integrability depends on the details of the model. A number of deformations are easily implemented by switching of fields in our computer programs.

To begin with let us consider the purely bosonic SO(3) × SO(6) model by setting the fermions to zero in (2). We calculate the effective vertex for this bosonic model similar to the above computation but subtract off the vacuum energy shift – which due to lack of

supersymmetry does not vanish – by hand[4]. Then the planar dilatation generator reads:

$$D_0 = \{\} \,, \tag{42a}$$

$$D_2 = 14\{\} - 2\{0\} \,, \tag{42b}$$

$$D_4 = -558\{\} + 122\{0\} - 4(\{0,1\} + \{1,0\}) \,, \tag{42c}$$

$$D_6 = \frac{246929}{6}\{\} - \frac{63683}{6}\{0\} + 584(\{0,1\} + \{1,0\})$$
$$+ 16\{0,2\} - 16(\{0,1,2\} + \{2,1,0\}) - 8(\{0,2,1\} - \{1,0,2\}) \,. \tag{42d}$$

It is important to note that the planar parity pairs are degenerate under the *individual* action of the operators

$$\{\}, \quad \{0\}, \quad \{0,1\} + \{1,0\} \quad \text{and}$$
$$2\{0,2\} - 2(\{0,1,2\} + \{2,1,0\}) - (\{0,2,1\} - \{1,0,2\}) \,.$$

Hence the purely bosonic $SO(3) \times SO(6)$ model displays integrability up to three-loops as well!

It turns out to be rather hard to break the hidden integrability structure of a large N matrix quantum mechanical model. For this we have studied a "brutal" deformation and considered the bosonic $SO(d)$ model ($i = 1, \ldots, d$)

$$H = \frac{M}{2} \operatorname{tr} a_i^\dagger a_i - \frac{\alpha}{2} \operatorname{tr} X_i X_j X_i X_j + \frac{\beta}{2} \operatorname{tr} X_i X_i X_j X_j \tag{43}$$

which for parameters $\alpha \neq \beta$ has lost all its traces of a higher dimensional gauge field theory origin. Then one finds (again dropping the vacuum energy shift)

$$D_0 = \{\} \,, \tag{44a}$$

$$D_2 = (-4\alpha + (2d+3)\beta)\{\} - 2\alpha\{0\} \,, \tag{44b}$$

$$D_4 = \tfrac{1}{2}\left((-100 - 12d)\alpha^2 + (168 + 96d)\alpha\beta + (-59 - 74d - 24d^2)\beta^2\right)\{\}$$
$$- 2\left(16\alpha^2 - (17 + 8d)\alpha\beta\right)\{0\} - 4\alpha^2(\{0,1\} + \{1,0\}) \,, \tag{44c}$$

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[4]The vacuum energy shift is actually dominant for large N.

$$D_6 = \tfrac{1}{2}\big((-560d - 2440)\alpha^3 + (280d^2 + 3668d + 6044)\alpha^2\beta$$
$$+ (-1536d^2 - 5274d - 4592)\alpha\beta^2 + (256d^3 + 1199d^2 + 1910d + 1029)\beta^3\big)\{\}$$
$$+ \big((-36d - 832)\alpha^3 + (768d + 1700)\alpha^2\beta + (-192d^2 - 797d - 791)\alpha\beta^2\big)\{0\}$$
$$+ \big(-164\alpha^3 + 4(20d + 37)\alpha^2\beta\big)\big(\{0,1\} + \{1,0\}\big)$$
$$+ 16\alpha^2\beta\{0,2\} - 16\alpha^3(\{0,1,2\} + \{2,1,0\}) - 8\alpha^3(\{0,2,1\} - \{1,0,2\}) \,. \tag{44d}$$

We see that the integrability of the model (43) only breaks down at the three-loop order, as the last line of (44d) destroys integrability for $\alpha \neq \beta$. If one chooses $\alpha = \beta$, however, integrability is stable up to three-loops.

We interpret the findings of this section as an indication that integrability might be a generic effect in large N *gauge* quantum mechanical models and independent of supersymmetry. In the same instance we learn from the example (43), that if integrability is to break down in a concrete model, it does so at rather high loop-orders (here three). This should caution us in our expectancy of an exact integrability of plane-wave matrix theory or $\mathcal{N} = 4$ Super Yang-Mills based on three-loop results, even if the idea seems too appealing to be false.

Acknowledgments

Appendix

Δ_0	rep	$\delta\Delta_+$	$\delta\Delta_-$
4	$[2,0,2]$	$12\Lambda^2 - 90\Lambda^4 + 1098\Lambda^6$	
5	$[2,1,2]$	$8\Lambda^2 - 52\Lambda^4 + 588\Lambda^6$	
6	$[2,2,2]$	$5.5279\Lambda^2 - 30.987\Lambda^4 + 326.57\Lambda^6$	
		$14.472\Lambda^2 - 107.01\Lambda^4 + 1327.4\Lambda^6$	
	$[3,0,3]$		$12\Lambda^2 - 78\Lambda^4 + 930\Lambda^6$
7	$[2,3,2]$	$4\Lambda^2 - 20\Lambda^4 + 202.5\Lambda^6$	
		$12\Lambda^2 - 84\Lambda^4 + 997.5\Lambda^6$	
	$[3,1,3]$	$10\Lambda^2 - 65\Lambda^4 + 765\Lambda^6$	$10\Lambda^2 - 65\Lambda^4 + 765\Lambda^6$
8	$[2,4,2]$	$3.0121\Lambda^2 - 13.863\Lambda^4 + 137.83\Lambda^6$	
		$15.208\Lambda^2 - 112.68\Lambda^4 + 1412.8\Lambda^6$	
		$9.7802\Lambda^2 - 63.455\Lambda^4 + 719.41\Lambda^6$	
	$[3,2,3]$	$8\Lambda^2 - 48\Lambda^4 + 536\Lambda^6$	$8\Lambda^2 - 48\Lambda^4 + 536\Lambda^6$
			$12\Lambda^2 - 78\Lambda^4 + 942\Lambda^6$
	$[4,0,4]$	$6.4912\Lambda^2 - 30.275\Lambda^4 + 293.55\Lambda^6$	
		$22.604\Lambda^2 - 167.80\Lambda^4 + 2059.3\Lambda^6$	
		$10.904\Lambda^2 - 69.927\Lambda^4 + 859.19\Lambda^6$	
9	$[2,5,2]$	$2.3432\Lambda^2 - 10.159\Lambda^4 + 100.35\Lambda^6$	
		$8\Lambda^2 - 48\Lambda^4 + 552\Lambda^6$	
		$13.657\Lambda^2 - 97.841\Lambda^4 + 1189.7\Lambda^6$	
	$[3,3,3]$	$6.4532\Lambda^2 - 35.770\Lambda^4 + 382.11\Lambda^6$	$6.4532\Lambda^2 - 35.770\Lambda^4 + 382.11\Lambda^6$
		$11.040\Lambda^2 - 72.143\Lambda^4 + 858.37\Lambda^6$	$11.040\Lambda^2 - 72.143\Lambda^4 + 858.37\Lambda^6$
		$16.507\Lambda^2 - 113.09\Lambda^4 + 1356.5\Lambda^6$	$16.507\Lambda^2 - 113.09\Lambda^4 + 1356.5\Lambda^6$
	$[4,1,4]$	$5.072\Lambda^2 - 22.574\Lambda^4 + 220.97\Lambda^6$	
		$10\Lambda^2 - 65\Lambda^4 + 785\Lambda^6$	$10\Lambda^2 - 65\Lambda^4 + 785\Lambda^6$
		$18.928\Lambda^2 - 133.43\Lambda^4 + 1591.0\Lambda^6$	

Table 1: Energy shifts for all unprotected states in SU(2) subsector up to dimension $\Delta_0 = 9$.

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