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Canonical Quantum Supergravity in Three-Dimensions

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We discuss the canonical treatment and quantization of matter coupled supergravity in three dimensions, with special emphasis on $N = 2$ supergravity. We then analyze the quantum constraint algebra; certain operator ordering ambiguities are found to be absent due to local supersymmetry. We show that the supersymmetry constraints can be partially solved by a functional analog of the method of characteristics. We also consider extensions of Wilson loop integrals of the type previously found in ordinary gravity, but now with connections involving the bosonic and fermionic matter fields in addition to the gravitational connection. In a separate section of this paper, the canonical treatment and quantization of non-linear coset space sigma models are discussed in a self contained way.

1 Introduction

The search for solutions of the Wheeler DeWitt equation [1] is one of the key issues of present day research in quantum gravity (for a recent review and many further references, see e.g. [2]). Unfortunately, progress has been severely hampered by technical problems, most notably the fact that the Wheeler DeWitt equation is a non-polynomial functional differential equation that is even difficult to *define* properly. The equation can be substantially simplified by retaining only a finite number of of degrees of freedom and thereby converting it into an ordinary partial differential equation (which is still not not easy to solve); this is the so-called "mini-superspace approximation", see e.g. [2] for further explanations. Another, and perhaps more promising attempt to come to grips with the Wheeler DeWitt equation, which does not involve any mutilation of the physical degrees of freedom, is based on Ashtekar's new variables (see [3] for a recent summary and many references). The main advantage of this approach, which so far works only in three and four space time dimensions, is that the canonical constraints become polynomial, which in turn facilitates the search for solutions. Indeed, it is then possible to construct formal solutions to all the constraints of pure quantum gravity in four dimensions [4, 5]. At a kinematical level, one can also incorporate matter in such a way that the constraints remain polynomial; however, little progress has been made so far in extending the results of [5] to matter coupled theories, but for special cases generalizations of the Wilson loop variables may be constructed [6]. Furthermore, it is not easy to see what has become of the singularities of perturbative quantum gravity in this approach. As a consequence, it is far from clear how the requirement of quantum mechanical consistency could possibly affect matter couplings in this approach, whereas experience with string theory [7] and $2d$ gravity [8] would make us expect such consistency requirements to impose stringent constraints on the allowed theories. In our opinion, the inclusion of matter couplings and their proper treatment beyond the purely kinematical aspects remains a major open problem. The present work constitutes an attempt to address this problem in the context of three dimensional supergravity.

This paper, then, deals with the canonical quantization of matter coupled supergravities in three dimensions. It is based on and considerably extends our previous results [9, 10], where mostly classical aspects were studied. In section 2, we review pure (topological) supergravities, which exist for any number of local supersymmetries; this section will also serve to set up our notations and conventions (see also [10]). The canonical treatment of non-linear sigma models is discussed in section 3. Since the results described there might also be of interest in other contexts, and because the literature on this topic seems to be scarce (see [9] for the canonical formulation of $N = 16$ supergravity and [11] for a discussion of flat space sigma models), we have aspired to make this section self contained as far as possible. Section 4 is devoted to a detailed study of the $N = 2$ theory, which represents the simplest non-trivial example of a locally supersymmetric theory with matter couplings in three dimensions. Since the generalization of these results to $N > 2$ is to a large extent straightforward, we have relegated the discussion of the higher N theories to an appendix, where we explain the redefinition of the gravitational connection required for the decoupling of the phase space variables. A central part of this paper is section 5, where we quantize the $N = 2$ theory and analyze its quantum constraint algebra. In particular, we will find that at least some of the operator ordering ambiguities present in the bosonic theories disappear due to local supersmmetry. Unfortunately, apart from the trivial solution $\Psi = 1$, we have so far not been able to find solutions to all of the constraints. Nonetheless, we can report some partial progress in this direction by demonstrating that at least one half of the supersymmetry constraints can be solved by a functional analog of the method of characteristics; this requires the exponentiation of an infinitesimal local supersymmetry transformation to a finite transformation. Furthermore, we discuss a class of partial solutions based on Wilson loop integrals over a connection constructed out of the gravitational fields *and* the matter fields, which can be regarded as a "supercovariant" extension of the Wilson loop functionals considered in [4].

As is well known, supergravity theories are generally characterized by rather complicated Lagrangians with non-polynomial scalar self-interactions and quartic fermionic terms. Readers might therefore wonder why one should choose to study them rather than models with simpler matter couplings such as scalar or Dirac fields without self-interactions. One of the main reasons why we prefer these models over simpler ones is the *geometrical structure* that is always present in the matter sectors of supergravity theories and that is at the origin of their "hidden symmetries" [12]. We believe that these symmetries may eventually play an important role in improving our understanding of the matter coupled Wheeler DeWitt equation for the following reason. Associated with the hidden symmetries, there are non-trivial observables (or conserved charges) in the sense of Dirac, which act on the space of solutions of the quantum constraints. These symmetries may therefore be interpreted as "solution generating symmetries" for the Wheeler DeWitt equation. An intriguing aspect is the emergence of infinite dimensional symmetries acting on the space of classical solutions of the gravitational field equations in the reduction to *two* dimensions [13] (for more recent developments, see [14]). If the theories could be quantized in a way compatible with these symmetries, the Wheeler DeWitt equation would become integrable in this reduction.

The fact that pure gravity in three dimensions is much easier to quantize than theories of gravity in higher dimensions has been fully appreciated only relatively recently, although classical aspects (absence of gravitational excitations, i.e. gravitons, in empty space, conical singularities at the locations of matter point sources, etc.) have been understood for a long time [15]. Since Einstein's action is superficially non-renormalizable in three dimensions, the theory was for a long time thought to make no more sense as a quantum theory than gravity in four dimensions. The discovery that the quantum theory can be solved exactly came thus as quite a surprise [16] (see also [17] for further studies of the quantum theory). An important ingredient in that work was the reformulation of Einstein's theory as a Chern Simons gauge theory. Here, we will, however, not make use of this formulation, but rather adopt an alternative and equivalent version based on [18], which is a direct extension of Ashtekar's formalism to three dimensions, and which provides an alternative route to solving the quantum theory[1]. Both formulations in an essential way exploit the fact that pure gravity in three dimensions is a topological theory, whose physical phase space is related to the moduli space of flat $\mathrm{SL}(2, \mathbf{R})$ connections and hence *finite*-dimensional for each genus. This result obviously relies on the use of the gravitational (or spin) connection as the primary canonical variable and would be much more difficult to obtain in the usual metric formulation of quantum gravity. Similar statements apply to pure supergravity in three dimensions, which has been discussed extensively in [10], where a solution to the quantum constraints of pure $N = 1$ supergravity has been presented, and in [20], where a Chern Simons formulation has been used. A common feature of the topological theories is the existence of a complete set of observables in the sense of Dirac, based on Wilson loops with or without dreibein and gravitino insertions. By means of these observables, the solutions to the quantum constraints can be obtained by applying the observables to a suitable "vacuum

[1] We note that there is no reality constraint on Ashtekar's variables in three dimensions unlike in four dimensions [19]. However, this feature, which may be viewed as another virtue of three dimensions, is lost when gravity is coupled to fermionic matter, as we will explain in section 4 and the appendix.

functional".

There are several reasons for studying locally supersymmetric theories rather than non-supersymmetric ones. Local supersymmetry leads to a constraint which can be thought of as the square root of the Wheeler DeWitt constraint, and is related to it in the same way as the Dirac equation is related to the Klein Gordon equation (as was first observed in [21]). However, due to the technical complexities, the early papers on canonical supergravity [22] make no attempt at exploiting this idea, but content themselves with setting up the canonical formalism and discussing the classical constraint algebra in terms of Poisson (or Dirac) brackets. The first investigation of the quantum theory appears to be [23], where the metric formulation is utilized. More recently, there have been several treatments of canonical quantum supergravity in the mini-superspace approximation [24]. A well known feature of supersymmetric theories is the absence of certain short distance singularities. From the analogy with the so-called non-renormalization theorems of perturbative supersymmetric quantum field theories [25] and explicit calculations in perturbative quantum supergravity [26] one would expect local supersymmetry to mitigate (if not eliminate) the singularities occurring in the canonical constraint operators as well, and thereby eliminate some of the operator ordering ambiguities that afflict the canonical treatment of non-supersymmetric theories [2]. In section 5, we will show that this is indeed the case for the constraint generators of supergravity. In particular, one of the supersymmetry generators is a first order functional differential operator and manifestly free of ordering ambiguities and short distance singularities, while the other is a second order operator, and the absence of ordering ambiguities is due to a non-trivial cancellation. These results provide a first glimpse as to what a non-perturbative non-renormalization theorem in canonical quantum supergravity might look like. However, it must be stressed that the question of non-perturbative divergences and operator singularities cannot be resolved until a scalar product in the space of physical states has been found.

While writing this paper, we received three preprints dealing with canonical quantum $N = 1$ supergravity in four dimensions. [36] discusses a supersymmetric extension of the solution found in [27] (with non-vanishing cosmological constant) and makes use of the Ashtekar formulation. [37] is

[2] For instance, the solutions of [4, 5] are based on the prescription that all functional differential operators should be moved to the right. If one chooses the opposite operator ordering prescription, one obtains very different, and presumably inequivalent solutions [27] (since the solutions of [27] require a non-vanishing constant in contrast to [5]).

based on the metric formulation and proposes a solution very similar to the Hartle-Hawking wave functional; see, however, [38] for a criticism of this ansatz.

2 Pure Gravity and Supergravity in Three Dimensions

The geometrical background for both pure and matter coupled supergravity is characterized by a general three-dimensional space time manifold, which is parametrized by local coordinates $x^\mu, y^\mu, \ldots$. We use Greek letters $\mu, \nu, ... = 0, 1, 2$ for curved indices in three dimensions and Latin letters $a, b, ...$ for tangent space indices transforming under the local Lorentz group $\mathrm{SO}(1,2) \cong \mathrm{SL}(2, \mathbf{R})$. With $e_\mu^{\ a}$ the usual dreibein, the space time metric is given by $g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab}$; it has has signature $(- + +)$. The Levi-Civita tensor with flat indices is defined by $\epsilon^{012} = -\epsilon_{012} = +1$; it is related to the Levi-Civita tensor density by $\varepsilon^{\mu\nu\rho} = e e_\mu^{\ a} e^\nu_{\ b} e^\rho_{\ c} \epsilon_{abc}$. Instead of the usual (first order) spin connection $\omega_{\mu bc}$, it is advantageous to use the dual connection

$$A_\mu{}^a = -\tfrac{1}{2}\varepsilon^{abc}\omega_{\mu bc} \tag{2.1}$$

in terms of which the Lorentz (i.e. $\mathrm{SO}(1,2)$) covariant derivative acting on a three-component vector V^a reads

$$D_\mu V_a = \partial_\mu V_a - \epsilon_{abc} A_\mu{}^b V^c \tag{2.2}$$

The use of $A_\mu{}^a$ rather than $\omega_{\mu bc}$ simplifies the canonical treatment considerably; in fact, as a canonical variable, this field is the direct analog of Ashtekar's variable in three dimensions. The field strength of the connection $A_\mu{}^a$ is related to the Riemann tensor by

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - \varepsilon_{abc} A_\mu{}^b A_\nu{}^c = -\tfrac{1}{2}\varepsilon_{abc} R_{\mu\nu}{}^{bc} \tag{2.3}$$

so that Einstein's action becomes

$$S = \tfrac{1}{4}\int \mathrm{d}^3 x\, e\, R = \tfrac{1}{4}\int \mathrm{d}^3 x \varepsilon^{\mu\nu\rho} e_\mu{}^a F_{\nu\rho a} \tag{2.4}$$

To introduce fermions, we make use of the real γ-matrices $\gamma_0 = i\sigma_2$, $\gamma_1 = \sigma_1$ and $\gamma_2 = \sigma_3$, which satisfy

$$\gamma_a \gamma_b = \eta_{ab}\mathbf{1} - \varepsilon_{abc}\gamma^c \tag{2.5}$$

The matrices γ^a generate the group $\mathrm{SL}(2,\mathbf{R})$ (the covering group of the Lorentz group $\mathrm{SO}(1,2)$); because this group is real, a reality constraint is not necessary unlike in four dimensions [3]. The Lorentz covariant derivative on a spinor ϵ reads

$$D_\mu \epsilon = \left(\partial_\mu + \tfrac{1}{2}\gamma_a A_\mu{}^a\right)\epsilon \tag{2.6}$$

In addition to the dreibein $e_\mu{}^a$ and the connection field $A_\mu{}^a$, we need N gravitino fields ψ_μ^I, where $I, J, ... = 1, ..., N$. The Rarita Schwinger action in three dimensions reads

$$S = \tfrac{1}{2}\int \mathrm{d}^3x\ \varepsilon^{\mu\nu\rho}\bar{\psi}_\mu^I D_\nu \psi_\rho^I, \qquad D_\nu\psi_\rho^I := \nabla_\nu\psi_\rho^I + \tfrac{1}{2}A_\mu{}^a\gamma_a\psi_\rho^I. \tag{2.7}$$

The sum of (2.7) and (2.4) is invariant under the local supersymmetry transformations

$$\delta_\epsilon e_\mu^{\ a} = \bar{\epsilon}^I\gamma_a\psi_\mu^I, \qquad \delta_\epsilon\psi_\mu^I = D_\mu\epsilon^I, \tag{2.8}$$

It may seem curious that the combined action is supersymmetric for arbitrary N, but this can be understood by noting that the topological bosonic and fermionic degrees of freedom need *not* balance in a supersymmetric theory unlike the propagating degrees of freedom. The fact that the above theories are topological is straightforward to verify. Namely, varying (2.4) with respect to the dreibein we immediately deduce that the field strength $F_{\mu\nu a}$ must vanish [4] (note that, in three dimensions, the Rarita Schwinger action is independent of the dreibein and therefore does not contribute to this variation). Hence, the connection $A_\mu{}^a$ is pure gauge, at least locally. However, $A_\mu{}^a$ may still be non-trivial in that there may not exist a globally defined function $g(x) \in \mathrm{SL}(2,\mathbf{R})$ such that $A_\mu \equiv \frac{1}{2}A_\mu{}^a\gamma_a = g^{-1}\partial_\mu g$. Similar conclusions hold for the gravitino fields ψ_μ^I. The Rarita Schwinger equation

[3] Our conventions regarding spinors are as follows. The charge conjugation matrix is $C = \gamma^0$ and obeys the usual properties $C^T = -C$ and $(C\gamma^a)^T = +C\gamma^a$. Majorana spinors satisfy $\bar{\chi} = \chi^T C$. Later on, we will make use of the Fierz identity

$$\bar{\chi}\varphi\,\bar{\lambda}\psi = -\tfrac{1}{2}\bar{\lambda}\varphi\,\bar{\chi}\psi - \tfrac{1}{2}\bar{\lambda}\gamma^a\varphi\,\bar{\chi}\gamma_a\psi$$

for anticommuting spinors χ, φ, λ and ψ. The underlying completeness relation can also be expressed directly in terms of γ-matrices

$$\gamma^a_{\alpha\beta}\gamma_{a\gamma\delta} = -\delta_{\alpha\beta}\delta_{\gamma\delta} + 2\delta_{\alpha\delta}\delta_{\gamma\beta}$$

[4] The variation with respect the connection $A_\mu{}^a$ tells us that the covariant derivative of the dreibein is equal to a fermionic bilinear (torsion); this equation can be solved for the connection in terms of the dreibein and the fermions ("second order formalism" [26]).

$\epsilon^{\mu\nu\rho} D_\nu \psi^I_\rho = 0$ implies that ψ^I_μ, too, is locally pure gauge: we can always find a locally defined spinor ϕ^I such that $\psi^I_\mu = D_\mu \phi^I$ (of course, this is only true if $F_{\mu\nu}(A) = 0$). Again, an obstruction only arises if the spinor ϕ^I cannot be defined globally.

As is customary, for the canonical treatment[5] we will assume the space time manifold to be the product of a spatial two-dimensional hypersurface and the real line, parametrized by the time coordinate t [16]. Derivatives with respect to t are denoted by a dot. Local coordinates on the spacelike hypersurface will be denoted by $x^i, y^i, ...,$ or simply just by bold face letters $\mathbf{x}, \mathbf{y}, ...,$ so that the three-dimensional coordinates decompose as $x^\mu = (t, x^i)$, etc. There is a corresponding split of the three-dimensional curved indices $\mu, \nu, \ldots$ into a time index t and spatial indices $i, j, \ldots$, so that $\mu = (t, i)$, etc. The Levi Civita tensor density splits as $\epsilon^{tij} = \epsilon^{ij}$, where $\epsilon^{ij} = \epsilon_{ij}$ is the tensor density on the spacelike manifold.

Finally, we explain the canonical decomposition of the dreibein and the metric. With $e = \det e_\mu{}^a$, we define the lapse and shift variables by [30]

$$n := eg^{tt}, \qquad n^i := g^{ti}/g^{tt}. \tag{2.9}$$

The dreibein is thus parametrized by the Lagrange multipliers n and n^i, and the remaining six components $e_i{}^a$, not all of which are physical phase space degrees of freedom since three of them can be eliminated in principle by local Lorentz rotations. The metric on the spatial hypersurface, its inverse and determinant are then given by

$$h_{ij} = g_{ij}, \qquad h^{ij} = g^{ij} - e^{-1} n n^i n^j, \qquad h = -en. \tag{2.10}$$

The following polynomial functions of the dreibein components $e_i{}^a$ will turn out to be useful

$$\widetilde{h}^{ij} = h h^{ij} = \varepsilon^{ik} \varepsilon^{jl} g_{kl}, \qquad e e^{ta} = -\tfrac{1}{2} \varepsilon^{abc} \varepsilon^{ij} e_{ib} e_{jc}, \tag{2.11}$$

Furthermore, it is convenient to employ a "curved" basis for the γ-matrices, which is given by $\gamma_i = e_i{}^a \gamma_a$ and $e\gamma^t = \frac{1}{2} \epsilon^{ij} \gamma_i \gamma_j$; observe that these, too, are *polynomial* functions of the dreibein components. Given two three-vectors X^μ and Y^μ, we have

$$eX^\mu Y_\mu = nX_n Y_n - n^{-1} \widetilde{h}^{ij} X_i Y_j,$$

[5]Standard references on the canonical formulation of gravity are [29, 30, 31]. For a general discussion of constrained Hamiltonian systems, see [29, 32].

$$
\begin{aligned}
eX_\mu\gamma^\nu\gamma^\mu Y_\nu = nX_nY_n - n^{-1}\widetilde{h}^{ij}X_iY_j \\
+ n^{-1}\,\varepsilon^{ij}X_i\,e\gamma^t\,Y_j - \varepsilon^{ij}(X_n\gamma_iY_j + X_i\gamma_jY_n), \\
e\gamma^\mu X_\mu = e\gamma^t X_n + n^{-1}\varepsilon^{ij}\,e\gamma^t\gamma_iX_j,
\end{aligned}
\tag{2.12}
$$

The index n here stands for the component normal to the spatial hypersurface. This component is defined by $X_n = X_t + n^iX_i$ and is related to $X^t = g^{t\mu}X_\mu$ by $nX_n = eX^t$.

3 Canonical Treatment of Non-Linear Sigma-Models

In the introduction, we have already mentioned the general result that the bosonic sectors of (extended) supergravities are governed by non-compact non-linear sigma-models. We will now describe the canonical treatment of these models. Since the results might also be useful in other contexts, we will temporarily ignore all fields other than the scalars so as to make the discussion self-contained. Because our main interest is the application of the canonical formalism to non-linear sigma models coupled to a non-trivial gravitational background characterized by the metric $g_{\mu\nu}$, we will, however, keep the dependence on the metric throughout; the flat space models are then easily recovered by putting $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere. Matter coupled supergravity theories in three dimensions have been completely classified recently [28]. In contrast to pure (topological) supergravity theories, which exist for any N, the number of local supersymmetries is bounded by $N \leq 16$ in the presence of matter couplings. The matter sectors of these theories are described by non-linear σ-models of the non-compact type [12], whose target spaces become more and more restricted with increasing N. More specifically, for three dimensional theories, we have the following results [28]: for $N = 1, 2$ and 3, the target manifolds $\mathcal{M}$ are Riemannian, Kaehler and quaternionic, respectively, whereas for $N = 4$, the target space is locally a product of two quaternionic manifolds associated with inequivalent $N = 4$ supermultiplets. Beyond $N = 4$, only homogeneous (and, in fact, symmetric) target spaces are allowed.

The standard sigma-model Lagrangian for an arbitrary Riemannian target manifold $\mathcal{M}$ is given by

$$
\mathcal{L} = -\tfrac{1}{2}eg^{\mu\nu}G_{mn}(\varphi)\partial_\mu\varphi^m\partial_\nu\varphi^n \tag{3.1}
$$

where $\mathcal{M}$ is parametrized by the coordinate fields $\varphi^m(x)$ with $m, n = 1, ...,$ $\dim\mathcal{M}$, and $G_{mn}(\varphi)$ is a Riemannian metric on $\mathcal{M}$. Obviously, the main problem here is posed by the non-linear interactions induced by the geometrical form of this Lagrangian, and this problem also makes its appearance in the canonical formalism. A first step in resolving the difficulties is to select (canonical) quantities that, despite their explicit dependence on the coordinate fields φ^m, transform as tensors under reparametrizations. Secondly, we will see that a further simplification can be achieved by utilizing tangent space tensors (tangent space, or just "flat", target space indices will be designated by $A, B, \ldots$). Accordingly, we introduce a vielbein $E_m{}^A(\varphi)$ satisfying

$$G_{mn}(\varphi) = E_m{}^A(\varphi) E_n{}^B(\varphi) \eta_{AB} \tag{3.2}$$

where η_{AB} is a flat metric in tangent space (which need not be unique); in the following, we will freely use this metric to raise and lower flat indices. We also define

$$P_\mu^A = \partial_\mu \varphi^m E_m{}^A \tag{3.3}$$

where $E_A{}^m$ is the inverse vielbein. The Lagrangian then takes the simple form

$$\mathcal{L} = -\tfrac{1}{2} e g^{\mu\nu} P_\mu^A P_\nu^B \eta_{AB} \tag{3.4}$$

This is also the form that appears in the supergravity Lagrangians to be used later.

The canonical momenta, which are conjugate to the coordinate fields φ^m, are now easily calculated

$$p_m := \frac{\delta \mathcal{L}}{\delta \dot{\varphi}^m} = -e g^{t\mu} G_{mn}(\varphi) \partial_\mu \varphi^n \tag{3.5}$$

The basic Poisson brackets are given by

$$\{p_m(\mathbf{x}), \varphi^n(\mathbf{y})\} = -\delta_m^n \delta(\mathbf{x}, \mathbf{y}) \tag{3.6}$$

Although the momenta do transform properly under reparametrizations (namely as vectors, i.e. elements of the tangent space $T_\varphi \mathcal{M}$), the coordinate fields φ^m do not; therefore, one must deal with non-covariant expressions at the intermediate stages of every calculation if one uses these variables. The Hamiltonian is given by

$$H := \int \mathrm{d}^2\mathbf{x}\ (p_m \dot{\varphi}^m - \mathcal{L}) \tag{3.7}$$

Canonical quantization will be awkward to carry out in terms of the variables φ^m and p_m due to operator ordering problems and the concomitant short distance singularities (which may also spoil general covariance in target space by "anomalies"). In any case, quantization will require a definite ordering prescription for the operators involving the momenta p_m. Here, we find it convenient to employ another set of canonical variables and to perform the quantization directly in terms of them rather than in terms of of the original variables φ^m and p_m. This procedure defines the quantum theory in an unambiguous way, as it corresponds to a definite choice of operator ordering.

As our basic canonical variables, we choose the "composite" quantities

$$P_A := \frac{\delta \mathcal{L}}{\delta P_t^A} = E_A{}^m(\varphi) p_m \quad , \quad P_i^A := \partial_i \varphi^m E_m{}^A(\varphi) \tag{3.8}$$

The Hamiltonian (3.7) can also be obtained from

$$H(P_A, \varphi) = \int \mathrm{d}^2\mathbf{x} \, \left(P_A P_t^A - \mathcal{L}\right) \tag{3.9}$$

The variables P_A, which we now regard as the momenta, evidently correspond to an anholonomic basis in tangent space (whereas the p_m are like a coordinate basis). Our choice is also motivated by the fact that the variables (3.8) are precisely the ones which will appear in the the supergravity constraints to be derived in later sections.

To compute the canonical brackets of P_A and P_i^A, we employ the basic Poisson brackets (3.6). Of course, it does not matter at this point whether or not we use the original fields φ^m and p_m for this purpose; afterwards, we can simply "forget" how the results were derived. A straightforward calculation yields

$$\begin{aligned}
\{P_A(\mathbf{x}), P_B(\mathbf{y})\} &= \Omega_{AB}{}^C(\mathbf{x}) P_C(\mathbf{x}) \, \delta(\mathbf{x}, \mathbf{y}) \\
\{P_A(\mathbf{x}), P_i^B(\mathbf{y})\} &= \left(\delta_A^B \partial_i - \Omega_{AC}{}^B(\mathbf{x}) P_i^C(\mathbf{x})\right) \delta(\mathbf{x}, \mathbf{y}) \\
\{P_i^A(\mathbf{x}), P_j^B(\mathbf{y})\} &= 0
\end{aligned} \tag{3.10}$$

where $\Omega_{AB}{}^C := 2E_{[A}{}^m E_{B]}{}^n \partial_m E_n{}^C$ are the coefficients of anholonomy (by ∂_m, we denote the derivative with respect to φ^m). Here and in the remainder, spatial derivatives ∂_i will always be understood to act on the *first* argument in the δ-function (i.e. $\mathbf{x}$ in (3.10)). The Poisson brackets (3.10) will be regarded as the basic relations from now on. If we parametrize phase space in terms of the variables φ^m and the momenta P_A, these brackets are reproduced by the general formula

$$\{f,g\} = \int \mathrm{d}^2\mathbf{x}\, \Big(E_A{}^m \frac{\delta f}{\delta\varphi^m(\mathbf{x})}\frac{\delta g}{\delta P_A(\mathbf{x})} - E_A{}^m \frac{\delta g}{\delta\varphi^m(\mathbf{x})}\frac{\delta f}{\delta P_A(\mathbf{x})}$$
$$+\Omega_{AB}{}^C(\mathbf{x}) P_C(\mathbf{x}) \frac{\delta f}{\delta P_A(\mathbf{x})}\frac{\delta g}{\delta P_B(\mathbf{x})}\Big) \tag{3.11}$$

where f and g are arbitrary functionals of φ^m and P_A.

The transition to the quantized theory is implemented by the replacement

$$P_A(\mathbf{x}) \longrightarrow \widehat{P}_A(\mathbf{x}) = -iE_A{}^m(\varphi(\mathbf{x}))\frac{\delta}{\delta\varphi^m(\mathbf{x})}$$
$$P_i^A(\mathbf{x}) \longrightarrow \widehat{P}_i^A(\mathbf{x}) = \partial_i\varphi^m(\mathbf{x}) E_m{}^A(\varphi(\mathbf{x})) \tag{3.12}$$

The ordering prescription implicit in this replacement ensures that the relations (3.10) can be directly replaced by quantum mechanical commutators (modulo factors of i), and the geometrical structure of (3.10) is thus preserved.

At this point, not much more can be said if the target space $\mathcal{M}$ is an arbitrary Riemannian manifold. For this reason and also in view of the fact that the target manifolds of supergravity are usually constrained by local supersymmetry to be of a very special type, we will now make further assumptions on the structure of $\mathcal{M}$. The simplest possibility is to assume that the target space is a group manifold, i.e. $\mathcal{M} = G$ for some Lie group G. Although the target spaces relevant to our investigation are *not* group manifolds in general, we discuss this case first since all relevant formulas can be derived from it. This is because, as we will explain below, we can formally treat the coset manifolds occurring in supergravity on the same footing as group manifolds if we add suitable gauge degrees of freedom.

For group manifolds, we assume the vielbein (3.2) to be a left invariant vector field; this means that

$$E_A{}^m(\tilde\varphi(\varphi)) = E_A{}^n(\varphi)\frac{\partial\tilde\varphi^m(\varphi)}{\partial\varphi^n}. \tag{3.13}$$

where $\varphi \to \tilde\varphi(\varphi)$ is the diffeomorphism induced by left multiplication. Then, the coefficients of anholonomy and the flat metric are given by the structure constants of G, viz.

$$\Omega_{AB}{}^C = -f_{AB}{}^C, \qquad \eta_{AB} = f_{AC}{}^D f_{BD}{}^C, \tag{3.14}$$

where the structure constants f_{ABC} are defined through the commutation relations

$$[Z_A, Z_B] = f_{AB}{}^C Z_C \tag{3.15}$$

for the generators Z_A of G. The vielbein $E_m{}^A$ can be explicitly computed by introducing a matrix representation $\mathcal{V} = \mathcal{V}(\varphi^m(x)) \in G$

$$\mathcal{V}^{-1}\partial_m \mathcal{V} = E_m{}^A Z_A \tag{3.16}$$

where as before ∂_m denotes the derivative with respect to the coordinate field φ^m. From (3.3), we get the identification

$$P_\mu^A Z_A = \partial_\mu \varphi^m \, \mathcal{V}^{-1}\partial_m \mathcal{V} = \mathcal{V}^{-1}\partial_\mu \mathcal{V} \tag{3.17}$$

The field theoretic model obtained in this way goes by the the name of "principal chiral model"; its Lagrangian is simply obtained by substituting (3.17) into (3.4). From the results listed above, we can immediately derive the relevant brackets by substituting the structure constants for the coefficients of anholonomy; in addition, we can determine the brackets between the momenta P_A and the matrices $\mathcal{V}(\varphi)$. The result is

$$\begin{aligned} \{P_A(\mathbf{x}), P_B(\mathbf{y})\} &= -f_{AB}{}^C P_C \, \delta(\mathbf{x}, \mathbf{y}), \\ \{\mathcal{V}(\mathbf{x}), P_A(\mathbf{y})\} &= \mathcal{V} Z_A \, \delta(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{3.18}$$

As a check, we can recalculate the brackets between P_A and P_i^B (cf. (3.10)) from these formulas and (3.17).

The above formulas are not yet quite what we want, since the relevant target spaces to be considered in the remainder are coset spaces rather than group manifolds; however, the above brackets will nonetheless prove useful in that they will enable us to compute the relevant Poisson brackets for coset space sigma models as well. As is well known, any symmetric space can be represented as a coset G/H; in the case at hand, the group G is non-compact, and H its maximally compact subgroup [12]. There are now two equivalent formulations. One either parametrizes the manifold $\mathcal{M} = G/H$ in terms of coordinates φ^m with $m = 1, ..., \dim G/H$ as described above; or one introduces extra coordinate fields $u^r(x)$ associated with the subgroup H (so that $r = 1, ..., \dim H$), in which case the coordinates (φ^m, u^r) parametrize the whole group G. If one uses only the physical fields $\varphi^m(x)$, part of the invariance under the isometry group G is realized non-linearly. In the second case, the invariance transformations under the isometry group can be realized linearly, at the expense of introducing an artificial gauge invariance necessary to remove the unphysical degrees of freedom corresponding to the fields $u^r(x)$. In the canonical formalism, this gauge invariance will lead to constraints.

Since we prefer to make use of the second formulation, let us introduce a matrix representation $\mathcal{V} = \mathcal{V}(\varphi^m(x), u^r(x)) \in G$. To get rid of the unwanted degrees of freedom which are represented by the fields $u^r(x)$, we postulate in addition that the Lagrangian should be invariant under the transformations

$$\mathcal{V}(x) \longrightarrow g\mathcal{V}(x)h(x) \tag{3.19}$$

with $g \in G$ and $h(x) \in H$, so that putting $u^r = 0$, we recover the description in terms of the physical fields φ^m (this gauge choice is sometimes referred to as the "unitary gauge"). We split the generators Z of G into the generators X^α of H ($\alpha, \beta, ... = 1, ..., r$) and the remaining coset generators Y^A [6]; the structure constants are decomposed accordingly. For a *symmetric* space, we have $f_{ABC} = f_{\alpha\beta C} = 0$, and (3.15) reads

$$[Y^A, Y^B] = f^{AB}{}_\gamma X^\gamma \quad , \quad [X^\alpha, Y^B] = f^{\alpha B}{}_C Y^C \quad , \quad [X^\alpha, X^\beta] = f^{\alpha\beta}{}_\gamma X^\gamma \tag{3.20}$$

To write down the Lagrangian, we decompose the Lie algebra valued expression $\mathcal{V}^{-1}\partial_\mu \mathcal{V}$ according to

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = P^A_\mu Y_A + Q^\alpha_\mu X_\alpha \tag{3.21}$$

The Lagrangian is then again given by (3.4). Note, however, that the sum over A now runs only over the coset generators. As a consequence, the fields Q^α_μ do not appear the Lagrangian (however, they do couple to the fermionic fields in its supersymmetric extension); they are just the gauge fields required by local H symmetry. So we see that it is the Lagrangian that determines which degrees of freedom are physical and which are not; we can convert the principal chiral model into a coset space sigma model simply by omitting those P^A_μ corresponding to a subgroup of H from the sum (3.4). Of course, for a non-compact group G, there is only one choice of the subgroup H for which the Hamiltonian is positive definite. If we define the canonical momenta by

$$P_A := \frac{\delta \mathcal{L}}{\delta P^A_t} \quad , \quad Q_\alpha := \frac{\delta \mathcal{L}}{\delta Q^\alpha_t} \tag{3.22}$$

[6] We hope that the dual use of the indices $A, B, ...$ will not cause confusion; they label either all group generators as in (3.15) or just the coset generators as here.

the absence of Q_t^α from the Lagrangian immediately implies the constraint $Q_\alpha = 0$; this must be interpreted as a weak equality in accordance with the general theory of constraints [29, 32]. The Hamiltonian is now given by

$$H(P_A, Q_\alpha, \varphi, u) = \int \mathrm{d}^2\mathbf{x} \, \Big(P_A P_t^A + Q_\alpha Q_t^\alpha - \mathcal{L} \Big) \tag{3.23}$$

We repeat that the main difference from the canonical point of view between the principal chiral model and the coset space sigma model characterized by this Hamiltonian is that the momenta Q_α corresponding to the subgroup H have become constraints. Nonetheless, the combined set of momenta P_A and Q_α still obeys the same Poisson brackets as before; consequently, we can read off the result directly from (3.18). So, we get

$$\begin{aligned}
\{P_A(\mathbf{x}), P_B(\mathbf{y})\} &= -f_{AB}{}^{\gamma} Q_\gamma(\mathbf{x}) \, \delta(\mathbf{x}, \mathbf{y}), \\
\{Q_\alpha(\mathbf{x}), P_B(\mathbf{y})\} &= -f_{\alpha B}{}^{C} P_C(\mathbf{x}) \, \delta(\mathbf{x}, \mathbf{y}), \\
\{Q_\alpha(\mathbf{x}), Q_\beta(\mathbf{y})\} &= -f_{\alpha\beta}{}^{\gamma} Q_\gamma(\mathbf{x}) \, \delta(\mathbf{x}, \mathbf{y})
\end{aligned} \tag{3.24}$$

Furthermore,

$$\begin{aligned}
\{\mathcal{V}(\mathbf{x}), P_A(\mathbf{y})\} &= \mathcal{V} Y_A \, \delta(\mathbf{x}, \mathbf{y}), \\
\{\mathcal{V}(\mathbf{x}), Q_\alpha(\mathbf{y})\} &= \mathcal{V} X_\alpha \, \delta(\mathbf{x}, \mathbf{y})
\end{aligned} \tag{3.25}$$

which shows that the Q_α generate local H transformations on $\mathcal{V}$.

To construct an operator representation for the P_A and Q_α, we could simply take over formula (3.12) with an appropriate split of indices. The matrix ${E_m}^A$ would then have to be decomposed accordingly. The general parametrization of $\mathcal{V}$ in terms of φ^m and u^r adopted so far would, however, lead to formulas which are somewhat unwieldy for practical calculations, as the physical and unphysical degrees are difficult to disentangle. For this reason, we choose a slightly different parametrization in terms of which the constraints are easier to solve. Locally, we can always assume that the matrix $\mathcal{V}(\varphi, u)$ can be written in the form

$$\mathcal{V}(\varphi, u) = \mathcal{V}_0(\varphi) h(u) \tag{3.26}$$

Then a straighforward calculation shows that

$$\begin{aligned}
\mathcal{V}^{-1}\partial_m \mathcal{V} &= Q_m^\alpha(\varphi, u) X_\alpha + {E_m}^A(\varphi, u) Y_A \\
\mathcal{V}^{-1}\partial_r \mathcal{V} &= {E_r}^\alpha(u) X_\alpha
\end{aligned} \tag{3.27}$$

where we have expanded the right hand side in terms of the subgroup and the coset generators, thereby defining the various submatrices. Consequently, the vielbein on G (which is a $\dim G \times \dim G$ matrix) is triangular

$$\mathrm{Vielbein} = \begin{pmatrix} E_m{}^A(\varphi, u) & Q_m^\alpha(\varphi, u) \\ 0 & E_r{}^\alpha(u) \end{pmatrix} \tag{3.28}$$

The advantage of this parametrization is that the $\dim G/H \times \dim G/H$ matrix $E_m{}^A(\varphi, u)$ can be identified with the vielbein on $\mathcal{M} = G/H$ after a u-dependent tangent space rotation. The inverse vielbein on G is given by

$$\mathrm{Inverse\,Vielbein} = \begin{pmatrix} E_A{}^m(\varphi, u) & -E_A{}^m Q_m^\beta E_\beta{}^r(\varphi, u) \\ 0 & E_\alpha{}^r(u) \end{pmatrix} \tag{3.29}$$

and $E_A{}^m$ can be identified with the inverse vielbein on G/H up to a u-dependent H rotation. Inserting these expressions into (3.12) and relabeling indices, one arrives at the following operator representations

$$\begin{aligned} \widehat{P}_A(\mathbf{x}) &= iE_A{}^m \left(\frac{\delta}{\delta\varphi^m(\mathbf{x})} - Q_m^\beta E_\beta{}^r \frac{\delta}{\delta u^r(\mathbf{x})} \right) \\ \widehat{P}_i^A(\mathbf{x}) &= \partial_i \varphi^m E_m{}^A \\ \widehat{Q}_i^\alpha(\mathbf{x}) &= \partial_i \varphi^m Q_m^\alpha + \partial_i u^r E_r{}^\alpha \end{aligned} \tag{3.30}$$

The constraint Q_α is realized by the operator

$$\widehat{Q}_\alpha(\mathbf{x}) = iE_\alpha{}^r \frac{\delta}{\delta u^r(\mathbf{x})} \tag{3.31}$$

and depends only on the gauge degrees of freedom. Observe that the momentum operator $\widehat{P}_A$ can be viewed as a connection on the principal fiber bundle $G \to G/H$ with base space G/H and fiber H (it defines a "horizontal subspace" of $T_{\{\varphi,u\}}G$ at each point); note, however, that we are dealing with *functional*, not ordinary derivatives here. We recall that in the quantized theory, any physical wave functional $\Psi[\varphi, u]$ must satisfy $\widehat{Q}_\alpha \Psi = 0$; with the above parametrization, this is simply solved by $\Psi = \Psi[\varphi]$. We emphasize however, that the u-dependence of $\widehat{P}_A$ cannot be dropped since otherwise the constraint algebra (3.24) would not be obeyed.

From (3.19) it follows that G acts as a group of isometry transformations on the target space $\mathcal{M} = G/H$. The associated charge density $\mathcal{J}(\mathbf{x})$ (which is a matrix with values in the Lie algebra of G) is obtained by sandwiching the momenta and constraints between the matrix $\mathcal{V}$ and its inverse. Thus

$$\mathcal{J}(\mathbf{x}) = \mathcal{V}\left(P_A Y^A + Q_\alpha X^\alpha\right)\mathcal{V}^{-1} \tag{3.32}$$

The charges

$$\mathcal{Q} = \int \mathrm{d}^2\mathbf{x}\, \mathcal{J}(\mathbf{x}) \tag{3.33}$$

constitute the canonical generators of the isometry group, and generate the isometry transformations on the fields, as can be verified from the relations (3.25). Incidentally, this formula also remains valid in supergravity since the rigid group G does not act on the fermions, or only via induced H-rotations. The above expressions for the charge differs from the one given in [9] by the constraint generator of H gauge transformations, which precisely removes the fermionic bilinears in the formulas given there.

To conclude this section, we give another and equivalent representation of the operators (3.12) and (3.30), to be used in section 5. When working with a concrete matrix representation $\mathcal{V}$, we can regard the elements of $\mathcal{V}$ as independent fields. To be sure, we would then have to introduce second class constraints to ensure that $\mathcal{V}$ remains an element of the group G. However, there is no need to enter into the details of this construction here, as long as $\mathcal{V}$ is always understood to be an element of the group G in all formulas below. Let us simply assume that the Lagrangian is given as a function of the matrix field $\mathcal{V}(x)$ and its inverse $\mathcal{V}^{-1}(x)$ as well as its "derivatives" $P_\mu^A(x)$. Similarly, the physical states are assumed to be represented by wave functionals Ψ which depend on $\mathcal{V}$, $\mathcal{V}^{-1}$ and their spatial derivatives. On any such wave functional, we define the momentum operators $\widehat{P}_A$ through their action on $\mathcal{V}$ and $\mathcal{V}^{-1}$, which is given by $\widehat{P}_A\mathcal{V} := i\mathcal{V}Z_A$ and $\widehat{P}_A\mathcal{V}^{-1} := -iZ_A\mathcal{V}^{-1}$. It follows immediately that $[\widehat{P}_A, \widehat{P}_B] = if_{AB}{}^C\widehat{P}_C$. Defining the matrix valued derivative operator $\delta/\delta\mathcal{V}$ by $(\delta/\delta\mathcal{V})_{pq} := \delta/\delta\mathcal{V}_{qp}$, the operator representation for the momenta becomes

$$\widehat{P}_A = i\,\mathrm{Tr}\Big(\mathcal{V}Z_A\frac{\delta}{\delta\mathcal{V}}\Big), \tag{3.34}$$

Since the matrices Z_A generate the Lie algebra of G, the action of this operator is tangent to the submanifold defined by the group G in the space of all matrices $\mathcal{V}$, and hence does not depend on how we define the functional $\Psi[\mathcal{V}]$ away from it. If we are dealing with a coset space sigma model, the same remarks as before apply; we simply have to split the group indices into subgroup and coset indices, and the momenta corresponding to the subgroup become constraints. However, the solution to the constraint $\widehat{Q}_\alpha \Psi = 0$ apparently cannot be cast into a simple form in this representation. Finally, we note the simple expression for the quantum mechanical charge operator (3.33) in this representation; it is

$$\mathcal{Q} = \mathcal{Q}_A Z^A, \qquad \mathcal{Q}_A = \int \mathrm{d}^2\mathbf{x}\, \mathrm{Tr}\Big(Z_A \mathcal{V} \frac{\delta}{\delta \mathcal{V}} \Big). \tag{3.35}$$

(If we are dealing with a coset space sigma model, the generators are $\mathcal{Q}_A$ and $\mathcal{Q}_\alpha$). It is easy to check that $\mathcal{Q}$ generates the global G-transformations acting from the left on $\mathcal{V}$ according to (3.19). Furthermore,

$$[\widehat{P}_A\,,\, \mathcal{Q}] = 0 \tag{3.36}$$

In particular, the global charges commute with the constraints $\widehat{Q}_\alpha$ for a coset space sigma model and are thus observables in the sense of Dirac, or "conserved charges".

4 The $N = 2$ Theory

The simplest nontrivial example of matter coupled supergravity in three dimensions is provided by the $N = 2$ theory [34, 10]. We will here follow the presentation in [10], where this model has been described in great detail, and only summarize its main features before moving on to the canonical formulation. In addition to the gravitational degrees of freedom, the theory contains two gravitinos ψ_μ^I ($I = 1, 2$), two matter fermions χ^1 and χ^2 and two (real) bosons that live on the coset space $\mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2)$. The fermions are real (Majorana) spinors, which we will combine into complex (Dirac) spinors by defining $\psi_\mu = \frac{1}{\sqrt{2}}(\psi_\mu^1 + i\psi_\mu^2)$ and $\chi = \frac{1}{\sqrt{2}}(\chi^1 + i\chi^2)$. If the $\mathrm{SL}(2,\mathbf{R})$ symmetry is linearly realized, the fermions transform only under the gauge group H. The case of abelian H is a little peculiar as the relative normalization between the H generators and the coset (i.e. G/H) generators is not fixed, unlike for non-abelian H. Consequently, the requirement of local supersymmetry and H invariance does not uniquely determine the fermionic $\mathrm{SO}(2)$ charges in contrast to the theories with $N \geq 3$ [7]. Our charge assignments agree with those used in our previous work [10] and coincide

[7] This distinction between abelian and non-abelian subgroup H was first emphasized by B. de Wit (private communication).

with the ones obtained by dimensional reduction of $N = 1$ supergravity in four dimensions [33, 26] to three dimensions (but differ from the ones that one would obtain from the Lagrangian given in [28]). Thus the matter fermion χ has charge $+\frac{3}{2}$ and the gravitino field ψ_μ has charge $-\frac{1}{2}$; the SO(2) group can then be interpreted as the helicity group of the four-dimensional ancestor theory.

As explained in the foregoing section, we will parametrize the bosonic fields by a matrix $\mathcal{V}$ which takes values in the group $\mathrm{SL}(2,\mathbf{R})$. The unphysical degree of freedom corresponding to the subgroup H is removed by postulating invariance under local H transformations as in (3.19). In accordance with the notation used in section 3, we denote the generator of the SO(2) subgroup by X, and the remaining generators by Y^1 and Y^2. Again, we find it convenient to switch to a complex basis $Y = \frac{1}{\sqrt{2}}(Y_1 + iY_2), Y^* = \frac{1}{\sqrt{2}}(Y^1 - iY^2)$. The $\mathrm{SL}(2,\mathbf{R})$ commutation relations then read

$$[X,Y] = 2iY, \qquad [X,Y^*] = -2iY^*, \qquad [Y,Y^*] = -2iX, \tag{4.1}$$

and formula (3.21) becomes

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = P_\mu^* Y + P_\mu Y^* + Q_\mu X, \tag{4.2}$$

Remembering our SO(2) charge assignments and (2.6), we can immediately write down the fully covariant derivatives on the spinors

$$\begin{aligned} D_\mu\chi &:= \partial_\mu\chi + \tfrac{1}{2}A_\mu{}^a\gamma_a\chi - \tfrac{3}{2}iQ_\mu\chi \\ D_\mu\psi_\nu &:= \nabla_\mu\psi_\nu + \tfrac{1}{2}A_\mu{}^a\gamma_a\psi_\mu + \tfrac{1}{2}iQ_\mu\psi_\nu, \\ D_\mu e_\nu{}^a &:= \nabla_\mu e_\nu{}^a - \varepsilon^{abc}A_{\mu b}e_{\nu c}, \end{aligned} \tag{4.3}$$

The Lagrangian of $N = 2$ supergravity is then given by

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{L}^{(0)} &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho}e_\mu{}^aF_{\nu\rho a} + \varepsilon^{\mu\nu\rho}\,\bar\psi_\mu D_\nu\psi_\rho, \\ \mathcal{L}^{(1)} &= -\,eg^{\mu\nu}P_\mu P_\nu^* + e\,\bar\psi_\mu\gamma^\nu\gamma^\mu\chi\,P_\nu^* + e\,\bar\chi\gamma^\mu\gamma^\nu\psi_\mu\,P_\nu \\ &\quad -\tfrac{1}{2}e\,\bar\psi_\mu\gamma^\nu\gamma^\mu\chi\,\bar\chi\psi_\nu - \tfrac{1}{2}e\,\bar\chi\gamma^\mu\gamma^\nu\psi_\mu\,\bar\psi_\nu\chi, \\ \mathcal{L}^{(2)} &= -\,\tfrac{1}{2}e\,\bar\chi\gamma^\mu D_\mu\chi + \tfrac{1}{2}e\,D_\mu\bar\chi\gamma^\mu\chi - \tfrac{3}{2}e\,\bar\chi\chi\,\bar\chi\chi. \end{aligned} \tag{4.5}$$

Apart from the contribution involving the gauge field Q_μ, $\mathcal{L}^{(0)}$ is identical with the topological Lagrangian introduced in section 2 (see (2.4) and (2.7)) for $N = 2$. The full Lagrangian (4.4) is invariant under the local supersymmetry transformations

$$\begin{aligned}
\delta_\epsilon \psi_\mu &= D_\mu \epsilon + \tfrac{1}{2}\varepsilon_{\mu\nu\rho}\gamma^\nu \epsilon\, \bar{\chi}\gamma^\rho \chi, \\
\delta_\epsilon e_\mu^{\ a} &= \bar{\epsilon}\gamma^a \psi_\mu - \bar{\psi}_\mu \gamma^a \epsilon, \\
\delta_\epsilon \mathcal{V} &= \bar{\chi}\epsilon\, Y + \bar{\epsilon}\chi Y^*, \\
\delta_\epsilon \chi &= \gamma^\mu \epsilon\, \hat{P}_\mu,
\end{aligned} \tag{4.6}$$

where $\hat{P}_\mu := P_\mu - \bar{\psi}_\mu \chi$ is the supercovariantization of P_μ. We refrain from giving the variation of the gravitational connection $A_\mu^{\ a}$, as the variation of the action under (4.6) is proportional to the torsion equation, and therefore $\delta A_\mu{}^a$ can be chosen so as to cancel this contribution (see e.g. [26]). The torsion equation reads

$$D_{[\mu} e_{\nu]}^{\ a} = \bar{\psi}_{[\mu}\gamma^a \psi_{\nu]} - \tfrac{1}{2}\varepsilon^{abc} e_{\mu b} e_{\nu c}\, \bar{\chi}\chi \tag{4.7}$$

In the canonical treament of this model, one encounters the following technical difficulty [10]. As it turns out, the Dirac brackets between the components of $A_i{}^a$ do *not* vanish, but rather commute to give a bilinear expression in the matter fermions χ; furthermore, the Dirac brackets between $A_i{}^a$ and χ do not vanish, either. This feature prevents straightforward quantization via the replacement of phase space variables by functional differential operators. It has already been pointed out in [10] that, for the $N = 2$ theory, the phase space variables can be decoupled through a redefinition of the gravitational connection by a fermionic bilinear[8]. In [10], this redefinition was performed *after* setting up the Hamiltonian formulation, but, as we shall now demonstrate, it is much more convenient to do so already at the level of the Lagrangian, since this will entail substantial simplifications. For this purpose, we define a new connection field $A'^{\ a}_\mu$ by

$$A'^{\ a}_\mu := A_\mu{}^a + \varepsilon^{abc} e_{\mu b}\, \bar{\chi}\gamma_c \chi \tag{4.8}$$

Observe that the fermionic bilinear is purely imaginary, and hence the connection becomes *complex*. The redefinition and the decoupling also work

[8] A very similar redefinition is necessary in the metric formulation [23].

for the higher N theories, although the details are more involved and are therefore explained in appendix A.

To see how this redefinition affects the Lagrangian, we now substitute the new connection (4.8) into the the above Lagrangian. For the gravitational curvature, one finds

$$F_{\mu\nu a} = F'_{\mu\nu a} - 2\varepsilon_{abc} D'_{[\mu}(e_{\nu]}^{\,b}\,\bar{\chi}\gamma^c\chi\,) - 3\varepsilon_{abc} e_\mu^{\,b} e_\nu^{\,c}\,\bar{\chi}\chi\,\bar{\chi}\chi, \tag{4.9}$$

Here, the prime on the covariant derivative indicates the replacement of ${A_\mu}^a$ by ${A'_\mu}^a$. Because ${A'_\mu}^a$ is complex, the Dirac conjugate of $D'_\mu\chi$ would involve the complex conjugate connection; it is thus different from the the derivative $D'_\mu\bar{\chi}$ used in the above equation, which is defined by $D'_\mu\bar{\chi} = \partial_\mu\bar{\chi} - \frac{1}{2}{A'_\mu}^a\bar{\chi}\gamma_a + \frac{3}{2}iQ_\mu\bar{\chi}$. On the other hand, $\mathcal{L}^{(1)}$ does not contain ${A_\mu}^a$ and is therefore unchanged. In $\mathcal{L}^{(2)}$ we may replace ${A_\mu}^a$ by ${A'_\mu}^a$, because the difference vanishes by simple symmetry arguments.

Insertion of the new connection into $\mathcal{L}^{(0)}$ yields

$$\begin{aligned} \tfrac{1}{4}\varepsilon^{\mu\nu\rho} e_\mu^{\,a} F_{\nu\rho a} &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho} e_\mu^{\,a} F'_{\nu\rho a} - \tfrac{1}{2} e e^\mu_{\,a}\, D'_\mu(\,\bar{\chi}\gamma^c\chi\,) + \tfrac{3}{2} e\,\bar{\chi}\chi\,\bar{\chi}\chi, \\ \varepsilon^{\mu\nu\rho}\,\bar{\psi}_\mu D_\nu\psi_\rho &= \varepsilon^{\mu\nu\rho}\,\bar{\psi}_\mu D'_\nu\psi_\rho - \tfrac{1}{2}\varepsilon^{\mu\nu\rho}(\,\bar{\psi}_\mu\gamma_\nu\chi\,\bar{\chi}\psi_\rho - \bar{\psi}_\mu\chi\,\bar{\chi}\gamma_\nu\psi_\rho\,), \end{aligned} \tag{4.10}$$

where the first equation holds up to a total derivative only. We observe two crucial results. First, most of the higher order fermionic terms in the action are canceled by the redefinition of the spin connection, which greatly simplifies the Lagrangian. Secondly, the Einstein term now contributes to the Dirac term in such a way that all derivatives on $\bar{\chi}$ disappear from the action. This is rather fortunate, because otherwise, either hermiticity would be lost (if all derivatives were defined with the complex connection ${A'_\mu}^a$, as we did above), or we would have to introduce the complex conjugate connection $({A_\mu}^a)^*$, which would not be subject to simple commutation relations (cf. (4.18) below) and spoil the decoupling of bosonic and fermionic fields in the canonical brackets.

The total Lagrangian in terms of the new connection is now the sum of

$$\begin{aligned} \mathcal{L}^{(0)} &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho} e_\mu^{\,a} F_{\nu\rho a} + \varepsilon^{\mu\nu\rho}\,\bar{\psi}_\mu D_\nu\psi_\rho, \\ \mathcal{L}^{(1)} &= -e g^{\mu\nu} P_\mu P^*_\nu + e\,\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi\,P^*_\nu + e\,\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\,(P_\nu - \bar{\psi}_\nu\chi), \\ \mathcal{L}^{(2)} &= -e\,\bar{\chi}\gamma^\mu D_\mu\chi. \end{aligned} \tag{4.11}$$

where we henceforth drop all primes, as the old spin connection will no longer be used.

For the canonical treatment, we must now perform a space time split as described at the end of section 2. After a little work, one arrives at the following decomposition of the Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(0)} = & -\tfrac{1}{2}\varepsilon^{ij} e_i{}^a \dot{A}_{ja} - \varepsilon^{ij}\,\bar{\psi}_i \dot{\psi}_j \\
& + \tfrac{1}{4}(n^{-1} e e^{at} - n^k e_k{}^a)\varepsilon^{ij} F_{ija} - \tfrac{1}{2} i \varepsilon^{ij}\,\bar{\psi}_i \psi_j\, Q_t \\
& + \tfrac{1}{2}\varepsilon^{ij}(D_i e_{ja} - \bar{\psi}_i \gamma_a \psi_j) A_t{}^a + \varepsilon^{ij}(D_i \bar{\psi}_j \psi_t + \bar{\psi}_t D_i \psi_j) \\
\mathcal{L}^{(1)} = & -n \hat{P}_n \hat{P}_n^* + n^{-1} \widetilde{h}^{ij} \hat{P}_i \hat{P}_j^* + n^{-1}\varepsilon^{ij}(\bar{\psi}_i\, e\gamma^t \chi\, P_j^* - \bar{\chi}\, e\gamma^t \psi_i\, \hat{P}_j) \\
& + \varepsilon^{ij}(\bar{\psi}_n \gamma_j \chi\, P_i^* - \bar{\psi}_i \gamma_j \chi\, P_n^* + \bar{\chi}\gamma_i \psi_n\, \hat{P}_j - \bar{\chi}\gamma_i \psi_j\, \hat{P}_n), \\
\mathcal{L}^{(2)} = & -\bar{\chi}\, e\gamma^t\, \dot{\chi} - \tfrac{1}{2}\,\bar{\chi}\, e\gamma^t \gamma_a \chi\, A_t{}^a + \tfrac{3}{2} i\, \bar{\chi}\, e\gamma^t \chi\, Q_t \\
& - n^{-1}\varepsilon^{ij}\,\bar{\chi}\, e\gamma^t \gamma_i D_j \chi - n^k\, \bar{\chi}\, e\gamma^t D_k \chi.
\end{aligned} \tag{4.12}
$$

which will serve as our starting point for the canonical treatment.

We next compute the canonical momenta, which lead to second class constraints (to make the formulas less cumbersome, we will not explicitly indicate the **x**-dependence below); they are

$$
\begin{aligned}
p_a{}^i &= \frac{\delta \mathcal{L}}{\delta \dot{e}_i{}^a} = 0, \qquad & \Pi_a{}^i &= \frac{\delta \mathcal{L}}{\delta \dot{A}_i{}^a} = \tfrac{1}{2}\varepsilon^{ij} e_{ja}, \\
\pi^i &= \frac{\delta \mathcal{L}}{\delta \dot{\bar{\psi}}_i} = 0, \qquad & \bar{\pi}^i &= \frac{\delta \mathcal{L}}{\delta \dot{\psi}_i} = -\varepsilon^{ij} \bar{\psi}_j, \\
\lambda &= \frac{\delta \mathcal{L}}{\delta \dot{\bar{\chi}}} = 0, \qquad & \bar{\lambda} &= \frac{\delta \mathcal{L}}{\delta \dot{\chi}} = \bar{\chi}\, e\gamma^t.
\end{aligned} \tag{4.13}
$$

We can now read off the second class constraints

$$
\begin{aligned}
P_a^i &:= p_a{}^i, \qquad & Z_a^i &:= \Pi_a{}^i - \tfrac{1}{2}\varepsilon^{ij} e_{ja}, \\
\Lambda &:= \lambda, \qquad & \bar{\Lambda} &:= \bar{\lambda} - \bar{\chi}\, e\gamma^t, \\
\Gamma^i &:= \pi^i, \qquad & \bar{\Gamma}^i &:= \bar{\pi}^i + \varepsilon^{ij} \bar{\psi}_j.
\end{aligned} \tag{4.14}
$$

As is well known, the Dirac brackets [29] are defined by

$$
\{A, B\}_* = \{A, B\} - \sum_{K,L} \{A, K\} C(K, L) \{L, B\}, \tag{4.15}
$$

where $C(\cdot,\cdot)$ is the inverse of the Poisson matrix defined by

$$\sum_L C(K,L)\{L,M\} = \delta(K,M), \tag{4.16}$$

Here $K, L, \ldots$ label the constraints (4.14). A little care has to be taken as we are dealing with fermions here, because they are anticommuting (Grassmann) variables. When defining the momenta by (4.13), $\bar\lambda$ is the *negative* momentum of χ and the Hamiltonian reads $\dot{\bar\chi}\lambda - \bar\lambda\dot\chi - \mathcal{L}$. To get the right equations of motion, the Poisson brackets, which are symmetric for fermions, must read $\{\lambda_\alpha, \bar\chi_\beta\} = -\delta_{\alpha\beta}$, which in this order corresponds to the bosonic bracket $\{p_a{}^i, e_j{}^b\} = -\delta^i_j\delta^b_a$. Thus the Poisson brackets are always negative if the momentum is the first entry.

Calculation of the $C(\cdot,\cdot)$ matrix now gives

$$\begin{aligned}
C(Z^i_a, P^j_b) &= 2\varepsilon_{ij}\eta^{ab}, \\
C(Z^i_a, \Lambda) &= 2h^{-1}\varepsilon_{abc}e_i{}^c\,\bar\chi\gamma^b\ e\gamma^t, \\
C(\bar\Lambda, \Lambda) &= -h^{-1}\,e\gamma^t, \\
C(\bar\Gamma^i, \Gamma^j) &= -\varepsilon_{ij}\mathbf{1},
\end{aligned} \tag{4.17}$$

all other components vanish. As a consistency check, we note that $C(\cdot,\cdot)$ is antisymmetric if and only if both entries are bosonic. From these formulas we can deduce the crucial result that the Dirac bracket between different components of the spin connection now vanishes, which is not the case for the original spin connection [9, 10]. This result follows essentially from the vanishing of $C(Z^i_a, \bar\Lambda)$ and the fact that Λ does not depend on the dreibein. Furthermore, it is also easy to check that the spin connection now commutes with χ. However, it does *not* commute with $\bar\chi$, which is therefore not a good canonical variable. For this reason, we will not use $\bar\chi$, but rather $\bar\lambda$ as an independent phase space variable; the two fields are related by $\bar\lambda = e\bar\chi\gamma^t$ by (4.14). All non-vanishing Dirac brackets are then numerical and given by

$$\begin{aligned}
\{A_i{}^a(\mathbf{x}), e_j{}^b(\mathbf{y})\}_* &= 2\varepsilon_{ij}\eta^{ab}\,\delta(\mathbf{x},\mathbf{y}) \\
\{\chi_\alpha(\mathbf{x}), \bar\lambda_\beta(\mathbf{y})\}_* &= -\delta_{\alpha\beta}\,\delta(\mathbf{x},\mathbf{y}) \\
\{\psi_{i\alpha}(\mathbf{x}), \bar\psi_{j\beta}(\mathbf{y})\}_* &= \varepsilon_{ij}\delta_{\alpha\beta}\,\delta(\mathbf{x},\mathbf{y})
\end{aligned} \tag{4.18}$$

We repeat that the absence of the complex conjugate connection $(A_\mu{}^a)^*$ from the constraints is an important consistency check on our results, since this field would have non-vanishing brackets with both $A_\mu{}^a$ and χ.

Next we proceed to the discussion of the first class constraints. Defining the momenta of the scalar fields as in (3.8), but with the complex notation introduced above, we have

$$\begin{aligned} P &= \frac{\delta\mathcal{L}}{\delta P_t^*} = \,-\,nP_n + n\,\bar\psi_n\chi - \varepsilon^{ij}\,\bar\psi_i\gamma_j\chi, \\ P^* &= \frac{\delta\mathcal{L}}{\delta P_t} = \,-\,nP_n^* + n\,\bar\chi\psi_n - \varepsilon^{ij}\,\bar\chi\gamma_i\psi_j, \\ Q &= \frac{\delta\mathcal{L}}{\delta Q_t} = \,-\,\tfrac{1}{2}i\varepsilon^{ij}\,\bar\psi_i\psi_j + \tfrac{3}{2}i\,\bar\chi\,e\gamma^t\chi. \end{aligned} \tag{4.19}$$

From (3.18) we obtain their Poisson (or Dirac) brackets

$$\begin{gathered} \{\mathcal{V},P\} = \mathcal{V}Z, \qquad \{\mathcal{V},P^*\} = \mathcal{V}Z^*, \qquad \{\mathcal{V},Q\} = \mathcal{V}Y, \\ \{P,P^*\} = 2iQ, \qquad \{Q,P\} = -2iP, \qquad \{Q,P^*\} = 2iP^*. \end{gathered} \tag{4.20}$$

These brackets together with (4.18) constitute the complete list of non-vanishing Dirac brackets of $N=2$ supergravity (brackets that have not been listed vanish). To summarize, our basic canonical variables are

$$e_i{}^a,\ A_i{}^a,\ \mathcal{V},\ P,\ P^*,\ Q,\ \bar\lambda,\ \chi,\ \bar\psi_i,\ \psi_i, \tag{4.21}$$

Observe that quantization is now straightforward to implement by replacing the momenta by functional differential operators. The Lagrange multipliers leading to first class constraints are

$$n^{-1},\ n^i,\ A_{ta},\ \bar\psi_t,\ \psi_t. \tag{4.22}$$

As the momentum Q does not contain any time derivative in (4.19), we have the first class constraint

$$T = \,-\,Q - \tfrac{1}{2}i\varepsilon^{ij}\,\bar\psi_i\psi_j + \tfrac{3}{2}i\,\bar\lambda\chi, \tag{4.23}$$

which is the generator of local U(1) transformations. The other first class constraints are obtained by varying the Lagrangian with respect to the multipliers (4.22). $A_t{}^a$ yields the Lorentz constraint

$$L_a = \tfrac{1}{2}\varepsilon^{ij}D_ie_{ja} - \tfrac{1}{2}\varepsilon^{ij}\,\bar\psi_i\gamma_a\psi_j - \tfrac{1}{2}\bar\lambda\gamma_a\chi. \tag{4.24}$$

The multipliers $\bar\psi_t$ and ψ_t correspond to the supersymmetry constraints

$$\begin{aligned} \mathcal{S} &= \varepsilon^{ij}D_i\psi_j - \chi\,P^* - \varepsilon^{ij}\,\gamma_i\chi\,P_j^*, \\ \bar{\mathcal{S}} &= \varepsilon^{ij}D_i\bar\psi_j - P\,\bar\chi + \varepsilon^{ij}P_j\,\bar\chi\gamma_i + \varepsilon^{ij}\,\bar\psi_i(\chi\,\bar\chi\gamma_j - \gamma_j\chi\,\bar\chi). \end{aligned} \tag{4.25}$$

The asymmetry in these expressions is caused by the fact that it is the redefined connection A_i^a which appears in the derivative on $\bar{\psi}_j$ in the second line (and not A_i^{a*}!), but in fact, the second expression is the Dirac conjugate of the first because of the Fierz identity

$$\varepsilon^{ij}\left(\bar{\psi}_i \chi\, \bar{\chi}\gamma_j - \bar{\psi}_i \gamma_j \chi\, \bar{\chi}\right) = -\varepsilon^{abc}\varepsilon^{ij} e_{jb} \bar{\chi}\gamma_c \chi\, \bar{\psi}_j \gamma_a \tag{4.26}$$

The first constraint in (4.25) is manifestly polynomial; to render the second polynomial as well, we multiply it by $e\gamma^t$ from the right so as to replace $\bar{\chi}$ by $\bar{\lambda}$. In this way, we get

$$\widetilde{\mathcal{S}} := \bar{\mathcal{S}}\, e\gamma^t = \varepsilon^{ij}\, D_i \bar{\psi}_j\, e\gamma^t - P\,\bar{\lambda} - \varepsilon^{ij} P_j\, \bar{\lambda}\gamma_i - \varepsilon^{ij}\, \bar{\psi}_i (\chi\, \bar{\lambda}\gamma_j + \gamma_j \chi\, \bar{\lambda}). \tag{4.27}$$

The derivative of the Lagrangian with respect to n^k and n^{-1} gives the diffeomorphism and Hamiltonian constraints:

$$\begin{aligned}
\mathcal{H}'_k &= -\tfrac{1}{4} e_k^{\,a} \varepsilon^{ij} F_{ija} - \bar{\lambda} D_k \chi + \hat{P}\hat{P}_k^* + \hat{P}_k \hat{P}^* \\
&\quad + \varepsilon^{ij} P_i^* \bar{\psi}_j \gamma_k \chi - \varepsilon^{ij} \hat{P}_i \bar{\chi}\gamma_k \psi_j, \\
\mathcal{H}' &= \tfrac{1}{4} e e^{ta}\, \varepsilon^{ij} F_{ija} - \varepsilon^{ij}\, \bar{\lambda}\gamma_i D_j \chi + \hat{P}\hat{P}^* + \widetilde{h}^{ij} \hat{P}_i \hat{P}_j^* \\
&\quad - \varepsilon^{ij} P_i^*\, \bar{\psi}_j e\gamma^t \chi + \varepsilon^{ij} \hat{P}_i\, \bar{\lambda}\psi_j,
\end{aligned} \tag{4.28}$$

where we used the supercovariant quantities

$$\begin{aligned}
\hat{P}_i &= P_i - \bar{\psi}_i \chi, \qquad \hat{P} = P + \varepsilon^{ij}\, \bar{\psi}_i \gamma_j \chi, \\
\hat{P}_i^* &= P_i^* - \bar{\chi}\psi_i, \qquad \hat{P}^* = P^* + \varepsilon^{ij}\, \bar{\chi}\gamma_i \psi_j.
\end{aligned} \tag{4.29}$$

The covariant momentum $\hat{P}$ is nothing but the time component of the quantity already defined in (4.6), as can be seen by (4.19).

At first sight it seems rather difficult to cast these constraints into a polynomial form, as $\bar{\chi}$ appears also implicitly through $\hat{P}^*$ and $\hat{P}_i^*$, but indeed all terms containing $\bar{\chi}$ can be eliminated by adding suitable multiples of other constraints. Let us first consider the diffeomorphism constraint. The terms containing $\bar{\chi}$ are

$$-\,\hat{P}\,\bar{\chi}\psi_k - \varepsilon^{ij} \hat{P}_i\, \bar{\chi}\gamma_k \psi_j + \varepsilon^{ij} \hat{P}_k\, \bar{\chi}\gamma_i \psi_j. \tag{4.30}$$

A short calculation shows that this is equal to

$$\bar{\mathcal{S}}\psi_k - \varepsilon^{ij}\, D_i \bar{\psi}_j \psi_k. \tag{4.31}$$

Thus if we subtract $\bar{\mathcal{S}}\psi_k$ from $\mathcal{H}'_k$, all the $\bar{\chi}$ terms disappear and we get the polynomial constraint

$$\begin{aligned}\mathcal{H}_k = &-\tfrac{1}{4}e_k{}^a\varepsilon^{ij}F_{ija} - \bar{\lambda}D_k\chi - \varepsilon^{ij}\,D_i\bar{\psi}_j\psi_k \\ &+ P_k^*\hat{P} + P^*\hat{P}_k + \varepsilon^{ij}P_i^*\,\bar{\psi}_j\gamma_k\chi \end{aligned}\tag{4.32}$$

As usual this is not the real generator of spatial diffeomorphisms, but it generates extra U(1), Lorentz, and supersymmetry transformations. The generator of pure translations is

$$\mathcal{D}_k = \mathcal{H}_k - Q_kT - A_k{}^aL_a - \bar{\psi}_k\mathcal{S},\tag{4.33}$$

and reads explicitly

$$\begin{aligned}\mathcal{D}_k = &-\tfrac{1}{2}\varepsilon^{ij}(\partial_iA_j{}^ae_{ka} + A_k{}^a\partial_ie_{ja}) - \varepsilon^{ij}(\partial_i\bar{\psi}_j\psi_k + \bar{\psi}_k\partial_i\psi_j) \\ &+ PP_k^* + P^*P_k + QQ_k - \bar{\lambda}\partial_k\chi.\end{aligned}\tag{4.34}$$

The situation is similar for the Hamiltonian constraint. The $\bar{\chi}$ terms to be subtracted are

$$\varepsilon^{ij}\hat{P}\,\bar{\chi}\gamma_i\psi_j - \widetilde{h}^{ij}\hat{P}_i\,\bar{\chi}\psi_j.\tag{4.35}$$

Again this nonpolynomial expression can be rendered polynomial by subtracting a suitable multiple of $\bar{\mathcal{S}}$. It is equal to

$$-\,\varepsilon^{ij}\,\bar{\mathcal{S}}\gamma_i\psi_j + \varepsilon^{ij}\varepsilon^{kl}\,D_k\bar{\psi}_l\gamma_i\psi_j + \varepsilon^{ij}\hat{P}_i\,\bar{\lambda}\psi_j.\tag{4.36}$$

Inserting this we get the Hamiltonian constraint

$$\begin{aligned}\mathcal{H} = &\tfrac{1}{4}ee^{ta}\,\varepsilon^{ij}F_{ija} + \varepsilon^{ij}\,\bar{\lambda}\gamma_jD_i\chi + P^*\hat{P} + \widetilde{h}^{ij}P_i^*\hat{P}_j \\ &+ 2\varepsilon^{ij}\hat{P}_i\,\bar{\lambda}\psi_j + \varepsilon^{ij}\varepsilon^{kl}\,D_i\bar{\psi}_j\gamma_k\psi_l - \varepsilon^{ij}P_i^*\,\bar{\psi}_j\,e\gamma^t\chi.\end{aligned}\tag{4.37}$$

We can now compute the classical Dirac bracket algebra of constraints and verify that it closes. The brackets between the "kinematical" constraints L, T, $\mathcal{H}_k$ and $\mathcal{S}$ are straightforward and yield the expected results. The brackets between the supersymmetry generators require more work, and, in particular, repeated use of the Fierz identities quoted in section 2. After a somewhat lengthy calculation, one obtains

$$\begin{aligned}&\{\mathcal{S}[\bar{\epsilon}],\widetilde{\mathcal{S}}[\eta]\}_* = \int \mathrm{d}^2\mathbf{x}\,(\,-\,\bar{\epsilon}\eta\,\mathcal{H} + \varepsilon^{kl}\mathcal{H}_k\,\bar{\epsilon}\gamma_l\eta + 2i\bar{\lambda}\eta\,\bar{\epsilon}\chi\,T + 2\bar{\lambda}\eta\,\bar{\epsilon}\gamma^a\chi\,L_a), \\ &\{\mathcal{S}[\bar{\epsilon}],\mathcal{S}[\bar{\epsilon}']\}_* = 0, \\ &\{\widetilde{\mathcal{S}}[\eta],\widetilde{\mathcal{S}}[\eta']\}_* = 2\int \mathrm{d}^2\mathbf{x}\,\varepsilon^{ij}e_i{}^a(\bar{\psi}_j\eta\,\widetilde{\mathcal{S}}\gamma_a\eta' + \bar{\psi}_j\gamma_a\eta\,\widetilde{\mathcal{S}}\eta').\end{aligned}\tag{4.38}$$

where, for convenience, the supersymmetry generators have been smeared with smooth spinorial (i.e. anti-commuting) test functions $\bar{\epsilon}(\mathbf{x})$ and $\eta(\mathbf{x})$ according to

$$\mathcal{S}[\bar{\epsilon}] := \int \mathrm{d}^2\mathbf{x}\, \bar{\epsilon}(\mathbf{x})\mathcal{S}(\mathbf{x}) \quad , \quad \widetilde{\mathcal{S}}[\eta] := \int \mathrm{d}^2\mathbf{x}\, \widetilde{\mathcal{S}}(\mathbf{x})\eta(\mathbf{x}) \tag{4.39}$$

The formulas (4.38) show that indeed all the bosonic constraints can be generated from the fermionic ones, and in this sense, the supersymmetry generators can be thought of as the square roots of the bosonic ones. We note that a complete check of closure would also require the determination of the brackets involving the Hamiltonian constraint, but we omit this consistency check.

Finally, one can verify that the constraints are the canonical generators of the associated space dependent gauge transformations on the fields as expected. Since this computation is completely straightforward, we refrain from giving further details.

5 Quantization and Constraint Algebra

We will now perform the canonical quantization and re-examine the relations (4.38) in the context of the quantum theory. After expressing all the first class constraints as polynomials in terms of the canonical variables (4.21), it is straightforward to give a quantum operator representation for them. We use the convention that the commutator is obtained by multiplying the Dirac bracket by $-i$. We will also define the quantum constraints to be i times the classical constraints. In this way they generate the same transformations on the fields as the classical constraints and no extra factors will appear in the constraint algebra.

For each pair of canonically conjugate variables, we have to choose one which is to be represented by a multiplication operator. The bosonic fields and their momenta P, P^* and Q will be represented by the matrix $\mathcal{V}$ and the differential operators (3.34), respectively. In this representation, the wave functional depends on the scalars via the matrix $\mathcal{V}(\mathbf{x})$. As we explained in section 3, we could equivalently use the coordinates (φ^m, u) on $\mathrm{SL}(2,\mathbf{R})$ (where $m = 1, 2$, and u parametrizes the subgroup SO(2)) as multiplication operators and represent the momenta by (3.30). For the remaining operators we are in principle free to choose $e_i{}^a$ or $A_i{}^a$, $\bar{\lambda}$ or χ, $\bar{\psi}_i$ or ψ_i as multiplication operators; of course, it is very likely that the resulting quantum theories will be inequivalent, depending on this choice. We will here adopt an operator representation which renders all the constraints homogeneous in the functional differential operators (see below), and which is given by

$$\begin{aligned} e_i{}^a &\longrightarrow -2i\varepsilon_{ij}\frac{\delta}{\delta A_{ja}}, \\ \bar\lambda_\alpha &\longrightarrow i\frac{\delta}{\delta\chi_\alpha}, \\ \psi_{i\alpha} &\longrightarrow -i\varepsilon_{ij}\frac{\delta}{\delta\bar\psi_{j\alpha}}. \end{aligned} \tag{5.1}$$

The wave functional $\Psi(\phi)$ will thus depend on the fields $A_i{}^a, \mathcal{V}, \chi, \bar\psi_i$, which we will collectively denote by ϕ. With the above operator representation the Lorentz, U(1) and diffeomorphism constraints become well defined if ordered in such a way that all differential operators appear *to the right*. In particular, they will then generate the respective space dependent gauge transformations on the fields (i.e. without extra anomalous contributions). Their algebra reads

$$\begin{aligned} [\mathcal{D}[n^k], \mathcal{D}[m^k]] &= \mathcal{D}[m^l\partial_l n^k - n^l\partial_l m^k], \\ [L[\omega^a], L[\upsilon^a]] &= L[\varepsilon^{abc}\omega_b\upsilon_c], \\ [\mathcal{D}[n^k], L[\omega^a]] &= L[-n^l\partial_l\omega^a], \\ [\mathcal{D}[\omega^a], T[q]] &= T[-n^l\partial_l q]. \end{aligned} \tag{5.2}$$

where $\mathcal{D}[n^k] := \int \mathrm{d}^2\mathbf{x}\, n^k(\mathbf{x})\mathcal{D}_k(\mathbf{x})$, etc. Remember that the quantum constraints are defined as i times the classical constraints, thus there are no extra factors of i in the algebra.

Next, we turn to the supersymmetry generators $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. In the representation (5.1), $\mathcal{S}$ becomes a first order differential operator; smearing with an anticommuting spinor paramater $\bar\epsilon$ as in (4.39), we get

$$\mathcal{S}[\bar\epsilon] = \int \mathrm{d}^2\mathbf{x}\left(D_i\bar\epsilon\frac{\delta}{\delta\bar\psi_i} + \bar\epsilon\chi\,\mathrm{Tr}\left(\mathcal{V}Y^*\frac{\delta}{\delta\mathcal{V}}\right) - 2\bar\epsilon\gamma^a\chi\,P_i\,\frac{\delta}{\delta A_i{}^a}\right). \tag{5.3}$$

From this expression, it is easy to see that there are no ordering ambiguities and no short distance singularities in $\mathcal{S}$. In this form $\mathcal{S}$ can be regarded as a "kinematical" constraint whose action on the wave functional is just given by the transformation of the fields under supersymmetry. Below we will see that it is even possible to exponentiate this generator to obtain a finite supersymmetry transformation.

The constraint $\widetilde{\mathcal{S}}$, on the other hand, is a second order operator in our representation and is therefore "dynamical" like the Hamiltonian constraint $\mathcal{H}$. Explicitly, we have

$$\widetilde{\mathcal{S}}[\eta] = \int \mathrm{d}^2\mathbf{x} \Big(4i\varepsilon^{abc} D_i \bar{\psi}_j \gamma_a \eta \frac{\delta}{\delta A_i{}^b} \frac{\delta}{\delta A_j{}^c} + i\eta_\alpha \operatorname{Tr}\left(\mathcal{V} Y \frac{\delta}{\delta \mathcal{V}} \right) \frac{\delta}{\delta \chi_\alpha}$$

$$- 2i \Big(\eta_\alpha \, (\bar{\psi}_i \gamma^a \chi) - (\gamma^a \eta)_\alpha \, (P_i - \bar{\psi}_i \chi) \Big) \frac{\delta}{\delta A_i{}^a} \frac{\delta}{\delta \chi_\alpha} \Big). \tag{5.4}$$

In contrast to $\mathcal{S}$, $\widetilde{\mathcal{S}}$ must be regularized because it contains products of functional differential operators at coincident points. This can be done for instance by smearing all operators with a regularized delta function $\delta_\Lambda(\mathbf{x}, \mathbf{y})$ according to

$$\mathcal{O}(\mathbf{x}) \longrightarrow \mathcal{O}_\Lambda(\mathbf{x}) := \int \mathrm{d}^2\mathbf{y} \, \delta_\Lambda(\mathbf{x}, \mathbf{y}) \mathcal{O}(\mathbf{y}), \tag{5.5}$$

where Λ is a regularization parameter such that $\lim_{\Lambda \to 0} \delta_\Lambda(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x}, \mathbf{y})$. Denoting the regularized constraint by $\widetilde{\mathcal{S}}_\Lambda$, we automatically obtain a regularized Hamiltonian from the regularized commutator of $\mathcal{S}$ with $\widetilde{\mathcal{S}}_\Lambda$ that is obtained from (4.38). In the remainder, we will simply assume that our formal manipulations can be made rigorous by means of a suitable regularization.

In addition to the problem of regularization there is also an operator ordering ambiguity in the definition of $\widetilde{\mathcal{S}}$, as one can place the differential operators either to the left or to the right [9]. Whichever prescription we choose, we then *define* the ordering of the bosonic constraint operators by the right hand side of the commutator (4.38), so all ambiguities disappear once we have fixed the ordering of $\widetilde{\mathcal{S}}$. Relying on formal manipulations, one can convince oneself that any change of ordering in $\widetilde{\mathcal{S}}$ produces a singular term proportional to

$$i\delta(\mathbf{x}, \mathbf{x}) \, \bar{\psi}_i \gamma^a \eta \frac{\delta}{\delta A_i{}^a} \tag{5.6}$$

where the overall factor depends on which terms are interchanged. In (5.4), all differential operators have been placed to the right. Remarkably, and in contrast to ordinary gravity, it turns out that the singular contributions

[9] Or in between.

precisely cancel when one inverts this ordering by placing all differential operators to the left, so that the two ordering prescriptions in fact coincide! This change of ordering also reverses the order of canonically conjugate operators on the right hand side of the commutators in (4.38).

In order to be able to interpret the quantum constraints $\mathcal{S}\Psi = \widetilde{\mathcal{S}}\Psi = 0$ as square roots of the quantized bosonic constraints, and especially the Wheeler DeWitt equation $\mathcal{H}\Psi = 0$, we would like to find an operator ordering prescription such that in the quantized version of (4.38), the constraint operators on the right hand side always appear *to the right* of the field dependent structure functions. Otherwise, there will be extra contributions from the bosonic constraint operators acting on the structure functions, when (4.38) acts on a wave functional, and the constraint algebra will be "anomalous". Let us thus put all differential operators to the right as in (5.4) and recalculate the commutator (4.38), now paying attention to the order in which the operators appear. After some algebra, we arrive at the same result with exactly the ordering indicated in (4.38). In particular, the operator T is properly ordered, i.e. with the differential operators to the right; this is important because otherwise T would not only generate U(1) transformations on the fields but extra singular terms (the ordering does not matter for the Lorentz generator L_a because γ_a is traceless). We also observe that the last commutator in (4.38) remains the same if $\widetilde{\mathcal{S}}$ is placed to the left because the singular terms again cancel by virtue of the Fierz identity

$$\varepsilon^{ij}\bar{\psi}_i\gamma^a\eta\,\bar{\psi}_j\gamma_a\eta' = \varepsilon^{ij}\bar{\psi}_i\eta\,\bar{\psi}_j\eta' \tag{5.7}$$

In summary, all constraint operators appear in the desired order, except for the diffeomorphism generator, which appears in the "wrong" order, i.e. to the left of a structure function. Consequently, the equations $\mathcal{S}\Psi = \widetilde{\mathcal{S}}\Psi = 0$ imply that $\mathcal{H}\Psi = L\Psi = T\Psi = 0$, but *not* $\mathcal{H}_k\Psi = 0$. This means that a solution of the supersymmetry constraints cannot be diffeomorphism invariant, although it would satisfy the other constraints. We note that a similar "anomaly" was already encountered in [4], and was there identified as the basic reason why the solutions of [4] fail to be diffeomorphism invariant, despite the fact that they are formally annihilated by the Hamiltonian constraint. Another somewhat bothersome feature is that the ordering inside $\mathcal{H}_k$ as obtained from this commutator is precisely as in (4.34), i.e. with P^* to the left of P_k (all the other operators in $\mathcal{H}_k$ are in the "correct" order). This means that in addition to diffeomorphisms $\mathcal{H}_k$ (or rather $\mathcal{D}_k$) will generate anomalous singular terms when acting on the scalar fields.

We could now try to cancel the anomalous contribution by some intermediate reordering of the constraint operators (note that simply reverting the order of the operators will not do, because both $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ remain the same as we already explained). Commuting $\mathcal{H}_k$ through $\gamma_l = e_l{}^a \gamma_a$ to the right, we pick up a singular term

$$\epsilon^{kl} \mathcal{H}_k \, \bar{\epsilon} \gamma_l \eta = \epsilon^{kl} \bar{\epsilon} \gamma_l \eta \, \mathcal{H}_k + \left(\bar{\epsilon} \gamma^a \eta \, \bar{\lambda} \gamma_a \chi - 2 \bar{\epsilon} \gamma^a \eta \, L_a \right) \delta(\mathbf{x}, \mathbf{x}) \tag{5.8}$$

where we disregard a term involving a derivative on $\delta(\mathbf{x}, \mathbf{x})$. Here the third term can be ignored, as it is proportional to the Lorentz constraint L_a, but the second term constitutes an anomaly. It can be cancelled by a suitable reordering "inside" the constraint operators. Unfortunately, apart from the rather artificial orderings required for this cancellation to work, this procedure leads to new "anomalies", which must be cancelled in turn. We have so far not found any ordering prescription that would remove *all* anomalies and maintain the desired ordering of the quantum constraint algebra, but we anyhow would not expect that this problem can be solved simply by a clever reordering of the operators. What is really needed, but unfortunately unavailable at this point, is a properly defined scalar product on the space of physical states.

Given this state of affairs, we believe that a more reasonable option is therefore to give up diffeomorphism invariance (at least in any conventional sense) of the wave functional, replacing the the constraints $\mathcal{H}$ and $\mathcal{H}_k$ by a single (matrix valued) new constraint $\mathcal{K} := \mathcal{H} - \varepsilon^{kl} \mathcal{H}_k \gamma_l$, whose ordering is defined by the commutator of $\mathcal{S}$ with $\widetilde{\mathcal{S}}$, but whose physical significance is obscure. In the remainder, we will tentatively adopt this point of view, and discuss possible ansätze to solve the supersymmetry constraints. We have already seen that $\mathcal{S}$ is homogeneous of the first degree in the functional differential operators, while the conjugate constraint operator $\widetilde{\mathcal{S}}$ is homogeneous of the second degree, and therefore much harder to solve. We will now demonstrate how the constraint $\mathcal{S}\Psi[\phi] = 0$ can be solved in full generality by the functional analogue of the method of characteristics known from the theory of first order partial differential equations [35]. Before going into the details, however, two remarks are in order. First of all, $\Psi \equiv 1$ trivially solves all the constraints if the differential operators are placed to the right (this is not true if another and inequivalent ordering is chosen). Secondly, given one non-trivial solution (i.e. $\Psi \neq 1$), we can construct many more solutions by application of the conserved charge $\mathcal{Q}$ constructed at the end of section

3 (cf. (3.35)). By use of (3.36), one shows that

$$\Big[\mathcal{Q}\,,\,\mathcal{S}[\bar\epsilon]\Big] = \Big[\mathcal{Q}\,,\,\widetilde{\mathcal{S}}[\eta]\Big] = 0 \tag{5.9}$$

Therefore $\mathcal{Q}\Psi$ solves the supersymmetry constraints if Ψ does. It is in this sense that $\mathrm{SL}(2,\mathbf{R})$ (and the corresponding hidden symmetry groups for higher N supergravities) can be viewed as "solution generating symmetries" of the quantum constraints [9]. It would be interesting to check whether this symmetry is unitarily realized on the space of physical states, in which case one could even generate infinitely many solutions out of a given non-trivial one. However, this question again hinges on the unresolved problem of the scalar product.

In order to use the method of characteristics, we must determine the orbits in functional space generated by the action of the operator $\mathcal{S}$. This is equivalent to exponentiating an infinitesimal local supersymmetry transformation with parameter $\bar\xi = \bar\xi(\mathbf{x})$ so as to obtain the corresponding *finite* local supersymmetry transformation. Although this would be a formidable problem in general, in the case at hand the solutions can be obtained in closed form, because repeated application of the supersymmetry generator $\mathcal{S}$ on any of the fields gives zero after at most three steps. Labeling the "initial" fields by the superscript $^{(0)}$, we thus find

$$\begin{aligned}
\chi(\bar\xi) &= \chi^{(0)}, \\
\mathcal{V}(\bar\xi) &= \mathcal{V}^{(0)}(1+\bar\xi\chi^{(0)}Y^*) \\
&\quad \Rightarrow P_i^*(\bar\xi) = P_i^{*(0)} \\
&\qquad\quad \hat P_i(\bar\xi) = \hat P_i^{(0)} + \bar\xi D_i^{(0)}\chi^{(0)} \\
&\qquad\quad Q_i(\bar\xi) = Q_i^{(0)} - 2iP_i^{*(0)}\,\bar\xi\chi^{(0)} \\
A_i{}^a(\bar\xi) &= A_i^{(0)a} - 2P_i^{*(0)}\bar\xi\gamma^a\chi^{(0)} \\
\bar\psi_i(\bar\xi) &= D_i^{(0)}\bar\xi - 2P_i^{*(0)}\,(\bar\xi\chi^{(0)})\bar\xi + \bar\psi_i^{(0)}
\end{aligned} \tag{5.10}$$

Hence P_i^* and χ are inert, whereas the other fields transform in a relatively simple fashion[10] since only the gravitino field evolves with terms quadratic in $\bar\xi$. The trajectories described by the fields $\phi(\bar\xi)$ in functional space as $\bar\xi$

[10]Here we assume that all the fields are complexified to make the *chiral* supersymmetry transformation well defined

is varied are the orbits under local supersymmetry. This fact is reflected in the identity

$$\mathcal{S}[\bar{\epsilon}, \phi(\bar{\xi})]\Psi[\phi(\bar{\xi})] = \int \mathrm{d}^2\mathbf{x}\, \bar{\epsilon}(\mathbf{x}) \frac{\delta}{\delta\bar{\xi}(\mathbf{x})} \Psi[\phi(\bar{\xi})] \tag{5.11}$$

The most general solution of the quantum constraint $\mathcal{S}\Psi = 0$ is now obtained by choosing Ψ such that $\frac{\delta}{\delta\bar{\xi}}\Psi[\phi(\bar{\xi})] = 0$, i.e. constant along the trajectories given by the "evolution equations" (5.10), and by prescribing arbitrary values of Ψ on some (infinite dimensional) hypersurface in functional space which is nowhere tangent to the supersymmetry orbits.

In practice, choosing a functional hypersurface amounts to imposing some gauge condition on the gravitino. For example, let us choose $c^i\bar{\psi}_i = 0$, where $c^i(\mathbf{x})$ is an arbitrary non-vanishing vector field. Given an arbitrary configuration of the fields ϕ (which, in general, will not satisfy the gauge condition), we must first determine to which supersymmetry orbit it belongs. This requires solving the first order partial differential equation for $\bar{\xi}$

$$c^i\bar{\psi}_i(\bar{\xi}) = c^i\Big(D_i\bar{\xi} - 2P_i^*(\bar{\xi}\chi)\bar{\xi} + \bar{\psi}_i\Big) = 0 \tag{5.12}$$

We note the following subtleties about this (and similar) gauge conditions. For arbitrary and topologically non-trivial space-like surfaces, the vector field $c^i(\mathbf{x})$ will in general have zeros, at which the gauge condition degenerates. Secondly, a given $\bar{\psi}_i$ is gauge equivalent to a configuration satisfying $c^i\bar{\psi}_i = 0$ if and only if the solution $\bar{\xi}$ is single valued on the spacelike surface, i.e. obeys $\oint_\gamma dx^i\partial_i\bar{\xi} = 0$ for any closed curve γ. Then $\bar{\xi}$, which depends on the initial fields ϕ, is the finite supersymmetry transformation parameter connecting the given configuration of fields to the gauge hypersurface. We now simply define

$$\Psi[\phi] := \Psi[\phi(\bar{\xi})] \tag{5.13}$$

with $\bar{\xi}$ from (5.12), where $\Psi[\phi(\bar{\xi})]$ is given by the previously assigned value of Ψ on the gauge hypersurface. Incidentally, the differential equation (5.12) is again solved by the method of characteristics, but now in ordinary, not in functional space. Consequently, the transformation parameter $\bar{\xi}$ is completely determined by (5.12) only after specification of suitable initial values.

To gain a somewhat different perspective (and possibly also to establish a connection with the work of [4, 5]), we will now consider wave functionals that do not depend on the fields on the whole spatial surface, but are supported only on given set of curves $c(s)$ determined from the equations

$\dot{x}^i(s) = c^i(\mathbf{x}(s))$ where $c^i(\mathbf{x})$ is the vector field introduced in (5.12) (the integral curves $x^i(s)$ are just the characteristics of (5.12)). Along $c(s)$, we now consider the $\mathrm{SL}(2,\mathbf{R})\times\mathrm{SO}(2)$ valued gauge potential (remember SO(2) is just the helicity group of the $N=1$ supergravity in four dimensions)

$$\mathcal{A}_i := \tfrac{1}{2}\gamma_a\left(A_i{}^a - 2P_i^*\bar{\xi}\gamma^a\chi\right) + \tfrac{1}{2}i\left(Q_i - 2iP_i^*\bar{\xi}\chi\right) \tag{5.14}$$

where $\bar{\xi} = \bar{\xi}(\phi)$ is determined from (5.12). Note that the connection $\mathcal{A}_i$ also depends on the bosonic and fermionic matter fields; it is just the gauge potential occurring in the covariant derivative on $\bar{\psi}_i$ shifted by a finite supersymmetry transformation according to (5.10). Next introduce the path ordered integral

$$T_c(\mathbf{a},\mathbf{b}) = \mathcal{P}\,\exp\int_0^1 ds\, c^i(\mathbf{x}(s))\mathcal{A}_i(\mathbf{x}(s)) \tag{5.15}$$

where the curve c is parametrized such that $c(0)=\mathbf{a}$ and $c(1)=\mathbf{b}$ are the initial and end points connected by the curve $c(s)$, respectively. Computing the variation of T_c under local supersymmetry, we obtain

$$\Big[\mathcal{S}[\bar{\epsilon}]\,,\,T_c(\mathbf{a},\mathbf{b})\Big] = \int_0^1 ds\, c^i(\mathbf{x}(s))T_c(\mathbf{a},\mathbf{x}(s))\,\delta\mathcal{A}_i(\mathbf{x}(s))\,T_k(\mathbf{x}(s),\mathbf{b}) \tag{5.16}$$

with

$$\delta\mathcal{A}_i = -\gamma_a P_i^*\bar{\epsilon}\gamma^a\chi + P_i^*\bar{\epsilon}\chi + P_i^*\left(\delta\bar{\xi}\chi - \delta\bar{\xi}\gamma^a\chi\,\gamma_a\right) \tag{5.17}$$

Here, the variation $\delta\bar{\xi}$ must be determined from equation (5.12). Namely, varying (5.12), we find that after a little algebra and use of the identity $2\bar{\xi}\chi\,\bar{\epsilon} = -\bar{\epsilon}\chi\,\bar{\xi} - \bar{\epsilon}\gamma^a\chi\,\bar{\xi}\gamma_a$, this equation reduces to

$$c^i D_i(\mathcal{A}_i(\bar{\xi}))\Big(\delta\bar{\xi} + \bar{\epsilon}\Big) = 0 \tag{5.18}$$

The solution is thus

$$\delta\bar{\xi}(\mathbf{x}) = -\bar{\epsilon}(\mathbf{x}) + (\delta\bar{\xi}+\bar{\epsilon})(\mathbf{a})T_c(\mathbf{a},\mathbf{x}) \tag{5.19}$$

where $\mathbf{x}=\mathbf{x}(s)$ is any point along the curve. If the initial value $\bar{\xi}_0 \equiv \bar{\xi}(\mathbf{a})$ is chosen in such a way that $\delta\bar{\xi}_0 + \bar{\epsilon}(\mathbf{a}) = 0$, the gauge potential $\mathcal{A}_i$ is invariant, and the right hand side of (5.16) vanishes. A solution, which is also invariant under the $U(1)$ and Lorentz constraints is then easily obtained by closing the curve into a loop and taking the trace. These solutions can be regarded as "supercovariant" extensions of the solutions found in [4]. We have not checked, however, whether they are also annihilated by the second order constraint $\widetilde{\mathcal{S}}$. We note that for $\delta\bar{\xi}(\phi) = -\bar{\epsilon}$, one can construct many other invariants: since all the $\phi(\bar{\xi})$ in (5.10) are invariant with this choice of $\bar{\xi}$, an arbitrary functional of them will also be invariant. In this respect, there is nothing special about loop functionals as opposed to functionals that are supported on the whole space-like surface.

Appendix A: Decoupling of Canonical Variables for $N > 2$

In this appendix we explain how the redefinition of the spin connection works for the higher N theories. We will not discuss these models in detail here, but refer the reader to [28] for further explanations. In analogy with (4.8), we proceed from the ansatz

$$A'^{a}_{\mu} := A_{\mu}^{\ a} + B_{\mu}^{\ a}, \qquad B_{\mu}^{\ a} := \tfrac{1}{2} i\varepsilon^{abc} e_{\mu b}\, \bar{\chi}^{\dot{A}} \gamma_c \chi^{\dot{B}}\, \mathcal{J}_{\dot{A}\dot{B}}, \tag{A.1}$$

where $\chi^{\dot{A}}$ are the matter fermions, which transform as spinors under H. Observe that the new spin connection will be *complex* just as for $N = 2$. $\mathcal{J}_{\dot{A}\dot{B}}$ is an antisymmetric matrix to be determined by requiring decoupling of the canonical variables; we will find that $\mathcal{J}_{\dot{A}\dot{B}}$ is a complex structure, i.e. $\mathcal{J}^2 = -\mathbf{1}$. Inserting this ansatz into (2.3), we obtain

$$F_{\mu\nu a} = F'_{\mu\nu a} - 2\, D'_{[\mu} B_{\nu]}^{\ a} - \varepsilon_{abc} B_{\mu}^{\ b} B_{\nu}^{\ c}. \tag{A.2}$$

where the derivative D'_{μ} is covariant with respect to the Lorentz group $\mathrm{SO}(2,1)$ and the gauge group H (see [28] for the precise definitions). As in section 4, we may replace the original spin connection by A'^{a}_{μ} in the Lagrangian. Up to a total derivative we then have

$$\begin{aligned} \tfrac{1}{4}\varepsilon^{\mu\nu\rho} e_{\mu}^{\ a} F_{\nu\rho a} &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho} e_{\mu}^{\ a} F'_{\nu\rho a} - \tfrac{1}{4} i e e^{\mu}_{\ a}\, D'_{\mu}(\, \bar{\chi}^{\dot{A}} \gamma^a \chi^{\dot{B}}\, \mathcal{J}_{\dot{A}\dot{B}}) \\ &\qquad - \tfrac{1}{4} e\, \bar{\chi}^{\dot{A}} \chi^{\dot{B}}\, \mathcal{J}_{\dot{B}\dot{C}}\, \bar{\chi}^{\dot{C}} \chi^{\dot{D}}\, \mathcal{J}_{\dot{D}\dot{A}}, \\ \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\, \bar{\psi}^I_{\mu} D_{\nu} \psi^I_{\rho} &= \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\, \bar{\psi}^I_{\mu} D'_{\nu} \psi^I_{\rho} - \tfrac{1}{4} i \varepsilon^{\mu\nu\rho}\, \bar{\chi}^{\dot{A}} \psi^I_{\mu}\, \bar{\chi}^{\dot{B}} \gamma_{\nu} \psi_{\rho}\, \mathcal{J}_{\dot{A}\dot{B}}. \end{aligned} \tag{A.3}$$

The contribution to the kinetic term of the fermion is slightly more complicated than in the $N = 2$ case, because the covariant derivative acts on $\mathcal{J}$, too:

$$
\begin{aligned}
D'_\mu(\bar\chi^{\dot A}\gamma^a\chi^{\dot B}\,\mathcal{J}_{\dot A\dot B}) &= 2\bar\chi^{\dot A}\gamma^a D'_\mu\chi^{\dot B}\,\mathcal{J}_{\dot A\dot B} + \bar\chi^{\dot A}\gamma^a\chi^{\dot B}\,D_\mu\mathcal{J}_{\dot A\dot B},\\
D_\mu\mathcal{J}_{\dot A\dot B} &= \tfrac{1}{4}Q_\mu^{IJ}\,[\Gamma^{IJ},\mathcal{J}]_{\dot A\dot B} + Q_\mu^\alpha\,[\Gamma^\alpha,\mathcal{J}]_{\dot A\dot B}.
\end{aligned} \tag{A.4}
$$

If the last expression does not vanish, i.e. if $\mathcal{J}$ does not commute with the whole gauge group H, there will be a mixing of complexified spinors and their Dirac conjugates. Since $\mathcal{J}$ generates a U(1) subgroup of H, $D_\mu\mathcal{J}_{\dot A\dot B}=0$ whenever H has a U(1) factor, i.e. whenever the target space is a Kaehler manifold. This is the case for the theories with $N=2$ mod 4 [28]. For $N\neq 2$ mod 4, manifest invariance under those generators of H which do not commute with $\mathcal{J}$ is lost. For instance, the manifest SO(16) invariance of $N=16$ supergravity is thereby broken to SU(8) $\times$ U(1).

Inserting everything into the Lagrangian given in [28] and dropping the primes on the redefined spin connection, we get

$$
\begin{aligned}
\mathcal{L}^{(0)} &= \tfrac{1}{4}\varepsilon^{\mu\nu\rho}e_\mu^{\ a}F_{\nu\rho a} + \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\bar\psi_\mu^I D_\nu\psi_\rho^I\,,\\
\mathcal{L}^{(1)} &= -\tfrac{1}{4}eg^{\mu\nu}\hat P_{A\mu}\hat P_{A\nu} + \tfrac{1}{2}\varepsilon^{\mu\nu\rho}\hat P_{A\mu}\Gamma^I_{A\dot C}\,\bar\chi^{\dot C}\gamma_\rho\psi_\nu^I\\
&\qquad + \tfrac{1}{4}\varepsilon^{\mu\nu\rho}\,(\Gamma^I_{A\dot B}\Gamma^J_{A\dot C} + i\delta^{IJ}\mathcal{J}_{\dot B\dot C})\,\bar\chi^{\dot B}\psi_\mu^I\,\bar\chi^{\dot C}\gamma_\rho\psi_\nu^J,\\
\mathcal{L}^{(2)} &= -\tfrac{1}{2}e\,\bar\chi^{\dot A}\gamma^\mu D_\mu\chi^{\dot B}\,(\delta_{\dot A\dot B} + i\mathcal{J}_{\dot A\dot B}) - \tfrac{1}{4}\,\bar\chi^{\dot A}\chi^{\dot B}\,\mathcal{J}_{\dot B\dot C}\,\bar\chi^{\dot C}\chi^{\dot D}\,\mathcal{J}_{\dot D\dot A}\\
&\qquad - \tfrac{1}{4}ie\,\bar\chi^{\dot A}\gamma^\mu\chi^{\dot B}\,D_\mu\mathcal{J}_{\dot A\dot B} + \text{ other } \chi^4 \text{ terms}\,.
\end{aligned} \tag{A.5}
$$

where we have decomposed the Lagrangian as in (4.5).

The momenta of the dreibein, spin connection, gravitinos and fermions are now

$$
\begin{aligned}
p_a^{\ i} &= \frac{\delta\mathcal{L}}{\delta\dot e_i^{\ a}} = 0,\\
\mathit{\Pi}_a^{\ i} &= \frac{\delta\mathcal{L}}{\delta\dot A_i^{\ a}} = \tfrac{1}{2}\varepsilon^{ij}e_{ja},\\
\bar\pi^{iI} &= \frac{\delta\mathcal{L}}{\delta\dot\psi_i^I} = -\tfrac{1}{2}\varepsilon^{ij}\bar\psi_j^I,\\
\bar\lambda_{\dot A} &= \frac{\delta\mathcal{L}}{\delta\dot\chi^{\dot A}} = \tfrac{1}{2}\bar\chi^{\dot B}\,e\gamma^t\,(\delta_{\dot B\dot A} + i\mathcal{J}_{\dot B\dot A}).
\end{aligned} \tag{A.6}
$$

Observe that $\bar\lambda_{\dot A}$ is complex and thus no longer a Majorana spinor. The Poisson brackets read

$$
\begin{aligned}
\{e_i{}^a, p_b{}^j\} = \delta^a_b \delta^j_i, \qquad \{A_i{}^a, \Pi_b{}^j\} = \delta^a_b \delta^j_i, \\
\{\bar\lambda_{\dot A\alpha}, \chi^{\dot B}_\beta\} = -\delta^{\dot B}_{\dot A}\delta_{\alpha\beta}, \qquad \{\bar\pi^{iI}_\alpha, \psi^I_{j\beta}\} = -\delta^i_j \delta^{IJ}\delta_{\alpha\beta},
\end{aligned}
\tag{A.7}
$$

and the full set of second class constraints is

$$
\begin{aligned}
P^i_a &:= p_a{}^i, \\
Z^i_a &:= \Pi_a{}^i - \tfrac{1}{2}\varepsilon^{ij} e_{ja}, \\
\bar\Lambda_{\dot A\alpha} &:= \bar\lambda_{\dot A\alpha} - \tfrac{1}{2}\,\bar\chi^{\dot B}_\beta (C\, e\gamma^t)_{\beta\alpha}\,(\delta_{\dot B\dot A} + i\mathcal{J}_{\dot B\dot A}), \\
\bar\Gamma^{iI}_\alpha &:= \bar\pi^{iI}_\alpha + \tfrac{1}{2}\varepsilon^{ij}\,\psi^I_{j\beta} C_{\beta\alpha}.
\end{aligned}
\tag{A.8}
$$

To obtain the Dirac brackets one has to invert the matrix of Poisson brackets of these constraints. The nonvanishing components of this matrix are

$$
\begin{aligned}
\{P^i_a, Z^j_b\} &= -\tfrac{1}{2}\eta_{ab}\varepsilon^{ij}, \\
\{P^i_a, \bar\Lambda_{\dot B\beta}\} &= -\tfrac{1}{2}\,\chi^{\dot A}_\alpha\,(C\gamma^c)_{\alpha\beta}\,(\delta_{\dot A\dot B} + i\mathcal{J}_{\dot A\dot B})\,\varepsilon_{abc}\varepsilon^{ij} e_j{}^b, \\
\{\bar\Lambda_{\dot A\alpha}, \bar\Lambda_{\dot B\beta}\} &= (C\, e\gamma^t)_{\alpha\beta}\,\delta_{\dot A\dot B}, \\
\{\bar\Gamma^{iI}_\alpha, \bar\Gamma^{jJ}_\beta\} &= \varepsilon^{ij}\delta^{IJ} C_{\alpha\beta}.
\end{aligned}
\tag{A.9}
$$

The inverse matrix is found to be

$$
\begin{aligned}
C(P^i_a, Z^j_b) &= 2\varepsilon_{ij}\eta^{ab}, \\
C(Z^i_a, Z^j_b) &= -\varepsilon_{ij}\, h^{-1}\, ee^{ta}\, ee^{tb}\,\bar\chi^{\dot A}\chi^{\dot B}\,(\delta_{\dot A\dot B} + \mathcal{J}_{\dot A\dot C}\mathcal{J}_{\dot C\dot B}), \\
C(Z^i_a, \bar\Lambda_{\dot A\alpha}) &= h^{-1}\varepsilon^{abc} e_{ib}\,(\delta_{\dot A\dot B} - i\mathcal{J}_{\dot A\dot B})(e\gamma^t\, C)_{\alpha\beta}\,\chi^{\dot B}_\beta, \\
C(\bar\Lambda_{\dot A\alpha}, \bar\Lambda_{\dot B\beta}) &= -\delta_{\dot A\dot B}\, h^{-1}\,(e\gamma^t\, C^{-1})_{\alpha\beta}, \\
C(\bar\Gamma^{iI}_\alpha, \bar\Gamma^{jJ}_\beta) &= -\varepsilon_{ij}\,\delta^{IJ}\, C^{-1}_{\alpha\beta}.
\end{aligned}
\tag{A.10}
$$

The bracket between $A_i{}^a$ and $A_j{}^b$ is easily seen to vanish if $C(Z^i_a, Z^j_b)$ vanishes. This is the case if and only if $\mathcal{J}_{\dot A\dot B}\mathcal{J}_{\dot B\dot C} = -\delta_{\dot A\dot C}$. Hence $\mathcal{J}_{\dot A\dot B}$ is indeed a complex structure as previously asserted. With $\mathcal{J}^2 = -\mathbf{1}$, the combinations

$$\mathcal{P}^{\pm}_{\dot{A}\dot{B}} := \tfrac{1}{2}\left(\delta_{\dot{A}\dot{B}} \pm i\mathcal{J}_{\dot{A}\dot{B}}\right) \tag{A.11}$$

become projection operators acting on the fermions. This is important because only then the number of physical fermionic degrees of freedom stays the same after complexification: we are simply trading d-dimensional real spinors for $\frac{d}{2}$-dimensional complex ones. However, as already pointed out above, the complexified spinors will no longer transform linearly under the group H unless $N = 2 \bmod 4$.

Using the formula (4.15) we obtain

$$\{A_i{}^a, A_j{}^b\}_* = 0, \qquad \{e_i{}^a, e_j{}^b\}_* = 0,$$
$$\{A_i{}^a, e_j{}^b\}_* = 2\varepsilon_{ij}\eta^{ab}. \tag{A.12}$$

Furthermore,

$$\{A_i{}^a, \bar{\lambda}_{\dot{A}\alpha}\}_* = 0, \quad \{\bar{\lambda}_{\dot{A}\alpha}, \bar{\lambda}_{\dot{B}\beta}\}_* = 0. \tag{A.13}$$

This shows that the $\bar{\lambda}_{\dot{A}}$'s are indeed good canonical variables as their brackets decouple from $A_i{}^a$. On the other hand, the original real spinors $\chi^{\dot{A}}$ are *not*, as they mix with each other and the spin connection under Dirac brackets. However, the correct variables are now easy to guess; they are

$$\eta^{\dot{A}} := \mathcal{P}^{+}_{\dot{A}\dot{B}}\chi^{\dot{B}} = \tfrac{1}{2}\left(\delta_{\dot{A}\dot{B}} + i\mathcal{J}_{\dot{A}\dot{B}}\right)\chi^{\dot{B}}, \tag{A.14}$$

and are related to $\bar{\lambda}_{\dot{A}}$ by $\bar{\lambda}_{\dot{A}} = \bar{\eta}^{\dot{A}} e\gamma^t$, where the bar on η denotes Dirac conjugation. It is now straightforward to check that

$$\{A_i{}^a, \eta^{\dot{A}}_\alpha\}_* = 0, \quad \{\eta^{\dot{A}}_\alpha, \eta^{\dot{B}}_\beta\}_* = 0, \quad \{\eta^{\dot{A}}_\alpha, \bar{\lambda}_{\dot{B}\beta}\}_* = -\delta^{\dot{A}}_{\dot{B}}\delta_{\alpha\beta}. \tag{A.15}$$

This completes the proof that the canonical variables can be decoupled by the redefinition (A.1)

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